ON LIN-NI'S CONJECTURE IN CONVEX DOMAINS

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ABSTRACT. We consider the following nonlinear Neumann problem

$$\begin{cases} -\Delta u + \mu u = u^{\frac{N+2}{N-2}}, \quad u > 0 & \text{in } \Omega; \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\mu > 0$, Ω is a bounded domain in $\mathbb{R}^N$, and n denotes the outward unit normal of $\partial\Omega$. Lin and Ni ([7]) conjectured that *for μ small, all solutions are constants*. It has been shown in [14], [13] that this conjecture is true if Ω is convex and $N = 3$. The main result of this paper is that if $N \geq 4$, Ω is convex and satisfies some symmetric conditions, then for any fixed μ, there are *infinitely many positive solutions*. As a corollary, the Lin-Ni's conjecture is false in some convex domains if $N \geq 4$.

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1. Introduction

This paper is concerned with the existence or nonexistence of positive solutions for the following nonlinear elliptic Neumann problem

$$\begin{cases} -\Delta u + \mu u - u^q = 0, \quad u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $1 < q < +\infty, \mu > 0, n$ denotes the outward unit normal vector of $\partial\Omega$, and Ω is a smooth and bounded domain in $\mathbb{R}^N, N \geq 3$.

Problem (1.1) has been studied intensively in recent years. When q is sub-critical, i.e. $q < \frac{N+2}{N-2}$, Lin, Ni and Takagi [8] proved that the only solution to (1.1), for *small* μ, is the constant $u \equiv \mu^{\frac{1}{p-1}}$. Based on this, Lin and Ni [7] made the following conjecture:

Lin-Ni's Conjecture: *For μ small and $q = \frac{N+2}{N-2}$, problem (1.1) admits only the constant solution.*

We recall below the main results towards proving or disproving Lin-Ni's conjecture. Adimurthi-Yadava [1], [2] and Budd-Knapp-Peletier [4] first considered the following problem:

$$\begin{cases} \Delta u - \mu u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in} \quad B_R(0), \quad u > 0 \quad \text{in} \quad B_R(0), \\ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial B_R(0). \end{cases} \tag{1.2}$$

The following results were proved:

Theorem A([1], [2], [3], [4]). *For μ sufficiently small,*
(1) if $N=3$ or $N\geq 7$, then any radial solution of (1.2) must be the constant solution;
(2) if $N=4,5$ or 6, problem (1.2) admits a non-constant radial solution.

Theorem A reveals that Lin-Ni's conjecture is not true in dimensions $4,5$ and 6. But for the other dimensions, Theorem A gives neither a positive answer, nor a negative answer to Lin-Ni's conjecture. In three and five dimensional cases, the following result was proved in [9], [14], [13] :

Theorem B: *The conjecture is true if $N=3$, $q=5$ and Ω is convex ([14], [13]). The conjecture is false if $N=5$, $q=\frac{7}{3}$ and Ω is any bounded domain ([9]).*

Recently, in [10], we gave a *negative answer* to Lin-Ni's conjecture in *all dimensions* for some *non-convex* domain Ω. More precisely, we assume that Ω is a smooth and bounded domain satisfying the following conditions

(H_1) $y\in\Omega$ if and only if $(y_1,y_2,y_3,\cdots,-y_i,\ldots,y_N)\in\Omega,\quad\forall\quad i=3,\ldots,N;$
(H_2) if $(r,0,y'')\in\Omega$, then $(r\cos\theta,r\sin\theta,y'')\in\Omega$, $\forall\ \theta\in(0,2\pi)$, where $y''=(y_3,\cdots,y_N)$;
(H_3) Let $T:=\partial\Omega\cap\{y_3=\cdots=y_N=0\}$. There exists a connected component Γ of T, such that $H(x)\equiv\gamma<0,\quad\forall x\in\Gamma$, where $H(x)$ is the mean curvature of $\partial\Omega$ at $x\in\partial\Omega$.

Theorem C [10]. *Suppose that $N\geq 3, q=\frac{N+2}{N-2}$ and Ω is a bounded smooth domain satisfying (H_1), (H_2) and (H_3). Let μ be any fixed positive number. Then problem (1.1) has infinitely many positive solutions, whose energy can be made arbitrarily large.*

By the result of [14] and [13], the assumption that Ω is non-convex is necessary to obtain the result of Theorem C for $N=3$. By Theorem A, we know that Lin-Ni's conjecture is not true in the dimensions $N=4,5,6$ even if the domain is convex. Now we can ask the following question: is Lin-Ni's conjecture true for convex domains when $N\geq 7$? The result in (1) of Theorem A seems to suggest that Lin-Ni's conjecture may be true for $N\geq 7$ and $\Omega=B_R(0)$. However this is *not* the case, as we shall prove in this paper.

The purpose of this paper is to give a *negative answer* to Lin-Ni's conjecture in some convex domain including the balls for all $N\geq 4$. First we make the assumptions on the domains.

For normalization reason, we consider throughout the paper the following problem:

$$\begin{cases} -\Delta u+\mu u-\alpha_N u^{\frac{N+2}{N-2}}=0, \quad u>0 & \text{in}\quad \Omega,\\ \frac{\partial u}{\partial n}=0, & \text{on}\quad \partial\Omega, \end{cases} \tag{1.3}$$

where $\alpha_N=N(N-2)$. The solutions are identical up to the multiplicative constant $(\alpha_N)^{-\frac{N-2}{4}}$.

Our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that $N \geq 4$ and Ω satisfies (H_1) and (H_2). Let μ be any fixed positive number. Then problem (1.3) has infinitely many positive solutions, whose energy can be made arbitrarily large.*

If $\Omega = B_R(0) \setminus B_r(0)$, $R > r > 0$, (H_1) and (H_2) hold. So, (H_1) and (H_2) do not implies the convexity of the domain. Certainly $\Omega = B_R(0)$ satisfies (H_1) and (H_2). Thus we have the following corollary, which complements Theorem A in the case of $N \geq 7$.

Corollary 1.2. *Suppose that $N \geq 4$, $\Omega = B_R(0)$. Let μ be any fixed positive number. Then problem (1.3) has infinitely many non-radially symmetric positive solutions.*

Theorem B suggests that for any $N \geq 3$, Lin-Ni's conjecture is not true for domains in R^N whose boundary admits a geodesics with negative curvature. On the other hand, Theorem C suggests that for any $N \geq 4$, Lin-Ni's conjecture is not true for domains in R^N whose boundary admits a geodesics with positive curvature.

Based on Theorems B, C and Theorem 1.1, we propose the following conjecture

Conjecture: *(a) Let $N \geq 3$ and $\mu > 0$ be fixed. If $\partial\Omega$ admits a geodesics with negative curvature, then there are infinitely many positive solutions to (1.3). (b) Let $N \geq 4$ and $\mu > 0$ be fixed. If $\partial\Omega$ admits a geodesics with negative curvature, there are infinitely many positive solutions to (1.3).*

This paper is arranged as follows. In section 2, we outline the proof of the main result. The reduction procedure is carried out in section 3, and the main result is proved in section 4. We put all the technical estimates in the appendices.

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2. Outline of Proofs

In this section, we will outline the main idea in the proof of Theorem 1.1.

It is well-known that the functions

$$U_{\lambda,a}(y) = \left(\frac{\lambda}{1 + \lambda^2|y-a|^2}\right)^{\frac{N-2}{2}}, \ \lambda > 0, \ a \in \mathbb{R}^N \tag{2.1}$$

are the only solutions to the problem

$$-\Delta u = \alpha_N u^{\frac{N+2}{N-2}}, \qquad u > 0, \quad \text{in} \quad \mathbb{R}^N.$$

As the scaling parameter $\lambda \to +\infty$, $U_{\lambda,a}$ is called a *single-bubble* centered at the point a. Our objective is to construct solutions with large number of bubbles. For this purpose, we first need to determine the location of the bubbles, and then the scaling parameter. Unlike a singular perturbation problem, where there is a parameter which can be used to determine the scaling parameter as well as the location of the bubbles, problem (1.3) has no parameter, since $\mu > 0$ is fixed. In this paper, we will create a parameter: the *number of bubbles*, so that we can use this number to determine the scaling parameter. Now the problem is where to put the bubbles. This is an important problem, since the location of the bubble affects its energy and thus the existence of the bubbling solutions. When Ω satisfies (H_1)–(H_3), we put all the bubbles exactly on the component Γ of the boundary [10]. In the construction of boundary bubble solutions, to determine the scaling parameter, it is necessary to assume that the mean curvature is negative at the places where the bubbles are put. But in a convex domain, such component does not exist. It is well known now that for Neumann problem, the energy of a bubble will decrease as the bubble moves toward to boundary. On the other hand, if all the bubbles move toward the outer boundary, the distance between bubbles will increase and so will the energy. From this observation, in the present situation, we can put many bubbles *in the domain but near an outer component of the boundary.*

Let us explain the precise ideas. We fix a positive integer

$$k \geq k_0,$$

where k_0 is a large positive integer, which is to be determined later. In this paper, we will show that for any $k \geq k_0$, (1.3) has a solution u with

$$u \approx \sum_{j=1}^{k} U_{x_{j,k},\lambda_k}, \tag{2.2}$$

where $x_{j,k} \in \Omega$, and as $k \to +\infty$, $d(x_{j,k}, \partial\Omega) \to 0$, $\lambda_k \to +\infty$.

The function $U_{x_{j,k},\lambda_k}$ can be regarded as an approximate solution of (1.3). To construct a solution of the form (2.2) for (1.3), it is crucial to find a better approximate solution for (1.3). Here, we need to take into account of the linear term μu in (1.3).

Integral estimates (see Appendix A) suggest to make the additional a priori assumption that the scaling parameter λ_k behaves like

$$\lambda_k = \frac{1}{\Lambda\varepsilon},$$

where

$$\begin{cases} \varepsilon = k^{-\frac{N-2}{N-4}} & \text{if } N \geq 5, \\ \varepsilon = e^{-D_k k^2} & \text{if } N = 4 \end{cases} \tag{2.3}$$

$\delta \leq \Lambda \leq \frac{1}{\delta}$, D_k satisfies $0 < \bar{D}_0 \leq D_k \leq \bar{D}_1$, δ is a small positive constant which is to be determined later. From now on, ε is given by (2.3).

Let $2^* = \frac{2N}{N-2}$. Using the transformation $u(y) \mapsto \varepsilon^{-\frac{N-2}{2}} u\big(\frac{y}{\varepsilon}\big)$, we find that (1.3) becomes

$$\begin{cases} -\Delta u + \mu \varepsilon^2 u = \alpha_N u^{2^*-1}, u > 0, & \text{in} \quad \Omega_\varepsilon, \\ \frac{\partial u}{\partial n} = 0, & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{2.4}$$

where $\Omega_\varepsilon = \{y : \ \varepsilon y \in \Omega\}$.

For any $a \in \Omega_\varepsilon$ with $d(a, \partial\Omega_\varepsilon)$ large, $U_{\frac{1}{\Lambda},a}$ is an approximate solution of (2.4). Because of the additional linear term $\mu\varepsilon^2 u$ in (2.4), $U_{\frac{1}{\Lambda},a}$ has to be improved. To this end, for $N \geq 5$, let $\Psi(|y|)$ be the radial solution of

$$\Delta\Psi + U_{1,0} = 0 \ \text{ in } \mathbb{R}^N, \quad \Psi \to 0 \quad \text{as} \quad |y| \to +\infty. \tag{2.5}$$

Then, it is easy to check that

$$\Psi(y) = \frac{1}{2(N-4)|y|^{N-4}}\Big(1 + O\big(\frac{1}{|y|^2}\big)\Big) \quad \text{as} \quad |y| \to +\infty. \tag{2.6}$$

For $a \in \mathbb{R}^N$, we set

$$\Psi_{\Lambda,a}(y) = \Lambda^{-\frac{N-6}{2}} \Psi\big(\frac{y-a}{\Lambda}\big).$$

Then

$$\Delta\Psi_{\Lambda,a} + U_{\frac{1}{\Lambda},a} = 0 \ \text{ in } \mathbb{R}^N.$$

It is easy to check that

$$|\Psi_{\Lambda,a}(y)|, \ |\partial_\Lambda \Psi_{\Lambda,a}(y)| \leq \frac{C}{(1+|y-a|)^{N-4}}, \quad |\partial_{a_i} \Psi_{\Lambda,a}(y)| \leq \frac{C}{(1+|y-a|)^{N-3}}. \tag{2.7}$$

For $N = 4$, we let $\bar{\Psi}(|y|)$ be the radial solution of

$$\Delta\bar{\Psi} + U_{1,0} = 0 \ \text{ in } \mathbb{R}^4, \quad \bar{\Psi}(0) = 1. \tag{2.8}$$

Then

$$\bar{\Psi}(|y|) = -\frac{1}{2}\ln|y| + I + O\big(\frac{1}{|y|}\big), \ \ \bar{\Psi}'(|y|) = -\frac{1}{2|y|}\Big(1 + O\big(\frac{\ln(1+|y|)}{|y|^2}\big)\Big), \quad \text{as } |y| \to \infty, \tag{2.9}$$

where I is a constant. Let

$$\Psi_{\Lambda,a}(y) = \frac{\Lambda}{2}\ln\frac{1}{\Lambda\varepsilon} + \Lambda\bar{\Psi}\big(\frac{y-a}{\Lambda}\big), \quad N = 4. \tag{2.10}$$

Then

$$\Delta\Psi_{\Lambda,a}+U_{\frac{1}{\Lambda},a}=0.$$

Note that we have

$$|\Psi_{\Lambda,a}(y)|,\ \ |\partial_\Lambda\Psi_{\Lambda,a}(y)|\le C|\ln\frac{1}{\varepsilon(1+|y-a|)}|,\quad |\partial_{a_i}\Psi_{\Lambda,a}(y)|\le\frac{C}{1+|y-a|}. \tag{2.11}$$

It is easy to see that

$$-\Delta\big(U_{\frac{1}{\Lambda},a}-\mu\varepsilon^2\Psi_{\Lambda,a}\big)+\mu\varepsilon^2\big(U_{\frac{1}{\Lambda},a}-\mu\varepsilon^2\Psi_{\Lambda,a}\big)=\alpha_NU_{\frac{1}{\Lambda},a}^{2^*-1}-\mu^2\varepsilon^4\Psi_{\Lambda,a}.$$

Now, we define the approximate solution $W_{\varepsilon,\Lambda,a}$ as the unique solution of

$$\begin{cases}-\Delta u+\mu\varepsilon^2u=\alpha_NU_{\frac{1}{\Lambda},a}^{2^*-1}-\mu^2\varepsilon^4\Psi_{\Lambda,a}, & \text{in } \Omega_\varepsilon;\\ \frac{\partial u}{\partial n}=0, & \text{on } \partial\Omega_\varepsilon.\end{cases} \tag{2.12}$$

Take any connected component Γ of $\partial\Omega\cap\big\{y_3=\cdots=y_N=0\big\}$ satisfying

$$\big\langle y,n\big\rangle>0,\quad y\in\Gamma. \tag{2.13}$$

Without loss of generality, throughout this paper, we always assume that

$$\Gamma=\big\{y_1^2+y_2^2=1,\ y_3=\cdots=y_N=0\big\}.$$

Define

$$\begin{aligned}H_s=\big\{u:&u\in H^1(\Omega_\varepsilon),u\text{ is even in }y_h,h=2,\cdots,N,\\ &u(r\cos\theta,r\sin\theta,y'')=u(r\cos(\theta+\frac{2\pi j}{k}),r\sin(\theta+\frac{2\pi j}{k}),y''),j=1,\dots,k-1\big\},\end{aligned}$$

and

$$x_j=\big(\frac{1-d}{\varepsilon}\cos\frac{2(j-1)\pi}{k},\frac{1-d}{\varepsilon}\sin\frac{2(j-1)\pi}{k},0\big),\quad j=1,\cdots,k,$$

where $d\in[\delta k^{-\frac{N-2}{N-1}},\frac{1}{\delta}k^{-\frac{N-2}{N-1}}]$, $\delta>0$ is to be determined later and 0 is the zero vector in $\mathbb{R}^{N-2}$. From (2.13), $x_j\in\Omega_\varepsilon$, $j=1,\cdots,k$.

We are now able to define the approximate k-bubble solution. Let $\mathbf{x}=(x_1,\cdots,x_k)$ and define our approximate solutions as

$$W_{\varepsilon,\Lambda,\mathbf{x}}(y)=\sum_{j=1}^kW_{\varepsilon,\Lambda,x_j}. \tag{2.14}$$

Theorem 1.1 is a direct consequence of the following result:

Theorem 2.1. *Suppose that Ω satisfies (H_1) – (H_2) and $N \geq 4$. Then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, problem* (2.4) *has a solution u_k of the form*

$$u_k = W_{\varepsilon,\Lambda,\mathbf{x}}(y) + \omega_k,$$

where $\omega_k \in H_s$, and as $k \to +\infty$, $\|\omega_k\|_{L^\infty} \to 0$.

Let us point out that the non-convex domain $\Omega = B_R(0) \setminus B_r(0)$, $0 < r < R < +\infty$, satisfies (H_1) and (H_2). From (2.13), for the solutions obtained in Theorem 2.1, the bubbles are closed to the outer boundary $|x| = R$. This is in contrast with the boundary bubble solutions obtained in [10], where all the bubbles are exactly on the inner boundary $|x| = r$.

The proof of Theorem 2.1 is by the method of *localized energy*. Associated with (2.4), there is an energy functional

$$I(u) = \frac{1}{2}\int_{\Omega_\varepsilon} (|Du|^2 + \mu\varepsilon^2 u^2) - \frac{(N-2)^2}{2}\int_{\Omega_\varepsilon} |u|^{2^*}.$$

We look for solutions of the following form:

$$u = W_{\varepsilon,\Lambda,\mathbf{x}} + \omega$$

where ω is small in some suitable norms. For each fixed $\Lambda, \mathbf{x}$, we solve a nonlinear projected problem for $\omega = \omega_{\varepsilon,\Lambda,\mathbf{x}}$ and then we find a critical point of the reduced energy functional

$$I(W_{\varepsilon,\Lambda,\mathbf{x}} + \omega_{\varepsilon,\Lambda,\mathbf{x}})$$

in some finite dimensional configuration space. Such method has been used in many papers for singular perturbation problems. But only recently, it was used to study non-singular perturbation problems. See [10, 11, 12]. In [10], the bubbles locate exactly on the boundary, while in this paper, we need to determine how far away the bubbles are from the boundary. So the estimates in this paper are more difficult than those in [10].

Before we close this section, let us mention that the techniques in this paper can be used to study the following Dirichlet problem:

$$\begin{cases} -\Delta u + \mu u - u^{\frac{N+2}{N-2}} = 0, \quad u > 0, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where $\mu > 0$ is a fixed constant, Ω is a bounded smooth domain in $\mathbb{R}^N$, satisfying (H_1), (H_2) and

(H_4) there is a connected component Γ of $\partial\Omega \cap \{y_3 = \cdots = y_N = 0\}$, such that

$$\langle y, n\rangle < 0, \quad \forall\, y \in \Gamma, \tag{2.16}$$

where n is the unit outward normal of $\partial\Omega$ at y.

In fact, for the Dirichlet problem, the energy of a bubble will increase as the bubble moves toward the boundary. On the other hand, if the bubbles move toward in "inner" boundary, the energy will be smaller. So we can construct solutions for (2.16) with large number of bubbles near Γ, where (2.16) holds.

It is easy to see that an annulus or a torus satisfies (H_1), (H_2) and (H_4). In particular, if $\Omega = B_R(0)\setminus B_r(0)$, $R > r > 0$, then (2.15) has infinitely many non-radial solutions, no matter how small $r > 0$ is. It is interesting to know whether (2.17) has non-radial solution if $\mu = 0$, $\Omega = B_R(0)\setminus B_r(0)$ and $r > 0$ is small. Unfortunately, the method in this paper can not be used to deal with (2.17) if $\mu = 0$. Note that similar conditions were used in [6] to construct bubble solutions for the following Dirichlet problem with slightly super-critical growth:

$$\begin{cases} -\Delta u - u^{\frac{N+2}{N-2}+\varepsilon} = 0, \quad u > 0, & \text{in } \Omega;\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.17}$$

where $\varepsilon > 0$ is a small parameter.

3. Finite-dimensional Reduction

In this section, we perform a finite-dimensional reduction.

We first introduce two norms for u and the error:

$$\|u\|_* = \sup_y \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1} |u(y)|, \tag{3.1}$$

$$\|f\|_{**} = \sup_y \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)^{-1} |f(y)|, \tag{3.2}$$

where we choose

$$\tau = \frac{N-4}{N-2}. \tag{3.3}$$

For this choice of τ, we have

$$\sum_{j=2}^k \frac{1}{|x_j - x_1|^\tau} \le C\varepsilon^\tau k^\tau \sum_{j=1}^k \frac{1}{j^\tau} \le Ck\varepsilon^\tau \le C, \quad \text{if } N \ge 5, \tag{3.4}$$

since $\varepsilon = k^{-\frac{N-2}{N-4}}$. If $N = 4$, then for any $\tau_1 > 0$,

$$\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\tau_1}} \leq Ck^{1+\tau_1}\varepsilon^{\tau_1} \leq C. \tag{3.5}$$

To simplify the notations, we use W_j to denote $W_{\varepsilon,\Lambda,x_j}$. Let

$$Y_{i,1} = \frac{\partial W_i}{\partial d}, \quad Y_{i,2} = \frac{\partial W_i}{\partial \Lambda}$$

and

$$Z_{i,1} = -\Delta Y_{i,1} + \mu\varepsilon^2 Y_{i,1}, \qquad Z_{i,2} = -\Delta Y_{i,2} + \mu\varepsilon^2 Y_{i,2}.$$

Then, it is easy to check

$$|Z_{i,j}| \leq \frac{C}{(1+|y-x_i|)^N}. \tag{3.6}$$

We consider

$$\begin{cases} -\Delta\phi_k + \mu\varepsilon^2\phi_k - N(N+2)W^{2^*-2}\phi_k = h + \sum\limits_{j=1}^{2}\sum\limits_{i=1}^{k} c_j Z_{i,j}, & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi_k}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ \phi_k \in H_s, \\ < \sum\limits_{i=1}^{k} Z_{i,j}, \phi_k >= 0, & j = 1, 2 \end{cases} \tag{3.7}$$

for some numbers c_1, c_2, where $< u, v >= \int_{\Omega_\varepsilon} uv$.

We recall the following result, whose proof is given in [9].

Lemma 3.1. *Let f satisfy $\|f\|_{**} < \infty$ and let u be the solution of*

$$-\Delta u + \mu\varepsilon^2 u = f \quad in \quad \Omega_\varepsilon, \qquad \frac{\partial u}{\partial n} = 0 \quad on \quad \partial\Omega_\varepsilon.$$

Then we have

$$|u(x)| \leq C\int_{\Omega_\varepsilon} \frac{|f(y)|}{|x-y|^{N-2}} dy.$$

Next, we need the following lemma to carry out the reduction.

Lemma 3.2. *Assume that ϕ_k solves (3.7) for $h = h_k$. If $\|h_k\|_{**}$ goes to zero as k goes to infinity, so does $\|\phi_k\|_*$.*

Proof. The proof of this lemma is similar to that of Lemma 3.2 in [10]. We thus just sketch it.

We argue by contradiction. Suppose that there are $k \to +\infty$, $h = h_k$, $\Lambda_k \in [\delta, \delta^{-1}], d_k \in [\delta k^{-\frac{N-2}{N-1}}, \frac{1}{\delta}k^{-\frac{N-2}{N-1}}]$, and ϕ_k solving (3.7) for $h = h_k$, $\Lambda = \Lambda_k, d = d_k$, with $\|h_k\|_{**} \to 0$, and $\|\phi_k\|_* \geq c' > 0$. We may assume that $\|\phi_k\|_* = 1$. For simplicity, we drop the subscript k.

According to Lemma 3.1, we have

$$
\begin{aligned}
|\phi(y)| \leq & C \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2} |\phi(z)| \, dz \\
& + C \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} \Big(|h(z)| + \Big| \sum_{j=1}^{2} \sum_{i=1}^{k} c_j Z_{i,j} \Big| \Big) \, dz
\end{aligned}
\tag{3.8}
$$

Using Lemma C.3, there is a strictly positive number θ such that

$$
\Big| \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2} \phi(z) \, dz \Big| \leq C \|\phi\|_* \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}. \tag{3.9}
$$

It follows from Lemma C.2 that

$$
\begin{aligned}
& \Big| \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} h(z) \, dz \Big| \\
\leq & C \|h\|_{**} \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}} \, dz \\
\leq & C \|h\|_{**} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}
\tag{3.10}
$$

On the other hand, using (3.6), we find

$$
\Big| \int_{\Omega_\varepsilon} \frac{1}{|z-y|^{N-2}} \sum_{i=1}^{k} Z_{i,j}(z) \, dz \Big| \leq C \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}, \qquad j = 1, 2. \tag{3.11}
$$

Next, we estimate c_j, $j = 1, 2$. Multiplying (3.7) by $Y_{1,l}$ for $l = 1, 2$, we see that c_j satisfies

$$
\sum_{j=1}^{2} \Big\langle \sum_{i=1}^{k} Z_{i,1}, Y_{1,l} \Big\rangle c_j = \Big\langle -\Delta\phi + \mu\varepsilon^2\phi - N(N+2) W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2} \phi, Y_{1,l} \Big\rangle - \Big\langle h, Y_{1,l} \Big\rangle \tag{3.12}
$$

It follows from Lemmas A.2 and C.1 that

$$
\Big| \Big\langle h, Y_{1,l} \Big\rangle \Big| \leq C \|h\|_{**} \int_{\mathbb{R}^N} \frac{1}{(1+|z-x_1|)^{N-2}} \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}} \, dz \leq C \|h\|_{**}.
$$

On the other hand,

$$
\begin{aligned}
&\big\langle -\Delta\phi+\mu\varepsilon^2\phi-N(N+2)W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2}\phi, Y_{1,l}\big\rangle\\
=&\big\langle -\Delta Y_{1,l}+\mu\varepsilon^2 Y_{1,l}-N(N+2)W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2}Y_{1,l},\phi\big\rangle \qquad (3.13)\\
=&N(N+2)\big\langle U_{\frac{1}{\Lambda},x_1}^{2^*-2}\partial U_{\frac{1}{\Lambda},x_1}-W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2}Y_{1,l},\phi\big\rangle-\varepsilon^4\mu^2\big\langle\partial\Psi_{\Lambda,x_1},\phi\big\rangle.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
&\big|\varepsilon^4\mu^2\big\langle\partial\Psi_{\Lambda,x_1},\phi\big\rangle\big|\le C\|\phi\|_*\int_{\Omega_\varepsilon}\frac{\varepsilon^4|\ln\frac{1}{\varepsilon(1+|y-x_1|)}|^m}{(1+|y-x_1|)^{N-4}}\sum_{j=1}^k\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\tau}}\\
\le&C\varepsilon^{\frac12}\|\phi\|_*\int_{\Omega_\varepsilon}\frac{1}{(1+|y-x_1|)^{N-\frac12}}\sum_{j=1}^k\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\tau}}=o(1)\|\phi\|_*.
\end{aligned}
$$

where $m=1$ if $N=4$, $m=0$ if $N\ge5$.

Using Lemmas A.2, A.3 and C.1, similar to (3.12) and (3.13) in [10], we can prove

$$
\Big|\big\langle U_{\frac{1}{\Lambda},x_1}^{2^*-2}\partial U_{\frac{1}{\Lambda},x_j}-W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2}Y_{1,l},\phi\big\rangle\Big|=o(1)\|\phi\|_*.
$$

But there is a constant $\bar c_l>0$,

$$
\big\langle\sum_{i=1}^k Z_{i,j},Y_{1,l}\big\rangle=\bar c_l\delta_{jl}+o(1).
$$

Thus we obtain that

$$
c_j=o(\|\phi\|_*)+O(\|h\|_{**}).
$$

So,

$$
\|\phi\|_*\le\Big(o(1)+\|h_k\|_{**}+\frac{\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}}\Big). \qquad (3.14)
$$

Since $\|\phi\|_*=1$, we obtain from (3.14) that there is $R>0$, such that

$$
\|\phi(y)\|_{B_R(x_i)}\ge c_0>0, \qquad (3.15)
$$

for some i. But $\bar\phi(y)=\phi(y-x_i)$ converges uniformly in any compact set of $\mathbb{R}^N_+$ to a solution u of

$$\Delta u + N(N+2)U_{\frac{1}{\Lambda},0}^{2^*-2}u = 0 \tag{3.16}$$

for some $\Lambda \in [\delta, \delta^{-1}]$, and u is perpendicular to the kernel of (3.16). So, $u = 0$. This is a contradiction to (3.15).

□

From Lemma 3.2, using the same argument as in the proof of Proposition 4.1 in [5], Proposition 3.1 in [9], we can prove the following result :

Proposition 3.3. *There exists $k_0 > 0$ and a constant $C > 0$, independent of k, such that for all $k \geq k_0$ and all $h \in L^\infty(\Omega_\varepsilon)$, problem (3.7) has a unique solution $\phi \equiv L_k(h)$. Besides,*

$$\|L_k(h)\|_* \leq C\|h\|_{**}, \qquad |c_j| \leq C\|h\|_{**}, \quad j = 1, 2. \tag{3.17}$$

Moreover, the map $L_k(h)$ is C^1 with respect to (Λ, d).

Now, we consider

$$\begin{cases} -\Delta\big(W_{\varepsilon,\Lambda,\mathbf{x}} + \phi\big) + \mu\varepsilon^2(W_{\varepsilon,\Lambda,\mathbf{x}} + \phi) = \alpha_N\big(W_{\varepsilon,\Lambda,\mathbf{x}} + \phi\big)^{2^*-1} + \sum\limits_{j=1}^{2}\sum\limits_{i=1}^{k} c_j Z_{i,j}, & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ \phi \in H_s, \\ < \sum\limits_{i=1}^{k} Z_{i,j}, \phi >= 0, \quad j = 1, 2. \end{cases} \tag{3.18}$$

We have

Proposition 3.4. *There is an integer $k_0 > 0$, such that for each $k \geq k_0$, $\delta \leq \Lambda \leq \delta^{-1}$, where δ is a fixed small constant, (3.18) has a unique solution ϕ, satisfying*

$$\|\phi\|_* \leq \begin{cases} C\varepsilon^{1+\sigma}, & N \geq 5; \\ C\varepsilon, & N = 4, \end{cases}$$

where $\sigma > 0$ is a fixed small constant. Moreover, $(\Lambda, d) \to \phi(\Lambda, d)$ is C^1.

Rewrite (3.18) as

$$\begin{cases} -\Delta\phi + \mu\varepsilon^2\phi - N(N+2)W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2}\phi = N(\phi) + l_k + \sum\limits_{j=1}^{2}\sum\limits_{i=1}^{k} c_j Z_{i,j}, & \text{in } \Omega_\varepsilon, \\ \frac{\partial\phi}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ \phi \in H_s, \\ < \sum\limits_{i=1}^{k} Z_{i,j}, \phi >= 0, \quad j = 1, 2, \end{cases} \tag{3.19}$$

where

$$N(\phi) = \alpha_N\Big(\big(W_{\varepsilon,\Lambda,\mathbf{x}} + \phi\big)^{2^*-1} - W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-1} - (2^*-1)W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2}\phi\Big),$$

and

$$l_k = \alpha_N\Big(W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-1} - \sum_{j=1}^{k} U_{\frac{1}{\Lambda},x_j}^{2^*-1}\Big) - \sum_{i=1}^{k} \mu^2\varepsilon^4\Psi_{\Lambda,x_i}.$$

In order to use the contraction mapping theorem to prove that (3.19) is uniquely solvable in the set that $\|\phi\|_*$ is small, we need to estimate $N(\phi)$ and l_k.

In the following, we always assume that $\|\phi\|_* \le \varepsilon|\ln\varepsilon|$.

Lemma 3.5. *We have*

$$\|N(\phi)\|_{**} \le C\|\phi\|_*^{\min(2^*-1,2)}.$$

Proof. We have

$$|N(\phi)| \le \begin{cases} C|\phi|^{2^*-1}, & N \ge 6; \\ C\big(W_{\varepsilon,\Lambda,\mathbf{x}}^{\frac{6-N}{N-2}}\phi^2 + |\phi|^{2^*-1}\big), & N = 4, 5. \end{cases}$$

Similar to the proof of Lemma 3.5 in [10], we can prove

$$|N(\phi)\|_{**} \le \begin{cases} C\|\phi\|_*^{2^*-1}, & N \ge 6; \\ C\|\phi\|_*^2, & N = 5. \end{cases}$$

Now, we discuss the case $N = 4$. In this case, $\tau = 0$. Similar to the proof of Lemma 3.5 in [10] for the case $N = 3$, using

$$\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\tau_1}} < +\infty,$$

for any $\tau_1 > 0$, we can also prove

$$\|N(\phi)\|_{**} \le C\|\phi\|_*^2.$$

□

Next, we estimate l_k.

Lemma 3.6. *We have*

$$\|l_k\|_{**} \le \begin{cases} C\varepsilon^{1+\sigma}, & N \ge 5; \\ C\varepsilon, & N = 4, \end{cases}$$

where $\sigma > 0$ is a fixed small constant.

Proof. Recall that $m = 1$ if $N = 4$, $m = 0$ if $N \ge 5$. Let

$$\Omega_j = \Big\{ y = (y', y'') \in \Omega_\varepsilon : \Big\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \Big\rangle \ge \cos\frac{\pi}{k} \Big\}.$$

By the symmetry, we can assume that $y \in \Omega_1$. We have

$$|y - x_j| \ge |y - x_1|, \quad \forall\, y \in \Omega_1.$$

Thus, for $y \in \Omega_1$,

$$\begin{aligned} |l_k| \le & \frac{C}{(1+|y-x_1|)^4} \sum_{j=2}^{k} \frac{1}{(1+|y-x_j|)^{N-2}} + C\Big(\sum_{j=2}^{k} \frac{1}{(1+|y-x_j|)^{N-2}} \Big)^{2^*-1} \\ & + C\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^4} |\bar\varphi_{\Lambda,x_j}| + C\sum_{j=1}^{k} \frac{\varepsilon^4 |\ln \frac{1}{\varepsilon(1+|y-x_j|)}|^m}{(1+|y-x_j|)^{N-4}}, \end{aligned} \tag{3.20}$$

where $\bar\varphi_{\Lambda,x_j} = U_{\frac{1}{\Lambda},x_j} - W_{\varepsilon,\Lambda,x_j}$.

Let us estimate the first term of (3.20). Using Lemma C.1 and $|y-x_j| \ge |y-x_1|$ in Ω_1, taking $1 < \alpha < \min(N-2, \frac{N+2}{2} - \tau)$, we obtain for $j > 1$,

$$\begin{aligned} & \frac{1}{(1+|y-x_1|)^4} \frac{1}{(1+|y-x_j|)^{N-2}} \le \frac{C}{(1+|y-x_1|)^{N+2-\alpha}} \frac{1}{|x_j - x_1|^\alpha} \\ \le & \frac{C}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \frac{1}{|x_j - x_1|^\alpha}. \end{aligned} \tag{3.21}$$

As a result

$$
\begin{aligned}
&\frac{1}{(1+|y-x_1|)^4}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\\
\le&\frac{C}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\big(k\varepsilon\big)^{\alpha}\le C\varepsilon^{1+\sigma}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}},
\end{aligned}
\tag{3.22}
$$

since, if $N=4$, $k\varepsilon=O(\varepsilon|\ln\varepsilon|)$ (so we take $\alpha>1$), while if $N\ge 5$, $k\varepsilon=\varepsilon^{\frac{2}{N-2}}$ (so we take $\alpha>\frac{N-2}{2}$).

Now, we estimate the second term of (3.20).

If $N\ge 5$, then $\frac{N-2}{2}-\frac{N-2}{N+2}\tau>1$. If $N=4$, then $\frac{N-2}{2}-\frac{N-2}{N+2}\tau=1$ since $\tau=0$. Using Lemma C.1 again, we find for $y\in\Omega_1$,

$$
\begin{aligned}
&\frac{1}{(1+|y-x_j|)^{N-2}}\le\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}}}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}}}\\
\le&\frac{C}{|x_j-x_1|^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau}}\Big(\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}+\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}\Big)\\
\le&\frac{C}{|x_j-x_1|^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau}}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}\\
\le&C|\ln k|^{m}\big(k\varepsilon\big)^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}
\end{aligned}
$$

which, gives for $y\in\Omega_1$

$$
\begin{aligned}
&\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}\\
\le&C|\ln k|^{m(2^*-1)}\big(k\varepsilon\big)^{\frac{N+2}{2}-\tau}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\le C\varepsilon^{1+\sigma}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

Now, we estimate the third term of (3.20). From Lemma A.3, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^4}|\bar{\varphi}_{\Lambda,x_j}|\\
\le&\sum_{j=1}^{k}\frac{C}{(1+|y-x_j|)^4}\Big(\frac{\varepsilon^2|\ln\frac{1}{\varepsilon(1+|y-x_j|)}|^m}{(1+|y-x_j|)^{N-4}}+\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_j)\Big)\\
\le& C\varepsilon^{1+\sigma}\sum_{j=1}^{k}\frac{C}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}+C\sum_{j=1}^{k}\frac{\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_j)}{(1+|y-x_j|)^4}.
\end{aligned}
$$

But from (A.2),

$$
\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_j)\le CU_{\frac{1}{\Lambda},x_j}.
$$

So if $\sigma>0$ is small,

$$
\begin{aligned}
&\sum_{j=1}^{k}\frac{\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_j)}{(1+|y-x_j|)^4}\\
\le&\sum_{j=1}^{k}\frac{C}{(1+|y-x_j|)^{4+(N-2)(\frac{1}{2}-\sigma)}}\big(\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_j)\big)^{\frac{1}{2}+\sigma}\\
\le&\Big(\frac{\varepsilon}{d}\Big)^{(N-2)(\frac{1}{2}+\sigma)}\sum_{j=1}^{k}\frac{C}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\le C\varepsilon^{1+\sigma}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

So, we obtain

$$
\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^4}|\bar{\varphi}_{\Lambda,x_j}|\le C\varepsilon^{1+\sigma}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.
$$

Finally, we estimate the last term of (3.20).

Suppose that $N\ge 5$. Since $N-1>\frac{N+2}{2}+\tau$, we obtain

$$
\sum_{j=1}^{k}\frac{\varepsilon^4}{(1+|y-x_j|)^{N-4}}\le\sum_{j=1}^{k}\frac{C\varepsilon^{1+\sigma}}{(1+|y-x_j|)^{N-1-\sigma}}\le\sum_{j=1}^{k}\frac{C\varepsilon^{1+\sigma}}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}},
$$

if $\sigma>0$ is small.

If $N=4$, then

$$
\begin{aligned}
&\sum_{j=1}^{k}\varepsilon^4|\Psi_{\Lambda,x_j}|\le\sum_{j=1}^{k}C\varepsilon^4\ln\frac{1}{\varepsilon(1+|y-x_j|)}\\
\le&C\sum_{j=1}^{k}\varepsilon^4\frac{1}{\varepsilon(1+|y-x_j|)}\le C\varepsilon\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^3}.
\end{aligned}
$$

Combining all the above estimates, we obtain the result.

□

Now, we are ready to prove Proposition 3.4.

Proof of Proposition 3.4. Let us recall that

$$
\varepsilon=k^{-\frac{N-2}{N-4}},\ \text{ if } N\ge 5;\quad \varepsilon=e^{-D_k k^2},\ \text{ if } N=4.
$$

Let

$$
E_N=\Big\{u:u\in C(\Omega_\varepsilon),\|u\|_*\le\varepsilon|\ln\varepsilon|,\int_{\Omega_\varepsilon}\sum_{i=1}^{k}Z_{i,j}u=0,\ j=1,2\Big\}.
$$

Then, (3.19) is equivalent to

$$
\phi=A(\phi)=:L(N(\phi))+L(l_k).
$$

Now we prove that A is a contraction map from E_N to E_N. Using Lemma 3.5, we have

$$
\begin{aligned}
&\|A\phi\|_*\le C\|N(\phi)\|_{**}+C\|l_k\|_{**}\le C\|\phi\|_*^{\min(2^*-1,2)}+C\|l_k\|_{**}\\
\le&C(\varepsilon|\ln\varepsilon|)^{\min(2^*-1,2)}+C\|l_k\|_{**}\\
\le&C\varepsilon^{1+\sigma}+C\|l_k\|_{**}.
\end{aligned}
\tag{3.23}
$$

Thus, by Lemma 3.6, we find that A maps E_N to E_N.

Next, we show that A is a contraction map.

$$
\|A(\phi_1)-A(\phi_2)\|_*=\|L(N(\phi_1))-L(N(\phi_2))\|_*\le C\|N(\phi_1)-N(\phi_2)\|_{**}.
$$

On the other hand, we have

$$
|N'(t)|\le\begin{cases}C|t|^{2^*-2}, & N\ge 6;\\ C\big(W_{\varepsilon,\Lambda,\mathbf{x}}^{\frac{6-N}{N-2}}|\phi|+|\phi|^{2^*-2}\big), & N=4,5,\end{cases}
$$

from which, we can deduce (see the proof of Proposition 3.4 in [10] for details)

$$\begin{aligned}&\|A(\phi_1)-A(\phi_2)\|_*\le C\|N(\phi_1)-N(\phi_2)\|_{**}\\ \le& C\big(\|\phi_1\|_*^{\min(2^*-2,1)}+\|\phi_2\|_*^{\min(2^*-2,1)}\big)\|\phi_1-\phi_2\|_*\le\frac{1}{2}\|\phi_1-\phi_2\|_*.\end{aligned}$$

Thus, A is a contraction map.

It follows from the contraction mapping theorem that there is a unique $\phi\in E_N$, such that

$$\phi=A(\phi).$$

Moreover, it follows from (3.23) that

$$\|\phi\|_*\le C\varepsilon^{1+\sigma}+C\|l_k\|_{**}.$$

So, the estimate for $\|\phi\|_*$ follows from Lemma 3.6.

□

4. Proof of the main result

Let

$$F(\Lambda,d)=I\big(W_{\varepsilon,\Lambda,\mathbf{x}}+\phi\big),$$

where ϕ is the function obtained in Proposition 3.4, and let

$$I(u)=\frac{1}{2}\int_{\Omega_\varepsilon}(|Du|^2+\mu\varepsilon^2u^2)-\frac{(N-2)^2}{2}\int_{\Omega_\varepsilon}|u|^{2^*}.$$

Using the symmetry, we can check that if (Λ,d) is a critical point of $F(\Lambda,d)$, then $W+\phi$ is a solution of (1.3).

Our next result shows that the estimate in Proposition 3.4 implies that $F(\Lambda,d)$ is a perturbation of $I\big(W_{\varepsilon,\Lambda,\mathbf{x}}\big)$,

Proposition 4.1. *For $N\ge4$, we have*

$$F(\Lambda,d)=I(W_{\varepsilon,\Lambda,\mathbf{x}})+\begin{cases}O\big(k\varepsilon^{2+\sigma}\big), & N\ge5;\\ O\big(k\varepsilon^2\big), & N=4.\end{cases}$$

Proof. There is $t\in(0,1)$, such that

$$\begin{aligned}F(\Lambda,d)=&I(W_{\varepsilon,\Lambda,\mathbf{x}})+\big\langle I'\big(W_{\varepsilon,\Lambda,\mathbf{x}}+\phi\big),\phi\big\rangle+\frac{1}{2}D^2I\big(W_{\varepsilon,\Lambda,\mathbf{x}}+t\phi\big)(\phi,\phi)\\ =&I(W_{\varepsilon,\Lambda,\mathbf{x}})+\int_{\Omega_\varepsilon}\big(|D\phi|^2+\varepsilon^2\mu\phi^2-N(N+2)\big(W_{\varepsilon,\Lambda,\mathbf{x}}+t\phi\big)^{2^*-2}\phi^2\big)\\ =&I(W_{\varepsilon,\Lambda,\mathbf{x}})-N(N+2)\int_{\Omega_\varepsilon}\Big(\big(W_{\varepsilon,\Lambda,\mathbf{x}}+t\phi\big)^{2^*-2}-W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2}\Big)\phi^2+\int_{\Omega_\varepsilon}\big(l_k+N(\phi)\big)\phi.\end{aligned}$$

But

$$\begin{aligned}&\int_{\Omega_\varepsilon}\big(|l_k|+|N(\phi)|\big)|\phi|\\ \le& C\big(\|l_k\|_{**}+\|N(\phi)\|_{**}\big)\|\phi\|_*\int_{\Omega_\varepsilon}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.\end{aligned}$$

Using Lemma C.1, we find

$$\begin{aligned}&\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\ =&\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2\tau}}+\sum_{j=1}^k\sum_{i\ne j}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\ \le&\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2\tau}}+C\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2\tau-\tau_1}}\sum_{i=2}^k\frac{1}{|x_i-x_1|^{\tau_1}}\\ \le& C\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{N+2\tau-\tau_1}},\end{aligned}$$

where $\tau_1=\tau+\theta$ with $\theta>0$ small. Thus, we obtain,

$$\int_{\Omega_\varepsilon}\big(|l_k|+|N(\phi)|\big)|\phi|\le Ck\big(\|l_k\|_{**}+\|N(\phi)\|_{**}\big)\|\phi\|_*\le\begin{cases}Ck\varepsilon^{2+\sigma}, & N\ge 5;\\ Ck\varepsilon^2, & N=4.\end{cases}$$

Now

$$\big(W_{\varepsilon,\Lambda,\mathbf{x}}+t\phi\big)^{2^*-2}-W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2}=\begin{cases}O\big(|\phi|^{2^*-2}\big), & N\ge 6;\\ O\big(W_{\varepsilon,\Lambda,\mathbf{x}}^{\frac{6-N}{N-2}}|\phi|+|\phi|^{2^*-2}\big), & N=4,5,\end{cases}$$

from which we can deduce that

$$\begin{aligned}&\Big|\int_{\Omega_\varepsilon}\Big(\big(W_{\varepsilon,\Lambda,\mathbf{x}}+t\phi\big)^{2^*-2}-W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*-2}\Big)\phi^2\Big|\\ \le& Ck\varepsilon^{-\theta}|\ln\varepsilon|^{m2^*}\|\phi\|_*^{\min(3,2^*)}\le Ck\varepsilon^{2+\sigma},\end{aligned}$$

where $\theta>0$ can be taken as any fixed small number. Thus, we obtain

$$F(\Lambda,d)=I(W_{\varepsilon,\Lambda,\mathbf{x}})+\begin{cases}O\big(k\varepsilon^{2+\sigma}\big), & N\ge 5;\\ O\big(k\varepsilon^2\big), & N=4.\end{cases}$$

□

Proof of Theorem 2.1: We just need to prove that $F(\Lambda, d)$ has a critical point.

Let

$$\bar{F}(\Lambda, d) = \frac{1}{k\varepsilon^2}(F(\Lambda, d) - \frac{1}{N}A_N).$$

Define $\Sigma = \{(\Lambda, d) \mid \delta \le \Lambda \le \frac{1}{\delta}, \ \ \delta k^{-\frac{N-2}{N-1}} \le d \le \frac{1}{\delta} k^{-\frac{N-2}{N-1}}\}$, where δ is small positive number. We consider the following maximum problem

$$\max_{\Sigma} \bar{F}(\Lambda, d).$$

We will prove that $\bar{F}(\Lambda, d)$ can't attain its maximum on $\partial\Sigma$ for suitable δ. Thus we get a critical point of $\bar{F}(\Lambda, d)$.

For $N \ge 5$, by Lemma A.4, we have

$$\bar{F}(\Lambda, d) = B_1\Lambda^2 + \Lambda^{N-2}\varepsilon^{N-4}\big(B_2 H(a_1, a_1) - \frac{k^{N-2}B}{(1-d)^{N-2}}\big) + o(1).$$

Since $\varepsilon = k^{-\frac{N-2}{N-4}}$, and

$$H(a_1, a_1) = -\frac{c_0}{d^{N-2}}\big(1 + O(d)\big),$$

where $c_0 > 0$ is a constant, we find

$$\bar{F}(\Lambda, d) = B_1\Lambda^2 + \Lambda^{N-2}\big(-\frac{c_0 B_2}{d^{N-2}k^{N-2}} - \frac{B}{(1-d)^{N-2}}\big) + o(1).$$

Let

$$\tilde{F}(\Lambda, d) = B_1\Lambda^2 + \Lambda^{N-2}\big(-\frac{c_0 B_2}{d^{N-2}k^{N-2}} - \frac{B}{(1-d)^{N-2}}\big).$$

Then, $\tilde{F}(\Lambda, d)$ has a maximum point (Λ_k, d_k) with

$$d_k = k^{-\frac{N-2}{N-1}}(d^* + o(1)), \quad \Lambda_k = \Lambda^* + o(1),$$

where d^* and Λ^* are positive constants, independent of k. Thus, if $\delta > 0$ is chosen small enough, $\bar{F}(\Lambda, d)$ can't attain its maximum on $\partial\Sigma$.

Finally, we consider the case $N = 4$. Now

$$\begin{aligned}\bar F(\Lambda,d)=&B_1\Lambda^2\ln\frac{1}{\Lambda\varepsilon}+\Lambda^2k^2\big(-\frac{B_2c_0}{d^2k^2}-\frac{B}{(1-d)^2}\big)+\beta_k\Lambda^2+O\big(\Lambda^2\big)\\=&B_1\Lambda^2\big(\ln\frac{1}{\Lambda\varepsilon}+\beta_k\big)+\Lambda^2k^2\big(-\frac{B_2c_0}{d^2k^2}-\frac{B}{(1-d)^2}\big)+O(\Lambda^2),\end{aligned}\tag{4.24}$$

where β_k is independent of Λ and satisfies $\beta_k=O(k\ln k)$ as $k\to+\infty$.

Let d_k be the solution of

$$\frac{B_2c_0}{d^3k^2}-\frac{B}{(1-d)^3}=0$$

Then, we have $d_k=(d^*+o(1))k^{-\frac{2}{3}}$ for some positive constant d^*, and d_k is a maximum point of the function

$$g_k(d)=-\frac{B_2c_0}{d^2k^2}-\frac{B}{(1-d)^2}.$$

Define

$$\varepsilon=e^{\frac{k^2g_k(d_k)+\beta_k}{B_1}}\tag{4.25}$$

Then,

$$\bar F(\Lambda,d)=B_1\Lambda^2\ln\frac{1}{\Lambda}+\Lambda^2k^2\big(g_k(d)-g_k(d_k)\big)+O(\Lambda^2).$$

Let $(\bar\Lambda_k,\bar d_k)$ be a maximum point of $\bar F(\Lambda,d)$ in Σ. Firstly, we claim

$$\max_{\Sigma}\bar F(\Lambda,d)\ge a_0>0.\tag{4.26}$$

In fact, take $d=d_k$ and $\Lambda_0>\delta$ small. Then

$$\max_{\Sigma}\bar F(\Lambda,d)\ge\bar F(\Lambda_0,d_k)=B_1\Lambda_0^2\ln\frac{1}{\Lambda_0}+O(\Lambda_0^2)=:a_0>0.$$

Secondly, we claim $\bar\Lambda_k\in(\delta,\frac{1}{\delta})$. Since $g_k(d)-g_k(d_k)\le 0$, we find from (4.26)

$$B_1\bar\Lambda_k^2\ln\frac{1}{\bar\Lambda_k}+O(\bar\Lambda_k^2)\ge\bar F(\bar\Lambda_k,\bar d_k)\ge a_0>0.\tag{4.27}$$

But $B_1\Lambda^2\ln\frac{1}{\Lambda}+O(\Lambda^2)\to-\infty$ as $\Lambda\to+\infty$, and $B_1\Lambda^2\ln\frac{1}{\Lambda}+O(\Lambda^2)\to 0$ as $\Lambda\to 0$. So we obtain from (4.27) that $\bar\Lambda_k\in(\delta,\frac{1}{\delta})$ if $\delta>0$ is small enough.

Lastly, it follows from (4.26) that

$$k^2\big(g_k(d)-g_k(d_k)\big)\ge -C.\tag{4.28}$$

Since $g''(d) = -\frac{6B_2c_0}{d^4k^2} - \frac{6B}{(1-d)^4}$, we find that if $d = \delta k^{-\frac{2}{3}}$, or $d = \frac{1}{\delta}k^{-\frac{2}{3}}$, then

$$k^2\big(g_k(d) - g_k(d_k)\big) \le -c'k^2d^2 \to -\infty, \quad \text{as } k \to +\infty.$$

This is a contradiction to (4.28). Thus we have proved that $(\bar\Lambda_k, \bar d_k)$ is an interior point of Σ.

□

Appendix A. Estimate of the approximate solutions

Recall that $N \ge 4$ and $W_{\varepsilon,\Lambda,a}$ is the solution of

$$\begin{cases} -\Delta u + \mu\varepsilon^2 u = \alpha_N U_{\frac{1}{\Lambda},a}^{2^*-1} - \mu^2\varepsilon^4\Psi_{\Lambda,a}, & \text{in } \Omega_\varepsilon; \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{A.1}$$

where $\Psi_{\Lambda,a}$ satisfies

$$\Delta\Psi_{\Lambda,a} + U_{\frac{1}{\Lambda},a} = 0, \quad \text{in } \mathbb{R}^N.$$

In this section, we will estimate $W_{\varepsilon,\Lambda,a}$.

Denote $d = d(\varepsilon a, \partial\Omega) > 0$. Let $K_\mu(|y-x|)$ be the solution of

$$-\Delta u + \mu u = \delta_x, \quad \text{in } \mathbb{R}^N.$$

Then, we have the following expansion

$$K_\mu(|y-x|) = \frac{c_\mu}{|y-x|^{N-2}}\big(1 + O(|y-x|)\big), \quad \text{as } y \to x,$$

and

$$K'_\mu(|y-x|) = -\frac{c_\mu(N-2)}{|y-x|^{N-1}}\big(1 + O(|y-x|)\big), \quad \text{as } y \to x,$$

where $c_\mu > 0$ is a constant. Let $H(y,x)$ be the regular part of the Green's function under the Neumann boundary condition. That is, $H(y,x)$ satisfies

$$\begin{cases} -\Delta H + \mu H = 0, & \text{in } \Omega; \\ \frac{\partial H}{\partial n} = \frac{\partial K_\mu(|y-x|)}{\partial n}, & \text{on } \partial\Omega. \end{cases}$$

Set $G(y,x) = K(|y-x|) - H(y,x)$. Then, it is easy to check that

$$|G(y,x)| \le \frac{C}{|y-x|^{N-2}}.$$

We have

Lemma A.1. *For any $a \in \Omega_\varepsilon$, then*

$$\begin{aligned} W_{\varepsilon,\Lambda,a}(y) =& U_{\frac{1}{\Lambda},a}(y) - \mu\varepsilon^2\Psi_{\Lambda,a}(y) - \frac{\Lambda^{\frac{N-2}{2}}\varepsilon^{N-2}}{c_\mu}H(\varepsilon y, \varepsilon a) \\ &+ O\Big(\frac{\varepsilon^{N-2}}{|\varepsilon y - \varepsilon a|^{N-3+\theta} + d^{N-3+\theta}} + \frac{\varepsilon^{N-2}}{|\varepsilon y - \varepsilon a|^{N-2+\theta} + d^{N-2+\theta}}\frac{\varepsilon^2}{d^2}\Big), \end{aligned}$$

and

$$\begin{aligned} \partial W_{\varepsilon,\Lambda,a}(y) =& \partial\Big(U_{\frac{1}{\Lambda},a}(y) - \mu\varepsilon^2\Psi_{\Lambda,a}(y) - \frac{\Lambda^{\frac{N-2}{2}}\varepsilon^{N-2}}{c_\mu}H(\varepsilon y, \varepsilon a)\Big) \\ &+ O\Big(\frac{\varepsilon^{N-2}}{|\varepsilon y - \varepsilon a|^{N-3+\theta} + d^{N-3+\theta}} + \frac{\varepsilon^{N-2}}{|\varepsilon y - \varepsilon a|^{N-2+\theta} + d^{N-2+\theta}}\frac{\varepsilon^2}{d^2}\Big), \end{aligned}$$

where ∂ denotes either ∂_d or ∂_Λ.

Proof. Let

$$\varphi_{\varepsilon,\Lambda,a} = U_{\frac{1}{\Lambda},a} - \mu\varepsilon^2\Psi_{\Lambda,a} - W_{\varepsilon,\Lambda,a}.$$

Then

$$\begin{cases} -\Delta\varphi_{\varepsilon,\Lambda,a} + \mu\varepsilon^2\varphi_{\varepsilon,\Lambda,a} = 0, & \text{in } \Omega_\varepsilon; \\ \frac{\partial\varphi_{\varepsilon,\Lambda,a}}{\partial n} = \frac{\partial}{\partial n}\big(U_{\frac{1}{\Lambda},a} - \mu\varepsilon^2\Psi_{\Lambda,a}\big), & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

It is convenient to carry out the estimates in Ω. For this reason, we let

$$\tilde{w}(y) = \varepsilon^{-\frac{N-2}{2}} w\Big(\frac{y}{\varepsilon}\Big), \quad y \in \Omega,$$

for any function w defined in Ω_ε. Then

$$\begin{cases} -\Delta\tilde{\varphi}_{\varepsilon,\Lambda,a} + \mu\tilde{\varphi}_{\varepsilon,\Lambda,a} = 0, & \text{in } \Omega; \\ \frac{\partial\tilde{\varphi}_{\varepsilon,\Lambda,a}}{\partial n} = \frac{\partial}{\partial n}\big(U_{\frac{1}{\varepsilon\Lambda},\varepsilon a} - \mu\varepsilon^2\tilde{\Psi}_{\Lambda,a}\big), & \text{on } \partial\Omega. \end{cases}$$

Let

$$\tilde{\varphi}_{\varepsilon,\Lambda,a} = \tilde{\varphi}_{\varepsilon,\Lambda,a,1} - \tilde{\varphi}_{\varepsilon,\Lambda,a,2},$$

where $\tilde{\varphi}_{\varepsilon,\Lambda,a,1}$ satisfies

$$\begin{cases} -\Delta\tilde{\varphi}_{\varepsilon,\Lambda,a,1} + \mu\tilde{\varphi}_{\varepsilon,\Lambda,a,1} = 0, & \text{in } \Omega; \\ \frac{\partial\tilde{\varphi}_{\varepsilon,\Lambda,a,1}}{\partial n} = \frac{\partial U_{\frac{1}{\varepsilon\Lambda},\varepsilon a}}{\partial n}, & \text{on } \partial\Omega. \end{cases}$$

and $\tilde{\varphi}_{\varepsilon,\Lambda,a,2}$ satisfies

$$\begin{cases} -\Delta\tilde{\varphi}_{\varepsilon,\Lambda,a,2} + \mu\tilde{\varphi}_{\varepsilon,\Lambda,a,2} = 0, & \text{in } \Omega; \\ \frac{\partial\tilde{\varphi}_{\varepsilon,\Lambda,a,2}}{\partial n} = \frac{\partial\mu\varepsilon^2\tilde{\Psi}_{\Lambda,a}}{\partial n}, & \text{on } \partial\Omega. \end{cases}$$

By (2.7) and (2.11),

$$\Big|\frac{\partial\mu\varepsilon^2\tilde{\Psi}_{\Lambda,a}}{\partial n}\Big| \le \frac{C\varepsilon^{1-\frac{N-2}{2}}}{(1+\frac{|z-\varepsilon a|}{\varepsilon})^{N-3}} \le \frac{C\varepsilon^{\frac{N-2}{2}}}{|z-\varepsilon a|^{N-3}}, \quad \forall\, z\in\partial\Omega,$$

So,

$$|\tilde{\varphi}_{\varepsilon,\Lambda,a,2}(y)| \le C\int_{\partial\Omega}\frac{1}{|z-y|^{N-2}}\frac{\varepsilon^{\frac{N-2}{2}}}{|z-\varepsilon a|^{N-3}}\,dz.$$

If $|y-\varepsilon a|\ge d$, then

$$\begin{aligned} |\tilde{\varphi}_{\varepsilon,\Lambda,a,2}(y)| \le & \frac{C\varepsilon^{\frac{N-2}{2}}}{|y-\varepsilon a|^{N-4+\theta}}\int_{\partial\Omega}\Big(\frac{1}{|z-y|^{N-1-\theta}}+\frac{1}{|z-\varepsilon a|^{N-1-\theta}}\Big)\,dz \\ \le & \frac{C\varepsilon^{\frac{N-2}{2}}}{|y-\varepsilon a|^{N-4+\theta}} \le \frac{C'\varepsilon^{\frac{N-2}{2}}}{|y-\varepsilon a|^{N-4+\theta}+d^{N-4+\theta}}. \end{aligned}$$

On the other hand, if $|y-\varepsilon a|\le d$, then $|z-y|\le|z-\varepsilon a|+d\le 2|z-\varepsilon a|$. As a result,

$$\begin{aligned} |\tilde{\varphi}_{\varepsilon,\Lambda,a,2}(y)| \le & \frac{C\varepsilon^{\frac{N-2}{2}}}{d^{N-4+\theta}}\int_{\partial\Omega}\frac{1}{|z-y|^{N-1-\theta}}\,dz \le \frac{C\varepsilon^{\frac{N-2}{2}}}{d^{N-4+\theta}} \\ \le & \frac{C'\varepsilon^{\frac{N-2}{2}}}{|y-\varepsilon a|^{N-4+\theta}+d^{N-4+\theta}}. \end{aligned}$$

So, we have proved

$$|\tilde{\varphi}_{\varepsilon,\Lambda,a,2}(y)| \le \frac{C\varepsilon^{\frac{N-2}{2}}}{|y-\varepsilon a|^{N-4+\theta}+d^{N-4+\theta}}.$$

where $\theta>0$ is any fixed small constant.

On the other hand, for any $z\in\partial\Omega$

$$\begin{aligned} \frac{\partial U_{\frac{1}{\varepsilon\Lambda},\varepsilon a}}{\partial n} = & (\Lambda\varepsilon)^{\frac{N-2}{2}}\frac{\partial\frac{1}{|z-\varepsilon a|^{N-2}}}{\partial n}\Big(1+O\big(\frac{\varepsilon^2}{|z-\varepsilon a|^2}\big)\Big) \\ = & \frac{(\Lambda\varepsilon)^{\frac{N-2}{2}}}{c_\mu}\frac{\partial K_\mu(|z-\varepsilon a|)}{\partial n}(1+O(|z-\varepsilon a|))\big(1+O(\frac{\varepsilon^2}{|z-\varepsilon a|^2})\big). \end{aligned}$$

So, similar to the estimate of $\tilde{\varphi}_{\varepsilon,\Lambda,a,2}$, we can deduce

$$\begin{aligned}&\tilde{\varphi}_{\varepsilon,\Lambda,a,1}(y)\\=&\frac{(\Lambda\varepsilon)^{\frac{N-2}{2}}}{c_\mu}H(y,\varepsilon a)+O\Big(\frac{\varepsilon^{\frac{N-2}{2}}}{|y-\varepsilon a|^{N-3+\theta}+d^{N-3+\theta}}+\frac{\varepsilon^{\frac{N-2}{2}}}{|y-\varepsilon a|^{N-2+\theta}+d^{N-2+\theta}}\frac{\varepsilon^2}{d^2}\Big),\end{aligned}$$

which gives

$$\begin{aligned}&\tilde{\varphi}_{\varepsilon,\Lambda,a}(y)\\=&\frac{(\Lambda\varepsilon)^{\frac{N-2}{2}}}{c_\mu}H(y,\varepsilon a)+O\Big(\frac{\varepsilon^{\frac{N-2}{2}}}{|y-\varepsilon a|^{N-3+\theta}+d^{N-3+\theta}}+\frac{\varepsilon^{\frac{N-2}{2}}}{|y-\varepsilon a|^{N-2+\theta}+d^{N-2+\theta}}\frac{\varepsilon^2}{d^2}\Big).\end{aligned}$$

Therefore, we have the following expansion for the approximate solution $W_{\varepsilon,\Lambda,a}$

$$\begin{aligned}W_{\varepsilon,\Lambda,a}=&U_{\frac{1}{\Lambda},a}-\mu\varepsilon^2\Psi_{\Lambda,a}-\frac{\Lambda^{\frac{N-2}{2}}\varepsilon^{N-2}}{c_\mu}H(\varepsilon y,\varepsilon a)\\&+O\Big(\frac{\varepsilon^{N-2}}{|\varepsilon y-\varepsilon a|^{N-3+\theta}+d^{N-3+\theta}}+\frac{\varepsilon^{N-2}}{|\varepsilon y-\varepsilon a|^{N-2+\theta}+d^{N-2+\theta}}\frac{\varepsilon^2}{d^2}\Big).\end{aligned}$$

Finally, since $\partial\varphi_{\varepsilon,\Lambda,a}$ satisfies

$$\begin{cases}-\Delta(\partial\varphi_{\varepsilon,\Lambda,a})+\mu\varepsilon^2(\partial\varphi_{\varepsilon,\Lambda,a})=0, & \text{in } \Omega_\varepsilon;\\ \frac{\partial(\partial\varphi_{\varepsilon,\Lambda,a})}{\partial n}=\frac{\partial}{\partial n}\big(\partial U_{\frac{1}{\Lambda},a}-\mu\varepsilon^2\partial\Psi_{\Lambda,a}\big), & \text{on } \partial\Omega_\varepsilon.\end{cases}$$

we can prove as above that

$$\begin{aligned}\partial\varphi_{\varepsilon,\Lambda,a}=&\partial\Big(\frac{(\Lambda\varepsilon)^{\frac{N-2}{2}}}{c_\mu}H(\varepsilon y,\varepsilon a)\Big)\\&+O\Big(\frac{\varepsilon^{\frac{N-2}{2}}}{|\varepsilon y-\varepsilon a|^{N-3+\theta}+d^{N-3+\theta}}+\frac{\varepsilon^{\frac{N-2}{2}}}{|\varepsilon y-\varepsilon a|^{N-2+\theta}+d^{N-2+\theta}}\frac{\varepsilon^2}{d^2}\Big).\end{aligned}$$

So, the result follows.

□

We also need the following result.

Lemma A.2. *There is a constant $C>0$, such that*

$$|W_{\varepsilon,\Lambda,a}|,\ |\partial W_{\varepsilon,\Lambda,a}|\le CU_{\frac{1}{\Lambda},a},$$

where ∂ denotes either ∂_d or ∂_Λ.

Proof. Note that $|y-a|\le \frac{C}{\varepsilon}$ for $y\in\Omega_\varepsilon$.

If $N\ge 5$, then from (2.7),

$$|\varepsilon^2\Psi_{\Lambda,a}(y)|\le \frac{C\varepsilon^2}{(1+|y-a|)^{N-4}}\le \frac{C}{(1+|y-a|)^{N-2}}\le CU_{\frac{1}{\Lambda},a}.$$

If $N=4$, then from (2.11) and $\ln t\le t$ for $t\ge 1$, we obtain

$$|\Psi_{\Lambda,a}(y)|\le C\Big|\ln\frac{1}{\varepsilon(1+|y-a|)}\Big|\le \frac{C}{\varepsilon(1+|y-a|)}.$$

As a result,

$$|\varepsilon^2\Psi_{\Lambda,a}(y)|\le \frac{C\varepsilon^2}{\varepsilon(1+|y-a|)}\le \frac{C}{(1+|y-a|)^{2}}\le CU_{\frac{1}{\Lambda},a}.$$

On the other hand, if $|\varepsilon y-\varepsilon a|\le d$, then $1+|y-a|\le 1+\frac{d}{\varepsilon}\le\frac{2d}{\varepsilon}$ (since $\frac{d}{\varepsilon}\to+\infty$). As a result

$$\varepsilon^{N-2}|H(\varepsilon y,\varepsilon a)|\le \frac{C\varepsilon^{N-2}}{d^{N-2}}\le \frac{C}{(1+|y-a|)^{N-2}}\le CU_{\frac{1}{\Lambda},a}.$$

Suppose that $|\varepsilon y-\varepsilon a|\ge d$. Then $|y-a|\ge\frac{d}{\varepsilon}\to+\infty$. So

$$\varepsilon^{N-2}|H(\varepsilon y,\varepsilon a)|\le \frac{C\varepsilon^{N-2}}{|\varepsilon y-\varepsilon a|^{N-2}}\le \frac{C}{|y-a|^{N-2}}\le \frac{C'}{(1+|y-a|)^{N-2}}\le CU_{\frac{1}{\Lambda},a},$$

since $\frac{1}{|y-a|}\le\frac{2}{1+|y-a|}$. Thus, we have proved

$$\varepsilon^{N-2}|H(\varepsilon y,\varepsilon a)|\le CU_{\frac{1}{\Lambda},a}. \tag{A.2}$$

Using (A.2), we can deduce

$$\begin{aligned}&\frac{\varepsilon^{N-2}}{|\varepsilon y-\varepsilon a|^{N-3+\theta}+d^{N-3+\theta}}+\frac{\varepsilon^{N-2}}{|\varepsilon y-\varepsilon a|^{N-2+\theta}+d^{N-2+\theta}}\frac{\varepsilon^2}{d^2}\\ =&O\big(\varepsilon^{N-2}H(\varepsilon y,\varepsilon a)|\big)=O\big(U_{\frac{1}{\Lambda},a}\big).\end{aligned} \tag{A.3}$$

So, we have proved $|W_{\varepsilon,\Lambda,a}|\le CU_{\frac{1}{\Lambda},a}$.

Finally, by (A.1), we find

$$-\Delta\partial W_{\varepsilon,\Lambda,a}+\mu\varepsilon^2\partial W_{\varepsilon,\Lambda,a}=\alpha_N(2^*-1)U^{2^*-2}_{\frac{1}{\Lambda},a}\partial U_{\frac{1}{\Lambda},a}-\mu^2\varepsilon^4\partial\Psi_{\Lambda,a},$$

and $\frac{\partial(\partial W_{\varepsilon,\Lambda,a})}{\partial n}=0$, on $\partial\Omega$. But from $\varepsilon\le\frac{C}{1+|y-a|}$ for any $y\in\Omega_\varepsilon$, (2.7) and (2.11), we obtain

$$|\alpha_N(2^*-1)U_{\frac{1}{\Lambda},a}^{2^*-2}\partial U_{\frac{1}{\Lambda},a}-\mu^2\varepsilon^4\partial\Psi_{\Lambda,a}|\le\frac{C}{(1+|y-a|)^N}$$

So, we can prove

$$|\partial W_{\varepsilon,\Lambda,a}|\le\frac{C|\ln\varepsilon|^m}{(1+|y-a|)^{N-2}}\le CU_{\frac{1}{\Lambda},a}.$$

□

Define

$$\bar\varphi_{\Lambda,a}=U_{\frac{1}{\Lambda},a}-W_{\varepsilon,\Lambda,a}. \tag{A.4}$$

Lemma A.3. *There is a constant $C>0$, such that*

$$|\bar\varphi_{\Lambda,a}|,\ |\partial\bar\varphi_{\Lambda,a}|\le C\Big(\frac{\varepsilon^2|\ln\frac{1}{\varepsilon(1+|y-a|)}|^m}{(1+|y-a|)^{N-4}}+\frac{\varepsilon^{(N-2)\sigma}}{d^{(N-2)\sigma}}U_{\frac{1}{\Lambda},a}^{1-\sigma}\Big),$$

where $\sigma>0$ is a fixed small constant, $m=1$ if $N=4$, and $m=0$ if $N\ge5$.

Proof. This lemma follows from Lemma A.1, (A.2), (A.3), (2.7),(2.11) and

$$\varepsilon^{N-2}|H(\varepsilon y,\varepsilon a)|\le C\frac{\varepsilon^{N-2}}{d^{N-2}}.$$

□

To end this section, we prove

Lemma A.4. *Suppose that $N\ge4$. Then*

$$\sum_{j=2}^k H(\varepsilon x_i,\varepsilon x_1)=o(k^{N-2}).$$

Proof. Let $a_j=\varepsilon x_j$. Then

$$H(a_j,a_1)=-\frac{c_\mu}{|a_j-\bar a_1|^{N-2}}\big(1+O(|a_j-\bar a_1|)\big),$$

where $\bar a_1=(1+d,0,\cdots,0)$ is the reflection point of a_1 with respect to $\partial\Omega$. But

$$|a_j-\bar a_1|=\sqrt{4(1-d)^2\sin^2\frac{(j-1)\pi}{k}+4d^2}.$$

So

$$\sum_{j=2}^{k} H(\varepsilon x_i,\varepsilon x_1) \le Ck^{N-2}\int_1^{+\infty}\frac{1}{(k^2d^2+t^2)^{\frac{N-2}{2}}}\,dt$$
$$\le Ck^{N-2}\int_1^{+\infty}\frac{1}{(k^{\frac{2}{N-1}}+t^2)^{\frac{N-2}{2}}}\,dt = o(k^{N-2}),$$

since $d\in\left[\delta k^{-\frac{N-2}{N-1}},\frac{1}{\delta}k^{-\frac{N-2}{N-1}}\right]$.

□

Appendix B. Energy Expansion

Recall that

$$I(u)=\frac{1}{2}\int_{\Omega_\varepsilon}(|Du|^2+\mu\varepsilon^2|u|^2)-\frac{\alpha_N}{2^*}\int_{\Omega_\varepsilon}|u|^{2^*},$$

$$U_{\frac{1}{\Lambda},x_j}(y)=\frac{(\frac{1}{\Lambda})^{\frac{N-2}{2}}}{(1+\frac{1}{\Lambda^2}|y-x_j|^2)^{\frac{N-2}{2}}},$$

and

$$W_{\varepsilon,\Lambda,\mathbf{x}}(y)=\sum_{j=1}^{k}W_{\varepsilon,\Lambda,x_j}(y),$$

where $W_{\varepsilon,\Lambda,a}$ is defined in (2.12),

$$a_j=\varepsilon x_j,\ x_j=\Big(\frac{1-d}{\varepsilon}\cos\frac{2(j-1)\pi}{k},\frac{1-d}{\varepsilon}\sin\frac{2(j-1)\pi}{k},0\Big),\quad j=1,\cdots,k,$$

0 is the zero vector in $\mathbb{R}^{N-2}$, $d\in\left[\delta k^{-\frac{N-2}{N-1}},\frac{1}{\delta}k^{-\frac{N-2}{N-1}}\right]$ and

$$\varepsilon=k^{-\frac{N-2}{N-4}},\ \text{ if } N\ge 5,\quad \varepsilon=e^{-D_k k^2},\ \text{ if } N=4.$$

In this section, we will estimate the energy of $W_{\varepsilon,\Lambda,\mathbf{x}}$.

Proposition B.1. *For $N\ge 5$, we have*

$$I(W_{\varepsilon,\Lambda,\mathbf{x}})=k\left[\frac{1}{N}A_N+B_1(\Lambda\varepsilon)^2+B_2(\Lambda\varepsilon)^{N-2}H(a_1,a_1)-\frac{B(\Lambda k\varepsilon)^{N-2}}{(1-d)^{N-2}}+o\big(\varepsilon^2\big)\right],$$

where A_N, B_1, B_2 and B are some positive constants.

Proof. Using the symmetry, we have

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\big(|\nabla W_{\varepsilon,\Lambda,\mathbf{x}}|^2+\mu\varepsilon^2 W_{\varepsilon,\Lambda,\mathbf{x}}^2\big)\\
=&\sum_{j=1}^k\int_{\Omega_\varepsilon}\big(\alpha_N U_{\frac{1}{\Lambda},x_j}^{2^*-1}+\varepsilon^4\mu^2\Psi_{\Lambda,x_j}\big)W_{\varepsilon,\Lambda,x_j}+2\sum_{j\neq i}^k\int_{\Omega_\varepsilon}\big(\alpha_N U_{\frac{1}{\Lambda},x_i}^{2^*-1}+\varepsilon^4\mu^2\Psi_{\Lambda,x_i}\big)W_{\varepsilon,\Lambda,x_j} \qquad \text{(B.5)}\\
=&k\int_{\Omega_\varepsilon}\big(\alpha_N U_{\frac{1}{\Lambda},x_1}^{2^*-1}+\varepsilon^4\mu^2\Psi_{\Lambda,x_1}\big)W_{\varepsilon,\Lambda,x_1}+2k\sum_{j=2}^k\int_{\Omega_\varepsilon}\big(\alpha_N U_{\frac{1}{\Lambda},x_1}^{2^*-1}+\varepsilon^4\mu^2\Psi_{\Lambda,x_1}\big)W_{\varepsilon,\Lambda,x_j}
\end{aligned}
$$

On the other hand, it follows from Lemma A.2 that

$$
\Big|\int_{\Omega_\varepsilon}\varepsilon^4\mu^2\Psi_{\Lambda,x_1}W_{\varepsilon,\Lambda,x_1}\Big|\le C\int_{\Omega_\varepsilon}\varepsilon^4|\Psi_{\Lambda,x_1}|U_{\frac{1}{\Lambda},x_1} \qquad \text{(B.6)}
$$

But

$$
\int_{\Omega_\varepsilon}\varepsilon^4\mu^2|\Psi_{\Lambda,x_1}|U_{\frac{1}{\Lambda},x_1}=O\Big(\int_{\Omega_\varepsilon}\frac{\varepsilon^4}{(1+|y-x_1|)^{2N-6}}\Big)=\begin{cases}O(\varepsilon^4), & N\ge 7;\\ O(\varepsilon^4|\ln\varepsilon|), & N=6;\\ O(\varepsilon^3), & N=5.\end{cases} \qquad \text{(B.7)}
$$

So, we obtain

$$
\int_{\Omega_\varepsilon}\varepsilon^4\mu^2\Psi_{\Lambda,x_1}W_{\varepsilon,\Lambda,x_1}=o(\varepsilon^2).
$$

We estimate

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\alpha_N U_{\frac{1}{\Lambda},x_1}^{2^*-1}W_{\varepsilon,\Lambda,x_1}\\
=&\int_{\Omega_\varepsilon}\alpha_N U_{\frac{1}{\Lambda},x_1}^{2^*-1}\Big(U_{\frac{1}{\Lambda},x_1}-\mu\varepsilon^2\Psi_{\Lambda,x_1}-\frac{\Lambda^{\frac{N-2}{2}}\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_1)}{c_\mu}+O\big(\frac{\varepsilon^{N-2}}{d^{N-3+\theta}}+\frac{\varepsilon^N}{d^{N+\theta}}\big)\Big)\\
=&A_N-B_1(\varepsilon\Lambda)^2-B_2(\varepsilon\Lambda)^{N-2}H(\varepsilon x_1,\varepsilon x_1)+O\big(\frac{\varepsilon^{N-2}}{d^{N-3+\theta}}+\frac{\varepsilon^{N-1}}{d^{N-1}}\big)\\
=&A_N-B_1(\varepsilon\Lambda)^2-B_2(\varepsilon\Lambda)^{N-2}H(\varepsilon x_1,\varepsilon x_1)+o(\varepsilon^2),
\end{aligned}
$$

where

$$
A_N=\alpha_N\int_{\mathbb{R}^N}U_{1,0}^{2^*},
$$

and

$$
B_1=\alpha_N\mu\int_{\mathbb{R}^N}U_{1,0}^{2^*-1}\Psi,\quad B_2=\frac{1}{c_\mu}\alpha_N\int_{\mathbb{R}^N}U_{1,0}^{2^*-1}.
$$

Now we estimate the interaction terms. Suppose that $i \geq 2$. From Lemma C.1, we find

$$
\begin{aligned}
|\int_{\Omega_\varepsilon} U^{2^*-1}_{\frac{1}{\Lambda},x_1}\varepsilon^2\Psi_{\Lambda,x_i}| \leq & C\varepsilon^2\int_{\Omega_\varepsilon}\frac{1}{(1+|y-x_1|)^{N+2}}\frac{1}{(1+|y-x_i|)^{N-4}} \\
\leq & \frac{C\varepsilon^2}{|x_i-x_1|^{N-4}}.
\end{aligned}
\tag{B.8}
$$

Thus

$$
\begin{aligned}
&\alpha_N\int_{\Omega_\varepsilon} U^{2^*-1}_{\frac{1}{\Lambda},x_1}W_{\varepsilon,\Lambda,x_i} \\
=&\alpha_N\int_{\Omega_\varepsilon} U^{2^*-1}_{\frac{1}{\Lambda},x_1}\Big[U_{\frac{1}{\Lambda},x_i}-\mu\varepsilon^2\Psi_{\Lambda,x_i}-\frac{\Lambda^{\frac{N-2}{2}}\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_i)}{c_\mu} \\
&\qquad +O\big(\frac{\varepsilon^{N-2}}{|\varepsilon y-\varepsilon x_i|^{N-3+\theta}}+\frac{\varepsilon^{N-2}}{|\varepsilon y-\varepsilon x_i|^{N-2+\theta}}\frac{\varepsilon^2}{d^2}\big)\Big] \\
=&B_2(\Lambda\varepsilon)^{N-2}\bar{G}(\varepsilon x_1,\varepsilon x_i) \\
&+O\Big(\frac{\varepsilon^2}{|x_i-x_1|^{N-4}}+\frac{\varepsilon^{N-2}}{|\varepsilon x_1-\varepsilon x_i|^{N-3+\theta}}+\frac{\varepsilon^{N-2}}{|\varepsilon x_1-\varepsilon x_i|^{N-2+\theta}}\frac{\varepsilon^2}{d^2}\Big),
\end{aligned}
\tag{B.9}
$$

where

$$
\bar{G}(y,z)=\frac{c_\mu}{|z-y|^{N-2}}-H(y,z).
$$

On the other hand, using $\varepsilon \leq \frac{1}{1+|y-x_i|}$ for any $y\in\Omega_\varepsilon$, and Lemma A.2, we can deduce

$$
\begin{aligned}
&|\int_{\Omega_\varepsilon}\varepsilon^4\Psi_{\varepsilon,\Lambda,x_1}W_{\varepsilon,\Lambda,x_i}| \\
=&C\int_{\Omega_\varepsilon}\varepsilon^4|\Psi_{\varepsilon,\Lambda,x_1}|U_{\frac{1}{\Lambda},x_i}\leq C\varepsilon^2\int_{\Omega_\varepsilon}\frac{1}{(1+|y-x_1|)^{N-2}}\frac{1}{(1+|y-x_i|)^{N-2}} \\
=&O\Big(\frac{\varepsilon^2|\ln\varepsilon|}{|x_i-x_1|^{N-4}}\Big).
\end{aligned}
\tag{B.10}
$$

So, we have proved

$$
\begin{aligned}
&\int_{\Omega_\varepsilon}\big(|\nabla W_{\varepsilon,\Lambda,\mathbf{x}}|^2+\mu\varepsilon^2W^2_{\varepsilon,\Lambda,\mathbf{x}}\big) \\
=&k\Big(A_N-B_1(\varepsilon\Lambda)^2-B_2(\varepsilon\Lambda)^{N-2}H(\varepsilon x_1,\varepsilon x_1)+B_2(\Lambda\varepsilon)^{N-2}\bar{G}(\varepsilon x_1,\varepsilon x_i)+o(\varepsilon^2)\Big).
\end{aligned}
$$

Let

$$
\Omega_j=\big\{y=(y',y'')\in\Omega_\varepsilon:\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\rangle\geq\cos\frac{\pi}{k}\big\}.
$$

Similarly, by symmetry, we have

$$
\begin{aligned}
&\frac{\alpha_N}{2^*}\int_{\Omega_\varepsilon} W^{2^*}_{\varepsilon,\Lambda,\mathbf{x}} = \frac{\alpha_N k}{2^*}\int_{\Omega_1} W^{2^*}_{\varepsilon,\Lambda,\mathbf{x}} \\
=&\frac{\alpha_N k}{2^*}\Big(\int_{\Omega_1} W^{2^*}_{\varepsilon,\Lambda,x_1} + 2^*\int_{\Omega_1}\sum_{i=2}^k W^{2^*-1}_{\varepsilon,\Lambda,x_1} W_{\varepsilon,\Lambda,x_i} + O\Big(\int_{\Omega_1} W^{2^*-2}_{\varepsilon,\Lambda,x_1}(\sum_{i=2}^k W_{\varepsilon,\Lambda,x_i})^2\Big)\Big).
\end{aligned}
\tag{B.11}
$$

It is easy to check

$$
\begin{aligned}
&\frac{1}{2^*}\alpha_N\int_{\Omega_1} W^{2^*}_{\varepsilon,\Lambda,x_1} = \frac{1}{2^*}\alpha_N\int_{\Omega_\varepsilon} W^{2^*}_{\varepsilon,\Lambda,x_1} + O\big(\varepsilon^N k^N \ln\frac{1}{\varepsilon}\big) \\
=&\frac{1}{2^*}\alpha_N\int_{\Omega_\varepsilon}\Big(U_{\frac{1}{\Lambda},x_1} - \mu\varepsilon^2\Psi_{\Lambda,x_1} - \frac{\Lambda^{\frac{N-2}{2}}\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_1)}{c_\mu} + O\big(\frac{\varepsilon^{N-2}}{d^{N-3+\theta}}\big)\Big)^{2^*} + O\big(\varepsilon^N k^N \ln\frac{1}{\varepsilon}\big) \\
=&\frac{1}{2^*}A_N - B_1(\varepsilon\Lambda)^2 - B_2(\varepsilon\Lambda)^{N-2}H(\varepsilon x_1,\varepsilon x_1) + O\big(\frac{\varepsilon^{N-2}}{d^{N-3+\theta}} + \frac{\varepsilon^{N-1}}{d^{N-1}} + \varepsilon^N k^N \ln\frac{1}{\varepsilon}\big).
\end{aligned}
\tag{B.12}
$$

On the other hand, by Lemma A.1,

$$
\begin{aligned}
&\alpha_N\int_{\Omega_1} W^{2^*-1}_{\varepsilon,\Lambda,x_1} W_{\varepsilon,\Lambda,x_i} \\
=&\alpha_N\int_{\Omega_1} W^{2^*-1}_{\varepsilon,\Lambda,x_1}\Big(U_{\frac{1}{\Lambda},x_i} - \mu\varepsilon^2\Psi_{\Lambda,x_i} - \frac{\Lambda^{\frac{N-2}{2}}\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_i)}{c_\mu} + o\big(\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_i)\big)\Big).
\end{aligned}
$$

From Lemma C.1, we obtain

$$
\begin{aligned}
&\mu\varepsilon^2\int_{\Omega_1} W^{2^*-1}_{\varepsilon,\Lambda,x_1}\Psi_{\Lambda,x_i} \le C\varepsilon^2\int_{\Omega_1} U^{2^*-1}_{\frac{1}{\Lambda},x_1}\Psi_{\Lambda,x_i} \\
\le& C\varepsilon^2\int_{\Omega_1}\frac{1}{(1+|y-x_1|)^{N+2}}\frac{1}{(1+|y-x_i|)^{N-4}} \le \frac{C\varepsilon^2}{|x_1-x_i|^{N-4}},
\end{aligned}
\tag{B.13}
$$

and, using (A.2)

$$
\begin{aligned}
&\int_{\Omega_1} W^{2^*-1}_{\varepsilon,\Lambda,x_1}\varepsilon^{N-2}H(\varepsilon y,\varepsilon x_i) \\
\le& C\varepsilon^{N-2}\int_{\Omega_1}\frac{1}{(1+|y-x_1|)^{N+2}}\frac{1}{(1+|y-x_i|)^{N-2}} \le \frac{C\varepsilon^{N-2}}{|x_1-x_i|^{N-2}}.
\end{aligned}
$$

So,

$$\begin{aligned}
&\alpha_N \sum_{i=2}^{k} \int_{\Omega_1} W_{\varepsilon,\Lambda,x_1}^{2^*-1} W_{\varepsilon,\Lambda,x_i} \\
=&\alpha_N \sum_{i=2}^{k} \int_{\Omega_1} W_{\varepsilon,\Lambda,x_1}^{2^*-1} \Big(U_{\frac{1}{\Lambda},x_i} - \frac{\Lambda^{\frac{N-2}{2}} \varepsilon^{N-2} H(\varepsilon y, \varepsilon x_i)}{c_\mu} \Big) + o(\varepsilon^2) \\
=&\alpha_N \sum_{i=2}^{k} \int_{\Omega_1} U_{\frac{1}{\Lambda},x_1}^{2^*-1} \Big(U_{\frac{1}{\Lambda},x_i} - \frac{\Lambda^{\frac{N-2}{2}} \varepsilon^{N-2} H(\varepsilon y, \varepsilon x_i)}{c_\mu} \Big) \\
&+ O\Big(\sum_{i=2}^{k} \int_{\Omega_1} U_{\frac{1}{\Lambda},x_1}^{2^*-2} \big(\varepsilon^2 |\Psi_{\Lambda,x_1}| + \varepsilon^{N-2} |H(\varepsilon y, \varepsilon x_1)| \big) U_{\frac{1}{\Lambda},x_i} \Big) + o(\varepsilon^2).
\end{aligned} \tag{B.14}$$

Moreover,

$$\begin{aligned}
\int_{\Omega_1} U_{\frac{1}{\Lambda},x_1}^{2^*-2} \varepsilon^2 |\Psi_{\Lambda,x_1}| U_{\frac{1}{\Lambda},x_i} \le& C\varepsilon^2 \int_{\Omega_1} \frac{1}{(1+|y-x_1|)^N} \frac{1}{(1+|y-x_i|)^{N-2}} \\
\le& \frac{C\varepsilon^2 |\ln \varepsilon|}{|x_1 - x_i|^{N-2}},
\end{aligned} \tag{B.15}$$

and

$$\begin{aligned}
\int_{\Omega_1} U_{\frac{1}{\Lambda},x_1}^{2^*-2} \varepsilon^{N-2} |H(\varepsilon y, \varepsilon x_1)| U_{\frac{1}{\Lambda},x_i} \le& \frac{C\varepsilon^{(N-2)\alpha}}{d^{(N-2)\alpha}} \int_{\Omega_1} U_{\frac{1}{\Lambda},x_1}^{2^*-2+(N-2)(1-\alpha)} U_{\frac{1}{\Lambda},x_i} \\
\le& \frac{C\varepsilon^{(N-2)\alpha}}{d^{(N-2)\alpha}} \frac{1}{|x_1 - x_i|^{N-2}},
\end{aligned} \tag{B.16}$$

where α is a constant with $\alpha \in (0, \frac{2}{N-2})$. Thus, we obtain from (B.11), (B.12), (B.14)–(B.16)

$$\begin{aligned}
\frac{\alpha_N}{2^*} \int_{\Omega_\varepsilon} W_{\varepsilon,\Lambda,\mathbf{x}}^{2^*} =& k \Big(A_N - B_1 (\varepsilon\Lambda)^2 - B_2 (\varepsilon\Lambda)^{N-2} H(\varepsilon x_1, \varepsilon x_1) \\
&+ B_2 \Lambda^{N-2} \varepsilon^{N-2} \sum_{i=2}^{k} \bar{G}(\varepsilon x_i, \varepsilon x_1) + o(\varepsilon^2) + O\Big(\int_{\Omega_1} U_{\frac{1}{\Lambda},x_1}^{2^*-2} (\sum_{i=2}^{k} U_{\frac{1}{\Lambda},x_i})^2 \Big) \Big).
\end{aligned} \tag{B.17}$$

Note that for $y \in \Omega_1$, $|y - x_i| \ge \frac{1}{2}|x_i - x_1|$. Thus

$$\begin{aligned}
\sum_{i=2}^{k} U_{\frac{1}{\Lambda},x_i} \quad &\le C \sum_{i=2}^{k} \frac{1}{(1+|y-x_1|)^{\frac{N-3}{2}}} \frac{1}{|x_1 - x_i|^{\frac{N-1}{2}}} \\
&\le \frac{1}{(1+|y-x_1|)^{\frac{N-3}{2}}} \sum_{i=2}^{k} \frac{1}{|x_1 - x_i|^{\frac{N-1}{2}}}.
\end{aligned}$$

As a result,

$$\int_{\Omega_\varepsilon} U^{2^*-2}_{\frac{1}{\Lambda},x_1}(\sum_{i=2}^{k} U_{\frac{1}{\Lambda}.x_i})^2 = O\big(\varepsilon^{N-1}k^{N-1}\big),$$

which, together with (B.17), gives

$$\begin{aligned}\frac{\alpha_N}{2^*}\int_{\Omega_\varepsilon} W^{2^*}_{\varepsilon,\Lambda,\mathbf{x}} =& k\Big(A_N - B_1(\varepsilon\Lambda)^2 - B_2(\varepsilon\Lambda)^{N-2}H(\varepsilon x_1,\varepsilon x_1) \\ &+ B_2\Lambda^{N-2}\varepsilon^{N-2}\sum_{i=2}^{k}\bar G(\varepsilon x_i,\varepsilon x_1) + o(\varepsilon^2)\Big).\end{aligned} \tag{B.18}$$

So, we have proved

$$\begin{aligned}I(W_{\varepsilon,\Lambda,\mathbf{x}}) = k\Big(&\frac{1}{N}A_N + B_1(\Lambda\varepsilon)^2 + B_2(\varepsilon\Lambda)^{N-2}H(\varepsilon x_1,\varepsilon x_1) \\ &- B_2(\Lambda\varepsilon)^{N-2}\sum_{i=2}^{k}\bar G(\varepsilon x_i,\varepsilon x_1) + o(\varepsilon^2)\Big).\end{aligned}$$

But from Lemma A.4,

$$\sum_{i=2}^{k}\bar G(\varepsilon x_i,\varepsilon x_1) = \sum_{i=2}^{k}\frac{c_\mu}{|\varepsilon x_i-\varepsilon x_1|^{N-2}} + o(k^{N-2}) = \frac{\bar B k^{N-2}}{(1-d)^{N-2}} + o(k^{N-2}).$$

Thus, the result follows.

□

Now, we consider the case $N = 4$. We have

Proposition B.2. *For $N = 4$, we have*

$$I(W_{\varepsilon,\Lambda,\mathbf{x}}) = k\left[\frac{1}{4}A_4 + B_1(\Lambda\varepsilon)^2\ln\frac{1}{\Lambda\varepsilon} + B_2(\Lambda\varepsilon)^2H(a_1,a_1) - \frac{B(\Lambda k\varepsilon)^2}{(1-d)^2} + \beta_k\Lambda^2\varepsilon^2 + O\big(\Lambda^2\varepsilon^2\big)\right],$$

where A_4, B_1, B_2 and B are some positive constants, and β_k is independent of Λ, satisfying $\beta_k = O(k\ln k)$.

Proof. The proof of the proposition is similar to that of Proposition B.1. So, we just point out the difference.

In the case $N = 4$, using Lemma A.2 and $\varepsilon \le \frac{1}{1+|y-x_1|}$ in Ω_ε, we obtain (compare with (B.6) and (B.7))

$$
\begin{aligned}
&|\varepsilon^4\int_{\Omega_\varepsilon}\Psi_{\Lambda,x_1}W_{\varepsilon,\Lambda,x_1}|\le|\varepsilon^4\int_{\Omega_\varepsilon\setminus B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}\Psi_{\Lambda,x_1}W_{\varepsilon,\Lambda,x_1}|+|\varepsilon^4\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}\Psi_{\Lambda,x_1}W_{\varepsilon,\Lambda,x_1}|\\
\le&C\varepsilon^4\frac{\Lambda}{2}\int_{\Omega_\varepsilon\setminus B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}|\ln\frac{1}{\varepsilon|y-x_1|}|U_{\frac{1}{\Lambda},x_1}+C\varepsilon^4|\ln\varepsilon|\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}U_{\frac{1}{\Lambda},x_1}\\
\le&C\Lambda^4\varepsilon^4\int_{\frac{1}{\Lambda\sqrt{\varepsilon}}}^{\frac{r_0}{\Lambda\varepsilon}}r|\ln\frac{1}{\Lambda\varepsilon r}|\,dr+O(\varepsilon^3|\ln\varepsilon|)\\
\le&C\Lambda^2\varepsilon^2\int_{\sqrt{\varepsilon}}^{r_0}r|\ln\frac{1}{r}|\,dr+O(\varepsilon^3|\ln\varepsilon|)=O(\Lambda^2\varepsilon^2+\varepsilon^3|\ln\varepsilon|),
\end{aligned}
\tag{B.19}
$$

where r_0 is a large fixed positive number such that $\Omega\subset B(\varepsilon x_1,r_0)$.

By (2.9) and (2.10), we can deduce

$$
\varepsilon^2\int_{\Omega_\varepsilon}U^3_{\frac{1}{\Lambda},x_1}\Psi_{\Lambda,x_1}=\frac{1}{2}(\Lambda\varepsilon)^2\ln\frac{1}{\Lambda\varepsilon}\int_{\mathbb{R}^4}U^3_{0,1}+O(\Lambda^2\varepsilon^2).
\tag{B.20}
$$

Moreover (compare with (B.8)),

$$
\begin{aligned}
&\varepsilon^2\int_{\Omega_\varepsilon}U^3_{\frac{1}{\Lambda},x_1}\Psi_{\Lambda,x_i}=\varepsilon^2\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}U^3_{\frac{1}{\Lambda},x_1}\Psi_{\Lambda,x_i}+O\big(\varepsilon^3|\ln\varepsilon|\big)\\
=&\frac{\Lambda\varepsilon^2}{2}\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}U^3_{\frac{1}{\Lambda},x_1}\ln\frac{1}{\varepsilon|y-x_i|}+\bar{B}\Lambda^2\varepsilon^2+O\big(\varepsilon^3|\ln\varepsilon|\big)\\
=&\Big(\frac{1}{2}\bar{B}_2\ln\frac{1}{|\varepsilon x_i-\varepsilon x_1|}+\bar{B}\Big)\Lambda^2\varepsilon^2+O\big(\varepsilon^3|\ln\varepsilon|\big),
\end{aligned}
\tag{B.21}
$$

where $\bar{B}_2$ and $\bar{B}$ are some constant.

We also need to estimate $\varepsilon^4\sum_{i=2}^k\int_{\Omega_\varepsilon}\Psi_{\Lambda,x_1}W_{\varepsilon,\Lambda,x_i}$.

Similar computation in (B.19) gives out

$$
\varepsilon^4|\int_{\Omega\setminus B_{\frac{1}{\sqrt{\varepsilon}}}(x_i)}\Psi_{\Lambda,x_1}W_{\varepsilon,\Lambda,x_i}|\le C\varepsilon^2|\ln\varepsilon|\int_{\Omega\setminus B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}U^3_{\frac{1}{\Lambda},x_i}=O(\varepsilon^3|\ln\varepsilon|)
$$

which, together with Lemma A.2, gives (compare with (B.10))

$$
\begin{aligned}
&|\varepsilon^4\int_{\Omega_\varepsilon}\Psi_{\Lambda,x_1}W_{\varepsilon,\Lambda,x_i}|\le\varepsilon^4|\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_i)}\Psi_{\Lambda,x_1}W_{\varepsilon,\Lambda,x_i}|+O\big(\varepsilon^3|\ln\varepsilon|\big)\\
\le&C\varepsilon^4|\ln\varepsilon|\ln\frac{1}{|\varepsilon x_1-\varepsilon x_i|}\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_i)}U_{\frac{1}{\Lambda},x_i}+O\big(\varepsilon^3|\ln\varepsilon|\big)\\
=&O\big(\varepsilon^3|\ln\varepsilon|^2\big).
\end{aligned}
$$

So, we obtain that for $N=4$,

$$\begin{aligned}&\int_{\Omega_\varepsilon}\left(|\nabla W_{\varepsilon,\Lambda,\mathbf{x}}|^2+\mu\varepsilon^2W_{\varepsilon,\Lambda,\mathbf{x}}^2\right)\\ =&k\Big(A_4-B_1(\varepsilon\Lambda)^2\ln\frac{1}{\Lambda\varepsilon}-B_2(\varepsilon\Lambda)^2H(\varepsilon x_1,\varepsilon x_1)+B_2(\Lambda\varepsilon)^2\sum_{i=2}^k\bar{G}(\varepsilon x_1,\varepsilon x_i)\\ &+\tilde{\beta}_k\Lambda^2\varepsilon^2+O\big(\Lambda^2\varepsilon^2\big)\Big),\end{aligned}$$

where

$$\tilde{\beta}_k=\sum_{i=2}^k\Big(\frac12\bar{B}_2\ln\frac{1}{|\varepsilon x_i-\varepsilon x_1|}+\bar{B}\Big)=O\Big(k+\sum_{i=2}^k\ln\frac{k}{i}\Big)=O(k\ln k)$$

To finish the proof, we need (compare with (B.13))

$$\begin{aligned}&\varepsilon^2\int_{\Omega_\varepsilon}W^3_{\varepsilon,\Lambda,x_1}\Psi_{\Lambda,x_i}=\varepsilon^2\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}W^3_{\varepsilon,\Lambda,x_1}\Psi_{\Lambda,x_i}+O\big(\varepsilon^3|ln\varepsilon|\big)\\ =&\varepsilon^2\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}U^3_{\frac{1}{\Lambda},x_1}\Psi_{\Lambda,x_i}+O\Big(\varepsilon^2|\ln\varepsilon|\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}U^2_{\frac{1}{\Lambda},x_1}\big(\varepsilon^2|\Psi_{\Lambda,x_1}|+\varepsilon^2H(\varepsilon y,\varepsilon x_1)\big)+\varepsilon^3|\ln\varepsilon|\Big)\\ =&\varepsilon^2\int_{B_{\frac{1}{\sqrt{\varepsilon}}}(x_1)}U^3_{\frac{1}{\Lambda},x_1}\Psi_{\Lambda,x_i}+O\big(\varepsilon^3|\ln\varepsilon|\big)\\ =&\Big(\frac12\bar{B}_2\ln\frac{1}{|\varepsilon x_i-\varepsilon x_1|}+\bar{B}\Big)\Lambda^2\varepsilon^2+O\big(\varepsilon^3|\ln\varepsilon|\big).\end{aligned}$$

In the last relation, we have used (B.21). The rest of the proof is similar to that in Proposition B.1.

□

Appendix C. Basic Estimates

In this section, we collect some lemmas, whose proof can be found in the appendices in [10] as well as [11].

Lemma C.1. *For any constant $0\le\sigma\le\min(\alpha,\beta)$, there is a constant $C>0$, such that*

$$\begin{aligned}&\frac{1}{(1+|y-x_j|)^\alpha}\frac{1}{(1+|y-x_i|)^\beta}\\ \le&\frac{C}{|x_i-x_j|^\sigma}\Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}}+\frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}}\Big),\end{aligned}$$

where $i\neq j$, $\alpha\ge1$ and $\beta\ge1$ are two constants.

Lemma C.2. *For any constant $0 < \sigma < N-2$, there is a constant $C > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}}\, dz \le \frac{C}{(1+|y|)^{\sigma}}.$$

Let us recall that

$$\varepsilon = k^{-\frac{N-2}{N-4}} \quad \text{if} \quad N \ge 5, \quad \varepsilon = e^{-D_k k^2}, \quad \text{if} \quad N = 4.$$

Lemma C.3. *Suppose that $\tau = \frac{N-4}{N-2}$. Then there is a small $\theta > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{\varepsilon,\Lambda,\mathbf{x}}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\, dz$$
$$\le C \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.$$

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