INFINITE MULTIPLICITY FOR AN INHOMOGENEOUS SUPERCRITICAL PROBLEM IN ENTIRE SPACE

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ABSTRACT. Let $K(x)$ be a positive function in $\mathbb{R}^N$, $N \geq 3$ and satisfy $\lim_{|x|\to\infty} K(x) = K_\infty$ where K_∞ is a positive constant. When $p > \frac{N+1}{N-3}, N \geq 4$, we prove the existence of infinitely many positive solutions to the following supercritical problem:

$$\Delta u(x) + K(x)u^p = 0, u > 0 \text{ in } \mathbb{R}^N, \lim_{|x|\to\infty} u(x) = 0.$$

If in addition we have, for instance, $\lim_{|x|\to\infty} |x|^\mu (K(x) - K_\infty) = C_0 \neq 0, 0 < \mu \leq N$, then this result still holds provided that $p > \frac{N+2}{N-2}$.

1. Introduction and Statement of the results

The purpose of this paper is to establish the existence of **infinitely many** positive solutions to the following inhomogeneous equation

$$\begin{cases} \Delta u + K(x)u^p = 0, \\ u > 0 \quad \text{in } \mathbb{R}^N, \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.1}$$

where $N \geq 3, p > \frac{N+2}{N-2}$ and $0 < a \leq K(x) \leq b < +\infty$.

Semilinear elliptic equations like above seem to arise naturally in many applied areas. We refer the interested readers to [2], [3] and [7] for a brief history and background of (1.1).

In [6], Ding and Ni showed that for $p \geq \frac{N+2}{N-2}$, if $x \cdot \nabla K(x) \geq 0$ and $K(x)$ is symmetric in $x_j, j = 1, ..., N$, then equation (1.1) admits infinitely many solutions. Using sub-super solution method, Gui [7]-[8] showed that there exists an exponent p_c (defined at (1.6) below) such that for $p \geq p_c, N \geq 11$, equation (1.1) has infinitely many (well-separated) solutions in the case when K is radially symmetric. Recent extensions can be found in Bae and Ni [3], and Bae [1]. However, in [1], [3], [7] and [8], it is always assumed that $p \geq p_c$ and $N \geq 11$. The case of $N \leq 10$ and $\frac{N+2}{N-2} < p < p_c$ has left open. Note that in this case, the method of sub-super solution breaks down. Other related results can be found in Wang-Wei [11], Yanagida-Yotsutani [12].

In this paper, under reasonable conditions on K, we establish that when $p > \frac{N+2}{N-2}$, equation (1.1) has a continuum of solutions. Our basic assumption is the following

$$(\mathbb{H}) \qquad K(x) \text{ is smooth, } \lim_{|x|\to\infty} K(x) = K_\infty > 0.$$

Our main result is the following:

Theorem 1.1. *Assume that $K(x)$ satisfies $(\mathbb{H})$ and $p > \frac{N+1}{N-3}, N \geq 4$. Then problem (1.1) has a continuum of solutions $u_\lambda(x)$ (parameterized by $\lambda \leq \lambda_0$) such that*

$$\lim_{\lambda \to 0} u_\lambda(x) = 0$$

uniformly in $\mathbb{R}^N$. The same result holds when $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$ provided that K is symmetric with respect to N coordinate axis, namely

$$K(x_1, \dots, x_i, \dots, x_N) = K(x_1, \dots, -x_i, \dots, x_N), \quad \text{for all } i = 1, \dots, N.$$

The basic obstruction to extend the result to the whole supercritical range is that the linearized operator around canonical approximation will no longer be onto if $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$. This problem can be overcome through a further condition on $K(x)$. We have the validity of the following result.

Theorem 1.2. *Assume that $K(x)$ satisfies $(\mathbb{H})$ and $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$ if $N \geq 4$, $p > \frac{N+2}{N-2}$ if $N = 3$. Then the result of Theorem 1.1 also holds true if*
(a) there exists $\mu > N$, such that

$$\int_{\mathbb{R}^N} \Big(K(x) - K_\infty\Big) \neq 0, \quad |K(x) - K_\infty| \leq C|x|^{-\mu}, \quad |x| \geq 1;$$

or
(b) there exist a bounded function $f : S^{N-1} \to \mathbb{R}$ and $N - \frac{2p+2}{p-1} < \mu \leq N$ such that

$$\lim_{|x| \to \infty} \Big(|x|^\mu (K(x) - K_\infty) - f(\frac{x}{|x|})\Big) = 0,$$

where $f(x)$ satisfies $\int_{\mathbb{R}^N} f(\frac{x}{|x|})|x|^{-\mu}|x+\omega|^{-\frac{2(p+1)}{p-1}} \neq 0$ for any $\omega \in \partial B_1(0)$ if $\mu < N$ and $\int_{S^{N-1}} f \neq 0$ if $\mu = N$;
or
(c) there exist $C_0 \neq 0$ and $\mu \in (0, N - \frac{2p+2}{p-1}]$, such that

$$\lim_{|x| \to \infty} |x|^\mu (K(x) - K_\infty) = C_0.$$

Instead of using sub-super solution method (which limits the applicability on the exponent p), we use asymptotic analysis and Liapunov-Schmidt reduction method to prove Theorem 1.1 and Theorem 1.2. This is based on the construction of a sufficiently good approximation solution. It is well known that the problem

$$\Delta W + W^p = 0 \quad \text{in} \quad \mathbb{R}^N \tag{1.2}$$

possesses a positive radially symmetric solution $W(|x|)$ whenever $p > \frac{N+2}{N-2}$. We fix in what follows the solution W of (1.2) such that

$$W(0) = 1. \tag{1.3}$$

Then all radial solutions to this problem can be expressed as

$$W_\lambda(x) = \lambda^{\frac{2}{p-1}} W(\lambda x). \tag{1.4}$$

At main order one has

$$W(r) \sim C_{p,N} r^{-\frac{2}{p-1}} \quad \text{as} \quad r \to \infty, \tag{1.5}$$

which implies that this behavior is actually common to all solutions $W_\lambda(x)$. In [10] and [9], it is shown that if $p = p_c$,

$$W(r) = \frac{\beta^{\frac{1}{p-1}}}{r^{\frac{2}{p-1}}} + \frac{a_1 \log r}{r^{\mu_0}} + o(\frac{\log r}{r^{\mu_0}}), \quad r \to \infty,$$

where $\beta = \frac{2}{p-1}(N - 2 - \frac{2}{p-1}), a_1 < 0, \mu_0 > 0$ and if $p > p_c$

$$W(r) = \frac{\beta^{\frac{1}{p-1}}}{r^{\frac{2}{p-1}}} + \frac{a_1}{r^{\mu_0}} + o(\frac{1}{r^{\mu_0}}), \quad r \to \infty,$$

where

$$p_c = \begin{cases} \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)}, & N > 10, \\ \infty, & N \leq 10. \end{cases} \tag{1.6}$$

The idea is to consider $W_\lambda(x)$ as an approximation for a solution of (1.1), provided that $\lambda > 0$ is chosen small enough. To this end, we need to study the solvability of the operator $\Delta + pW^{p-1}$ in suitable weighted Sobolev space. Recently, this issue has been studied in Davila-del Pino-Musso [4] and Davila-del Pino-Musso-Wei [5]. In particular, our method here is closely related to [5] where standing wave solutions are constructed for nonlinear Schrödinger equations

$$\Delta u - V(x)u + u^p = 0, u > 0 \text{ in } \mathbb{R}^N, \lim_{|x|\to+\infty} u(x) = 0 \tag{1.7}$$

with

$$V(x) = o(\frac{1}{|x|^2}), V(x) \geq 0. \tag{1.8}$$

Throughout the paper, the symbol C denotes always a positive constant independent of λ, which could be changed from one line to another. Denote $A \sim B$ if and only if there exist two positive numbers a, b such that $aA \leq B \leq bA$.

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2. The solvability of linearized operator $\Delta + pW^{p-1}$

Our main concern in this section is to study the existence of solution in certain weighted spaces for

$$\Delta\phi + pW^{p-1}\phi = h \text{ in } \mathbb{R}^N, \tag{2.1}$$

where W is the radial solution to (1.2), (1.3) and h is a known function having a specific decay at infinity.

We work in weighted L^∞ spaces adjusted to the nonlinear problem (1.1) and in particular take into account the behavior of W at infinity. We are looking for a solution ϕ to (2.1) that is small compared to W at infinity, thus it is natural to require that it has a decay of the form $O(|x|^{-\frac{2}{p-1}})$ as $|x| \to +\infty$. As a result we shall assume that h behaves like this but with two powers subtracted, that is, $h = O(|x|^{-\frac{2}{p-1}-2})$ at infinity. These remarks motivate the definitions

$$\|\phi\|_* = \sup_{|x|\le 1} |x|^{\sigma}|\phi(x)| + \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}}|\phi(x)|, \tag{2.2}$$

and

$$\|h\|_{**} = \sup_{|x|\le 1} |x|^{2+\sigma}|h(x)| + \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}+2}|h(x)|, \tag{2.3}$$

where $\sigma > 0$ will be fixed later as needed.

The following lemmas and remarks on the solvability are due to Davila-del Pino-Musso [4] and Davila-del Pino-Musso-Wei [5]:

Lemma 2.1. *Assume that $p > \frac{N+1}{N-3}, N \ge 4$. For $0 < \sigma < N-2$ there exists a constant $C > 0$ such that for any h with $\|h\|_{**} < \infty$, equation (2.1) has a solution $\phi = T(h)$ such that T defines a linear map and*

$$\|T(h)\|_* \le C\|h\|_{**}.$$

An obstruction arises if $\frac{N+2}{N-2} < p \le \frac{N+1}{N-3}$, which can be handled by considering suitable orthogonality conditions with respect to translations of W. Let us define

$$Z_i = \eta \frac{\partial W}{\partial x_i} \tag{2.4}$$

and $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$,

$$\eta(x) = 1 \quad \text{for } |x| \le R_0, \quad \eta(x) = 0 \quad \text{for } |x| \ge R_0 + 1.$$

We work with $R_0 > 0$ fixed large enough.

Then we have

Lemma 2.2. *Assume $N \ge 3$, $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ and let $0 < \sigma < N-2$. There is a linear map $(\phi, c_1, \dots, c_N) = T(h)$ defined whenever $\|h\|_{**} < \infty$ such that*

$$\Delta\phi + pW^{p-1}\phi = h + \sum_{i=1}^{N} c_i Z_i \quad \text{in } \mathbb{R}^N \tag{2.5}$$

and

$$\|\phi\|_* + \sum_{i=1}^{N} |c_i| \le C\|h\|_{**}.$$

Moreover, $c_i = 0$ for all $1 \le i \le N$ if and only if h satisfies

$$\int_{\mathbb{R}^N} h \frac{\partial W}{\partial x_i} = 0 \quad \forall 1 \le i \le N. \tag{2.6}$$

Remark 2.3. *If $p = \frac{N+1}{N-3}$, the conclusion of Lemma 2.2 still holds if one redefines the norms as*

$$\|\phi\|_* = \sup_{|x|\le 1} |x|^{\sigma}|\phi(x)| + \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}+\alpha}|\phi(x)|,$$

$$\|h\|_{**} = \sup_{|x|\le 1} |x|^{\sigma+2}|h(x)| + \sup_{|x|\ge 1} |x|^{\frac{2}{p-1}+\alpha+2}|h(x)|,$$

where $\alpha > 0$ is fixed small.

The operator T in Lemmas 2.1-2.2, Remark 2.3 are constructed "by hand" by decomposing h and ϕ into sums of spherical harmonics where the coefficients are radial functions. The nice property is of course that since W is radial, the problem decouples into an infinite collection of ODEs. We omit the details for the construction and refer to [4] and [5] for the details.

3. The proof of Theorem 1.1

Let $p > \frac{N+1}{N-3}$. We prove Theorem 1.1 in this section. The main idea is to use Lemma 2.1 and a contraction mapping principle.

By a change of variables $K_\infty^{\frac{1}{p-1}} \lambda^{-\frac{2}{p-1}} u(\frac{x}{\lambda})$, equation(1.1) is equivalent to

$$\Delta u + u^p + \bar{K}(\frac{x}{\lambda}) u^p = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{3.1}$$

where

$$\bar{K}(\frac{x}{\lambda}) = \frac{K(\frac{x}{\lambda})}{K_\infty} - 1. \tag{3.2}$$

Note that by our assumption on K, for any fixed $x \neq 0$, we have $\bar{K}(\frac{x}{\lambda}) = o(1)$.

We look for a solution of (3.1) of the form $u = W + \phi$, which yields the following equation for ϕ

$$\Delta\phi + pW^{p-1}\phi = N(\phi) - \bar{K}(\frac{x}{\lambda})(W+\phi)^p,$$

where

$$N(\phi) = -(W+\phi)^p + W^p + pW^{p-1}\phi. \tag{3.3}$$

Using the operator T defined in Lemma 2.1 we are led to solving the fixed point problem

$$\phi = T\Bigg(N(\phi) - \bar{K}(\frac{x}{\lambda})(W+\phi)^p\Bigg). \tag{3.4}$$

We use a fixed-point argument: Consider the set

$$\mathbb{F} = \{\phi : \mathbb{R}^N \to R \quad | \quad \|\phi\|_* \leq \rho\},$$

where $\rho \in (0,1)$ is to be chosen (suitably small) and the operator $\mathbb{A}(\phi) = T\Big(N(\phi) - \bar{K}(\frac{x}{\lambda})(W+\phi)^p\Big)$. We now prove that $\mathbb{A}$ has a fixed point in $\mathbb{F}$.

For any $\phi \in \mathbb{F}$, by the arguments in [4]-[5], we know that for $0 < \sigma < \frac{2}{p-1}$ chosen in (2.2), (2.3), it holds

$$\|N(\phi)\|_{**} \leq C(\|\phi\|_*^2 + \|\phi\|_*^p). \tag{3.5}$$

Next we estimate $\|\bar{K}(\frac{x}{\lambda})(W+\phi)^p\|_{**}$. Let $R > 0$. Observe that

$$\begin{aligned} \sup_{|x|\leq 1} |x|^{2+\sigma} |\bar{K}(\tfrac{x}{\lambda})(W+\phi)^p| &\leq C \sup_{|x|\leq 1} |x|^{2+\sigma} |\bar{K}(\tfrac{x}{\lambda})|(\|W\|_\infty^p + |\phi|^p) \\ &\leq C \sup_{|x|\leq \lambda R} \cdots + C \sup_{\lambda R \leq |x| \leq 1} \cdots \end{aligned} \tag{3.6}$$

But

$$\sup_{|x|\leq\lambda R} |x|^{2+\sigma} |\bar{K}(\frac{x}{\lambda})|(\|W\|_\infty^p + |\phi|^p) \leq C(\lambda R)^{2+\sigma} + C\|\phi\|_*^p, \tag{3.7}$$

$$\sup_{\lambda R\le|x|\le1}|x|^{2+\sigma}|\bar{K}(\frac{x}{\lambda})|(\|W\|_\infty^p+|\phi|^p)\le Ca(R)(1+\|\phi\|_*^p)\le Ca(R),\tag{3.8}$$

where

$$a(R)=\sup_{|x|\ge R}|\bar{K}(x)|,\quad\text{then}\quad\lim_{R\to\infty}a(R)=0.\tag{3.9}$$

On the other hand,

$$\begin{aligned}\sup_{|x|\ge1}|x|^{2+\frac{2}{p-1}}|\bar{K}(\tfrac{x}{\lambda})(W+\phi)^p| &\le Ca(\tfrac{1}{\lambda})\sup_{|x|\ge1}|x|^{2+\frac{2}{p-1}}(|W|^p+|\phi|^p)\\ &\le Ca(\tfrac{1}{\lambda})(1+\|\phi\|_*^p)\le Ca(\tfrac{1}{\lambda}).\end{aligned}\tag{3.10}$$

Thus by (3.7), (3.8), (3.10), we get

$$\|\bar{K}(\frac{x}{\lambda})(W+\phi)^p\|_{**}\le C\Big(a(R)+a(\frac{1}{\lambda})+(\lambda R)^{2+\sigma}+\|\phi\|_*^p\Big).\tag{3.11}$$

By Lemma 2.1, (3.5) and (3.11), we have

$$\begin{aligned}\|\mathbb{A}(\phi)\|_* &\le C\|N(\phi)\|_{**}+C\|\bar{K}(\tfrac{x}{\lambda})(W+\phi)^p\|_{**}\\ &\le C\Big(\|\phi\|_*^2+\|\phi\|_*^p+a(R)+a(\tfrac{1}{\lambda})+(\lambda R)^{2+\sigma}\Big)\\ &\le C\Big(\rho^2+\rho^p+a(R)+a(\tfrac{1}{\lambda})+(\lambda R)^{2+\sigma}\Big).\end{aligned}\tag{3.12}$$

Now we choose ρ small enough, such that $C(\rho^2+\rho^p)\le\frac{1}{4}\rho$. Then choose R large enough such that $Ca(R)\le\frac{1}{4}\rho$. Finally we choose λ small enough such that $Ca(\frac{1}{\lambda})+C(\lambda R)^{2+\sigma}\le\frac{1}{2}\rho$. All yield that $\mathbb{A}(\mathbb{F})\subset\mathbb{F}$.

It remains to prove that $\mathbb{A}$ is contractible.

Similar to arguments in [4], we see that $\forall\phi_1,\phi_2\in\mathbb{F}$,

$$\|N(\phi_1)-N(\phi_2)\|_{**}\le C(\rho+\rho^{p-1})\|\phi_1-\phi_2\|_*.\tag{3.13}$$

Observe that

$$|\bar{K}(\frac{x}{\lambda})||(W+\phi_1)^p-(W+\phi_2)^p|\le C|\bar{K}(\frac{x}{\lambda})||\phi_1-\phi_2|(|W|^{p-1}+|\phi_1|^{p-1}+|\phi_2|^{p-1}).$$

Similarly we obtain

$$\begin{aligned}&\sup_{|x|\le\lambda R}|x|^{2+\sigma}|\bar{K}(\tfrac{x}{\lambda})||(W+\phi_1)^p-(W+\phi_2)^p|\\ \le\ & C\|\phi_1-\phi_2\|_*\sup_{|x|\le\lambda R}|x|^2(|W|^{p-1}+|\phi_1|^{p-1}+|\phi_2|^{p-1})\\ \le\ & C\|\phi_1-\phi_2\|_*\Big((\lambda R)^2+\rho^{p-1}\Big),\end{aligned}\tag{3.14}$$

$$\sup_{\lambda R\le|x|\le1}|x|^{2+\sigma}|\bar{K}(\frac{x}{\lambda})||(W+\phi_1)^p-(W+\phi_2)^p|\le Ca(R)\|\phi_1-\phi_2\|_*,\tag{3.15}$$

$$\sup_{|x|\ge1}|x|^{2+\frac{2}{p-1}}|\bar{K}(\frac{x}{\lambda})||(W+\phi_1)^p-(W+\phi_2)^p|\le Ca(\frac{1}{\lambda})\|\phi_1-\phi_2\|_*.\tag{3.16}$$

Hence by Lemma 2.1, (3.13)–(3.16), we have

$$\begin{aligned}\|\mathbb{A}(\phi_1)-\mathbb{A}(\phi_2)\|_* &\le C\Big((\|N(\phi_1)-N(\phi_2)\|_{**}+\|\bar{K}(\tfrac{x}{\lambda})[(W+\phi_1)^p-(W+\phi_2)^p]\|_{**}\Big)\\ &\le C\|\phi_1-\phi_2\|_*\Big(\rho+\rho^{p-1}+(\lambda R)^2+a(R)+a(\tfrac{1}{\lambda})\Big)\\ &\le \tfrac{1}{2}\|\phi_1-\phi_2\|_*,\end{aligned}\tag{3.17}$$

provided that ρ small enough, R large enough and λ small enough.

By (3.12) and (3.17), $\mathbb{A}$ is a contraction mapping. By contraction-mapping principle, it follows that $\mathbb{A}$ has a fixed point ϕ_λ in $\mathbb{F}$. Since $W+\phi_\lambda$ is a solution of

$$\begin{cases} \Delta u + u^p + \bar{K}(\frac{x}{\lambda})u^p = 0, \\ u > 0 \quad \text{in } \mathbb{R}^N, \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{3.18}$$

For x such that $|x| = 1, W + \phi_\lambda$ remains bounded because $\phi_\lambda(x) \le C$. Then uniform upper bound for $W + \phi_\lambda$ follows from (3.18) by observing that $\|(1 + \bar{K}(\frac{x}{\lambda}))(W+\phi_\lambda)^p\|_{L^q(B_1)}$ remains bounded as $\lambda \to 0$ for $q > \frac{N}{2}$. In fact,

$$\int_{B_1} (1+\bar{K}(\frac{x}{\lambda}))^q (W+\phi_\lambda)^{pq} \le C\int_{B_1} W^{pq} + C\int_{B_1} |\phi_\lambda|^{pq} \le C + C\int_{B_1} |x|^{-\sigma pq} \le C$$

provided that $\sigma > 0$ small. Hence

$$|W + \phi_\lambda| \le C \quad \text{for all} \quad |x| \le 1. \tag{3.19}$$

It follows from then that

$$|\phi_\lambda(x)| \le C \quad \text{for all} \quad x. \tag{3.20}$$

Thus $u_\lambda(x) = K_\infty^{-\frac{1}{p-1}} \lambda^{\frac{2}{p-1}}(W(\lambda x) + \phi_\lambda(\lambda x))$ is a continuum solutions of (1.1) and

$$\lim_{\lambda\to 0} u_\lambda(x) = 0$$

uniformly in $\mathbb{R}^N$. This ends the proof of Theorem 1.1. □

Remark 3.1. *We observe that the above proof actually applies with no changes to the case $\frac{N+2}{N-2} < p \le \frac{N+1}{N-3}$ provided that K is symmetric with respect to N coordinate axis, namely*

$$K(x_1, \dots, x_i, \dots, x_N) = K(x_1, \dots, -x_i, \dots, x_N), \qquad \text{for all } i = 1, \dots, N.$$

In this case the problem is invariant with respect to the above reflections, and we can formulate the fixed point problem in the space of functions with these even symmetries with the linear operator defined in Lemma 2.2. Indeed, the orthogonality conditions in Lemma 2.2 are automatically satisfied, so that the associated numbers c_i's are all zero.

4. The proof of Theorem 1.2

In this section, we consider the case when $p \in (\frac{N+2}{N-2}, \frac{N+1}{N-3}]$ and prove Theorem 1.2. We need to use Lemma 2.2 and a Liapunov-Schmidt reduction argument.

By Lemma 2.2 and Remark 2.3, there is an obstruction in the solvability of the linearized operator. To overcome the obstruction, we introduce a new parameter ξ where W achieves its maximum. For this reason we make the change of variables $K_\infty^{\frac{1}{p-1}} \lambda^{-\frac{2}{p-1}} u(\frac{x-\xi}{\lambda})$ and look for a solution of the form $u = W + \phi$, leading to the following equation for ϕ:

$$\Delta\phi + pW^{p-1}\phi = N(\phi) - \bar{K}(\frac{x-\xi}{\lambda})(W+\phi)^p,$$

where

$$N(\phi) = -(W+\phi)^p + W^p + pW^{p-1}\phi.$$

We will change slightly the previous notations to make the dependence of the norms on σ explicit. Hence we set

$$\|\phi\|_{*,\xi}^{(\sigma)} = \sup_{|x-\xi|\le 1} |x-\xi|^\sigma |\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}} |\phi(x)|$$

and

$$\|h\|_{**,\xi}^{(\sigma)} = \sup_{|x-\xi|\le 1} |x-\xi|^{\sigma+2}|h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}+2}|h(x)|.$$

In the rest of the section we assume that

$$\frac{N+2}{N-2} < p < \frac{N+1}{N-3}.$$

The case $p = \frac{N-1}{N+3}$ can be handled similarly, with a slight modification of the norms where it is more convenient to define

$$\|\phi\|_{*,\xi}^{(\sigma)} = \sup_{|x-\xi|\le 1} |x-\xi|^{\sigma}|\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}+\alpha}|\phi(x)|$$

and

$$\|h\|_{**,\xi}^{(\sigma)} = \sup_{|x-\xi|\le 1} |x-\xi|^{\sigma+2}|h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}+\alpha+2}|h(x)|$$

for some small fixed $\alpha > 0$. See Remark 2.3 and Remark 4.2.

The proof of Theorem 1.2 is through a Liapunov-Schmidt reduction procedure. This will be achieved in **two steps.** In the first step, we solve (3.1) modulo Z_i, using Lemma 2.2. That is, we have the following lemma.

Lemma 4.1. *Assume that* $N \ge 3$, $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ *and* $K(x)$ *satisfies* ($\mathbb{H}$) *and* $\lambda > 0$. *Then there is* $\lambda_0 > 0$ *such that for* $|\xi| \le \Lambda$ *and* $\lambda < \lambda_0$ *there exist* $\phi_\lambda, c_1(\lambda), \dots, c_N(\lambda)$ *solution to*

$$\begin{cases} \Delta\phi + pW^{p-1}\phi = N(\phi) - \bar{K}(\frac{x-\xi}{\lambda})(W+\phi)^p + \sum_{i=1}^N c_i Z_i, \\ \lim_{|x|\to\infty} \phi(x) = 0. \end{cases} \tag{4.1}$$

If $K(x)$ *also satisfies*

$$|K(x) - K_\infty| \le C|x|^{-\mu}, \quad |x| \ge 1, \tag{4.2}$$

for some $\mu > 0$, *then for* $0 < \theta < N-2$,

$$\|\phi_\lambda\|_{*,\xi}^{(\theta)} + \sum_{i=1}^N |c_i(\lambda)| \le C\lambda^{\min\{\mu, 2+\theta\}} \quad \textit{for all} \quad 0 < \lambda < \lambda_0. \tag{4.3}$$

Proof: Similar to the proof of Theorem 1.1, we fix $0 < \sigma < \min\{2, \frac{2}{p-1}\}$ and define for small $\rho > 0$

$$\mathbb{F} = \{\phi : \mathbb{R}^N \to R \quad | \quad \|\phi\|_{*,\xi}^{(\sigma)} \le \rho\}$$

and the operator $\phi_1 = \mathbb{A}_\lambda(\phi)$ to

$$\begin{cases} \Delta\phi_1 + pW^{p-1}\phi_1 = N(\phi) - \bar{K}(\frac{x-\xi}{\lambda})(W+\phi)^p + \sum_{i=1}^N c_i Z_i, \\ \lim_{|x|\to\infty} \phi_1(x) = 0. \end{cases} \tag{4.4}$$

By the same proof as in those of Theorem 1.1, we have for any $\phi, \phi_1, \phi_2 \in \mathbb{F}$

$$\|N(\phi)\|_{**,\xi}^{(\sigma)} \le C(\|\phi\|_{*,\xi}^{(\sigma)})^2 + C(\|\phi\|_{*,\xi}^{(\sigma)})^p \le C(\rho^2 + \rho^p), \tag{4.5}$$

$$\|N(\phi_1) - N(\phi_2)\|_{**,\xi}^{(\sigma)} \le C(\rho + \rho^{p-1})\|\phi_1 - \phi_2\|_{*,\xi}^{(\sigma)}, \tag{4.6}$$

$$\|\bar{K}(\frac{x-\xi}{\lambda})(W+\phi)^p\|_{**,\xi}^{(\sigma)} \le C\Big(a(R) + a(\frac{1}{\lambda}) + (\lambda R)^{2+\sigma} + (\|\phi\|_{*,\xi}^{(\sigma)})^p\Big), \tag{4.7}$$

$$\|\bar{K}(\frac{x-\xi}{\lambda})\Big((W+\phi_1)^p-(W+\phi_2)^p\Big)\|_{**,\xi}^{(\sigma)}\le C\Big(a(R)+a(\frac{1}{\lambda})+(\lambda R)^2+\rho^{p-1}\Big)\|\phi_1-\phi_2\|_{*,\xi}^{(\sigma)}. \tag{4.8}$$

Using Lemma 2.2 and fixed point theorem we get a solution $\phi_\lambda, c_1(\lambda), \dots, c_N(\lambda)$ of (4.1) provided ρ small enough, R large enough and λ small enough.

As in the proof of (3.20), we can obtain

$$\|\phi_\lambda\|\le C \quad \text{for all} \quad x. \tag{4.9}$$

Under the assumption of (4.2) and for $0<\theta<N-2$, we can estimate $\bar{K}(\frac{x-\xi}{\lambda})(W+\phi_\lambda)^p$ as follows: for R fixed large enough,

$$\sup_{|x-\xi|\le 1}|x-\xi|^{2+\theta}|\bar{K}(\frac{x-\xi}{\lambda})(W+\phi_\lambda)^p|\le \sup_{|x-\xi|\le\lambda R}\cdots+\sup_{\lambda R\le|x-\xi|\le 1}\cdots$$

$$\begin{aligned}\sup_{|x-\xi|\le\lambda R}|x-\xi|^{2+\theta}|\bar{K}(\tfrac{x-\xi}{\lambda})(W+\phi_\lambda)^p| &\le C\sup_{|x-\xi|\le\lambda R}|x-\xi|^{2+\theta}(|W|^p+|\phi_\lambda|^p)\\ &\le C\Big((\lambda R)^{2+\theta}+\|\phi_\lambda\|_{*,\xi}^{(\theta)}\sup_{|x-\xi|\le\lambda R}|x-\xi|^2\Big)\\ &\le C\Big((\lambda R)^{2+\theta}+\|\phi_\lambda\|_{*,\xi}^{(\theta)}(\lambda R)^2\Big),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\sup_{\lambda R\le|x-\xi|\le 1}|x-\xi|^{2+\theta}|\bar{K}(\tfrac{x-\xi}{\lambda})(W+\phi_\lambda)^p| &\le C\lambda^\mu\sup_{\lambda R\le|x-\xi|\le 1}|x-\xi|^{2+\theta-\mu}(|W|^p+|\phi_\lambda|^p)\\ &\le C\lambda^{\min\{\mu,2+\theta\}}+C\lambda^{\min\{\mu,2\}}\|\phi_\lambda\|_{*,\xi}^{(\theta)},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\sup_{|x-\xi|\ge 1}|x-\xi|^{2+\frac{2}{p-1}}|\bar{K}(\tfrac{x-\xi}{\lambda})(W+\phi_\lambda)^p| &\le C\lambda^\mu\Big(1+(\|\phi_\lambda\|_{*,\xi}^{(\theta)})^p\Big)\sup_{|x-\xi|\ge 1}|x-\xi|^{-\mu}\\ &\le C\lambda^\mu\Big(1+(\|\phi_\lambda\|_{*,\xi}^{(\theta)})^p\Big).\end{aligned} \tag{4.12}$$

Thus

$$\|\bar{K}(\frac{x-\xi}{\lambda})(W+\phi_\lambda)^p\|_{**,\xi}^{(\theta)}\le C\lambda^{\min\{\mu,2+\theta\}}+C\lambda^{\min\{\mu,2\}}\|\phi_\lambda\|_{*,\xi}^{(\theta)}. \tag{4.13}$$

If $0<\theta\le\frac{2}{p-1}$, according to (3.5), Lemma 2.2 yields

$$\|\phi_\lambda\|_{*,\xi}^{(\theta)}+\sum_{i=1}^N|c_i(\lambda)|\le C\lambda^{\min\{\mu,2+\theta\}}. \tag{4.14}$$

provided ρ,λ small enough.

Now consider $\frac{2}{p-1}<\theta<N-2$ and let $0<\sigma\le\frac{2}{p-1}$.
If $p\ge 2$, then $0<\sigma\le 2$ and

$$|N(\phi_\lambda)|\le C(W^{p-2}|\phi_\lambda|^2+|\phi_\lambda|^p).$$

Observe that

$$\sup_{|x-\xi|\le 1}|x-\xi|^{2+\theta}|N(\phi_\lambda)|\le\sup_{|x-\xi|\le\lambda}\cdots+\sup_{\lambda\le|x-\xi|\le 1}\cdots$$

Thanks to (4.9), we have

$$\sup_{|x-\xi|\le\lambda}|x-\xi|^{2+\theta}|N(\phi_\lambda)|\le C\lambda^{2+\theta}$$

and

$$\begin{aligned}\sup_{\lambda\le|x-\xi|\le 1}|x-\xi|^{2+\theta}|N(\phi_\lambda)| &\le C(\|\phi_\lambda\|_{*,\xi}^{(\sigma)})^2\sup_{\lambda\le|x-\xi|\le 1}|x-\xi|^{2+\theta-2\sigma}\\ &\le C(\|\phi_\lambda\|_{*,\xi}^{(\sigma)})^2\sup_{\lambda\le|x-\xi|\le 1}|x-\xi|^{2-\sigma}\\ &\le C(\|\phi_\lambda\|_{*,\xi}^{(\sigma)})^2\le C\lambda^{\min\{2(2+\sigma),2\mu\}},\end{aligned}$$

$$\sup_{|x-\xi|\ge 1}|x-\xi|^{2+\frac{2}{p-1}}|N(\phi_\lambda)|\le C(\|\phi_\lambda\|_{*,\xi}^{(\sigma)})^2\le C\lambda^{\min\{2(2+\sigma),2\mu\}}.$$

Thus

$$\|N(\phi_\lambda)\|_{**,\xi}^{(\theta)}\le C\lambda^{\min\{2+\theta,2(2+\sigma),2\mu\}}\quad\text{if}\quad p\ge 2. \tag{4.15}$$

Similarly, if $1<p<2$, using $|N(\phi_\lambda)|\le C|\phi_\lambda|^p$, we can get

$$\|N(\phi_\lambda)\|_{**,\xi}^{(\theta)}\le C\lambda^{\min\{2+\theta,p(2+\sigma),p\mu\}}. \tag{4.16}$$

After finite steps we get for any $p>1$,

$$\|N(\phi_\lambda)\|_{**,\xi}^{(\theta)}\le C\lambda^{\min\{2+\theta,\mu\}}. \tag{4.17}$$

According to (4.13),(4.17) we get

$$\|\phi_\lambda\|_{*,\xi}^{(\theta)}+\sum_{i=1}^N|c_i(\lambda)|\le C\lambda^{\min\{\mu,2+\theta\}}. \tag{4.18}$$

provided λ small. □

In the second step, we need to vary ξ so that $c_i=0, i=1,...,N$, therefore proving Theorem 1.2.

By Lemma 4.1, we have found a solution $\phi_\lambda, c_1(\lambda),\dots,c_N(\lambda)$ to (4.1). By Lemma 2.2 the solution constructed satisfies for all $1\le j\le N$:

$$\int_{\mathbb{R}^N}\Big(N(\phi_\lambda)-\bar K(\frac{x-\xi}{\lambda})(W+\phi_\lambda)^p\Big)\frac{\partial W}{\partial x_j}=0$$

if and only if $c_j=0$. We divide it into three cases.

Case (a): $\mu>N$. In this case, we have

$$\int_{\mathbb{R}^N}-\bar K(\frac{x}{\lambda})W^p\frac{\partial W}{\partial x_j}(x+\xi)=-\lambda^N\int_{\mathbb{R}^N}\Big(K(x)-K_\infty\Big)W^p(\xi)\frac{\partial W}{\partial x_j}(\xi)+o(\lambda^N)\quad\text{as}\quad\lambda\to 0,$$

where the convergence is uniform with respect to $|\xi|\le\delta_0$.

Indeed, in the case $p\ge 2$, if we choose $\frac{N-2}{2}<\theta<\min\{\frac{N}{2},N-2\}$, then we obtain

$$\int_{\mathbb{R}^N}|N(\phi_\lambda)\frac{\partial W}{\partial x_j}|=\int_{B_1(\xi)}\cdots+\int_{\mathbb{R}^N\setminus B_1(\xi)}\cdots$$

$$\int_{B_1(\xi)}|N(\phi_\lambda)\frac{\partial W}{\partial x_j}|\le C(\|\phi_\lambda\|_{*,\xi}^{(\theta)})^2\int_{B_1(\xi)}|x-\xi|^{-2\theta}\le C\lambda^{2\min\{2+\theta,\mu\}}\le C\lambda^{\min\{N+2,2\mu\}},$$

$$\int_{\mathbb{R}^N\setminus B_1(\xi)}|N(\phi_\lambda)\frac{\partial W}{\partial x_j}|\le C(\|\phi_\lambda\|_{*,\xi}^{(\theta)})^2\int_{\mathbb{R}^N\setminus B_1(\xi)}|x-\xi|^{-3-\frac{4}{p-1}}\le C\lambda^{2\min\{2+\theta,\mu\}}\le C\lambda^{\min\{N+2,2\mu\}}.$$

Thus

$$\int_{\mathbb{R}^N}|N(\phi_\lambda)\frac{\partial W}{\partial x_j}|=o(\lambda^N). \tag{4.19}$$

Similarly, in the case $1 < p < 2$, we get

$$\int_{\mathbb{R}^N} |N(\phi_\lambda)\frac{\partial W}{\partial x_j}| \le C\lambda^{p\min\{2+\theta,\mu\}} = o(\lambda^N), \tag{4.20}$$

since we can choose θ such that $p\theta = N-1$, which is possible since then $\theta = \frac{N-1}{p} < \frac{(N-1)(N-2)}{N+2} < N-2$.

Next we need the following important claim

Claim: $0 < c \le U_\lambda \le C$ in $B_1(\xi)$, where $U_\lambda = W + \phi_\lambda$.
The upper bound has been given by (4.9) and we only need to prove that U_λ has a lower bound. Let $\chi(r) = \frac{1}{2N}(1-r^2)$, so that

$$\Delta\chi \equiv -1, \quad \chi \equiv 0 \quad \text{on} \quad \partial B_1$$

and consider $z = U_\lambda + (\sum_{i=1}^N |c_i(\lambda)|\|Z_i\|_\infty)\chi$. Then z satisfies

$$\Delta z \le 0.$$

By maximum principle we have

$$U_\lambda + (\sum_{i=1}^N |c_i(\lambda)|\|Z_i\|_\infty)\chi \ge U_\lambda\,|_{\partial B_1} \ge \frac{W(1)}{2} \quad \text{in} \quad B_1,$$

since the convergence $\phi_\lambda \to 0$ as $\lambda \to 0$ is uniform on any compact set of $\mathbb{R}^N\backslash\{0\}$. Then $U_\lambda \ge \frac{W(1)}{4} > 0$ in B_1 since $c_i(\lambda) \to 0$ as $\lambda \to 0$ and $\chi(r), Z_i (1 \le i \le N)$ are bounded. Thus we proved the claim. □

Define now $F_\lambda(\xi)$ by

$$\begin{aligned} F_\lambda^{(j)}(\xi) : &= \int_{\mathbb{R}^N} -\bar{K}(\tfrac{x}{\lambda})U_\lambda^p \tfrac{\partial W}{\partial x_j}(x+\xi) + \int_{\mathbb{R}^N} N(\phi_\lambda)\tfrac{\partial W}{\partial x_j} \\ &\sim -\lambda^N \int_{\mathbb{R}^N}\Big(K(x) - K_\infty\Big)\tfrac{\partial W}{\partial x_j}(\xi) + o(\lambda^N) \end{aligned} \tag{4.21}$$

and $F_\lambda = (F_\lambda^{(1)}, \dots, F_\lambda^{(N)})$. Fix now $\delta > 0$ small. Then from (4.19)–(4.21), we have for small λ

$$\langle F_\lambda(\xi), \xi\rangle \ne 0 \quad \text{for all} \quad |\xi| = \delta.$$

By degree theory we deduce that $F_\lambda(\xi)$ has a zero point in B_δ.

Case (b.1): $N - \frac{2p+2}{p-1} < \mu < N$.

Obviously

$$\int_{\mathbb{R}^N} \bar{K}(\frac{x-\xi}{\lambda})W^p\frac{\partial W}{\partial x_j} \sim \lambda^\mu \int_{\mathbb{R}^N} f(\frac{x}{|x|})|x|^{-\mu}W^p(x+\xi)\frac{\partial W}{\partial x_j}(x+\xi).$$

By the above computation, for $\theta_1, \theta_2 \in (0, N-2)$, we have

$$\int_{\mathbb{R}^N} |N(\phi_\lambda)\frac{\partial W}{\partial x_j}| = O\Big(\lambda^{2\min\{2+\theta_1,\mu\}} + \lambda^{p\min\{2+\theta_2,\mu\}}\Big).$$

If we choose $2\theta_1 = \mu - 2$ and $p\theta_2 = \mu$ which are possible since $\mu < N$ and $p > \frac{N+2}{N-2}$, then

$$\int_{\mathbb{R}^N} |N(\phi_\lambda)\frac{\partial W}{\partial x_j}| = o(\lambda^\mu), \tag{4.22}$$

$$\int_{\mathbb{R}^N} |\bar{K}(\frac{x-\xi}{\lambda})\big((W+\phi_\lambda)^p - W^p\big)\frac{\partial W}{\partial x_j}| = \int_{\mathbb{R}^N\backslash B_1(\xi)} \cdots + \int_{B_1(\xi)\backslash B_{\lambda R}(\xi)} \cdots + \int_{B_{\lambda R}(\xi)} \cdots$$

$$\begin{aligned}\int_{B_{\lambda R}(\xi)}|\bar K(\tfrac{x-\xi}{\lambda})\big((W+\phi_\lambda)^p-W^p\big)\tfrac{\partial W}{\partial x_j}| &\le C\int_{B_{\lambda R}(\xi)}(|\phi_\lambda|^p+|\phi_\lambda|)\\ &\le C\|\phi_\lambda\|_{*,\xi}^{(\theta)}\int_{B_{\lambda R}(\xi)}|x-\xi|^{-\theta}\\ &\le C\|\phi_\lambda\|_{*,\xi}^{(\theta)}(\lambda R)^{N-\theta}\\ &\le C\lambda^{\min\{2+\theta,\mu\}+N-\theta}=o(\lambda^\mu),\end{aligned}\tag{4.23}$$

$$\begin{aligned}\int_{B_1(\xi)\setminus B_{\lambda R}(\xi)}|\bar K(\tfrac{x-\xi}{\lambda})\big((W+\phi_\lambda)^p-W^p\big)\tfrac{\partial W}{\partial x_j}| &\le C\lambda^\mu\int_{B_1(\xi)\setminus B_{\lambda R}(\xi)}|x-\xi|^{-\mu}(|\phi_\lambda|^p+|\phi_\lambda|)\\ &\le C\lambda^\mu\|\phi_\lambda\|_{*,\xi}^{(\theta)}\int_{B_1(\xi)\setminus B_{\lambda R}(\xi)}|x-\xi|^{-\mu-\theta}\\ &\le C\lambda^\mu\|\phi_\lambda\|_{*,\xi}^{(\theta)}(\lambda R)^{N-\mu-\theta}\\ &\le C\lambda^{\mu+\min\{2+\theta,\mu\}+N-\mu-\theta}=o(\lambda^\mu),\end{aligned}\tag{4.24}$$

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_1(\xi)}|\bar K(\tfrac{x-\xi}{\lambda})\big((W+\phi_\lambda)^p-W^p\big)\tfrac{\partial W}{\partial x_j}| &\le C\lambda^\mu\|\phi_\lambda\|_{*,\xi}^{(\theta)}\int_{\mathbb{R}^N\setminus B_1(\xi)}|x-\xi|^{-3-\frac{4}{p-1}-\mu}\\ &\le C\lambda^{\mu+\min\{2+\theta,\mu\}}=o(\lambda^\mu).\end{aligned}\tag{4.25}$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^N}\big(N(\phi_\lambda)-\bar K(\tfrac{x-\xi}{\lambda})(W+\phi_\lambda)^p\big)\tfrac{\partial W}{\partial x_j} &= -\int_{\mathbb{R}^N}\bar K(\tfrac{x-\xi}{\lambda})W^p\tfrac{\partial W}{\partial x_j}+o(\lambda^\mu)\\ &\sim -\lambda^\mu\int_{\mathbb{R}^N}|x|^{-\mu}f(\tfrac{x}{|x|})W^p(x+\xi)\tfrac{\partial W(x+\xi)}{\partial x_j}+o(\lambda^\mu)\end{aligned}\tag{4.26}$$

Define now $\tilde F$ to be given by

$$\tilde F(\xi):=-\frac{1}{p+1}\int_{\mathbb{R}^N}|x|^{-\mu}f(\frac{x}{|x|})W(x+\xi)^{p+1}.$$

By Dominate Convergence Theorem we get

$$\tilde F(\xi)=-\frac{\beta^{\frac{p+1}{p-1}}}{p+1}|\xi|^{N-\mu-\frac{2(p+1)}{p-1}}\int_{\mathbb{R}^N}|x|^{-\mu}f(\frac{x}{|x|})|x+\frac{\xi}{|\xi|}|^{-\frac{2(p+1)}{p-1}}+o\Big(|\xi|^{N-\mu-\frac{2(p+1)}{p-1}}\Big)$$

and

$$\begin{aligned}\nabla\tilde F(\xi)\cdot\xi &= -\tfrac{\beta^{\frac{p+1}{p-1}}}{p+1}(N-\mu-\tfrac{2(p+1)}{p-1})|\xi|^{N-\mu-\frac{2(p+1)}{p-1}}\int_{\mathbb{R}^N}|x|^{-\mu}f(\tfrac{x}{|x|})|x+\tfrac{\xi}{|\xi|}|^{-\frac{2(p+1)}{p-1}}\\ &\quad+o\Big(|\xi|^{N-\mu-\frac{2(p+1)}{p-1}}\Big).\end{aligned}$$

Therefore $\nabla\tilde F(\xi)\cdot\xi\neq 0$ for all $|\xi|=R$ where R large. Using this and degree theory we obtain the existence of ξ such that $c_j=0, 1\le j\le N$ provided λ small enough.

Case (b.2): $\mu=N$.

In this case, we will have

$$\begin{aligned}G_j(\xi): &= \int_{\mathbb{R}^N}-\bar K(\tfrac{x}{\lambda})U_\lambda^p\tfrac{\partial W}{\partial x_j}(x+\xi)+\int_{\mathbb{R}^N}N(\phi_\lambda)\tfrac{\partial W}{\partial x_j}\\ &= \int_{\mathbb{R}^N}-\bar K(\tfrac{x}{\lambda})U_\lambda^p\tfrac{\partial W}{\partial x_j}(x+\xi)+o(\lambda^N)\end{aligned}\tag{4.27}$$

uniformly for ξ on compact sets of $\mathbb{R}^N$.

Similar to case (a), we derive that for small fixed ρ

$$\langle G(\xi),\xi\rangle\neq 0\quad\text{for all}\quad|\xi|=\rho.\tag{4.28}$$

Indeed, for $\rho>0$ small it holds

$$\langle\nabla W(\xi),\xi\rangle<0\quad\text{for all}\quad|\xi|=\rho.$$

Thus, for $\delta>0$ small and fixed

$$\gamma\equiv\sup_{x\in B_\delta}\langle\nabla W(x+\xi),\xi\rangle<0\quad\text{for all}\quad|\xi|=\rho.\tag{4.29}$$

We decompose

$$\int_{\mathbb{R}^N} -\bar{K}(\frac{x}{\lambda})U_\lambda^p\langle\nabla W(x+\xi),\xi\rangle = \int_{B_\delta}\cdots + \int_{\mathbb{R}^N\setminus B_\delta}\cdots$$

where

$$\begin{aligned} |\int_{\mathbb{R}^N\setminus B_\delta} -\bar{K}(\tfrac{x}{\lambda})U_\lambda^p\langle\nabla W(x+\xi),\xi\rangle| &\le C\lambda^N \int_{|x|\ge\delta} |x|^{-N}|x|^{-\frac{2}{p-1}-1} \\ &\le C\lambda^N. \end{aligned} \tag{4.30}$$

On the other hand, for $R>0$ we may write

$$\int_{B_\delta} -\bar{K}(\frac{x}{\lambda})U_\lambda^p\langle\nabla W(x+\xi),\xi\rangle = \int_{B_\delta\setminus B_{\lambda R}}\cdots + \int_{B_{\lambda R}}\cdots$$

We have

$$\int_{B_{\lambda R}} -\bar{K}(\frac{x}{\lambda})U_\lambda^p\langle\nabla W(x+\xi),\xi\rangle = O(\lambda^N). \tag{4.31}$$

Since, by Claim, we can get that

$$\int_{B_\delta\setminus B_{\lambda R}} -\bar{K}(\frac{x}{\lambda})U_\lambda^p\langle\nabla W(x+\xi),\xi\rangle \sim \int_{B_\delta\setminus B_{\lambda R}} \bar{K}(\frac{x}{\lambda}). \tag{4.32}$$

But

$$\begin{aligned} \int_{B_\delta\setminus B_{\lambda R}} \bar{K}(\tfrac{x}{\lambda}) &= \lambda^N \int_{B_\delta\setminus B_{\lambda R}} |x|^{-N} f(\tfrac{x}{|x|}) \\ &\quad + \lambda^N \int_{B_\delta\setminus B_{\lambda R}} |x|^{-N}\Big(\lambda^{-N}|x|^N\bar{K}(\tfrac{x}{\lambda}) - f(\tfrac{x}{|x|})\Big) \end{aligned} \tag{4.33}$$

and

$$\lambda^N \int_{B_\delta\setminus B_{\lambda R}} |x|^{-N} f(\frac{x}{|x|}) = \lambda^N \log\frac{1}{\lambda}\int_{S^{N-1}} f + O(1) \tag{4.34}$$

while given any $\varepsilon>0$ there is $R>0$ such that

$$|\lambda^N \int_{B_\delta\setminus B_{\lambda R}} |x|^{-N}\Big(\lambda^{-N}|x|^N\bar{K}(\frac{x}{\lambda}) - f(\frac{x}{|x|})\Big)| \le \varepsilon\lambda^N\log\frac{1}{\lambda}. \tag{4.35}$$

From (4.30)–(4.35) we deduce the validity of (4.28). Applying again degree theory we conclude that for some $|\xi|<\rho$ we have $G(\xi)=0$.

Case (c): $0<\mu\le N-\frac{2p+2}{p-1}$. For simplicity we assume that $C_0>0$.

As in the proof of part (b), we get that

$$\int_{\mathbb{R}^N} |N(\phi_\lambda)\frac{\partial W}{\partial x_j}| = o(\lambda^\mu). \tag{4.36}$$

Observe that

$$\int_{\mathbb{R}^N} \bar{K}(\frac{x-\xi}{\lambda})U_\lambda^p\frac{\partial W}{\partial x_j} = \int_{\mathbb{R}^N\setminus B_1(\xi)}\cdots + \int_{B_1(\xi)\setminus B_{\lambda R}(\xi)}\cdots + \int_{B_{\lambda R}(\xi)}\cdots$$

Obviously,

$$\int_{B_{\lambda R}(\xi)} |\bar{K}(\frac{x-\xi}{\lambda})U_\lambda^p\frac{\partial W}{\partial x_j}| \le C(\lambda R)^N = o(\lambda^\mu). \tag{4.37}$$

In $\mathbb{R}^N\setminus B_{\lambda R}(\xi), \bar{K}(\frac{x-\xi}{\lambda})$ doesn't change the sign provided R fixed large enough. Thus $\int_{B_1(\xi)\setminus B_{\lambda R}(\xi)}\cdots$ and $\int_{\mathbb{R}^N\setminus B_1(\xi)}\cdots$ have the same sign.

By the condition in Theorem 1.2 and the claim,

$$\int_{B_1(\xi)\setminus B_{\lambda R}(\xi)} \bar{K}(\frac{x-\xi}{\lambda})U_\lambda^p \frac{\partial W}{\partial x_j} \sim \lambda^\mu \int_{B_1(\xi)\setminus B_{\lambda R}(\xi)} \frac{\partial W}{\partial x_j}|x-\xi|^{-\mu}, \tag{4.38}$$

where $\int_{B_1(\xi)\setminus B_{\lambda R}(\xi)} |x-\xi|^{-\mu} = O(1)$ since $\mu < N$.

In $\mathbb{R}^N \setminus B_1(\xi)$, $\lambda^{-\mu}|\bar{K}(\frac{x-\xi}{\lambda})| \le C|x-\xi|^{-\mu}$, $|x-\xi|^{-\mu}U_\lambda^p \frac{\partial W}{\partial x_j} \in L^1(\mathbb{R}^N \setminus B_1(\xi))$ since

$$|U_\lambda| \le W + |\phi_\lambda| \le C|x-\xi|^{-\frac{2}{p-1}}.$$

By Dominated Convergence Theorem we get that

$$\lim_{\lambda\to 0} \lambda^{-\mu} \int_{\mathbb{R}^N\setminus B_1(\xi)} \bar{K}(\frac{x-\xi}{\lambda})U_\lambda^p \frac{\partial W}{\partial x_j} dx = \alpha. \tag{4.39}$$

Define now $F_\lambda(\xi)$ by

$$F_\lambda^{(j)}(\xi) := \int_{\mathbb{R}^N} -\bar{K}(\frac{x}{\lambda})U_\lambda^p \frac{\partial W}{\partial x_j}(x+\xi) + \int_{\mathbb{R}^N} N(\phi_\lambda)\frac{\partial W}{\partial x_j}$$

and $F_\lambda = (F_\lambda^{(1)}, \dots, F_\lambda^{(N)})$. Fix now $\delta > 0$ small. Then from (4.36)–(4.39), we have for small λ

$$\langle F_\lambda(\xi), \xi\rangle \ne 0 \quad \text{for all} \quad |\xi| = \delta.$$

By degree theory we deduce that $F_\lambda(\xi)$ has a zero point in B_δ. □

Remark 4.2. *The proof of Theorem 1.2 in the case $p = \frac{N+1}{N-3}$ follows exactly the same lines with the modified norms as defined in Remark 2.3. The argument works because we assume that $K(x) - K_\infty$ has decay, which implies that even the modified norms, the error $\|\bar{K}(W+\phi)^p\|_{**,\xi}^{(\sigma)}$ converges to 0. Indeed, we have*

$$\sup_{|x|\ge 1} |x-\xi|^{2+\frac{2}{p-1}+\alpha}|\bar{K}(\frac{x-\xi}{\lambda})(W+\phi)^p| \le C\lambda^\mu \sup_{|x|\ge 1}|x-\xi|^{\alpha-\mu} = C\lambda^\mu$$

provided that $\alpha < \mu$.

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