Eigenvalue Optimization, Spikes, and the Neumann Green's Function

An optimization problem for the fundamental eigenvalue of the Laplacian in a planar simply-connected domain that contains N small identically-shaped holes, each of a small radius $\varepsilon \ll 1$, is considered. The boundary condition on the domain is assumed to be of Neumann type, and a Dirichlet condition is imposed on the boundary of each of the holes. The reciprocal of this eigenvalue is proportional to the expected lifetime for Brownian motion in a domain with a reflecting boundary that contains N small traps. For small hole radii ε, we derive an asymptotic expansion for this eigenvalue in terms of certain properties of the Neumann Green's function for the Laplacian. This expansion depends on the locations x_i, for $i = 1, \ldots, N$, of the small holes. For the unit disk, ring-type configurations of holes are constructed to optimize the eigenvalue with respect to the hole locations. For arbitrary symmetric dumbbell-shaped domains containing exactly one hole, it is shown that there is a unique hole location that maximizes the fundamental eigenvalue. For an asymmetric dumbbell-shaped domain, it is shown that there can be two hole locations that locally maximize λ_0. This eigenvalue optimization problem is shown to be closely related to determining equilibrium locations of particle-like solutions, called spikes, to certain singularly perturbed reaction-diffusion systems. Some interesting properties of the equilibria, bifurcation behavior, and dynamics of these particle-like solutions are discussed.

joint work with Theodore Kolokolnikov, and Michele Titcombe.