18.100B Problem Set 7

Due Thursday March 31 by 2:30pm. The problem set has been divided into three parts (A,B,C) to facilitate grading. Please hand in three separate packets with your name, the homework number (i.e. Homework 7), and the part (A, B,C) clearly labeled at the top of each packet. Don't forget to staple your homework (but don't staple the three packets together!).

When solving homework problems, you may cite the theorems proved in class. However, you may not cite theorems from Apostol that were not discussed/proved in class unless noted in the problem description.

Part A

1. (5 points) Define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q}, \\ 1/n^\alpha, & \text{if } x = m/n \text{ in lowest terms, with } n > 0. \end{cases}$$

For which values of $\alpha \in \mathbb{R}$ does f have bounded variation? (and prove that your answer is correct) You may have to do some research on this one; for this problem, you may cite without proof any results about the primes that you need.

Part B

2. Let $f\colon [a,b] \to \mathbb{R}$. Define

$$\mathrm{Lip}_f(a,b) = \sup \left\{ \frac{|f(y) - f(x)|}{y - x} : a \le x < y \le b \right\}.$$

(A) (5 points) Prove that $\mathrm{Var}_f(a,b) \le (b-a) \, \mathrm{Lip}_f(a,b)$.

(B) (3 points) Write down an example where $\mathrm{Var}_f(a,b) \le 1$ but $\mathrm{Lip}_f(a,b)$ is infinite.

Part C

3. (5 points) Functions for which $\mathrm{Lip}_f(a,b)$ is finite are called Lipschitz continuous. More generally, for $\alpha > 0$ define
$$\mathrm{Lip}_f^\alpha(a,b) = \sup \left\{ \frac{|f(y) - f(x)|}{(y - x)^\alpha} : a \le x < y \le b \right\}.$$

Functions for which $\mathrm{Lip}_f^\alpha(a,b)$ is finite are called Lipschitz of order α.

Prove that if $\alpha > 1$, $f\colon [a,b] \to \mathbb{R}$, then $\mathrm{Lip}_f^\alpha(a,b)$ is either 0 or infinite. For what functions does $\mathrm{Lip}_f^\alpha(a,b) = 0$, and for what functions is $\mathrm{Lip}_f^\alpha(a,b)$ infinite?

Remark: This exercise shows that if $\alpha > 1$, then $\mathrm{Lip}_f^\alpha(a,b)$ isn't a very interesting quantity.

4. (5 points) (this is Apostol 5.28)
Theorem 5.20 from Apostol is a generalization of Taylor's theorem:

Theorem 1 *Let $f, g \colon [a, b] \to \mathbb{R}$ have finite n–th derivatives on (a, b), and continuous (and thus finite) k–th derivatives on $[a, b]$, for $k = 1, \ldots, n - 1$. Let $c \in [a, b]$ and let $x \in [a, b]$, $x \neq c$. Then there exists a point y in the interval between x and c so that*

$$\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!} (x - c)^k \right] g^{(n)}(y) = f^{(n)}(y) \left[g(x) - \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!} (x - c)^k \right].$$

The proof is essentially the same as the proof of the (usual) Taylor's theorem with remainder. You do not need to prove Theorem 1.

Using Theorem 1, prove the following version of L'Hopital's rule: Let $f, g \colon (a, b) \to \mathbb{R}$. Assume that f and g are n times differentiable, and have finite n–th derivatives on (a, b). Let $c \in (a, b)$ and suppose that $f^{(k)}(c) = 0$, $k = 0, \ldots, n - 1$, and similarly $g^{(k)}(c) = 0$, $k = 0, \ldots, n - 1$. (Recall that $f^{(0)}(x) = f(x)$). Suppose as well that $g^{(n)}(x) \neq 0$ for all $x \in (a, b)$. Prove that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

Hint: define

$$F(x) = f(x) - \frac{(x - c)^{n-1} f^{(n-1)}(c)}{(n - 1)!},$$

$$G(x) = g(x) - \frac{(x - c)^{n-1} g^{(n-1)}(c)}{(n - 1)!},$$

and use Theorem 1.