Mathematical Cell Biology Graduate Summer Course
University of British Columbia, May 1-31, 2012
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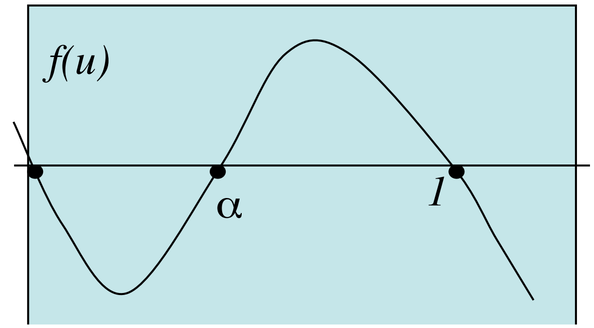
Traveling Waves in a bistable Reaction-Diffusion System

Wave phenomena in RD systems

Keener & Sneyd (1998) Mathematical Physiology; p 270-275

$$\frac{\partial u}{\partial t} = f(u) + D\frac{\partial^2 u}{\partial x^2}, \qquad \left.\frac{\partial u}{\partial x}\right|_{\pm\infty} = 0$$



With *f* cubic: $f(u) = au(u-1)(\alpha - u) \quad 0 < \alpha < 1$

or

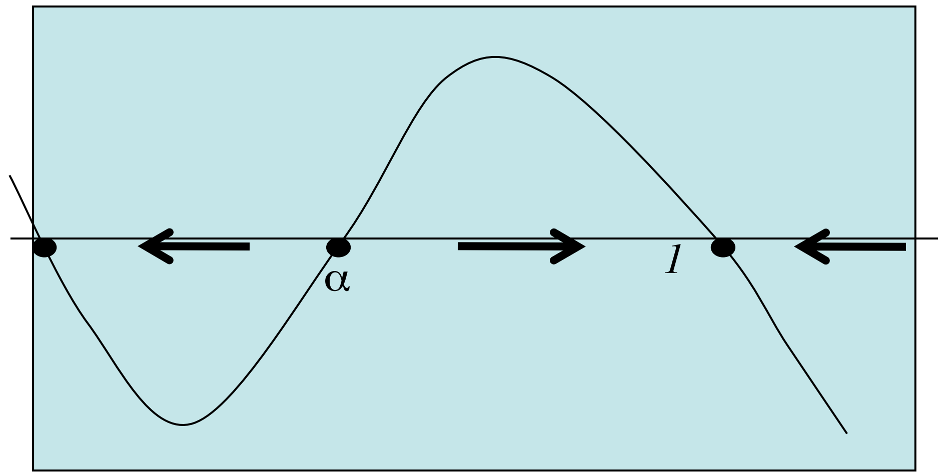
U is the shape of the wave, *z* is the position along the wave front

Bistable well-mixed system

$$\frac{\partial u}{\partial t} = f(u)$$

Steady states:

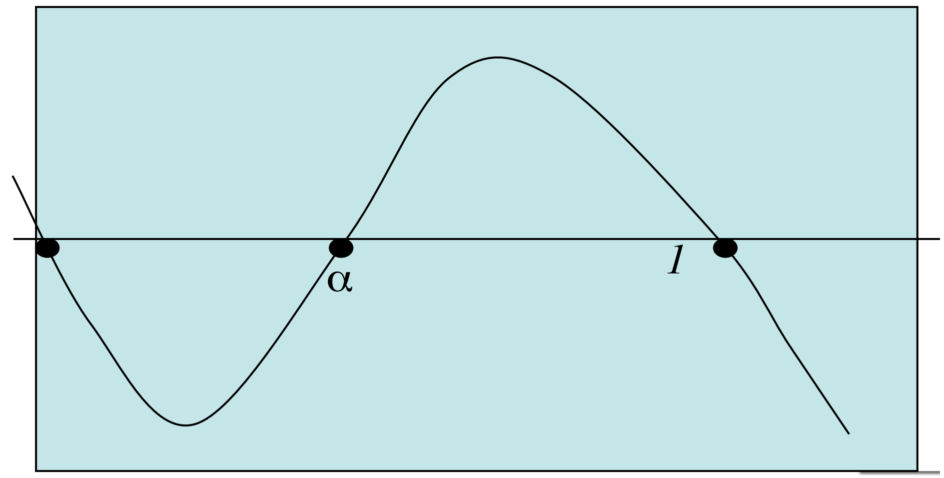
u = 0, α, 1



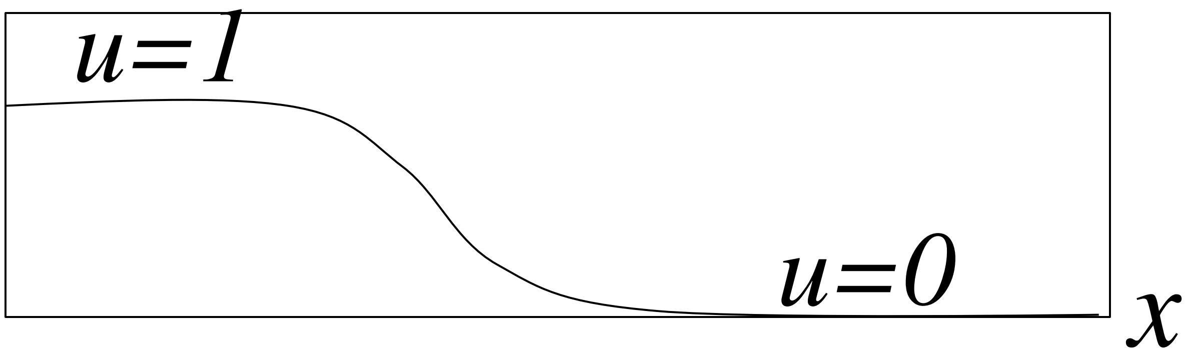
$$f(u) = au(u-1)(\alpha - u) \quad 0 < \alpha < 1$$

With diffusion

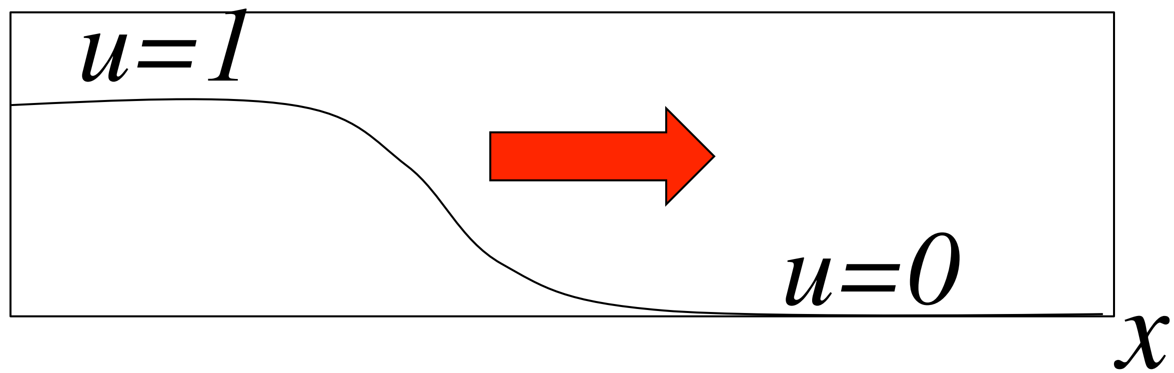
$$\frac{\partial u}{\partial t} = f(u) + D\frac{\partial^2 u}{\partial x^2}$$



$$f(u) = au(u-1)(\alpha - u) \quad 0 < \alpha < 1$$

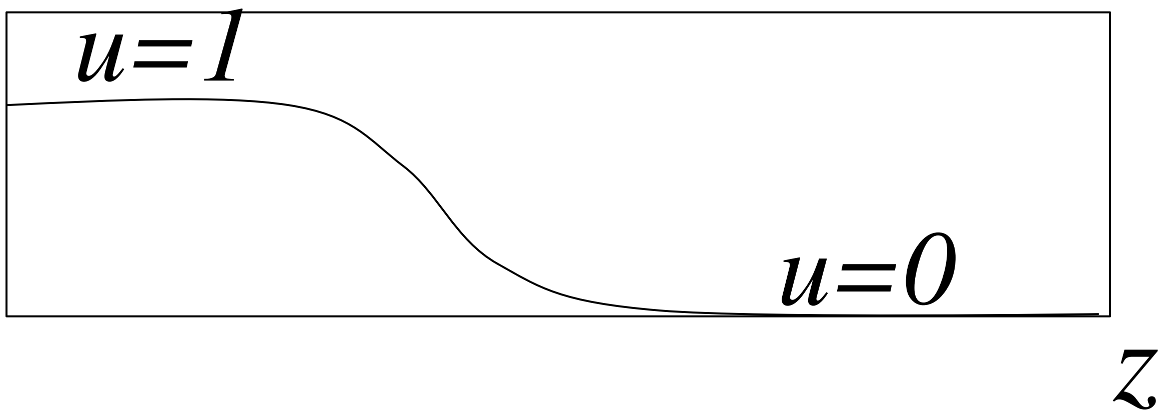


Moving Wave



Wave profile

Traveling wave coordinates: $z = x - ct$, $U(z) = u(x,t)$



What is the wave profile?

Traveling wave coordinates: *z=x-ct*, *U(z)=u(x,t)*

Scaling and transformation of coordinates:

$$\frac{\partial}{\partial t} \to -c\frac{d}{dz} \qquad \frac{\partial}{\partial x} \to \frac{d}{dz}$$

PDE

$$\frac{\partial u}{\partial t} = f(u) + \frac{\partial^2 u}{\partial x^2} \to -c\frac{dU}{dz} = f(U) + \frac{d^2 U}{dz^2}$$

ODE

2nd order (nonlin) ODE $U_{zz} - cU_z + f(U) = 0$

Equivalent ODE system:

$$\begin{cases} U_z = W \\ W_z = cW - f(U) \end{cases}$$

We can study this qualitatively in the UW phase plane to get insight into the shape of the wave.

$$\frac{\partial u}{\partial t} = f(u) + \frac{\partial^2 u}{\partial x^2}$$

(Rescaled)

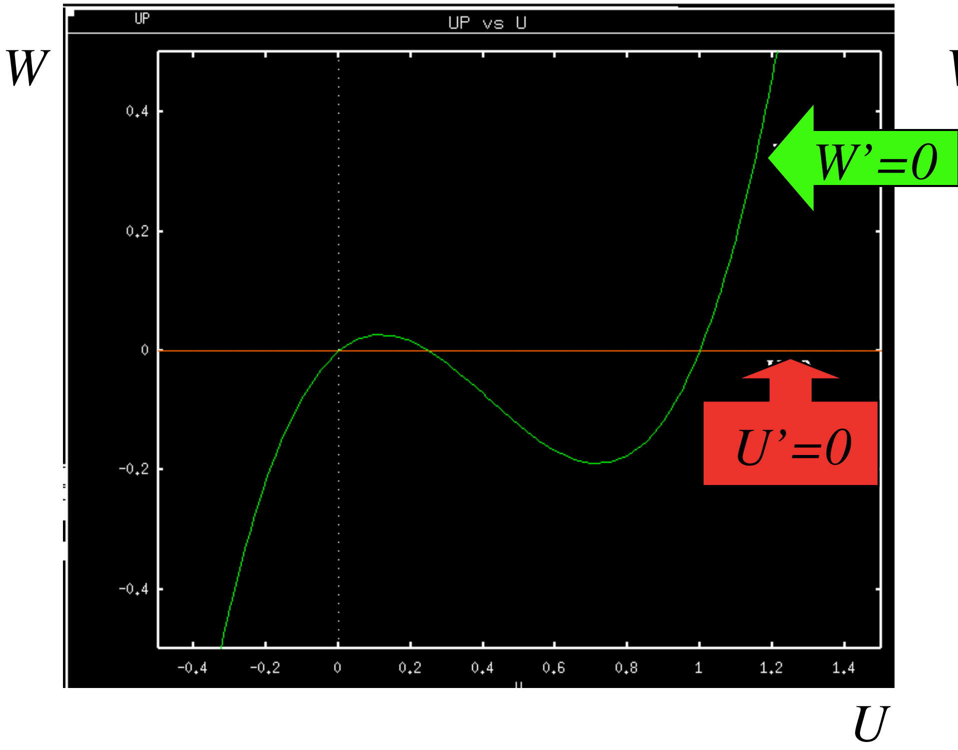
Example:

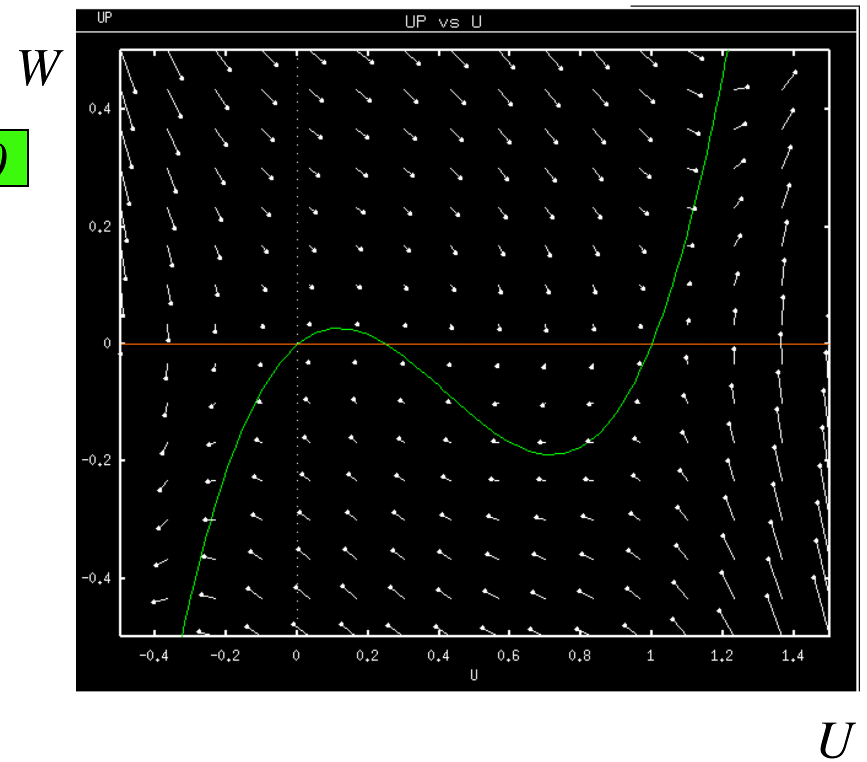
$f(u)=u(1-u)(u-\alpha)$

$$U_{zz} - cU_z + f(U) = 0$$

$$U_z = W$$

$$W_z = cW - f(U)$$





Bard Ermentrout's XPP file

Bistable.ode

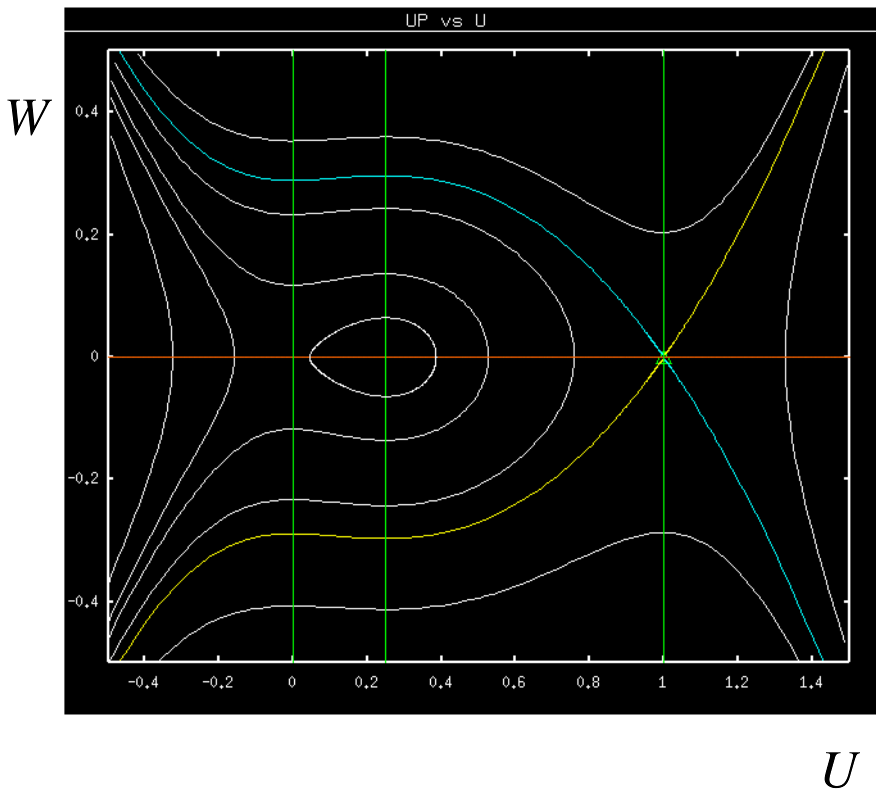
Xpp file

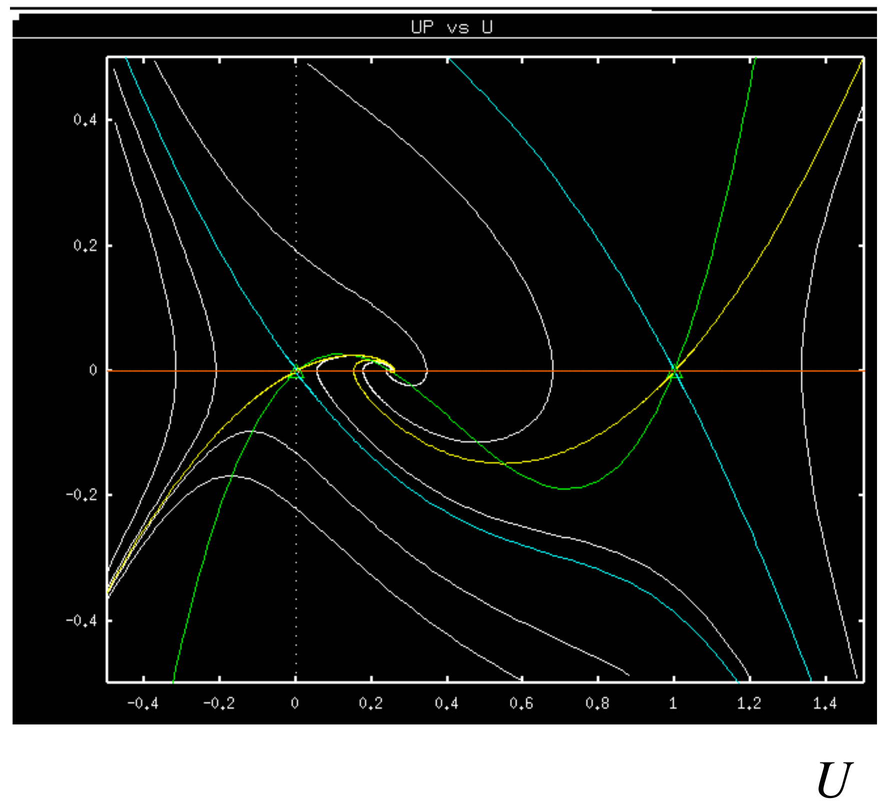
```
# bistable.ode
# classic example of a wave joining 0 and 1
#  u_t = u_xx + u(1-u)(u-a)
#
# -cu'=u"+u(1-u)(u-a)
f(u)=u*(1-u)*(u-a)
par c=0,a=.25
u'=up
up'=-c*up-f(u)
init u=1
@ xp=u,yp=up,xlo=-.5,xhi=1.5,ylo=-.5,yhi=.5
done
```

Interpretation:

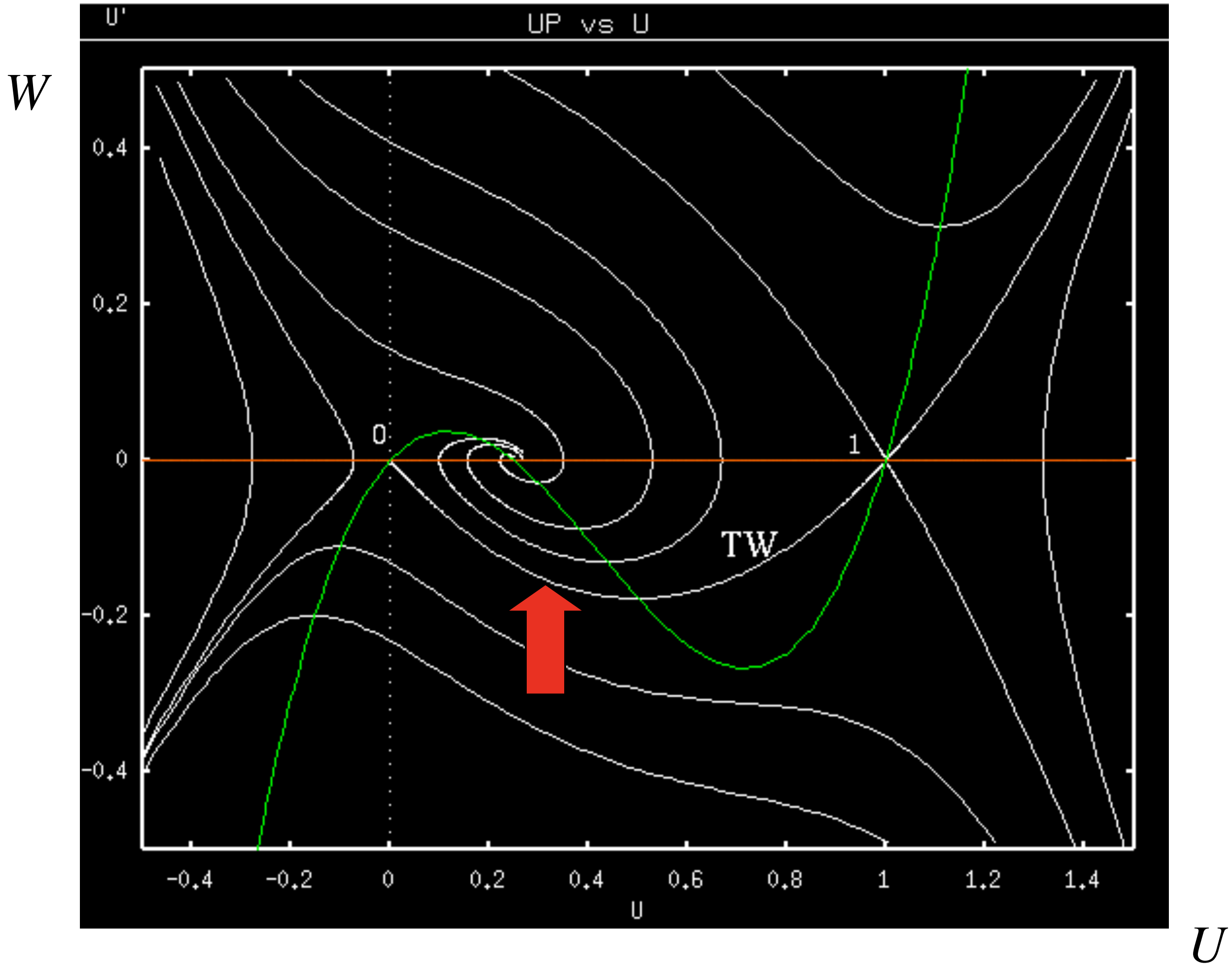
Each trajectory describes how U (and W) vary as z increases over range $-\infty < z < \infty$

In bio applications: u= density>0, so we want: a positive bounded wave: Look for a positive bounded trajectory connecting two steady states.

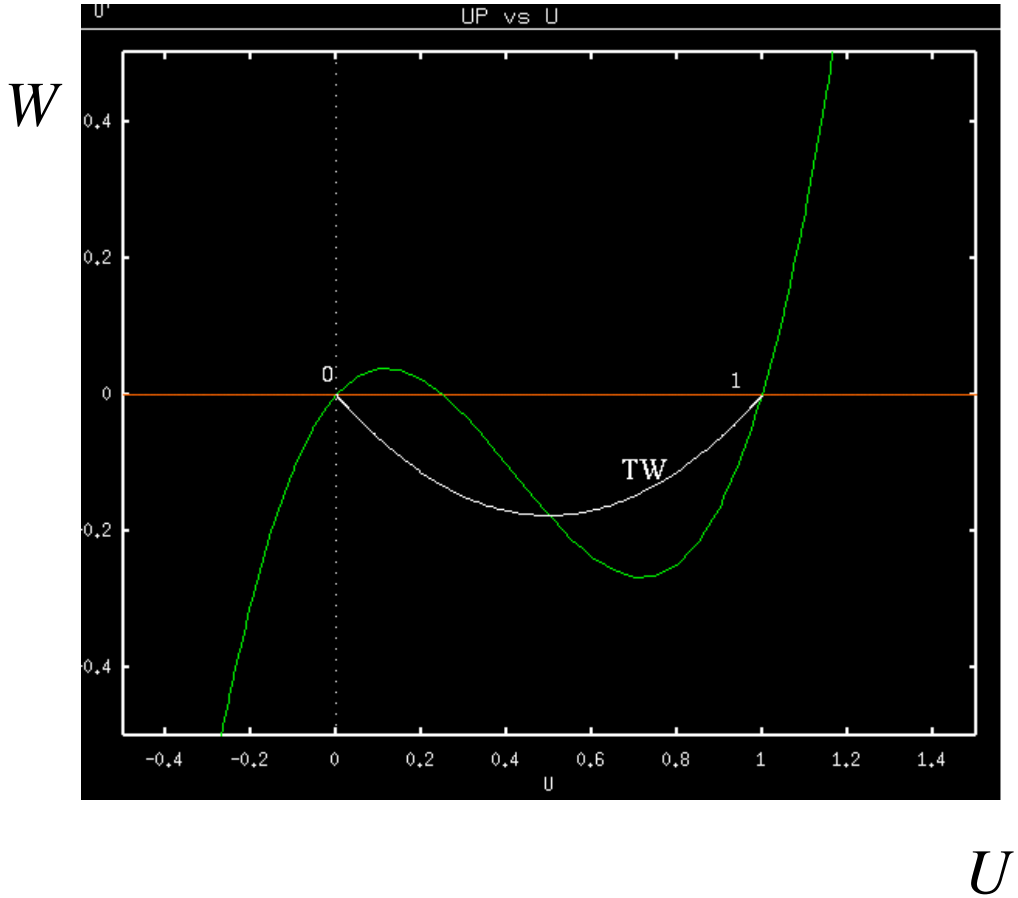




Here is a heteroclinic trajectory:

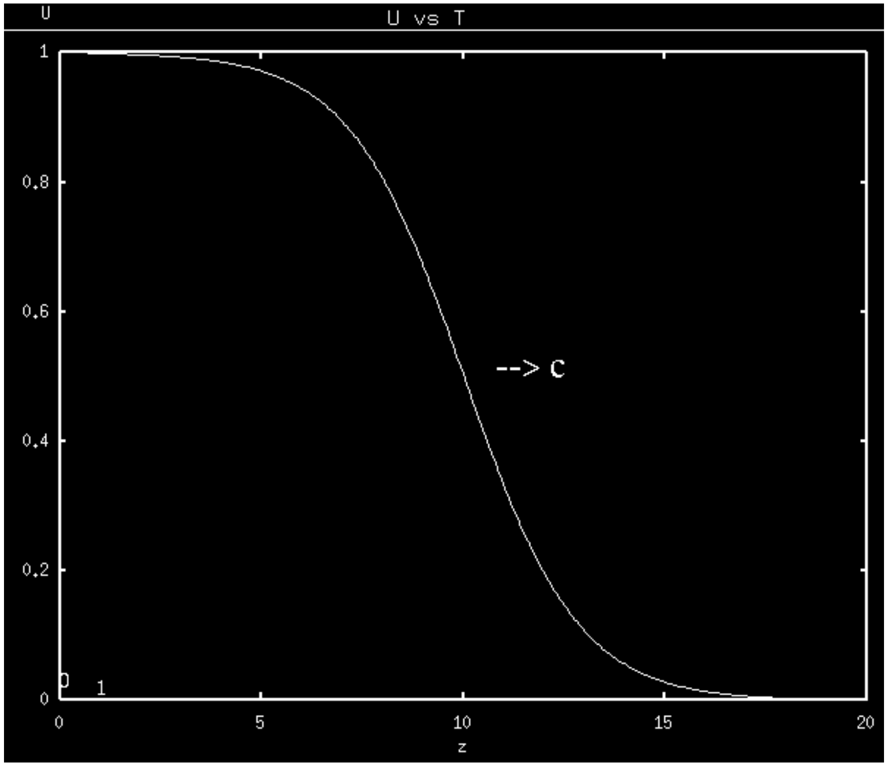


Here it is on its own:



And here is the shape of the wave it represents:

U



z

What is the speed of the wave?

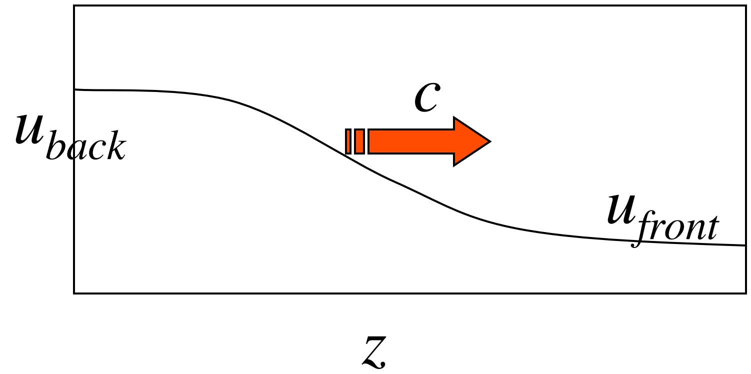
- A reaction-diffusion equation with “bistable” kinetics will admit a traveling wave that looks like a moving “front”, with high level at back, and low level ahead.
- But how fast does the wave move and does it ever stop?

There is a cute trick that allows us to answer this question for arbitrary function $f(u)$.

Determining wave speed

$$\frac{\partial u}{\partial t} = f(u) + \frac{\partial^2 u}{\partial x^2}, \quad z = x - ct$$

$$-c\frac{dU}{dz} - f(U) = \frac{\partial^2 U}{dz^2}$$



Multiply by dU/dz,

$$-c\left(\frac{dU}{dz}\right)^2 - f(U)\left(\frac{dU}{dz}\right) = \frac{1}{2}\frac{d}{dz}\left(\frac{dU}{dz}\right)^2 \quad \text{integrate,}$$

$$-c\int_{-\infty}^{\infty}\left(\frac{dU}{dz}\right)^2 dz - \int_{uback}^{ufront} f(U)dU = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{dz}\left(\frac{dU}{dz}\right)^2 = 0$$

$$-c\int_{-\infty}^{\infty}\left(\frac{dU}{dz}\right)^2 dz = \int_{uback}^{ufront} f(U)dU$$

The wave speed:

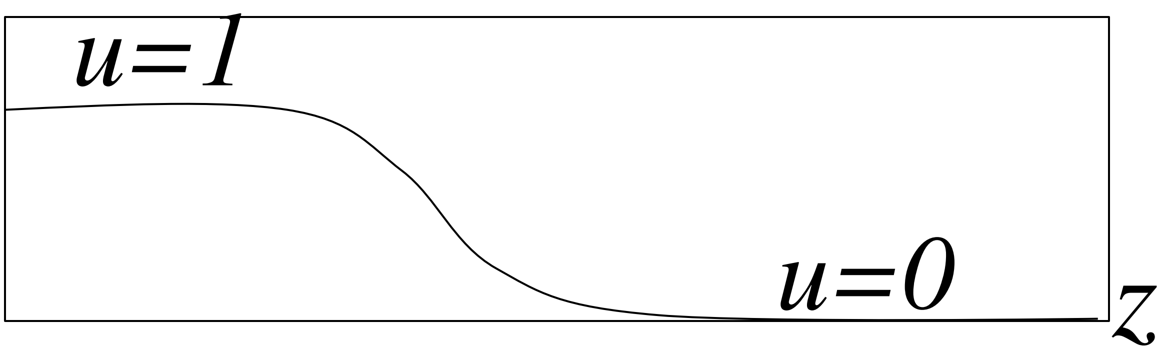
$$-c\int_{-\infty}^{\infty}\left(\frac{dU}{dz}\right)^2 dz - \int_{uback}^{ufront} f(U)dU = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{dz}\left(\frac{dU}{dz}\right)^2 = 0$$

$$-c\int_{-\infty}^{\infty}\left(\frac{dU}{dz}\right)^2 dz = \int_{uback}^{ufront} f(U)dU$$

Finally..

$$c = -\int_{uback}^{ufront} f(U)dU \Big/ \int_{-\infty}^{\infty} \left(\frac{dU}{dz}\right)^2 dz$$

$f(u)=u(1-u)(u-\alpha)$



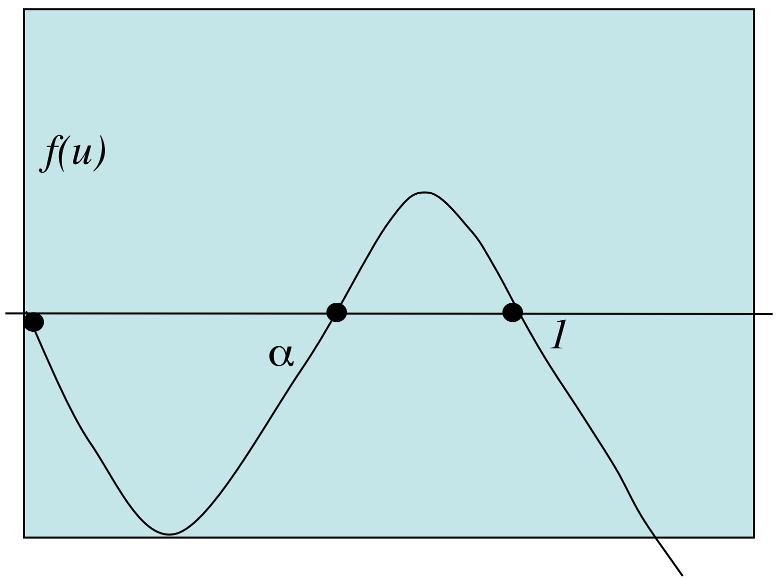
$$c = -\int_{0}^{1} f(U)dU \Big/ \int_{-\infty}^{\infty} \left(\frac{dU}{dz}\right)^2 dz$$

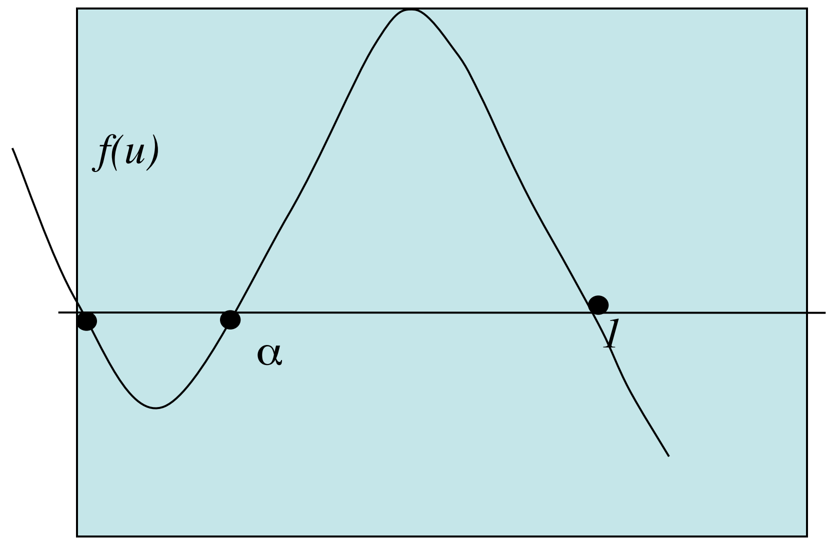
Result!

$$c = -\int_0^1 f(U)\,dU \bigg/ \int_{-\infty}^{\infty} \left(\frac{dU}{dz}\right)^2 dz$$

But what does this tell us????

Which way does it move?

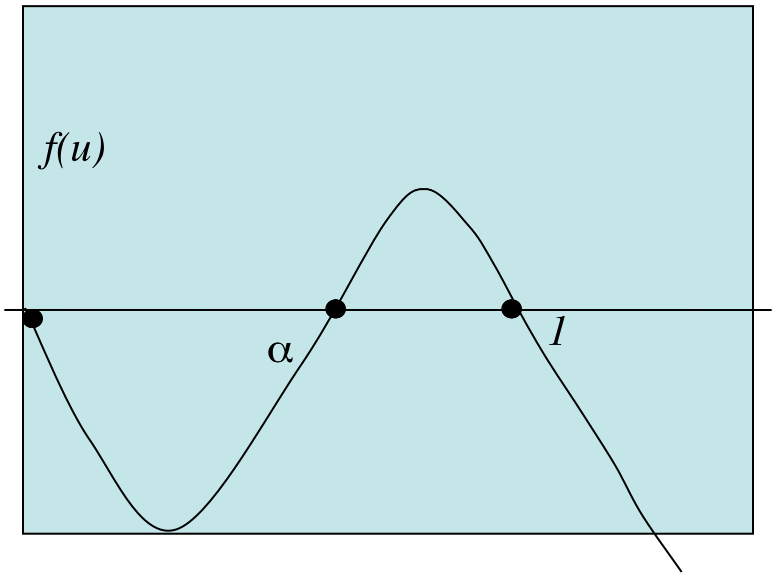


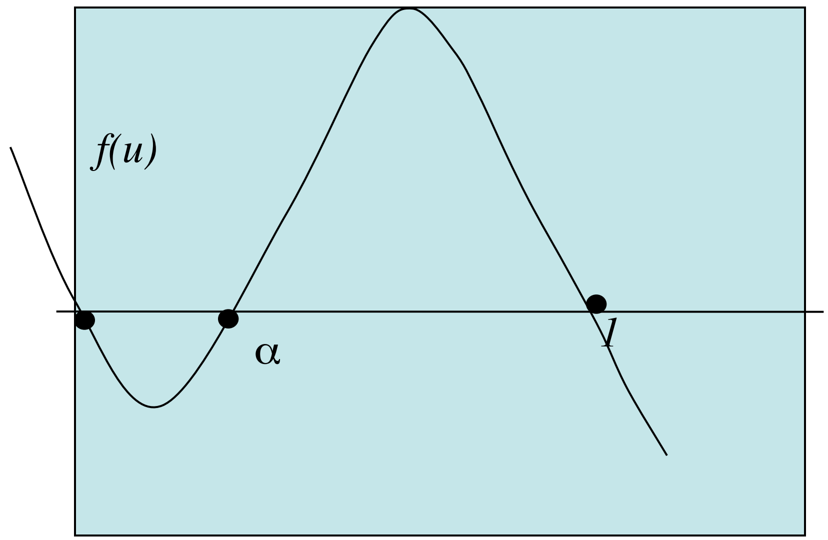


$$c = -\int_0^1 f(U)dU \Bigg/ \int_{-\infty}^{\infty} \left(\frac{dU}{dz}\right)^2 dz$$

Always positive

Which way does it move?





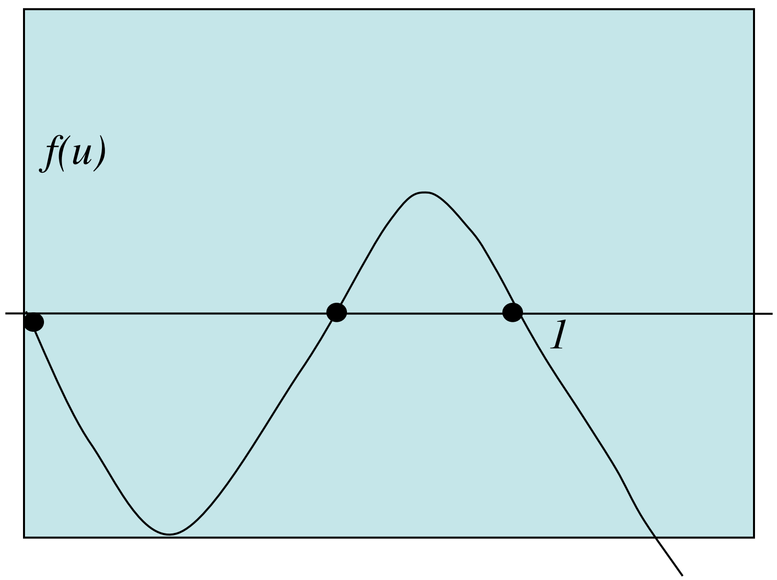
$$-\int_0^1 f(U)dU > 0$$

$$c > 0$$

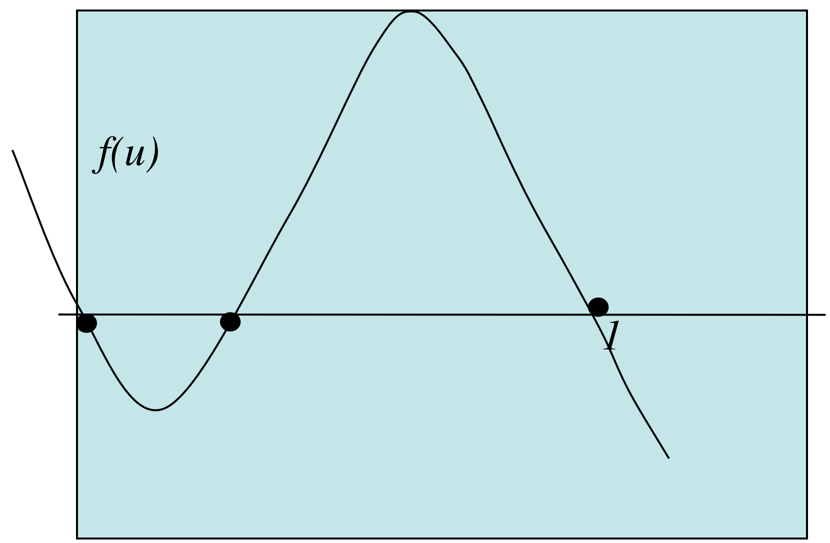
$$-\int_0^1 f(U)dU < 0$$

$$c < 0$$

Which way does it move?

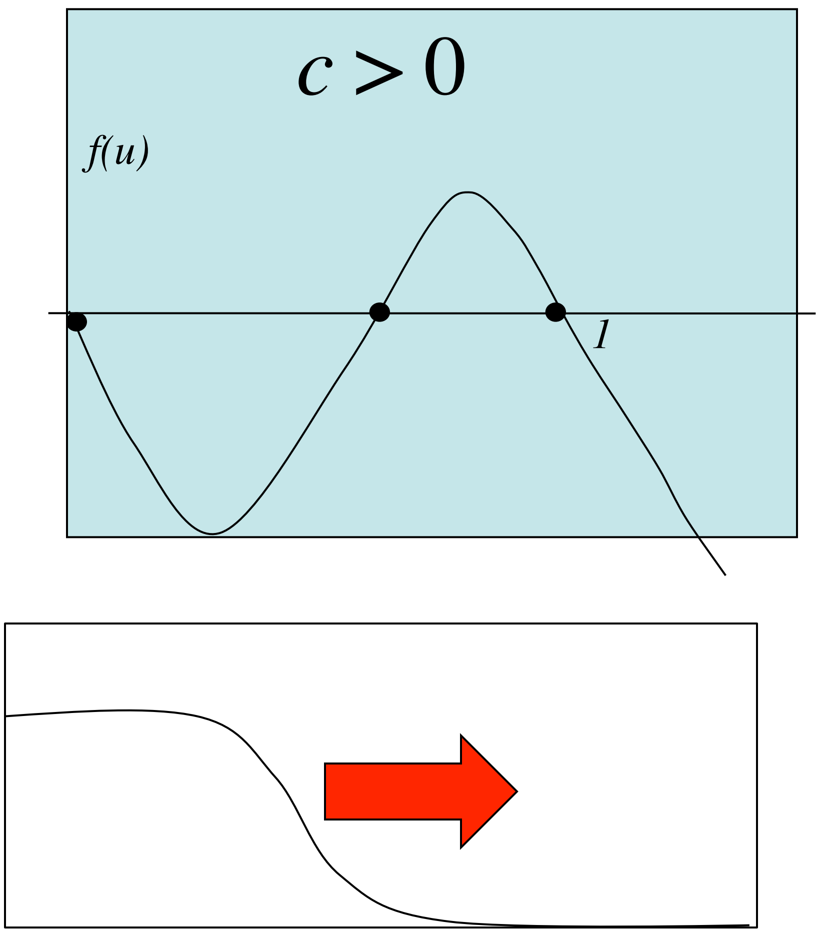


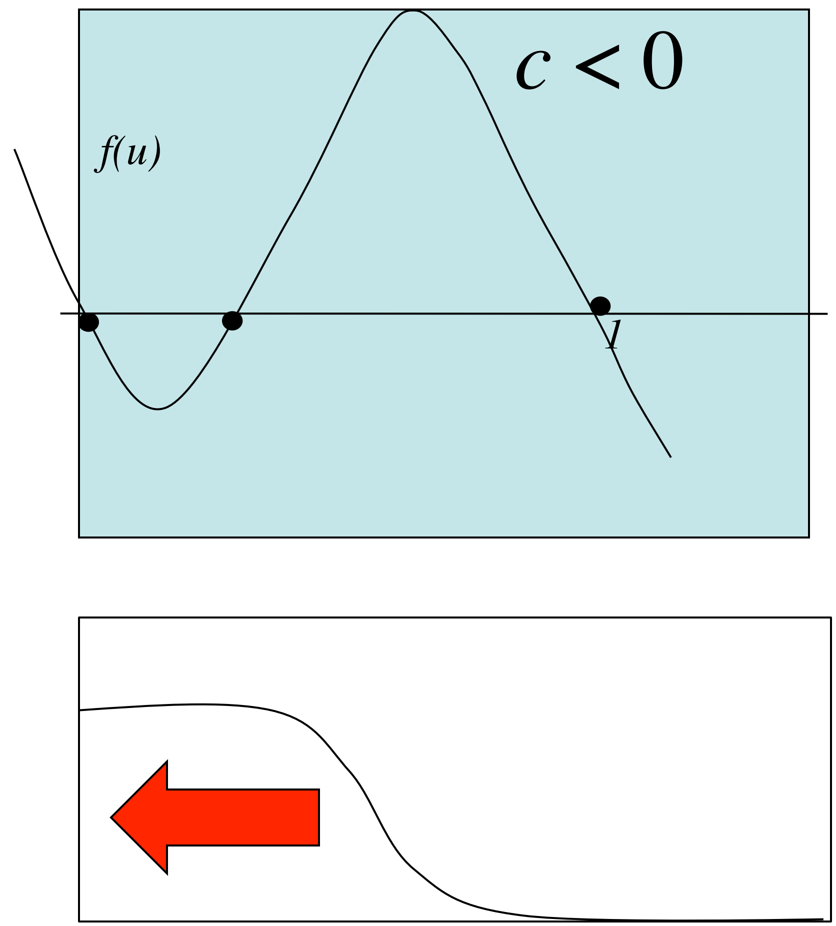
$c > 0$



$c < 0$

Which way does it move?





We can now answer the question:

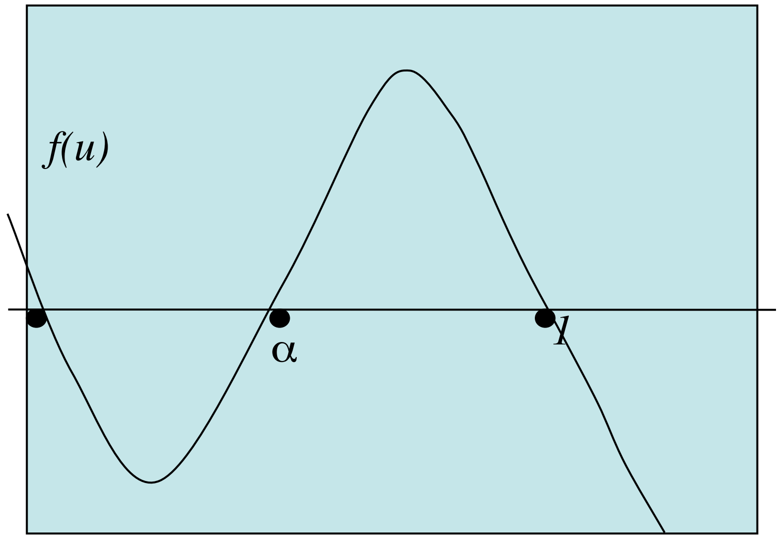
Under what conditions would the wave stop?

Wave stops:

Speed is zero if: $$0 = c = -\int_0^1 f(U)dU \Big/ \int_{-\infty}^{\infty} \left(\frac{dU}{dz}\right)^2 dz$$

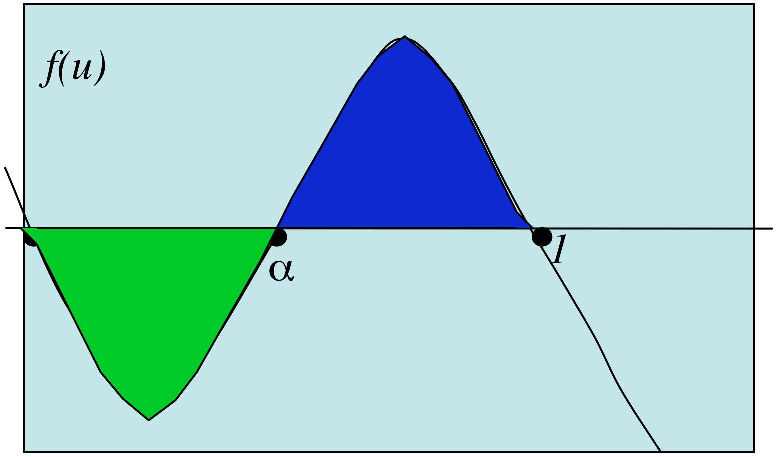
i.e.: if : $$\int_0^1 f(U)dU = 0$$

Geometry of stalled wave:



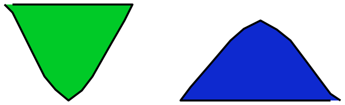
$$\int_0^1 f(U)\,dU = 0$$

Maxwell condition:



"Maxwell condition": $$\int_0^1 f(U)dU = 0$$

"Equal areas"



$$\int_0^\alpha f(U)dU = -\int_\alpha^1 f(U)dU$$

Can we get an explicit solution for the wave speed?

Explicit solution?

In some special cases, e.g. *f(u)* cubic or piecewise linear, can calculate wave speed fully by this method.

(See Keener & Sneyd p 274, Murray p 305)

For

$$f(u) = au(u-1)(\alpha - u) \quad 0 < \alpha < 1$$

Speed of the wave is:

$$c = \sqrt{\frac{a}{2}}(1-2\alpha)$$

$$f(u) = au(u-1)(\alpha - u) \quad 0 < \alpha < 1$$

Implications:

Wave moves right ($c>0$) if $\alpha<1/2$ Moves left ($c<0$) if $\alpha>1/2$

Wave stops ($c=0$) precisely for one value of the parameter, $\alpha=1/2$