LPA : Finer Points

MCB Course, UBC
William R. Holmes

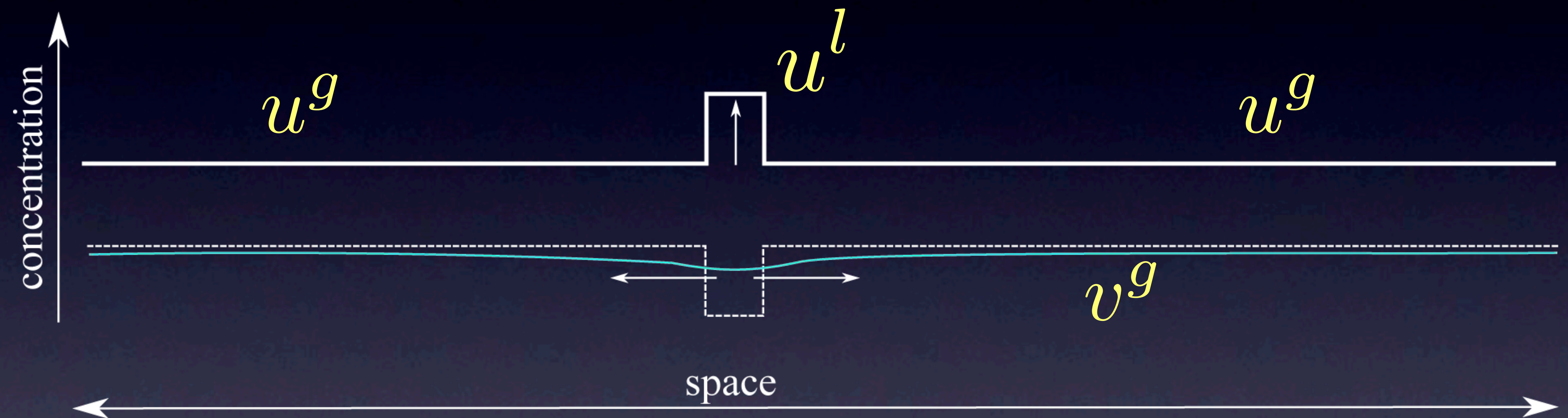
Topics of Discussion

- Relationship between LPA results and Turing analysis.
- How the LPA informs knowledge of long term evolution of spatial patterns.

LPA Setup

Slow Diffusing	$\frac{\partial u}{\partial t}(x,t) = f(u,v;p) + \epsilon u_{xx}$
Fast Diffusing	$\frac{\partial v}{\partial t}(x,t) = g(u,v;p) + D v_{xx}$

Local Perturbation Analysis



- ‘g’ indicates a global variable, ‘l’ indicates local.

LP - System

Global: $$\frac{du^g}{dt}(x,t) = f(u^g, v^g; p),$$

Global: $$\frac{dv^g}{dt}(x,t) = g(u^g, v^g; p),$$

Local: $$\frac{du^l}{dt}(x,t) = f(u^l, v^g; p)$$

LP Perturbation Restriction

- The applied 'local' perturbation can be of ARBITRARY HEIGHT, but it must be of SMALL AREA.

Why?

Consider: $$\frac{dv^g}{dt}(x,t) = g(u, v^g; p)$$

- Since v^g is spatially homogeneous, integrate.

$$\frac{dv^g}{dt} = \frac{1}{2}\int_{-1}^{1} g(u, v; p)\, dx$$

Continued

$u \approx u^g$ $\quad\quad$ $u \approx u^l$

$$\frac{dv^g}{dt} = \frac{1}{2}\int_{|x|>\epsilon} g(u, v^g; p)\, dx + \int_{|x|<\epsilon} g(u, v^g; p)\, dx$$

Away from the perturbation $\quad\quad$ Near Perturbation

- Now assume,
  - ϵ is small
  - $g(u^l, v^g; p) \sim O(1)$

Then

$$\frac{du^g}{dt} \approx g(u^g, v^g; p) + \epsilon(g(u^l, v^g; p) - g(u^g, v^g; p))$$

Small

- So those two assumptions are enough to ensure the equation for v^g is valid.
- Together, they imply the applied perturbation must be of SMALL AREA.

LPA vs Linearization Methods

- Well mixed and Turing analysis require a perturbation to be small in height, but spatially spread.
- LPA requires a perturbation of SMALL AREA, but it can be tall!

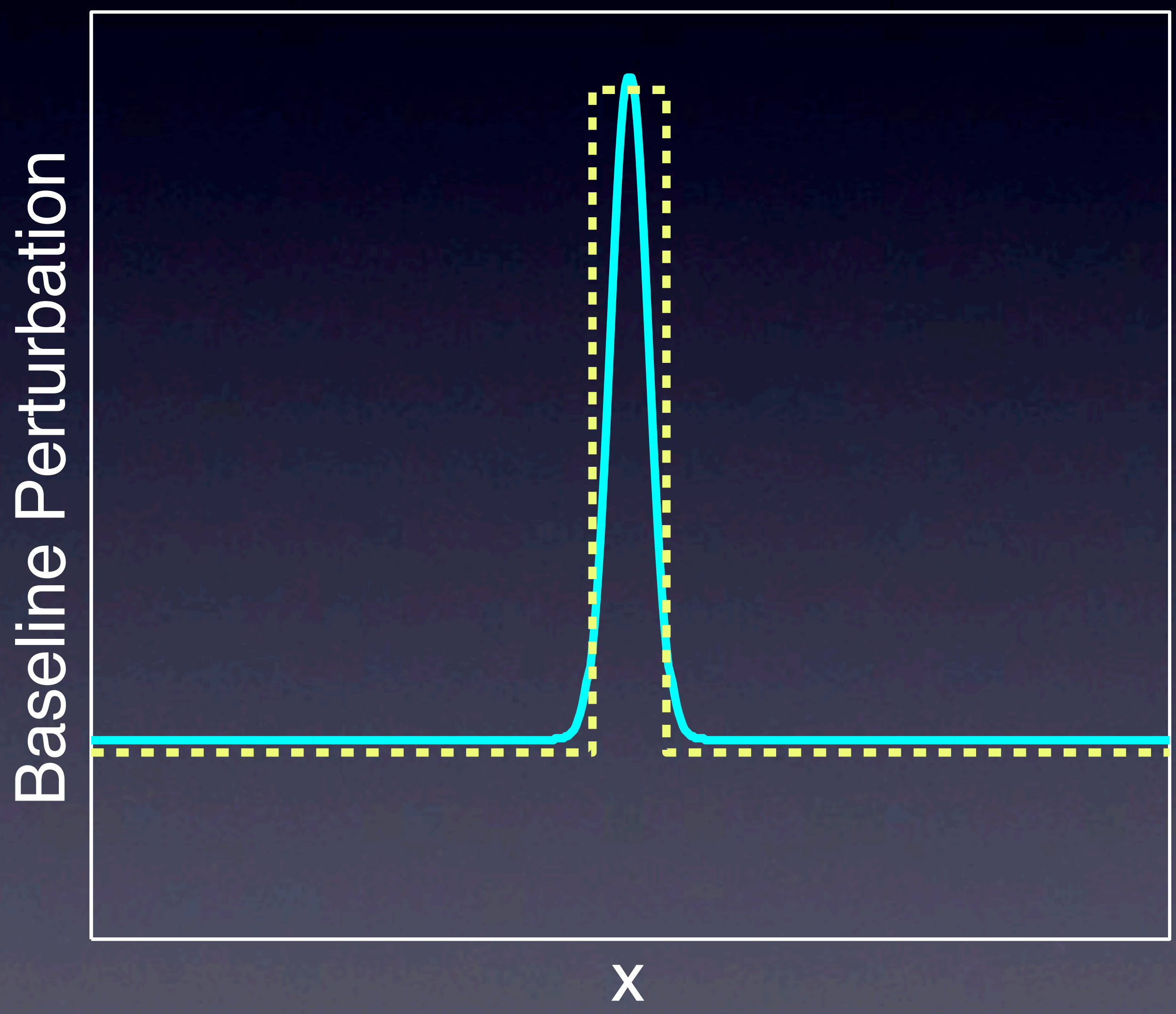
LPA - Linear and Nonlinear stability

- The LPA recovers stability properties of the
  1. Well mixed system - previously discussed
  2. Turing system - to be discussed

Properties of a Local Perturbation

$\delta-$ function like

$$\delta_\sigma = \exp\left(\frac{x^2}{\sigma^2}\right)$$



- Our local perturbation is akin to a delta function

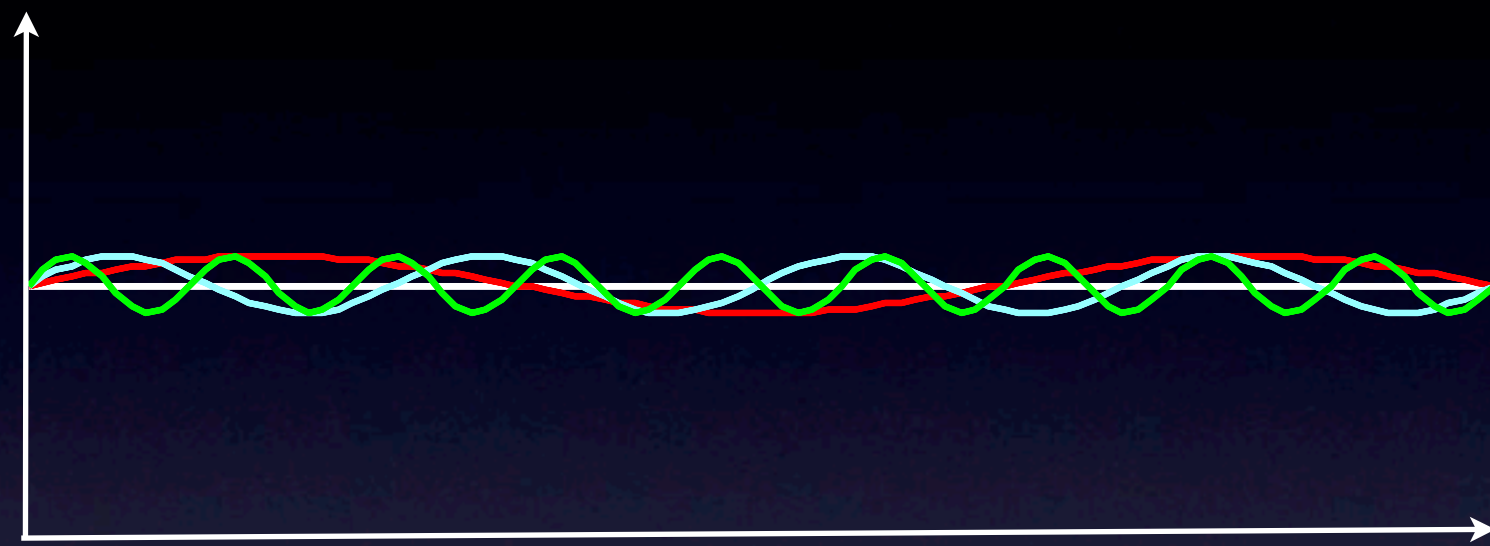
Frequency Spectrum

$$\mathcal{F}[\delta_\sigma](k) = \sqrt{\sigma k}\exp\left(-\sigma^2 k^2\right)$$

- The spatially local perturbation has a broad Fourier spectrum.
- In the infinitely localized limit, the delta function is the superposition of all wave numbers:

$$\mathcal{F}[\delta](k) = 1$$

Implication



- The LPA tests stability against all wave numbers at once.
- In this way it provides a wave number independent Turing analysis.

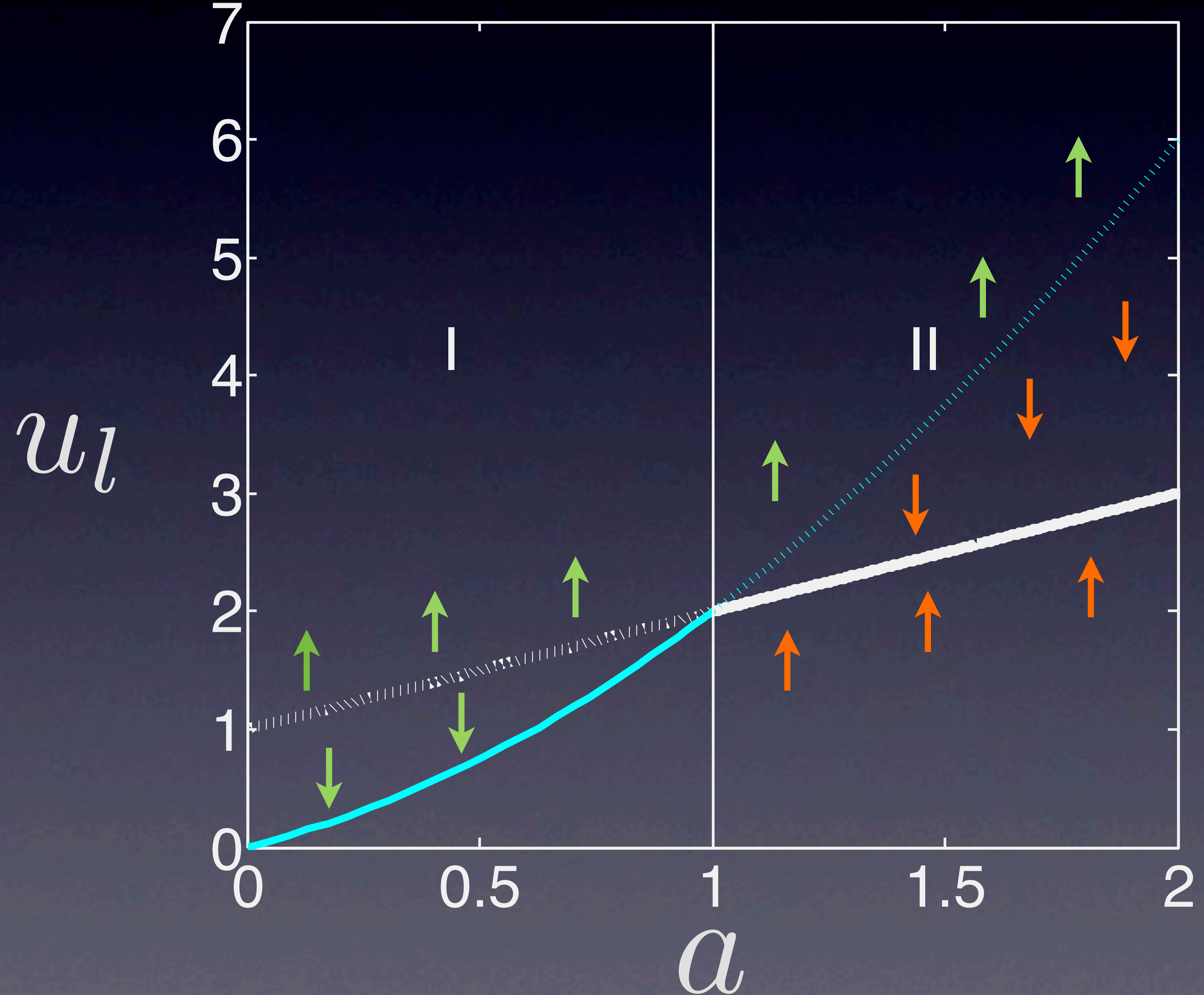
Conclusion

- The LPA really does recapitulate
    1. Well mixed analysis results
    2. AND Turing analysis results

LP Diagram Interpretation

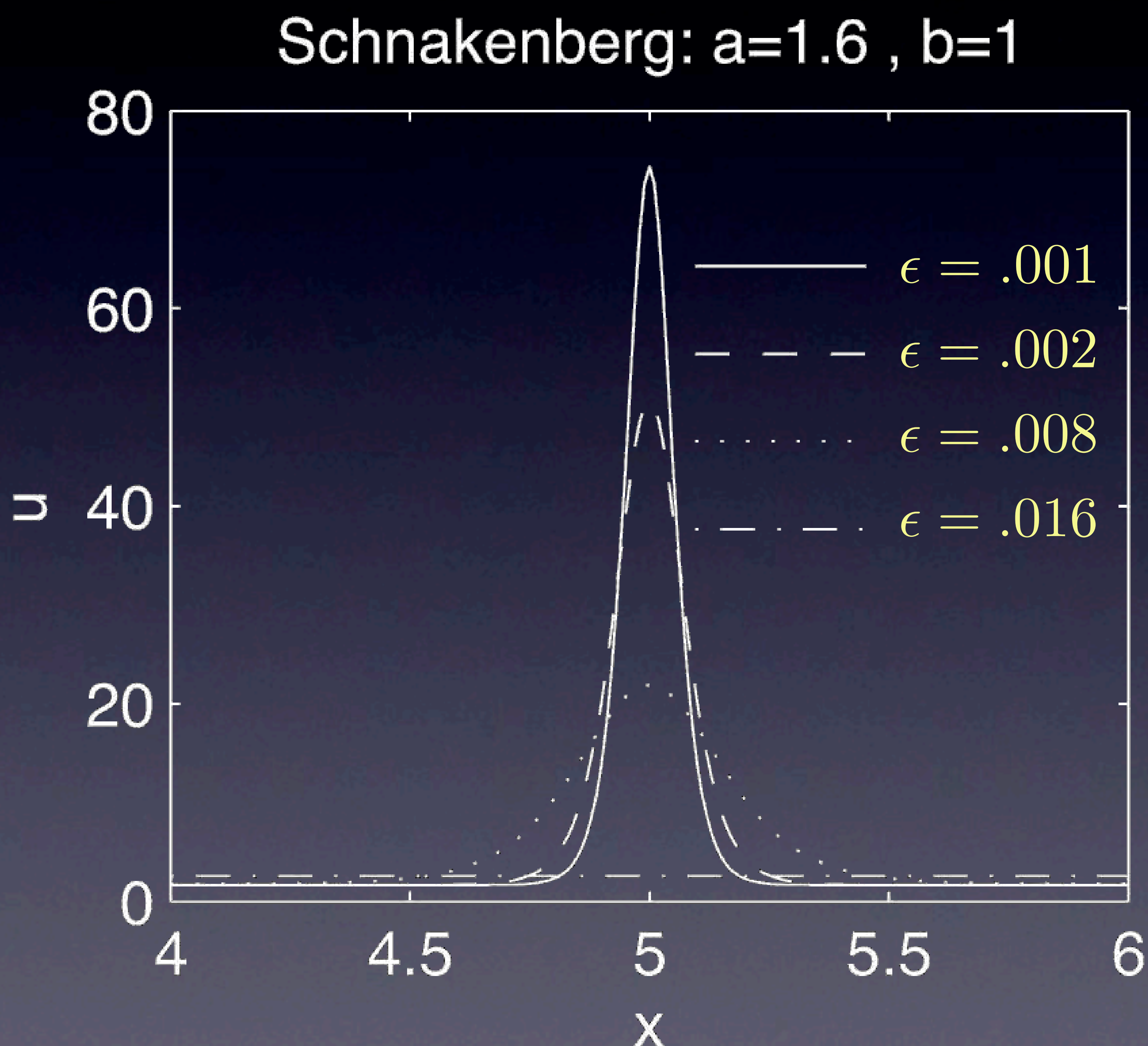
- The LPA does not directly predict the long term evolution of patterns.
- BUT, it can help infer the structure of a pattern.

Schnakenberg LPA

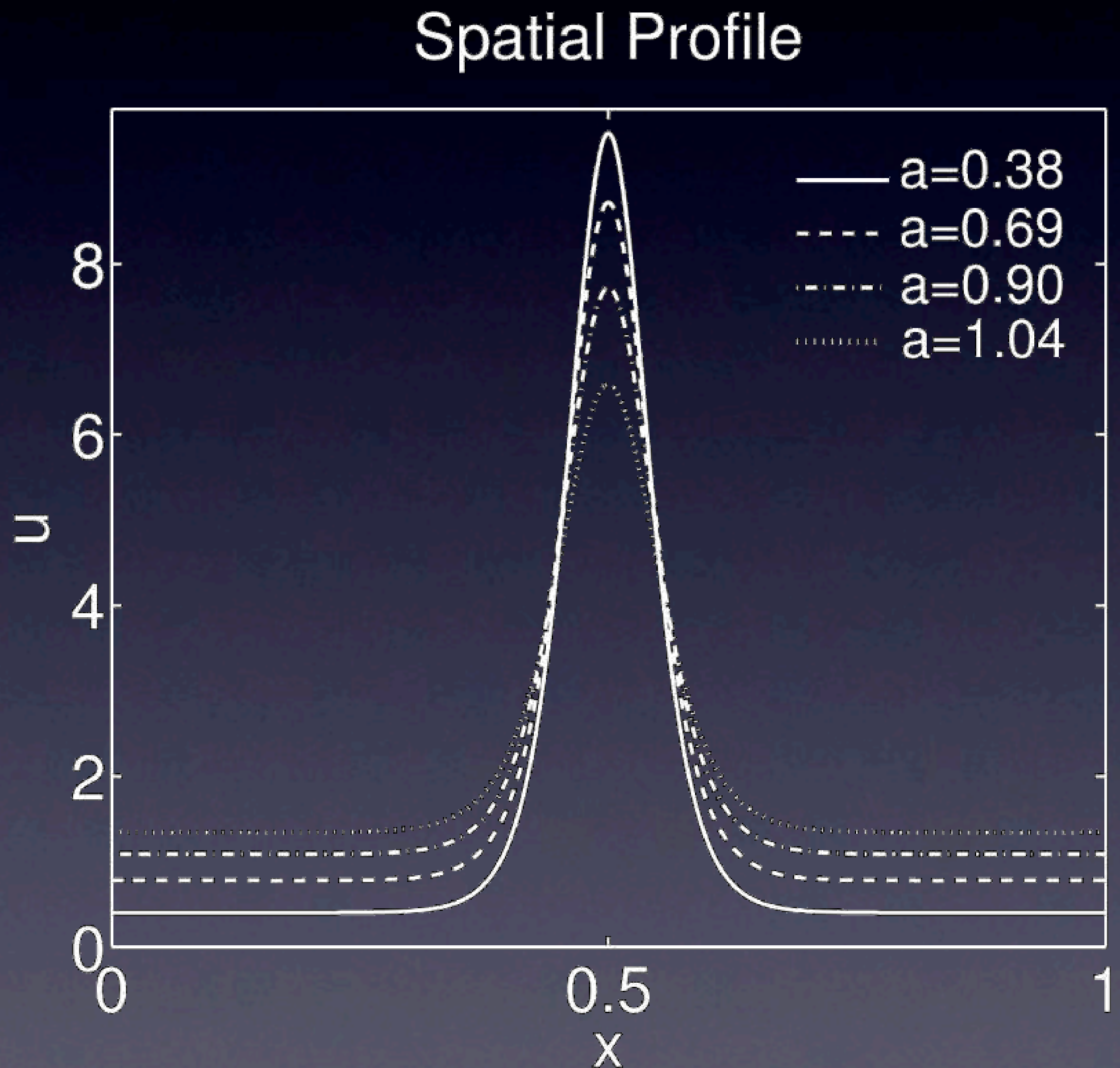


- Consider region II.
- Sufficiently large perturbations grow to infinity.
- Diffusion mitigates growth and produces spikes.

Spike Height Dependence



Turing vs Excitable



- The pattern resulting in the Turing and excitable regimes is the same!

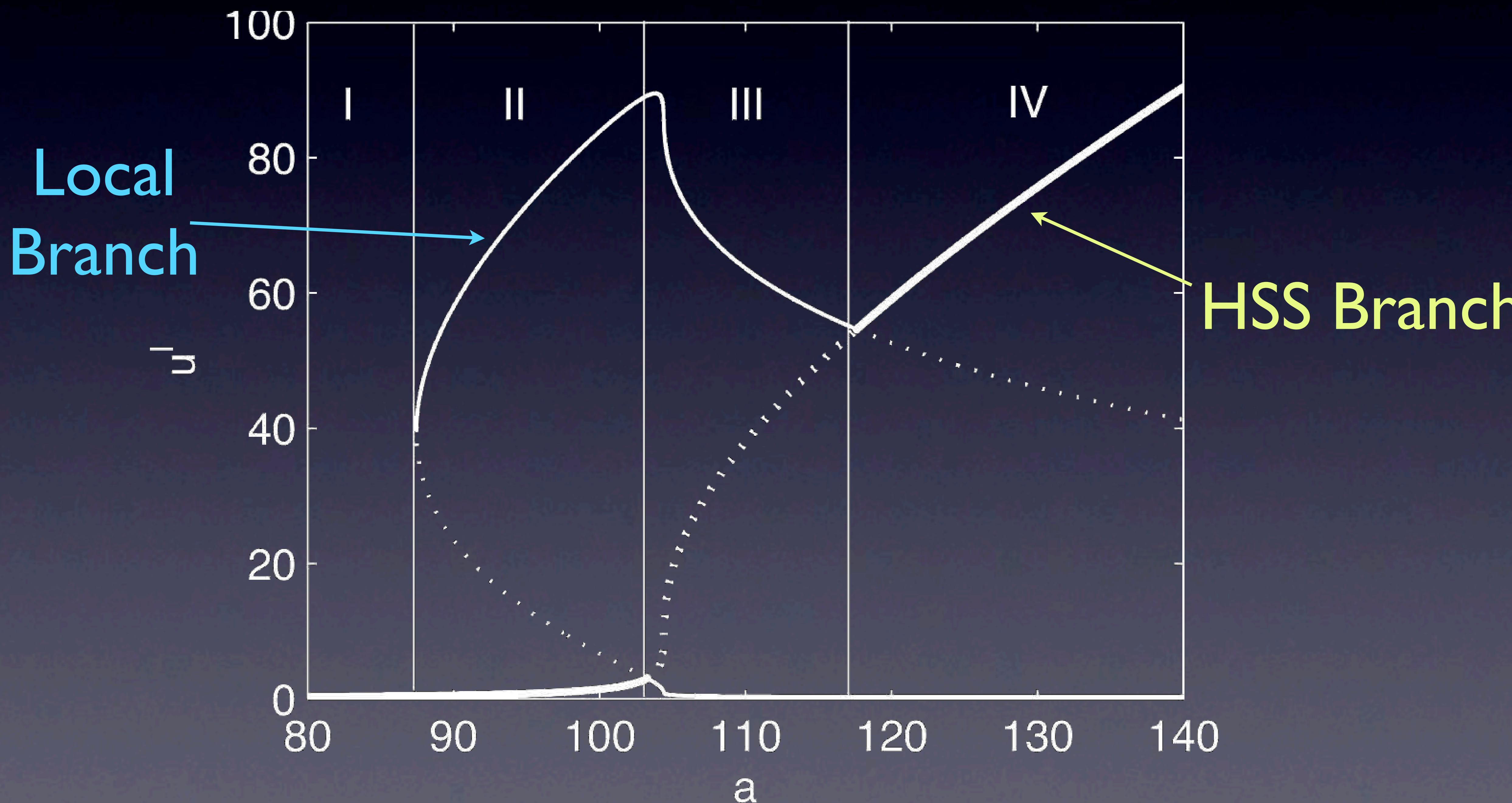
Substrate Inhibition

$$u_t(x,t) = a - u - \frac{\rho u v}{1 + u + K u^2} + \epsilon \triangle u,$$

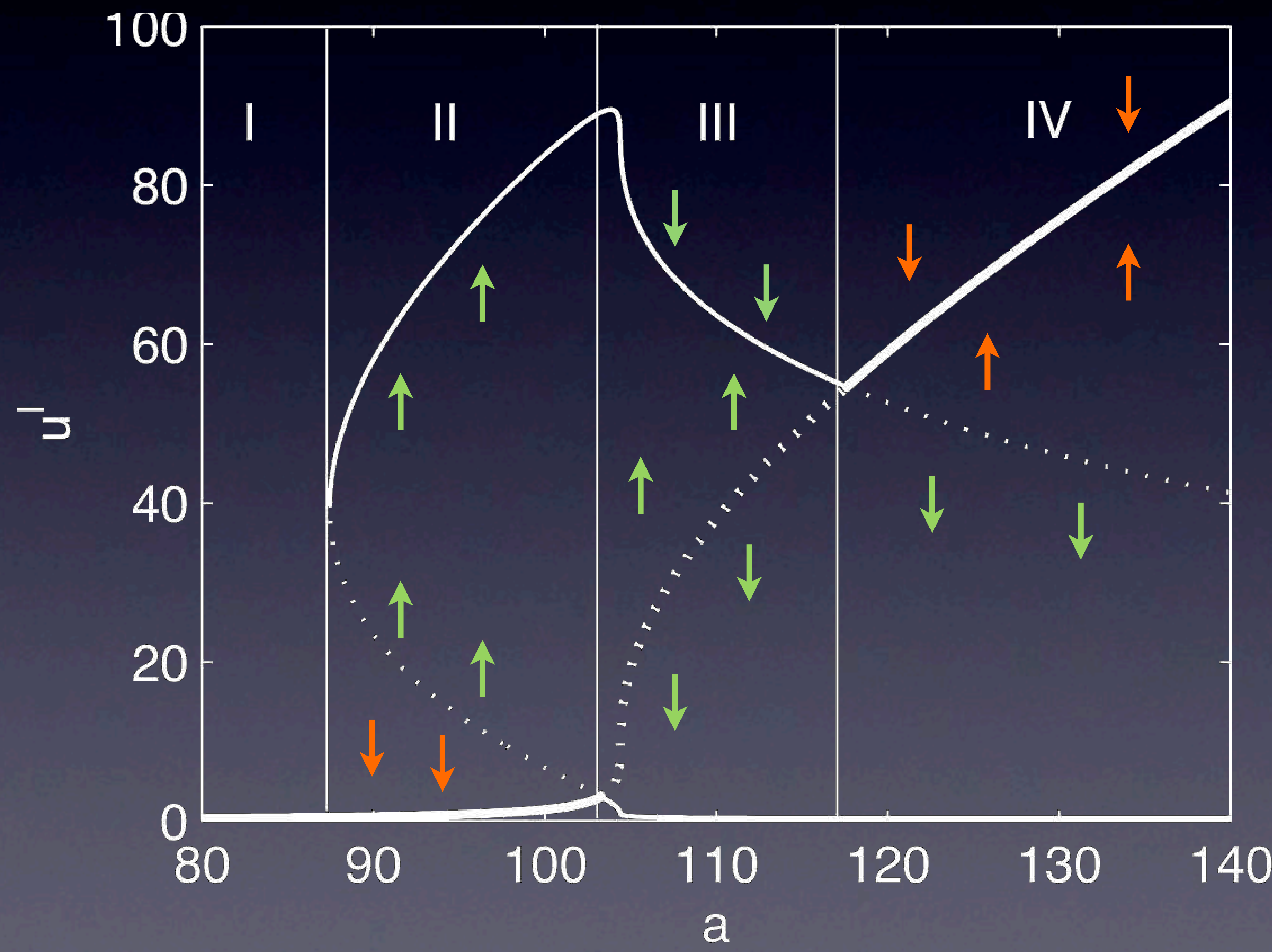
$$v_t(x,t) = \alpha(b - v) - \frac{\rho u v}{1 + u + K u^2} + D \triangle v$$

- 'u' and 'v' are co-substrates that consume each other in a enzymatic reaction.
- Nonlinearity indicative of 'u' binding to enzyme and rendering it inert.

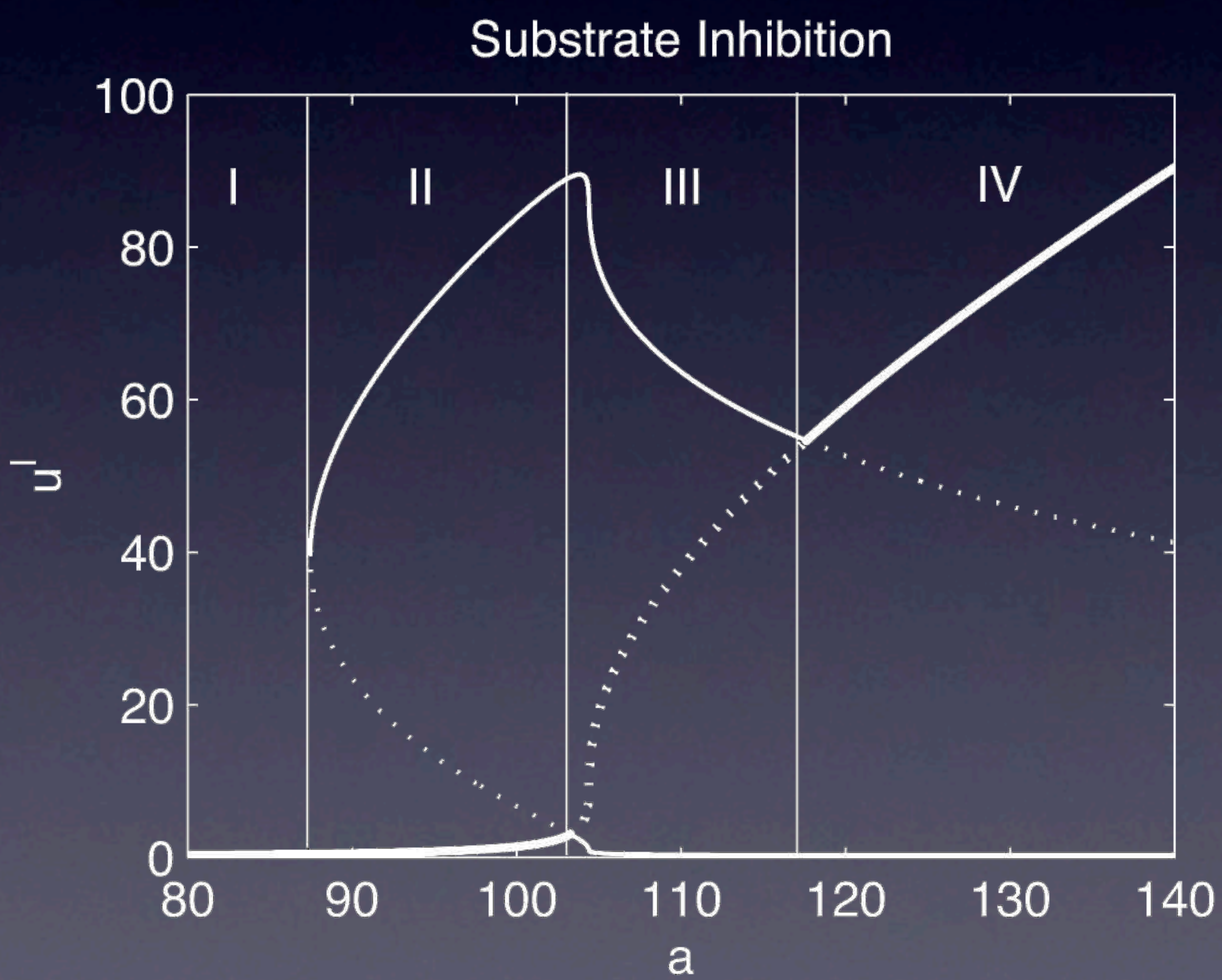
Substrate Inhibition
LPA



Substrate Inhibition
LPA

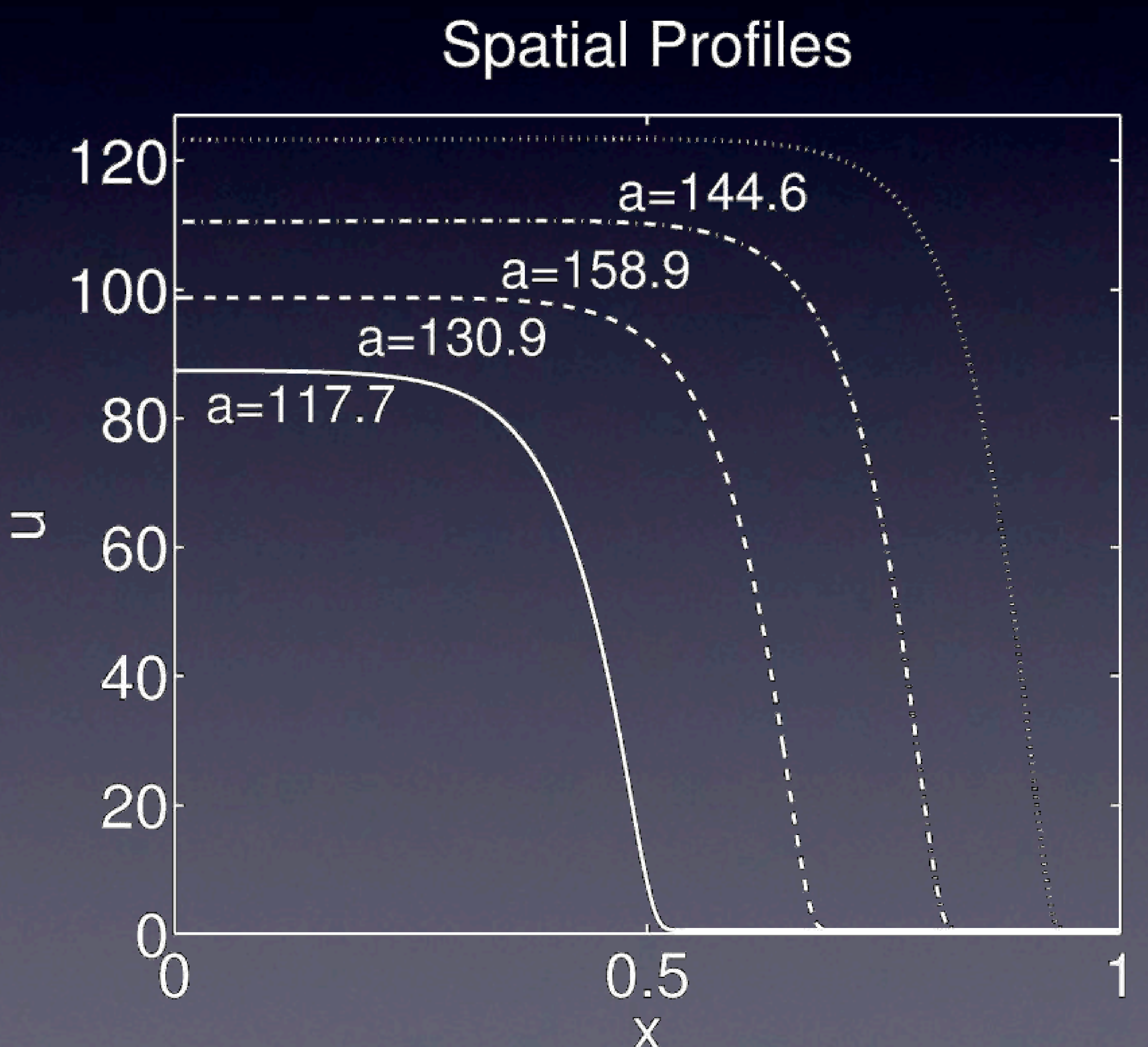


Substrate Inhibition LPA



- Region I - No patterning.
- Region II - Excitable
- Region III - Turing
- Region IV - Excitable

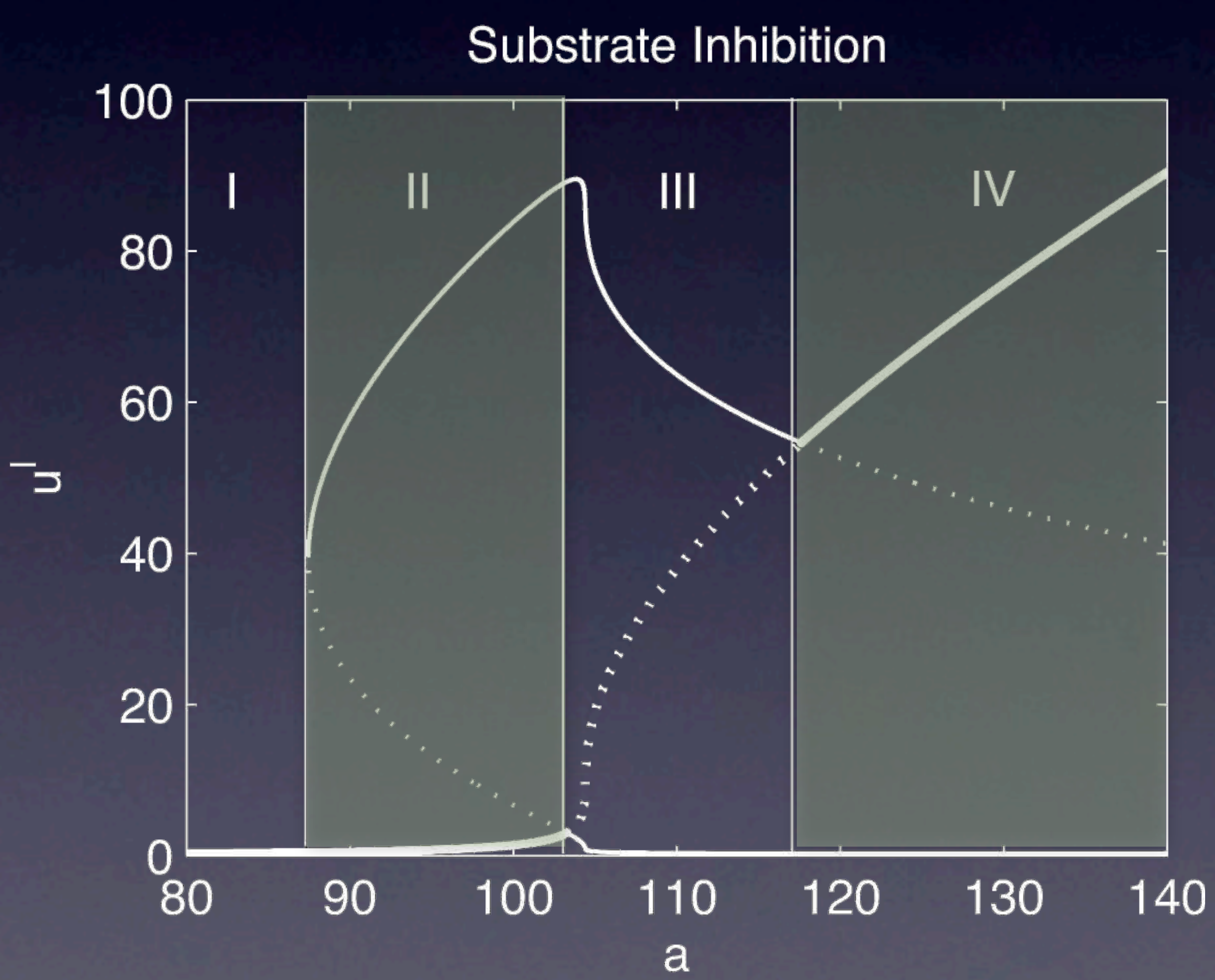
Turing vs Excitable



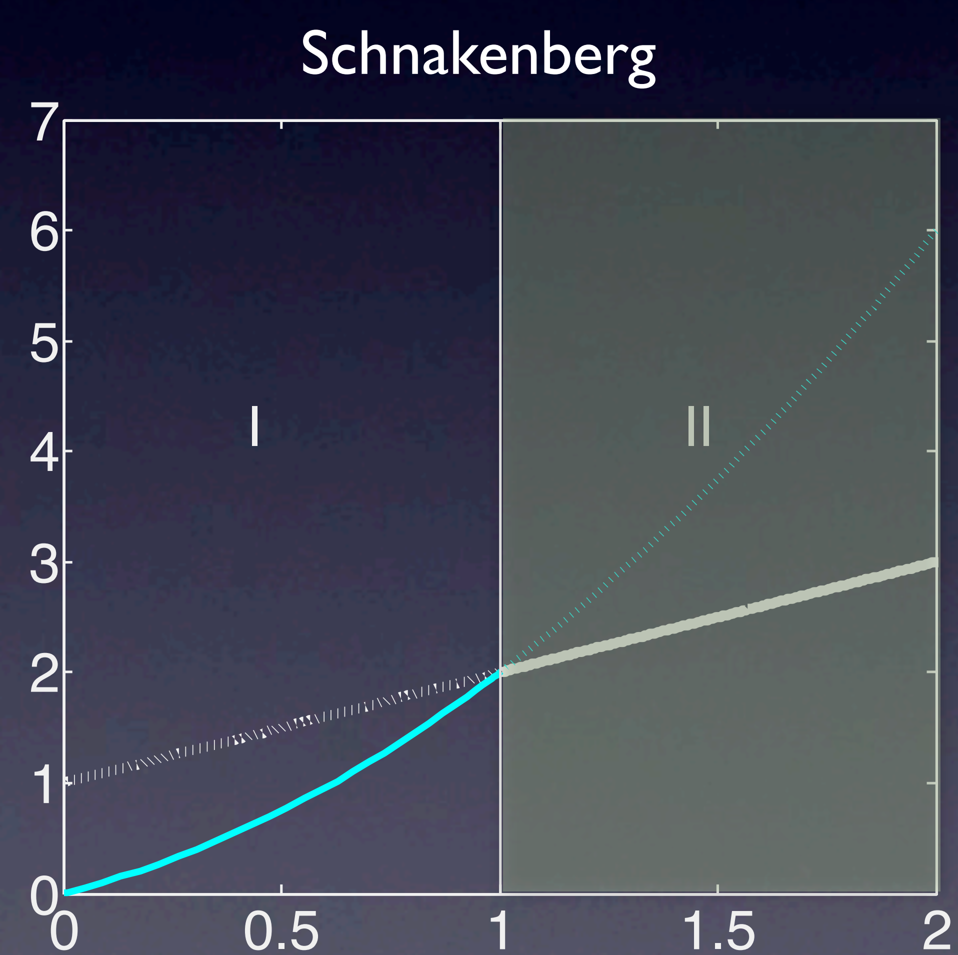
- Again, excitable and Turing generated patterns have the same character.

Types of excitable patterning.

δ- bistable



δ- unstable



- Interface Solutions
- Spike Solutions

LP Eigenvalue

$$J_{LP} = \begin{bmatrix} f_u(u^g, v^g) & f_v(u^g, v^g) & 0 \\ g_u(u^g, v^g) & g_v(u^g, v^g) & 0 \\ 0 & f_v(u^l, v^g) & f_u(u^l, v^g) \end{bmatrix}$$

- The LP eigenvalue that determines stability can be interpreted as a Turing growth rate / eigenvalue.