# 18.100B Problem Set 11

Due Thursday May 5 by 2:30pm. When solving homework problems, you may cite the theorems proved in class. However, you may not cite theorems from Apostol that were not discussed / proved in class unless noted in the problem description.

**Part A**

**1** (5 points). Prove that every uniformly convergent sequence of bounded functions is uniformly bounded. I.e. if $\{f_n\} \subset C([a,b])$ converges uniformly to $f$, then there is a constant $A$ so that $|f_n(x)| \leq A$ for each $x \in [a,b]$ and each $n$.

**2** (5 points). Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions that converge uniformly on a set $M \subset \mathbb{R}$. Define $h_n = f_n + g_n$. Prove that $\{h_n\}$ converges uniformly on $M$.

**3**. (5 points) (Apostol 9.15). Prove or disprove the following statement: Let $\{f_n\}$ be a sequence of continuous functions on $[0,1]$, and suppose that that $f_n \to f$ uniformly on $[0,1]$. Then

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)dx = \int_0^1 f(x)dx.$$

**Part B**

**4**. (5 points). Let $\{f_n\}$ be a sequence of functions that are Riemann integrable on $[a,b]$. Suppose that $f_n \to f$ uniformly on $[a,b]$. Prove that $f$ is Riemann integrable on $[a,b]$, and that $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

Remark: This is Lemma 1 from lecture (we used this lemma but did not prove it).

**5**. (5 points). Let $f\colon [0,1] \times [-a,a] \to \mathbb{R}$ be continuous. Let $\{g_n\}$ be a sequence of functions from $[0,1]$ to $[-a,a]$, and define $h_n(x) = f(x, g_n(x))$ for $x \in [0,1]$. Prove that if $g_n \to g$ uniformly on $[0,1]$, then $\{h_n\}$ converges uniformly to $h$, with $h(x) = f(x, g(x))$.

Remark: This is Lemma 2 from lecture (we used this lemma but did not prove it).

**6**. (5 points) Let $f\colon [-1,1] \to \mathbb{R}$. Suppose that $f$ is continuous on $[-1,1]$, and that for each $k = 1, \ldots,$ $f^{(k)}$ exists and is continuous on $(-1,1)$. For $n = 1, 2, \ldots,$ define

$$f_n(x) = f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

This is the sum of the first $k$ terms of the Taylor expansion of $f$ around the point 0.

Is it always true that $f_n \to f$ pointwise on $(-1,1)$? What about $f_n \to f$ uniformly? Prove that the answer is yes, or provide a counter-example.