18.100B Problem Set 10

Due Thursday April 28 by 2:30pm. When solving homework problems, you may cite the theorems proved in class. However, you may not cite theorems from Apostol that were not discussed / proved in class unless noted in the problem description.

Part A

For the next problem, you may find it helpful to use Theorem 7.35 from Apostol:

Theorem 1 (Aposol, 7.35) *Let f be Riemann integrable on $[a, b]$. Let $\alpha\colon [a, b] \to \mathbb{R}$ be continuous, and suppose that α' exists and is Riemann integrable on $[a, b]$. Then the following integrals exist and are equal:*

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$$

1 (8 points). Describe the set of all functions $\alpha\colon [a, b] \to \mathbb{R}$ that are monotonic increasing, whose derivative α' exists and is continuous $[a, b]$, and that have the following property:

$$\{f\colon [a, b] \to \mathbb{R}\colon f \in \mathcal{R}(\alpha) \text{ on } [a, b]\} = \{f\colon [a, b] \to \mathbb{R}\colon f \text{ is Riemann integrable on } [a, b]\}.$$

(As always, prove that your answer is correct).

For the next problem, you may find Theorem 7.39 from Apostol to be useful:

Theorem 2 (Apostol, Theorem 7.39) *Let $f\colon [a, b] \times [c, d] \to \mathbb{R}$ be continuous, and let g be Riemann integrable on $[a, b]$. Define*

$$F(y) = \int_a^b g(x)f(x, y)dx.$$

Then $F\colon [c, d] \to \mathbb{R}$ is continuous.

2. Recall the definition of $C([0, 1])$ from HW 5 #2. Let $g \in C([0, 1])$, let $K\colon [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function, and let $\lambda \in \mathbb{R}$. Define the function T as follows. If $f \in C([0, 1])$, then $T(f)$ is the function from $[0, 1]$ to $\mathbb{R}$ given by

$$T(f)(x) = g(x) + \lambda \int_0^1 K(x, t)f(t)dt.$$

a (3 points). Prove that if $f \in C([0, 1])$, then $T(f) \in C([0, 1])$.

b (7 points) Suppose that $|K(x, y)| \leq M$ for all $(x, y) \in [0, 1] \times [0, 1]$ (here $M > 0$), and that $|\lambda| < 1/M$. Prove that $T\colon C([0, 1]) \to C([0, 1])$ is a contraction map.

Remark: The unique fixed point of this contraction map is a function f satisfying $f(x) = g(x) + \lambda \int_0^1 K(x, t)f(t)dt$.

Part B

3 a (5 points). Let f and g be bounded and Riemann integrable integrable on $[0,1]$. If the set $\{x \in [0,1] \colon f(x) \neq g(x)\}$ has measure 0, is it always true that $\int_0^1 f(x)dx = \int_0^1 g(x)dx$? Prove that your answer is correct.

3 b (3 points). Let $\alpha \colon [0,1] \to [0,1]$ be the Cantor-Lebesgue function defined in lecture. Let f and g be bounded, with $f \in \mathcal{R}(\alpha)$ on $[0,1]$ and $g \in \mathcal{R}(\alpha)$ on $[0,1]$. If the set $\{x \in [0,1] \colon f(x) \neq g(x)\}$ has measure 0, is it always true that $\int_0^1 f(x)d\alpha = \int_0^1 g(x)d\alpha$? Prove that your answer is correct.

NOTE: there was a typo in a previous version of this problem; the condition $\int_0^1 f(x)d\alpha = \int_0^1 g(x)d\alpha$ was written as $\int_0^1 f(x)dx = \int_0^1 g(x)dx$. If you already solved the old version, you can hand the old version in for full credit.

5 (a, 3 points) In this exercise we will show that if f and g are Riemann integrable, then $f \circ g$ need not be Riemann integrable.

Define $f \colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x = 0, \\ 1, & x \neq 0. \end{cases}$$

and define $g \colon [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0, & x \notin \mathbb{Q}, \\ 1/p, & x \in \mathbb{Q},\ x = q/p \text{ in lowest terms, with } p > 0. \end{cases}$$

Prove that f and g are Riemann integrable (hint: look through your past homework, this should be easy!), but that $f \circ g$ is not Riemann integrable.

(b, 7 points) Prove that if $g \colon [a,b] \to [c,d]$ is Riemann integrable and if $f \colon [c,d] \to \mathbb{R}$ is continuous, then $f \circ g$ is Riemann integrable on $[a,b]$. Hint: since $[c,d]$ is compact, f is uniformly continuous on $[c.d]$.

Something to think about (ungraded bonus problem). If instead we have that $g \colon [a,b] \to [c,d]$ is continuous and $f \colon [c,d] \to \mathbb{R}$ is Riemann integrable, must it be true that $f \circ g$ is Riemann integrable on $[a,b]$? (hint: the Cantor set and fat Cantor set from HW 8 may come in handy).