18.100B Problem Set 1

Due Thursday February 11 by 3:30pm. Be sure to review the course collaboration policies, and don't forget to staple your homework if it is more than one page long.

Ordered Fields

1. Using the Field Axioms (Axiom 1-5 from lecture, or from Apostol §1.2) and the order axioms (axiom 6-9 from lecture, or from Apostol §1.3), prove that $1 > 0$. Here 0 and 1 are regarded as real numbers.

2. In this problem we will study a set that satisfies the field axioms but does not satisfy the order axioms. Consider the field $\mathbb{F}_3$. This field has three elements, which we will call $0, 1, 2$ (do not confuse these elements with real numbers. We're just using the labels 0, 1, and 2 for convenience. We could just as easily call the three elements a, b, c). Addition and multiplication are defined by the following addition and multiplication tables:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Using a proof by contradiction, show that it is impossible to define an operation "$<$" that satisfies the order axioms.

Remark. $\mathbb{F}_3$ is an example of a finite field. Finite fields play an important role in algebra, number theory, and computer science. We will mainly be interested in them because they behave very differently from $\mathbb{R}$, so they help illustrate some of the things that make $\mathbb{R}$ special.

3. Consider the set $S = \{x \in \mathbb{Q} : \ x^2 < 2\}$. Prove that if $a \in \mathbb{Q}$ is an upper bound for S, then there exists a number $b \in \mathbb{Q}$ with $b < a$ so that b is also an upper bound for S (this shows that the rational numbers do not satisfy the least upper bound property).

Sets and cardinality

4. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the natural numbers. Write down a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

5. A binary string is a string of 0s and 1s. Prove that the set of all finite binary strings is countable, but the set of all infinite binary strings is uncountable.