ON AMBROSETTI-MALCHIODI-NI CONJECTURE FOR GENERAL HYPERSURFACES

LIPING WANG, JUNCHENG WEI, AND JUN YANG

ABSTRACT. We consider the nonlinear problem

$$\varepsilon^2\Delta\tilde{u} - V(\tilde{y})\tilde{u} + \tilde{u}^p = 0, \quad \tilde{u}>0, \quad \tilde{u}\in H^1(\mathbb{R}^n),$$

where $p>1$, ε is a small parameter and V is a uniformly positive, smooth potential. Assume that $\mathcal{K}\subset\mathbb{R}^n$ is a smooth closed, stationary and non-degenerate hypersurface relative to the functional $\int_{\mathcal{K}} V^\sigma$ with $\sigma=\frac{p+1}{p-1}-\frac{1}{2}$. We prove the existence of solutions $\tilde{u}_\varepsilon$, at least for some sequence $\{\varepsilon_\ell\}_\ell$, which concentrate along smooth surfaces Γ_ε close to $\mathcal{K}$. This result confirms the validity of the conjecture of Ambrosetti, Malchiodi and Ni in [2] for concentration of Schrödinger equation on general hypersurfaces.

1. Introduction and the main result

1.1. Ambrosetti-Malchiodi-Ni Conjecture. We consider the problem

$$\varepsilon^2\Delta\tilde{u} - V(\tilde{y})\tilde{u} + \tilde{u}^p = 0, \quad \tilde{u}>0, \quad \tilde{u}\in H^1(\mathbb{R}^n), \tag{1.1}$$

where $p>1, n\geq 2$, ε is a small parameter and V is a smooth potential with

$$\inf_{\tilde{y}\in\mathbb{R}^n} V(\tilde{y})>0. \tag{1.2}$$

The above nonlinear problem arises from standing waves for a nonlinear Schrödinger equation in $\mathbb{R}^n$. For more details we refer to [2] and [13]. Considerable attention has been paid in recent years to the problem of construction of standing waves in the so-called *semi-classical limit* of (1.1) as $\varepsilon\to 0$. In the pioneering work [17], Floer and Weinstein constructed positive solutions to this problem when $p=3$ and $n=1$ with concentration taking place near a given point $\tilde{y}_0$ with $V'(\tilde{y}_0)=0, V''(\tilde{y}_0)\neq 0$, being exponentially small in ε outside any neighborhood of $\tilde{y}_0$. This result has been subsequently extended to higher dimensions to the construction of solutions exhibiting high concentration around one or more points of space under various assumptions on the potential and the nonlinearity by many authors. We refer the reader for instance to [1, 3, 7, 9, 10, 11, 12, 16] and the reference therein. An important question is whether solutions exhibiting concentration on higher dimensional set exists. In [2], Ambrosetti, Malchiodi and Ni considered the case of $V=V(|\tilde{y}|)$ and constructed radial solutions $u_\varepsilon(|\tilde{y}|)$ exhibiting concentration on a sphere $|\tilde{y}|=r_0$ in the form

$$u_\varepsilon(r)\sim V^{\frac{1}{p-1}}(r_0)w\big(V^{\frac{1}{2}}(r_0)\varepsilon^{-1}(r-r_0)\big),$$

under the assumption that $r_0>0$ is a non-degenerate critical point of

$$M(r)=r^{n-1}V^\sigma(r), \tag{1.3}$$

1991 *Mathematics Subject Classification.* 35J25, 35J20, 35B33, 35B40.

Key words and phrases. Concentration; Nonlinear Schrödinger equation; Ambrosetti-Malchiodi-Ni Conjecture; Infinite-dimensional reduction.

where w is the unique (even) solution of

$$w'' - w + w^p = 0, \quad w > 0, \quad w'(0) = 0, \quad w(\pm\infty) = 0, \tag{1.4}$$

and

$$\sigma = \frac{p+1}{p-1} - \frac{1}{2}. \tag{1.5}$$

Based on heuristic arguments, Ambrosetti, Malchiodi and Ni raised the following conjecture ([p.465, [2]]): *Let $\mathcal{K}$ be a non-degenerate $k-$dimensional stationary manifold of the following functional*

$$\int_{\mathcal{K}} V^{\frac{p+1}{p-1} - \frac{1}{2}(n-k)}.$$

Then there exists a solution to (1.1) concentrating near $\mathcal{K}$, at least along a subsequence $\varepsilon_\ell \to 0$.

For $n = 2, k = 1$, del Pino, Kowalczyk and Wei [13] proved the validity of this conjecture under some gap condition. Namely, they proved that if $\mathcal{K}$ in $\mathbb{R}^2$ is a non-degenerate, stationary curve for the weighted length functional $\int_{\mathcal{K}} V^\sigma$, then given $c > 0$ there exists ε_0 such that for all $\varepsilon < \varepsilon_0$ satisfying the gap condition

$$|\varepsilon^2 \ell^2 - \lambda_0| \geq c\varepsilon, \quad \forall\ \ell \in \mathbb{N},$$

where λ_0 is a fixed positive constant in (2.3), problem (1.1) has a positive solution u_ε which will concentrate on $\mathcal{K}$. Moreover, for some positive number c_0 independent of ε, u_ε satisfies globally

$$u_\varepsilon(\tilde{y}) \leq \exp\big(- c_0 \varepsilon^{-1} \mathrm{dist}(\tilde{y}, \mathcal{K})\big).$$

Recently, Mahmoudi, Malchiodi and Montenegro [23] established the validity of the Ambrosetti-Malchiodi-Ni conjecture in the case of $n = 3, k = 1$. They also considered the complex solutions of (1.1) carrying momentums.

The main purpose of this paper is to prove the conjecture when $n \geq 3, k = n-1$, i.e., when $\mathcal{K}$ is a non-degenerate stationary hypersurface under the functional $\int_{\mathcal{K}} V^{\frac{p+1}{p-1} - \frac{1}{2}}$.

1.2. **Geometric Background.** To state our main result, we need to introduce the definition of a hypersurface being stationary and non-degenerate for the weighted area functional $\int_{\mathcal{K}} V^\sigma$. We will also introduce the so-called Fermi coordinates which play important role in the computations.

Notation 1: *We shall always use the convention that indices $i, j, k, l \in \{1, 2, \cdots, n-1\}$ and indices $a, b, c \in \{1, 2, \cdots, n\}$.*

Assume that $\mathcal{K}$ is a smooth closed hypersurface in $\mathbb{R}^n$. Also let $\bar{g}_{ij}$ be the coefficients of the metric, denoted by $\bar{g}$, on $\mathcal{K}$ induced from the standard metric of $\mathbb{R}^n$. Using some local coordinates $\theta = (\theta_1, \theta_2, \cdots, \theta_{n-1})$ and letting φ be the corresponding immersion of $\mathcal{K}$ into $\mathbb{R}^n$, then we obtain

$$\bar{g}_{ij} = \left(\frac{\partial \varphi}{\partial \theta_i}, \frac{\partial \varphi}{\partial \theta_j} \right).$$

Here and in the sequel, by $(\cdot, \cdot)$, we have denoted the standard inner product in $\mathbb{R}^n$. The *Laplace-Beltrami operator* and *gradient operator* on $\mathcal{K}$ are defined in local coordinates by

$$\triangle_{\mathcal{K}} \equiv \frac{1}{\sqrt{\det \bar{g}}}\, \partial_i \Big(\sqrt{\det \bar{g}}\, \bar{g}^{ij} \partial_j \Big), \quad \nabla_{\mathcal{K}} h \equiv \bar{g}^{ij} \partial_i h \partial_j, \tag{1.6}$$

where h is any smooth function and the coefficients $\bar{g}^{ij}$ are the entries of the inverse matrix of $\bar{g}$. The differential of the unit normal ν of $\mathcal{K}$ is given by

$$\mathrm{d}\nu_\theta[v] = \mathbf{H}(\theta)[v], \quad \theta \in \mathcal{K}, v \in T_\theta\mathcal{K}, \tag{1.7}$$

where $\mathbf{H}(\theta) : T_\theta\mathcal{K} \to T_\theta\mathcal{K}$, identified with the corresponding bilinear form with coefficients $\{H^i_j\}$, is a symmetric operator. The eigenvalues of the matrix (H^i_j)(with respect to the metric $\bar{g}$) are called the *principal curvatures* of $\mathcal{K}$ and will be denoted by $\kappa_i, i = 1, 2, \cdots, n-1$. In the following, we let

$$H(\theta) = \kappa_1(\theta) + \cdots + \kappa_{n-1}(\theta), \tag{1.8}$$

denote *the mean curvature* (scaled by a factor $n-1$) of $\mathcal{K}$. We also define

$$|A_\mathcal{K}|^2 = \big|\big|\mathbf{H}\big|\big|^2 \equiv \kappa_1^2 + \cdots + \kappa_{n-1}^2, \tag{1.9}$$

to denote the square of the norm of the shape operator $A_\mathcal{K}$ defined by (1.7).

In a small δ_0-neighborhood of $\mathcal{K}$ in $\mathbb{R}^n$, we choose *Fermi coordinates* (θ, t) defined by

$$\tilde{y} = \Phi^0(\theta, t) = \varphi(\theta) + t\nu(\theta) \quad \text{with } (\theta, t) \in \mathcal{K} \times \big(-\delta_0, \delta_0\big), \tag{1.10}$$

where $\varphi(\theta) + t\nu(\theta)$ is understood as the sum of vectors in $\mathbb{R}^n$. Then we have

$$\frac{\partial\Phi^0}{\partial\theta_i}(\theta, t) = \frac{\partial\varphi}{\partial\theta_i}(\theta) \,+\, t\frac{\partial\nu}{\partial\theta_i}(\theta) = \frac{\partial\varphi}{\partial\theta_i}(\theta) + tH^j_i(\tilde{y})\frac{\partial\varphi}{\partial\theta_j}(\theta), \quad \frac{\partial\Phi^0}{\partial t}(\theta, t) = \nu(\tilde{y}).$$

Then by $\tilde{g}$ denoting the metric on $\mathbb{R}^n$, we have in Fermi coordinates

$$\{\tilde{g}_{ab}\} = \begin{pmatrix} \{\tilde{g}_{ij}\} & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.11}$$

where

$$\begin{aligned} \tilde{g}_{ij} &= \Big(\frac{\partial\varphi}{\partial\theta_i}(\theta) + tH^k_i(\theta)\frac{\partial\varphi}{\partial\theta_k}(\theta),\ \frac{\partial\varphi}{\partial\theta_j}(\theta) + tH^l_j(\theta)\frac{\partial\varphi}{\partial\theta_l}(\theta)\Big) \\ &= \bar{g}_{ij} + t\left(H^k_i\bar{g}_{kj} + H^l_j\bar{g}_{il}\right) + t^2H^k_iH^l_j\bar{g}_{kl}. \end{aligned} \tag{1.12}$$

Note also that the inverse matrix $\{\tilde{g}^{ab}\}$ can be decomposed as

$$\{\tilde{g}^{ab}\} = \begin{pmatrix} \{\tilde{g}^{ij}\} & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, we get

$$\det(\tilde{g}) = \det(\bar{g})\Big(1 + 2Ht + Gt^2 + \sum_{i=3}^n G_it^i\Big), \tag{1.13}$$

where $G, G_3, \cdots, G_n$ are smooth functions on $\mathcal{K}$.

Notation 2: *In the sequel, by slight abuse of notation we denote $V(\theta, t)$ to actually mean $V(\varphi(\theta) + t\nu(\theta))$ in Fermi coordinate system. The same way is understood to its derivatives with respect to θ and t.*

Any surface sufficiently close to $\mathcal{K}$ can be parameterized by

$$\gamma_f(\theta) = \varphi(\theta) + f(\theta)\nu(\theta),$$

where f is a smooth function with small $L^\infty-$norm. Call $\mathcal{K}_f$ the surface defined this way. Then letting $\bar{g}^f$ denote the metric on $\mathcal{K}_f$ induced from $\mathbb{R}^n$, we have in Fermi coordinates

$$\bar{g}^f_{ij} = \left(\frac{\partial \gamma_f}{\partial \theta_i}, \frac{\partial \gamma_f}{\partial \theta_j}\right),$$

whence

$$\begin{aligned}\bar{g}^f_{ij} = &\Big(\frac{\partial \varphi}{\partial \theta_i}(\theta) + \frac{\partial f}{\partial \theta_i}(\theta)\nu(\theta) + f(\theta)H_i^k(\theta)\frac{\partial \varphi}{\partial \theta_k}(\theta), \\ &\quad \frac{\partial \varphi}{\partial \theta_j}(\theta) + \frac{\partial f}{\partial \theta_j}(\theta)\nu(\theta) + f(\theta)H_j^l(\theta)\frac{\partial \varphi}{\partial \theta_l}(\theta)\Big) \\ = &\ \bar{g}_{ij} + f\left(H_i^k \bar{g}_{kj} + H_j^l \bar{g}_{li}\right) + f^2 H_i^k H_j^l \bar{g}_{kl} + \frac{\partial f}{\partial \theta_i}\frac{\partial f}{\partial \theta_j}.\end{aligned}$$

Moreover, we have

$$\det\left(\bar{g}^f\right) = \det(\bar{g})\Big(1 + 2Hf + Gf^2 + \left|\nabla_{\mathcal{K}} f\right|^2 + O(f^3) + O(f|\nabla_{\mathcal{K}} f|^2)\Big).$$

Then the weighted area of the surface $\mathcal{K}_f$ is given by the functional of f

$$J(f) = \int_{\mathcal{K}_f} V^\sigma = \int_{\mathcal{K}} V^\sigma(\gamma_f(\theta))\sqrt{\det\left(\bar{g}^f\right)}, \tag{1.14}$$

where σ is defined by (1.5).

The surface $\mathcal{K}$ is said to be *stationary* for the weighted area if the first variation of the functional (1.14) at $f = 0$ is equal to zero. That is, for any smooth function h defined on $\mathcal{K}$

$$0 = J^{'}(0)[h] = \int_{\mathcal{K}} (V^\sigma)_t\, h\sqrt{\det \bar{g}} + \int_{\mathcal{K}} V^\sigma H\, h\sqrt{\det \bar{g}}\,,$$

which is equivalent to the relation

$$\sigma V_t(\theta, 0) = -V(\theta, 0)H(\theta), \quad \theta \in \mathcal{K}. \tag{1.15}$$

We assume the validity of this relation at $\mathcal{K}$. Let us consider the second variation quadratic form

$$\begin{aligned}J^{''}(0)[h,h] = &\int_{\mathcal{K}} \left[(V^\sigma)_{tt} - 3V^\sigma (H)^2\right] h^2\sqrt{\det \bar{g}} + \int_{\mathcal{K}} V^\sigma G\, h^2\sqrt{\det \bar{g}} \\ &+ \int_{\mathcal{K}} V^\sigma \left|\nabla_{\mathcal{K}} h\right|^2 \sqrt{\det \bar{g}}.\end{aligned}$$

We say that $\mathcal{K}$ is *non-degenerate* if so is this quadratic form in the space of all functions $h \in H^1(\mathcal{K})$. This is equivalent to the statement that the differential equation

$$\triangle_{\mathcal{K}} h + \frac{\sigma}{V}\nabla_{\mathcal{K}} V \cdot \nabla_{\mathcal{K}} h + \left[|A_{\mathcal{K}}|^2 + \sigma^{-1}H^2 - \frac{\sigma}{V}V_{tt}\right]h = 0 \quad \text{on } \mathcal{K}, \tag{1.16}$$

has only the trivial solution.

1.3. **Main Theorem.** Our main theorem can be stated as the following:

Theorem 1.1. *Assume that the smooth closed hypersurface $\mathcal{K}$ satisfies the stationary and non-degenerate condition relative to the weighted functional $\int_{\mathcal{K}} V^{\frac{p+1}{p-1}-\frac{1}{2}}$ and $n \geq 3$. There exists a sequence of small parameters $\{\varepsilon_\ell\}_\ell$ such that problem (1.1) has a positive solution u_ε, still denoting ε_ℓ by ε, concentrating along a hypersurface Γ_ε near $\mathcal{K}$. Near $\mathcal{K}$ for $\tilde{y}$ given by (1.10), u_ε takes the form*

$$u_\varepsilon(\tilde{y}) \,=\, V(\theta,0)^{1/(p-1)}\, w\left(\frac{t}{\varepsilon}\cdot\sqrt{V(\theta,0)}\right)\,\big(1\,+\,o(1)\big), \tag{1.17}$$

where w denotes the unique positive solution of problem (1.4). Moreover, there exists some number c_0 such that u_ε satisfies globally,

$$u_\varepsilon(\tilde{y}) \leq \exp\Big[\,\frac{-c_0}{\varepsilon}\, dist(\tilde{y},\Gamma_\varepsilon)\,\Big],$$

and the surfaces Γ_ε will collapse to $\mathcal{K}$ as $\varepsilon \to 0$.

Remark 1: *Combining the result in* [13] *and Theorem 1.1, the validity of the conjecture raised in* [2] *is confirmed for hypersurfaces, at least along a sequence $\{\varepsilon_\ell\}_\ell$.*

To prove Theorem 1.1, not only the same difficulties as that in [13] are encountered but also more obstruction appears. More precisely, by the rescaling of the form $(z,s)=\varepsilon^{-1}(\theta,t)$, the solution to the full problem we take is roughly decomposed in the form,

$$v(z,s) = w\big(s - f(\varepsilon z)\big) \,+\, \varepsilon e(\varepsilon z) Z\big(s - f(\varepsilon z)\big) \,+\, \tilde{\phi}(z,s),$$

where Z is defined in (2.1), f and e are left as parameters, while $\tilde{\phi}(z,s)$ is L^2-orthogonal for each z both to $w_s(s-f(\varepsilon z))$ and to $Z(s-f(\varepsilon z))$. Solving first in $\tilde{\phi}$ a natural projected problem, the resolution of the full problem becomes reduced to a nonlinear, nonlocal second order system of differential equations in (f,e). Although the linear operator is solvable, some norm of e becomes big in some sense at the right hand side of this linear operator when $n \geq 3$. This shows that the approximation we construct doesn't work well as n becomes large. Hence we must improve our approximation as in [23], [24]-[26]. The principle is: the better the approximation, higher the chances of a correct inversion of the full problem to obtain a contraction mapping formulation of the nonlinear, nonlocal second order differential equations. To do that, we try the following form as our new approximation, (see [26])

$$v(z,s) = w\big(s - f(\varepsilon z)\big) \,+\, \varepsilon e(\varepsilon z) Z\big(s - f(\varepsilon z)\big) \,+\, \sum_{l=1}^{k-1} \varepsilon^l \phi_l(z,s).$$

The aim of adding the term $\sum_{l=1}^{k-1}\varepsilon^l\phi_l(z,s)$ is to cancel the error term till order $O(\varepsilon^k)$ such that our approximation is good enough. After very tedious but necessary computations we find that such ϕ_l may not exist since we will get some nonhomogeneous differential equation of ϕ_l. So we need to improve our approximation further, namely we take the following form, (see [22])

$$\mathcal{V} = w(x) \,+\, \varepsilon e(\varepsilon z) Z(x) \,+\, \sum_{l=1}^{k-1} \varepsilon^l \phi_l(\varepsilon z, x),$$

where $x = \sqrt{V(\theta,0)}\Big(s - \sum_{l=0}^{k-2} \varepsilon^l f_l(\varepsilon z)\Big)$ for some positive k. Now we can choose proper $f_l, 0 \le l \le k-3$ such that the corresponding nonhomogeneous differential equation of ϕ_l is solvable. To do this we need to analyze the Laplace-Beltrami operator very carefully and conduct lots of computations. Finally the resolution of the full problem becomes solving a nonlinear, nonlocal second order differential equations of (f_{k-2}, e) which turns out to be directly handled due to the assumption to $\mathcal{K}$.

We believe that the procedure in this paper may be used to solve the full Ambrosetti-Malchiodi-Ni conjecture for all $1 \le k \le n-1$.

In the rest paper we carry out the program outlined above, which leads to the complete proof of Theorem 1.1.

2. Preliminaries and setting up of the problem

2.1. Asymptotic behavior of w and its linear problem. It is well known that the associated linearized eigenvalue problem of (1.4),

$$h'' - h + pw^{p-1}h = \lambda h \text{ in } \mathbb{R}, \quad h \in H^1(\mathbb{R}), \tag{2.1}$$

possesses a unique positive eigenvalue λ_0 with a unique even and positive eigenfunction Z which we normalize so that $\int_{\mathbb{R}} Z^2 = 1$ (this follows for instance from the analysis in [29]). In fact, we have

$$w(x) \;=\; C_p \Big\{ \exp\Big[\frac{(p-1)\,x}{2}\Big] + \exp\Big[\frac{-(p-1)\,x}{2}\Big] \Big\}^{\frac{-2}{p-1}}, \tag{2.2}$$

$$Z \;=\; \Big[\int_{\mathbb{R}} w^{p+1}\,\mathrm{d}x\Big]^{-\frac{1}{2}} w^{p+1}, \quad \lambda_0 = \frac{1}{4}(p-1)(p+3). \tag{2.3}$$

It is easy to see that for $|x| \gg 1$

$$w(x) = C_p\, e^{-|x|} - \frac{2C_p}{p-1}\, e^{-p|x|} + O(e^{-(2p-1)|x|}), \tag{2.4}$$

$$w'(x) = -C_p\, e^{-|x|} + \frac{2\,p\,C_p}{p-1}\, e^{-p|x|} + O(e^{-(2p-1)|x|}), \tag{2.5}$$

$$Z(x) = C_p'\, e^{-(p+1)|x|} - \frac{2(p+1)C_p'}{p-1}\, e^{-2p|x|} + O(e^{-(3p-1)|x|}), \tag{2.6}$$

where

$$C_p = \Big[\frac{(p+1)}{2}\Big]^{\frac{1}{p-1}}, \quad C_p' = \Big[\frac{(p+1)}{2}\Big]^{\frac{p+1}{p-1}} \Big[\int_{\mathbb{R}} w^{p+1}\,\mathrm{d}x\Big]^{-\frac{1}{2}}.$$

As a consequence of the above analysis, we have the following inequality: there exists a constant $\gamma > 0$ such that whenever $\int_{\mathbb{R}} \psi w_x = \int_{\mathbb{R}} \psi Z = 0$ with $\psi \in H^1(\mathbb{R})$ we have that

$$\int_{\mathbb{R}} (|\psi'|^2 + |\psi|^2 - pw^{p-1}\psi^2)dx \ge \gamma\Big(\int_{\mathbb{R}} (|\psi'|^2 + |\psi|^2 dx\Big). \tag{2.7}$$

We also recall the Weyl's asymptotic formula, referring for example to [6], or to [21] and [28] for further details. Let ρ_i, $\omega_i, i = 1, 2, \cdots$, denote the eigenvalues and eigenfunctions of $-\Delta_{\mathcal{K}}$(ordered to be non-decreasing in i and counted with the multiplicity), then we have that

$$\rho_i \to \frac{C\, i^{2/(n-1)}}{\mathrm{Vol}(\mathcal{K})} \quad \text{as } i \to \infty, \tag{2.8}$$

where $\mathrm{Vol}(\mathcal{K})$ is the volume of $(\mathcal{K}, \bar{g})$ and C is a constant depending only on the dimension $n-1$.

2.2. **Laplace-Beltrami Operator in stretched Fermi Coordinates.** Here and in the following we always apply the notations in the previous section.

To construct the approximation to a solution of (1.1), which concentrates near $\mathcal{K}$, after rescaling, in $\mathbb{R}^n$ we also introduce the stretched Fermi coordinates in the neighborhood of $\mathcal{K}_\varepsilon = \mathcal{K}/\varepsilon$ by

$$\Phi_\varepsilon(z,s) = \frac{1}{\varepsilon}\Phi^0(\varepsilon z, \varepsilon s), \quad (z,s) = (z_1, \cdots, z_{n-1}, s) \in \mathcal{K}_\varepsilon \times \Big(-\frac{\delta_0}{\varepsilon}, \frac{\delta_0}{\varepsilon} \Big). \tag{2.9}$$

Obviously, the new coefficients g_{ab}'s of the Riemannian metric of $\mathbb{R}^n$, in the stretched Fermi coordinates, can be written as

$$\begin{aligned} g_{ij}(z,s) = &\ \bar{g}_{ij}(\varepsilon z) + \varepsilon s\Big(H_i^k(\varepsilon z)\bar{g}_{kj}(\varepsilon z) + H_j^l(\varepsilon z)\bar{g}_{il}(\varepsilon z)\Big) \\ &+ \varepsilon^2 s^2 H_i^k(\varepsilon z) H_j^l(\varepsilon z)\bar{g}_{kl}(\varepsilon z), \end{aligned}$$

$$g_{i,n} = g_{n,i} = 0, \quad g_{nn} = 1, \quad i,j = 1,2,\cdots, n-1.$$

Note also that the inverse matrix $\{g^{ab}\}$ decomposes as

$$\{g^{ab}\} = \begin{pmatrix} \{g^{ij}\} & 0 \\ 0 & 1 \end{pmatrix}.$$

We also let g_0 denote the metric of $\mathcal{K}_\varepsilon$ induced from the standard metric of $\mathbb{R}^n$ with corresponding Laplace-Beltrami operator defined as the form

$$\Delta_{\mathcal{K}_\varepsilon} = \frac{1}{\sqrt{\det \bar{g}(\varepsilon z)}} \frac{\partial}{\partial z_i}\Big(\sqrt{\det \bar{g}(\varepsilon z)}\, \bar{g}^{ij}(\varepsilon z) \frac{\partial}{\partial z_j} \Big).$$

For further references, this subsection focus on the expansion of the Laplace-Beltrami operator defined by

$$\begin{aligned} \Delta_g &= \frac{1}{\sqrt{\det g}}\, \partial_a \Big(g^{ab} \sqrt{\det g}\, \partial_b \Big) \\ &= g^{ab}\, \partial_a \partial_b + (\partial_a g^{ab})\, \partial_b + \frac{1}{2}\, \partial_a \big(\log(\det g) \big)\, g^{ab}\, \partial_b. \end{aligned} \tag{2.10}$$

Using (1.12), direct computation gives that

$$\det(g) = \det(\bar{g}) \Big(1 + 2\varepsilon H(\varepsilon z) s + \varepsilon^2 G(\varepsilon z) s^2 + \sum_{i=3}^{n} \varepsilon^i G_i(\varepsilon z) s^i \Big),$$

where we have used (1.8) and (1.13) for the definitions of H and $G, G_3, \cdots, G_n$. This gives

$$\log(\det g) = \log\big(\det(\bar{g}) \big) + \log\Big[1 + 2\varepsilon H(\varepsilon z) s + \varepsilon^2 G(\varepsilon z) s^2 + \sum_{i=3}^{n} \varepsilon^i G_i(\varepsilon z) s^i \Big].$$

Hence, the following formula can be checked by careful calculation

$$\Delta_g = \frac{\partial^2}{\partial s^2} + \triangle_{\mathcal{K}_\varepsilon} + B_0(\varepsilon z, s) + B_1(\varepsilon z, s) + B_2(\varepsilon z, s) + B_3(\varepsilon z, s),$$

where

$$B_0(\varepsilon z, s) = \varepsilon H \frac{\partial}{\partial s} - \varepsilon^2 |A_{\mathcal{K}}|^2 s \frac{\partial}{\partial s} + \sum_{l=3}^{n} \varepsilon^l b_l(\varepsilon z) s^{l-1} \frac{\partial}{\partial s},$$

$$B_1(\varepsilon z, s) = \sum_{l=2}^{n-1} \sum_{j=1}^{n-1} \varepsilon^l b_l^j(\varepsilon z, s) \frac{\partial}{\partial z_j}, \quad B_2(\varepsilon z, s) = \sum_{l=1}^{n-2} \sum_{i,j=1}^{n-1} \varepsilon^l b_l^{ij} \frac{\partial^2}{\partial z_i \partial z_j},$$

$$B_3(\varepsilon z, s) = \varepsilon^{n+1} b_{n+1}(\varepsilon z, s) \frac{\partial}{\partial s} + \varepsilon^n \sum_{j=1}^{n-1} b_n^j(\varepsilon z, s) \frac{\partial}{\partial z_j}$$
$$+ \varepsilon^{n-1} \sum_{j=1}^{n-1} b_{n-1}^{ij}(\varepsilon z, s) \frac{\partial^2}{\partial z_i \partial z_j},$$

and functions $b_{n+1}, b_2^j, \cdots, b_n^j, b_1^{ij} \cdots, b_{n-1}^{ij}, i, j = 1, \cdots, n-1$ satisfy:

$$\left| b_{n+1}, b_2^j, \cdots, b_n^j, b_1^{ij} \cdots, b_{n-1}^{ij} \right| \le C\big(1 + |s|^{n+1}\big).$$

Moreover, we have that $b_l^{ij} = b_l^{ji}$, $i, j = 1, 2, \cdots, n-1, l = 1, \cdots, n-1$.

2.3. **Local formulation of the problem.** If we set $u(y) = \tilde{u}(\varepsilon y)$, then problem (1.1) is thus equivalent to

$$\Delta u - V(\varepsilon y) u + u^p = 0 \quad \text{in } \mathbb{R}^n. \tag{2.11}$$

We assume that, in the (z, s) coordinates, the location of concentration of the solution is characterized by the surface

$$\tilde{\Gamma}_\varepsilon : s = \sum_{l=0}^{k-2} \varepsilon^l f_l(\varepsilon z), \tag{2.12}$$

Remark 2: *Here the k in (2.12) can be any positive integer. In the next section, we will find an approximate solution by a recurrence procedure, which will solve (2.11) up to $O(\varepsilon^k)$. In the last section, to handle the resonance phenomenon we will choose $k \ge n+1$. The reader can refer to Proposition 8.1.*

Remark 3: *The smooth functions $f_0, \cdots, f_{k-3}$ are to be determined in next section. While the unknown parameter f_{k-2} is to be chosen by a type of reduction procedure, which is equivalent to solving a system of differential equations in the last section. In the sequel, we assume that f_{k-2} satisfies the uniform constraint*

$$||f_{k-2}||_a = ||f_{k-2}||_{L^\infty(\mathcal{K})} + ||\nabla_{\mathcal{K}} f_{k-2}||_{L^\infty(\mathcal{K})} + ||\triangle_{\mathcal{K}} f_{k-2}||_{L^q(\mathcal{K})} \le \varepsilon^{\frac{1}{2}}. \tag{2.13}$$

Here and in the following we always assume $q > n$ and q is fixed.

We consider a further changing of variables with the property that replaces at main order the potential V by 1. By setting

$$\alpha(\theta) = V(\theta, 0)^{1/(p-1)}, \quad \beta(\theta) = V(\theta, 0)^{1/2} \quad \text{with } \theta = (\theta_1, \ldots, \theta_{n-1}), \tag{2.14}$$

define a new function $v(z,x)$ as follows

$$u(z,s)=\alpha(\varepsilon z)v(z,x),\quad x=\beta(\varepsilon z)\left(s-\sum_{l=0}^{k-2}\varepsilon^l f_l(\varepsilon z)\right),\quad z=z. \tag{2.15}$$

We now want to express the problem in the new coordinates. Whence we need the following formulas

$$u_s=\alpha\beta v_x,\quad u_{ss}=\alpha\beta^2 v_{xx}, \tag{2.16}$$

$$\frac{\partial u}{\partial z_i}=\varepsilon\frac{\partial\alpha}{\partial\theta_i}v+\alpha\frac{\partial v}{\partial z_i}+\alpha v_x\frac{\partial\Big(\beta\big(s-\sum_{l=0}^{k-2}\varepsilon^l f_l\big)\Big)}{\partial z_i}, \tag{2.17}$$

$$\begin{aligned}\triangle_{\mathcal{K}_\varepsilon}u=&\alpha v_{xx}\left|\nabla_{\mathcal{K}_\varepsilon}\Big(\beta\big(s-\sum_{l=0}^{k-2}\varepsilon^l f_l\big)\Big)\right|^2+2\alpha\,\nabla_{\mathcal{K}_\varepsilon}v_x\cdot\nabla_{\mathcal{K}_\varepsilon}\Big(\beta\big(s-\sum_{l=0}^{k-2}\varepsilon^l f_l\big)\Big)\\&+\varepsilon^2\,\triangle_{\mathcal{K}}\alpha v+\alpha\,\triangle_{\mathcal{K}_\varepsilon}v+\alpha v_x\,\triangle_{\mathcal{K}_\varepsilon}\Big(\beta\big(s-\sum_{l=0}^{k-2}\varepsilon^l f_l\big)\Big)\\&+2\varepsilon\,\nabla_{\mathcal{K}}\alpha\cdot\left[v_x\,\nabla_{\mathcal{K}_\varepsilon}\Big(\beta\big(s-\sum_{l=0}^{k-2}\varepsilon^l f_l\big)\Big)+\nabla_{\mathcal{K}_\varepsilon}v\right],\end{aligned} \tag{2.18}$$

where we have denoted

$$\nabla_{\mathcal{K}_\varepsilon}\left(\beta\big(s-\sum_{l=0}^{k-2}\varepsilon^l f_l\big)\right)=\varepsilon\Big[\big(s-\sum_{l=0}^{k-2}\varepsilon^l f_l\big)\,\nabla_{\mathcal{K}}\beta-\beta\sum_{l=0}^{k-2}\varepsilon^l\,\nabla_{\mathcal{K}}f_l\Big],$$

$$\begin{aligned}\triangle_{\mathcal{K}_\varepsilon}\left(\beta\big(s-\sum_{l=0}^{k-2}\varepsilon^l f_l\big)\right)=&\,\varepsilon^2\big(s-\sum_{l=0}^{k-2}\varepsilon^l f_l\big)\,\triangle_{\mathcal{K}}\beta-2\varepsilon^2\,\nabla_{\mathcal{K}}\beta\cdot\sum_{l=0}^{k-2}\varepsilon^l\,\nabla_{\mathcal{K}}f_l\\&-\varepsilon^2\beta\sum_{l=0}^{k-2}\varepsilon^l\,\triangle_{\mathcal{K}}f_l.\end{aligned}$$

We also have similar expressions for the operators $\frac{\partial^2 u}{\partial z_i\partial z_j}, i,j=1,2,\cdots,n-1$, which are omitted here. It is convenient to expand

$$\begin{aligned}V(\varepsilon z,\varepsilon s)=&\,V(\varepsilon z,0)+V_t(\varepsilon z,0)\varepsilon s+\frac{1}{2}V_{tt}(\varepsilon z,0)\varepsilon^2 s^2+\sum_{m=3}^{k}a_m(\varepsilon z,0)\varepsilon^m s^m\\&+a_{k+1}(\varepsilon z,\varepsilon s)\varepsilon^{k+1}s^{k+1},\end{aligned}$$

with the notation

$$a_m(\varepsilon z,0)=\frac{1}{m!}\frac{\partial^m V(\varepsilon z,0)}{\partial t^m},$$

and a smooth function $a_{k+1}(\theta,t)$.

Locally, this gives that u solves (2.11) if and only if v defined in (2.14) solves the following problem

$$S(v)\equiv v_{xx}-v+v^p+\beta^{-2}\,\triangle_{\mathcal{K}_\varepsilon}v+B(v)=0. \tag{2.19}$$

In the above we have denoted the linear operator

$$B(v)=B_4(v)+B_5(v)+B_6(v).$$

The linear operator B_4 comes from (2.18) and can be expressed explicitly by

$$\begin{aligned}
B_4(v) &= \varepsilon\beta^{-1}Hv_x \;-\; \varepsilon^2\beta^{-1}|A_{\mathcal{K}}|^2\left(\frac{x}{\beta}+\sum_{l=0}^{k-2}\varepsilon^l f_l\right)v_x \\
&\quad+\;\beta^{-1}\sum_{m=3}^{n}\varepsilon^m b_m\left(\frac{x}{\beta}+\sum_{l=0}^{k-2}\varepsilon^l f_l\right)^{m-1} v_x \;+\; \varepsilon^2(\alpha\beta^2)^{-1}\,\triangle_{\mathcal{K}}\,\alpha v \\
&\quad+\;2\varepsilon^2(\alpha\beta^2)^{-1}\,\nabla_{\mathcal{K}}\,\alpha\cdot\left(\frac{x}{\beta}\,\nabla_{\mathcal{K}}\,\beta-\beta\sum_{l=0}^{k-2}\varepsilon^l\,\nabla_{\mathcal{K}}\,f_l\right)v_x \\
&\quad+\;2\varepsilon(\alpha\beta^2)^{-1}\,\nabla_{\mathcal{K}}\,\alpha\cdot\nabla_{\mathcal{K}_\varepsilon}v \;+\; \varepsilon^2\left|x\beta^{-2}\,\nabla_{\mathcal{K}}\,\beta-\sum_{l=0}^{k-2}\varepsilon^l\,\nabla_{\mathcal{K}}\,f_l\right|^2 v_{xx} \\
&\quad+\;\varepsilon^2\beta^{-2}\left(\frac{x}{\beta}\,\triangle_{\mathcal{K}}\,\beta\;-\;2\sum_{l=0}^{k-2}\varepsilon^l\,\nabla_{\mathcal{K}}\,\beta\cdot\nabla_{\mathcal{K}}f_l\;-\;\beta\sum_{l=0}^{k-2}\varepsilon^l\,\triangle_{\mathcal{K}}\,f_l\right)v_x \\
&\quad+\;2\varepsilon\beta^{-2}\left(\frac{x}{\beta}\,\nabla_{\mathcal{K}}\,\beta\;-\;\beta\sum_{l=0}^{k-2}\varepsilon^l\,\nabla_{\mathcal{K}}\,f_l\right)\cdot\nabla_{\mathcal{K}_\varepsilon}v_x \\
&\quad-\;\varepsilon\beta^{-2}V_t\left(\frac{x}{\beta}+\sum_{l=0}^{k-2}\varepsilon^l f_l\right)v\;-\;\frac{\varepsilon^2}{2}\beta^{-2}V_{tt}\left(\frac{x}{\beta}+\sum_{l=0}^{k-2}\varepsilon^l f_l\right)^2 v \\
&\quad-\;\frac{1}{\beta^2}\sum_{m=3}^{k}a_m(\varepsilon z,0)\varepsilon^m\left(\frac{x}{\beta}+\sum_{l=0}^{k-2}\varepsilon^l f_l\right)^m v.
\end{aligned}$$

At the meantime, B_5 is the linear operator corresponding operator B_1+B_2 expressed in the new coordinates (z,x)

$$B_5(v)\;=\;B_{51}\;+\;B_{52}, \tag{2.20}$$

where we have decomposed B_{51} in the form

$$\begin{aligned}
B_{51} \;=\; &\frac{1}{\alpha\beta^2}\sum_{m=2}^{n-1}\sum_{j=1}^{n-1}\varepsilon^m b_m^j\left[\varepsilon\frac{\partial\alpha}{\partial\theta_j}v+\alpha\frac{\partial v}{\partial z_j}+\varepsilon\alpha\Big(\frac{x}{\beta}\frac{\partial\beta}{\partial\theta_j}-\beta\sum_{l=0}^{k-2}\varepsilon^l\frac{\partial f_l}{\partial\theta_j}\Big)v_x\right] \\
&+\;\left(\alpha\beta^2\right)^{-1}\sum_{m=1}^{n-2}\sum_{i,j=1}^{n-1}\varepsilon^m b_m^{ij}\left[\varepsilon^2\frac{\partial^2\alpha}{\partial\theta_i\partial\theta_j}v+2\varepsilon\frac{\partial\alpha}{\partial\theta_i}\frac{\partial v}{\partial z_j}+\alpha\frac{\partial^2 v}{\partial z_i\partial z_j}\right] \\
&+\;\left(\alpha\beta^2\right)^{-1}\sum_{m=1}^{n-2}\sum_{i,j=1}^{n-1}2\varepsilon^{m+2}b_m^{ij}\frac{\partial\alpha}{\partial\theta_i}\left(\frac{x}{\beta}\frac{\partial\beta}{\partial\theta_j}-\beta\sum_{l=0}^{k-2}\varepsilon^l\frac{\partial f_l}{\partial\theta_j}\right)v_x
\end{aligned}$$

and B_{52} in the form

$$\begin{aligned} B_{52} \;=\; & \beta^{-2}\sum_{m=1}^{n-2}\sum_{i,j=1}^{n-1} 2\varepsilon^{m+1} b_m^{ij}\left(\frac{x}{\beta}\frac{\partial\beta}{\partial\theta_j} - \beta\sum_{l=0}^{k-2}\varepsilon^l\frac{\partial f_l}{\partial\theta_j}\right)\frac{\partial^2 v}{\partial x\partial z_i} \\ & + \beta^{-2}\sum_{m=1}^{n-2}\sum_{i,j=1}^{n-1}\varepsilon^{m+2} b_m^{ij}\left(\frac{x}{\beta}\frac{\partial\beta}{\partial\theta_j} - \beta\sum_{l=0}^{k-2}\varepsilon^l\frac{\partial f_l}{\partial\theta_j}\right) \\ & \qquad\qquad \times\left(\frac{x}{\beta}\frac{\partial\beta}{\partial\theta_i} - \beta\sum_{l=0}^{k-2}\varepsilon^l\frac{\partial f_l}{\partial\theta_i}\right)\frac{\partial^2 v}{\partial x^2} \\ & + \beta^{-2}\sum_{m=1}^{n-2}\sum_{i,j=1}^{n-1}\varepsilon^{m+2} b_m^{ij}\left[\frac{x}{\beta}\frac{\partial^2\beta}{\partial\theta_i\partial\theta_j} - 2\frac{\partial\beta}{\partial\theta_i}\sum_{l=0}^{k-2}\varepsilon^l\frac{\partial f_l}{\partial\theta_j}\right. \\ & \qquad\qquad \left. - \beta\sum_{l=0}^{k-2}\varepsilon^l\frac{\partial^2 f_l}{\partial\theta_i\partial\theta_j}\right] v_x. \end{aligned}$$

Moreover, the operator with higher order of ε has the form

$$B_6(v) = (\alpha\beta^2)^{-1}B_3(u) + \beta^{-2}a_{k+1}(\varepsilon z, \varepsilon s)\varepsilon^{k+1}s^{k+1}v, \tag{2.21}$$

with all derivatives expressed in terms of (2.15), (2.16)-(2.18).

3. **Local approximate solution**

The main objective of this section is to use coordinates (z, x) defined in (2.15) to construct a suitable local approximation to a solution expressed by the form,

$$\mathcal{V} = w(x) + \varepsilon e(\varepsilon z)Z(x) + \sum_{l=1}^{k-1}\varepsilon^l\phi_l(\varepsilon z, x), \tag{3.1}$$

where w and Z are two functions defined by (1.4) and (2.1). In the above expression, we have denoted ϕ_l, $l = 1, \cdots, k-1$, smooth bounded functions to be determined in the sequel. As we have mentioned, the unknown parameters f_{k-2} (c.f. (2.12)) and e will be chosen in the last section by solving a system of differential equations. In all what follows, we shall assume the validity of the following uniform constraints on the parameter e

$$\|e\|_b = \|e\|_{L^\infty(\mathcal{K})} + \varepsilon\|\nabla_{\mathcal{K}}\, e\|_{L^q(\mathcal{K})} + \varepsilon^2\|\triangle_{\mathcal{K}}\, e\|_{L^q(\mathcal{K})} \le \varepsilon^{\frac{1}{2}}. \tag{3.2}$$

For simplicity of notations, define

$$\mathcal{F} = \{\ (f_{k-2}, e) | f_{k-2} \text{ and } e \text{ satisfy (2.13) and (3.2) respectively }\}. \tag{3.3}$$

Now, the key point is to choose suitable correction terms $\phi_1, \cdots, \phi_{k-1}$, and then prove that the approximate solution $\mathcal{V}$ solves problem (2.19) up to order $O(\varepsilon^k)$ for the given integer k in (2.12).

Formally, we have

$$\begin{aligned} (w+\Theta)^p \;=\; w^p\Big(1+\frac{\Theta}{w}\Big)^p \;=\; & w^p\bigg[1 + p\frac{\Theta}{w} + \frac{p(p-1)}{2}\Big(\frac{\Theta}{w}\Big)^2 + \cdots \\ & + C_{k,p}\Big(\frac{\Theta}{w}\Big)^k + O\Big(\big|\frac{\Theta}{w}\big|^{k+1}\Big)\bigg]. \end{aligned} \tag{3.4}$$

Setting $\Theta = \varepsilon(eZ+\phi_1)+\sum_{l=2}^{k-1}\varepsilon^l\phi_l$ and separating the powers of ε, we get

$$\begin{aligned}\Big(w+\varepsilon eZ+\sum_{l=1}^{k-1}\varepsilon^l\phi_l\Big)^p &= w^p\sum_{i=0}^{k}\varepsilon^i\sum_{j_1,\cdots,j_{k-1},\sum lj_l=i}C_{i,j_1,\cdots,j_{k-1}}\frac{(eZ+\phi_1)^{j_1}\phi_2^{j_2}\cdots\phi_{k-1}^{j_{k-1}}}{w^{j_1+\cdots+j_{k-1}}}\\ &\quad+w^pO\Big(\big|\frac{\Theta}{w}\big|^{k+1}\Big).\end{aligned}$$

Whence, using elementary calculation, we collect the powers of ε up to order k in the last formula, and then get the estimate

$$\begin{aligned}\mathfrak{B}_0 &= \Bigg|\Big(w+\varepsilon eZ+\sum_{l=1}^{k-1}\varepsilon^l\phi_l\Big)^p\\ &\qquad - w^p\sum_{i=0}^{k}\varepsilon^i\sum_{j_1,\cdots,j_{k-1},\sum lj_l=i}C_{i,j_1,\cdots,j_{k-1}}\frac{(eZ+\phi_1)^{j_1}\phi_2^{j_2}\cdots\phi_{k-1}^{j_{k-1}}}{w^{j_1+\cdots+j_{k-1}}}\Bigg|\\ &\le C_{k,p}w^p\Big[\varepsilon^{k+1}\big(1+\frac{|\Theta|}{w}\big)^k+\big|\frac{\Theta}{w}\big|^{k+1}\Big].\end{aligned}$$

More precisely, using (3.4), we make a decomposition

$$\begin{aligned}&w^p\sum_{i=0}^{k}\varepsilon^i\sum_{j_1,\cdots,j_{k-1},\sum lj_l=i}C_{i,j_1,\cdots,j_{k-1}}\frac{(eZ+\phi_1)^{j_1}\phi_2^{j_2}\cdots\phi_{k-1}^{j_{k-1}}}{w^{j_1+\cdots+j_{k-1}}}\\ &\qquad= w^p+pw^{p-1}\varepsilon eZ+\sum_{l=1}^{k-1}\varepsilon^lpw^{p-1}\phi_l+\frac12\varepsilon^2p(p-1)w^{p-2}\big(eZ+\phi_1\big)^2\\ &\qquad\quad+p(p-1)w^{p-2}\big(eZ+\phi_1\big)\sum_{l=3}^{k}\varepsilon^l\phi_{l-1}+\mathfrak{B}_3\\ &\qquad\equiv \mathfrak{B}_1+\mathfrak{B}_2+B_3,\end{aligned}\tag{3.5}$$

where we have denoted

$$\mathfrak{B}_1 = w^p+pw^{p-1}\varepsilon eZ+\sum_{l=1}^{k-1}\varepsilon^lpw^{p-1}\phi_l,\tag{3.6}$$

$$\begin{aligned}\mathfrak{B}_2 &= \frac12\varepsilon^2p(p-1)w^{p-2}\big(eZ+\phi_1\big)^2\\ &\quad+p(p-1)w^{p-2}\big(eZ+\phi_1\big)\sum_{l=3}^{k}\varepsilon^l\phi_{l-1}.\end{aligned}\tag{3.7}$$

In the above, we have denoted that

$$\mathfrak{B}_3=\sum_{l=3}^{k}\varepsilon^l\mathfrak{D}_l,\tag{3.8}$$

where for every $l=3,\cdots,k$, the component $\mathfrak{D}_l$ is independent of the terms $\phi_{l-1},\cdots,\phi_{k-1}$. The reader can refer Remark 2 for the criterion to arrange the error terms with the correction terms $\phi_1, \cdots, \phi_{k-1}$ involved.

Putting $\mathcal{V}$ into (2.19) and expanding formally, we derive that

$$
\begin{aligned}
&B(w) + \varepsilon(\varepsilon^2\beta^{-2} \triangle_{\mathcal{K}} e + \lambda_0 e)Z + \sum_{l=1}^{k-1} \varepsilon^l \Big[\phi_{l,xx} - \phi_l + pw^{p-1}\phi_l\Big] \\
&+ B(\varepsilon e Z) + \beta^{-2}\sum_{l=1}^{k-1} \varepsilon^l \triangle_{\mathcal{K}_\varepsilon} \phi_l + \sum_{l=1}^{k-1} \varepsilon^l B(\phi_l) + \mathfrak{B}_0 + \mathfrak{B}_2 + \mathfrak{B}_3 = 0.
\end{aligned}
\tag{3.9}
$$

3.1. **Local error.** In this subsection, we compute all error terms in (3.9). First, we calculate the error

$$
B(w) = B_4(w) + B_5(w) + B_6(w). \tag{3.10}
$$

Direct calculation gives that

$$
B_4(w) = \sum_{i=1}^{k} \varepsilon^i \hat{\mathbf{A}}_i + \sum_{i=1}^{k} \varepsilon^i \check{\mathbf{A}}_i + \varepsilon^{k+1}\mathbf{A}_{k+1},
$$

with expressions defined by

$$
\begin{aligned}
\hat{\mathbf{A}}_1 &= \beta^{-1}H(\varepsilon z)w_x - \beta^{-3}V_t(\varepsilon z, 0)xw, \qquad \check{\mathbf{A}}_1 = -\beta^{-2}V_t(\varepsilon z, 0)f_0 w, \\
\hat{\mathbf{A}}_2 &= -\beta^{-1}|A_{\mathcal{K}}|^2 f_0 w_x - \beta^{-1} \triangle_{\mathcal{K}} f_0 w_x - 2\beta^{-2} \nabla_{\mathcal{K}} \beta \cdot \nabla_{\mathcal{K}} f_0 w_x \\
&\quad - 2\beta^{-1}\alpha^{-1} \nabla_{\mathcal{K}} \alpha \cdot \nabla_{\mathcal{K}} f_0 w_x - 2\beta^{-2} \nabla_{\mathcal{K}} \beta \cdot \nabla_{\mathcal{K}} f_0 x w_{xx} - \beta^{-3} V_{tt} f_0 x w, \\
\check{\mathbf{A}}_2 &= -\beta^{-2}|A_{\mathcal{K}}|^2 x w_x + \beta^{-4}|\nabla_{\mathcal{K}} \beta|^2 x^2 w_{xx} + |\nabla_{\mathcal{K}} f_0|^2 w_{xx} + \beta^{-3} \triangle_{\mathcal{K}} \beta x w_x \\
&\quad + \beta^{-2}\alpha^{-1} \triangle_{\mathcal{K}} \alpha w + 2\alpha^{-1}\beta^{-3} \nabla_{\mathcal{K}} \alpha \cdot \nabla_{\mathcal{K}} \beta x w_x \\
&\quad - \frac{1}{2\beta^2} V_{tt} x^2 w - \frac{1}{2} V_{tt} f_0^2 w - \beta^{-2} V_t f_1 w.
\end{aligned}
$$

Note that $\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2$ are odd functions in the variable x, while $\check{\mathbf{A}}_1, \check{\mathbf{A}}_2$ are even functions in the variable x. For l running from 3 to k, we have that the odd parts can be expressed by

$$
\begin{aligned}
\hat{\mathbf{A}}_l &= -\beta^{-1}|A_{\mathcal{K}}|^2 f_{l-2} w_x - \beta^{-1} \triangle_{\mathcal{K}} f_{l-2} w_x - 2\beta^{-2} \nabla_{\mathcal{K}} \beta \cdot \nabla_{\mathcal{K}} f_{l-2} w_x \\
&\quad - 2\beta^{-1}\alpha^{-1} \nabla_{\mathcal{K}} \alpha \cdot \nabla_{\mathcal{K}} f_{l-2} w_x - 2\beta^{-2} \nabla_{\mathcal{K}} \beta \cdot \nabla_{\mathcal{K}} f_{l-2} x w_{xx} \\
&\quad - \beta^{-3} V_{tt} f_{l-2} x w + \hat{\mathfrak{b}}_l(f_0, \cdots, f_{l-3}).
\end{aligned}
$$

On the other hand, the even parts and high order terms have the form

$$
\begin{aligned}
\check{\mathbf{A}}_l &= -\beta^{-2} V_t f_{l-1} w + \check{\mathfrak{b}}_l(f_0, \cdots, f_{l-2}), \quad l = 3, \cdots, k-1, \\
\check{\mathbf{A}}_k &= \check{\mathfrak{b}}_k(f_0, \cdots, f_{k-2}), \qquad \mathbf{A}_{k+1} = \mathfrak{b}(f_0, \cdots, f_{k-2}).
\end{aligned}
$$

In the above, the terms $\hat{\mathfrak{b}}_l(f_0, \cdots, f_{l-3}), \check{\mathfrak{b}}_l(f_0, \cdots, f_{l-2}), \mathfrak{b}(f_0, \cdots, f_{k-2})$, are combination of powers of the parameters $f_0, \cdots, f_{k-2}$ and their derivatives with smooth bounded coefficients. Moreover, $\hat{\mathfrak{b}}_l(f_0, \cdots, f_{l-3}), l = 3, \cdots, k$ are odd functions in the variable x, while $\check{\mathfrak{b}}_l(f_0, \cdots, f_{l-2}), l = 3, \cdots, k$ are even functions in the variable x.

Remark 4: *In the above, we do not write* $\hat{\mathfrak{b}}_l(f_0, \cdots, f_{l-3})$ *and* $\check{\mathfrak{b}}_l(f_0, \cdots, f_{l-2})$

in an explicit form. In fact, we will use a recurrence procedure to find the correction terms $\phi_1, \cdots, \phi_{k-1}$ and parameters $f_1, \cdots, f_{k-3}$, and then cancel the error terms with order lower than k. For that purpose, for any chosen parameters $f_0, \cdots, f_{l-3}$ and correction terms $\phi_1, \cdots, \phi_{l-1}$, by solving the differential equation (3.55) for f_{l-2} under the non-degeneracy condition (1.16), we choose a suitable parameter f_{l-2} to make the sum of odd parts of all errors with the same order $O(\varepsilon^l)$, such as $\hat{\mathbf{A}}_l$, do not lie in the kernel (spanned by the function w_x) of the operator $\frac{\partial^2}{\partial x^2} - 1 + pw^{p-1}$ (c.f.(3.25)-(3.26)). This is equivalent to the orthogonal condition like (3.54). *On the other hand, since w_x is an odd function, and $\check{\mathfrak{b}}_l(f_0, \cdots, f_{l-2})$ is an even function in the variable x, the term $\check{\mathfrak{b}}_l(f_0, \cdots, f_{l-2})$ automatically satisfies the orthogonal condition for any chosen parameters $f_0, \cdots, f_{l-2}$. After that, we then solve a differential equation (3.52)-(3.53) again and find the correction term ϕ_l to cancel the error terms of order $O(\varepsilon^l)$. In the sequel we will write the error terms as this form.*

By using (2.20), it is also derived that

$$
\begin{aligned}
B_5(w) = {} & \beta^{-2} \sum_{m=1}^{n-2} \sum_{i,j=1}^{n-1} \varepsilon^{m+2} b_m^{ij} \left[\frac{x}{\beta} \frac{\partial^2 \beta}{\partial \tilde{y}_i \partial \tilde{y}_j} - 2 \frac{\partial \beta}{\partial \tilde{y}_i} \sum_{l=0}^{k-2} \varepsilon^l \frac{\partial f_l}{\partial \tilde{y}_j} - \beta \sum_{l=0}^{k-2} \varepsilon^l \frac{\partial^2 f_l}{\partial \tilde{y}_i \partial \tilde{y}_j} \right] w_x \\
& + 2(\alpha\beta^2)^{-1} \sum_{m=1}^{n-2} \sum_{i,j=1}^{n-1} \varepsilon^{m+2} b_m^{ij} \frac{\partial \alpha}{\partial \tilde{y}_i} \left(\frac{x}{\beta} \frac{\partial \beta}{\partial \tilde{y}_j} - \beta \sum_{l=0}^{k-2} \varepsilon^l \frac{\partial f_l}{\partial \tilde{y}_j} \right) w_x \\
& + \beta^{-2} \sum_{m=2}^{n-1} \sum_{j=1}^{n-1} \varepsilon^{m+1} b_m^{j} \Big(\frac{x}{\beta} \frac{\partial \beta}{\partial \tilde{y}_j} - \beta \sum_{l=0}^{k-2} \varepsilon^l \frac{\partial f_l}{\partial \tilde{y}_j} \Big) w_x \\
& + (\alpha\beta^2)^{-1} \sum_{m=1}^{n-2} \sum_{i,j=1}^{n-1} \varepsilon^{m+2} b_m^{ij} \frac{\partial^2 \alpha}{\partial \tilde{y}_i \partial \tilde{y}_j} w + (\alpha\beta^2)^{-1} \sum_{m=2}^{n-1} \sum_{j=1}^{n-1} \varepsilon^{m+1} b_m^{j} \frac{\partial \alpha}{\partial \tilde{y}_j} w \\
& + \beta^{-2} \sum_{m=1}^{n-2} \sum_{i,j=1}^{n-1} \varepsilon^{m+2} b_m^{ij} \left(\frac{x}{\beta} \frac{\partial \beta}{\partial \tilde{y}_j} - \beta \sum_{l=0}^{k-2} \varepsilon^l \frac{\partial f_l}{\partial \tilde{y}_j} \right) \\
& \qquad \times \left(\frac{x}{\beta} \frac{\partial \beta}{\partial \tilde{y}_i} - \beta \sum_{l=0}^{k-2} \varepsilon^l \frac{\partial f_l}{\partial \tilde{y}_i} \right) \frac{\partial^2 w}{\partial x^2}.
\end{aligned} \tag{3.11}
$$

There is a similar form for $B_6(w)$ as above. Whence, we write $B_5(w) + B_6(w)$ as the form

$$
\begin{aligned}
B_5(w) + B_6(w) = {} & \sum_{l=3}^{k} \varepsilon^l \hat{\mathbf{B}}_l(f_0, \cdots, f_{l-3}) + \sum_{l=3}^{k} \varepsilon^l \check{\mathbf{B}}_l(f_0, \cdots, f_{l-3}) \\
& + \varepsilon^{k+1} \mathbf{B}_{k+1}(f_0, \cdots, f_{k-2}),
\end{aligned}
$$

where the terms $\mathbf{B}_{k+1}(f_0, \cdots, f_{k-2})$ and $\hat{\mathbf{B}}_l(f_0, \cdots, f_{l-3}), \check{\mathbf{B}}_l(f_0, \cdots, f_{l-3}), l = 3, \cdots, k$ are combination of powers of the parameters $f_0, \cdots, f_{k-2}$ and their derivatives with smooth bounded coefficients. Moreover, $\hat{\mathbf{B}}_l(f_0, \cdots, f_{l-3}), l = 3, \cdots, k$ are odd functions in the variable x, while $\check{\mathbf{B}}_l(f_0, \cdots, f_{l-3}), l = 3, \cdots, k$ are even functions in the variable x. As a conclusion, we get

$$
\begin{aligned}
B(w) = & \sum_{i=1}^{k} \varepsilon^i \hat{\mathbf{A}}_i + \sum_{i=1}^{k} \varepsilon^i \check{\mathbf{A}}_i + \varepsilon^{k+1}\mathbf{A}_{k+1} + \sum_{i=3}^{k} \varepsilon^i \hat{\mathbf{B}}_i \\
& + \sum_{i=3}^{k} \varepsilon^i \check{\mathbf{B}}_i + \varepsilon^{k+1}\mathbf{B}_{k+1}.
\end{aligned} \tag{3.12}
$$

Second, we compute the error

$$
\varepsilon\Big(\varepsilon^2\beta^{-2} \triangle_{\mathcal{K}} eZ + \lambda_0 eZ\Big) + B(\varepsilon eZ), \tag{3.13}
$$

with $B(\varepsilon eZ) = B_4(\varepsilon eZ) + B_5(\varepsilon eZ) + B_6(\varepsilon eZ)$. There also holds

$$
B_4(\varepsilon eZ) + B_5(\varepsilon eZ) + B_6(\varepsilon eZ) = \sum_{i=2}^{k} \varepsilon^i \hat{\mathbf{C}}_i \, + \, \sum_{i=2}^{k} \varepsilon^i \check{\mathbf{C}}_i \, + \, \varepsilon^{k+1}\mathbf{C}_{k+1}.
$$

In the above, we have denoted the following forms

$$
\begin{aligned}
\hat{\mathbf{C}}_2 &= \beta^{-1} H eZ_x - \beta^{-3} V_t exZ, & \check{\mathbf{C}}_2 &= -\beta^{-2} V_t f_0 eZ, \\
\hat{\mathbf{C}}_3 &= \mathfrak{b}_6(f_0), & \check{\mathbf{C}}_3 &= -\beta^{-2} V_t f_1 eZ + \mathfrak{b}_7(f_0), \\
\hat{\mathbf{C}}_l &= \hat{\mathbf{C}}_l(f_0, \cdots, f_{l-3}, e), & \check{\mathbf{C}}_l &= \check{\mathbf{C}}_l(f_0, \cdots, f_{l-2}, e), \quad l = 4, \cdots, k, \\
\mathbf{C}_{k+1} &= \mathbf{C}_{k+1}(f_0, \cdots, f_{k-2}, e).
\end{aligned}
$$

Moreover, $\hat{\mathbf{C}}_l, l = 2, 3, \cdots, k$ are odd functions in the variable x, while $\check{\mathbf{C}}_l, l = 2, 3, \cdots, k$ are even functions in the variable x.

In summary, we have that

$$
\begin{aligned}
S(\mathcal{V}) = & \sum_{i=1}^{k} \varepsilon^i \hat{\mathbf{A}}_i \, + \, \sum_{i=1}^{k} \varepsilon^i \check{\mathbf{A}}_i \, + \, \varepsilon^{k+1}\mathbf{A}_{k+1} \, + \, \sum_{i=3}^{k} \varepsilon^i \hat{\mathbf{B}}_i \, + \, \sum_{i=3}^{k} \varepsilon^i \check{\mathbf{B}}_i \\
& + \, \varepsilon^{k+1}\mathbf{B}_{k+1} \, + \, \varepsilon^3\beta^{-2} \triangle_{\mathcal{K}} eZ \, + \, \varepsilon\lambda_0 eZ \, + \, \sum_{i=2}^{k} \varepsilon^i \hat{\mathbf{C}}_i \, + \, \sum_{i=2}^{k} \varepsilon^i \check{\mathbf{C}}_i \\
& + \, \varepsilon^{k+1}\mathbf{C}_{k+1} \, + \, \sum_{l=1}^{k-1} \varepsilon^l \Big[\phi_{l,xx} \, - \, \phi_l \, + \, pw^{p-1}\phi_l\Big] \\
& + \, \sum_{l=1}^{k-1} \varepsilon^l \beta^{-2} \triangle_{\mathcal{K}_\varepsilon} \phi_l \, + \, \sum_{l=1}^{k-1} \varepsilon^l B(\phi_l) \, + \, \mathfrak{B}_0 \, + \, \mathfrak{B}_2 \, + \, \mathfrak{B}_3.
\end{aligned} \tag{3.14}
$$

Now, we shall write the error terms involving correction terms $\phi_1, \cdots, \phi_{k-1}$ in a suitable form. In the next subsection, for any given $l = 1, 2, \cdots, k-1$, we will choose ϕ_l as the form

$$
a_{l1}(\varepsilon z) b_{l1}(x) + a_{l2}(\varepsilon z) b_{l2}(x),
$$

for some generic smooth functions a_{l1}, a_{l2}, b_{l1}(odd) and b_{l2}(even) (c.f.(3.33)). Moreover, the terms a_{l1} and a_{l2} do not depend on the unknown parameters $f_l, \cdots, f_{k-2}$. Whence, we make a decomposition as

$$
\sum_{l=1}^{k-1} \varepsilon^l \beta^{-2} \triangle_{\mathcal{K}_\varepsilon} \phi_l \, = \, \sum_{i=3}^{k} \varepsilon^i \hat{\mathcal{H}}_i \, + \, \sum_{i=3}^{k} \varepsilon^i \check{\mathcal{H}}_i \, + \, \varepsilon^{k+1}\mathcal{H}_{k+1}. \tag{3.15}
$$

where

$$\hat{\mathcal{H}}_i = \hat{\mathcal{H}}_i(\phi_1, \cdots, \phi_{i-2}, f_0, \cdots, f_{i-3}), \tag{3.16}$$

$$\check{\mathcal{H}}_i = \check{\mathcal{H}}_i(\phi_1, \cdots, \phi_{i-2}, f_0, \cdots, f_{i-3}), \tag{3.17}$$

$$\mathcal{H}_{k+1} = \mathcal{H}_{k+1}(\phi_1, \cdots, \phi_k, f_0, \cdots, f_{k-2}). \tag{3.18}$$

Moreover, $\hat{\mathcal{H}}_i$ is an odd function in the variable x, while $\check{\mathcal{H}}_i$ is an even functions in the variable x.

Remark 5: *Based on the same reason as we stated in Remark 4, we do not write $\hat{\mathcal{H}}_i$ and $\check{\mathcal{H}}_i$ in an explicit form in the above formulas. In fact, as we stated in the above, for any $i = 3, \cdots, k$, the term ϕ_i does not depend on the parameters $f_i, \cdots, f_{k-2}$. Whence, for any chosen parameters $f_0, \cdots, f_{i-3}$ and correction terms $\phi_1, \cdots, \phi_{i-1}$, the terms $\hat{\mathcal{H}}_i$ and $\check{\mathcal{H}}_i$ with the form in (3.16)-(3.17) do not depend on the parameters $f_{i-2}, \cdots, f_{k-2}$ and correction terms $\phi_i, \cdots, \phi_{k-1}$. We can solve the differential equation (3.52)-(3.53) and find the correction term ϕ_l, which has two components to cancel the error terms $\varepsilon^i \hat{\mathcal{H}}_i$ and $\varepsilon^i \check{\mathcal{H}}_i$. In the sequel we will write the error terms as this form.*

From the definition of the operator B in (2.19), we also write

$$\begin{aligned}
\sum_{l=1}^{k} \varepsilon^l B(\phi_l) &= \varepsilon^2 \Big(\frac{H}{\beta} \phi_{1,x} - \frac{V_t}{\beta^3} x \phi_1 - \frac{V_t}{\beta^2} f_0 \phi_1 \Big) \\
&+ \sum_{i=3}^{k} \varepsilon^i \Big(\frac{H}{\beta} \phi_{i-1,x} - \frac{V_t}{\beta^3} x \phi_{i-1} - \frac{V_t}{\beta^2} (f_{i-2}\phi_1 + f_0 \phi_{i-1}) \Big) \\
&+ \sum_{i=3}^{k} \varepsilon^i \hat{\mathfrak{F}}_i + \sum_{i=3}^{k} \varepsilon^i \check{\mathfrak{F}}_i + \varepsilon^{k+1} \mathfrak{F}_{k+1},
\end{aligned}$$

where, for every $i = 3, \cdots, k$, the components $\hat{\mathfrak{F}}_i$ and $\check{\mathfrak{F}}_i$ do not depend on the correction terms $\phi_{i-1}, \cdots, \phi_k$ and the unknown parameters $f_{i-2}, \cdots, f_{k-2}$. In other words, we have

$$\begin{aligned}
\hat{\mathfrak{F}}_i &= \hat{\mathfrak{F}}_i(\phi_1, \cdots, \phi_{i-2}, f_0, \cdots, f_{i-3}), \\
\check{\mathfrak{F}}_i &= \check{\mathfrak{F}}_i(\phi_1, \cdots, \phi_{i-2}, f_0, \cdots, f_{i-3}).
\end{aligned}$$

Moreover, $\hat{\mathfrak{F}}_i$ is an odd function in the variable x, while $\check{\mathfrak{F}}_i$ is an even function in the variable x. For further references, using (3.7) and (3.8), we decompose $\mathfrak{B}_3$ into even parts and odd parts, and then write $\mathfrak{B}_2 + \mathfrak{B}_3$ as

$$\begin{aligned}
\mathfrak{B}_2 + \mathfrak{B}_3 &= \frac{1}{2} \varepsilon^2 p(p-1) w^{p-2} \big(eZ + \phi_1 \big)^2 \\
&+ p(p-1) w^{p-2} \big(eZ + \phi_1 \big) \sum_{l=3}^{k} \varepsilon^l \phi_{l-1} + \sum_{i=2}^{k} \varepsilon^i \Big(\hat{\mathfrak{D}}_i + \check{\mathfrak{D}}_i \Big),
\end{aligned} \tag{3.19}$$

For $i = 2, \cdots, k$, the components $\hat{\mathfrak{D}}_i$ and $\check{\mathfrak{D}}_i$ are independent of the terms $\phi_{i-1}, \cdots, \phi_{k-1}$, i. e.

$$\hat{\mathfrak{D}}_i = \hat{\mathfrak{D}}_i(\phi_1, \cdots, \phi_{i-2}), \qquad \check{\mathfrak{D}}_i = \check{\mathfrak{D}}_i(\phi_1, \cdots, \phi_{i-2}). \tag{3.20}$$

Moreover, $\hat{\mathfrak{D}}_i$ is an odd function in the variable x, while $\check{\mathfrak{D}}_i$ is an even function in the variable x.

3.2. **Further improvement.** We will find the unknown parameters $f_0, \cdots, f_{k-3}$ and the correction layers $\phi_1, \cdots, \phi_{k-1}$, and then improve the approximation by a recurrence procedure. It is worth to mention that the term $\varepsilon^3\beta^{-2} \triangle_{\mathcal{K}} eZ + \varepsilon\lambda_0 eZ$ lies in the approximate kernel of the linearized problem(c.f.(6.1)). We ignore this term for the moment and then cancel other components of the error with order lower than k by choosing suitable correction terms $\phi_1, \cdots, \phi_{k-1}$. Using (3.14), for given $z \in \mathcal{K}$, we then consider the problems

$$\phi_{1,xx} - \phi_1 + pw^{p-1}\phi_1 = -\hat{\mathbf{A}}_1 - \check{\mathbf{A}}_1 \quad \text{in } \mathbb{R}, \tag{3.21}$$

$$\phi_1(\pm\infty) = 0, \quad \int_{\mathbb{R}} \phi_1 w_x \mathrm{d}x = 0; \tag{3.22}$$

$$\begin{aligned} \phi_{2,xx} - \phi_2 + pw^{p-1}\phi_2 = &- \hat{\mathbf{A}}_2 - \check{\mathbf{A}}_2 - \frac{1}{2}p(p-1)w^{p-2}\phi_1\phi_1 - \mathcal{G}_2 \\ &- \Big(\frac{H}{\beta}\phi_{1,x} - \frac{V_t}{\beta^3}x\phi_1 - \frac{V_t}{\beta^2}f_0\phi_1\Big) \quad \text{in } \mathbb{R}, \end{aligned} \tag{3.23}$$

$$\phi_2(\pm\infty) = 0, \quad \int_{\mathbb{R}} \phi_2 w_x \mathrm{d}x = 0; \tag{3.24}$$

and for $l = 3, \cdots, k-1$

$$\begin{aligned} \phi_{l,xx} - \phi_l + pw^{p-1}\phi_l = &- \hat{\mathbf{A}}_l - \check{\mathbf{A}}_l - p(p-1)w^{p-2}\phi_1\phi_{l-1} - \mathcal{G}_l - \frac{H}{\beta}\phi_{l-1,x} \\ &+ \frac{V_t}{\beta^3}x\phi_{l-1} + \frac{V_t}{\beta^2}(f_{l-2}\phi_1 + f_0\phi_{l-1}) \quad \text{in } \mathbb{R}, \end{aligned} \tag{3.25}$$

$$\phi_l(\pm\infty) = 0, \quad \int_{\mathbb{R}} \phi_l w_x \mathrm{d}x = 0. \tag{3.26}$$

In the above, we have denoted

$$\mathcal{G}_2 = \hat{\mathcal{G}}_2 + \check{\mathcal{G}}_2 + p(p-1)w^{p-2}eZ\phi_1 + \frac{1}{2}p(p-1)w^{p-2}e^2Z^2, \tag{3.27}$$

with the odd part and even part given by

$$\hat{\mathcal{G}}_2 = \hat{\mathbf{C}}_2 + \hat{\mathcal{H}}_2 + \hat{\mathfrak{F}}_2 + \hat{\mathfrak{D}}_2, \qquad \check{\mathcal{G}}_2 = \check{\mathbf{C}}_2 + \check{\mathcal{H}}_2 + \check{\mathfrak{F}}_2 + \check{\mathfrak{D}}_2, \tag{3.28}$$

and also $\mathcal{G}_l = \hat{\mathcal{G}}_l + \check{\mathcal{G}}_l + p(p-1)w^{p-2}eZ\phi_{l-1}$ with

$$\hat{\mathcal{G}}_l = \hat{\mathbf{B}}_l + \hat{\mathbf{C}}_l + \hat{\mathcal{H}}_l + \hat{\mathfrak{F}}_l + \hat{\mathfrak{D}}_l, \qquad \check{\mathcal{G}}_l = \check{\mathbf{B}}_l + \check{\mathbf{C}}_l + \check{\mathcal{H}}_l + \check{\mathfrak{F}}_l + \check{\mathfrak{D}}_l. \tag{3.29}$$

To cancel the first order terms $\varepsilon\hat{\mathbf{A}}_1$, $\varepsilon\check{\mathbf{A}}_1$ and improve the approximation, we should choose the correction layer $\varepsilon\phi_1$ with suitable form. For this purpose, given $z \in \mathcal{K}$, we consider the problem

$$-\phi_{1,xx} + \phi_1 - pw^{p-1}\phi_1 = \hat{\mathbf{A}}_1 + \check{\mathbf{A}}_1 \quad \text{in } \mathbb{R}, \tag{3.30}$$

$$\phi_1(\pm\infty) = 0, \quad \int_{\mathbb{R}} \phi_1 w_x \mathrm{d}x = 0, \tag{3.31}$$

as it is well known, which is uniquely solvable provided that

$$\int_{\mathbb{R}} (\hat{\mathbf{A}}_1 + \check{\mathbf{A}}_1) w_x \mathrm{d}x = 0. \tag{3.32}$$

In fact, using the fact that w is an even function in the variable x, we have

$$\int_{\mathbb{R}} \check{\mathbf{A}}_1 w_x \mathrm{d}x = 0.$$

Moreover, the assumption that $\mathcal{K}$ is stationary in (1.15) and the identity

$$\int_{\mathbb{R}} w^2 \mathrm{d}x = 2\sigma \int_{\mathbb{R}} w_x^2 \mathrm{d}x,$$

will lead to

$$\int_{\mathbb{R}} \hat{\mathbf{A}}_1 w_x \mathrm{d}x = 0.$$

Whence, the solution to (3.30)-(3.31) can be expressed as

$$\phi_1 = \phi_{11} + \phi_{12}, \tag{3.33}$$

where

$$\phi_{11}(z,x) = a_{11}(\varepsilon z) w_1(x), \quad \phi_{12}(z,x) = f_0(\varepsilon z) a_{12}(\varepsilon z) w_2(x), \tag{3.34}$$

with

$$a_{11} = \beta^{-1} H, \quad a_{12} = -\beta^{-2} V_t = \sigma^{-1} H. \tag{3.35}$$

In fact, function w_1 is the unique odd function satisfying

$$-w_{1,xx} + w_1 - p w^{p-1} w_1 = w_x + \sigma^{-1} x w, \quad \int_{\mathbb{R}} w_1 w_x \mathrm{d}x = 0, \tag{3.36}$$

and function w_2 is the unique even function satisfying

$$-w_{2,xx} + w_2 - p w^{p-1} w_2 = w, \quad \int_{\mathbb{R}} w_2 w_x \mathrm{d}x = 0. \tag{3.37}$$

Moreover, w_2 has an explicit expression

$$w_2 = -\frac{1}{p-1} w - \frac{1}{2} x w_x. \tag{3.38}$$

For more details of the functions w_1 and w_2, the reader can refer to [13].

In order to cancel the error terms of order $O(\varepsilon^2)$ and improve the approximation by solving problem (3.23)-(3.24), we collect all terms of order $O(\varepsilon^2)$ in $S(\mathcal{V})$, which has the form $\varepsilon^2 S_2$ with

$$\begin{aligned} S_2 = \hat{\mathbf{A}}_2 + \check{\mathbf{A}}_2 + \frac{1}{2} p(p-1) w^{p-2} \phi_1 \phi_1 + \Big(\frac{H}{\beta} \phi_{1,x} - \frac{V_t}{\beta^3} x \phi_1 - \frac{V_t}{\beta^2} f_0 \phi_1 \Big) \\ + p(p-1) w^{p-2} e Z \phi_1 + \frac{1}{2} p(p-1) w^{p-2} e^2 Z^2 + \hat{\mathcal{G}}_2 + \check{\mathcal{G}}_2. \end{aligned} \tag{3.39}$$

We denote the odd part and even part respectively by $\varepsilon^2 \hat{S}_2$ and $\varepsilon^2 \check{S}_2$. As the arguments in solving (3.30)-(3.31), we need an orthogonal condition like (3.32). Hence, we compute the projection of $\hat{S}_2$ and $\check{S}_2$ onto the kernel of the operator

$\phi_{xx} - \phi + pw^{p-1}\phi$, which is spanned by w_x. In fact, we obtain

$$
\begin{aligned}
\hat{S}_2 =& -\frac{1}{\beta}|A_{\mathcal{K}}|^2 f_0 w_x - \frac{1}{\beta}\triangle_{\mathcal{K}} f_0 w_x - \frac{2}{\beta^2}\nabla_{\mathcal{K}}\beta\cdot\nabla_{\mathcal{K}} f_0 w_x + \beta^{-1}HeZ_x \\
& - \frac{2}{\alpha\beta}\nabla_{\mathcal{K}}\alpha\cdot\nabla_{\mathcal{K}} f_0 w_x - \frac{2}{\beta^2}\nabla_{\mathcal{K}}\beta\cdot\nabla_{\mathcal{K}} f_0 x w_{xx} - \frac{V_{tt}}{\beta^3} f_0 x w \\
& + \beta^{-1}Ha_{12} f_0 w_{2,x} - \beta^{-3}V_t a_{12} f_0 x w_2 - \beta^{-2}V_t a_{11} f_0 w_1 - \beta^{-3}V_t e x Z \\
& + p(p-1)w^{p-2}a_{11}w_1 eZ + p(p-1)w^{p-2}a_{11}a_{12}f_0 w_1 w_2 + \hat{\mathcal{G}}_2,
\end{aligned}
$$

$$
\begin{aligned}
\check{S}_2 =& -\beta^{-2}V_t f_1 w - \beta^{-2}|A_{\mathcal{K}}|^2 x w_x + \beta^{-4}|\nabla_{\mathcal{K}}\beta|^2 x^2 w_{xx} + |\nabla_{\mathcal{K}} f_0|^2 w_{xx} \\
& + \beta^{-3}\triangle_{\mathcal{K}}\beta x w_x + (\alpha\beta^{-2})^{-1}\triangle_{\mathcal{K}}\alpha w + (\alpha\beta^3)^{-1}\nabla_{\mathcal{K}}\alpha\cdot\nabla_{\mathcal{K}}\beta x w_x \\
& - (2\beta^4)^{-1}V_{tt}x^2 w - (2\beta^2)^{-1}V_{tt}f_0^2 w + \beta^{-1}Ha_{11}w_{1,x} - \beta^{-3}V_t a_{11} x w_1 \\
& - \beta^{-2}V_t a_{12} f_0^2 w_2 - \beta^{-2}V_t f_0 Z + \frac{1}{2}p(p-1)w^{p-2}\Big[\big(\phi_{11}\big)^2 + \big(\phi_{12}\big)^2\Big] \\
& + p(p-1)w^{p-2}\Big[a_{12}w_2 eZ + \frac{1}{2}e^2 Z^2\Big] + \check{\mathcal{G}}_2,
\end{aligned}
$$

where the terms $\hat{\mathcal{G}}_2$ and $\check{\mathcal{G}}_2$ are defined in (3.28). Since the term $\check{S}_2$ is even in the variable x and w_x is odd in the variable x, there holds

$$
\int_{\mathbb{R}} \check{S}_2 w_x \mathrm{d}x = 0. \tag{3.40}
$$

On the other hand, using the relations (3.35) and the identity

$$
\int_{\mathbb{R}} x w w_x \, \mathrm{d}x = -\sigma \int_{\mathbb{R}} w_x^2 \, \mathrm{d}x,
$$

we get

$$
\begin{aligned}
\int_{\mathbb{R}} \hat{S}_2 w_x \mathrm{d}x =& -\beta^{-1}\triangle_{\mathcal{K}} f_0 \int_{\mathbb{R}} w_x^2\,\mathrm{d}x - 2\beta^{-2}\nabla_{\mathcal{K}}\beta\cdot\nabla_{\mathcal{K}} f_0 \int_{\mathbb{R}}\big(x w_{xx} w_x + w_x^2\big)\mathrm{d}x \\
& - 2\alpha^{-1}\beta^{-1}\nabla_{\mathcal{K}}\alpha\cdot\nabla_{\mathcal{K}} f_0 \int_{\mathbb{R}} w_x^2\,\mathrm{d}x - \beta^{-1}|A_{\mathcal{K}}|^2 f_0 \int_{\mathbb{R}} w_x^2\,\mathrm{d}x \\
& - \beta^{-3}V_{tt} f_0 \int_{\mathbb{R}} x w w_x\,\mathrm{d}x + \beta^{-1}Ha_{12} f_0 \int_{\mathbb{R}} w_{2,x} w_x \mathrm{d}x \\
& - \beta^{-3}V_t a_{12} f_0 \int_{\mathbb{R}} x w_2 w_x \mathrm{d}x - f_0\beta^{-2}a_{11}V_t \int_{\mathbb{R}} w_1 w_x \mathrm{d}x \\
& + p(p-1)a_{11}a_{12} f_0 \int_{\mathbb{R}} w^{p-2} w_1 w_2 w_x \mathrm{d}x + \beta^{-1}He\int_{\mathbb{R}} Z_x w_x \mathrm{d}x \\
& - \beta^{-3}V_t e\int_{\mathbb{R}} x Z w_x \mathrm{d}x + p(p-1)a_{11}e\int_{\mathbb{R}} w^{p-2} w_1 Z w_x \mathrm{d}x + \int_{\mathbb{R}} \hat{\mathcal{G}}_2 w_x \mathrm{d}x \\
=& -\frac{\delta_1}{\beta}\Big[\triangle_{\mathcal{K}} f_0 + \beta^{-1}\nabla_{\mathcal{K}}\beta\cdot\nabla_{\mathcal{K}} f_0 + 2\alpha^{-1}\nabla_{\mathcal{K}}\alpha\cdot\nabla_{\mathcal{K}} f_0 \\
& \qquad + \big(|A_{\mathcal{K}}|^2 - \beta^{-2}\sigma V_{tt}\big) f_0\Big] \\
& + a_{11}a_{12} f_0 \int_{\mathbb{R}} \big(w_{2,x} w_x + \frac{x}{\sigma} w_x w_2 + w_1 w_x\big)\,\mathrm{d}x \\
& + p(p-1)a_{11}a_{12} f_0 \int_{\mathbb{R}} w^{p-2} w_1 w_2 w_x \mathrm{d}x + \mathbf{d}_0 e,
\end{aligned}
$$

where $\delta_1 = \int_{\mathbb{R}} w_x^2 \,\mathrm{d}x$ and $\mathbf{d}_0$ is a smooth bounded function independent of the unknown parameters $f_0, \cdots, f_{k-2}, e$ and the correction terms $\phi_1, \cdots, \phi_{k-1}$. Note that by differentiating the equation (3.37) and using the equation (3.36), we get

$$p(p-1)\int_{\mathbb{R}} w^{p-2} w_1 w_2 w_x \mathrm{d}x = -\int_{\mathbb{R}} w_1 w_x \mathrm{d}x + \int_{\mathbb{R}} \big(w_x + \sigma^{-1} x w\big) w_{2,x} \,\mathrm{d}x.$$

The last relation leads to

$$\begin{aligned} a_{11}a_{12}f_0 \int_{\mathbb{R}} \big(w_{2,x}w_x + \sigma^{-1} x w_x w_2 + w_1 w_x\big) + p(p-1)a_{11}a_{12}f_0 \int_{\mathbb{R}} w^{p-2}w_1 w_2 w_x \\ = a_{11}(\varepsilon z)a_{12}(\varepsilon z)f_0(\varepsilon z)\int_{\mathbb{R}} \big(2w_x w_{2,x} + \sigma^{-1}x(ww_2)_x\big)\,\mathrm{d}x \\ = -\sigma^{-1}\beta^{-1}H^2 f_0 \int_{\mathbb{R}} w_x^2 \,\mathrm{d}x, \end{aligned}$$

where we have used (3.35) and the following integral identities

$$2\int_{\mathbb{R}} w_{2,x} w_x \mathrm{d}x = -\Big(\frac{2}{p-1} + \frac{1}{2}\Big)\int_{\mathbb{R}} w_x^2 \,\mathrm{d}x,$$
$$\sigma^{-1}\int_{\mathbb{R}} w w_2 \,\mathrm{d}x = \Big(\frac{1}{2} - \frac{2}{p-1}\Big)\int_{\mathbb{R}} w_x^2 \,\mathrm{d}x.$$

Using the last relation, we add up all components together and get

$$\begin{aligned} \int_{\mathbb{R}} \hat{S}_2 w_x \mathrm{d}x \;=\; &-\frac{\delta_1}{\beta}\Big[\,\triangle_{\mathcal{K}} f_0 + \frac{\sigma}{V}\,\nabla_{\mathcal{K}} V \cdot \nabla_{\mathcal{K}} f_0\,\Big] \\ &-\frac{\delta_1}{\beta}\big(\,|A_{\mathcal{K}}|^2 + \sigma^{-1}H^2 - \frac{\sigma}{V}V_{tt}\,\big)f_0 + \mathbf{d}_0 e, \end{aligned} \tag{3.41}$$

where we have used the relation

$$\beta^{-1}\,\nabla_{\mathcal{K}}\,\beta + 2\alpha^{-1}\,\nabla_{\mathcal{K}}\,\alpha = \frac{\sigma}{V}\,\nabla_{\mathcal{K}}\,V.$$

Note that there is only one term $-\varepsilon^2 V_t f_1 w$ in $\varepsilon^2 \check{S}_2$, which is relative to the unknown parameter f_1. We choose (c.f.(3.34))

$$\varepsilon^2\phi_{22} \;=\; \varepsilon^2 f_1 a_{12} w_2 \;=\; \varepsilon^2 f_1 \sigma^{-1} H w_2, \tag{3.42}$$

as a further correction layer to cancel the term $-\varepsilon^2 V_t f_1 w$. For the solvent of the problem (3.23)-(3.24), for any given $z \in \mathcal{K}$, we consider the problem

$$-\psi_{2,xx} + \psi_2 - p w^{p-1}\psi_2 = \hat{S}_2 + \check{S}_2 + V_t f_1 w \text{ in } \mathbb{R}, \tag{3.43}$$

$$\psi_2(\pm\infty) = 0, \quad \int_{\mathbb{R}} \psi_2 w_x \mathrm{d}x = 0, \tag{3.44}$$

as it is well known, which is uniquely solvable provided that

$$\int_{\mathbb{R}} (\hat{S}_2 + \check{S}_2 + V_t f_1 w) w_x \mathrm{d}x = 0.$$

In fact, using (3.40) we have

$$\int_{\mathbb{R}} (\check{S}_2 + V_t f_1 w) w_x \mathrm{d}x = 0.$$

While (3.41) implies that

$$\int_{\mathbb{R}} \hat{S}_2 w_x \mathrm{d}x = 0,$$

is equivalent to the following differential equation

$$\triangle_{\mathcal{K}} f_0 + \gamma_1 \nabla_{\mathcal{K}} V \cdot \nabla_{\mathcal{K}} f_0 + \gamma_2 f_0 = -\frac{\mathbf{d}_0 \beta}{\delta_1} e, \tag{3.45}$$

where γ_1 and γ_2 are some smooth functions defined by

$$\gamma_1 = \frac{\sigma}{V}, \qquad \gamma_2 = \big(|A_{\mathcal{K}}|^2 + \sigma^{-1} H^2 - \frac{\sigma}{V} V_{tt} \big).$$

By using the non-degeneracy condition (1.16), for any given e, we can solve problem (3.45) and determine the parameter $f_0(e)$, which is obviously Lipschitz continuous with respect to e. Whence, the solution to (3.43)-(3.44) can be expressed as

$$\psi_{21}(\varepsilon z, x) + \psi_{22}(\varepsilon z, x), \tag{3.46}$$

where ψ_{21} is an odd function in the variable x and ψ_{22} is an even function in the variable x. The components in ψ_{21} and ψ_{22} are Lipschitz continuous with respect to unknown parameter e and independent of the parameters $f_1, \cdots, f_{k-2}$. Finally, we choose

$$\phi_2(z, x) = \psi_{21}(\varepsilon z, x) + \psi_{22}(\varepsilon z, x) + \phi_{22}. \tag{3.47}$$

3.3. **Recurrence procedure.** For $l = 3, \cdots, k-1$, in order to cancel the error terms of order $O(\varepsilon^l)$ and improve the approximation by solving problem (3.25)-(3.26), we collect all terms of order $O(\varepsilon^l)$ in $S(\mathcal{V})$, and denote their sum as $\varepsilon^l S_l$ with

$$\begin{aligned} S_l =& \hat{\mathbf{A}}_l + \check{\mathbf{A}}_l + \frac{1}{2} p(p-1) w^{p-2} \phi_1 \phi_{l-1} + \frac{H}{\beta} \phi_{l-1,x} - \frac{V_t}{\beta^3} x \phi_{l-1} \\ & - \frac{V_t}{\beta^2} f_{l-2} \phi_1 - \frac{V_t}{\beta^2} f_0 \phi_{l-1} + \mathcal{G}_l. \end{aligned}$$

Here, the function ϕ_{l-1} has the form

$$\phi_{l-1}(z, x) = \psi_{l-1,1}(\varepsilon z, x) + \psi_{l-1,2}(\varepsilon z, x) + \phi_{l-1,2}. \tag{3.48}$$

with

$$\phi_{l-1,2} \;=\; f_{l-2} a_{12} w_2 \;=\; f_{l-2} \sigma^{-1} H w_2. \tag{3.49}$$

The components in $\psi_{l-1,1}$ and $\psi_{l-1,2}$ are independent of the parameters $f_{l-2}, \cdots, f_{k-2}$. Moreover, $\psi_{l-1,1}$ is an odd function in the variable x and $\psi_{l-1,2}$ is an even function in the variable x.

We denote the odd part and even part respectively by $\varepsilon^l \hat{S}_l$ and $\varepsilon^l \check{S}_l$. As the arguments in solving (3.30)-(3.31), we need an orthogonal condition like (3.32). Hence, we compute the projection of $\check{S}_l$ and $\hat{S}_l$ onto the kernel of the operator

$\phi_{xx} - \phi + pw^{p-1}\phi$, which is spanned by w_x. In fact, we obtain

$$\begin{aligned}\hat{S}_l &= -\frac{1}{\beta}|A_{\mathcal{K}}|^2 f_{l-2}w_x - \frac{1}{\beta}\triangle_{\mathcal{K}} f_{l-2}w_x - \frac{2}{\beta^2}\nabla_{\mathcal{K}}\beta\cdot\nabla_{\mathcal{K}}f_{l-2}w_x \\ &\quad - \frac{2}{\alpha\beta}\nabla_{\mathcal{K}}\alpha\cdot\nabla_{\mathcal{K}}f_{l-2}w_x - \frac{2}{\beta^2}\nabla_{\mathcal{K}}\beta\cdot\nabla_{\mathcal{K}}f_{l-2}xw_{xx} - \frac{V_{tt}}{\beta^3}f_{l-2}xw \\ &\quad + \beta^{-1}Ha_{12}f_{l-2}w_{2,x} - \beta^{-3}V_t a_{12}f_{l-2}xw_2 - \beta^{-2}V_t a_{11}f_{l-2}w_1 - \beta^{-2}V_t f_0\psi_{l-1,1} \\ &\quad + p(p-1)w^{p-2}a_{11}a_{12}f_{l-2}w_1w_2 + p(p-1)w^{p-2}a_{11}w_1\psi_{l-1,2} \\ &\quad + p(p-1)w^{p-2}f_0a_{12}w_2\psi_{l-1,1} + \beta^{-1}H\psi_{l-2,2,x} - \beta^{-3}V_t x\psi_{l-2,2} + \hat{\mathcal{G}}_l,\end{aligned}$$

$$\begin{aligned}\check{S}_l &= -\beta^{-2}V_t f_{l-1}w + p(p-1)w^{p-2}a_{11}w_1\psi_{l-1,1} + p(p-1)w^{p-2}f_0a_{12}w_1\psi_{l-1,2} \\ &\quad + p(p-1)w^{p-2}f_0f_{l-2}a_{12}^2w_2^2 - \beta^{-2}V_t f_{l-2}f_0a_{12}w_2 + \beta^{-1}H\psi_{l-1,1,x} \\ &\quad - \beta^{-2}V_t f_0\psi_{l-1,2} - \beta^{-2}V_t f_{l-2}a_{12}w_2 - \beta^{-3}V_t x\psi_{l-1,1} + \check{\mathcal{G}}_l.\end{aligned}$$

Since the term $\check{S}_l$ is even in the variable x and w_x is odd in the variable x, there holds

$$\int_{\mathbb{R}} \check{S}_l w_x \mathrm{d}x = 0. \tag{3.50}$$

On the other hand, using the same arguments as in (3.41), we get

$$\begin{aligned}\int_{\mathbb{R}} \hat{S}_l w_x \mathrm{d}x &= -\frac{\delta_1}{\beta}\Big[\triangle_{\mathcal{K}} f_{l-2} + \frac{\sigma}{V}\nabla_{\mathcal{K}}V\cdot\nabla_{\mathcal{K}}f_{l-2}\Big] \\ &\quad -\frac{\delta_1}{\beta}\big(|A_{\mathcal{K}}|^2 + \sigma^{-1}H^2 - \frac{\sigma}{V}V_{tt}\big)f_{l-2} \\ &\quad + \mathbf{d}_{l-2}(f_0,\cdots,f_{l-3},e,\phi_1,\cdots,\phi_{l-1}).\end{aligned} \tag{3.51}$$

Here $\mathbf{d}_{l-2}$ is a smooth bounded function independent of the unknown parameters $f_{l-2},\cdots,f_{k-2}$ and the correction terms $\phi_l,\cdots,\phi_{k-1}$ and is Lipschitz continuous with respect to its parameters.

Note that there is only one term $-\varepsilon^l V_t f_{l-1}w$ in $\varepsilon^l\check{S}_l$, which is relative to the unknown parameter f_{l-1}. Similarly as in previous subsection, we choose (c.f.(3.34) and (3.42))

$$\varepsilon^l\phi_{l2} = \varepsilon^l f_{l-1}a_{12}w_2 = \varepsilon^l f_{l-1}\sigma^{-1}Hw_2,$$

as a further correction layer to cancel the term $-\varepsilon^l V_t f_{l-1}w$. Moreover, for given $z\in\mathcal{K}$, we consider the problem

$$-\psi_{l,xx} + \psi_l - pw^{p-1}\psi_l = \hat{S}_l + \check{S}_l + V_t f_{l-1}w \text{ in } \mathbb{R}, \tag{3.52}$$

$$\psi_l(\pm\infty) = 0, \quad \int_{\mathbb{R}}\psi_l w_x\mathrm{d}x = 0, \tag{3.53}$$

as it is well known, which is uniquely solvable provided that

$$\int_{\mathbb{R}}(\hat{S}_l + \check{S}_l + V_t f_{l-1}w)w_x\mathrm{d}x = 0. \tag{3.54}$$

In fact, using (3.50) we have

$$\int_{\mathbb{R}}(\check{S}_l + V_t f_{l-1}w)w_x\mathrm{d}x = 0.$$

While (3.51) implies that

$$\int_{\mathbb{R}} \hat{S}_l w_x \mathrm{d}x = 0,$$

is equivalent to the following differential equation

$$\triangle_{\mathcal{K}} f_{l-2} + \gamma_1 \nabla_{\mathcal{K}} V \cdot \nabla_{\mathcal{K}} f_{l-2} + \gamma_2 f_{l-2} = \frac{\mathbf{d}_{l-2}\beta}{\delta_1}, \tag{3.55}$$

where γ_1 and γ_2 are some smooth functions defined by

$$\gamma_1 = \frac{\sigma}{V}, \qquad \gamma_2 = \big(|A_{\mathcal{K}}|^2 + \sigma^{-1} H^2 - \frac{\sigma}{V} V_{tt} \big).$$

By using the non-degeneracy condition (1.16), for any given e, we can solve problem (3.55) and determine the parameter $f_{l-2}(e)$. Whence, the solution to (3.52)-(3.53) can be expressed as

$$\psi_{l1}(\varepsilon z, x) + \psi_{l2}(\varepsilon z, x), \tag{3.56}$$

where ψ_{l1} is an odd function in the variable x and ψ_{l2} is an even function in the variable x. The components in ψ_{l1} and ψ_{l2} are independent of the parameter $f_{l-1}, \cdots, f_{k-2}$. We choose

$$\phi_l(z,x) = \psi_{l1}(\varepsilon z, x) + \psi_{l2}(\varepsilon z, x) + \phi_{l2}. \tag{3.57}$$

3.4. **Summary.** We conclude that for any given parameter pair $(f_{k-2}, e) \in \mathcal{F}$, the basic local approximate solution is expressed by $\mathcal{V}$ in (3.1). Near the surface $\mathcal{K}_\varepsilon$, if we set $\mathcal{V} + \phi$ as the solution to (2.19), the problem takes the form as

$$L_1(\phi) + B(\phi) + N_1(\phi) + E_1 = 0, \tag{3.58}$$

where the two operators L_1 and N_1 defined by

$$L_1(\phi) = \beta^{-2} \triangle_{\mathcal{K}_\varepsilon} \phi + \phi_{xx} - \phi + p\mathcal{V}^{p-1}\phi,$$
$$N_1(\phi) = (\mathcal{V} + \phi)^p - \mathcal{V}^p - p\mathcal{V}^{p-1}\phi.$$

The corresponding error is defined by

$$\begin{aligned} E_1 &= S(\mathcal{V}) \\ &= \varepsilon^3 \beta^{-2} \triangle_{\mathcal{K}} eZ + \varepsilon\lambda_0 eZ + \varepsilon^k \Big[\hat{\mathbf{A}}_k + \check{\mathbf{A}}_k + \frac{1}{2}p(p-1)w^{p-2}\phi_1\phi_{k-1} \Big] \\ &\quad + \varepsilon^k \Big(\frac{H}{\beta}\phi_{k-1,x} - \frac{V_t}{\beta^3} x \phi_{k-1} - \frac{V_t}{\beta^2} f_{k-2}\phi_1 - \beta^{-2} V_t f_0 \phi_{k-1} \Big) + \varepsilon^k \mathcal{G}_k \\ &\quad + \varepsilon^{k+1} \Big[\mathbf{A}_{k+1} + \mathbf{B}_{k+1} + \mathbf{C}_{k+1} + \mathcal{H}_{k+1} + \mathfrak{F}_{k+1} \Big] + \mathfrak{B}_0, \end{aligned} \tag{3.59}$$

where $\mathcal{G}_k = \hat{\mathcal{G}}_k + \check{\mathcal{G}}_k + p(p-1)w^{p-2} eZ \phi_{k-1}$ with

$$\hat{\mathcal{G}}_k = \hat{\mathbf{B}}_k + \hat{\mathbf{C}}_k + \hat{\mathcal{H}}_k + \hat{\mathfrak{F}}_k + \hat{\mathfrak{D}}_k, \quad \check{\mathcal{G}}_k = \check{\mathbf{B}}_k + \check{\mathbf{C}}_k + \check{\mathcal{H}}_k + \check{\mathfrak{F}}_k + \check{\mathfrak{D}}_k. \tag{3.60}$$

The components in $\hat{\mathcal{G}}_k$ and $\check{\mathcal{G}}_k$ are independent of the parameter f_{k-2}. Here, the function ϕ_{k-1} has the form

$$\phi_{k-1}(z,x) = \psi_{k-1,1}(\varepsilon z, x) + \psi_{k-1,2}(\varepsilon z, x) + \phi_{k-1,2}, \tag{3.61}$$

where

$$\phi_{k-1,2} = f_{k-2} a_{12} w_2 = f_{k-2} \sigma^{-1} H w_2. \tag{3.62}$$

The reader can refer (3.34) for the definition of a_{12} and w_2. The components in $\psi_{k-1,1}$ and $\psi_{k-1,2}$ are independent of the parameters f_{k-2}. Moreover, $\psi_{k-1,1}$ is an odd function in the variable x and $\psi_{k-1,2}$ is an even function in the variable x.

For further references, we decompose the error as two components

$$E_1 = E_{11} + E_{12}, \tag{3.63}$$

where we have denoted

$$E_{11} = \varepsilon^3\beta^{-2} \bigtriangleup_{\mathcal{K}} eZ + \varepsilon\lambda_0 eZ, \quad E_{12} = E_1 - E_{11}. \tag{3.64}$$

3.5. **Size of the error in weighted Sobolev norms.** To estimate the size of error, we have to introduce some suitable weighted Sobolev norms. Here we use the same norm as those in [14] and [15]: for a function $h(z,x)$ defined on a set $E \in \mathbb{R}^n$, and for $0 < \sigma < \frac{1}{100}$ and $n+1 < q \leq +\infty$, we set

(3.65)

$$||h||_{q,\sigma;E} = \sup_{(z,x)\in E} e^{\sigma|x|}||h||_{L^q(B((z,x),1))}, \; ||h||_{2,q,\sigma;E} = \sum_{j=0}^{2} \sup_{(z,x)\in E} e^{\sigma|x|}||D^j h||_{L^q(B((z,x),1))}.$$

Here $B((z,x),1)$ denotes the ball of radius 1 centered at (z,x).

Let $\bar{\mathfrak{S}} := \mathcal{K}_\varepsilon \times (-\frac{\delta_0}{\varepsilon}, \frac{\delta_0}{\varepsilon})$. From the uniform bound of e in (3.2), it is easy to see that

$$||E_{11}||_{q,\sigma;\bar{\mathfrak{S}}} \leq C\varepsilon^{1/2+1-\frac{n}{q}}. \tag{3.66}$$

All terms in E_{12} carry ε^k in front, we then claim that

$$||E_{12}||_{q,\sigma;\bar{\mathfrak{S}}} \leq C\varepsilon^{k+\frac{1}{2}-\frac{n}{q}}. \tag{3.67}$$

A rather delicate term in E_{12} is the one carrying $\bigtriangleup_{\mathcal{K}} f_{k-2}$ since we only assume a uniform bound on $|| \bigtriangleup_{\mathcal{K}} f_{k-2}||_{L^q(\mathcal{K})}$. For example, we have a term $K_1 = \beta^{-2}\varepsilon^k \bigtriangleup_{\mathcal{K}} f_{k-2} w_x$ in $S(w)$ which has bound like

$$||K_1||_{q,\sigma;\bar{\mathfrak{S}}} \leq C\varepsilon^{k+\frac{1}{2}-\frac{n}{q}}.$$

Other terms can be estimated in the similar way. Moreover, for the Lipschitz dependence of the term of error E_{12} on the parameters f_{k-2} and e for the norm defined in (2.13) and (3.2), we have the validity of the estimate

$$\begin{aligned} &||E_{12}(f_{k-2}, e) - E_{12}(\tilde{f}_{k-2}, \tilde{e})||_{q,\sigma;\bar{\mathfrak{S}}} \\ &\qquad \leq C\varepsilon^{k+\frac{1}{2}-\frac{n}{q}}\big(||f_{k-2} - \tilde{f}_{k-2}||_a + ||e - \tilde{e}||_b\big). \end{aligned} \tag{3.68}$$

4. The gluing procedure

In this section, we use a gluing technique (as in [15]) to reduce problem (2.11) defined on $\mathbb{R}^n$ to a projected problem on the infinite strip $\mathfrak{S}$, where:

$$\mathfrak{S} \equiv \mathcal{K}_\varepsilon \times \mathbb{R}. \tag{4.1}$$

Let $\delta < \delta_0/100$ be a fixed number, where δ_0 is a constant defined in (2.9). We consider a smooth cut-off function $\eta_\delta(t)$ where $t \in \mathbb{R}_+$ such that $\eta_\delta(t) = 1$ for $0 \leq t \leq \delta$ and $\eta(t) = 0$ for $t > 2\delta$. Set $\eta_\delta^\varepsilon(s) = \eta_\delta(\varepsilon|s|)$, where s is the normal coordinate to $\mathcal{K}_\varepsilon$. Let $\mathcal{V}(z,x)$ denote the approximate solution constructed near

$\mathcal{K}_\varepsilon$ in the coordinates (z, x), which was introduced in (2.15). We define our global approximation by

$$W(y) \,=\, \eta_{3\delta}^{\varepsilon}(s)\alpha(\varepsilon z)\mathcal{V}\Big(z, \beta\big(s - \sum_{l=0}^{k-2}\varepsilon^l f_l\big)\Big). \tag{4.2}$$

Obviously, W is a function defined in the whole space $\mathbb{R}^n$, which is extended as 0 outside the $6\delta/\varepsilon$-neighborhood of $\mathcal{K}_\varepsilon$.

For $u = W + \hat\phi$ where $\hat\phi$ is globally defined in $\mathbb{R}^n$, denote

$$\Upsilon(u) = \Delta u \,-\, Vu \,+\, u^p \quad \text{in } \mathbb{R}^n.$$

Then u satisfies (2.11) if and only if

$$\widetilde{\mathcal{L}}(\hat\phi) \,=\, -\widetilde{E} \,-\, \widetilde{N}(\hat\phi) \quad \text{in } \mathbb{R}^n, \tag{4.3}$$

where we have denoted

$$\widetilde{\mathcal{L}}(\hat\phi) = \Delta\hat\phi \,-\, V\hat\phi + pW^{p-1}\hat\phi,$$
$$\widetilde{N}(\hat\phi) = (W+\hat\phi)^p - W^p - pW^{p-1}\hat\phi, \quad \widetilde{E} = \Upsilon(W).$$

We will look for $\hat\phi$ in the following form

$$\hat\phi \,=\, \eta_{3\delta}^{\varepsilon}(s)\phi \,+\, \psi,$$

where, in the coordinates (z, s) in (2.9), we assume that ϕ is defined in a neighborhood of $\mathcal{K}_\varepsilon$. Let $\bar{\mathcal{L}}$ be also an extension of the operator $\widetilde{\mathcal{L}}$ defined on the whole strip $\mathfrak{S}$. More specifically we set:

$$\bar{\mathcal{L}}(\phi) = \eta_{6\delta}^{\varepsilon}\Big[\Delta\phi - V\phi + pW^{p-1}\phi\Big] + (1-\eta_{6\delta}^{\varepsilon})\big(\partial_{ss}\phi + \Delta_{\mathcal{K}_\varepsilon}\phi - V\phi\big). \tag{4.4}$$

With this definition $\hat\phi$ is a solution of (4.3) if the pair (ϕ, ψ) satisfies the following coupled system:

$$\bar{\mathcal{L}}(\phi) \,=\, \eta_{\delta}^{\varepsilon}\Big[-\widetilde{N}(\eta_{3\delta}^{\varepsilon}\phi + \psi) - \widetilde{E} - pW^{p-1}\psi\Big], \tag{4.5}$$

$$\begin{aligned} \Delta\psi - V\psi + (1-\eta_{\delta}^{\varepsilon})pW^{p-1}\psi \,=\, &-(\Delta\eta_{3\delta}^{\varepsilon})\phi - 2(\nabla\eta_{3\delta}^{\varepsilon})\cdot(\nabla\phi) - (1-\eta_{\delta}^{\varepsilon})\widetilde{E} \\ &- (1-\eta_{\delta}^{\varepsilon})\widetilde{N}(\eta_{3\delta}^{\varepsilon}\phi + \psi), \end{aligned} \tag{4.6}$$

where ϕ is defined globally on $\mathfrak{S}$ and ψ is defined in $\mathbb{R}^n$.

The key observation is that, after solving (4.6), the problem can be transformed to the following nonlinear, nonlocal problem involving $\psi = \psi(\phi)$

$$\bar{\mathcal{L}}(\phi) = \eta_{\delta}^{\varepsilon}\Big[-\widetilde{N}(\eta_{3\delta}^{\varepsilon}\phi + \psi) - \widetilde{E} - pW^{p-1}\psi\Big]. \tag{4.7}$$

To solve (4.7) we will set up a fixed point argument by first solving (4.6) for a given ϕ. We assume that ϕ satisfies the following decay property

$$\big|\nabla\phi(z,s)\big| + \big|\phi(z,s)\big| \,\leq\, e^{-\tau/\varepsilon} \quad \text{if } |s| > \delta/\varepsilon, \tag{4.8}$$

for certain constant $\tau > 0$. Let us observe that V is uniformly positive and W is exponentially small for $|s| > \delta/\varepsilon$, where s is the normal coordinate to $\mathcal{K}_\varepsilon$. Then the problem

$$\Delta\psi - V\psi + (1-\eta_{\delta}^{\varepsilon})pW^{p-1}\psi = h \quad \text{in } \mathbb{R}^n,$$

has a unique bounded solution ψ whenever $||h||_\infty \leq +\infty$. Moreover, there holds

$$||\psi||_\infty \leq C||h||_\infty.$$

Since $\tilde{N}$ is power-like with power greater than one, a direct application of contraction mapping principle yields that (4.6) has a unique (small) solution $\psi = \psi(\phi)$ with

$$||\psi(\phi)||_{L^\infty} \le C\varepsilon\big(||\phi||_{L^\infty(|s|>\delta/\varepsilon)} + ||\nabla\phi||_{L^\infty(|s|>\delta/\varepsilon)} + e^{-\tau/\varepsilon} \big), \tag{4.9}$$

where $|s| > \delta/\varepsilon$ denotes the complement in $\mathbb{R}^n$ of δ/ε-neighborhood of $\mathcal{K}_\varepsilon$. Moreover, the nonlinear operator ψ satisfies Lipschitz condition of the form

$$\begin{aligned} ||\psi(\phi_1) - \psi(\phi_2)||_{L^\infty} \le & C\varepsilon\big(||\phi_1 - \phi_2||_{L^\infty(|s|>\delta/\varepsilon)} \\ & + ||\nabla\phi_1 - \nabla\phi_2||_{L^\infty(|s|>\delta/\varepsilon)} \big). \end{aligned} \tag{4.10}$$

From the above discussion, the full problem has been reduced to solving (4.7) for ϕ satisfying the condition (4.8). We make changing of variables as defined in (2.15), and then define an operator on the whole strip $\mathfrak{S}$ as the form

$$\begin{aligned} \mathcal{L}(\phi) \equiv & \phi_{xx} + \beta^{-2} \triangle_{\mathcal{K}_\varepsilon} \phi - \phi + p\mathcal{V}^{p-1}\phi + \eta^\varepsilon_{6\delta} B(\phi) \\ & - (1 - \eta^\varepsilon_{6\delta})p\mathcal{V}^{p-1}\phi. \end{aligned} \tag{4.11}$$

Rather than solving problem (4.7) directly, we deal with the following projected problem: given parameter pair $(f_{k-2}, e) \in \mathcal{F}$, finding function ϕ, such that

$$\begin{aligned} \mathcal{L}(\phi) = & \eta^\varepsilon_\delta(\alpha\beta^2)^{-1}\Big[-\tilde{N}\big(\eta^\varepsilon_{3\delta}\phi + \psi(\phi)\big) - \tilde{E} - pW^{p-1}\psi(\phi)\Big] \\ & + \eta^\varepsilon_\delta c\, w_x + \eta^\varepsilon_\delta dZ \quad \text{in } \mathfrak{S}, \end{aligned} \tag{4.12}$$

$$\int_{\mathbb{R}} \phi(z,x)\, w_x(x)\eta^\varepsilon_\delta \,\mathrm{d}x = \int_{\mathbb{R}} \phi(z,x)\, Z(x)\eta^\varepsilon_\delta \,\mathrm{d}x = 0, \quad z \in \mathcal{K}_\varepsilon. \tag{4.13}$$

In Proposition 6.2, we will prove that this problem has a unique solution ϕ whose norm is controlled by the $\|\cdot\|_{q,\sigma}$-norm of $E_2 = \eta^\varepsilon_\delta E_{12}$, the component of $(\alpha\beta^2)^{-1}\eta^\varepsilon_\delta\tilde{E}$. Moreover, ϕ will satisfy the constraint (4.8). After this has been done, our task is to choose suitable parameters f_{k-2} and e, possessing all properties in (2.13) and (3.2), such that the functions c and d are identically zero. It is equivalent to solving a nonlocal, nonlinear system of second order differential equations for the unknown parameters f_{k-2} and e defined on the manifold $\mathcal{K}$.

5. **Linear Theory**

Recall the definition that

$$\mathcal{L}(\phi) \equiv \phi_{xx} + \beta^{-2} \triangle_{\mathcal{K}_\varepsilon} \phi - \phi + p\mathcal{V}^{p-1}\phi + \eta^\varepsilon_{6\delta} B(\phi) - (1 - \eta^\varepsilon_{6\delta})p\mathcal{V}^{p-1}\phi.$$

This section will be devoted to the resolution of the basic linear problem for $\mathcal{L}$. Given function h, we consider the problem of finding ϕ such that we have

$$\mathcal{L}(\phi) = h + c\,\eta^\varepsilon_\delta w_x + d\eta^\varepsilon_\delta Z \quad \text{in } \mathfrak{S}, \tag{5.1}$$

$$\int_{\mathbb{R}} \phi(z,x)\, w_x(x)\eta^\varepsilon_\delta \,\mathrm{d}x = \int_{\mathbb{R}} \phi(z,x)\, Z(x)\eta^\varepsilon_\delta \,\mathrm{d}x = 0, \quad z \in \mathcal{K}_\varepsilon. \tag{5.2}$$

Our main result in this section is the following.

Proposition 5.1. *There exists a constant $C > 0$, independent of ε and uniform for the parameters f_{k-2} and e in (2.13) and (3.2) such that for all small ε problem (5.1)-(5.2) has a solution $\phi = T_{f_{k-2},e}(h)$, which defines a linear operator of its arguments and satisfies the estimate*

$$\|\phi\|_{q,\sigma} \le C\,\|h\|_{q,\sigma}.$$

We remark that we have omitted the dependence of the norm on the domain $\mathfrak{S}$. We will use this convention in the rest of the paper.

For the proof of Proposition 5.1, we need to show existence result for a simpler problem. Let us define the linear operator

$$L(\phi) = \phi_{xx} + \beta^{-2}\,\Delta_{\mathcal{K}_\varepsilon}\,\phi - \phi + pw^{p-1}\phi,$$

and consider the problem: given h, finding functions ϕ and c, d to

$$L(\phi) = h + c\,w_x + dZ \quad \text{in } \mathfrak{S}, \tag{5.3}$$

$$\int_{\mathbb{R}} \phi(z,x)\,w_x(x)\,\mathrm{d}x = \int_{\mathbb{R}} \phi(z,x)\,Z(x)\,\mathrm{d}x = 0, \quad z \in \mathcal{K}_\varepsilon. \tag{5.4}$$

Certainly, if ϕ satisfies (5.3)-(5.4), the functions c and d are given by

$$c(y) = -\frac{\int_{\mathbb{R}} h(z,x) w_x dx}{\int_{\mathbb{R}} w_x^2}, \quad d(y) = -\frac{\int_{\mathbb{R}} h(z,x) Z(x) dx}{\int_{\mathbb{R}} Z^2}. \tag{5.5}$$

Lemma 5.2. *Problem (5.3)-(5.4) possesses a unique solution, denoted by $\phi = T_0(h)$. Moreover, we have*

$$\|D^2\phi\|_{q,\sigma} + \|D\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma} \le C\|h\|_{q,\sigma}.$$

Proof. To this end let ϕ be a solution of (5.3)-(5.4). We observe that for the purpose of the a priori estimate we can assume that $c \equiv d \equiv 0$ in (5.3). We follow the proof of Proposition 4.1 in [15]. Since the arugment is similar, we just give a sketch.

We first claim that there exists a constant $C > 0$ such that for all small ε and every solution ϕ to Problem (5.3) with $\|\phi\|_{\infty,\sigma} < +\infty$ and right hand side h satisfying $\|h\|_{q,\sigma} < +\infty$ we have

$$\|D^2\phi\|_{2,q,\sigma} + \|D\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma} \le C\|g\|_{q,,\sigma}. \tag{5.6}$$

By local elliptic estimates and Sobolev embedding (since $q > n+1$), it is enough to show that

$$\|\phi\|_{\infty,\sigma} \le C\|g\|_{q,\sigma}. \tag{5.7}$$

Let us assume by contradiction that (5.7) does not hold. Then we have sequences $\varepsilon = \varepsilon_k \to 0$, h_k with $\|h_k\|_{q,\sigma} \to 0$, ϕ_k with $\|\phi_k\|_{\infty,\sigma} = 1$ such that

$$\begin{aligned} \partial_{xx}\phi_k + \beta^{-2}\Delta_{\mathcal{K}_\varepsilon}\phi_k - \phi_k + pw^{p-1}\phi_k &= h_k \quad \text{in } \mathcal{K}_\varepsilon \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi_k(z,x)\,w'(t)\,\mathrm{d}x &= 0 \;\; \forall z \in \mathcal{K}_\varepsilon \end{aligned}$$

By blowing up argument similar to [15], we arrive at a non-zero and bounded function $\tilde{\phi}(z,x)$ that satisfies

$$\tilde{\phi}_{xx} + \Delta_{\mathbb{R}^{n-1}}\tilde{\phi} + -1 + pw^{p-1}\tilde{\phi} = 0 \text{ in } \mathbb{R}^n$$

and

$$0 = \int_{\mathbb{R}} \tilde{\phi}(z,x)\, w_x = \int_{\mathbb{R}} \tilde{\phi}(z,x)\, Z(x).$$

We obtain a conclusion that $\tilde{\phi} = 0$ by following the same proof as in Lemma 4.1 of of [15] and using the inequality (2.7).

Thus we obtain a contradiction. This proves the a priori estimates. The existence follows exactly the same as in Proposition 1.1 of [15], replacing the orthogonality condition $\int_{\mathbb{R}} \phi w_x = \int_{\mathbb{R}} \phi Z = 0$.

□

Proof of Proposition 5.1. We will reduce Problem (5.1)-(5.2) to a small perturbation of a problem of the form (5.3)-(5.4), in which Lemma 5.2 is applicable. The key point is that the operator

$$B_8(\phi) = \eta^{\varepsilon}_{6\delta}\, B(\phi)\; -\; (1-\eta^{\varepsilon}_{6\delta}) p \mathcal{V}^{p-1}\phi + p(\mathcal{V}^{p-1} - w^{p-1})\phi$$

is small in the sense that

$$||B_8(\phi)||_{q,\sigma} \le C\delta ||\phi||_{q,\sigma}.$$

Hence, the results can be derived by the invertibility conclusion of Lemma 5.2 if we choose δ sufficiently small. □

6. Solving the Nonlinear Problem

In this section, we will solve (4.12)-(4.13) in $\mathfrak{S}$. Note that we have locally $(\alpha\beta^2)^{-1}\widetilde{E} = E_1$. A first elementary, but crucial observation is the following: the term

$$E_{11} = \varepsilon^3 \beta^{-2}\, \Delta_{\mathcal{K}}\, eZ + \varepsilon\lambda_0 eZ, \tag{6.1}$$

in the decomposition of E_1, has precisely the form dZ and can be absorbed in that term $\eta^{\varepsilon}_{\delta} dZ$. Then, the equivalent equation of (4.12) is

$$\mathcal{L}_2(\phi) = -\eta^{\varepsilon}_{\delta} E_{12} - \eta^{\varepsilon}_{\delta} N_2(\phi) + c\,\eta^{\varepsilon}_{\delta} w_x + d\,\eta^{\varepsilon}_{\delta} Z,$$

where we have denoted

$$N_2(\phi) = (\alpha\beta^2)^{-1}\Big[\, \widetilde{N}\big(\eta^{\varepsilon}_{3\delta}\phi + \psi(\phi)\big) + pW^{p-1}\psi(\phi)\Big].$$

Let T be the bounded operator defined by Proposition 5.1. Then the problem (4.12)-(4.13) is equivalent to the following fixed point problem

$$\phi = T\Big(-\eta^{\varepsilon}_{\delta} E_{12} - \eta^{\varepsilon}_{\delta} N_2(\phi)\Big) \equiv \mathcal{A}(\phi). \tag{6.2}$$

We collect some useful facts to find the domain of the operator $\mathcal{A}$ such that $\mathcal{A}$ becomes a contraction mapping. Firstly, the big difference between E_{11} and E_{12} is their sizes. From (3.66) and (3.67)

$$||E_{12}||_{q,\sigma} \le c_*\ \varepsilon^{k+\frac{1}{2}-\frac{n}{q}}, \tag{6.3}$$

while E_{11} is only of size $O(\varepsilon^{1/2})$. Secondly, the operator T has a useful property: assume $\hat{h}$ has a support contained in $|x| \le 20\delta/\varepsilon$, then $\phi = T(\hat{h})$ satisfies the estimate

$$\big|\phi(x,z)\big| + \big|D\phi(x,z)\big| \le ||\phi||_{L^\infty}\ e^{-2\delta/\varepsilon} \quad \text{for } |x| > 40\delta/\varepsilon. \tag{6.4}$$

For more details readers can refer to [13]. Thirdly, recall that the operator $\psi(\phi)$ satisfies, as seen directly from its definition

$$||\psi(\phi)||_{\infty,\sigma} \leq C\varepsilon\Big(\big|\big| \, |\phi| + |D\phi| \, \big|\big|_{L^\infty(|x|>20\delta/\varepsilon)} + e^{-\delta/\varepsilon} \Big), \tag{6.5}$$

and Lipschitz condition of the form

$$||\psi(\phi_1) - \psi(\phi_2)||_{\infty,\sigma} \leq C\varepsilon\Big(\big|\big| \, |\phi_1 - \phi_2| + |D(\phi_1 - \phi_2)| \, \big|\big|_{L^\infty(|x|>20\delta/\varepsilon)} \Big). \tag{6.6}$$

Whence, the facts above will allow us to construct a region where contraction mapping principle applies and then solve the problem (4.12)-(4.13).

Consider the following closed, bounded subset

$$\mathfrak{D} = \left\{ \phi \;\middle|\; \begin{array}{l} ||\phi||_{2,q,\sigma} \leq \tau\varepsilon^{k+\frac{1}{2}-\frac{n}{q}}, \\ \big|\big| \, |\phi| + |D\phi| \, \big|\big|_{L^\infty(|x|>40\delta/\varepsilon)} \leq ||\phi||_{2,q,\sigma} \; e^{-\delta/\varepsilon} \end{array} \right\}. \tag{6.7}$$

Lemma 6.1. *If the constant τ is sufficiently large, then the map $\mathcal{A}$ defined in (6.2) is a contraction from $\mathfrak{D}$ into itself such that problem (6.2) is solvable.*

Proof. Let us analyze the analytic character of the nonlinear operator involved in $\mathcal{A}$ with respect to functions in $\mathfrak{D}$

$$\begin{aligned} \eta_\delta^\varepsilon N_2(\phi) \; &= \; \eta_\delta^\varepsilon N_1(\phi + \psi(\phi)) \; + \; \eta_\delta^\varepsilon p W^{p-1}\psi(\phi) \\ &\equiv \; \bar{N}_2(\phi) \; + \; \eta_\delta^\varepsilon p W^{p-1}\psi(\phi). \end{aligned} \tag{6.8}$$

Note that $N_1(\varphi) = p\big[(W + t\varphi)^{p-1} - W^{p-1}\big]\varphi$ for some $t \in (0,1)$. thus

$$|N_1(\varphi)| \leq C|\varphi|^{\min(p,2)}.$$

Denoting $S_\delta = \mathfrak{S} \cap \{|x| < 10\delta/\varepsilon\}$, we have that for $\phi \in \mathfrak{D}$

$$||\bar{N}_2(\phi)||_{q,\sigma} \leq \begin{cases} C\Big[\, ||\phi||^p_{qp,\sigma} + ||\psi(\phi)||^p_{qp,\sigma;S_\delta} \Big], & p \leq 2; \\ \\ C\Big[\, ||\phi||^2_{2q,\sigma} + ||\psi(\phi)||^2_{2q,\sigma;S_\delta} \Big], & p \geq 2. \end{cases}$$

We may assume now $p \leq 2$ if we take the following form

$$||\bar{N}_2(\phi)||_{q,\sigma} \leq C\big[|||\phi||^p_{qp,\sigma} + ||\psi(\phi)||^p_{qp,S_\delta} + ||\phi||^2_{2q,\sigma} + ||\psi(\phi)||^2_{2p,\sigma;S_\delta} \; \big].$$

Using Sobolev's embedding, we derive that

$$||\phi||^p_{qp,\sigma} + ||\phi||^2_{2q,\sigma} \leq C\big(||\phi||^p_{2,q,\sigma} + ||\phi||^2_{2,q,\sigma} \; \big).$$

Using estimates (4.9), the facts that $\phi \in \mathfrak{D}$, (6.4), the area of S_δ is of order $O(\delta/\varepsilon)$ and Sobolev's embedding, we get

$$||\psi(\phi)||^p_{qp,\sigma;S_\delta} + ||\psi(\phi)||^2_{2q,\sigma;S_\delta} \leq Ce^{-\delta/4\varepsilon}\Big[\; 1 + ||\phi||^p_{2,q,\sigma} + ||\phi||^2_{2,q,\sigma} \; \Big].$$

Hence, from the properties of W and $\psi(\phi)$ we obtain

$$||\eta_\delta^\varepsilon \bar{N}_2(\phi)||_{qp,\sigma} \leq C(\varepsilon^{(k+\frac{1}{2}-\frac{n}{q})p}\tau^p + \varepsilon^{2k+1-\frac{2n}{q}}\tau^2). \tag{6.9}$$

As for Lipschitz condition, we find after a similar calculations

$$\begin{aligned} &||N_1(\varphi_1) - N_1(\varphi_2)||_{q,\sigma} \\ &\quad \leq \; C\Big[||\varphi_1||^{p-1}_{qp,\sigma} + ||\varphi_1||_{2q,\sigma} + ||\varphi_2||^{p-1}_{qp,\sigma} + ||\varphi_2||_{2q,\sigma}\Big] \\ &\qquad\quad \times \big(||\varphi_1 - \varphi_2||_{qp,\sigma} + ||\varphi_1 - \varphi_2||_{2q,\sigma}\big). \end{aligned}$$

Hence, there holds

$$\begin{aligned}
&||\bar{N}_2(\phi_1)-\bar{N}_2(\phi_2)||_{q,\sigma}\\
&\quad\le ||N_1(\phi_1+\psi(\phi_1))-N_1(\phi_2+\psi(\phi_1))||_{q,\sigma;S_\delta}\\
&\qquad+||N_1(\phi_2+\psi(\phi_1))-N_1(\phi_2+\psi(\phi_2))||_{q,\sigma;S_\delta}\\
&\quad\le Cv\Big(||\phi_1-\phi_2||_{qp,\sigma;S_\delta}+||\phi_1-\phi_2||_{qp,\sigma;S_\delta}\Big)\\
&\qquad+Cv\Big(||\psi(\phi_1)-\psi(\phi_2)||_{2q,\sigma;S_\delta}+||\psi(\phi_1)-\psi(\phi_2)||_{2q,\sigma;S_\delta}\Big),
\end{aligned}$$

where $v=v_1+v_2$ with

$$v_l=||\phi_l||^{p-1}_{qp,\sigma;S_\delta}+||\psi(\phi_l)||^{p-1}_{qp,\sigma;S_\delta}+||\phi_l||_{2q,\sigma;S_\delta}+||\psi(\phi_l)||_{2q,\sigma;S_\delta}.$$

Arguing as above and using the Lipschitz dependence of ψ on ϕ, it can be derived

$$\begin{aligned}
||\eta^\varepsilon_\delta N_2(\phi_1)-\eta^\varepsilon_\delta N_2(\phi_2)||_{q,\sigma}\ &\le C\big[\varepsilon^{(k+\frac12-\frac nq)(p-1)}\tau^{p-1}+\varepsilon^{(k+\frac12-\frac nq)}\tau\big]\\
&\qquad\times||\phi_1-\phi_2||_{2,q,\sigma}.
\end{aligned}\tag{6.10}$$

Now, we can find the solution of (6.2) in the sequel. Let $\phi\in\mathfrak{D}$ and $\nu=\mathcal{A}(\phi)$, then from (6.3) and (6.9)

$$||\nu||_{2,q,\sigma}\le||T_2||\cdot\Big[\ c_*\varepsilon^{k+\frac12-\frac nq}+C\tau^p\varepsilon^{(k+\frac12-\frac nq)p}+C\tau^2\varepsilon^{2k+1-\frac{2n}{q}}\ \Big].$$

Choosing any number $\tau>C_*||T||$, we get that for small ε

$$||\nu||_{2,q,\sigma}\le\tau\varepsilon^{k+\frac12-\frac nq}.$$

From (6.4)

$$\Big|\Big||\nu|+|D\nu|\Big|\Big|_{L^\infty(|x|>40\delta/\varepsilon)}\le||\nu||_\infty\ e^{\frac{-2\delta}{\varepsilon}}\le||\nu||_{2,q,\sigma}\ e^{\frac{-\delta}{\varepsilon}}.$$

Therefore, $\nu\in\mathfrak{D}$. $\mathcal{A}$ is clearly a contraction thanks to (6.10) and we can conclude that (6.2) has a unique solution in $\mathfrak{D}$. □

The error E_{12} and the operator T itself carry the functions f_{k-2} and e as parameters. For future reference, we should consider their Lipschitz dependence on these parameters. (3.68) is just the formula about the Lipschitz dependence of error E_{12} on these two parameters. The other task can be realized by careful and direct computations of all terms involved in the differential operator which will show this dependence is indeed Lipschitz with respect to the $\|\cdot\|_{2,q,\sigma}$-norm (for all ε).

Within the operator, consider for instance the following term involving $\triangle_{\mathcal{K}}f_{k-2}$ as the form

$$Q_f(\phi)=\varepsilon^k\,\triangle_{\mathcal{K}}\,f_{k-2}\phi_x.$$

Then we have

$$||Q_f(\phi)||^q_{L^q(B)}\le\varepsilon^{qk+1-n}\int_{\mathcal{K}}|\triangle_{\mathcal{K}}f|^q\left(\sup_z\int_{\mathbb{R}}|\phi_x(z,x)|^q\mathrm{d}x\right).$$

Let $\mu(z)=\int_{\mathbb{R}}|\phi_x(z,x)|^q\mathrm{d}x$. Then there holds

$$\begin{aligned}
\sup_z\mu(z)\ &\le\ \frac12\varepsilon^{n-1}\int_{\mathfrak{S}}|\phi_x|^q\mathrm{d}x+C\int_{\mathfrak{S}}|\phi_x|^{\mathbf{q-1}}|\,\nabla_{\mathcal{K}}\,\phi_x|\mathrm{d}x\\
&\le\ \frac12\sup_z\mu(z)+\frac{C}{\varepsilon^{n-1}}\int_{\mathfrak{S}}|\,\nabla_{\mathcal{K}}\,\phi_x|^q\mathrm{d}x,
\end{aligned}$$

which implies that

$$\mu(z) \leq C\varepsilon^{1-n}||\phi||^q_{2,q,\sigma}.$$

Therefore,

$$||Q_{f_{k-2}}(\phi)||_{q,\sigma} \;\leq\; C\varepsilon^{2k+\frac{1}{2}-\frac{3n-2}{q}}||f_{k-2}||_a \leq C\varepsilon||f_{k-2}||_a,$$

provided that $k \geq n$.

Similar estimates can be applied to other terms in the operator involving $\varepsilon^k \triangle_{\mathcal{K}} f_{k-2}$.

For the linear operator T, we have the following Lipschitz dependence

$$||T(f_{k-2}) - T(\tilde{f}_{k-2})||_{2,q,\sigma} \leq C\varepsilon^{-1}||f_{k-2} - \tilde{f}_{k-2}||_a.$$

Moreover, the operator N_2 also has Lipschitz dependence on (f_{k-2}, e). It is easily checked that for $\phi \in \mathfrak{D}$ we have, with obvious notation

$$\left\|\eta^\varepsilon_\delta N_{2,(f_{k-2},e)}(\phi) - \eta^\varepsilon_\delta N_{2,(\tilde{f}_{k-2},\tilde{e})}(\phi)\right\|_{q,\sigma} \;\leq\; C\varepsilon^{k+\frac{1}{2}-\frac{n}{q}}\Big[\; ||f_{k-2} - \tilde{f}_{k-2}||_a + ||e-\tilde{e}||_b \;\Big].$$

Hence, from the fixed point characterization we get that

$$(6.11)\qquad \begin{aligned} &\left\|\phi(f_{k-2},e) - \phi(\tilde{f}_{k-2},\tilde{e})\right\|_{2,q,\sigma} \\ &\quad \leq C\varepsilon^{k+\frac{1}{2}-\frac{n}{q}}\Big[\; \left\|f_{k-2} - \tilde{f}_{k-2}\right\|_a + \left\|e-\tilde{e}\right\|_b \;\Big]. \end{aligned}$$

The conclusion of this section reads

Proposition 6.2. *There is a number $\tau > 0$ such that for all ε small enough and all parameters (f_{k-2}, e) in $\digamma$, problem (4.12)-(4.13) has a unique solution of the form*

$$(\phi, c, d) = \big(\phi(f_{k-2},e), c(f_{k-2},e), d(f_{k-2},e)\big),$$

which satisfies

$$||\phi||_{2,q,\sigma} \leq \tau\varepsilon^{k+\frac{1}{2}-\frac{n}{q}},$$

$$\Big|\Big|\; |\phi| + |D\phi| \;\Big|\Big|_{L^\infty(|x|>40\delta/\varepsilon)} \leq ||\phi||_{2,q,\sigma}e^{-\delta/\varepsilon}.$$

Moreover, the function $\phi(f_{k-2}, e)$ depends Lipschitz-continuously on the unknown parameters f_{k-2} and e in the sense of the estimate (6.11). □

Next we carry out the second part of the program which is to set up equations for f_{k-2} and e which are equivalent to making c, d identically zero. These equations are obtained by simply integrating the equation (4.12)(only in x) against respectively w_x and Z. It is therefore of crucial importance to carry out computations of the terms $\int_{\mathbb{R}} E_1 w_x \mathrm{d}x$ and $\int_{\mathbb{R}} E_1 Z \mathrm{d}x$. We do that in the next section.

7. The Nonlinear System

Clearly Proposition 6.2 and the gluing procedure in Section 4 yield a solution to our original problem (1.1) if we can find f_{k-2} and e such that

$$(7.1)\qquad c(f_{k-2}, e) \;=\; d(f_{k-2}, e) \;=\; 0.$$

As we will see this leads to a system of nonlinear partial differential equations. It is easy to see that the identities (7.1) is equivalent to the following equations

$$
\begin{aligned}
&\int_{\mathbb{R}} (\alpha\beta^2)^{-1}\eta_\delta^\varepsilon \Big[\widetilde{N}\big(\eta_{3\delta}^\varepsilon\phi+\psi(\phi)\big)+\widetilde{E}+pW^{p-1}\psi(\phi)\Big]w_x\mathrm{d}x \\
&\qquad\qquad\qquad\qquad + \int_{\mathbb{R}} \mathcal{L}(\phi)w_x\,\mathrm{d}x \,=\, 0,
\end{aligned}
\tag{7.2}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}} (\alpha\beta^2)^{-1}\eta_\delta^\varepsilon \Big[\widetilde{N}\big(\eta_{3\delta}^\varepsilon\phi+\psi(\phi)\big)+\widetilde{E}+pW^{p-1}\psi(\phi)\Big]Z\mathrm{d}x \\
&\qquad\qquad\qquad\qquad + \int_{\mathbb{R}} \mathcal{L}(\phi)Z\,\mathrm{d}x \,=\, 0.
\end{aligned}
\tag{7.3}
$$

Since the cut-off function η_δ^ε appears in (7.2) and (7.3), and the facts that w_x and Z have exponential decay, we have that $(\alpha\beta^2)^{-1}\eta_\delta^\varepsilon\widetilde{E}=E_1$ with the error term E_1 defined in (3.59). Whence, it is crucial to estimate the terms

$$
\int_{\mathbb{R}} E_1 w_x \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}} E_1 Z \mathrm{d}x.
$$

The same arguments can be applied to other terms in (7.2) and (7.3). Now, we divide the estimates for the components in (7.2) and (7.3) into three parts.

7.1. **Part I.** First, multiplying (3.59) by w_x and integrating over the variable x, using the decomposition of E_1 in (3.64) and the fact that w_x is an odd function in x, we obtain

$$
\int_{\mathbb{R}} E_1 w_x\,\mathrm{d}x \,=\, \int_{\mathbb{R}} E_{12} w_x\,\mathrm{d}x.
$$

More precisely, there holds

$$
\begin{aligned}
&\int_{\mathbb{R}} E_{12}w_x\mathrm{d}x \\
&= \varepsilon^k \int_{\mathbb{R}} \Big[\hat{\mathbf{A}}_k + \frac{1}{2}p(p-1)w^{p-2}\phi_1\phi_{k-1} + \frac{H}{\beta}\phi_{k-1,x} - \frac{V_t}{\beta^3}x\phi_{k-1} - \frac{V_t}{\beta^2}f_{k-2}\phi_1\Big]w_x\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \Big[\frac{V_t}{\beta^2}f_0\phi_{k-1} - \hat{\mathcal{G}}_k\Big]w_x\mathrm{d}x + \varepsilon^k p(p-1)\int_{\mathbb{R}} w^{p-2}eZ\phi_{k-1}w_x\mathrm{d}x + \int_{\mathbb{R}} \mathfrak{B}_0 w_x\mathrm{d}x \\
&\quad + \varepsilon^{k+1}\int_{\mathbb{R}} \Big[\mathbf{A}_{k+1} + \mathbf{B}_{k+1} + \mathbf{C}_{k+1} + \mathcal{H}_{k+1} + \mathfrak{F}_{k+1}\Big]w_x\mathrm{d}x.
\end{aligned}
$$

Using the same arguments as (3.51), we derive that

$$
\begin{aligned}
\int_{\mathbb{R}} E_{12}w_x\mathrm{d}x = &-\varepsilon^k\frac{\delta_1}{\beta}\Big[\,\triangle_{\mathcal{K}}\, f_{k-2} + \gamma_1\,\nabla_{\mathcal{K}}\,V\cdot\nabla_{\mathcal{K}}f_{k-2} + \gamma_2 f_{k-2}\,\Big] \\
&+ \gamma_3\varepsilon^k e + \gamma_4\varepsilon^{k+2}\,\triangle_{\mathcal{K}}\,e + \varepsilon^{k+1}b_{1\varepsilon}\,\triangle_{\mathcal{K}}\,e \\
&+ \varepsilon^{k+1}b_{2\varepsilon}\,\triangle_{\mathcal{K}}\,f_{k-2} + \varepsilon^{k+1}b_{3\varepsilon},
\end{aligned}
\tag{7.4}
$$

where γ_3,γ_4 are two constants,

$$
\gamma_1 = \frac{\sigma}{V}, \qquad \gamma_2 = \big(\,|A_{\mathcal{K}}|^2 + \sigma^{-1}H^2 - \frac{\sigma}{V}V_{tt}\,\big).
\tag{7.5}
$$

Here and below we denote by $b_{l\varepsilon}, l=1,2,3$, generic, uniformly bounded continuous functions of the form

$$
b_{l\varepsilon} \,=\, b_{l\varepsilon}(f_{k-2}, e, \nabla f_{k-2}, \nabla e),
$$

where additionally $b_{l\varepsilon}$ is uniformly Lipschitz in its arguments.

7.2. **Part II.** We will estimate other terms that involve ϕ in (7.2),

$$\int_{\mathbb{R}} (\alpha\beta^2)^{-1}\eta_\delta^\varepsilon\Big[\tilde{N}\big(\eta_{3\delta}^\varepsilon\phi + \psi(\phi)\big) + pW^{p-1}\psi(\phi)\Big]w_x \mathrm{d}x + \int_{\mathbb{R}} \mathcal{L}(\phi)w_x\,\mathrm{d}x. \tag{7.6}$$

Using the orthogonality condition (4.13) and the definition of $\mathcal{L}$ in (4.11), the main components of the last term in (7.6) are

$$\int_{\mathbb{R}} B(\phi)w_x \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}} p\big(\mathcal{V}^{p-1} - w^{p-1}\big)\phi w_x \mathrm{d}x.$$

Here we recalled the definitions of the operator B in (2.19) and the local approximation $\mathcal{V}$ in (3.1).

Let $\Upsilon_1(\varepsilon z) = \int_{\mathbb{R}} B(\phi)w_x \mathrm{d}x$. We make the following observation: all terms in $B(\phi)$ carry ε and involve powers of x times derivatives of $0, 1$ or two orders of ϕ. The conclusion is that since w_x has exponential decay then

$$\int_{\mathcal{K}} |\Upsilon_1(\theta)|^q \mathrm{d}\tilde{y} \leq C\varepsilon^{n+1}\|\phi\|_{2,q,\sigma}^q.$$

Hence there holds

$$\|\Upsilon_1\|_{L^q(\mathcal{K})} \leq C\varepsilon^{k+\frac{1}{2}+\frac{1}{q}}.$$

In $B(\phi)$ we single out two less regular terms. The one whose coefficient depends on $\Delta_{\mathcal{K}} f_{k-2}$ explicitly has the form

$$\begin{aligned}\Upsilon_{1*} &= \varepsilon^k \Delta_{\mathcal{K}} f_{k-2} \int_{\mathbb{R}} \phi_x Z\left(1 + H\varepsilon\beta\Big(x - \sum_{l=0}^{k-2}\varepsilon^l f_l\Big)\right)^{-2} \\ &= -\varepsilon^k \Delta_{\mathcal{K}} f_{k-2} \int_{\mathbb{R}} \phi\left\{Z\Big(1 + H\varepsilon\beta\big(x - \sum_{l=0}^{k-2}\varepsilon^l f_l\big)\Big)^{-2}\right\}_x.\end{aligned}$$

Since ϕ has Lipschitz dependence on (f_{k-2}, e) in the form (6.11), we see that

$$\begin{aligned}&\|\Upsilon_{1*}(f_{k-2}, e) - \Upsilon_{1*}(\tilde{f}_{k-2}, \tilde{e})\|_{L^q(\mathcal{K})} \\ &\quad\leq C\varepsilon^{2k+\frac{1}{2}-\frac{n}{q}}\big(\|f_{k-2} - \tilde{f}_{k-2}\|_a + \|e - \tilde{e}\|_b\big).\end{aligned} \tag{7.7}$$

The other arising from second derivative in z for ϕ is

$$\Upsilon_{1**} = \int_{\mathbb{R}} \Delta_{\mathcal{K}_\varepsilon}\phi Z\left[1 - \Big(1 + H\varepsilon\beta\big(x - \sum_{l=0}^{k-2}\varepsilon^l f_l\big)\Big)^{-2}\right]\mathrm{d}x.$$

We readily see that

$$\begin{aligned}&\|\Upsilon_{1**}(f_{k-2}, e) - \Upsilon_{1**}(\tilde{f}_{k-2}, \tilde{e})\|_{L^q(\mathcal{K})} \\ &\quad\leq C\varepsilon^{k+\frac{1}{2}+\frac{1}{q}}\big(\|f_{k-2} - \tilde{f}_{k-2}\|_a + \|e - \tilde{e}\|_b\big).\end{aligned} \tag{7.8}$$

The remainder $\Upsilon_1 - \Upsilon_{1*} - \Upsilon_{1**}$ actually defines for fixed ε a compact operator of the pair (f_{k-2}, e) into $L^q(\mathcal{K})$. This is a consequence of the fact that weak convergence in $W^{2,q}(\mathcal{K})$ implies local strong convergence in $W^{1,q}(\mathcal{K})$. If $f_{k-2,j}$ and e_j are weakly convergent sequences in $W^{2,q}(\mathcal{K})$ then clearly the functions $\phi(f_{k-2,j}, e_j)$ constitute a bounded sequence in $W^{1,q}(\mathcal{K})$. In the above remainder one can integrate by parts if necessary once in x. Averaging against w_x which decays exponentially localizes the situation and the desired fact follows.

Let us consider now the term

$$\Upsilon_2(\varepsilon z) = \int_{\mathbb{R}} p\big(\mathcal{V}^{p-1} - w^{p-1}\big)\phi w_x \mathrm{d}x.$$

Since the term $\varepsilon e(\varepsilon z)Z(x) + \sum_{l=1}^{k-1} \varepsilon^l \phi_l(\varepsilon z, x)$ can be estimated as

$$\varepsilon |e(\varepsilon z)Z(x)| + \sum_{l=1}^{k-1} |\varepsilon^l \phi_l(\varepsilon z, x)| \le C\varepsilon(1+|x|^2)e^{-|x|},$$

we easily see that for some $\tau > 0$ the uniform bound holds

$$|\mathcal{V}^{p-1} - w^{p-1}| \cdot |w_x| \le C\varepsilon e^{-\tau|x|}.$$

From here we readily find that

$$\|\Upsilon_2\|_{L^q(\mathcal{K})} \;\le\; C\varepsilon^{\frac{n+1}{q}} \|\phi\|_{2,q,\sigma} \;\le\; C\varepsilon^{k+\frac{1}{2}+\frac{1}{q}}.$$

We observe also that other terms in (7.6) such as

$$\Upsilon_3(\varepsilon z) = \int_{\mathbb{R}} \widetilde{N}\big(\eta^{\varepsilon}_{3\delta}\phi + \psi(\phi)\big) w_x \mathrm{d}x,$$

can be estimated similarly. In fact, using the definition of $\widetilde{N}\big(\eta^{\varepsilon}_{3\delta}\phi + \psi(\phi)\big)$ and the exponential decay of w_x we obtain

$$\|\Upsilon_3\|_{L^q(\mathcal{K})} \le C\|\phi\|^2_{2,q,\sigma} \le C\varepsilon^{2k+1-\frac{2n}{q}}.$$

These terms define compact operators similarly as before.

7.3. **Part III.** We observe that exactly the same estimates can be carried out in the terms obtained from integration against Z. So the remaining thing is to compute the term $\int_{\mathbb{R}} E_1 Z \mathrm{d}x$.

Multiplying (3.59) by Z and integrating over the variable x and using the decomposition of E_1 in (3.63), we get

$$\int_{\mathbb{R}} E_1 Z \mathrm{d}x = \int_{\mathbb{R}} E_{11} Z \mathrm{d}x + \int_{\mathbb{R}} E_{12} Z \mathrm{d}x,$$

where

$$\int_{\mathbb{R}} E_{11} Z \mathrm{d}x \;=\; \varepsilon(\varepsilon^2\beta^{-2} \bigtriangleup_{\mathcal{K}} e + \lambda_0 e)\int_{\mathbb{R}} Z^2 \,\mathrm{d}x \;=\; \varepsilon^3\beta^{-2} \bigtriangleup_{\mathcal{K}} e + \varepsilon\lambda_0 e.$$

On the other hand, we have

$$\begin{aligned}
&\int_{\mathbb{R}} E_{12} Z \mathrm{d}x \\
&= \varepsilon^k \int_{\mathbb{R}} \Big[\check{\mathbf{A}}_k + \frac{1}{2}p(p-1)w^{p-2}\phi_1\phi_{k-1} + \frac{H}{\beta}\phi_{k-1,x} - \frac{V_t}{\beta^3}x\phi_{k-1} - \frac{V_t}{\beta^2}f_{k-2}\phi_1\Big] Z\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \Big[\frac{V_t}{\beta^2}f_0\phi_{k-1} - \hat{\mathcal{G}}_k\Big] Z\mathrm{d}x + \varepsilon^k p(p-1)\int_{\mathbb{R}} w^{p-2} e Z \phi_{k-1} Z \mathrm{d}x + \int_{\mathbb{R}} \mathfrak{B}_0 Z \mathrm{d}x \\
&\quad + \varepsilon^{k+1}\int_{\mathbb{R}} \Big[\mathbf{A}_{k+1} + \mathbf{B}_{k+1} + \mathbf{C}_{k+1} + \mathcal{H}_{k+1} + \mathfrak{F}_{k+1}\Big] Z\mathrm{d}x.
\end{aligned}$$

The components in $\check{\mathcal{G}}_k$ are even functions in the variable x and independent of the parameters f_{k-2}. Here, the function ϕ_{k-1} has the form

$$\phi_{k-1}(z,x) = \psi_{k-1,1}(\varepsilon z, x) + \psi_{k-1,2}(\varepsilon z, x) + \phi_{k-1,2}, \tag{7.9}$$

where

$$\phi_{k-1,2} \;=\; f_{k-2}a_{12}w_2 \;=\; f_{k-2}\sigma^{-1}Hw_2. \tag{7.10}$$

The reader can refer (3.34) for the definition of a_{12} and w_2. The components in $\psi_{k-1,1}$ and $\psi_{k-1,2}$ are independent of the parameters f_{k-2}. Moreover, $\psi_{k-1,1}$ is an odd function in the variable x and $\psi_{k-1,2}$ is an even function in the variable x. Therefore, adding up all terms together, we conclude that

$$\begin{aligned}\int_{\mathbb{R}} E_{12}Z\mathrm{d}x \;=&\varepsilon^{k+1}b^1_{1\varepsilon}\,\triangle_{\mathcal{K}}\, e+\varepsilon^{k+1}b^2_{1\varepsilon}\,\triangle_{\mathcal{K}}\, f_{k-2}\\ &+\varepsilon^{k+1}b_{2\varepsilon}(f_{k-2},e,\bigtriangledown f_{k-2},\bigtriangledown e).\end{aligned}$$

As a consequence, we give the following proposition.

Proposition 7.1. *The condition (7.1) is equivalent to the following system of differential equations*

$$\begin{aligned}&\varepsilon^k\frac{\delta_1}{\beta}\Big[\,\triangle_{\mathcal{K}}\, f_{k-2}+\gamma_1\,\nabla_{\mathcal{K}}\, V\cdot\nabla_{\mathcal{K}}f_{k-2}+\gamma_2f_{k-2}\Big]\\ &\qquad\qquad=\;\gamma_3\varepsilon^ke+\gamma_4\varepsilon^{k+2}\,\triangle_{\mathcal{K}}\, e+\varepsilon^kM_1(f_{k-2},e),\end{aligned} \tag{7.11}$$

$$\varepsilon^3\beta^{-2}\,\triangle_{\mathcal{K}}\, e+\varepsilon\lambda_0e+\varepsilon M_2(f_{k-2},e)\;=\;0. \tag{7.12}$$

with the estimates

$$\big\|M_1(f_{k-2},e)\big\|_{L^q(\mathcal{K})}\le C\varepsilon^{\frac12+\frac1q},\quad \big\|M_2(f_{k-2},e)\big\|_{L^q(\mathcal{K})}\;\le\; C\varepsilon^{k-1}. \tag{7.13}$$

Moreover, the functions M_1 and M_2 are Lipschitz functions of their arguments

$$\Big\|M_1(f_{k-2},e)-M_1(\tilde f_{k-2},\tilde e)\Big\|_{L^q(\mathcal{K})}\;\le\; C\varepsilon^{\frac12+\frac1q}\Big(\big\|f_{k-2}-\tilde f_{k-2}\big\|_a+\big\|e-\tilde e\big\|_b\Big),$$

$$\Big\|M_2(f_{k-2},e)-M_2(\tilde f_{k-2},\tilde e)\Big\|_{L^q(\mathcal{K})}\;\le\; C\varepsilon^{k-1}\Big(\big\|f_{k-2}-\tilde f_{k-2}\big\|_a+\big\|e-\tilde e\big\|_b\Big).$$

□

Finally, we make a conclusion that (7.2)-(7.3) is equivalent to the following nonlinear, nonlocal system of differential equations for the parameters (f_{k-2},e) in the variable $\theta=\varepsilon z$

$$\begin{aligned}\mathcal{L}^*_1(f_{k-2})\;\equiv&\;\triangle_{\mathcal{K}}\, f_{k-2}+\gamma_1\,\nabla_{\mathcal{K}}\, V\cdot\nabla_{\mathcal{K}}f_{k-2}+\gamma_2f_{k-2}\\ =&\;\gamma_3e+\varepsilon^2\,\triangle_{\mathcal{K}}\, e+M_{1\varepsilon}\quad\text{on }\mathcal{K},\end{aligned} \tag{7.14}$$

$$\mathcal{L}^*_{2\varepsilon}(e)\;\equiv\;-\,\varepsilon^2\,\triangle_{\mathcal{K}}\, e-\lambda_0\beta^2e\;=\;M_{2\varepsilon}\quad\text{on }\mathcal{K}, \tag{7.15}$$

the functions γ_1 and γ_2 are defined in (7.5). The operators $M_{1\varepsilon}$ and $M_{2\varepsilon}$ can be decomposed in the following form

$$M_{l\varepsilon}(f_{k-2},e)=A_{l\varepsilon}(f_{k-2},e)+K_{l\varepsilon}(f_{k-2},e),\quad l=1,2,$$

where $K_{l\varepsilon}$ is uniformly bounded in $L^q(\mathcal{K})$ for (f_{k-2},e) in $\mathcal{F}$ and is also compact. The operator $A_{l\varepsilon}$ is Lipschitz in this region, see (7.7)-(7.8),

$$\begin{aligned}&||A_{l\varepsilon}(f_{k-2},e)-A_{l\varepsilon}(\tilde f_{k-2},\tilde e)||_{L^q(\mathcal{K})}\\ &\quad\le\; C\varepsilon^{\frac12+\frac1q}\big[\,||f_{k-2}-\tilde f_{k-2}||_a+||e-\tilde e||_b\,\big].\end{aligned} \tag{7.16}$$

8. Location of concentration set

For the resolution of the system (7.14)-(7.15), we shall consider the invertibility of the operators $\mathcal{L}_1^*$ and $\mathcal{L}_{2\varepsilon}^*$ on $\mathcal{K}$. The first observation is that, by using of non-degeneracy condition (1.16), if $h_1 \in L^q(\mathcal{K})$ then there is a unique solution $f_{k-2} \in W^{2,q}(\mathcal{K})$ of $\mathcal{L}_1^*(f_{k-2}) = h_1$ with the property

$$||f_{k-2}||_{L^\infty(\mathcal{K})} + ||\nabla_{\mathcal{K}} f_{k-2}||_{L^\infty(\mathcal{K})} + ||\Delta_{\mathcal{K}} f_{k-2}||_{L^q(\mathcal{K})} \leq C||h_1||_{L^q(\mathcal{K})}.$$

We now deal with the invertibility theory of $\mathcal{L}_{2\varepsilon}^*$, which is stated as

Proposition 8.1. *If $h_2 \in L^q(\mathcal{K})$ then there exists a sequence $(\varepsilon_\ell)_\ell$ such that the problem*

$$\mathcal{L}_{2\varepsilon_\ell}^*(e) = h_2 \quad on\ \mathcal{K}, \tag{8.1}$$

has a unique solution $e \in W^{2,q}(\mathcal{K})$ with the property

$$||e||_{L^\infty(\mathcal{K})} + \varepsilon_\ell||\nabla_{\mathcal{K}} e||_{L^\infty(\mathcal{K})} + \varepsilon_\ell^2||\Delta_{\mathcal{K}} e||_{L^q(\mathcal{K})} \leq C\varepsilon_\ell^{-(n-1)}||h_2||_{L^q(\mathcal{K})}. \tag{8.2}$$

Using Proposition 8.1, the proof of Theorem 1.1 is almost the same as that of Theorem 1.1 in [13]. For completeness we sketch the proof.

Proof of Theorem 1.1: Let us observe now that the linear operator

$$\mathbb{L}(f_{k-2}, e) = \big(\mathcal{L}_1^*(f_{k-2}) - \gamma_3 e - \varepsilon^2\gamma_4 e'',\ \mathcal{L}_{2\varepsilon}^*(e)\big),$$

is invertible with bounds for $\mathcal{L}(f_{k-2}, e) = (g, d)$ given by

$$\|f\|_a + \|e\|_b \leq C\|g\|_{L^q(\mathcal{K})} + \varepsilon^{-1}\|d\|_{L^q(\mathcal{K})}.$$

It then follows from contraction mapping principle that the problem

$$\big[\mathbb{L} + (\varepsilon A_{1\varepsilon}, \varepsilon A_{2\varepsilon})\big](f_{k-2}, e) = (g, d)$$

is uniquely solvable for f_{k-2}, e satisfying (2.13) and (3.2) provided that

$$\|g\|_{L^q(\mathcal{K})} < \varepsilon^{\frac{1}{2}+\rho}, \|d\|_{L^q(\mathcal{K})} < \varepsilon^{\frac{3}{2}+\rho},$$

for some $\rho > 0$. The desired result for the full problem (7.14)-(7.15) then follows directly from Schauder's fixed point theorem. In fact, refining the fixed point region, we can actually get $\|e\|_b + \|f\|_a = O(\varepsilon^{\frac{1}{2}+\frac{1}{q}})$ for the solution. The location of the concentration set is settled down, which completes the proof of Theorem 1.1. □

It remains to prove Proposition 8.1. We follow the method introduced in [15], which relies only on elementary considerations on the variational characterization of the eigenvalues of the operator $\mathcal{L}_{2\varepsilon}^*$ and the Weyl's asymptotic formula in (2.8). We remark this approach is a slightly different from [24]-[25] where Kato's theorems were the main tools.

First, we consider the following eigenvalue problem

$$\mathcal{L}_{2\varepsilon}^*(v) = -\varepsilon^2 \Delta_{\mathcal{K}} v - \lambda_0\beta^2 v = \Lambda_\varepsilon\beta^2 v \quad \text{on } \mathcal{K}, \tag{8.3}$$

where the weighted function β^2 is defined in (2.14) with explicit form

$$\beta(\theta) = V(\theta, 0)^{1/2} \quad \text{for any } \theta \in \mathcal{K}.$$

Since $\mathcal{K}$ is a compact Riemannian manifold, by using the assumption (1.2), we can choose certain positive numbers γ_+ and γ_- such that

$$\gamma_- \leq \beta^2(\theta)\lambda_0 \leq \gamma_+ \quad \text{for any } \theta \in \mathcal{K}.$$

We denote its eigenvalues $\Lambda_{\varepsilon,j}$ in non-decreasing order and counting them with multiplicity. Here λ_0 is the unique positive eigenvalue to the eigenvalue problem (2.1), which implies the spectrum of $\mathcal{L}^*_{2\varepsilon}$ contains negative or zero eigenvalues. From the Courant-Fisher characterization we can write $\Lambda_{\varepsilon,j}$ in two different ways:

$$\Lambda_{\varepsilon,j} = \sup_{\Xi\,\in\,\Xi_{j-1}} \left[\inf_{v\perp\Xi, v\neq 0} \frac{\int_{\mathcal{K}} v\mathcal{L}^*_{2\varepsilon}v}{\int_{\mathcal{K}}\beta^2 v^2} \right], \tag{8.4}$$

$$\Lambda_{\varepsilon,j} = \inf_{\Xi\,\in\,\Xi_{j}} \left[\sup_{v\,\in\,\Xi, v\neq 0} \frac{\int_{\mathcal{K}} v\mathcal{L}^*_{2\varepsilon}v}{\int_{\mathcal{K}}\beta^2 v^2} \right]. \tag{8.5}$$

Here Ξ_j (resp.Ξ_{j-1}) represents the family of j dimensional (resp. $j-1$ dimensional) subspaces of $H^2(\mathcal{K})$, and the symbol $\perp$ denotes orthogonality with respect to the L^2 scalar product. There holds the following result for the estimates of gap between two successive eigenvalues.

Lemma 8.2. *There exits a number $\varepsilon_0 > 0$ such that for all $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$ and all $j \geq 1$ the following estimate holds.*

$$\Lambda_{\varepsilon_1,j} = \frac{\varepsilon_1^2}{\varepsilon_2^1}\Lambda_{\varepsilon_2,j} - \lambda_0\Big(1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}\Big). \tag{8.6}$$

In particular, the functions $\varepsilon \in (0, \varepsilon_0) \mapsto \Lambda_{\varepsilon,j}$ are continuous and increasing.

Proof. Let us consider small numbers $0 < \varepsilon_1 < \varepsilon_2$. We observe that for any v with $\int_{\mathcal{K}}\beta^2 v^2 = 1$, we have

$$\varepsilon_2^{-2}\int_{\mathcal{K}} v\mathcal{L}^*_{2\varepsilon_2}v - \varepsilon_1^{-2}\int_{\mathcal{K}} v\mathcal{L}^*_{2\varepsilon_1}v = (\varepsilon_1^{-2} - \varepsilon_2^{-2})\lambda_0\int_{\mathcal{K}}\beta^2 v^2 = \lambda_0(\varepsilon_1^{-2} - \varepsilon_2^{-2}).$$

Then the result follows. $\square$

Proof of Proposition 8.1: For $\ell \in \mathbb{N}$, choose $\sigma_\ell = 2^{-\ell}$. In order to find a sequence of values $\varepsilon_\ell \in (\sigma_{\ell+1}, \sigma_\ell)$ such that the spectrums of the operators $\mathcal{L}^*_{2\varepsilon_\ell}$, for large ℓ, stay away from 0, we define

$$\digamma^1_\ell = \{\varepsilon \in (\sigma_{\ell+1}, \sigma_\ell) : \ \ker\mathcal{L}^*_{2\varepsilon} \neq \varnothing\}, \qquad \digamma^2_\ell = (\sigma_{\ell+1}, \sigma_\ell) \setminus \digamma^1_\ell.$$

It is crucial to estimate the cardinality of $\digamma^1_\ell$. If $\varepsilon \in \digamma^1_\ell$ then for some j we have that $\Lambda_{\varepsilon,j} = 0$. The monotonicity of the function $\varepsilon \in (0, \varepsilon_0) \mapsto \Lambda_{\varepsilon,j}$ implies that $\Lambda_{\sigma_{\ell+1},j} < 0$. Hence,

$$\mathrm{card}(\digamma^1_\ell) \leq N_{\sigma_{\ell+1}}, \tag{8.7}$$

where $\mathbf{N}_\varepsilon$ is the number of negative eigenvalues of the operator $\mathcal{L}^*_{2\varepsilon}$.

We now give an asymptotic estimate on the number $\mathbf{N}_\varepsilon$ of negative eigenvalues of the differential operator $\mathcal{L}^*_{2\varepsilon}$. By $(\rho_i)_i$ we will denote the set of eigenvalues of the eigenvalue problem

$$-\,\triangle_{\mathcal{K}}\,\omega = \rho\beta^2\omega \quad \text{on } \mathcal{K}.$$

From the Weyl asymptotic formula as in (2.8) and the formula in (8.5), one derives

$$\mathbf{N}_\varepsilon \geq C_{\mathcal{K}}\big(1 + o(1)\big)\varepsilon^{-(n-1)},$$

where $C_{\mathcal{K}}$ is a fixed constant depending on the volume of the manifold $\mathcal{K}$ and its dimension. To prove a similar upper bound, we choose i to be the first index such that $\varepsilon^2\rho_i - \lambda_0 > 0$. Then from the Weyl formula we find that

$$i = C_{\mathcal{K}}\big(1 + o(1)\big)\varepsilon^{-(n-1)}.$$

Define $\Xi_{j-1} = \text{span}\{\omega_\ell : l = 1, 2, \cdots, j-1\}$. For an arbitrary function $v \in H^2(\mathcal{K})$ and $v \perp \Xi_{j-1}$, we can write

$$v = \sum_{l \geq j} \kappa_\ell \varphi_\ell.$$

Plugging this v into (8.4) and using the Weyl formula, we also have

$$\mathbf{N}_\varepsilon \ \leq \ C_{\mathcal{K}}\big(1 + o(1)\big)\varepsilon^{-(n-1)}.$$

Hence we get that

$$\mathbf{N}_\varepsilon \sim C_{\mathcal{K}}\varepsilon^{-(n-1)} \quad \text{as } \varepsilon \to 0.$$

The last inequality and (8.7) imply that $\text{card}(F_\ell^1) \leq C{\sigma_\ell}^{-(n-1)}$, and hence there exists an interval (a_ℓ, b_ℓ) such that

$$(a_\ell, b_\ell) \subset F_\ell^2, \quad |b_\ell - a_\ell| \geq \frac{\text{meas}(F_\ell^2)}{\text{card}(F_\ell^1)} \geq 2C_0\sigma_\ell^n, \tag{8.8}$$

for a universal positive constant C_0, independent of ℓ. By setting $\varepsilon_\ell = (a_\ell + b_\ell)/2$ for all large $\ell \in \mathbb{N}$, we conclude that $\mathcal{L}^*_{2\varepsilon_\ell}$ is invertible and there exists a number $C > 0$, independent of ℓ, such that for all $j \in \mathbb{N}$ there holds

$$\big|\Lambda_{\varepsilon_\ell, j}\big| \ \geq \ C\varepsilon_\ell^{n-1}. \tag{8.9}$$

Assume the opposite, namely that for some j we have

$$\big|\Lambda_{\varepsilon_\ell, j}\big| \ < \ \delta\varepsilon_\ell^{n-1},$$

with δ arbitrarily small. Since $\varepsilon_\ell \in F_\ell^2$, then $\big|\Lambda_{\varepsilon_\ell, j}\big| > 0$. Let us assume that

$$0 < \Lambda_{\varepsilon_\ell, j} \ < \ \delta\varepsilon_\ell^{n-1}. \tag{8.10}$$

Then from Lemma 8.2, we have

$$\Lambda_{a_\ell, j} = \Lambda_{\varepsilon_\ell, j} - \frac{(\varepsilon_\ell^2 - a_\ell^2)}{\varepsilon_\ell^2}(\Lambda_{\varepsilon_\ell, j} + \lambda_0). \tag{8.11}$$

The inequalities in (8.8) and (8.10) imply that

$$\Lambda_{a_\ell, j} \leq \delta\varepsilon_\ell^{n-1} - C_0\sigma_\ell^{n-1}\frac{\sigma_\ell(\varepsilon_\ell + a_\ell)}{\varepsilon_\ell^2}(\Lambda_{\varepsilon_\ell, j} + \lambda_0) < 0,$$

if δ is chosen a priori sufficiently small. It follows from the continuity of the function $\varepsilon \in (0, \varepsilon_0) \mapsto \Lambda_{\varepsilon, j}$ that $\Lambda_{\varepsilon, j}$ must vanish at some $\varepsilon \in (a_\ell, \varepsilon_\ell)$, and we get a contradiction with the choice of the interval (a_ℓ, b_ℓ). The case

$$-\delta\sigma_\ell^{(n-1)} \ < \ \Lambda_{\varepsilon_\ell, j} < 0$$

can be handled similarly. In fact, we have the inequality

$$\Lambda_{b_\ell, j} = \Lambda_{\varepsilon_\ell, j} + \frac{(b_\ell^2 - \varepsilon_\ell^2)}{\varepsilon_\ell^2}(\Lambda_{\varepsilon_\ell, j} + \lambda_0) > 0.$$

Hence, the proof of (8.9) for the spectral gap between critical eigenvalues was complete.

As a consequence, the solution to (8.1) exists and satisfies

$$\|e\|_{L^2(\mathcal{K})} \le C\varepsilon_\ell^{-(n-1)} \|h_2\|_{L^2(\mathcal{K})}. \tag{8.12}$$

From (8.12) by a standard elliptic argument one can show

$$\varepsilon_\ell^2 \| \triangle_{\mathcal{K}} e\|_{L^q(\mathcal{K})} + \varepsilon_\ell \| \nabla_{\mathcal{K}} e\|_{L^\infty(\mathcal{K})} + \|e\|_{L^\infty(\mathcal{K})} \le C\varepsilon_\ell^{-(n-1)} \|h_2\|_{L^q(\mathcal{K})}.$$

The reader can refer to [15] for proof of further estimate in (8.2). □

Acknowledgments. The first author is supported by NSFC 10901053 and ECNU 78210052. The second author is supported by an Earmarked Grant from RGC of Hong Kong. The third author is supported by the foundations: NSFC(No.10901108), NSF of Guangdong(No.10451806001004770) and 801-000012 of SZU R/F. Part of this work was done when the third author visited the Chern Institute of Mathematics, Nankai University: he is very grateful to the institution for the kind hospitality. We thank Prof. M. del Pino and Prof. M. Kowalczyk for useful conversations. Last but not the least, we are very grateful to the referees for pointing out a gap in the first version of the manuscript.

References

[1] A. Ambrosetti, M. Badiale and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rational Mech. Anal. 140 (1997), 285-300.

[2] A. Ambrosetti, A. Malchiodiand W.-M. Ni, *Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres*, Part I. Comm. Math. Phys. 235 (2003), 427-466.

[3] A. Ambrosetti, A. Malchiodi, A. and S. Secchi, *Multiplicity results for some nonlinear singularly perturbed elliptic problems on $\mathbb{R}^n$*, Arch. Rat. Mech. Anal. 159 (2001), 253-271.

[4] J. Byeon and Z.-Q. Wang, *Standing waves with a critical frequency for nonlinear Schrödinger equations*, Arch. Rat. Mech. Anal. 65 (2002), 295-316.

[5] L. Caffarelli, A. Córdoba, *Uniform convergence of a singular perturbation problem,* Comm. Pure Appl. Math. XLVII (1995), 1-12.

[6] I. Chavel, *Riemannian Geometry – A Modern Introduction*, Cambridge Tracts in Math. 108, Cambridge Univ. Press, Cambridge, 1993.

[7] S. Cingolani and M. Lazzo, *Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions*, J. Differential Equations, 160 (2000) 118-138.

[8] E. N. Dancer and S. Yan, *Multi-layer solutions for an elliptic problem*, J. Diff. Eqns. 194 (2003), 382-405.

[9] M. del Pino and P. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations, 4 (1996), 121-137.

[10] M. del Pino and P. Felmer, *Semi-classcal states for nonlinear Schrödinger equations*, J. Funct. Anal. 149 (1997), 245-265.

[11] M. del Pino and P. Felmer, *Multi-peak bound states for nonlinear Schrödinger equations*, Ann. Inst. H. Poincare Anal. Non Lineaire 15 (1998), no. 2, 127-149.

[12] M. del Pino and P. Felmer, *Semi-classical states of nonlinear Schrödinger equations: a variational reduction method*, Math. Ann. 324 (2002), no. 1, 1-32.

[13] M. del Pino, M. Kowalczyk and J. Wei, *Concentration on curves for nonlinear Schrödinger equations*, Comm. Pure Appl. Math. 70 (2007), 113-146.

[14] M. del Pino, M. Kowalczyk and J. Wei, *On De Giorgi conjecture in dimension $n \ge 9$*, Annals of Mathematics 2011, to appear.

[15] M. del Pino, M. Kowalczyk, J. Wei and J. Yang, *Interface foliation near minimal submanifolds in Riemannian manifolds with positive Ricci curvature*, Geom. Funct. Anal. 20 (2010), no. 4, 918-957.

[16] P. Felmer and J. J. Torres, *Semi classical limits for the one dimensional nonlinear Schrödinger equation*, Comm. Contemp. Math. 4 (2002) 481-512.

[17] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. 69 (1986), 397-408.

[18] C. E. Garza-Hume and P. Padilla, *Closed geodesic on oval surfaces and pattern formation*, Comm. Anal. Geom. 11 (2003), no. 2, 223-233.

[19] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[20] B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac operator*. Mathematics and its application (Soviet Series), 59. Kluwer Acadamic Publishers Group, Dordrecht, 1991.

[21] Peter Li and Shing-Tung Yau, *On the Schrödinger equation and the eigenvalue problem*, Commun. Math. Phys. 88 (1983), 309-318.

[22] F. Mahmoudi and A. Malchiodi, *Concentration on minimal submanifolds for a singularly perturbed Neumann problem*, Adv. Math. 209 (2007), no. 2, 460-525.

[23] F. Mahmoudi, A. Malchiodi and M. Montenegro, *Solutions to the nonlinear Schrödinger equation carrying momentum along a curve*, Comm. Pure Appl. Math. 62 (2009), no.9, 1155-1264.

[24] A. Malchiodi, *Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains*, Geom. Funct. Anal. 15 (2005), no. 6, 1162-1222.

[25] A. Malchiodi and M. Montenegro, *Boundary concentration phenomena for a singularly perturbed elliptic problem*, Commun. Pure Appl. Math. 55 (2002), 1507-1568.

[26] A. Malchiodi and M. Montenegro, *Multidimensional boundary layers for a singularly perturbed Neumann problem*, Duke Math. J. 124 (2004), no. 1, 105-143.

[27] R. Mazzeo, F. Pacard, *Foliations by constant mean curvature tubes*, Comm. Anal. Geom. 13 (2005), no. 4, 633-670.

[28] S. Minakshisundaram and A. Pleijel, *Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds*, Canad. J. Math. 1 (1949), 242-256.

[29] W.-M. Ni and I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. 70 (2003), 247-281.

[30] M. Spivak, *A comprehensive introduction to differential geometry*, Second edition, Publish or Perish Inc. wilmington, Del., (1979).

Department of Mathematics, East China Normal University, 500 Dongchuan Road, Shanghai, China, 200241.

E-mail address: `lpwang@math.ecnu.edu.cn`

(Corresponding Author) Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong.

E-mail address: `wei@math.cuhk.edu.hk`

College of mathematics and computational sciences, Shenzhen University, Shenzhen, China, 518060.

E-mail address: `jyang@szu.edu.cn`