Math 101 - 951 Quiz #4 (August 6, 2014)

Show all your work. Use back of page if necessary. Calculators are not allowed.

Last Name: **First Name:** **Student No.:**

---

1. (4 marks each) Determine if the following series are convergent or divergent:

 (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^{68}}$

> **Solution:** Using the *Ratio Test*: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{68}} \cdot \frac{n^{68}}{n!}$ ① $= \left(\frac{n}{n+1}\right)^{68} (n+1) \to \infty$.②
> And the series diverges.①

 (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - \sqrt{n}}$

> **Solution:** Using the *Limit Comparison Test*, we can compare this series with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ①. Noting that $\lim_{n\to\infty} \frac{n^2}{n^2-\sqrt{n}} = 1$, ① and that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ①, so the series converges ① (absolutely, since all terms are positive). [Alternatively, we can use the *Comparison Test*. Note that for $n > 2$, $\sqrt{n} < \frac{n^2}{2}$ ① and $n^2 - \sqrt{n} > n^2 - \frac{n^2}{2} = \frac{n^2}{2}$. Thus $\frac{1}{n^2-\sqrt{n}} < \frac{1}{\frac{n^2}{2}} = \frac{2}{n^2}$.① But $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges. ① Thus $\sum_{n=1}^{\infty} \frac{1}{n^2-\sqrt{n}}$ converges as well.①]

2. (2 mark) For what values of p is the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{(n-1)^p}$ convergent? Justify your answer.

> **Solution:** This is an alternating series. Denoting $b_n = \frac{n}{(n-1)^p}$, we check if b_n satisfies the following two conditions:
>
> (i) $b_{n+1} < b_n$ for sufficiently large n. This can be checked by differentiating $f(x) = x(x-1)^{-p}$: $f'(x) = (x-1)^{-p} + x(-p)(x-1)^{-p-1} = (x-1)^{-p-1}[(1-p)x - 1] < 0$ is satisfied for large x if $p \geq 1$. ①
>
> (ii) $\lim_{n\to\infty} b_n = 0$ if the exponent p in the denominator is greater than that in the numerator $p > 1$. ①
>
> Based on the *Alternative Series Test*, this series converges for $p > 1$. [Deduct ① if the correct conclusion is reached but no detailed check of the 2 conditions are given.]
>
> [To be rigorous, one might want to show that the series converges *only* for $p > 1$. But this is not required. If $p = 1$, $b_n \to 1$. If $p < 1$, $b_n \to \infty$. In either case $a_n = (-1)^{n-1} b_n$ does not tend to zero, and the series diverges.]