Math 101 - 951 Quiz #3 (July 30, 2014)

Show all your work. Use back of page if necessary. Calculators are not allowed.

Last Name: **First Name:** **Student No.:**

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1. (4 marks) Find the solution of the differential equation $\dfrac{dy}{dx} = e^{x-y}$ that satisfies $y(0) = 1$. Express your final answer in the form $y = f(x)$.

> **Solution:** We write $e^y\,dy = e^x\,dx$ ① and integrate both sides:
>
> $$e^y = e^x + C.①$$
>
> Putting in the initial condition gives us $e = 1 + C$ and thus $C = e - 1$. ① Finally the solution can be written as
>
> $$y = \ln(e^x + e - 1).①$$

2. (4 marks) Determine if the sequence $\{a_n\}$, with $a_n = \dfrac{\sin\sqrt{n}}{\sqrt{n}}$, converges. Justify your answer. If it does, compute the limit.

> **Solution:** Note that $-1 < \sin\sqrt{n} < 1$ for all integer n ①. Therefore
>
> $$\frac{-1}{\sqrt{n}} < \frac{\sin\sqrt{n}}{\sqrt{n}} < \frac{1}{\sqrt{n}}①.$$
>
> Both $\dfrac{-1}{\sqrt{n}}$ and $\dfrac{1}{\sqrt{n}}$ converge to 0 ①. By the Squeeze Theorem, our sequence $\{a_n\}$ converges to 0 as well ①.
>
> Alternatively, one can consider the function $f(x) = \dfrac{\sin\sqrt{x}}{\sqrt{x}}$ ①, and note that as $x \to \infty$, the numerator is bounded between -1 and 1, while the denominator tends to ∞ ①. Thus, $\lim_{x\to\infty} f(x) = 0$ ①. By Theorem 3 of §11.1, we know that $\lim_{n\to\infty} a_n = 0$①.

3. (2 marks) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{n\pi}{2n+1}\right)$ diverges.

> **Solution:** Theorem 6 of §11.2 says that if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. In this case,
>
> $$\lim_{n\to\infty} \sin^2\left(\frac{n\pi}{2n+1}\right) = \sin^2(\pi/2) = 1 \neq 0.①$$
>
> Using the theorem, the series must diverge ①. [Note that we are using the "converse-negative" of Theorem 6. This has been written out as Theorem 7 on p. 709 of the textbook. Full credit for citing Theorem 7 directly.]