CLASSIFICATION OF SOLUTIONS OF HIGHER ORDER CONFORMALLY INVARIANT EQUATIONS

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ABSTRACT. We study higher order conformally invariant equations involving the operator $(-\Delta u)^p, p > 1$ which arises naturally from conformal geometry and Sobolev embedding. We classify all possible solutions without any condition on the lower order derivatives of u. Our main idea is that we first derive *a prior* estimates of $(-\Delta)^j u, 1 \leq j \leq p-1$ and then use Kelvin transform as well as moving plane method.

1. INTRODUCTION

Recently, there have been much analytic work on the conformally invariant operators as well as its associated differential equations. A well known second order conformally invariant operator comes from the Yamabe problem or, more generally, the problem of prescribed scalar curvature. Given a smooth positive function K defined on a compact Riemannian manifold (M, g_0) of dimension $n \geq 2$, we ask whether there exists a metric g conformal to g_0 such that K is the scalar curvature of the new metric g. Let $g = e^{2u} g_0$ for $n = 2$ or $g = u^{\frac{4}{n-2}} g_0$ for $n \geq 3$, then the problem is reduced to find solutions of the following nonlinear elliptic equations:

$$\Delta_{g_0} u + K e^{2u} = k_0 \tag{1.1}$$

for $n = 2$, or

$$\begin{cases} \frac{4(n-1)}{n-2} \Delta_{g_0} u + K u^{\frac{n+2}{n-2}} = k_0 u \\ u > 0 \text{ on } M \end{cases} \tag{1.2}$$

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for $n \geq 3$, where Δ_{g_0} denotes the Laplace-Beltrami operator of (M, g_0) and k_0 is the scalar curvature of g_0.

In studying equations (1.1) and (1.2) as well as other problems, it is very important to understand the solution set of the following equations

$$\begin{cases} \Delta u + e^{2u} = 0 \text{ in } R^2 \\ \int_{R^2} e^{2u} < \infty \text{ in } R^2, \end{cases} \tag{1.3}$$

for $n = 2$, or

$$\begin{cases} \Delta u + u^{\frac{n+2}{n-2}} = 0 \text{ in } R^n \\ u > 0 \text{ in } R^n, \end{cases} \tag{1.4}$$

for $n \geq 3$.

By employing the method of moving planes, Caffarelli-Gidas-Spruck [4] was able to classify all solutions of (1.4) for $n \geq 3$, and Chen-Li [3] did the same thing for both equations (1.3) and (1.4).

The natural generalizations of (1.3) and (1.4) are the following higher order conformally invariant equations

$$\begin{cases} (-\Delta)^p u = (n-1)! e^{nu} \text{ on } R^n, n = 2p, \\ \int_{R^n} e^{nu} < \infty, p > 0, p \in \mathbb{Z}, \end{cases} \tag{1.5}$$

and

$$\begin{cases} (-\Delta)^p u = u^{\frac{n+2p}{n-2p}} \text{ in } R^n, n > 2p, p > 0, p \in \mathbb{Z}, \\ u > 0 \text{ in } R^n. \end{cases} \tag{1.6}$$

Equation (1.5) with $p = 2$ arises from the difference of log-determinants of a conformally covariant operator with respect to two conformal metrics. For background material and other related problems, we refer [5] and the references therein. Equation (1.6) can be derived from the Sobolev's embedding of H^{2p} into $L^{\frac{2n}{n-2p}}$:

$$\sup_{u \in H^{2p}(R^n)} \frac{\int_{R^n} |(-\Delta)^{\frac{p}{2}} u|^2}{\left(\int_{R^n} u^{\frac{2n}{n-2p}}\right)^{\frac{n-2p}{n}}}. \tag{1.7}$$

As concerned with the classification of solutions of (1.5) and (1.6), the method of moving planes being used to study equations (1.5) and (1.6) naturally comes to one's mind. However, the central difficulty is that the Maximum Principle can not be directly appplied to u if one does not know enough information about $(-\Delta)^i u, i = 1, ..., p-1$. There exist several works on this classification problem but up to our limit knowege, it seems that the said information was taken as granted in most previous works as one will comment it later.

The main point of present paper is to show that the equations (1.5) and (1.6) provided enough information for applying the maximum principle. More precisely, one states it in the following theorem which is one of main results in this article.

Theorem 1.1. *Let u be a solution of (1.5) with*

$$u(x) = o(|x|^2) \text{ at } \infty \tag{1.8}$$

or a solution of (1.6). Then necessarily we have

$$(-\Delta)^i u > 0, i = 1, ..., p-1. \tag{1.9}$$

Remark: Condition (1.8) is needed in our proof and in the classification of solutions of (1.5) which has been pointed out in [10].

Once we have (1.9), we can use Kelvin transform and moving planes method to prove symmetric property of the solution. (Here we apply moving plane method to the function $(-\Delta)^{p-1}u$.)

We first have

Theorem 1.2. *Suppose u is a solution of (1.5) satisfying (1.8). Then $u(x)$ is symmetric w.r.t. some point $x_0 \in R^n$, and there exists some $\lambda > 0$ so that*

$$u(x) = \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2} \text{ for all } x \in R^n. \tag{1.10}$$

Next we consider equation (1.6). We prove that all solutions of (1.6) are radial:

Theorem 1.3. *Suppose u is a smooth positive solution of (1.6). Then u is radially symmetric about some point $x_0 \in R^n$ and u has the following form*

$$u(x) = (\frac{2\lambda}{1 + \lambda^2|x - x_0|^2})^{\frac{n-2p}{2}} \tag{1.11}$$

for some constant $\lambda > 0$.

We also prove a nonexistence result.

Theorem 1.4. *Suppose u is a nonnegative solution of*

$$(-\Delta)^p u = u^q \text{ in } R^n \tag{1.12}$$

for $1 < q < \frac{n+2p}{n-2p}$, then $u \equiv 0$ in R^n.

Finally we prove a converse of Theorems 1.3 and 1.4.

Theorem 1.5. *Let u be a solution of*

$$\begin{cases} (-\Delta)^p u = f(u) \text{ in } R^n, \\ u > 0 \text{ in } R^n \end{cases} \tag{1.13}$$

where $f(t)$ is locally Lipschitz satisfying

(f1) $f(t) \geq 0$ and is nondecreasing for $t > 0$,

(f2) $t^{-q} f(t)$ is nonincreasing for $t > 0$, and

(f3) $\lim_{t\to\infty} t^{-q} f(t) = 1$, where $1 < q \leq \frac{n+2p}{n-2p}$.

Then $n > 2p, f(t) = ct^{\frac{n+2p}{n-2p}}$ for $0 \leq t \leq \max_{x\in R^n} u(x)$.

We remark that the method of moving planes was first invented by A.D. Alexandrov [1], and was shown to be a powerful tool in studying equations (1.3) and (1.4) by Gidas-Ni-Nirenberg [8], Caffarelli-Gidas-Spruck [4] and Chen-Li [3].

We note that for $p = 2$, similar results are obtained independently by C. S. Lin [10] and the second author [21]. Lin uses the method of moving planes for both equations (1.5) and (1.6) while the second author uses the method of moving spheres, a variant of the method of moving planes for equation (1.5). We remark that this second method can also be applied to the equation(1.6) [22]. We also note that in [6], Chang and Yang used the method of moving planes to prove Theorem 1.1 under the condition that $u(x) = \log \frac{2}{1+|x|^2} + w(\xi(x))$ for some smooth function w defined on S^n (in [6], they also consider the case when $p = n/2$ for n odd). Here our condition is much weaker than theirs. We also remark that in [20], Troy studied symmetry problems for system in bounded domains. For equation (1.6), if u is the maximizer of the energy (1.7) which was ensured to exist by Lions' existence theorem [11], then also by Lions' Theorem [11], u is radially symmetric and has property (1.9). Finally, u was shown to be of the form (1.11) in this case by C. Swanson [19]. Our Theorem 1.4 has been existed for several years in different form but all with the assumption that u is radial and inequality (1.9) holds true for u, see [12] and [16].

The organization of the paper is the following: In Section 2, we first study the asymptotic behavior of solutions of (1.5) satisfying (1.8), especially we establish Theorem 1.1 for equation (1.5). Then we go on to prove Theorem 1.2 by applying the method of moving planes to reduce the solution to the radially symmetric case. And finally, we show that the radial solutions of (1.5) necessarily have form (1.11). We prove Theorem 1.1 for equation (1.6) in Section 3 and then we use moving planes method to prove Theorem 1.3 and Theorem 1.4 in Section 4. Section 5 is devoted to the proof of Theorem 1.5.

Throughout this paper, the constant C will denote various generic constants. $B = O(A)$ means $|B| \leq CA$.

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2. Asymptotic Behavior of Solutions of (1.5)

In this section, we study the asymptotic behavior for a solution u of (1.5). First, we note that the fundamental solution of the operator $(-\Delta)^p, n = 2p$, is

$$P(x,y) = \frac{1}{\beta_0(n)} \log \frac{1}{|x-y|}$$

where $\beta_0(n) = 2^{2p-1}p!(p-1)!\omega_n, \omega_n$ is the volume of the unit ball in R^n.

Let u be a solution of (1.5). Set

$$\alpha = \frac{(n-1)!}{\beta_0(n)} \int_{R^n} e^{nu(y)} dy \tag{2.1}$$

and

$$v(x) = \frac{(n-1)!}{\beta_0(n)} \int_{R^n} \log \frac{|x-y|}{|y|} e^{nu(y)} dy. \tag{2.2}$$

Obviously, $v(x)$ satisfies

$$(-\Delta)^p v(x) = -(n-1)! e^{nu(x)} \text{ in } R^n. \tag{2.3}$$

We first have some preliminary analysis of u.

Lemma 2.1. *Let u be a solution of (1.5) satisfying (1.8). Then we have*

$$(-\Delta)^i u \geq 0, i = 1, ..., p-1. \tag{2.4}$$

Proof: Let $v_i = (-\Delta)^i u, i = 1, ..., p-1$. We first prove that

$$v_{p-1} \geq 0. \tag{2.5}$$

Suppose not, there exists $x_0 \in R^n$ such that

$$v_{p-1}(x_0) < 0.$$

Without loss of generality, we assume that $x_0 = 0$. We introduce the average of a function

$$\bar{f}(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} f d\sigma.$$

Then we have by Jensen's inequality (see [14] in the case $p = 1$),

$$\Delta \bar{u} + \bar{v}_1 = 0,$$

$$\Delta \bar{v}_1 + \bar{v}_2 = 0,$$

$$...,$$

$$\Delta \bar{v}_{p-1} + e^{\bar{u}} \leq 0.$$

Since $\bar{v}_{p-1}(0) < 0$ and $\bar{v}_{p-1}^{'} < 0$, we have

$$\bar{v}_{p-1}(r) \leq \bar{v}_{p-1}(0) < 0 \text{ for all } r. \tag{2.6}$$

Then it is easy to see that

$$\bar{v}_{p-2}^{'} > \frac{(-\bar{v}_{p-1}(0))}{n} r.$$

Hence

$$\bar{v}_{p-2}(r) \geq c_2 r^2, \quad \text{for } r \geq r_1. \tag{2.7}$$

Same arguments shows that

$$\bar{v}_{p-3}(r) \leq -c_3 r^4, \quad \text{for } r \geq r_2 > r_1, \tag{2.8}$$

and

$$(-1)^i \bar{v}_{p-i}(r) \geq c_i r^{2(i-1)}, \quad \text{for } r \geq r_{i-1}, i = 1, ..., p. \tag{2.9}$$

Hence

$$(-1)^{p-1} \bar{u}(r) \geq c_0 r^{2(p-1)}, \text{ for } r \geq r_{p-1}. \tag{2.10}$$

Since $p \geq 2$, we have $|\bar{u}(r)| \geq Cr^2$ for $r \geq r_{p-1}$. This is in contradiction with assumption that $u(x) = o(|x|^2)$.

Hence $v_{p-1} \geq 0$.

Now we show that $v_{p-i} > 0$ for all $1 \leq i \leq p-1$ by mathematical induction. For $i = 1$, we have shown that $v_{p-1} > 0$ above. Assume for $k \leq i$, $v_{p-k} > 0$ and $v_{p-(i+1)} < 0$ at some point x_0. Without loss of generality, we can assume that $x_0 = 0$. Thus we can form the above system again. Now since by assumption, $v_{p-i} > 0$, we get $\Delta v_{p-(i+1)} < 0$. Integrate it to see that $\bar{v}_{p-(i+1)} \leq v_{p-(i+1)}(0) < 0$. Then it follows that $v_{p-(i+2)} \geq c_2 r^2$ for some constant c_2. Repeatedly integrate the system to conclude that

$$(-1)^j \bar{v}_{p-(i+j)} \geq c_j r^{2(j-1)}, i + j \leq p.$$

If $j \geq 2$, again we are done since then $|u| \geq cr^2$ at infinity and $c > 0$. If $j = 1$, then $i = p - 1$. Then $\bar{v}_1(r) \leq \bar{v}_1(0) < 0$. Clearly this implies that $\bar{u} \geq cr^2$ which contradicts to assumption again. This finishes the proof of Lemma 2.1.

□

We now study properties of v. We have

Lemma 2.2. *Suppose v is given by (2.2) and Let α be defined by (2.1). Then*

$$v(x) \leq \alpha \log |x| + C \tag{2.11}$$

and

$$\lim_{|x|\to\infty}(-\Delta)^i v(x) = 0, i = 1, ..., p-1. \tag{2.12}$$

Moreover,

$$\lim_{|x|\to\infty}(-\Delta)^k v(x)|x|^{2k} = a_k, k = 1, ..., p \tag{2.13}$$

where a_k is given by

$$a_1 = \alpha(2-n), a_{k+1} = (2k)(2k-n+2)a_k.$$

Proof:

The proof of (2.11) is elementary, see Lemma 2.1 in [10].

To prove (2.12) and (2.13), we first note that if u is a solution of (1.5) and satisfies (1.8), then $u \le C$ for some constant $C > 0$. In fact, by Lemma 2.1, we know that $-\Delta u \ge 0$. Assume that there is a constant $\epsilon_0 > 0$ such that $\Delta u \le -\epsilon_0$, then it is easy to show that $\bar{u} \le u(0) - \epsilon_0/2nr^2$ which contradicts to the assumption that $u = o(|x|^2)$. Thus $\lim_{|x|\to\infty}\Delta u = 0$. Hence, $\Delta u \ge -C$, i.e., Δu is bounded. Similar argument shows that $(\Delta)^i u$ are also bounded for $i = 2, 3, \cdots, p-1$. Therefore u has upper bound by the following the proof of Theorem 1 of [21]. Hence we can differentiate the integral in (2.2) to obtain

$$(-\Delta)^i v(x) = \frac{(n-1)!}{\beta_0(n)}\int_{R^n}(-\Delta_x)^i \log(|x-y|)e^{nu(y)}dy.$$

Thus

$$\int_{R^n}|(-\Delta_x)^i \log(|x-y|)|e^{nu(y)}dy \le C\int_{R^n}\frac{e^{nu(y)}}{|x-y|^{2i}}dy.$$

Hence

$$(-\Delta)^i v(x) \to 0 \text{ as } |x| \to \infty.$$

Observe that

$$(-\Delta)^k \log|x-y| = c_k|x-y|^{-2k}$$

where $c_1 = (2-n), c_{k+1} = (2k)(2k-n+2)c_k$. (2.13) follows by elementary calculations and an observation that $a_k = \alpha c_k$.

□

Lemma 2.3. *We have*

$$(-\Delta)^i(u+v)=0, i=1,...,p-1. \tag{2.14}$$

Proof: We first prove it for $i=p-1$.

Let $w(x)=(-\Delta)^{p-1}(u+v)(x)$. Then $\Delta w(x)=0$ and by Lemma 2.1 and Lemma 2.2, we have $\lim_{|x|\to\infty} w(x)\geq 0$ and by maximum principle $w(x)\geq 0$.

By Liouville's Theorem, we have $w(x)=C_0$ for some nonnegative constant C_0.

We claim that $C_0=0$. In fact, suppose not. Then $(-\Delta)^{p-1}w(x)\geq C_0$. Let $\bar{w}$ be the average of w as in Lemma 2.1. Let $w_i=(-\Delta)^i w, i=1,...,p-2$. Then as in Lemma 2.1, we will have

$$(-1)^{p-1}\bar{w}(x)\geq C_0 r^{2(p-1)}$$

for $r\geq \bar{r}$.

Clearly it contradicts to the fact that $w(x)=o(|x|^2)$.

Now that $(-\Delta)^{p-1}(w(x))=0$. Then we have $(-\Delta)w_{p-2}=0$. Similar arguments as before show that $w_{p-2}=0$. The rest of the proof follows then.

□

Lemma 2.4. *For any $\epsilon>0$, there exists a constant $C_\epsilon>0$ such that*

$$v(x)\geq(\alpha-\epsilon)\log|x|-C_\epsilon \tag{2.15}$$

for $|x|$ large.

Proof: This follows exactly from the arguments of [10] (note that now $(-\Delta)^i u\leq C, i=1,...,p-1$ by Lemma 2.3). We omit the details. □

Next lemma is the key result in this section.

Lemma 2.5. *Suppose $|u(x)|=o(|x|^2)$ at ∞. Then*

$$u(x)=\frac{(n-1)!}{\beta_0(n)}\int_{R^n}\log(\frac{|y|}{|x-y|})e^{nu(y)}dy+C_0 \tag{2.16}$$

where C_0 is a constant. Furthermore, we have

$$\lim_{|x|\to\infty}\frac{u(x)}{\log(|x|)} = -\alpha, \tag{2.17}$$

$$\lim_{|x|\to\infty}(-\Delta)^k u(x)|x|^{2k} = a_k, k = 1, ..., p, \tag{2.18}$$

$$|\nabla^k u(x)| = O(|x|^{-k}) \quad \text{for all } 1 \le k \le 2p, \tag{2.19}$$

where a_k is given by

$$a_1 = \alpha(2-n), a_{k+1} = a_k(2k)(2k-n+2).$$

Proof: By Lemma 2.3, we have $\Delta(u+v) = 0$ in R^n. By the assumption and Lemma 2.2 and Lemma 2.4, we have $|u+v|(x) = o(|x|^2)$. Since $u+v$ is a harmonic function, by the gradient estimates of harmonic functions, we have $u(x)+v(x) = \sum_{j=1}^n a_j x_j + a_0$ for some constants $a_j \in R, 0 \le j \le n$. Thus

$$e^{nu(x)} = e^{a_0}e^{-nv(x)}e^{\sum_{j=1}^n a_j x_j} \ge C|x|^{-n\alpha}e^{\sum_{j=1}^n a_j x_j}.$$

Since $e^{nu(x)} \in L^1(R^n)$, we have $a_j = 0$ for $1 \le j \le n$. Hence we have proved (2.16).

The other claims follows from Lemma 2.2 and Lemma 2.3 and integral representation (2.16) of u.

□

Next we shall prove that $\alpha = 2$. To this end, we need the following Pohozaev's identity.

Lemma 2.6. *For any function u so that $(-\Delta)^p u = f(u)$ and $F(u) = \int^u f(t)dt$, we have*

$$\int_\Omega [nF(u) - \frac{n-2p}{2}uf(u)]dx = -\int_{\partial\Omega} B_p(u)d\sigma \tag{2.20}$$

where when $p = 2m$,

$$B_p(u) = (2-\frac{n}{2})\sum_{k=1}^m(-\Delta)^{2m-k}u\frac{\partial(-\Delta)^{k-1}u}{\partial\nu} - F(u) < x,\nu >$$

$$-(2-\frac{n}{2})\sum_{k=1}^m\frac{\partial(-\Delta)^{p-k}u}{\partial\nu}(-\Delta)^{k-1}u + 1/2((-\Delta)^m)^2 < x,\nu >$$

$$+2\sum_{k=1}^{m-1}\sum_{j=1}^{k}[(-\Delta)^{p-j}u\frac{\partial(-\Delta)^{j-1}u}{\partial\nu}-\frac{\partial(-\Delta)^{p-j}u}{\partial\nu}(-\Delta)^{j-1}u]$$

$$+\sum_{k=1}^{m}< x,\nabla(-\Delta)^{k-1}u > \frac{\partial(-\Delta)^{p-k}u}{\partial\nu}$$

$$-\sum_{k=1}^{m}[(-\Delta)^{p-k}u]\frac{\partial < x,\nabla(-\Delta)^{k}u >}{\partial\nu}$$

when $p = 2m+1$,

$$-B_p(u) = F(u) < x,\nu > -2\sum_{k=1}^{m}\sum_{j=1}^{k}[(-\Delta)^{p-j}u\frac{\partial((-\Delta)^{j-1}u)}{\partial\nu}$$

$$-((-\Delta)^{j-1}u)\frac{\partial(-\Delta)^{p-j}u}{\partial\nu}] - 1/2|\nabla[(-\Delta)^m u]|^2 < x,\nu >$$

$$+(1-n/2)\sum_{k=1}^{m}((-\Delta)^{k-1}u)\frac{\partial(-\Delta)^{p-k}u}{\partial\nu}$$

$$-(1-n/2)\sum_{k=1}^{m}(-\Delta)^m u\frac{\partial(-\Delta)^m u}{\partial\nu}$$

$$+\sum_{k=1}^{m}< x,\nabla[(-\Delta)^{k-1}u > \frac{\partial(-\Delta)^{p-k}u}{\partial\nu}$$

$$-\sum_{k=1}^{m}[(-\Delta)^{p-k}u]\frac{\partial < x,\nabla(-\Delta)^{k-1}u >}{\partial\nu}$$

$$+ < x,\nabla(-\Delta)^m u > \frac{\partial(-\Delta)^m u}{\partial\nu},$$

where ν *is outward normal vector along the boundary* $\partial\Omega$*.*

Proof: Notice that

$$(-\Delta)[< x,\nabla(-\Delta)^i u >] = 2(-\Delta)u + < x,\nabla(-\Delta)^{i+1}u > .$$

By repeatly using this fact and the second Green's identity, we can get above formula easily. □

Lemma 2.7. *We have*

$$\alpha = 2. \tag{2.21}$$

Proof: Let u be a solution of (1.5) with $u(x) = o(|x|^2)$.

Let
$$u_0 = \frac{\alpha}{2}\psi_0$$
where
$$\psi_0(x) = \log\frac{2\lambda}{\lambda^2 + |x|^2}.$$

Note that by Lemma 2.4, choose λ such that $u(0) = \psi_0(0)$, then $u = u_0 + o(1), |x|^{2i}(\Delta)^i(u - u_0) = o(1)$. By pluging the asymptotic formula of both u and u_0, we have
$$B_p(u) = B_p(u_0) + o(R^{1-n}).$$

Hence
$$\begin{aligned}\int_{R^n}(-\Delta)^p u(x,\nabla u)dx &= \int_{R^n}(-\Delta)^p u_0(x,\nabla u_0)dx\\ &= (\frac{\alpha}{2})^2\int_{R^n}(-\Delta)^p\psi_0(x,\nabla\psi_0)dx\\ &= \frac{\alpha^2}{4}\int_{R^n}(n-1)!e^{n\psi_0}(x,\nabla\psi_0)\\ &= \frac{\alpha^2}{4}(n-1)!(-\int_{R^n}e^{n\psi_0}).\end{aligned}$$

Similarly we have
$$\int_{R^n}(-\Delta)^p u(x,\nabla u)dx = -(n-1)!\int_{R^n}e^{nu} = -\beta_0(n)\alpha.$$

Hence
$$\frac{\alpha^2}{4}(n-1)!\int_{R^n}e^{n\psi_0} = \beta_0(n)\alpha.$$

So
$$\alpha = 2.$$

□

Proof Of Theorem 1.2: By the proof of Theorem 1.2 in [6], since u satisfies Lemma 2.5 and Lemma 2.7, u is radially symmetric with respect to some point x_0. Without loss of generality, we can assume $x_0 = 0$. Thus $u = u(r)$ where $r = |x|$. Choose λ so that $u(0) = \log(2/\lambda)$. And set $u_\lambda(r) = \log\frac{2\lambda}{\lambda^2+r^2}$. It is well-known that u_λ is a solution of (1.5). Set $\phi(r) = u(r) - u_\lambda(r)$. Then by our choice of λ,

$\phi(0) = 0$. And since $\phi(r)$ is smooth at $r = 0$, we have $\phi^{(2k+1)}(0) = 0$ for $k = 0, 1, 2 \cdots, p-1$. Now since u and u_λ are both solution of (1.5), we get

$$(-\Delta)^p \phi = g(r)\phi$$

where $g(r) = n[(1-\theta)e^{nu} + \theta e^{nu_\lambda}](n-1)!$ for some $\theta = \theta(r)$ between 0 and 1. Lemma 2.5 and the implicit expression of u_λ imply that

$$\lim_{r\to\infty} \sup r^{2p} g(r) = 0.$$

This implies that ϕ cannot be oscillatory at ∞. Thus by the theory of ordinary differential equation, ϕ can only have at most finitely many zeros on $(0, \infty)$. If ϕ is not identically zero, then by simple counting the zeros of $(-\Delta)^k \phi$, we can see that the number of zeros of ϕ must be zero and similarly $\Delta\phi$ cannot have zero either on $(0, \infty)$. But the fact that $\phi(0) = \phi(\infty) = 0$ forces ϕ identically zero. The interesting reader can find the details of this argument in [19]. Therefore the proof of Theorem 1.2 is complete.

□

3. Proof of Theorem 1.1 For Equation (1.6)

In this section, we prove Theorem 1.1 for equation (1.6). In fact, we can prove more. Namely, we shall prove

Theorem 3.1. *Let u be a solution of*

$$\begin{cases} (-\Delta)^p u = u^q \text{ in } R^n, \\ u > 0 \text{ in } R^n, 1 < q. \end{cases} \tag{3.1}$$

Then we have

$$(-\Delta)^i u > 0, i = 1, 2, ..., p-1. \tag{3.2}$$

Proof: Let $v_i = (-\Delta)^i u, i = 0, 1, 2, ..., p-1$ with $v_0 = u$. We first prove the following

$$v_{p-1} > 0. \tag{3.3}$$

Suppose not, there exists $x_0 \in R^n$ such that

$$v_{p-1}(x_0) < 0.$$

Without loss of generality, we assume that $x_0 = 0$. As in the proof of Lemma 2.3, we introduce the average of a function

$$\bar{f}(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} f d\sigma.$$

Then we have by Jensen's inequality (see [14] in the case $p = 1$).

$$\Delta \bar{u} + \bar{v}_1 = 0,$$

$$\Delta \bar{v}_1 + \bar{v}_2 = 0,$$

$$\ldots,$$

$$\Delta \bar{v}_{p-1} + (\bar{u})^q \le 0.$$

Since $\bar{v}_{p-1}(0) < 0$ and $\bar{v}_{p-1}^{'} < 0$, we have

$$\bar{v}_{p-1}(r) \le \bar{v}_{p-1}(0) < 0, \text{ for all } r > \bar{r}_1 = 0. \tag{3.4}$$

Then it is easy to see that

$$\bar{v}_{p-2}^{'} > \frac{(-\bar{v}_{p-1})(0)}{n} r.$$

Hence

$$\bar{v}_{p-2}(r) \ge c_2 r^2, \quad \text{for } r \ge \bar{r}_2 > \bar{r}_1. \tag{3.5}$$

Same arguments shows that

$$\bar{v}_{p-3}(r) \le -c_3 r^4, \quad \text{for } r \ge \bar{r}_3 > \bar{r}_2 \tag{3.6}$$

and

$$(-1)^i \bar{v}_{p-i}(r) \ge c_i r^{2(i-1)}, \quad \text{for } r \ge \bar{r}_i, i = 1, ..., p. \tag{3.7}$$

Hence if p is odd, we have a contradiction with the fact that $u > 0$.

So p must be even and we have

$$\bar{u}(r) \ge c_0 r^{\sigma_0}, \sigma_0 = 2(p-1) \tag{3.8}$$

and

$$(-1)^i \bar{v}_{p-i} > 0$$

for $r > \bar{r}_0 > 0$.

Setting $A = (2q(p-1) + n + 2p)$ and suppose now that

$$\bar{u}(r) \ge \frac{c_0^{q^k} r^{\sigma_k}}{A^{b_k}}, \text{for } r \ge r_k. \tag{3.9}$$

Then we have

$$r^{n-1}(\bar{v}_{p-1})' \leq r_k^{n-1}(\bar{v}_{p-1})'(r_k) - \int_{r_k}^{r} s^{n-1}u^q(s)ds,$$

$$\bar{v}_{p-1}' \leq -\frac{r^{q\sigma_k+1} - r_k^{q\sigma_k+1}}{b_k^q(q\sigma_k+n)} c_0^{q^{k+1}}.$$

Hence

$$\bar{v}_{p-1}' \leq -\frac{c_0^{q^{k+1}} r^{q\sigma_k+1}}{2A^{qb_k}(q\sigma_k+n)}$$

for $r \geq 2^{\frac{1}{q\sigma_k+1}} r_k$.

Similarly

$$\bar{v}_{p-1} \leq -\frac{c_0^{q^{k+1}} r^{q\sigma_k+2}}{4A^{qb_k}(q\sigma_k+n)(q\sigma_k+2)}$$

for $r \geq 2^{\frac{1}{q\sigma_k+2}} 2^{\frac{1}{q\sigma_k+1}} r_k$.

Hence

$$\bar{v}_{p-1} \leq -\frac{c_0^{q^{k+1}} r^{q\sigma_k+2}}{A^{qb_k}4(q\sigma_k+n)^2}$$

for $r \geq 2^{\frac{2}{q\sigma_k+1}} r_k$.

By induction, we have

$$(-1)^i \bar{v}_{p-i}(r) \geq \frac{c_0^{q^{k+1}} r^{q\sigma_k+2i}}{(q\sigma_k+n+2p)^{2i} A^{qb_k} 4^i}, r \geq 2^{\frac{2i}{q\sigma_k+1}} r_k. \tag{3.10}$$

Hence

$$\bar{u}(r) \geq \frac{c_0^{q^{k+1}} r^{q\sigma_k+2p}}{2^{2p} A^{qb_k}(q\sigma_k+n+2p)^{2p}}, r \geq 2^{\frac{2p}{q\sigma_k+1}} r_k. \tag{3.11}$$

Set

$$\sigma_0 = 2(p-1), r_0 = \bar{r}_0,$$

$$\sigma_{k+1} = q\sigma_k + 2p,$$

$$r_{k+1} = 2^{\frac{2p}{q\sigma_k+1}} r_k.$$

First of all, by mathematical induction, it is easy to see that

$$2^{2p}(q\sigma_k+n+2p)^{2p} \leq A^{2p(k+1)}$$

by noticing that

$$q\sigma_k + n + 2p \leq A(q\sigma_{k-1} + n + 2p).$$

Hence we also can set

$$b_0 = 0, b_{k+1} = qb_k + 2p(k+1).$$

Then we have

$$\bar{u}(r) \geq \frac{c_0^{q^{k+1}} r^{\sigma_{k+1}}}{A^{b_{k+1}}}, r \geq r_{k+1}.$$

Notice that

$$r_{k+1} \leq cr_0$$

where c can be chosen to be $2^{\sum_{k=0}^{\infty} \frac{2p}{q\sigma_k + 1}}$.

Also notice that, by using the iteration formulas above, we have

$$\sigma_k = 2\frac{p(q^{k+1}-1) - q + 1}{q-1},$$

and

$$b_k = 2p\frac{q^{k+1} - (k+1)q^2 + k}{(q-1)^2}.$$

Hence, if we take $M > 1$ is large enough so that $MA^{2/(q-1)} \geq 2cr_0$ if $c_0 \geq 1$ and $MA^{2/(q-1)}c_0^{-1} \geq 2cr_0$ if $c_0 < 1$, and then take $r_1 = MA^{2/(q-1)}$ or $MA^{2/(q-1)}c_0^{-1}$ depending on whether c_0 is greater than or less than 1, then we have

$$\bar{u}(r_1) \geq [A^{1/(q-1)^2}]^{2pq^{k+1} - 4(p+q) + 4 + 2p(k+1)q^2 - 2pk} \to \infty \text{ as } k \to \infty.$$

Since r_1 is independent of k, a contradiction is reached.

Hence

$$v_{p-1} \geq 0.$$

Next we claim that

$$v_{p-i} \geq 0, i = 2, 3, \cdots, p-1.$$

The proof is exactly the same as before except now that we need take extra care about the case that p is odd if i is even. We omit the details.

$\square$

Next we recall the following lemma.

Lemma 3.2. (i) *Let $u \in C^{2p}(R^n), p \geq 1$ be a positive solution of (3.1). Then q is necessarily greater than or equal to $n/(n-2p)$;*

(ii) *Let* $u \in C^{2p}(R^n), p \geq 1$ *be radially symmetric satisfying the inequalities*

$$(-\Delta)^k u \geq 0 \text{ in } R^n \text{ for } 0 \leq k \leq p$$

where $2p < n$. *Then necessarilly we have*

$$(ru^{'}(r) + (n-2p)u(r))^{'} < 0. \tag{3.12}$$

Proof:

(i) The argument for this is standard. We refer interesting reader to [22] for the proof of case $p = 2$. With help of Theorem 3.1, it is not hard to generalize it to present case. We omit the detail here.

(ii) This is a well-known fact. There are several different forms for it. The details can be found, for example, in [2]. □

Let u be a smooth positive solution of

$$(-\Delta)^p u = u^q \text{ in } R^n$$

for $1 < q \leq \frac{n+2p}{n-2p}$. We define the Kelvin transform

$$u^*(x) = |x|^{2p-n} u(\frac{x}{|x|^2}). \tag{3.13}$$

By a direction computation, u^* satisfies

$$(-\Delta)^p u = |x|^{-\tau} u^q \text{ in } R^n \backslash \{0\} \tag{3.14}$$

where $\tau = n + 2p - q(n-2p) \geq 0$.

Let $v_{p-1} = (-\Delta)^{p-1} u$. Then v has the following asymptotic behavior at ∞.

$$\begin{cases} v_{p-1}(x) = c_0 |x|^{2-n} + \sum_{j=1}^n \frac{a_j x_j}{|x|^n} + O(\frac{1}{|x|^n}), \\ v_{p-1,x_i} = -(n-2)c_0 |x|^{-n} x_i + O(\frac{1}{|x|^n}), \\ v_{p-1,x_i x_j} = O(\frac{1}{|x|^n}) \end{cases} \tag{3.15}$$

where $c_0 > 0$ and $a_j \in R$. In particular, we have for large $|x|$,

$$v_{p-1}(x) > 0. \tag{3.16}$$

The key fact in using moving plane method is the following lemma.

Lemma 3.3.

$$v_{p-1}(x) > 0 \ in \ R^n \backslash \{0\}$$

Proof: We first prove that $|x|^{-\tau}(u^*)^q \in L^1(B_{\frac{1}{2}})$. In fact, suppose not, then we have

$$\int_{\partial B_r} \frac{\partial v_{p-1}}{\partial r} - \int_{\partial B_s} \frac{\partial v_{p-1}}{\partial s} + \int_{B_r \backslash B_s} |x|^{-\tau}(u^*)^q = 0.$$

Take $r = \frac{1}{2}$. Then we have

$$-\frac{1}{|\partial B_r|} \int_{\partial B_r} \frac{\partial v_{p-1}}{\partial r} \leq -c_1 r^{1-n} \int_{B_{\frac{1}{2}} \backslash B_r} |x|^{-\tau}(u^*)^q$$

which implies

$$\bar{v}_{p-1}(r) \leq -c_2 r^{2-n}$$

for $0 < r \leq r_1$.

Similar as in the proof of Theorem 3.1, we have by induction

$$(-1)^i \bar{v}_{p-i}(r) \geq c_i r^{2i-n}, \ 0 < r \leq r_i.$$

If p is odd, then $\bar{v}_0 = \bar{u}(r) < 0$ which is a contradiction. Hence p must be even. In this case, we have

$$\bar{v}_1(r) < 0 \text{ for } r < r_{p-1}.$$

Namely

$$-\Delta \bar{u}^*(r) < 0.$$

This implies that $(\bar{u}^*)'(r) < 0$ for r small otherwise $\bar{u}^*(r)$ is increasing for r small and hence $\bar{u}^*(r) \leq C$ which is impossible by noticing Lemma 3.2(i) and assumption that $|x|^{-\tau}(u^*)^q$ is not in $L^1(B_{1/2})$.

On the other hand, if we let $\tilde{u}(s) = s^{2p-n}\bar{u^*}(\frac{1}{s})$. Then

$$(-\Delta)^p \tilde{u}(s) \geq s^{\tau} \tilde{u}^q(s).$$

By the argument in the proof of Theorem 3.1, we have

$$(-\Delta)^i \tilde{u}(s) \geq 0, i = 1, ..., p.$$

By Lemma 3.2,

$$(s\tilde{u}'(s) + (n-2p)\tilde{u}(s))' < 0,$$

i.e.

$$\tilde{u}'' + \frac{n-2p+1}{s}\tilde{u}'(s) < 0.$$

Note that an easy computation shows that

$$\Delta\bar{u^*}(r) = r^{2p-n-4}[\tilde{u}'' + \frac{n-2p+1}{s}\tilde{u}'(s)] + \frac{2p-2}{r}(\bar{u}^*)'(r) < 0$$

(since $(\bar{u}^*)'(r) < 0$ and $p > 1$).

Thus we get the desired contradiction.

Hence $|x|^{-\tau}u^q \in L^1(B_1)$. Then we can prove that $v > 0$ in the distribution sense. The proof is standard, we include it here for the sake of completeness.

In fact, let $\varphi \in C_0^\infty(B_{\frac{1}{2}})$ be a nonnegative function. We want to prove that

$$\int_{R^n} \Delta\varphi v dx \leq 0. \tag{3.17}$$

Let $\eta_\epsilon \in C_0^\infty(B_{\frac{1}{2}})$ satisfy $\eta_\epsilon(x) \equiv 1$ for $|x| > 2\epsilon$, and $\eta_\epsilon(x) \equiv 0$ for $|x| \leq \epsilon$. We also assume that

$$|D^j\eta_\epsilon(x)| \leq \frac{C}{\epsilon^j}$$

for $1 \leq j \leq n$. Multiplying (3.14) by $\varphi(x)\eta_\epsilon$, we have

$$\begin{aligned} 0 &< \int \varphi\eta_\epsilon(x)|x|^{-\tau}u^q(x)dx && (3.18)\\ &= \int(-\Delta)(\varphi(x)\eta_\epsilon(x))(-\Delta)^{p-1}u(x)dx && (3.19)\\ &= \int v(x)\{(-\Delta)\varphi(x)\eta_\epsilon(x) - 2\nabla\varphi(x)\nabla\eta_\epsilon - \varphi(x)\Delta\eta_\epsilon\}dx && (3.20)\end{aligned}$$

Let $\psi(x) = 2\nabla\varphi(x)\nabla\eta_\epsilon + \varphi(x)\Delta\eta_\epsilon(x)$. We have $\psi(x) \equiv 0$ for $|x| \leq \epsilon$ and for $|x| \geq 2\epsilon$, and $|\Delta\psi^i(x)| \leq C\epsilon^{-2i}$.

Since $\frac{n}{s}+\frac{\tau}{p}=\frac{n}{q}+n>n$ where $\frac{1}{s}=1-\frac{1}{q}$, we have

$$\begin{aligned} & |\int(-\Delta)^{p-1}v(x)\psi(x)dx| \\ \leq & \int v(x)|(-\Delta)^{p-1}\psi(x)|dx \\ \leq & C\epsilon^{-n}(\int_{\epsilon\leq|x|\leq 2\epsilon}|x|^{-\tau}u^q(x)dx)^{\frac{1}{q}}\epsilon^{\frac{n}{s}+\frac{\tau}{q}} \\ \leq & C\epsilon^{\frac{n}{s}+\frac{\tau}{q}-n}\to 0 \end{aligned}$$

as $\epsilon\to 0$. Therefore, by (3.18), we have

$$\begin{aligned} \int(-\Delta)^{p-1}u(x)(-\Delta)\varphi(x)dx &= \lim_{\epsilon\to 0}\int\eta_\epsilon(x)(-\Delta)^{p-1}u(x)(-\Delta)\varphi(x)dx \\ &= \int\varphi(x)|x|^{-\tau}u^p(x)dx>0. \end{aligned}$$

Thus $v>0$ in $B_{\frac{1}{2}}(0)$.

$\square$

4. proof of Theorems 1.3 and 1.4: Moving Plane Method

In this section, we apply the well-known method of moving plane to prove Theorem 1.3 and Theorem 1.4. Since the method is standard, we shall only sketch the proof. For more details, please see [6] and [10]. The key point is that we apply moving plane method to the function $(-\Delta)^{p-1}u$.

Let u be a solution of (1.12). Let $u^*(x)=|x|^{2p-n}u(\frac{x}{|x|^2})$. Let $v^*_{p-1}(x)=(-\Delta)^{p-1}u^*(x)$. By Lemma 3.3, $v^*_{p-1}>0$ in $R^n\backslash\{0\}$ and $v^*_{p-1}(x)$ satisfies for any $r>0$,

$$v^*_{p-1}(x)\geq\inf_{\partial B_r(0)}v^*_{p-1}(x)>0,\ \text{for } x\in B_r(0). \tag{4.1}$$

Since $u^*(x)$ is a superharmonic function in $B_r(0)\backslash\{0\}$ (by the proof of Lemma 3.3) and $u^*>0$, then we have

$$u(x)\geq\inf_{\partial B_r(0)}u(x)>0,\ \text{for } x\in B_r(0). \tag{4.2}$$

Following conventional notations, for any λ, and $x=(x_1,x_2,\cdots,x_n)$, we let $T_\lambda=\{x\in R^n|x_1=\lambda\},\Sigma_\lambda=\{x|x_1<\lambda\}$ and $x^\lambda=(2\lambda-x_1,x_2,...,x_n)$ be the reflection point of x with respect to T_λ. To start the process of moving planes along the x_1-direction, we need two lemmas below.

Lemma 4.1. *Let v be a positive function defined in a neighborhood at infinity satisfying the asymptotic expansion (3.15). then there exists $\bar{\lambda}$ and $R>0$ such that the inequality*

$$v(x) > v(x^{\lambda})$$

holds true for $\lambda \geq \bar{\lambda}, |x| \geq R$ and $x \in \Sigma_{\lambda}$.

Lemma 4.2. *Suppose v satisfies the assumption of Lemma 4.1, and $v(x) > v(x_{\lambda_0})$ for $x \in \Sigma_{\lambda_0}$. Assume that $v(x) - v(x^{\lambda_0})$ is superharmonic in Σ_{λ_0}. Then there exists ϵ and $S > 0$ such that the following hold true.*
(i) $v_{x_1} < 0$ in $|x_1 - \lambda_0| < \epsilon$ and $|x| > S$.
(ii) $v(x) > v(x^{\lambda})$ in $x_1 \geq \lambda_0 + \epsilon/2 > \lambda$ and $|x| > S$
for all $x \in \Sigma_{\lambda}, \lambda_1 \leq \lambda$ with $|\lambda_1 - \lambda_0| < \epsilon$.

The proofs of both lemmas are contained in [4].

Now let $w_{\lambda}(x) = u^*(x) - u^*(x^{\lambda})$ in Σ_{λ}. Since $v^*_{p-1}(x) = (-\Delta)^{p-1}u^*$ has a harmonic expansion (3.15) at infinity, by Lemma 4.1 and (4.1), there exists a $\bar{\lambda}_0 > 0$ such that

$$(-\Delta)^{p-1} w_{\lambda} > 0 \text{ in } \Sigma_{\lambda}$$

for all $\lambda \geq \bar{\lambda}_0$. By the maximum principle, we have

$$w_{\lambda}(x) > 0 \text{ in } \Sigma_{\lambda}$$

for all $\lambda \geq \bar{\lambda}_0$. (There is a subtle estimate, please see [6] and [10] for similar arguments.)

We consider the case $q < \frac{n+2p}{n-2p}$ first. Let

$$\lambda_0 = inf\ \{\lambda > 0 | (-\Delta)^{p-1} w_{\mu}(x) < 0 \ in\ \Sigma_{\mu} \text{ for } \mu \geq \lambda\}.$$

Although u^* may be singular at 0, by (4.1) and (4.2), we still can apply the same arguments as in Theorem 1.2 to prove $w_{\lambda_0}(x) \equiv 0$ in Σ_{λ_0}. Since $\tau < 0$, we must have $\lambda_0 = 0$. Since we can move the hyperplane along any direction in $R^n, u^*(x)$ is radially symmetric w.r.t. 0. Since we can take any point in R^n as the origin, we conclude that if u is a positive smooth solution of R^n, then $u \equiv constant$ in R^n which implies $u \equiv 0$ in R^n. Thus, Theorem 1.4 is proved.

The proof of the case $q = \frac{n+2p}{n-2p}$ is similar. So the solutions to equation (1.12) with $q = \frac{n+2p}{n-2p}$ are radial. Theorem 1.3 is proved by the following lemma.

Lemma 4.3. *Let $u = u(r)$ be a radial solution of $(-\Delta)^p u = u^{\frac{n+2p}{n-2p}}$. Then*

$$u(r) = u_{\lambda_0,0}(r)$$

for some $\lambda_0 > 0$ where

$$u_{\lambda,0}(r) = (\frac{2\lambda}{1+\lambda^2 r^2})^{\frac{n-2p}{2}}.$$

Proof: Notice that $u_{\lambda,0}(r)$ is a solution of equation (1.6). It is not hard to see that if $y(r) > 0$ satisfies

$$y'' + \frac{n-1}{r} y' + \phi(r) \leq 0, \quad r > 0$$

with ϕ non-negative and non-increasing, and y' bounded for r near 0, then

$$y(r) \geq c r^2 \phi(r)$$

where c is a dimensional constant. The interesting reader can find the proof for it in [17].

From this, we can easily get that $u^q(r) \leq c r^{-(n+2p)/2}$. Since u is smooth on R^n, $u \in L^2(R^n)$. Hence the Liouville's theorem will imply that u has the integral representation

$$u(x) = c_{n,p} \int_{R^n} \frac{u^q}{|x-y|^{n-2p}} dy.$$

Therefore, $u(r) = O(r^{2p-n})$ at ∞.

Let $u(r)$ be a solution of equation (1.6). Let λ_0 be such that $u(0) = u_{\lambda_0,0}(0)$. We now claim that $u(r) = u_{\lambda_0,0}(r)$. In fact, let $\phi(r) = u(r) - u_{\lambda_0,0}(r)$. First we have the equation

$$(-\Delta)^p \phi = g(r)\phi$$

where $g(r) = q[(1-\theta)u + \theta u_{\lambda_0,0}]^{q-1}$. According to our estimates above and the implicit expression of $u_{\lambda_0,0}$, we clearly have $g(r) = o(r^{-2p})$.

Therefore we can argue exactly as in the proof of Theorem 1.2 to conclude that ϕ is identically zero. Thus we finish the proof of Lemma 4.3 and hence the proof of Theorem 1.3.

□

5. Proof of Theorem 1.5

In this section, we prove Theorem 1.5.

Let q be defined by (f2) and (f3). By condition (f2) and (f3), we have

$$c_1 t^q \leq f(t) \leq C t^q.$$

Then the argument in the proof of Theorem 3.1 can be adopted to show that u satisfies

$$(-\Delta)^p u \geq c_1 u^q, (-\Delta)^i u \geq 0, i = 1, ..., p-1.$$

From this, by taking the spherical average, it is standard that $n > 2p$ and $q \geq \frac{n}{n-2p}$.

If we set $v(x) = |x|^{2p-n} u(\frac{x}{|x|^2})$, then v satisfies

$$(-\Delta)^p v = G(x, v)$$

in $R^n \backslash \{0\}$ where $G(x, v) = \frac{1}{|x|^{n+2p}} f(|x|^{n-2p} v)$. And it is clear that

$$\lim_{x \to \infty} |x|^{n-2p} v(x) = u(0).$$

According to our assumption (f2) and (f3), $G(x, t_1) \leq G(y, t_2)$ holds when $t_1 \leq t_2$ and $|y| \leq |x|$, hence the moving planes method can apply to $(-\Delta)^{p-1}$ again. Therefore we conclude that u is radial symmetric about some point x_0. Again, we assume that $x_0 = 0$.

Now the same argument in Lemma 4.3 shows that $f(u) \in L^2(R^n)$, hence $u = O(r^{2p-n})$ at ∞.

By using the integral representation of u, we have

$$(-\Delta)^i u = v_i \leq C r^{2p-2i-n},$$

$$|r^{n-1}((-\Delta)^j u)'(-\Delta)^s u| \leq C r^{4p-2(j+s)-2-n}, 1 \leq j, s \leq 2m-1,$$

$$|r^{n-1}(\Delta^k u)'(\Delta^s u)'| \leq C r^{4p-2(k+s+1)-1-n}, 1 \leq k, s \leq 2m-1,$$

$$|r^n(\Delta^m u)^2| \leq C r^{4p-4m-n},$$

$$r^n|\Delta^k u \Delta^{p-k} u| \leq C r^{4p-2(p-k)-2k-n}, 1 \leq k \leq 2m-1.$$

This clearly implies that

$$\int_{\partial B_r} B_p d\sigma \to 0 \text{ as} r \to \infty.$$

Applying the above inequality to the Pahozaev identity (see Lemma 2.5), we have

$$\int_{R^n} (nF(u) - \frac{n-2p}{2} f(u)) = 0.$$

Since $nF(u) - \frac{n-2p}{2} u f(u)$ never changes sign by condition (f2), we have

$$[nF(u) - \frac{n-2p}{2} u f(u)](x) = 0 \text{ for all } x \in R^n.$$

Hence

$$f(t) = c t^{\frac{n+2p}{n-2p}}, \text{ for } 0 \leq t \leq \max_{x \in R^n} u(x).$$

References

[1] A.D. Alexandrov, Uniqueness theorems for surfaces in the large, Vestnik Leningrad, Uni. Mat. Mekh. Astronom. 13(1958), 5-8;Ameri. Math. Soc. Transl. Ser. 2 21(1962), 412-416.

[2] P. Clement, R. Manasevich, and E. Mitidieri, Positive solutions for a quasilinear system via blow up, *Comm. PDE* 18(12) 1993, 2071-2106.

[3] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.* 63(1991), 615-622.

[4] L. Caffarelli, B. Gidas, and J. Spruck, Asymptotic symmetry and local behavior of semilinear equations with critical Sobolev growth, *Comm. Pure Appl. Math.* 42(1989), 271-289.

[5] S-Y A. Chang and P.C. Yang, Extremal metrics of zeta function determinants on 4-manifolds, *Ann. Math.* 142 (1995), 171-212.

[6] S-Y A. Chang and P. C. Yang, On uniqueness of solutions of n-th order differential equations in conformal geometry, *Math. Research Letters* 4 (1997), 91-102.

[7] C.R. Graham, R. Jenne, L. Mason and G. Sparling, Conformally invariant powers of the Laplacian, I: Existence, *J. London Math. Soc.* (2) 46(1992), 557-565.

[8] Gidas, B., Ni, W.-M., and Nirenberg, L., Symmetry and related properties via the Maximum Principle, *Comm. Math. Phys.* 68 (3) 1979, 209-243.

[9] D. Gilbarg and N.S.Trudinger, *Elliptic partial differential equations of second order*, Second Edition, Springer-Verlag (1983).

[10] C.-S. Lin, A classification of solutions of a conformally invariant fourth order equation in R^n, *Comment. Math. Helv.* 73(1998), 206-231.

[11] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. Part I, *Rev. Mat. Iberoamericana*, 1(1985), 145-201; Part II, 1(1985), 45-121.

[12] E. Mitidieri, A Rellich type identity and applications, *Comm. PDE* 18(1993), 125-151.

[13] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in R^N, *Diff. Int. Eqns* 9(3) 1996,465-479.

[14] W.-M. Ni, On the elliptic equation $\Delta u + Ke^{\frac{n+2}{n-2}} = 0$, its generalizations and applications in geometry, *Indiana Univ. Math. J.* 31(1982), 493-529.

[15] S. Paneitz, A quadratic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds, preprint (1983).

[16] P. Pucci and J. Serrin, Critical exponents and critical dimensions for polyharmonic operators, *J. Math. Pures Appl.*, 69(1990), 55-83.

[17] J. Serrin and H. Zhou, Non-existence of positive solutions of Lane-Emden systems, *Advance in Differential equations*, 1(1996), 1-19.

[18] E. M. Stein, Singular integrals and differentiability properties of functions, *Princeton University Press,* Princeton, New Jersey, 1970.

[19] C.A. Swanson, The Best Sobolev Constant, *Applicable Analysis*, 47(1992), 227-239.

[20] W. Troy, Symmetry properties in systems of semilinear elliptic equations, *J. Diff. Eqns* 42(1981), 400-413.

[21] X. Xu, Classification of solutions of certain fourth order nonlinear elliptic equations in R^4, preprint (1996).

[22] X. Xu, Uniqueness Theorem for the Entire Positive solutions of biharmonic equations in R^n, preprint (1997).

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