MATH 101 Section 951
Midterm Exam, July 23, 2014)
Duration of exam: 4 - 5 pm

Last Name: **First Name:** **Student No.:**

Rules of Conduct

1. This is a close-book, close-notes exam, and calculators are not allowed. Turn off cell-phones and put away personal belongings, textbook and notes.
2. Each student must be prepared to produce a UBC card for identification upon request.
3. Students suspected of cheating may be immediately dismissed from the examination by the invigilator, and may be subject to disciplinary actions. Cheating includes:
   - speaking or communicating with other students, unless otherwise authorized;
   - purposely exposing written papers to the view of other students or imaging devices;
   - purposely viewing the written papers of other students;
   - using or having visible at the place of writing any unauthorized books or papers;
   - using or operating any unauthorized electronic devices.
4. Follow additional instructions given by the invigilators.

This midterm exam has 5 questions on 6 pages including this cover page.

Q1		12
Q2		4
Q3		5
Q4		5
Q5		4
Total		30

1. (12 marks, 2 for each part below) The following are short-answer problems. You earn full mark for a correct answer in the box or 0 for an incorrect answer. Leave irrational numbers such as π, e or $\sqrt{2}$ in the answer and do not approximate them by decimals.

   (a) Find the indefinite integral $\displaystyle\int \frac{dx}{x^2-1}$.

Answer

   Solution: Integrating rational function gives $\displaystyle\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$. For intermediate steps see Example 3 on p. 487. It's ok if students omit $+C$ in the answer.

   (b) Calculate the approximate area under the curve $f(x) = x^2$ in the interval $[0, 2]$ using the trapezoidal rule with $n = 4$.

Answer

   Solution: The interval is divided into 4 segments, with $\Delta x = 0.5$ and $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, and $x_4 = 2$. The corresponding functional values are $f(x_i) = [0, 0.25, 1, 2.25, 4]$. Thus

   $$\int_0^2 x^2\,dx \approx \frac{0.5}{2}(0 + 2\times 0.25 + 2\times 1 + 2\times 2.25 + 4) = \frac{11}{4} = 2.75.$$

   The answer can be given either as fraction or decimal.

   (c) Find the average value of the function on the given interval: $f(x) = x\sin x$, $[0, \pi]$.

Answer

   Solution: The average is $\bar{f} = \dfrac{1}{\pi}\displaystyle\int_0^\pi x\sin x\,dx$. The integral $\int x\sin x\,dx$ can be integrated by parts: $\displaystyle\int x\sin x\,dx = -\int x\,d(\cos x) = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x$. Finally $\bar{f} = \dfrac{1}{\pi}(-x\cos x + \sin x)|_0^\pi = 1$.

   (d) Calculate the indefinite integral $\displaystyle\int \sqrt{\sec x}\ \tan^3 x\,dx$.

Answer

   Solution: As the power of $\tan x$ is odd, we use the substitution $u = \sec x$ and $du = \sec x\tan x\,dx$:

   $$\int \sqrt{\sec x}\ \tan^3 x\,dx = \int \tan^2 x(\sec x)^{-1/2}\,(\tan x\sec x\,dx) = \int \frac{u^2-1}{\sqrt{u}}\,du = \int (u^{3/2} - u^{-1/2})\,du$$
   $$= \frac{2}{5}u^{5/2} - 2u^{1/2} + C.$$

Replacing u by $\sec x$ gives the answer $\displaystyle\int \sqrt{\sec x}\ \tan^3 x\, dx = \frac{2}{5}\sec^{5/2} x - 2\sec^{1/2} x + C$.

(e) Find the anti-derivative of $f(x) = x^3\sqrt{1+x^4}$.

Answer

Solution: $F(x) = \dfrac{1}{6}(1+x^4)^{3/2} + C$ using substitution $u(x) = x^4 + 1$.

(f) Calculate the definite integral using trigonometric substitution: $A = \displaystyle\int_0^1 \frac{dx}{\sqrt{3+2x-x^2}}$

Answer

Solution: Complete the square under the square-root sign: $3 + 2x - x^2 = 4 - (x-1)^2$. Now we can use the substitution $x - 1 = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$ to calculate the indefinite integral as: $\int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{2\cos\theta\, d\theta}{2\cos\theta} = \theta + C$. Noting the lower bound $x = 0$ corresponds to $\theta = \sin^{-1}(-1/2) = -\pi/6$ and the upper bound $x = 1$ to $\theta = 0$, we finally get $A = \theta|_{-\pi/6}^{0} = \frac{\pi}{6}$. It's ok to leave answer as $\sin^{-1}(1/2)$ or $\sin^{-1}(0.5)$.

2. (4 marks) Find the function f and the number a that satisfy $1 + \displaystyle\int_a^x e^{f(t)}\, dt = e^x$.

Solution: Differentiating the equation and using FTC1, we have $e^{f(x)} = e^x$ 1. Thus $f(x) = x$ 1. Now $\displaystyle\int_a^x e^t\, dt = e^x - e^a$. 1 To satisfy the given equation we must have $a = 0$. 1

3. (5 marks) A comet passes around the Sun on a parabolic trajectory. The Sun is fixed at $(0, 1)$, and the comet's trajectory is described by $y = x^2$.

   (a) Calculate the area that is swept by the line connecting the comet and the Sun as the comet travels from (0,0) to (2,4).

   Solution: First, sketch the trajectory and the line connecting the Sun and the comet. The equation for the line between (0,1) and (2,4) is $y = 1 + \frac{3}{2}x$. Thus the area is

$$A = \int_0^2 \left[\left(1 + \frac{3}{2}x \right) - (x^2) \right] dx \text{①} = \frac{7}{3}. \text{①}$$

   (b) Calculate the volume of the solid generated by rotating the above area around the y axis.

   Solution: To calculate this volume we need to integrate with respect to y, from $y = 0$ to $y = 4$. This may be done in two segments, $\int_0^1 dy + \int_1^4 dy$ as the straight line $y = 1 + \frac{3}{2}x$ ends at the Sun ($y = 1$) and does not go all the way to (0,0). But it is perhaps simpler to take the difference between the volume generated by the parabola and the short cone generated by that line segment.

   For the parabola, now written as $x = \sqrt{y}$, the radius is simply $\sqrt{y}$. For the cone, the radius is $x = \frac{2}{3}(y - 1)$. ① Thus:

$$V = \int_0^4 \pi y \, dy - \int_1^4 \pi \left[\frac{2}{3}(y - 1) \right]^2 dy \text{①} = 8\pi - 4\pi = 4\pi. \text{①}$$

4. (5 marks) Pharaoh Khufu of ancient Egypt wanted to build a grand pyramid with a square base of side L and height H. His queen Henutsen argued instead for a cone-shaped monument with a round base. She said they would save much labor and material for a cone of the same height H and a base diameter L. Can you determine how much work they would have saved with the cone, percentage-wise?

Solution: For the square-based design, a horizontal slice situated at x above ground is a square whose side $l(x)$ can be determined from ratios of similar triangles (the student will have sketched this graph to see the ratio):

$$\frac{l}{L} = \frac{H-x}{H}.$$

Thus, the cross-sectional area is $A(x) = [l(x)]^2 = \frac{L^2}{H^2}(H-x)^2$. ① This slice has to be elevated by the distance x. So the total amount of work for building the pyramid is

$$W_p = \int_0^H \rho g A(x)\, dx\, x = \rho g \frac{L^2}{H^2} \int_0^H x(H-x)^2\, dx = \frac{1}{12}\rho g L^2 H^2, \text{①}$$

where ρ is the density of the limestone material, and g is gravitational acceleration.

For the cone, the calculation is similar. The radius of the cone's base is $R = L/2$, and the radius of a slice at a distance x above ground is $r(x) = \frac{R}{H}(H-x)$. The cross-sectional area is $A(x) = \pi[r(x)]^2 = \frac{\pi}{4}\frac{L^2}{H^2}(H-x)^2$. ① The rest of the calculation is essentially the same, and the final answer, for the total amount of work needed to build the cone, is

$$W_c = \frac{\pi}{48}\rho g L^2 H^2. \text{①}$$

The ratio between W_c and W_p is $\pi/4 \approx 0.785$. The Pharaoh would have saved about 21.5% ① (Either number is fine) with the cone-based design advocated by Queen Henutsen. Maybe this did not impress the Pharaoh and he went with the square-based design that has survived to this day.

5. (4 marks, 2 for each part below)

(a) Determine if this improper integral is convergent: $\displaystyle\int_1^\infty \frac{dx}{x - x^{1/3} + x^{1/5}}$. If yes, evaluate it.

Solution: For $x \geq 1$, $x^{1/3} \geq x^{1/5}$. Thus the integrand $\dfrac{1}{x - x^{1/3} + x^{1/5}} \geq \dfrac{1}{x} \geq 0$. 1 But we know that $\displaystyle\int_1^\infty \frac{dx}{x}$ diverges, thus the integral here also diverges from the Comparison Test. 1

(b) Evaluate the improper integral $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$.

Solution: Let's get the indefinite integral first by substituting $x = u^2$ and $dx = 2udu$:

$$\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{2u\,du}{u(1+u^2)} = 2\int \frac{du}{(1+u^2)} = 2\tan^{-1} u. 1$$

Now the definite integral is

$$2\tan^{-1} u\Big]_0^\infty = 2 \times \frac{\pi}{2} = \pi. 1$$

Note: this integral is improper for two reasons: the integrand is discontinuous at the lower bound $x = 0$, and the upper bound is ∞. So in principle one needs to break it up into the sum of a Type I and a Type II integral:

$$\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}(1+x)}.$$

Now each can be calculated. The final result is the same as above.