ASYMPTOTIC BEHAVIOR OF THE LEAST ENERGY SOLUTION OF A PROBLEM WITH COMPETING POWERS

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ABSTRACT. We consider the problem $\varepsilon^2\Delta u - u^q + u^p = 0$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$. Here Ω is a smooth bounded domain in $\mathbb{R}^N$, $1 < q < p < \frac{N+2}{N-2}$ if $N \geq 3$ and ε is a small positive parameter. We study the asymptotic behavior of the least energy solution as ε goes to zero in the case $q \leq \frac{N}{N-2}$. We show that the limiting behavior is dominated by the singular solution $\Delta G - G^q = 0$ in $\Omega\backslash\{P\}$, $G = 0$ on $\partial\Omega$. The reduced energy is of nonlocal type.

1. INTRODUCTION

There has been considerable interest in understanding the asymptotic behavior of positive solutions of the elliptic problem

$$\begin{cases} \varepsilon^2\Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 \text{ in } \Omega,\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, f is a superlinear nonlinearity with $f(0) = 0$ and Ω is a smooth bounded domain in $\mathbb{R}^N$. The existence and asymptotic behavior of solutions to (1.1) depend crucially on the behavior of f near 0. It is easy to check that problem (1.1) admits solutions on Ω if $f'(0) < 0$, while there may be no nontrivial solutions for small $\varepsilon > 0$ if $f'(0) > 0$. The case of $f'(0) < 0$ is called *positive mass case* and has been studied by many authors. We refer to the papers Berestycki-Lions [2], del Pino-Felmer [8], Ni-Wei [23] and the references therein.

In this paper, we consider the problems in the zero mass case i.e. when $f(0) = 0$ and $f'(0) = 0$. This problem (1.1) can be viewed as *borderline* problems. Berestycki and Lions in [2] proved the existence of ground state solutions if $f(u)$ behaves like $|u|^p$ for large u and $|u|^q$ for small u where p and q are respectively supercritical and subcritical. We remark that this type of equations arises in the Yang-Mills theory and are much harder to handle; see Gidas [13] and Gidas-Ni-Nirenberg [14].

Flucher-Wei [12] considered the case when $f(u) = (u-1)_+^p$ with $p \in (1, \frac{N+2}{N-2})$ and showed that the least energy solution concentrates at a harmonic center of the domain. On the other hand, Dancer-Santra [7] considered another prototype zero-mass problem

$$\begin{cases} \varepsilon^2\Delta u - u^q + u^p = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

1991 *Mathematics Subject Classification.* 35J10, 35J65.
Key words and phrases. least energy solution, asymptotic behavior, zero mass.

where $1 < q < p < \frac{N+2}{N-2}$ and $N \geq 3$. They have proved that for $q \in (q_\star, \frac{N+2}{N-2})$, the least energy solution concentrates at a harmonic center of Ω. Here $q_\star = \frac{N}{N-2}$ is called *zero mass exponent*. Therefore for $q > q_\star$, problem (1.2) behaves similar to the case of $f(u) = (u-1)_+^p$. An open problem is the case of $q \in (1, q_\star]$.

In this paper, we show that when q is below the *zero mass exponent* the asymptotic behavior of the least energy solution of problem (1.2) is not determined by harmonic centers, instead it is determined by a nonlinear singular problem. We state this singular problem first. For any $\xi \in \Omega$ and $q < q_\star$, let $G_q(\cdot, \xi)$ be the unique positive weakly singular solution (see Brezis–Oswald [3]) to the problem

$$\begin{cases} \Delta_x G_q(x,\xi) - G_q(x,\xi)^q = 0 & \text{in } \Omega \setminus \{\xi\}, \\ G_q(x,\xi) \sim \dfrac{\omega_q}{|x-\xi|^{\frac{2}{q-1}}} & \text{for } x \sim \xi, \\ G_q(x,\xi) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

where ω_q is defined as

$$\omega_q^{q-1} = \begin{cases} \frac{2}{q-1}\left[\frac{2}{q-1} - (N-2)\right] & \text{if } q < q_\star \\ \left(\frac{N-2}{\sqrt{2}}\right)^{N-2} & \text{if } q = q_\star \end{cases} \tag{1.4}$$

Surprisingly, it turns out that the least energy solution u_ε concentrates at a global minimum ξ of the *renormalized energy*

$$\Phi_q(\xi) := \lim_{\delta \to 0} \int_{\Omega \setminus B_\delta(\xi)} \left\{ \frac{1}{2}|\nabla G_q(x,\xi)|^2 + \frac{1}{q+1} G_q^{q+1}(x,\xi) - \frac{(q-1)}{2(q+1)} \delta^{N-2-2\alpha} \omega_q^{q+1} \right\}. \tag{1.5}$$

Note that a similar kind of renormalized energy

$$W(\xi) = \lim_{\rho \to 0} \left[\int_{\Omega_\rho} |\nabla_x w|^2 - k\pi \log \frac{1}{\rho} \right]. \tag{1.6}$$

arises when we study the minimization problem

$$E = \inf_{v \in \mathcal{E}} \int_{\Omega_\rho} |\nabla v|^2.$$

where $\Omega_\rho = B_1 \setminus \cup_{i=1}^d \overline{B(a_i, \rho)} \subset \mathbb{R}^2$ and a_i are points in B_1 such that $B(a_i,\rho) \cap B(a_j, \rho) = \emptyset$ for $i \neq j$. Furthermore,

$$\mathcal{E} = \{ v \in H^1(\Omega_\rho; S^1); v = g \text{ on } \partial B \text{ and } deg(v, \partial B(a_i, \rho)) = +1 \text{ for } i = 1, 2, \cdots, d \}$$

see Bethuel-Brezis-Hélein [1]. In other words Φ_q is the remaining energy after a removal of the singular core energy which arises in theoretical physics, see Kleman [17].

In this paper we study the asymptotic behavior of the least energy solution when $q \in (1, q_\star]$.

Let us consider the entire problem

$$\begin{cases} \Delta U - U^q + U^p = 0 & \text{in } \mathbb{R}^N, \\ U > 0 & \text{in } \mathbb{R}^N, \\ U \to 0 & \text{as } |x| \to \infty, \\ U \in C^2(\mathbb{R}^N). \end{cases} \tag{1.7}$$

By Li-Ni [19] and Kwong–Zhang [18], (1.7) has a unique radial solution U such that $U \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \cap \mathrm{L}^{q+1}\left(\mathbb{R}^N\right)$.

Let $\alpha = \max\{\frac{2}{q-1}, N-2\}$. Our first result concerns $q < q_\star$.

Theorem 1.1. *For $N \geq 2$ and $q < q_\star$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, the least energy positive solution of (1.2) $u_\varepsilon \in H_0^1(\Omega)$ has a unique point of maximum x_ε. Moreover, u_ε concentrates at the global minimum of Φ_q, where Φ_q satisfies (1.5).*

For $q = q_\star$, the statement is more complicated. First we need to derive the decay estimates of the solution of the entire problem when $q = q_\star$. This estimate is also obtained in Veron [27]. We obtain this in Section 5 by a slightly different method.

Lemma 1.1. *Let $q = q_\star$. Then the solution U of (1.7) satisfies*

$$U(r) \sim \frac{1}{r^{N-2}(\log r)^{\frac{N-2}{2}}} \tag{1.8}$$

as $r \to +\infty$. Moreover,

$$\lim_{|x|\to\infty} |x|^{N-2} \left(\log|x|\right)^{\frac{N-2}{2}} U(|x|) = \omega_{q_\star} \tag{1.9}$$

where $\omega_{q_\star}$ satisfies (1.4).

Let T be a positive real number such that $T > \operatorname{diam}\Omega$, then $\frac{T}{|x-\xi|} > 1$ for any two points x and ξ in Ω. Without loss of generality, we consider $T = 1$. Furthermore, if $q = q_\star$ and for any $\xi \in \Omega$, we let $H_{q_\star}(\cdot, \xi)$ be the solution to the problem

$$\begin{cases} \Delta_x H_{q_\star}(x,\xi) = 0 & \text{in } \Omega, \\ H_{q_\star}(x,\xi) = \dfrac{1}{|x-\xi|^{N-2}|\log|x-\xi||^{\frac{N-2}{2}}} & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

Our second theorem concerns the borderline case $q = q_\star$.

Theorem 1.2. *For $N \geq 3$ and $q = q_\star$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, the least energy positive solution of (1.2) $u_\varepsilon \in H_0^1(\Omega)$ has a unique point of maximum x_ε. Furthermore, u_ε concentrates at the global minima of $\Psi_{q_\star}$, where $\Psi_{q_\star}$ is defined by*

$$\begin{aligned} \Psi_{q_\star}(\xi) : &= \int_\Omega |\nabla H_{q_\star}(x,\xi)|^2\,dx \\ &+ (N-2)^2 \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-\xi|^{2(N-1)}|\log|x-\xi||^{N-2}}dx \\ &+ \frac{1}{2}(N-2)^2 \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-\xi|^{2(N-1)}|\log|x-\xi||^{N-1}}dx \\ &+ \frac{(N-1)(N-2)}{2} \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-\xi|^{2(N-1)}|\log|x-\xi||^{N}}dx. \end{aligned}$$

In the case $q = 1$, the existence of a single spike solution was first studied by Ni–Wei [23] and they proved that the least energy solution u_ε has a unique (local) maximum, and it is achieved exactly at one point P_ε . Furthermore, u_ε tends to 0 except at its peak P_ε, thereby exhibiting a single spike-layer, and $d(P_\varepsilon, \Omega) \to max_{P\in\Omega} d(P,\Omega)$ as $\varepsilon \to 0$, where d denotes the distance function. A simplified proof was given by Del Pino–Felmer in [8], without using the non-degeneracy condition. But in our case all the terms are non local.

There are two difficulties in the case of $q \leq q_\star$. First since both the inner part of the spike and the outer part of the spike contribute to the second term of the expansion of the energy, we have to glue the inner and outer part at some *neck region*. We achieve this by introducing a *nonlinear projection* in Section 3. This seems to be *new*. Secondly, it seems quite difficult to exclude the boundary spikes since the reduced energy is *nonlocal*.

We summarize the asymptotic behavior of the least energy solution to (1.2) in the following table.

$\varepsilon << 1$	$q = 1$	$1 < q < q_\star$	$q = q_\star$	$q_\star < q < \frac{N+2}{N-2}$
location of maximum points	$\max d(P, \partial\Omega)$	$\max \Phi_q(\xi)$	$\max \Psi_{q_\star}(\xi)$	harmonic center

Locating the concentrating points is an intriguing problem in nonlinear elliptic equations. As far as we know, there are three functions identified for concentration. The first one is the distance function for singularly perturbed problems with nonzero mass (Ni-Wei [23], Del Pino-Felmer [8], Gui-Wei [16]). This is mainly due to the exponential decaying of the ground states. The second function is the mean curvature function for singularly perturbed Neumann problems (Ni-Takagi [22]). This is the effect of boundary condition. The last one, which is commonly found in most of the concentration phenomena for Dirichlet problems, is the Green function and its diagonal part. We refer to Bethuel-Brezis-Helen [1], Del Pino-Kowalczyk-Musso [10] for Ginzburg-Landau equations, Del Pino-Felmer-Musso [9] and Rey [24] for Brezis-Nirenberg type problems, Del Pino-Kowalczyk-Musso [11] for Liouville equations. As far as we know, the nonlocal renormalized energy for zero mass is new, and the nonlinear singular function represents a new type of concentration locations.

In this paper, we have only studied the concentration behavior for least energy solutions. It may be possible to construct concentrating solutions at other critical points of the renormalized energy, by using the nonlinear projection. The main problem is the topological properties of the renormalized energy. Another interesting question is the existence of multiple concentrations (for example, on topologically nontrivial domains).

Finally, we mention two interesting papers by Merle-Peletier [20]-[21] in which they studied problem (1.2) when $q > p \geq \frac{N+2}{N-2}$. The asymptotic behavior of the blow-up solutions is determined by the harmonic centers.

The paper is organized as follows: In Section 2, we prove existence of the least energy solution and give a preliminary analysis of its asymptotic behavior. Section 3 contains the main part of the proof of Theorem 1.1: following [23], we obtain the upper and lower bound for the energy. To this end, we need to introduce a nonlinear projection and study the difference between the least energy solution and its nonlinear projection. The proofs will be quite involved. In Section 4, we show that there is no boundary spikes. We show that when the spikes move toward the boundary its energy increases. Section 5 and Section 6 are devoted to the borderline case $q = q_\star$.

2. Preliminaries

Let us modify the problem (1.2) to

$$\begin{cases} \varepsilon^2\Delta u - (u^+)^q + (u^+)^p = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $u^+ = \max\{u, 0\}$. It is easy to show that any solution of (2.1) is positive and is in fact a positive solution to (1.2). Note that the associated functional to the problem (2.1) is

$$J_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2 - \frac{1}{p+1}(u^+)^{p+1} + \frac{1}{q+1}(u^+)^{q+1}\right)dx.$$

Moreover, J_ε satisfies Palais-Smale condition and all the conditions of the mountain pass theorem and hence there exists a mountain pass solution $u_\varepsilon > 0$ and a mountain pass critical value

$$0 < c_\varepsilon = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\varepsilon(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}.$$

Here $e \in H_0^1(\Omega)$ is such that $J_\varepsilon(e) < 0$.

With a change of variable the problem (1.2) takes the form

$$\begin{cases} \Delta u - u^q + u^p = 0 & \text{in } \Omega_\varepsilon \\ u > 0 & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{2.2}$$

where $\Omega_\varepsilon = \frac{\Omega}{\varepsilon}$ is a re-scaled version of Ω. The functional associated to the problem (2.2) is

$$I_\varepsilon(u) = \int_{\Omega_\varepsilon} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}(u^+)^{p+1} + \frac{1}{q+1}(u^+)^{q+1}\right)dx$$

Note that $I_\varepsilon(0) = 0$, $I_\varepsilon(tu) \to -\infty$ as $t \to +\infty$ and I_ε satisfies the Palais-Smale condition on $H_0^1(\Omega)$. Hence we obtain a positive solution v_ε for each $\varepsilon > 0$ obtained by the mountain pass theorem. Then the mountain pass critical value b_ε is given by

$$b_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} I_\varepsilon(\gamma(t))$$

where

$$\Gamma_\varepsilon = \{\gamma \in C([0,1], H_0^1(\Omega_\varepsilon)) : \gamma(0) = 0, \gamma(1) \neq 0, I_\varepsilon(\gamma(1)) \leq 0\}$$

Note that as 0 is a strict local minima of I_ε, $b_\varepsilon > 0$, $\forall\varepsilon > 0$. Also note that $J_\varepsilon(u) = \varepsilon^N I_\varepsilon(u)$ which implies that $c_\varepsilon = \varepsilon^N b_\varepsilon$. Let

$$\mathcal{N}_\varepsilon(\Omega_\varepsilon) = \left\{u \in H_0^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} |\nabla u|^2 + \int_{\Omega_\varepsilon} (u^+)^{q+1} = \int_{\Omega_\varepsilon} (u^+)^{p+1}\right\}.$$

Lemma 2.1. *(a) For all* $\varepsilon > 0$

$$b_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon}\max_{t\in[0,1]} I_\varepsilon(\gamma(t)) = \inf_{u\in\mathcal{N}_\varepsilon(\Omega_\varepsilon)} I_\varepsilon(u) = \inf_{u\in H_0^1(\Omega_\varepsilon), u\neq 0}\max_{t\geq 0} I_\varepsilon(tu).$$

(b) $0 < b_\varepsilon \le C$ *for sufficiently small* ε *for some* $C > 0$ *where* C *is independent of* ε. *Hence along a subsequence* b_ε *converges as* $\varepsilon \to 0$.

(c) Let z_ε *be a point of local maximum of* v_ε. *Then* $\lim_{\varepsilon\to 0} d(z_\varepsilon, \partial\Omega_\varepsilon) = +\infty$.

(d) $Ker(\Delta + f'(U)) \cap \mathcal{D}^{1,2}(\mathbb{R}^N) = \{\frac{\partial U}{\partial x_1}, \cdots \frac{\partial U}{\partial x_N}\}$.

Proof. This follows from [7]. Since

$$b_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon(\Omega_\varepsilon)} I_\varepsilon(u) = I_\varepsilon(v_\varepsilon)$$

we have

$$b_\varepsilon = I_\varepsilon(v_\varepsilon) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 + \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\int_{\Omega_\varepsilon} v_\varepsilon^{q+1} \tag{2.3}$$

which implies that $\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2$, $\int_{\Omega_\varepsilon} v_\varepsilon^{p+1}$ and $\int_{\Omega_\varepsilon} v_\varepsilon^{q+1}$ are uniformly bounded. First note that from (1.2), $\max_{x\in\Omega} u_\varepsilon \ge 1$. Also note that by Gidas-Spruck [15] we obtain $\|v_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \le C$ and from Schauder estimates, it follows that there exists $C > 0$ such that $\|v_\varepsilon\|_{C^{2,\beta}_{loc}(\mathbb{R}^N)} \le C$ for some $0 < \beta \le 1$. Hence by the Ascoli-Arzela's theorem there exists an $U \ne 0$ such that

$$\|v_\varepsilon - U\|_{C^2_{loc}(\mathbb{R}^N)} \to 0 \text{ as } \varepsilon \to 0.$$

Blowing up around z_ε (where z_ε is a point of maximum of v_ε) we easily see by a limit argument and the strong maximum principle U satisfies (1.7). Note $U \to 0$ as $|x| \to +\infty$ follows from [7]. The only case we have difficulty if z_ε is within order 1 of $\partial\Omega_\varepsilon$. In this case, we obtain a non-trivial solution of the half space problem

$$\begin{cases} \Delta u - u^q + u^p = 0 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } y_N = 0 \\ u \in C^2(\mathbb{R}^N_+) & . \end{cases} \tag{2.4}$$

Suppose $\tilde{U}$ is a solution of (2.4) which achieves its maximum, then by [6] it follows that $\frac{\partial \tilde{U}}{\partial y_N} > 0$ in $\mathbb{R}^N_+$ and hence $\tilde{U}$ cannot achieve a maximum, a contradiction. Using the above argument, it is easy to show that $d(z_\varepsilon, \partial\Omega_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. □

Moreover, if $\alpha := \max\{\frac{2}{q-1}, N-2\}$ we have from Dancer–Santra [7]

$$\lim_{|x|\to\infty} |x|^\alpha U(x) = \omega_q > 0, \text{ if } q \ne q_\star. \tag{2.5}$$

It is easy to check that if

$$q < q_\star \tag{2.6}$$

then $\alpha > N-2$ and

$$U(x) = \frac{\omega_q}{|x|^\alpha} + \mathcal{O}\left(\frac{1}{|x|^{(p-q)\alpha+\alpha}}\right) \text{ as } |x| \to \infty, \tag{2.7}$$

where $\alpha = -\frac{N-2}{2} + \frac{\sqrt{(N-2)^2+4\omega_q^2}}{2}$. Moreover,

$$\lim_{r\to\infty} r^{\alpha(q+1)} U_r^2(r) = \omega_q^{q+1}.$$

Recall that for any $\xi \in \Omega$, let $G_q(\cdot, \xi)$ be defined at (1.3). We consider the singular solution of G_q which behaves like

$$G_q(x,\xi) \sim \frac{\omega_q}{|x-\xi|^\alpha}$$

near ξ. Note that there exist another singular solution which behaves like the fundamental solution but it does not match with the inner solution: the inner solution is like $\varepsilon^{\alpha}|x|^{-\alpha}$, while the other singular solution is like $\varepsilon^{N-2}|x|^{-(N-2)}$ for $N \geq 3$ and in the case $N = 2$, the other singular solution behaves like $\log \frac{\varepsilon}{|x|}$.

Then we can obtain a first order asymptotic of G_q

$$G_q(x,\xi) = \frac{\omega_q}{|x-\xi|^{\alpha}} + \mathcal{O}(|x-\xi|^{\gamma})$$

where $\gamma > \alpha + (2-N) > 0$. Moreover, for $q \geq q_\star$, there exists no singular solutions of (1.3) see Brezis-Veron[4] and the choice of γ follows from Veron [25].

Lemma 2.2. *The function* $H_{q_\star} : \Omega \times \Omega \to \mathbb{R}$; $H_{q_\star}(x,\xi)$ *is positive and* $x \mapsto H_{q_\star}(x,x)$ *is continuous in* $\Omega \times \Omega$. *Furthermore,* $H_{q_\star}(x,x) \to +\infty$ *as* $d(x,\partial\Omega) \to 0$.

Proof. Let $(x_0,y_0) \in \Omega \times \Omega$, let $B = B(x_0,r)$ be a ball centered at x_0 with closures in Ω. Since $H_{q_\star}(x,y)$ is a harmonic function, then by the maximum principle we have

$$0 \leq H_{q_\star}(x,y) \leq \sup_{y\in\partial\Omega} \frac{1}{|x-y|^{N-2}|\log|x-y||^{\frac{N-2}{2}}} \leq \sup_{x\in B}\sup_{y\in\partial\Omega} \frac{1}{|x-y|^{N-2}|\log|x-y||^{\frac{N-2}{2}}} < +\infty.$$

Hence the set $\{H_{q_\star}(x,y)|x \in B\}$ is uniformly bounded in Ω and therefore is uniformly equicontinuous on compact sets of Ω. As a result, we have

$$|H_{q_\star}(x,y) - H_{q_\star}(x_0,y_0)| \leq |H_{q_\star}(x,y) - H_{q_\star}(x,y_0)| + |H_{q_\star}(x,y_0) - H_{q_\star}(x_0,y_0)|$$

and the first term can be made arbitrarily small by the equicontinuity of the family $\{H_{q_\star}(x,y)|x \in B\}$ at y_0 and the smallness of the last term follows by the continuity of $H_{q_\star}(.,y_0)$ at x_0.

Let $H_{q_\star,B}$ be a solution of

$$\begin{cases} \Delta_x H_{q_\star,B}(x,\xi) = 0 & \text{in } B, \\ H_{q_\star,B}(x,\xi) = \dfrac{1}{|x-\xi|^{N-2}|\log|x-\xi||^{\frac{N-2}{2}}} & \text{on } \partial B. \end{cases} \tag{2.8}$$

Since

$$H_{q_\star,B}(x,x_0) < H_{q_\star}(x,x_0) \text{ on } \partial B \tag{2.9}$$

hence by the maximum principle we have

$$H_{q_\star,B}(x,x_0) < H_{q_\star}(x,x_0) \text{ in } B.$$

Hence $H_{q_\star}(x_0,x_0) \geq H_{q_\star,B}(x_0,x_0) = \frac{1}{d(x_0,\partial\Omega)^{N-2}|\log d(x_0,\partial\Omega)|^{\frac{N-2}{2}}} \to +\infty$ if $d(x_0,\partial\Omega) \to 0$. □

For any $\xi \in \mathbb{R}^N$ and for any $\varepsilon > 0$ set

$$U_{\varepsilon,\xi}(x) := U\left(\frac{x-\xi}{\varepsilon}\right) \quad x \in \mathbb{R}^N.$$

It is clear that $U_{\varepsilon,\xi}$ solves

$$-\varepsilon^2 \Delta U_{\varepsilon,\xi} = U^p_{\varepsilon,\xi} - U^q_{\varepsilon,\xi} \text{ in } \mathbb{R}^N. \tag{2.10}$$

3. Profile of Spikes $q < q_\star$

Choose $\delta > 0$ sufficiently small. In course of the proof, we will choose $\delta = \varepsilon^{\sigma_0}$ where $\sigma_0 \ll 1$. Choose a $\eta \in C_0^\infty(\Omega)$ such that $0 \leq \eta \leq 1$

$$\eta(x) = \begin{cases} 1 & \text{in } |x-\xi| \leq \delta, \\ 0 & \text{in } |x-\xi| > 2\delta. \end{cases} \tag{3.1}$$

We define a *nonlinear projection* in the following way: $PU_{\varepsilon,\xi} \in \mathrm{H}_0^1(\Omega)$ is defined as

$$PU_{\varepsilon,\xi} = \eta U_{\varepsilon,\xi} + (1-\eta)\varepsilon^\alpha G_q(x,\xi). \tag{3.2}$$

Note that this kind of projection is *new*: unlike [23] or Dancer-Santra [7], here we have to match the inner and outer solutions in a special way. The reason for this being simple: both inner and outer part of the solutions contribute *equally* to the reduced energy.

First note that

$$G_q(x,\xi) - \frac{\omega_q}{|x-\xi|^\alpha} = \mathcal{O}(|x-\xi|^\gamma).$$

So the difference is regular. First we define

$$f(x,\xi) = G_q(x,\xi) - \frac{\omega_q}{|x-\xi|^\alpha}$$

Lemma 3.1. *Then the following happens*

$$|\nabla f(x,\xi)| = \mathcal{O}(|x-\xi|^{\gamma-1}) \tag{3.3}$$

and

$$|\Delta f(x,\xi)| = \mathcal{O}(|x-\xi|^{\gamma-2}) \tag{3.4}$$

near ξ.

Proof. Without loss of generality, we consider $\xi = 0$. Then

$$\Delta f - \frac{\omega_q^{q-1}}{|x|^2} f = \mathcal{O}(|x|^{\gamma-2}). \tag{3.5}$$

It is easy to check that there exists a $R > 0$ such that

$$|f(x)| \leq C|x|^\nu \text{ in } B_R(0) \subset \Omega.$$

Let $x \in B(\frac{R}{2})$ and $r = \frac{|x|}{2}$. For any $\in B_1$ define $\tilde{f}(y) = f(x+ry)$. Then from (3.5) we have

$$\Delta \tilde{f} = r^2 \Delta f = \omega_q^{q-1}\tilde{f} + \mathcal{O}(|x+ry|^\gamma).$$

Hence by elliptic estimates

$$\begin{aligned} |\nabla \tilde{f}(0)| &\leq C(\|\tilde{f}\|_{L^\infty(B_1(0))} + \|\Delta \tilde{f}\|_{L^\infty(B_1(0))}) \\ &\leq C\|\tilde{f}\|_{L^\infty(B_1(0))} \\ &\leq C\|f\|_{L^\infty(B_1(x))}. \end{aligned}$$

As a result $|\nabla f(x)| \leq C|x|^{\nu-1}$. Similarly

$$|\Delta \tilde{f}(0)| \leq C\|\tilde{f}\|_{L^\infty B_1(0)}$$

and hence we have

$$|\Delta f(x)| \leq C|x|^{\nu-2}.$$

□

Lemma 3.2. *The function*

$$\Phi_{q,\delta}(\xi) := \int_{\Omega\setminus B_\delta(\xi)} \left\{ \frac{1}{2}|\nabla G_q(x,\xi)|^2 + \frac{1}{q+1} G_q^{q+1}(x,\xi) - \frac{(q-1)}{2(q+1)} \delta^{N-2-2\alpha} \omega_q^{q+1} \right\}$$

is uniformly bounded and non-decreasing as $\delta \downarrow 0$. Hence $\Phi_q(\xi)$ exists finitely.

Proof. Let $\delta_1 > \delta_2$, then $\Omega \setminus B_{\delta_1}(\xi) \subset \Omega \setminus B_{\delta_2}(\xi)$ and hence

$$\int_{\Omega\setminus B_{\delta_1}(\xi)} \left\{ \frac{1}{2}|\nabla G_q(x,\xi)|^2 + \frac{1}{q+1} G_q^{q+1}(x,\xi) \right\} \le \int_{\Omega\setminus B_{\delta_2}(\xi)} \left\{ \frac{1}{2}|\nabla G_q(x,\xi)|^2 + \frac{1}{q+1} G_q^{q+1}(x,\xi) \right\}$$

and $-\delta_1^{N-2-2\alpha} < -\delta_2^{N-2-2\alpha}$. As a result, we have

$$\Phi_{q,\delta_1}(\xi) \le \Phi_{q,\delta_2}(\xi).$$

Furthermore, using Lemma 3.1 we have

$$\Phi_{q,\delta}(\xi) \le \frac{(q-1)}{2(q+1)} \omega_q^{q+1} \left[\int_{\Omega\setminus B_\delta(\xi)} \frac{1}{|x-\xi|^{2+2\alpha}} dx - \delta^{N-2-2\alpha} \right] + \mathcal{O}(1).$$

Hence we have

$$|\Phi_{q,\delta}(\xi)| \le C$$

where $C > 0$ independent of δ. Thus along a subsequence $\Phi_{q,\delta}(\xi)$ converges as $\delta \to 0$. □

Lemma 3.3. *The following expansion holds*

$$J_\varepsilon(PU_{\varepsilon,\xi}) = \varepsilon^N I_\infty + \varepsilon^{2\frac{q+1}{q-1}} \Phi_q(\xi) + o\left(\varepsilon^{2\frac{q+1}{q-1}}\right) \tag{3.6}$$

uniformly with respect to ξ in compact sets of Ω, where

$$I_\infty(U) := \int_{\mathbb{R}^N} \left[\frac{p-1}{2(p+1)} U^{p+1}(x) - \frac{q-1}{2(q+1)} U^{q+1}(x) \right] dx \tag{3.7}$$

and the renormalized energy

$$\begin{aligned} \Phi_q(\xi) : &= \lim_{\delta\to 0} \left[\frac{1}{2} \int_{\Omega\setminus B_\delta(\xi)} |\nabla G(x,\xi)|^2 dx + \frac{1}{q+1} \int_{\Omega\setminus B_\delta(\xi)} |G_q(x,\xi)|^{q+1} dx \right. \\ &\left. - \frac{q-1}{2(q+1)} \delta^{N-(2\alpha+2)} \omega_q^{q+1} \right]. \end{aligned} \tag{3.8}$$

Proof. Set $F(s) := \frac{1}{p+1}(s^+)^{p+1} - \frac{1}{q+1}(s^+)^{q+1}$. Here $\alpha = \frac{2}{q-1}$. Let us compute the reduced energy.

$$J_\varepsilon(PU_{\varepsilon,\xi}) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla(PU_{\varepsilon,\xi}(x))|^2 dx + \frac{1}{q+1} \int_\Omega (PU_{\varepsilon,\xi}(x))^{q+1} dx - \frac{1}{p+1} \int_\Omega (PU_{\varepsilon,\xi}(x))^{p+1} dx.$$

We estimate

$$\begin{aligned}
\int_\Omega (PU_{\varepsilon,\xi}(x))^{q+1}\,dx &= \int_{B_\delta(\xi)} U^{q+1}_{\varepsilon,\xi}(x) + \varepsilon^{\alpha(q+1)}\int_{\Omega\setminus B_{2\delta}(\xi)} G^{q+1}_q(x,\xi)\\
&+ \int_{\delta<|x-\xi|<2\delta} (\varepsilon^\alpha G_q + (U_{\varepsilon,\xi}-\varepsilon^\alpha G_q)\eta)^{q+1}\\
&= \int_{B_\delta(\xi)} U_{\varepsilon,\xi}(x)^{q+1} + \varepsilon^{\alpha(q+1)}\int_{\Omega\setminus B_\delta(\xi)} G^{q+1}(x,\xi)\\
&+ \int_{\delta<|x-\xi|<2\delta} [(\varepsilon^\alpha G_q + (U_{\varepsilon,\xi}-\varepsilon^\alpha G_q)\eta)^{q+1} - (\varepsilon^\alpha G_q)^{q+1}]dx\\
&= \int_{\mathbb{R}^N} U_{\varepsilon,\xi}(x)^{q+1} + \varepsilon^{\alpha(q+1)}\left[\int_{\Omega\setminus B_\delta(\xi)} G^{q+1}_q(x,\xi) - \delta^{N-2\alpha-2}\omega_q^{q+1}\right]\\
&+ \int_{\delta<|x-\xi|<2\delta} \left[(\varepsilon^\alpha G_q + (U_{\varepsilon,\xi}-\varepsilon^\alpha G_q)\eta)^{q+1} - (\varepsilon^\alpha G_q)^{q+1}\right]dx\\
&= \varepsilon^N\int_{\mathbb{R}^N} U^{q+1} + \varepsilon^{\alpha(q+1)}\left[\int_{\Omega\setminus B_\delta(\xi)} G^{q+1}_q(x,\xi) - \delta^{N-2\alpha-2}\omega_q^{q+1}\right]\\
&+ \mathcal{O}(1)\varepsilon^{\alpha(q+1)+1}\int_{\delta<|x-\xi|<2\delta} G^q_q(x,\xi)\left\{\frac{\varepsilon^{\alpha(p-q)}}{|x-\xi|^{\alpha(p-q)+\alpha}} + |x-\xi|^\gamma\right\}dx\\
&= \varepsilon^N\int_{\mathbb{R}^N} U^{q+1} + \varepsilon^{\alpha(q+1)}\left[\int_{\Omega\setminus B_\delta(\xi)} G^{q+1}_q(x,\xi) - \delta^{N-2\alpha-2}\omega_q^{q+1}\right]\\
&+ \mathcal{O}(1)\varepsilon^{\alpha(q+1)}\int_{\delta<|x-\xi|<2\delta}\left(\frac{\varepsilon^{\alpha(p-q)}}{|x-\xi|^{\alpha+\alpha p}} + |x-\xi|^{\nu-\alpha q}\right)dx.
\end{aligned}$$

First note that

$$\nabla PU_{\varepsilon,\xi}(x) = \begin{cases}\nabla U_{\varepsilon,\xi} & \text{in } |x-\xi|\le\delta,\\ \varepsilon^\alpha\nabla G_q & \text{in } |x-\xi|>2\delta.\end{cases} \tag{3.9}$$

and in the annulus $\delta<|x-\xi|<2\delta$ we have

$$\nabla PU_{\varepsilon,\xi}(x) = \varepsilon^\alpha\nabla G_q(x,\xi) + 2\nabla\eta(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi}) + \eta\nabla(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi}).$$

Hence we obtain

$$\begin{aligned}
&\int_\Omega |\nabla PU_{\varepsilon,\xi}|^2 = \int_{|x-\xi|<\delta} |\nabla U_{\varepsilon,\xi}|^2 + \varepsilon^{2\alpha}\int_{|x-\xi|>\delta} |\nabla G_q(x,\xi)|^2\\
&+ 4\int_{\delta<|x-\xi|<2\delta} |\nabla\eta|^2|\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi}|^2\\
&+ \int_{\delta<|x-\xi|<2\delta} |\eta|^2|\nabla(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi})|^2 + 2\varepsilon^\alpha\int_{\delta<|x-\xi|<2\delta} \eta\nabla G_q\nabla(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi})\\
&+ 4\varepsilon^\alpha\int_{\delta<|x-\xi|<2\delta} \nabla\eta\nabla G_q(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi}) + 4\int_{\delta<|x-\xi|<2\delta} \eta\nabla\eta\nabla(\varepsilon^\alpha G_q - U_{\varepsilon,\xi})(\varepsilon^\alpha G_q - U_{\varepsilon,\xi}).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\varepsilon^2\int_\Omega |\nabla(PU_{\varepsilon,\xi}(x))|^2dx &= \varepsilon^N\int_{\mathbb{R}^N} |\nabla U|^2 + \varepsilon^{\alpha(q+1)}\left[\int_{\Omega\setminus B_\delta(\xi)} |\nabla G_q(x,\xi)|^2 - \delta^{N-2\alpha-2}\omega_q^{q+1}\right]\\
&+ O(1)\varepsilon^{\alpha(q+1)+1}\delta^{-\alpha(q+1)-1+N}
\end{aligned}$$

and similarly we have

$$\begin{aligned}\int_\Omega (PU_{\varepsilon,\xi}(x))^{p+1}\,dx &= \varepsilon^N \int_{\mathbb{R}^N} U^{q+1} + \varepsilon^{\alpha(q+1)}\left[\int_{\Omega\setminus B_\delta(\xi)} G_q^{q+1}(x,\xi) - \delta^{N-2\alpha-2}\omega_q^{q+1}\right]\\ &+ \mathcal{O}(1)\varepsilon^{\alpha(q+1)}\int_{\delta<|x-\xi|<2\delta}\left(\frac{\varepsilon^{\alpha(p-q)}}{|x-\xi|^{\alpha+\alpha p}} + |x-\xi|^{\nu-\alpha q}\right)dx.\end{aligned}$$

Hence we have

$$J_\varepsilon\left(PU_{\varepsilon,\xi}\right) = \varepsilon^N I_\infty + \varepsilon^{2\alpha+2}\Phi_q(\xi) + o(1)\varepsilon^{2\alpha+2}. \tag{3.10}$$

□

Let

$$E_\varepsilon[u] = \varepsilon^2\Delta u - u^q + u^p.$$

Now we estimate the error due to $PU_{\varepsilon,\xi}(x)$.

Lemma 3.4. *For $\delta > 0$, sufficiently small, there exists $\sigma' > 0$ such that*

$$E_\varepsilon[PU_{\varepsilon,\xi}(x)] = \begin{cases} 0 & \text{in } |x-\xi|<\delta,\\ \mathcal{O}\left(\varepsilon^{2+\alpha+\sigma'}\dfrac{1}{|x-\xi|^{2+\alpha}}\right) & \text{in } \delta<|x-\xi|<2\delta\\ \varepsilon^{\alpha p}G_q^p & \text{in } |x-\xi|>2\delta.\end{cases} \tag{3.11}$$

Proof. First it is easy to note that

$$E_\varepsilon[PU_{\varepsilon,\xi}(x)] = \begin{cases} 0 & \text{in } |x-\xi|<\delta,\\ \varepsilon^{\alpha p}G_q^p & \text{in } |x-\xi|>2\delta.\end{cases} \tag{3.12}$$

So we need to calculate the error when $\delta < |x-\xi| < 2\delta$. We write

$$PU_{\varepsilon,\xi}(x) = U_{\varepsilon,\xi}(x) + (1-\eta)(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi}(x)).$$

Hence we have

$$\begin{aligned}\Delta PU_{\varepsilon,\xi}(x) &= \Delta U_{\varepsilon,\xi}(x) + \Delta(1-\eta)(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi}(x))\\ &= \Delta U_{\varepsilon,\xi}(x) + (1-\eta)\Delta(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi}(x))\\ &- 2\nabla\eta\nabla(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi}(x)) + \Delta\eta(\varepsilon^\alpha G_q(x,\xi) - U_{\varepsilon,\xi}(x)).\end{aligned}$$

As a result, we have

$$\begin{aligned}\varepsilon^2\Delta PU_{\varepsilon,\xi}(x) &= \varepsilon^2\Delta U_{\varepsilon,\xi}(x) + \mathcal{O}\Bigg(\varepsilon^{2+\alpha}|x-\xi|^{\gamma-2} + \frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-\xi|^{\alpha(p-q)+\alpha+2}}\\ &+ \varepsilon^{2+\alpha}|x-\xi|^{\gamma-1} + \frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-\xi|^{\alpha(p-q)+\alpha+1}}\\ &+ \varepsilon^{2+\alpha}|x-\xi|^{\gamma} + \frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-\xi|^{\alpha(p-q)+\alpha}}\Bigg);\end{aligned}$$

$$\begin{aligned}(PU_{\varepsilon,\xi}(x))^q &= (U_{\varepsilon,\xi}(x))^q + \mathcal{O}(U_{\varepsilon,\xi}^{q-1}(\varepsilon^\alpha G_q - U_{\varepsilon,\xi}))\\ &= U_{\varepsilon,\xi}^q + \mathcal{O}\left(\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-\xi|^{\alpha p}} + \varepsilon^{2+\alpha}|x-\xi|^{\gamma-2}\right);\end{aligned}$$

and

$$\begin{aligned}(PU_{\varepsilon,\xi}(x))^p &= (U_{\varepsilon,\xi}(x))^p + \mathcal{O}(U_{\varepsilon,\xi}^{p-1}(\varepsilon^\alpha G_q - U_{\varepsilon,\xi})) \\ &= U_{\varepsilon,\xi}^p + \mathcal{O}\left(\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-\xi|^{\alpha p}} + \varepsilon^{2+\alpha}|x-\xi|^{\gamma-2}\right).\end{aligned}$$

Summing up all the terms and using the fact (2.10) we obtain

$$\begin{aligned}E_\varepsilon[PU_{\varepsilon,\xi}(x)] &= \mathcal{O}\Bigg(\varepsilon^{2+\alpha}|x-\xi|^{\gamma-2} + \frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-\xi|^{\alpha(p-q)+\alpha+2}} \\ &+ \varepsilon^{2+\alpha}|x-\xi|^{\gamma-1} + \frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-\xi|^{\alpha(p-q)+\alpha+1}} \\ &+ \varepsilon^{2+\alpha}|x-\xi|^{\gamma} + \frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-\xi|^{\alpha(p-q)+\alpha}}\Bigg) + \mathcal{O}\left(\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-\xi|^{\alpha p}} + \varepsilon^{2+\alpha}|x-\xi|^{\gamma-2}\right).\end{aligned}$$

Hence we have

$$\begin{aligned}E_\varepsilon[PU_{\varepsilon,\xi}(x)] &= \mathcal{O}\left(\varepsilon^{2+\alpha}\delta^{(\gamma-2)} + \varepsilon^{\alpha(p-q)+\alpha+2}\delta^{-(\alpha(p-q)+\alpha+2)}\right) \\ &= \varepsilon^\alpha\mathcal{O}\left(\varepsilon^2\delta^{-2}(\delta^\gamma + \varepsilon^{\alpha(p-q)}\delta^{-(\alpha(p-q)+\alpha)})\right).\end{aligned}$$

As a result, we can choose $\sigma' \in (0,1)$ sufficiently small such that

$$E_\varepsilon[PU_{\varepsilon,\xi}(x)] = \mathcal{O}\left(\frac{\varepsilon^{2+\alpha+\sigma'}}{|x-\xi|^{2+\alpha}}\right). \tag{3.13}$$

□

Lemma 3.5. *Moreover, if $\xi \in \Omega$, then*

$$c_\varepsilon \le \varepsilon^N I_\infty + \varepsilon^{\frac{2(q+1)}{q-1}}\Phi_q(\xi) + o(\varepsilon^{\frac{2(q+1)}{q-1}}).$$

Proof. Let ξ be a point in Ω. From Lemma 3.3, we obtain

$$J_\varepsilon(PU_{\varepsilon,\xi}) = \varepsilon^N I_\infty + \frac{1}{2}\varepsilon^{2\frac{q+1}{q-1}}\Phi_q(\xi) + o\left(\varepsilon^{2\frac{q+1}{q-1}}\right).$$

Let $t_\varepsilon \in (0,+\infty)$ be the unique constant such that

$$J_\varepsilon(t_\varepsilon PU_{\varepsilon,\xi}) = \max_{t\ge 0} J_\varepsilon(tPU_{\varepsilon,\xi})$$

hence

$$\langle J_\varepsilon'(t_\varepsilon PU_{\varepsilon,\xi}), PU_{\varepsilon,\xi}\rangle = 0. \tag{3.14}$$

We claim that $t_\varepsilon \to 1 + \mathcal{O}(\varepsilon^\alpha)$ as $\varepsilon \to 0$. We have

$$\begin{aligned}\langle J_\varepsilon'(PU_{\varepsilon,\xi}), PU_{\varepsilon,\xi}\rangle &= \int_\Omega \left(\varepsilon^2|\nabla PU_{\varepsilon,\xi}|^2 - (PU_{\varepsilon,\xi})_+^{p+1} + (PU_{\varepsilon,\xi})_+^{q+1}\right) \\ &= \int_\Omega E_\varepsilon[PU_{\varepsilon,x_\varepsilon}]PU_{\varepsilon,x_\varepsilon} = \mathcal{O}(\varepsilon^{\alpha(q+1)}).\end{aligned} \tag{3.15}$$

and analyzing the higher order terms, and using the fact that

$$\int_{\mathbb{R}^N}|\nabla U|^2 = \int_{\mathbb{R}^N}U^{p+1} - \int_{\mathbb{R}^N}U^{q+1}.$$

There exists a $c>0$ such that

$$
\begin{aligned}
J_\varepsilon''(PU_{\varepsilon,\xi})\langle PU_{\varepsilon,\xi},PU_{\varepsilon,\xi}\rangle &= \int_{\Omega_\varepsilon}\Big(\varepsilon^2|\nabla PU_{\varepsilon,\xi}|^2-p(PU_{\varepsilon,\xi})_+^{p+1}+q(PU_{\varepsilon,\xi})_+^{q+1}\Big)\\
&= \varepsilon^N\int_{\mathbb{R}^N}\Big(-(p-1)U^{p+1}+(q-1)U^{q+1}\Big)+O(1)\varepsilon^{\alpha(q+1)}\\
&= \varepsilon^N\Big(-(p-q)\int_{\mathbb{R}^N}U^{p+1}-(q-1)\int_{\mathbb{R}^N}|\nabla U|^2+o(1)\Big)\\
&\le -c'\varepsilon^N. \qquad (3.16)
\end{aligned}
$$

Since $\langle J_\varepsilon'(t_\varepsilon PU_{\varepsilon,\xi}),PU_{\varepsilon,\xi}\rangle=0$ and $\langle J_\varepsilon'(PU_{\varepsilon,\xi}),PU_{\varepsilon,\xi}\rangle=\mathcal{O}(1)\varepsilon^{\alpha(q+1)}$, we have

$$\langle J_\varepsilon'(t_\varepsilon PU_{\varepsilon,\xi})-J_\varepsilon'(PU_{\varepsilon,\xi}),PU_{\varepsilon,\xi}\rangle=\mathcal{O}(1)\varepsilon^{\alpha(q+1)}$$

which implies

$$(t_\varepsilon^2-1)\int_\Omega\varepsilon^2|\nabla PU_{\varepsilon,\xi}|^2-(t_\varepsilon^{p+1}-1)\int_\Omega(PU_{\varepsilon,\xi})_+^{p+1}+(t_\varepsilon^{q+1}-1)\int_\Omega(PU_{\varepsilon,\xi})_+^{q+1}=\mathcal{O}(1)\varepsilon^{\alpha(q+1)}$$

and letting $\tilde{PU}_{\varepsilon,\xi}(x)=PU_{\varepsilon,\xi}(\varepsilon x+\xi)$ in Ω_ε we have

$$(t_\varepsilon^2-1)\int_{\Omega_\varepsilon}|\nabla\tilde{PU}_{\varepsilon,\xi}|^2-(t_\varepsilon^{p+1}-1)\int_{\Omega_\varepsilon}(\tilde{PU}_{\varepsilon,\xi})_+^{p+1}+(t_\varepsilon^{q+1}-1)\int_{\Omega_\varepsilon}(\tilde{PU}_{\varepsilon,\xi})_+^{q+1}=\mathcal{O}(1)\varepsilon^{\alpha(q+1)-N}$$

which implies that $t_\varepsilon-1=\mathcal{O}(1)\varepsilon^\alpha$. Hence we obtain

$$
\begin{aligned}
J_\varepsilon(u_\varepsilon) &\le \max_{t>0}J_\varepsilon(tPU_{\varepsilon,\xi})=J_\varepsilon(t_\varepsilon PU_{\varepsilon,\xi})\\
&= J_\varepsilon(PU_{\varepsilon,\xi})+(t_\varepsilon-1)\langle J_\varepsilon'(PU_{\varepsilon,\xi}),PU_{\varepsilon,\xi}\rangle+\frac{1}{2}(t_\varepsilon-1)^2J_\varepsilon''(\eta_\varepsilon PU_{\varepsilon,\xi})\langle PU_{\varepsilon,\xi},PU_{\varepsilon,\eta}\rangle\\
&\le J_\varepsilon(PU_{\varepsilon,\xi})+o(1)\varepsilon^{\alpha(q+1)}\\
&\le \varepsilon^NI_\infty+\varepsilon^{2\frac{q+1}{q-1}}\Phi_q(\xi)+o\left(\varepsilon^{2\frac{q+1}{q-1}}\right)
\end{aligned}
$$

where η_ε lies in between t_ε and 1. □

Lemma 3.6. *For sufficiently small $\varepsilon>0$, u_ε has a unique maximum.*

Proof. First note by an application of mountain pass theorem, $\varepsilon^2\int_\Omega|\nabla u_\varepsilon|^2\le C$ and hence by Moser iteration, $u_\varepsilon(x)$ is uniformly bounded. Thus applying Schauder estimates we obtain a $C>0$ such that $\|\varepsilon Du_\varepsilon\|_{L^\infty}\le C$. If possible, let $\xi_{\varepsilon,1}$ and $\xi_{\varepsilon,2}$ are two distinct local maxima of u_ε. Then it easily follows that $u_\varepsilon(\xi_{\varepsilon,1})\ge 1$ and $u_\varepsilon(\xi_{\varepsilon,2})\ge 1$. Suppose $\xi_\varepsilon=\frac{\xi_{\varepsilon,1}-\xi_{\varepsilon,2}}{\varepsilon}$. Suppose along a subsequence $|\xi_\varepsilon|\to\delta\in[0,+\infty)$. Let $\xi=\lim_{\varepsilon\to0}\frac{\xi_{\varepsilon,1}-\xi_{\varepsilon,2}}{\varepsilon}$. Then if $\delta>0$, then define $v_\varepsilon(y)=u_\varepsilon(\varepsilon y+\xi_{\varepsilon,2})$ then it follows that, $v_\varepsilon\to U$ in $C^2_{loc}(\mathbb{R}^N)$ and satisfies

$$
\begin{cases}
-\Delta U=U^p-U^q & \text{in } \mathbb{R}^N\\
U'(0)=U'(\delta)=0 & \\
\quad U\to 0 & \text{as } |x|\to\infty
\end{cases}
$$

which is a contradiction as $U'(r)<0$ for $r\in(0,+\infty)$. Now suppose $\delta=0$. Then $v_\varepsilon\to U$ in $C^2_{loc}(\mathbb{R}^N)$ and U has a unique critical point at 0 (since $U(0)>1$ and U is a radial). Thus v_ε has a critical point in a neighborhood of zero which is a contradiction. Hence $|\xi_\varepsilon|\to+\infty$ as $\varepsilon\to0$.

We claim that u_ε has exactly one maximum for sufficiently small $\varepsilon>0$. First note that as u_ε is a mountain pass solution and hence it has Morse index at most one. By the above result $\frac{|\xi_{1,\varepsilon}-\xi_{2,\varepsilon}|}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Now by [7] the principal eigenvalue $\lambda_1 > 0$ such that $-\Delta\psi - f'(U)\psi = \lambda_1\psi$ and is easy to check that $\psi_1 \in D^{1,2}(\mathbb{R}^N)$ hence $\int_{\mathbb{R}^N} |\nabla\psi|^2 - f'(U)\psi^2 < 0$. Now using an appropriate cut-off function, we can obtain the same property for ψ with compact support. Now define a two dimensional subspace spanned by $\psi_1(x) = \psi(\frac{x-\xi_1}{\varepsilon})$ and $\psi_2(x) = \psi(\frac{x-\xi_2}{\varepsilon})$ where $x \in \Omega$. Note that the support *supp* $\psi_1 \cap$ *supp* $\psi_2 = \emptyset$ as $\frac{|\xi_1-\xi_2|}{\varepsilon} \to +\infty$. Hence we obtain a two dimensional space on which $\varepsilon^2 \int_\Omega |\nabla\psi_i|^2 - f'(u_\varepsilon)\psi_i^2 = \int_{\mathbb{R}^N} |\nabla\psi_i|^2 - f'(U)\psi_i^2 < 0$ for $i = 1, 2$. As $u_\varepsilon \to U$ in $C^2_{loc}(\mathbb{R}^N)$ and ψ_i has compact support. Hence u_ε has Morse index at least two, a contradiction. □

First we prove that

Lemma 3.7. *There exists constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1\varepsilon^\alpha G_q(x, x_\varepsilon) \le u_\varepsilon \le C_2\varepsilon^\alpha G_q(x, x_\varepsilon) \text{ in } \Omega \setminus B_{\varepsilon R}(x_\varepsilon). \tag{3.17}$$

Proof. Note that the singularity does not matter on $\Omega\setminus B_{\varepsilon R}(x_\varepsilon)$ as u_ε and $\varepsilon^\alpha G_q(., x_\varepsilon)$ are bounded. We have $\varepsilon^2\Delta u_\varepsilon - u_\varepsilon^q = -u_\varepsilon^p \le 0$ and $\Delta G_q - G_q^q = 0$. Note that $\|u_\varepsilon\|_\infty \ge 1$. Choose $0 < \eta < 1$ such that

$$u_\varepsilon \ge \eta\varepsilon^\alpha G_q(x, x_\varepsilon) \text{ on } \partial B_{\varepsilon R}(x_\varepsilon). \tag{3.18}$$

Then we have

$$\Delta(\eta G_q) - (\eta G_q)^q = \eta\Delta G_q - \eta^q G_q^q = (\eta - \eta^q)G_q^q \ge 0. \tag{3.19}$$

Hence

$$\varepsilon^2\Delta(u_\varepsilon - \eta\varepsilon^\alpha G_q) - u_\varepsilon^q + (\eta\varepsilon^\alpha G_q)^q \le 0$$

which implies that

$$\varepsilon^2\Delta(u_\varepsilon - \eta\varepsilon^\alpha G_q) - \frac{u_\varepsilon^q - (\eta\varepsilon^\alpha G_q)^q}{u_\varepsilon - \eta\varepsilon^\alpha G_q}(u_\varepsilon - \eta\varepsilon^\alpha G_q) \le 0.$$

Hence by the maximum principle we have $u_\varepsilon \ge \eta\varepsilon^\alpha G_q$ in $\Omega \setminus B_{\varepsilon R}(x_\varepsilon)$.
For the upper bound, let $0 < \theta < 1$ such that $u_\varepsilon < \theta$ in $\Omega \setminus B_{\varepsilon R}(x_\varepsilon)$ and $\eta_1 \gg 1$ such that

$$u_\varepsilon \le \eta_1\varepsilon^\alpha G_q(x, x_\varepsilon) \text{ on } \partial B_{\varepsilon R}(x_\varepsilon) \tag{3.20}$$

then we have

$$\Delta(\eta_1 G_q) - (\eta_1 G_q)^q = \eta_1\Delta G_q - \eta_1^q G_q^q = (\eta_1 - \eta_1^q)G_q^q. \tag{3.21}$$

Then u_ε satisfies

$$\varepsilon^2\Delta u_\varepsilon - u_\varepsilon^q \ge -\theta^p \text{ in } \Omega \setminus B_{\varepsilon R}(x_\varepsilon).$$

As a result we obtain

$$\varepsilon^2\Delta(u_\varepsilon - \eta_1\varepsilon^\alpha G_q) - \frac{u_\varepsilon^q - (\eta_1\varepsilon^\alpha G_q)^q}{u_\varepsilon - \eta_1\varepsilon^\alpha G_q}(u_\varepsilon - \eta_1\varepsilon^\alpha G_q) \ge -\theta^p - (\eta_1 - \eta_1^q)G_q^q \ge 0.$$

Hence we obtain by the maximum principle in $\Omega \setminus B_{\varepsilon R}(x_\varepsilon)$

$$u_\varepsilon \le C_2\varepsilon^\alpha G_q(x, x_\varepsilon).$$

□

We write
$$u_\varepsilon = PU_{\varepsilon,x_\varepsilon} + \varepsilon^\alpha \varphi_\varepsilon.$$
If we plug this in equation (1.2) then $\varphi_\varepsilon \in H_0^1(\Omega)$ satisfies
$$\varepsilon^2 \Delta\varphi_\varepsilon + f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon = -\varepsilon^{-\alpha} E_\varepsilon(PU_{\varepsilon,x_\varepsilon}) + N_\varepsilon[\varphi_\varepsilon]. \tag{3.22}$$
where
$$N_\varepsilon[\varphi_\varepsilon] = \varepsilon^{-\alpha}\{f(PU_{\varepsilon,x_\varepsilon} + \varepsilon^\alpha\varphi_\varepsilon) - f(PU_{\varepsilon,x_\varepsilon}) - \varepsilon^\alpha f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon\}.$$

Lemma 3.8. *For sufficiently small $\varepsilon > 0$, there exists $C > 0$ such that*
$$\|\varphi_\varepsilon\|_\infty \le C. \tag{3.23}$$

Proof. We claim that φ_ε is uniformly bounded. If possible, let there exist a sequence ε_k such that $\|\varphi_{\varepsilon,k}\|_\infty \to \infty$. Let $|\varphi_\varepsilon|$ have it maximum at a point k_ε.

Claim $\frac{|k_\varepsilon - x_\varepsilon|}{\varepsilon} < R$.
Suppose not. Then $\frac{|k_\varepsilon - x_\varepsilon|}{\varepsilon} \to +\infty$. Then we have three cases $|x_\varepsilon - k_\varepsilon| \le \delta$, $\delta < |x_\varepsilon - k_\varepsilon| \le 2\delta$ or $|x_\varepsilon - k_\varepsilon| \ge 2\delta$.

When $|x_\varepsilon - k_\varepsilon| \ge 2\delta$, then $-\Delta\varphi_\varepsilon(k_\varepsilon) \ge 0$ and there exists a $c > 0$ such that $\varphi_\varepsilon(k_\varepsilon) \ge c$. We have from (3.22)
$$0 \le -\varepsilon^{2+\alpha}\Delta\varphi_\varepsilon(k_\varepsilon) = \{f(PU_{\varepsilon,x_\varepsilon} + \varepsilon^\alpha\varphi_\varepsilon(k_\varepsilon)) - f(PU_{\varepsilon,x_\varepsilon})\} - E_\varepsilon[PU_{\varepsilon,x_\varepsilon}]$$
which reduces to
$$(G_q + c)^q \le G_q^q + o(1)$$
and hence a contradiction.

The cases $|x_\varepsilon - k_\varepsilon| \le \delta$ and $\delta < |x_\varepsilon - k_\varepsilon| < 2\delta$ are similar to above.
Then we consider $\varphi_\varepsilon(x) = \varphi_\varepsilon(k_\varepsilon + \varepsilon x)$
$$\psi_\varepsilon = \frac{\varphi_\varepsilon}{\|\varphi_\varepsilon\|_\infty}.$$
By the Schauder estimates we obtain $\|\psi_\varepsilon\|_{C^{1,\theta}_{loc}}$ is bounded for some $\theta \in (0,1]$ and hence by the Arzela-Ascoli's theorem there exists $\psi_0 \in C^1$ such that $\|\psi_\varepsilon - \psi_0\|_{C^1_{loc}} \to 0$ as $\varepsilon \to 0$. Using the fact that $\frac{d(k_\varepsilon,\partial\Omega)}{\varepsilon} \to +\infty$, ψ_0 satisfies
$$\begin{cases} \Delta\psi_0 + f'(U)\psi_0 = 0 & \text{in } \mathbb{R}^N \\ \psi_0(k_0) = 1 \end{cases} \tag{3.24}$$
where k_0 is a sequential limit of k_ε. Let us write
$$\psi = \sum_{k=1}^{\infty} \phi_k(r) S_k(\theta)$$
where $r = |x|$, $\theta = \frac{x}{|x|} \in \mathbb{S}^{N-1}$; and $-\Delta_{\mathbb{S}^{N-1}} S_k = \lambda S_k$ where $\lambda_k = k(N-2+k)$; $k \in \mathbb{Z}^+ \cup \{0\}$ and whose multiplicity is given by $M_k - M_{k-2}$ where $M_k = \frac{(N+k-1)!}{(N-1)!k!}$ for $k \ge 2$. Note that $\lambda_0 = 0$ has algebraic multiplicity one and $\lambda_1 = (N-1)$ has algebraic multiplicity N. Then ϕ_k satisfy an infinite system of ODE given by,
$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pU^{p-1} - qU^{q-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \ r \in (0,\infty). \tag{3.25}$$

Also note that (3.25) has two linearly independent solutions $z_{1,k}$ and $z_{2,k}$. Let

$$A_k(\phi) = \phi'' + \frac{N-1}{r}\phi' + \left(pU^{p-1} - qU^{q-1} - \frac{\lambda_k}{r^2}\right)\phi$$

Also recall that if one solution $z_{1,k}$ to (3.25) is known, a second linearly independent solution can be found in any interval where $z_{1,k}$ does not vanish as

$$z_{2,k}(r) = z_{1,k}(r) \int z_{1,k}^{-2} r^{1-N} dr$$

where $\int$ denotes antiderivatives. One can obtain the asymptotic behavior of any solution z as $r \to \infty$ by examining the indicial roots of the associated Euler equation. The limiting equation becomes

$$r^2\phi'' + (N-1)r\phi' - (q\omega_q^{q-1} + \lambda_k)\phi = 0 \tag{3.26}$$

whose indicial roots are given by

$$\mu_k^{\pm} = \begin{cases} \dfrac{N-2}{2} \pm \dfrac{\sqrt{(N-2)^2 + 4(q\omega_q^{q-1} + \lambda_k)}}{2} & \text{if } k \neq 0 \\ \dfrac{N-2}{2} \pm \dfrac{\sqrt{(N-2)^2 + 4q\omega_q^{q-1}}}{2} & \text{if } k = 0 \end{cases}$$

In this way we see that the asymptotic behavior is ruled by $z(r) \sim r^{-\mu}$ as $r \to +\infty$; where μ satisfies the problem

$$\mu^2 - (N-2)\mu - (q\omega_q^{q-1} + \lambda_k) = 0 \text{ if } \alpha = \frac{2}{q-1}. \tag{3.27}$$

This implies that any bounded solution of (3.24) decays and hence $|\psi_0| \leq C|x|^{-(N-2)}$ and hence $\psi_0 \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. Thus $\psi_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and by lemma 2.1, $\psi_0 = \sum_{i=1}^N a_i \frac{\partial U}{\partial x_i}$ where $a_i \in \mathbb{R}$ where not all a_i's are zero. Since U is radial, $U'(0)$ and $\Delta U(0)$ are non-zero, it follows that $\nabla\psi_0(0) \neq 0$.

We obtain a contradiction by proving $\nabla\psi_0(0) = 0$. Note that $\nabla u_\varepsilon(x_\varepsilon) = 0$ and this implies

$$\nabla\psi_\varepsilon(0) = \frac{\nabla u_\varepsilon(x_\varepsilon) - \nabla PU_{\varepsilon,x_\varepsilon}(x_\varepsilon)}{\varepsilon^\alpha\|\varphi_\varepsilon\|_\infty}$$

which implies that $\nabla\psi_\varepsilon(0) \to 0$ as $\varepsilon \to 0$. This implies that $\nabla\psi_0(0) = 0$ by pointwise convergence and hence $\nabla(\sum_{i=1}^N a_i \frac{\partial U}{\partial x_i})(0) = 0$ and this implies that $a_i = 0$ for all i. □

Lemma 3.9. *We have,*

$$c_\varepsilon = \varepsilon^N I_\infty + \varepsilon^{2\alpha+2}\Phi_q(x_\varepsilon) + o(\varepsilon^{2(\alpha+1)}). \tag{3.28}$$

Proof. We want to write $u_\varepsilon = PU_{\varepsilon,x_\varepsilon} + \varepsilon^\alpha\varphi_\varepsilon$. So we have

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &= J_\varepsilon(PU_{\varepsilon,x_\varepsilon}) \\ &+ \varepsilon^\alpha \int_\Omega (\varepsilon^2\nabla PU_{\varepsilon,x_\varepsilon}\nabla\varphi_\varepsilon - f(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon)dx \\ &+ \frac{\varepsilon^{2\alpha}}{2}\left(\int_\Omega \varepsilon^2|\nabla\varphi_\varepsilon|^2 dx - f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon^2\right) \\ &- \int_\Omega \left[F(PU_{\varepsilon,x_\varepsilon} + \varepsilon^\alpha\varphi_\varepsilon) - F(PU_{\varepsilon,x_\varepsilon}) - \varepsilon^\alpha f(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon^2\right]. \end{aligned}$$

which can be expressed as

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &= J_\varepsilon(PU_{\varepsilon,x_\varepsilon}) \\ &+ \varepsilon^\alpha \int_\Omega E_\varepsilon(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon dx \\ &+ \frac{\varepsilon^{2\alpha}}{2}\Big(\varepsilon^2\int_\Omega |\nabla\varphi_\varepsilon|^2 dx - f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon^2\Big) \\ &- \int_\Omega \Big[F(PU_{\varepsilon,x_\varepsilon}+\varepsilon^\alpha\varphi_\varepsilon) - F(PU_{\varepsilon,x_\varepsilon}) - \varepsilon^\alpha f(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon - \frac{\varepsilon^{2\alpha}}{2} f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon^2\Big]. \end{aligned}$$

Now we estimate the following terms

$$\begin{aligned} \int_\Omega E_\varepsilon(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon dx &= \int_{\delta<|x-x_\varepsilon|<2\delta} E_\varepsilon(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon + \int_{|x-x_\varepsilon|>2\delta} E_\varepsilon(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon \\ &\le C\varepsilon^{2+\alpha+\sigma'}\int_{\delta<|x-x_\varepsilon|<2\delta}\frac{1}{|x-x_\varepsilon|^{2+\alpha}} + \varepsilon^{\alpha p}\int_{|x-x_\varepsilon|>2\delta} G_q^p\varphi_\varepsilon \\ &\le o(1)\varepsilon^{\alpha+2}. \end{aligned}$$

Note that $\alpha+2-N>0$.

From (3.22)

$$\int_\Omega \{\varepsilon^2|\nabla\varphi_\varepsilon|^2 dx - f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon^2\} = \varepsilon^{-\alpha}\int_\Omega E_\varepsilon(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon - \int_\Omega N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon.$$

As a result, we only estimate

$$\begin{aligned} \int_\Omega N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon &= \int_{|x-x_\varepsilon|\le\varepsilon R} N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon + \int_{\varepsilon R<|x-x_\varepsilon|\le\delta} N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon \\ &+ \int_{\delta<|x-x_\varepsilon|<2\delta} N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon + \int_{|x-x_\varepsilon|\ge 2\delta} N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon \\ &= I_1 + I_2 + \int_{\delta<|x-x_\varepsilon|<2\delta} N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon + \int_{|x-x_\varepsilon|\ge 2\delta} N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon. \end{aligned}$$

We compute I_1. If $q>2$, then we obtain

$$I_1 = \varepsilon^\alpha \mathcal{O}\Big(\int_{B_{\varepsilon R}(x_\varepsilon)} U_{\varepsilon,x_\varepsilon}^{q-2}\varphi_\varepsilon^3\Big) = \mathcal{O}(\varepsilon^{\alpha+N}).$$

We calculate I_2.

$$\begin{aligned} I_2 &= \varepsilon^\alpha\mathcal{O}\Big(\int_{B_\delta(x_\varepsilon)\setminus B_{\varepsilon R}(x_\varepsilon)} U_{\varepsilon,x_\varepsilon}^{q-2}\varphi_\varepsilon^3\Big) \\ &= \varepsilon^\alpha\mathcal{O}\Big(\int_{B_\delta(x_\varepsilon)\setminus B_{\varepsilon R}(x_\varepsilon)} \frac{\varepsilon^{\alpha(q-2)}}{|x-x_\varepsilon|^{\alpha(q-2)}}\Big) = \mathcal{O}(\varepsilon^2\delta^{N-\alpha(q-2)}). \end{aligned}$$

When $q\le 2$ we obtain

$$I_1 = \mathcal{O}\Big(\int_{B_{\varepsilon R}(x_\varepsilon)} U_{\varepsilon,x_\varepsilon}^{q-1}\Big(\frac{\varepsilon^\alpha\varphi_\varepsilon}{PU_{\varepsilon,x_\varepsilon}}\Big)\varphi_\varepsilon^2\Big) = \mathcal{O}(\varepsilon^N)$$

and noting the fact that $\left|\frac{\varepsilon^\alpha \varphi_\varepsilon}{PU_{\varepsilon,x_\varepsilon}}\right| \leq C\varepsilon^\alpha$ whenever $\varepsilon R \leq |x-x_\varepsilon| \leq \delta$ we obtain,

$$\begin{aligned} I_2 &= \mathcal{O}\Big(\int_{B_\delta(x_\varepsilon)\setminus B_{\varepsilon R}(x_\varepsilon)} U^{q-1}_{\varepsilon,x_\varepsilon}\Big(\frac{\varepsilon^\alpha\varphi_\varepsilon}{PU_{\varepsilon,x_\varepsilon}}\Big)\varphi_\varepsilon^2\Big) \\ &= \varepsilon^{\alpha(q-1)}\mathcal{O}\Big(\int_{B_\delta(x_\varepsilon)\setminus B_{\varepsilon R}(x_\varepsilon)} \frac{1}{|x-x_\varepsilon|^{\alpha(q-1)}}\Big) \\ &= \mathcal{O}(\varepsilon^2\delta^\alpha\delta^{N-\alpha(q-1)}) = \mathcal{O}(\varepsilon^2\delta^{N-2+\alpha}). \end{aligned}$$

Estimating in the neck region

$$\int_{\delta<|x-x_\varepsilon|<2\delta} N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon = \mathcal{O}\Big(\varepsilon^\alpha\int_{\delta<|x-x_\varepsilon|<2\delta} PU^{q-2}_{\varepsilon,x_\varepsilon}\varphi_\varepsilon^3\Big).$$

In the neck region we have

$$PU_{\varepsilon,x_\varepsilon} = U_{\varepsilon,x_\varepsilon} + (1-\eta)(\varepsilon^\alpha G_q - U_{\varepsilon,x_\varepsilon}).$$

In order to estimate

$$\begin{aligned} \varepsilon^\alpha\int_{\delta<|x-x_\varepsilon|<2\delta} PU^{q-2}_{\varepsilon,x_\varepsilon}\varphi_\varepsilon^3 &= \varepsilon^2\int_{\delta<|x-x_\varepsilon|<2\delta}\frac{1}{|x-x_\varepsilon|^{\alpha(q-2)}}\varphi_\varepsilon^3 \\ &\leq C\varepsilon^2\int_{\delta<|x-x_\varepsilon|<2\delta}\frac{1}{|x-x_\varepsilon|^{\alpha(q-2)}} \\ &= \mathcal{O}(\varepsilon^2\delta^{N-\alpha(q-2)}). \end{aligned}$$

Whenever $|x-x_\varepsilon| > 2\delta$, we have

$$\int_{|x-x_\varepsilon|\geq 2\delta} N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon = \mathcal{O}(\varepsilon^{\alpha q}).$$

Similarly, we show that

$$\begin{aligned} &\int_\Omega\Big[F(PU_{\varepsilon,x_\varepsilon}+\varepsilon^\alpha\varphi_\varepsilon) - F(PU_{\varepsilon,x_\varepsilon}) - \varepsilon^\alpha f(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon - \frac{\varepsilon^{2\alpha}}{2}f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon^2\Big] \\ = \;& o(\varepsilon^{2+2\alpha}). \end{aligned}$$

The estimate follows exactly as the previous estimate. This completes the proof. □

4. Exclusion of boundary spikes

Now we are required to show that the blow-up does not occur near the boundary. That is we claim that $\Phi_q(\xi) \to +\infty$ if $\xi \to \partial\Omega$. Let $x_\varepsilon \in \Omega$ be a sequence such that $d_\varepsilon = d(x_\varepsilon, \partial\Omega) \to 0$ as $\varepsilon \to 0$. Now we define a scaling of the form

$$x \mapsto \frac{x}{d_\varepsilon}$$

then the original equation reduces to

$$\begin{cases} \bar{\varepsilon}^2\Delta u - u^q + u^p = 0 & \text{in } \Omega_{d_\varepsilon} \\ u > 0 & \text{in } \Omega_{d_\varepsilon} \\ u = 0 & \text{on } \partial\Omega_{d_\varepsilon}, \end{cases} \tag{4.1}$$

where $\bar{\varepsilon} = \frac{\varepsilon}{d_\varepsilon}$. Let $\xi_\varepsilon = \frac{x_\varepsilon}{d_\varepsilon}$ and note that $\Omega_{d_\varepsilon} \mapsto \mathbb{R}^N_+$ (any compact subset of C of $\mathbb{R}^N_+$ can be embedded in Ω_{d_ε} for ε sufficiently small) as $\varepsilon \to 0$.

Let us consider the problem

$$(4.2)\qquad \begin{cases} \Delta_x G_q(x,\xi_\varepsilon) - G_q(x,\xi_\varepsilon)^q = 0 & \text{in } \Omega_{d_\varepsilon}\setminus\{\xi_\varepsilon\}, \\ G_q(x,\xi_\varepsilon) \geq 0 & \text{in } \Omega_{d_\varepsilon} \\ G_q(x,\xi_\varepsilon) = 0 & \text{on } \partial\Omega_{d_\varepsilon}. \end{cases}$$

$$(4.3)\qquad \begin{cases} \Delta_x G_{q,\mathbb{R}^N_+}(x,\xi) - G_{q,\mathbb{R}^N_+}(x,\xi)^q = 0 & \text{in } \mathbb{R}^N_+\setminus\{\xi\}, \\ G_{q,\mathbb{R}^N_+}(x,\xi) \geq 0 & \text{in } \mathbb{R}^N_+ \\ G_{q,\mathbb{R}^N_+}(x,\xi) = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

Note that $\xi_\varepsilon \to \xi$ where ξ does not lie on $\partial\mathbb{R}^N_+$.

Lemma 4.1. *The solution to (4.2) converges uniformly on compact subsets to the solution of (4.3). Moreover, there exists a $C > 0$ such that*

$$|G_{q,\mathbb{R}^N_+}(x,\xi)| \leq C|x-\xi|^{-\alpha}$$

and

$$|\nabla G_{q,\mathbb{R}^N_+}(x,\xi)| \leq C|x-\xi|^{-(\alpha+1)}$$

hold. Note that the second estimate holds away from the singularity.

Proof. Let $G_{q,0}(x,\xi) = \frac{\omega_q}{|x-\xi|^\alpha}$. Consider the solution to the problem (4.2). Note that $G_{q,0}(x,\xi) - G_q(x,\xi) > 0$ on $\partial\Omega_{d_\varepsilon}$. Furthermore, $(G_{q,0}(x,\xi) - G_q(x,\xi)) \to 0$ as $x \to \xi$. Let $S = G_{q,0}(x,\xi) - G_q(x,\xi))$. Then S satisfies

$$\Delta S - a(x)S = 0$$

where $a(x) > 0$. Hence by maximum principle we have $S \geq 0$ in Ω_{d_ε}. This implies $G_{q,0}(x,\xi) \geq G_q(x,\xi)$. Let $r > 0$ such that $B_r(\xi) \subset \Omega$ and consider

$$(4.4)\qquad \begin{cases} \Delta_x G_{q,B}(x,\xi) - G_{q,B}(x,\xi)^q = 0 & \text{in } B_r(\xi)\setminus\{\xi\}, \\ G_{q,B}(x,\xi) \geq 0 & \text{in } B_r(\xi) \\ G_{q,B}(x,\xi) = 0 & \text{on } \partial B_r(\xi). \end{cases}$$

By similar method it can be prove that $G_{q,B}(x,\xi) \leq G_q(x,\xi)$ in $B_r(\xi)$. Thus we have $G_{q,B}(x,\xi) \leq G_q(x,\xi) \leq G_{q,0}(x,\xi)$, hence by the Schauder estimate we conclude that $G_q(.,\xi_\varepsilon) \to G_{q,\mathbb{R}^N_+}(.,\xi)$ uniformly in compact subsets. Note that ξ_ε is not on the boundary and it is of distance $\mathcal{O}(1)$ from the boundary and hence $\xi \notin \partial\mathbb{R}^N_+$. Moreover far away from the ξ we can use the boundary estimates (note that there is no singularity) to obtain

$$(4.5)\qquad |\nabla G_{q,\mathbb{R}^N_+}(x,\xi)| \leq \frac{C}{|x-\xi|^{\alpha+1}}.$$

Note that using the local estimates in Brezis–Oswald [3], we have near ξ the following estimate holds

$$(4.6)\qquad G_{q,\mathbb{R}^N_+}(x,\xi) = G_{q,0}(x,\xi) + O(|x-\xi|^\gamma).$$

and moreover, (4.6) implies that the solution of (4.3) is unique. □

Corollary 4.1. *Then we have for all $\delta > 0$, as $\varepsilon \to 0$*

$$\begin{aligned}&\int_{\Omega_{d_\varepsilon}\setminus B_\delta(\xi_\varepsilon)} \frac{1}{2}|\nabla_x G_q(x,\xi_\varepsilon)|^2 + \frac{1}{q+1}G_q^{q+1}(x,\xi_\varepsilon)\\ =& \int_{\mathbb{R}^N_+\setminus B_\delta(\xi)} \frac{1}{2}|\nabla_x G_q(x,\xi)|^2 + \frac{1}{q+1}G_q^{q+1}(x,\xi) + o_\varepsilon(1).\end{aligned}$$

Proof. This follows from the dominated convergence theorem. □

Lemma 4.2. *Then the least energy solution of (4.1) satisfies*

$$J_\varepsilon(u_\varepsilon) \le \bar{\varepsilon}^N I_\infty + \bar{\varepsilon}^{2\alpha+2}\Phi_q(\xi) + o(\bar{\varepsilon}^{2\alpha+2}) \tag{4.7}$$

and

$$\begin{aligned}J_\varepsilon(u_\varepsilon) &= \bar{\varepsilon}^N I_\infty + \bar{\varepsilon}^{2\alpha+2}\lim_{\delta\to 0}\Big[\int_{\Omega_{d_\varepsilon}\setminus B_\delta(\xi_\varepsilon)} \frac{1}{2}|\nabla_x G_q(x,\xi_\varepsilon)|^2\\ &+ \frac{1}{q+1}G_q^{q+1}(x,\xi_\varepsilon) - \frac{q-1}{2(q+1)}\delta^{N-(2\alpha+2)}\omega_q^{q+1}\Big] + o(\bar{\varepsilon}^{2\alpha+2}).\end{aligned}$$

Proof. Follows exactly as the upper and lower estimate. Moreover, note that

$$\begin{aligned}J_\varepsilon(u_\varepsilon) &\le \bar{\varepsilon}^N I_\infty + \bar{\varepsilon}^{2\alpha+2}\Phi_q(\xi) + o(\bar{\varepsilon}^{2\alpha+2})\\ &\le \bar{\varepsilon}^N I_\infty + \bar{\varepsilon}^{2\alpha+2}\lim_{\delta\to 0}\Big[\int_{\Omega_{d_\varepsilon}\setminus B_\delta(\xi)} \frac{1}{2}|\nabla_x G_q(x,\xi)|^2\\ &+ \frac{1}{q+1}G_q^{q+1}(x,\xi) - \frac{q-1}{2(q+1)}\delta^{N-(2\alpha+2)}\omega_q^{q+1}\Big] + o(\bar{\varepsilon}^{2\alpha+2})\\ &\le \bar{\varepsilon}^N I_\infty + \bar{\varepsilon}^{2\alpha+2}\mathcal{O}(d_\varepsilon^{2+2\alpha-N}) + o(\bar{\varepsilon}^{2\alpha+2})\\ &\le \bar{\varepsilon}^N I_\infty + \varepsilon^{2\alpha+2}\mathcal{O}(d_\varepsilon^{-N}) + o(\bar{\varepsilon}^{2\alpha+2})\end{aligned}$$

and we have

$$\begin{aligned}J_\varepsilon(u_\varepsilon) &= \bar{\varepsilon}^N I_\infty + \bar{\varepsilon}^{2\alpha+2}\lim_{\delta\to 0}\Big[\int_{\Omega_{d_\varepsilon}\setminus B_\delta(\xi_\varepsilon)} \frac{1}{2}|\nabla_x G_q(x,\xi_\varepsilon)|^2\\ &+ \frac{1}{q+1}G_q^{q+1}(x,\xi_\varepsilon) - \frac{q-1}{2(q+1)}\delta^{N-(2\alpha+2)}\omega_q^{q+1} + o(1)\Big]\\ &= \bar{\varepsilon}^N I_\infty + \varepsilon^{2\alpha+2}d_\varepsilon^{-2-2\alpha}\lim_{\delta\to 0}\Big[\int_{\mathbb{R}^N_+\setminus B_\delta(\xi)} \frac{1}{2}|\nabla_x G_q(x,\xi)|^2\\ &+ \frac{1}{q+1}G_q^{q+1}(x,\xi) - \frac{q-1}{2(q+1)}\delta^{N-(2\alpha+2)}\omega_q^{q+1} + o(1)\Big] + o(\bar{\varepsilon}^{2\alpha+2}).\end{aligned} \tag{4.8}$$

□

Lemma 4.3. *In order to show that there is no boundary spikes it is enough to show that there exists a $c_0 > 0$ such that*

$$\lim_{\delta\to 0}\Big[\int_{\mathbb{R}^N_+\setminus B_\delta(\xi)} \frac{1}{2}|\nabla_x G_q(x,\xi)|^2 + \frac{1}{q+1}G_q(x,\xi)^{q+1} - \frac{q-1}{2(q+1)}\delta^{N-(2\alpha+2)}\omega_q^{q+1}\Big] \ge c_0. \tag{4.9}$$

Proof. We consider equation for $G_q(x,\xi)$. From the Green identity we deduce that

$$\int_{\Omega\setminus B_\delta(\xi)} |\nabla G_q(x,\xi)|^2 + G^{q+1}(x,\xi) = -\int_{\partial B_\delta(\xi)} G_q(x,\xi)\frac{\partial G_q(x,\xi)}{\partial \nu}. \tag{4.10}$$

Again by the Pohozaev identity we have

$$
\begin{aligned}
\int_{\Omega\setminus B_\delta(\xi)}\Big[-\frac{N}{q+1}G_q^{q+1}+\frac{N-2}{2}G_q^{q+1}\Big] &= \int_{\partial(\Omega\setminus B_\delta(\xi))}\Big[\langle x-\xi,\nabla G_q\rangle\frac{\partial G_q}{\partial\nu}-\langle x-\xi,\nabla G_q\rangle\frac{1}{2}|\nabla G_q|^2\\
&+ \langle x-\xi,\nu\rangle(-\frac{1}{q+1}G_q^{q+1})+\frac{N-2}{2}G_q\frac{\partial G_q}{\partial\nu}\Big]\\
&= \frac{1}{2}\int_{\partial\Omega}\langle x-\xi,\nu\rangle|\nabla G_q|^2\\
&- \int_{\partial B_\delta(\xi)}\Big[\langle x-\xi,\nabla G_q\rangle\frac{\partial G_q}{\partial\nu}-\langle x-\xi,\nu\rangle\frac{1}{2}|\nabla G_q|^2\\
&- \frac{1}{q+1}\langle x-\xi,\nu\rangle G_q^{q+1}+\frac{N-2}{2}G_q\frac{\partial G_q}{\partial\nu}\Big].
\end{aligned}
$$

But we also have from the equation (1.3)

$$
\begin{aligned}
\int_{\Omega\setminus B_\delta(\xi)}\Big[|\nabla G_q|^2+\frac{2}{q+1}G_q^{q+1}\Big] &= -\Big(1+\frac{N-2}{2}\beta\Big)\int_{\partial B_\delta(\xi)}G_q\frac{\partial G_q}{\partial\nu}\\
&= \beta\int_{\partial B_\delta(\xi)}\Big[\langle x-\xi,\nabla G_q\rangle\frac{\partial G_q}{\partial\nu}-\langle x-\xi,\nu\rangle\frac{1}{2}|\nabla G_q|^2\\
&- \frac{1}{q+1}\langle x-\xi,\nu\rangle G_q^{q+1}\Big].
\end{aligned}
$$

where

$$
\beta\Big(\frac{N}{q+1}-\frac{N-2}{2}\Big)=\frac{q-1}{q+1}.
$$

Let $\Omega=\mathbb{R}^N_+$. Then we have

$$
\begin{aligned}
\int_{\mathbb{R}^N_+\setminus B_\delta(\xi)}\Big[|\nabla G_{q,\mathbb{R}^N_+}|^2+\frac{2}{q+1}G_{q,\mathbb{R}^N_+}^{q+1}\Big] &= -\Big(1+\frac{N-2}{2}\beta\Big)\int_{\partial B_\delta(\xi)}G_{q,\mathbb{R}^N_+}\frac{\partial G_{q,\mathbb{R}^N_+}}{\partial\nu}\\
&+ \frac{\beta}{2}\int_{\mathbb{R}^N_+}\langle x-\xi,\nu\rangle|\nabla G_{q,\mathbb{R}^N_+}|^2\\
&- \beta\int_{\partial B_\delta(\xi)}\Big[\langle x-\xi,\nabla G_{q,\mathbb{R}^N_+}\rangle\frac{\partial G_{q,\mathbb{R}^N_+}}{\partial\nu}-\langle x-\xi,\nu\rangle\frac{1}{2}|\nabla G_{q,\mathbb{R}^N_+}|^2\\
&- \frac{1}{q+1}\langle x-\xi,\nu\rangle G_{q,\mathbb{R}^N_+}^{q+1}(x,\xi)\Big].
\end{aligned}
$$

Then up to a rotation we can assume that $\mathbb{R}^N_+=\{(x',-1):x'\in\mathbb{R}^{N-1}\}$. Note that on $\mathbb{R}^N_+$ we have $\langle x-\xi,\nu\rangle=1$. As a result we have

$$
\begin{aligned}
\int_{\mathbb{R}^N_+\setminus B_\delta(\xi)}\Big[|\nabla G_{q,\mathbb{R}^N_+}|^2+\frac{2}{q+1}G_{q,\mathbb{R}^N_+}^{q+1}\Big] &= -\Big(1+\frac{N-2}{2}\beta\Big)\int_{\partial B_\delta(\xi)}G_{q,\mathbb{R}^N_+}\frac{\partial G_{q,\mathbb{R}^N_+}}{\partial\nu}+\frac{\beta}{2}\int_{\mathbb{R}^N_+}|\nabla G_{q,\mathbb{R}^N_+}|^2\\
&- \beta\int_{\partial B_\delta(\xi)}\Big[\langle x-\xi,\nabla G_{q,\mathbb{R}^N_+}\rangle\frac{\partial G_q}{\partial\nu}-\langle x-\xi,\nu\rangle\frac{1}{2}|\nabla G_{q,\mathbb{R}^N_+}|^2\\
&- \frac{1}{q+1}\langle x-\xi,\nu\rangle G_{q,\mathbb{R}^N_+}^{q+1}\Big]. \qquad (4.11)
\end{aligned}
$$

Moreover, note that $G_{q,0} = \frac{\omega_q}{|x-\xi|^\alpha}$ is a unique solution of

$$\Delta G_q(x,\xi) - G_q(x,\xi)^q = 0 \text{ in } \mathbb{R}^N$$

and so we have

$$\int_{\mathbb{R}^N\setminus B_\delta(\xi)} \frac{1}{2}|\nabla G_{q,0}(x,\xi)|^2 + \frac{1}{q+1}G_{q,0}^{q+1}(x,\xi) = \frac{q-1}{2(q+1)}\omega_q^{q+1}\delta^{N-2\alpha-2} \tag{4.12}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_\delta(\xi)} \Big[|\nabla G_{q,0}|^2 + \frac{2}{q+1}G_{q,0}^{q+1}\Big] &= -\Big(1+\frac{N-2}{2}\beta\Big)\int_{\partial B_\delta(\xi)} G_{q,0}\frac{\partial G_{q,0}}{\partial \nu} \\ &- \beta\int_{\partial B_\delta(\xi)}\Big[\langle x-\xi_\varepsilon, \nabla G_{q,0}\rangle\frac{\partial G_{q,0}}{\partial\nu} - \langle x-\xi,\nu\rangle\frac{1}{2}|\nabla G_{q,0}|^2 \\ &- \frac{1}{q+1}\langle x-\xi,\nu\rangle G_{q,0}^{q+1}\Big].\end{aligned} \tag{4.13}$$

Hence using the estimates for $G_{q,\mathbb{R}^N_+}$ in (4.6) we obtain as $\delta \to 0$

$$\int_{\partial B_\delta(\xi)} G_{q,\mathbb{R}^N_+}\frac{\partial G_{q,\mathbb{R}^N_+}}{\partial\nu} - \int_{\partial B_\delta(\xi)} G_{q,0}\frac{\partial G_{q,0}}{\partial\nu} = o_\delta(1)$$

and

$$\begin{aligned}&\int_{\partial B_\delta(\xi)}\Big[\langle x-\xi,\nabla G_{q,0}\rangle\frac{\partial G_{q,0}}{\partial\nu} - \langle x-\xi,\nu\rangle\frac{1}{2}|\nabla G_{q,0}|^2 - \frac{1}{q+1}\langle x-\xi,\nu\rangle G_{q,0}^{q+1}\Big] \\ -&\int_{\partial B_\delta(\xi)}\Big[\langle x-\xi,\nabla G_{q,\mathbb{R}^N_+}\rangle\frac{\partial G_{q,\mathbb{R}^N_+}}{\partial\nu} - \langle x-\xi,\nu\rangle\frac{1}{2}|\nabla G_{q,\mathbb{R}^N_+}|^2 - \frac{1}{q+1}\langle x-\xi,\nu\rangle G_{q,\mathbb{R}^N_+}^{q+1}\Big] = o_\delta(1).\end{aligned}$$

Subtracting (4.11) from (4.13) we obtain

$$\int_{\mathbb{R}^N_+\setminus B_\delta(\xi)}\Big[\frac{1}{2}|\nabla G_{q,\mathbb{R}^N_+}|^2 + \frac{1}{q+1}G_{q,\mathbb{R}^N_+}^{q+1} - \frac{q-1}{2(q+1)}\omega_q^{q+1}\delta^{N-2\alpha-2}\Big] = \frac{\beta}{2}\int_{\partial\mathbb{R}^N_+}|\nabla G_{q,\mathbb{R}^N_+}|^2 > 0.$$

This proves the claim. □

Remark 4.1. *Lemma 4.2 implies* $d_\varepsilon^{N-2-2\alpha} \le C$ *where* $C>0$ *independent of* ε *and* $N-2\alpha-2<0$. *Hence we will obtain a contradiction.*

Remark 4.2. *Hence from the upper bound and the lower bound of* c_ε *we infer that*

$$\lim_{\varepsilon\to 0}\Phi_q(x_\varepsilon) = \Phi_q(\xi)$$

and ξ *minimizes the renormalized energy which is characterized by*

$$\Phi_q(\xi) = \lim_{\delta\to 0}\Big[\frac{1}{2}\int_{\Omega\setminus B_\delta(\xi)}|\nabla_x G_q(x,\xi)|^2 + \frac{1}{q+1}G_q^{q+1}(x,\xi) - \frac{q-1}{2(q+1)}\omega_q^{q+1}\delta^{N-2\alpha-2}\Big].$$

Remark 4.3. *Note that we can easily choose domains such that the points of maximal distance to the boundary are unchanged by smooth small perturbations of* Ω *on an open set of the boundary. On the other hand, the perturbations almost certainly move the locations of where* Φ_q *have their minimum. Thus it seems almost certain the location of peak of the mountain pass solution is different from the case when* $q=1$.

5. Decay Estimate $q = q_\star$

Proof of Lemma 1.1. Since U is radial (1.7) satisfies

$$U_{rr} + \frac{(N-1)U_r}{r} = U^{q_\star} - U^p. \tag{5.1}$$

Define $V(r) = r^{N-2}U(r)$. Then $U(r) = r^{-(N-2)}V(r)$. Then (5.1) reduces to

$$V_{rr} - \frac{(N-3)V_r}{r} = \frac{V^{q_\star}}{r^2} - \frac{V^p}{r^{(N-2)p-(N-2)}}. \tag{5.2}$$

Hence at infinity (5.2) reduces to

$$V_{rr} - \frac{(N-3)V_r}{r} = \frac{V^{q_\star}}{r^2}(1+o(1)). \tag{5.3}$$

Let us define $V(r) = W(t)$ where $r = e^t$. Then (5.3) reduces to

$$W''(t) - (N-2)W'(t) - W^{q_\star}(t)(1+o(1)) = 0. \tag{5.4}$$

Note from (5.4), we obtain

$$(W'(t)e^{-(N-2)t})' = e^{-(N-2)t}W^{q_\star}(t)(1+o(1)).$$

Integrating between $(t, +\infty)$ we obtain

$$-W'(t)e^{-(N-2)t} = \int_t^\infty e^{-(N-2)s}W^{q_\star}(s)(1+o(1))ds$$

and this implies that

$$W'(t) = -e^{(N-2)t}\int_t^\infty e^{-(N-2)s}W^{q_\star}(s)(1+o(1))ds. \tag{5.5}$$

Hence $W' < 0$ for sufficiently large t. Hence from (5.5) and using the fact W is decreasing we have

$$-W'(t) \le e^{(N-2)t}W^{q_\star}(t)\int_t^\infty e^{-(N-2)s}(1+o(1))ds.$$

As a result, we have

$$-W'(t) \le W^{q_\star}(t)(1+o(1)).$$

This implies that

$$(W^{-(q_\star-1)}(t))' \le C.$$

Integrating between $(1, t)$ we obtain

$$W^{-(q_\star-1)}(t) \le W^{-(q_\star-1)}(1) + C(t-1)$$
$$W^{-(q_\star-1)}(t) \le Ct$$

for $t \gg 1$ which implies that

$$W(t) \ge Ct^{-\frac{1}{(q_\star-1)}} = Ct^{-\frac{N-2}{2}} \tag{5.6}$$

for some $C > 0$ independent of t. Hence we obtain the lower bound.

For the upper bound we let $0 < \theta < 1$ such that $U < \theta$ in $\mathbb{R}^N \setminus B_R(0)$ and $\eta_1 \gg 1$ such that

$$U \le \eta_1 S \text{ on } \partial B_R(0). \tag{5.7}$$

where $S = \omega_{q_\star} r^{2-N}(\log r)^{\frac{2-N}{2}}$ satisfies

$$\Delta S = S^{q_\star} - \omega_{q_\star}\frac{N(N-2)}{4}\frac{S^{q_\star}}{r^2(\log r)}. \tag{5.8}$$

then we have
(5.9)
$$\Delta(\eta_1 S) - (\eta_1 S)^{q_\star} = \eta_1 \Delta S - \eta_1^{q_\star} S^{q_\star} = (\eta_1 - \eta_1^{q_\star}) S^{q_\star} - \eta_1 \omega_{q_\star} \frac{N(N-2)}{4} \frac{S^{q_\star}}{r^2(\log r)}.$$
Then U satisfies
$$\Delta U - U^{q_\star} \geq -\theta^p \text{ in } \mathbb{R}^N \setminus B_R.$$
As a result, we obtain
$$\Delta(U - \eta_1 S) - \frac{(U^{q_\star} - \eta_1 S^{q_\star})}{U - \eta_1 S}(U - \eta_1 S) \quad \geq \quad -\theta^p - (\eta_1 - \eta_1^q) S^{q_\star} + \eta_1 \omega_{q_\star} \frac{N(N-2)}{4} \frac{S^{q_\star}}{r^2(\log r)} \geq 0.$$
Hence we obtain by the maximum principle in $\mathbb{R}^N \setminus B_R$
$$U \leq \eta_1 S.$$
Now we define
$$Z(t) = t^{\frac{N-2}{2}} W(t)$$
and is positive everywhere. Then we have
$$W(t) = t^{-\frac{N-2}{2}} Z(t).$$
Moreover, from (5.4) we have
$$\begin{aligned} Z''(t) \quad &- \quad (N-2)Z'(t) - \frac{Z^{q_\star}(t)(1+o(1))}{t} \\ &+ \quad \frac{(N-2)Z'(t)}{t} + \frac{N(N-2)}{4}\frac{Z(t)}{t^2} + \frac{(N-2)^2}{2}\frac{Z(t)}{t} = 0. \end{aligned}$$
Now we suppose that $Z(t) > 1$ and $Z'(t) \geq 0$. Then $Z''(t) > 0$ and we obtain a contradiction.
Suppose that $Z(t) < 1$ and $Z'(t) \leq 0$, then $Z''(t) < 0$ and this implies that Z' decreases faster as Z does. Hence Z crosses the axes which is impossible. Suppose that $Z(t) > 1$ and $Z'(t) < 0$ for large t, then $Z(t)$ has a limit; if $Z(t) < 1$ and $Z'(t) > 0$ for all $t \geq t_0$; then also Z has a limit. Hence the most awkward case is to consider $Z'(t) < 0$ when $Z(t) > 1$ and $Z'(t) > 0$ when $Z(t) < 1$. Suppose that there exists a point t_0 such that $Z(t_0) < 1$ and $Z'(t_0) > 0$, then we have $Z'(t) > 0$ for al $t > t_0$ and hence $Z(t)$ hits 1. Hence $Z(t) \to l$ as $t \to +\infty$.

In order to estimate $\omega_{q_\star}$, we basically solve (1.7) at infinity; that is we are required to solve an equation of the form

(5.10) $$\Delta U = U^{q_\star}$$

in an exterior domain. Also we have
$$U(|x|) = \omega_{q_\star} r^{-(N-2)} (\log r)^{-\frac{N-2}{2}}.$$
Then
$$\Delta U = \omega_{q_\star} \frac{N(N-2)}{4} r^{-N} (\log r)^{-\frac{N+2}{2}} + \omega_{q_\star} \frac{(N-2)^2}{2} r^{-N} (\log r)^{-\frac{N}{2}}$$

(5.11) $$\Delta U = \omega_{q_\star} \frac{(N-2)^2}{2} r^{-N} (\log r)^{-\frac{N}{2}} (1 + o(1))$$

and

(5.12) $$U^{q_\star} = \omega_{q_\star}^{q_\star} r^{-N} (\log r)^{-\frac{N}{2}}.$$

Equating (5.11) and (5.12) we obtain

$$\omega_{q_\star}^{q_\star - 1} = \left(\frac{(N-2)}{\sqrt{2}} \right)^2$$

and using the fact $q_\star - 1 = \frac{2}{N-2}$, we obtain

$$\omega_{q_\star} = \left(\frac{N-2}{\sqrt{2}} \right)^{N-2}. \tag{5.13}$$

□

6. Profile of Spikes $q = q_\star$.

Let $PU_{\varepsilon,\xi} \in \mathrm{H}_0^1(\Omega)$ be the solution to the problem

$$\begin{cases} \Delta PU_{\varepsilon,\xi} = \Delta U_{\varepsilon,\xi} & \text{in } \Omega, \\ PU_{\varepsilon,\xi} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Lemma 6.1.

$$PU_{\varepsilon,\xi}(x) = U_{\varepsilon,\xi}(x) - \omega_{q_\star} \frac{\varepsilon^{N-2}}{|\log \varepsilon|^{\frac{N-2}{2}}} H_{q_\star}(x,\xi) + o\left(\frac{\varepsilon^{N-2}}{|\log \varepsilon|^{\frac{N-2}{2}}} \right) \quad \textit{if } q = \frac{N}{N-2}$$

uniformly with respect to $x \in \overline{\Omega}$ *and* ξ *in compact sets of* Ω.

Proof. Let $q = q_\star := \frac{N}{N-2}$. The function $w(x) := PU_{\varepsilon,\xi}(x) - U_{\varepsilon,\xi}(x) + \omega_{q_\star} \frac{\varepsilon^{N-2}}{|\log \varepsilon|^{\frac{N-2}{2}}} H_{q_\star}(x,\xi)$ solves the problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ w(x) = \dfrac{\varepsilon^{N-2}}{|x-\xi|^{N-2} |\log \varepsilon|^{\frac{N-2}{2}} |\log|x-\xi||^{\frac{N-2}{2}}} \left(\omega_{q_\star} - \dfrac{|x-\xi|^{N-2} |\log|x-\xi||^{\frac{N-2}{2}}}{\varepsilon^{N-2} |\log \varepsilon|^{-\frac{N-2}{2}}} U\left(\dfrac{x-\xi}{\varepsilon} \right) \right) & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle it follows that

$$\max_{x \in \Omega} |w(x)| \le \max_{x \in \partial\Omega} |w(x)| = \frac{\varepsilon^{N-2}}{|\log \varepsilon|^{\frac{N-2}{2}}} o(1),$$

because of (1.8). □

Lemma 6.2. *The following expansion holds*

$$J_\varepsilon(PU_{\varepsilon,\xi}) = \varepsilon^N I_\infty + \frac{1}{2} \omega_{q_\star}^2 \frac{\varepsilon^{2(N-1)}}{|\log \varepsilon|^{N-2}} \Psi_{q_\star}(\xi) + o\left(\frac{\varepsilon^{2(N-1)}}{|\log \varepsilon|^{N-2}} \right) \tag{6.2}$$

uniformly with respect to ξ *in compact sets of* Ω, *where*

$$I_\infty(U) := \int_{\mathbb{R}^N} \left[\frac{p-1}{2(p+1)} U^{p+1}(x) - \frac{q-1}{2(q+1)} U^{q+1}(x) \right] dx \tag{6.3}$$

and $\Psi_{q_\star}(\xi)$ *satisfies (1.11).*

Proof. As usual we set $F(s) := \frac{1}{p+1}(s^+)^{p+1} - \frac{1}{q_\star+1}(s^+)^{q_\star+1}$. Let us compute the reduced energy.

$$
\begin{aligned}
&J_\varepsilon(PU_{\varepsilon,\xi}) \\
&= \frac{1}{2}\varepsilon^2 \int_\Omega |\nabla (PU_{\varepsilon,\xi}(x))|^2 dx - \int_\Omega F(PU_{\varepsilon,\xi}(x))\,dx \\
&= \frac{1}{2}\int_\Omega f(U_{\varepsilon,\xi}(x))\,PU_{\varepsilon,\xi}(x)dx - \int_\Omega F(PU_{\varepsilon,\xi}(x))\,dx \\
&= \int_\Omega \left[\frac{1}{2}f(U_{\varepsilon,\xi}(x))\,U_{\varepsilon,\xi}(x) - F(U_{\varepsilon,\xi}(x))\right]dx - \frac{1}{2}\int_\Omega f(U_{\varepsilon,\xi}(x))\,[PU_{\varepsilon,\xi}(x) - U_{\varepsilon,\xi}(x)]\,dx \\
&- \int_\Omega \{F(PU_{\varepsilon,\xi}(x)) - F(U_{\varepsilon,\xi}(x)) - f(U_{\varepsilon,\xi}(x))\,[PU_{\varepsilon,\xi}(x) - U_{\varepsilon,\xi}(x)]\}\,dx \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{6.4}
$$

Let us estimate I_1. Using the fact that $q_\star + 1 = \frac{2(N-1)}{N-2}$ we have

$$
\begin{aligned}
&\int_\Omega \left[\frac{1}{2}f(U_{\varepsilon,\xi}(x)) - F(U_{\varepsilon,\xi}(x))\right]dx = \int_\Omega \left[\frac{p-1}{2(p+1)}U_{\varepsilon,\xi}^{p+1}(x) - \frac{q_\star-1}{2(q_\star+1)}U_{\varepsilon,\xi}^{q_\star+1}(x)\right]dx \\
&= \varepsilon^N \int_{\mathbb{R}^N}\left[\frac{p-1}{2(p+1)}U^{p+1}(x) - \frac{q_\star-1}{2(q_\star+1)}U^{q_\star+1}(x)\right]dx \\
&- \frac{p-1}{2(p+1)}\int_{\mathbb{R}^N\setminus\Omega} U_{\varepsilon,\xi}^{p+1}(x)dx + \frac{q_\star-1}{2(q_\star+1)}\int_{\mathbb{R}^N\setminus\Omega} U_{\varepsilon,\xi}^{q_\star+1}(x)dx \\
&= \varepsilon^N \int_{\mathbb{R}^N}\left[\frac{p-1}{2(p+1)}U^{p+1}(x) - \frac{q_\star-1}{2(q_\star+1)}U^{q_\star+1}(x)\right]dx \\
&+ \frac{q_\star-1}{2(q_\star+1)}\omega_{q_\star}^{q_\star+1}\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-1}}\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|x-\xi|^{2(N-1)}|\log|x-\xi||^{N-1}}dx + o\left(\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-1}}\right) \\
&= \varepsilon^N I_\infty + \frac{q_\star-1}{2(q_\star+1)}\omega_{q_\star}^{q_\star+1}\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-1}}\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|x-\xi|^{2(N-1)}|\log|x-\xi||^{N-1}}dx \\
&+ o\left(\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-1}}\right)
\end{aligned} \tag{6.5}
$$

Let us estimate I_2.

$$-\frac{1}{2}\int_\Omega f\left(U_{\varepsilon,\xi}(x)\right)\left[PU_{\varepsilon,\xi}(x)-U_{\varepsilon,\xi}(x)\right]dx=\frac{1}{2}\omega_{q_\star}\frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\int_\Omega f\left(U_{\varepsilon,\xi}(x)\right)H_{q_\star}(x,\xi)dx$$

$$=-\frac{1}{2}\omega_{q_\star}\frac{\varepsilon^{N}}{|\log\varepsilon|^{\frac{N-2}{2}}}\int_\Omega \Delta U_{\varepsilon,\xi}(x)H_{q_\star}(x,\xi)dx$$

$$=\frac{1}{2}\omega_{q_\star}\frac{\varepsilon^{N}}{|\log\varepsilon|^{\frac{N-2}{2}}}\int_{\partial\Omega}\left[U_{\varepsilon,\xi}(x)\partial_\nu H_{q_\star}(x,\xi)-\partial_\nu U_{\varepsilon,\xi}(x)H_{q_\star}(x,\xi)\right]dx$$

$$=\frac{1}{2}\omega_{q_\star}^2\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-2}}\left(\int_{\partial\Omega}\frac{1}{|x-\xi|^{N-2}|\log|x-\xi||^{\frac{N-2}{2}}}\left[\partial_\nu H_{q_\star}(x,\xi)-\partial_\nu\frac{1}{|x-\xi|^{N-2}|\log|x-\xi||^{\frac{N-2}{2}}}\right]dx\right)$$

(6.6)

$$+o(1)\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-2}}.$$

Here ν is the outward unit normal at the boundary $\partial\Omega$.

We deduce that

$$-\int_{\partial\Omega}\frac{1}{|x-\xi|^{N-2}|\log|x-\xi||^{\frac{N-2}{2}}}\partial_\nu\frac{1}{|x-\xi|^{N-2}|\log|x-\xi||^{\frac{N-2}{2}}}dx$$

$$=-\frac{1}{2}\int_{\partial\Omega}\partial_i\frac{1}{|x-\xi|^{2(N-2)}|\log|x-\xi||^{(N-2)}}$$

$$=(N-2)^2\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|x-\xi|^{2N-2}|\log|x-\xi||^{N-2}}dx$$

$$+\frac{(N-2)^2}{2}\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|x-\xi|^{2N-2}|\log|x-\xi||^{N-1}}dx$$

$$+\frac{(N-1)(N-2)}{2}\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|x-\xi|^{2N-2}|\log|x-\xi||^{N}}dx. \tag{6.7}$$

Moreover, from (1.10), Green's formula yields

$$\int_{\partial\Omega}\frac{1}{|x-\xi|^{N-2}|\log|x-\xi||^{\frac{N-2}{2}}}\partial_\nu H_{q_\star}(x,\xi)=\int_\Omega\left|\nabla H_{q_\star}(x,\xi)\right|^2dx. \tag{6.8}$$

From I_3 we have by

$$\begin{aligned}I_3 &= -\int_\Omega\left\{F\left(PU_{\varepsilon,\xi}(x)\right)-F\left(U_{\varepsilon,\xi}(x)\right)-f\left(U_{\varepsilon,\xi}(x)\right)\left[PU_{\varepsilon,\xi}(x)-U_{\varepsilon,\xi}(x)\right]\right\}dx\\ &= \int_\Omega\mathcal{O}(f'(U_{\varepsilon,\xi}(x))\left[PU_{\varepsilon,\xi}(x)-U_{\varepsilon,\xi}\right]^2)\\ &\le C\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-1}}\int_\Omega\frac{H_{q_\star}^2(x,\xi)}{|x-\xi|^2|\log|x-\xi||}dx.\end{aligned}$$

Hence the result follows. □

Lemma 6.3. *There exists* $\xi_{q_\star}\in\Omega$ *such that*

$$\Psi_{q_\star}(\xi_{q_\star}):=\min_{\xi\in\Omega}\Psi_{q_\star}(\xi).$$

Proof. By (1.11) we deduce that

$$\Psi_{q_\star}(\xi) \geq (N-2)^2\omega_{q_\star}^2 \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-\xi|^{2N-2}|\log|x-\xi||^{N-2}}dx \to +\infty \text{ as } \xi \text{ approaches } \partial\Omega.$$

Therefore, the claim follows. □

Lemma 6.4. *Then*

$$J_\varepsilon(u_\varepsilon) \leq \varepsilon^N I_\infty(U) + \frac{1}{2}\omega_{q_\star}^2 \frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-2}}\Psi_{q_\star}(\xi) + o\left(\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-2}}\right).$$

Proof. Follows exactly as Lemma 3.5. □

Lemma 6.5. *We have*

$$S_\varepsilon(x) = \frac{H_{q_\star}(x,x_\varepsilon)}{H_{q_\star}(x_\varepsilon,x_\varepsilon)} \tag{6.9}$$

is uniformly bounded.

Proof. Note that if $d(x_\varepsilon,\partial\Omega)\geq c$ for some $c>0$, then by Lemma 2.2, we have S_ε is uniformly bounded. Suppose that $d(x_\varepsilon,\partial\Omega)\to 0$ as $\varepsilon\to 0$. Then $H_{q_\star}(x_\varepsilon,x_\varepsilon)\to+\infty$ as $\varepsilon\to 0$. Suppose x is close to x_ε. We consider two points x and y and compute the difference $H_{q_\star}(x,x_\varepsilon)$ and $H_{q_\star}(y,x_\varepsilon)$ on the boundary. So it is enough to prove the uniform bound when x is a point close to y. Then on the boundary, we have

$$H_{q_\star}(x,x_\varepsilon)-H_{q_\star}(y,x_\varepsilon) = \frac{1}{|x-x_\varepsilon|^{N-2}|\log|x-x_\varepsilon||^{\frac{N-2}{2}}} - \frac{1}{|y-x_\varepsilon|^{N-2}|\log|y-x_\varepsilon||^{\frac{N-2}{2}}}.$$

Let $y=x-h$ where $h=\mathcal{O}(\varepsilon)$ is small; then we have

$$|x-x_\varepsilon+h|^{2-N} = |x-x_\varepsilon|^{2-N}+\mathcal{O}(h)|x-x_\varepsilon|^{1-N}$$

and

$$|\log|x-x_\varepsilon+h||^{\frac{2-N}{2}} = |\log|x-x_\varepsilon||^{\frac{2-N}{2}} + \mathcal{O}(h)\frac{1}{|x-x_\varepsilon||\log|x-x_\varepsilon||^{\frac{N}{2}}}.$$

and hence multiplying the above two expression we have

$$\begin{aligned}
|x-x_\varepsilon+h|^{2-N}|\log|x-x_\varepsilon+h||^{\frac{2-N}{2}} &= |x-x_\varepsilon|^{2-N}|\log|x-x_\varepsilon||^{\frac{2-N}{2}} \\
&+ \mathcal{O}(h)\left(\frac{1}{|x-x_\varepsilon|^{N-1}|\log|x-x_\varepsilon||^{\frac{N-2}{2}}}\right) \\
&+ \mathcal{O}(h)\left(\frac{1}{|x-x_\varepsilon|^{N-1}|\log|x-x_\varepsilon||^{\frac{N}{2}}}\right) \\
&+ \mathcal{O}(h^2)\left(\frac{1}{|x-x_\varepsilon|^{N}|\log|x-x_\varepsilon||^{\frac{N}{2}}}\right).
\end{aligned}$$

As a result, we have

$$\begin{aligned}
H_{q_\star}(x,x_\varepsilon)-H_{q_\star}(y,x_\varepsilon) &= \mathcal{O}\left(\frac{\varepsilon}{d(x_\varepsilon,\partial\Omega)}\right)\left(\frac{1}{d(x_\varepsilon,\partial\Omega)^{N-2}|\log|d(x_\varepsilon,\partial\Omega)||^{\frac{N-2}{2}}}\right) \\
&+ \mathcal{O}\left(\frac{\varepsilon}{d(x_\varepsilon,\partial\Omega)|\log|d(x_\varepsilon,\partial\Omega)|}\right)\left(\frac{1}{d(x_\varepsilon,\partial\Omega)^{N-2}|\log|d(x_\varepsilon,\partial\Omega)||^{\frac{N-2}{2}}}\right) \\
&+ \mathcal{O}\left(\frac{\varepsilon^2}{d(x_\varepsilon,\partial\Omega)^2|\log|d(x_\varepsilon,\partial\Omega)||}\right)\left(\frac{1}{d(x_\varepsilon,\partial\Omega)^{N-2}|\log|d(x_\varepsilon,\partial\Omega)||^{\frac{N-2}{2}}}\right) \\
&= o(1)H_{q_\star}(x_\varepsilon,x_\varepsilon).
\end{aligned}$$

using the estimates from Lemma 2.2 and $\frac{d(x_\varepsilon,\partial\Omega)}{\varepsilon}\to 0$. Also note that

$$\Delta(H_{q_\star}(x,x_\varepsilon)-H_{q_\star}(y,x_\varepsilon))=0$$

Hence by the maximum principle we have for all $x,y\in\Omega$

$$|H_{q_\star}(x,x_\varepsilon)-H_{q_\star}(y,x_\varepsilon)|\le o(1)H_{q_\star}(x_\varepsilon,x_\varepsilon).$$

Choosing $x=x_\varepsilon$ and $y=x$ we have

$$|H_{q_\star}(x_\varepsilon,x_\varepsilon)-H_{q_\star}(x,x_\varepsilon)|\le o(1)H_{q_\star}(x_\varepsilon,x_\varepsilon).$$

This implies that S_ε is uniformly bounded. □

Lemma 6.6. *For $R\gg 1$, the error satisfies*

(6.10)

$$E_\varepsilon[PU_{\varepsilon,x_\varepsilon}]=\begin{cases}\mathcal{O}\Big(\frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}U_{\varepsilon,x_\varepsilon}^{q_\star-1}H_{q_\star}(x,x_\varepsilon)\Big) & \text{in } |x-x_\varepsilon|\le\varepsilon R;\\ \mathcal{O}\Big(\frac{\varepsilon^{N}}{|\log\varepsilon|^{\frac{N}{2}}}\frac{1}{|x-x_\varepsilon|^2|\log|x-x_\varepsilon||}H_{q_\star}(x,x_\varepsilon)\Big) & \text{in } |x-x_\varepsilon|\ge\varepsilon R.\end{cases}$$

Proof. Follows from the estimate (6.1) and using the fact that $\frac{dist(x_\varepsilon,\partial\Omega)}{\varepsilon}\to\infty$. □

By Lemma 3.6 we write

$$u_\varepsilon=PU_{\varepsilon,x_\varepsilon}+\frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star}H_{q_\star}(x_\varepsilon,x_\varepsilon)\varphi_\varepsilon.$$

Then φ_ε satisfies

$$\varepsilon^2\Delta\varphi_\varepsilon+f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon=-\frac{|\log\varepsilon|^{\frac{N-2}{2}}}{\varepsilon^{N-2}H_{q_\star}(x_\varepsilon,x_\varepsilon)}E_\varepsilon[PU_{\varepsilon,x_\varepsilon}]+N_\varepsilon[\varphi_\varepsilon]. \tag{6.11}$$

where

$$E_\varepsilon[PU_{\varepsilon,x_\varepsilon}]=\varepsilon^2\Delta PU_{\varepsilon,x_\varepsilon}-PU^q_{\varepsilon,x_\varepsilon}+PU^p_{\varepsilon,x_\varepsilon}$$

and

$$\begin{aligned}N_\varepsilon[\varphi_\varepsilon] &= \frac{|\log\varepsilon|^{\frac{N-2}{2}}}{\varepsilon^{N-2}H_{q_\star}(x_\varepsilon,x_\varepsilon)}\Big[f(PU_{\varepsilon,x_\varepsilon}+\frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star}H_{q_\star}(x_\varepsilon,x_\varepsilon)\varphi_\varepsilon)-f(PU_{\varepsilon,x_\varepsilon})\\ &- \frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}H_{q_\star}(x_\varepsilon,x_\varepsilon)\omega_{q_\star}\varphi_\varepsilon f'(PU_{\varepsilon,x_\varepsilon})\Big]\end{aligned}$$

where $f(u)=(u^+)^p-(u^+)^q$. Also note that $q>2$ if $N<4$ and $q\le 2$ if $N\ge 5$.

Moreover, note that (6.11) can also be written as

$$\begin{cases}\Delta(\tilde u_\varepsilon-\tilde{PU}_\varepsilon)+f'(\tilde W_\varepsilon)(\tilde u_\varepsilon-\tilde{PU}_\varepsilon)=-f'(\tilde W_\varepsilon)(\tilde{PU}_\varepsilon-U) & \text{in } \Omega_\varepsilon\\ (\tilde u_\varepsilon-\tilde{PU}_\varepsilon)=0 & \text{on } \partial\Omega_\varepsilon\end{cases} \tag{6.12}$$

where Ω_ε is a inflated domain around x_ε, $\tilde u_\varepsilon=u_\varepsilon(x_\varepsilon+\varepsilon x)$; $\tilde{PU}_\varepsilon=PU_\varepsilon(x_\varepsilon+\varepsilon x)$ and $\tilde W_\varepsilon$ is a point lying between $\tilde u_\varepsilon$ and U. Then

$$\begin{cases}\Delta\tilde\varphi_\varepsilon+f'(\tilde W_\varepsilon)\tilde\varphi_\varepsilon=-f'(\tilde W_\varepsilon)\frac{H_{q_\star}(x_\varepsilon+\varepsilon x,x_\varepsilon)}{H_{q_\star}(x_\varepsilon,x_\varepsilon)} & \text{in } \Omega_\varepsilon\\ \tilde\varphi_\varepsilon=0 & \text{on } \partial\Omega_\varepsilon.\end{cases} \tag{6.13}$$

Lemma 6.7. *For sufficiently small $\varepsilon>0$, φ_ε is uniformly bounded.*

Proof. It is enough to prove that $\tilde{\varphi}_\varepsilon$ is bounded. We have $\frac{dist(x_\varepsilon,\partial\Omega)}{\varepsilon} \to +\infty$ and hence we obtain $\frac{H_{q_\star}(x_\varepsilon+\varepsilon x,x_\varepsilon)}{H_{q_\star}(x_\varepsilon,x_\varepsilon)}$ is uniformly bounded by Lemma 6.5. Note that by the decay property of $\tilde{u}_\varepsilon$ and U, $\tilde{W}_\varepsilon \leq \frac{C}{|x|^{N-2}|\log|x||^{\frac{N-2}{2}}}$ for $|x|$ sufficiently large. Hence $f'(\tilde{W}_\varepsilon) \leq 0$ for $|x| \geq R_0$ and $f'(\tilde{W}_\varepsilon) \leq \frac{c_0}{|x|^2|\log|x||}$. We can choose $C|x|^{-\eta}$ as a super-solution of (6.13) for $|x| \geq R_0$; if we choose $C > 0$ is large and $\eta > 0$ sufficiently small. Hence we can bound $C > 0$, if we have a uniform bound $\tilde{\varphi}_\varepsilon$ on $|x| = R_0$. Thus we have a uniform decay for $\tilde{\varphi}_\varepsilon$ if we can bound $\tilde{\varphi}_\varepsilon$ on $|x| = R_0$.

If possible let $\tilde{\varphi}_\varepsilon$ be unbounded. Then $\|\tilde{\varphi}_\varepsilon\|_\infty \to \infty$ (up to a subsequence). Define $\psi_\varepsilon = \frac{\tilde{\varphi}_\varepsilon}{\|\tilde{\varphi}_\varepsilon\|_\infty}$. Then $\|\psi_\varepsilon\|_\infty = 1$. Note that the right hand term in (6.13) is uniformly small and thus by the argument in the previous paragraph ψ_ε has a uniform decay for large $|x|$. Thus the maximum of ψ_ε must occur at k_ε where $|k_\varepsilon| \leq R$ for sufficiently small ε. Let k be a subsequential limit of k_ε. By Schauder estimates we obtain $\|\psi_\varepsilon\|_{C^{1,\theta}_{loc}}$ is bounded for some $\theta \in (0,1]$ and hence by the Arzela-Ascoli's theorem there exists $\psi \in C^1$ such that $\|\psi_\varepsilon - \psi\|_{C^1_{loc}} \to 0$ as $\varepsilon \to 0$ where ψ satisfies

$$\left\{\begin{array}{rl} \Delta\psi + f'(U)\psi = 0 & \text{in } \mathbb{R}^N \\ \psi(k) = 1 & \\ \psi(y) \to 0 & \text{as } |y| \to \infty. \end{array}\right. \tag{6.14}$$

Using the fact that $\frac{dist(k_\varepsilon,\partial\Omega)}{\varepsilon} \to +\infty$, we conclude that $\Omega_\varepsilon \mapsto \mathbb{R}^N$ i.e. given any compact subset C of $\mathbb{R}^N$; $C \subset \Omega_\varepsilon$ for ε sufficiently small. Also note that $\psi(y) \to 0$ as $|y| \to +\infty$ by the super-solution technique, discussed in the first paragraph.

Then by a simple comparison argument, we can show that $|\psi| \leq Cr^{-(N-2)}$. This implies that $\psi \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. On the other hand, there exists a $C > 0$ such that,

$$\int_{\mathbb{R}^N} |\nabla\psi|^2 = \int_{\mathbb{R}^N} f'(U)\psi^2 \leq C + \int_{\mathbb{R}^N\setminus B_R} \frac{1}{|x|^2(\log|x|)}\psi^2 \leq C + \int_{\mathbb{R}^N\setminus B_R} \frac{1}{|x|^{2N-2}(\log|x|)} < +\infty.$$

As a result, $\psi \in \mathcal{D}^{1,2}(\mathbb{R}^N) \cap ker(\Delta + f'(U))$. Since $\psi \not\equiv 0$ and since by Lemma 2.1, $\ker(\Delta + f'(U)) = \left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \cdots \frac{\partial U}{\partial x_N}\right\}$, we have

$$\psi = \sum_{j=1}^N a_j \frac{\partial U}{\partial x_i}$$

where not all a_j's are zero. Since U is radial $U'(0) = 0$ and $\Delta U(0) \neq 0$, it follows that $\nabla\psi(0) \neq 0$. Note that $\nabla u_\varepsilon(x_\varepsilon) = 0$ and this implies

$$\nabla\psi_\varepsilon(0) = \frac{\nabla\varphi_\varepsilon(0)}{\frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star} H_{q_\star}(x_\varepsilon,x_\varepsilon)\|\varphi_\varepsilon\|_{L^\infty}} \to 0$$

and by pointwise convergence $\nabla\psi(0) = 0$ and hence $\nabla(\sum_{i=1}^N a_i \frac{\partial U}{\partial x_i})(0) = 0$ and this implies that $a_i = 0$ for all i. This gives a contradiction. □

Lemma 6.8. *We have,*

$$c_\varepsilon = \varepsilon^N I_\infty + \frac{1}{2}\omega_{q_\star}^2 \frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-2}} \Psi_{q_\star}(x_\varepsilon) + o\left(\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-2}}\right). \tag{6.15}$$

Proof. From $u_\varepsilon = PU_{\varepsilon,x_\varepsilon} + \frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star} H_{q_\star}(x_\varepsilon, x_\varepsilon)\varphi_\varepsilon$, we obtain

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &= J_\varepsilon(PU_{\varepsilon,x_\varepsilon}) \\ &+ \frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star} H_{q_\star}(x_\varepsilon, x_\varepsilon)\int_\Omega E_\varepsilon(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon dx \\ &+ \frac{1}{2}\frac{\varepsilon^{2(N-2)}}{|\log\varepsilon|^{N-2}}\omega_{q_\star}^2 H_{q_\star}^2(x_\varepsilon, x_\varepsilon)\int_\Omega \{\varepsilon^2|\nabla\varphi_\varepsilon|^2 dx - f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon^2\} \\ &- \int_\Omega \Bigg[F\Bigg(PU_{\varepsilon,x_\varepsilon} + \frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star} H_{q_\star}(x_\varepsilon, x_\varepsilon)\varphi_\varepsilon\Bigg) - F(PU_{\varepsilon,x_\varepsilon}) \\ &- \frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star} H_{q_\star}(x_\varepsilon, x_\varepsilon)\varphi_\varepsilon f(PU_{\varepsilon,x_\varepsilon}) - \frac{1}{2}\frac{\varepsilon^{2(N-2)}}{|\log\varepsilon|^{N-2}}\omega_{q_\star}^2 H_{q_\star}(x_\varepsilon, x_\varepsilon)^2\varphi_\varepsilon^2 f'(PU_{\varepsilon,x_\varepsilon})\Bigg]. \end{aligned}$$

We estimate

$$\begin{aligned} &\frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star} H_{q_\star}(x_\varepsilon, x_\varepsilon)\int_\Omega E_\varepsilon(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon dx \\ &= \mathcal{O}\Bigg(\frac{\varepsilon^{2N-2}}{|\log\varepsilon|^{N-1}}\frac{\varepsilon^{N-2}}{|\log\varepsilon|}\Bigg) \\ &+ \mathcal{O}\Bigg(\frac{\varepsilon^{2N-2}}{|\log\varepsilon|^{N-1}}H_{q_\star}^2(x_\varepsilon, x_\varepsilon)\int_{\Omega\setminus B_{\varepsilon R}(x_\varepsilon)}\frac{|\varphi_\varepsilon|}{|x-x_\varepsilon|^2|\log|x-x_\varepsilon||}\Bigg) \\ &= \mathcal{O}\Bigg(\frac{\varepsilon^{2N-2}}{|\log\varepsilon|^{N}}H_{q_\star}^2(x_\varepsilon, x_\varepsilon)\int_{\Omega\setminus B_{\varepsilon R}(x_\varepsilon)}\frac{|\varphi_\varepsilon|}{|x-x_\varepsilon|^2}dx\Bigg) \\ &= \mathcal{O}\Bigg(\frac{\varepsilon^{2N-2}}{|\log\varepsilon|^{N}}H_{q_\star}^2(x_\varepsilon, x_\varepsilon)\int_\Omega \frac{1}{|x-x_\varepsilon|^2}dx\Bigg) \\ &= o\Bigg(\frac{\varepsilon^{2(N-1)}}{|\log\varepsilon|^{N-2}}\Bigg). \end{aligned}$$

using (6.6). From (6.11), we obtain by integrating

$$\begin{aligned} \int_\Omega \{\varepsilon^2|\nabla\varphi_\varepsilon|^2 dx - f'(PU_{\varepsilon,x_\varepsilon})\varphi_\varepsilon^2\}dx &= \frac{|\log\varepsilon|^{\frac{N-2}{2}}}{\varepsilon^{N-2}H_{q_\star}(x_\varepsilon, x_\varepsilon)}\int_\Omega E_\varepsilon[PU_{\varepsilon,x_\varepsilon}]\varphi_\varepsilon - \int_\Omega N_\varepsilon[\varphi_\varepsilon]\varphi_\varepsilon \\ &= \mathcal{O}\Bigg(\frac{\varepsilon^2}{|\log\varepsilon|}\Bigg). \end{aligned}$$

Using Taylor's expansion, as in Lemma 3.9 we obtain

$$\begin{aligned} \int_\Omega \Bigg[F\Bigg(PU_{\varepsilon,x_\varepsilon} &+ \frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star} H_{q_\star}(x_\varepsilon, x_\varepsilon)\varphi_\varepsilon\Bigg) - F(PU_{\varepsilon,x_\varepsilon}) \\ &- \frac{\varepsilon^{N-2}}{|\log\varepsilon|^{\frac{N-2}{2}}}\omega_{q_\star} H_{q_\star}(x_\varepsilon, x_\varepsilon)\varphi_\varepsilon f(PU_{\varepsilon,x_\varepsilon}) - \frac{1}{2}\frac{\varepsilon^{2(N-2)}}{|\log\varepsilon|^{N-2}}\omega_{q_\star}^2 H_{q_\star}(x_\varepsilon, x_\varepsilon)^2\varphi_\varepsilon^2 f'(PU_{\varepsilon,x_\varepsilon})\Bigg]. \\ &= \mathcal{O}\Bigg(\frac{\varepsilon^{2N-2}}{|\log\varepsilon|^{N-1}}\Bigg). \end{aligned}$$

This proves the theorem. □

Remark 6.1. *Note that $H_q(x_\varepsilon, x_\varepsilon)$ is bounded by Lemma (6.4) and by Lemma (6.3). Hence we have from Lemma (6.4) and Lemma (6.8) we obtain $\Psi_{q_\star}(x_\varepsilon) \to \Psi_{q_\star}(\xi)$. Hence Theorem 1.2 is proved.*

ACKNOWLEDGEMENTS

The first and second author were supported by the Australian Research Council. Part of the work was done while the first and the second author were visiting Angela Pistoia, while on an exchange agreement between The University of Sydney and Università di Roma "La Sapienza" funded by MUIR. The second author would also like to thank The Chinese University of Hong Kong, for their kind hospitality. The third author was supported by an Earmarked Grant from RGC of Hong Kong and a Joint Research Grant from NSFC.

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