18.100B Problem Set 4

Due Thursday March 3 by 2:30pm. The problem set has been divided into three parts (A,B, C) to facilitate grading. Please hand in three separate packets with your name, the homework number (i.e. Homework 4), and the part (A, B, C) clearly labeled at the top of each packet. Don't forget to staple your homework (but don't staple the three packets together!).

Note: in this problem set (and throughout the course), the notation $A \subset B$ means the same thing as $A \subseteq B$, i.e. A is contained in B; this includes the possibility that $A = B$. If we want to say that A is contained in B and $A \neq B$, we will write $A \subsetneq B$.

Part A

1. A set $S \subset \mathbb{R}^n$ is called *convex* if whenever $x \in S$ and $y \in S$, the line segment joining x and y is contained in S. A set $S \subset \mathbb{R}^n$ is called *symetric* if whenever $x = (x_1, \ldots, x_n) \in S$, the point $-x = (-x_1, \ldots, -x_n)$ is also in S. If $S \subset \mathbb{R}^n$ is a set and if $t > 0$ be a real number, define

$$tS = \{tx \colon x \in S\} = \{(tx_1, \ldots, tx_n) \colon (x_1, \ldots, x_n) \in S\};$$

this is the dilate of S by the number t.

Let $S \subset \mathbb{R}^2$ be convex and symmetric, and suppose $B(0,1) \subset S \subset B(0,2)$. For each $x \in \mathbb{R}^2$, define

$$\|x\|_S = \inf\{t > 0 : x \in tS\}$$

(recall that if $A \subset \mathbb{R}$ is a set, then $\inf A = -\sup(-A)$). Finally, define $d_S(x,y) = \|x - y\|_S$.

(A) (5 pts) Prove that $(\mathbb{R}^2, d_S)$ is a metric space.

(B) (5 pts) Suppose we require $S \subset B(0,2)$, but we don't require $B(0,1) \subset S$. Is $(\mathbb{R}^2, d_S)$ still a metric space? (either prove the answer is yes, or give a counter-example).

2. (5 pts) Let M be a set, and define

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

What are the open sets in the metric space (M, d)? What are the closed sets? (prove that your answer is correct).

Remark: The metric d is called the discrete metric.

Part B

3. (5 pts) Let $M = \mathbb{R}^n$. Define $d_2(x,y) = \big((x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\big)^{1/2}$ and define $d_1(x,y) = \big(|x_1 - y_1| + \ldots + |x_n - y_n|\big)$. Prove that $S \subset \mathbb{R}^n$ is an open set in the metric space (M, d_1) if and only if S is an open set in the metric space (M, d_2).

Remark: This shows that the metrics d_1 and d_2 induce the same topology on $\mathbb{R}^n$.

4. **(A)** (3 points) Let d be the usual (Euclidean) metric on $\mathbb{R}^n$. Let $S \subset \mathbb{R}^n$ be a set that is not closed. Find a sequence $\{x_i\} \subset S$ that has no convergent sub-sequence in the metric space (S, d).

(B) (3 pts) Let d be the usual (Euclidean) metric on $\mathbb{R}^n$. Let $S \subset \mathbb{R}^n$ be a set that is not bounded. Find a sequence $\{x_i\} \subset S$ that has no convergent sub-sequence in the metric space (S, d).

Remark: The above two statements imply that if $S \subset \mathbb{R}^n$ and if every sequence in (S, d) has a convergent sub-sequence, then S is closed and bounded, and thus by Heine-Borel, S is compact.

Part C

5. Let $M = \mathbb{R}[x]$ be the set of all polynomials in the variable x (with real coefficients). If $f = f(x) \in \mathbb{R}[x]$, define

$$e(f) = \begin{cases} \infty, & \text{if } f = 0, \\ \max\{n : x^n \text{ divides } f(x)\}, & \text{if } f \neq 0. \end{cases}$$

For example, if $f(x) = 3x^2 + x + 1$, then $e(f) = 0$, while if $f(x) = x^{11} + 9x^7$, then $e(f) = 7$.

If $f, g \in M$, define $f - g$ to be the polynomial $(f - g)(x) = f(x) - g(x)$ (this is an element of $M = \mathbb{R}[x]$), and define $d(f, g) = 2^{-e(f-g)}$, with the convention $2^{-\infty} = 0$.

(A) (5 pts) Prove that (M, d) is a metric space.

(B) (5 pts) Show that the metric d satisfies the following inequality: if $f, g, h \in M$, then $d(f, g) \leq \max\{d(f, h), d(g, h)\}$. This is called the ultrametric inequality.

(C) (3 pts) The open balls in this metric space look a bit weird—prove that if $f, g \in M$, $r > 0$ is a real number, and if $f \in B(g, r)$, then $B(g, r) = B(f, r)$.

Remark: this shows that every point in the ball $B(g, r)$ can be thought of as the "center" of the point $B(g, r)$. Balls in (M, d) behave very differently from balls in Euclidean space!