

NEW BLOW-UP PHENOMENA FOR $SU(N+1)$ TODA SYSTEM

MONICA MUSSO, ANGELA PISTOIA, AND JUNCHENG WEI

ABSTRACT. We consider the $SU(n+1)$ Toda system

$$(S_\lambda) \quad \begin{cases} \Delta u_1 + 2\lambda e^{u_1} - \lambda e^{u_2} - \dots - \lambda e^{u_k} = 0 & \text{in } \Omega, \\ \Delta u_2 - \lambda e^{u_1} + 2\lambda e^{u_2} - \dots - \lambda e^{u_k} = 0 & \text{in } \Omega, \\ \vdots & \ddots & \vdots \\ \Delta u_k - \lambda e^{u_1} - \lambda e^{u_2} - \dots + 2\lambda e^{u_k} = 0 & \text{in } \Omega, \\ u_1 = u_2 = \dots = u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

If $0 \in \Omega$ and Ω is symmetric with respect to the origin, we construct a family of solutions $(u_{1\lambda}, \dots, u_{k\lambda})$ to (S_λ) such that the i -th component $u_{i\lambda}$ blows-up at the origin with a mass $2^{i+1}\pi$ as λ goes to zero.

1. INTRODUCTION

Systems of elliptic equations in two dimensional spaces with exponential nonlinearity arise in many pure and applied disciplines such as Physics, Geometry, Chemistry and Biology (see Chern and Wolfson [9], Chipot, Shafrir and Wolansky [9], Guest [18] and Yang [41]). Recently there is also considerable interest in the study of Toda-like systems, due to the importance in differential and algebraic geometry, and also mathematical physics.

We start with the single component Liouville equation

$$\Delta u + \lambda e^u = 0 \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial\Omega \quad (1.1)$$

which has been extensively studied by many authors. Particular attention will be paid to the analysis of bubbling solutions. Let (u_k, λ_k) be a bubbling sequence to (1.1), namely a family of solutions to (1.1) with $\lambda_k \int_\Omega e^{u_k} \leq C$, for some constant C , and $\max_{x \in \Omega} u_k(x) \rightarrow +\infty$, as $k \rightarrow \infty$. Then it has been proved that all bubbles are simple, (see Brezis-Merle [3], Nagasaki-Suzuki [34], Li-Shafrir [23]), i.e. the local mass $\lim_{r \rightarrow 0} \lim_{k \rightarrow +\infty} \lambda_k \int_{B_r(x_k)} e^{u_k}$ equals 8π exactly. In fact in this case there is only one bubbling profile: after some rescaling, the bubble approaches to a solution of the Liouville equation

$$\Delta w + e^w = 0 \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^w < +\infty. \quad (1.2)$$

On the other hand, it is also possible to construct bubbling solutions with multiple concentrating points (see Baraket and Pacard [1], del Pino, Kowalczyk and Musso [13], Esposito, Grossi and Pistoia [14]). Degree formula has been obtained in Chen and Lin [6, 7]. Similar results can also be obtained when there are Dirac sources at the right hand side of (1.1) (see also Bartolucci, Chen, Lin and Tarantello [2]).

Let us now turn to systems of Liouville type equations. In particular, we concentrate on the so-called $SU(3)$ Toda system

$$\begin{cases} \Delta u_1 + 2\lambda e^{u_1} - \lambda e^{u_2} = 0 & \text{in } \Omega, \\ \Delta u_2 + 2\lambda e^{u_2} - \lambda e^{u_1} = 0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Date: March 19, 2013.

1991 Mathematics Subject Classification. 35J60, 35B33, 35J25, 35J20, 35B40.

Key words and phrases. Toda system, blow-up solutions, multiple blow-up points.

Systems of the above type (1.3) as well as its counterpart on a Riemannian surface M

$$\begin{cases} \Delta u_1 + 2\rho_1\left(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - \frac{1}{|M|}\right) - \rho_2\left(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - \frac{1}{|M|}\right) = 0 \\ \Delta u_2 + 2\rho_2\left(\frac{h_2 e^{u_2}}{\int h_2 e^{u_2}} - \frac{1}{|M|}\right) - \rho_1\left(\frac{h_1 e^{u_1}}{\int h_1 e^{u_1}} - \frac{1}{|M|}\right) = 0, \end{cases} \quad (1.4)$$

arise from many different research areas in geometry and physics. In physics, it is related to the relativistic version of non-abelian Chern-Simons models (see Dunne [11], Nolasco and Tarantello [36], Yang [40], Yang [41] and references therein). In geometry, the $SU(3)$ Toda system is closely related to holomorphic curves (or harmonic sequence) of M into $\mathbb{C}\mathbb{P}^2$ (see Bolton, Jensen, Rigoli and Woodward [4], Chern and Wolfson [9], Griffiths and Harris [17] and Guest [18]). When $M = S^2$, it was proved that the solution space of the $SU(3)$ Toda system is identical to the space of holomorphic curves of S^2 into $\mathbb{C}\mathbb{P}^3$. We refer to Lin, Wei and Ye [26] and the references therein.

For equation (1.3) or (1.4), the first main issue is to determine the set of critical masses, i.e, the limits of local masses $(\lambda_k \int_{B_r(x_k)} e^{u_{1,k}}, \lambda_k \int_{B_r(x_k)} e^{u_{2,k}})$ when $u_{1,k}(x_k) = \max_{B_{r_0}(x_k)} \max(u_{1,k}(x), u_{2,k}(x)) \rightarrow +\infty$ and $r_0 > 0$ is small radius.

In [19] (see Lin, Wei and Zhang [27] for another proof), Jost-Lin-Wang proved the following

Theorem 1.1. *Let p_j be a bubbling point, i.e., $\max_{B_{r_0}(p_j)} \max(u_{1,k}(x), u_{2,k}(x)) \rightarrow +\infty$ for some $r_0 > 0$. Define the local mass at p_j as*

$$\sigma_i(p_j) = \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \lambda_k \int_{B_r(p_j)} e^{u_{i,k}} \quad (1.5)$$

Then there are only five possibility for (σ_1, σ_2) , i.e., (σ_1, σ_2) could be one of $(4\pi, 0)$, $(0, 4\pi)$, $(8\pi, 4\pi)$, $(4\pi, 8\pi)$ and $(8\pi, 8\pi)$.

Unlike single equations, according to Theorem 1.1, there are five possible blow-up scenarios. A natural question is whether or not *all* these blow-up scenarios are possible. Note that if we take $u = v$, this reduces to the single Liouville equation. By the construction in [13] or in [14], $(8\pi, 8\pi)$ is possible for any domain.

The last blow-up scenario is called *fully blow-up* case. The limiting equation becomes the $SU(3)$ Toda system in \mathbb{R}^2

$$\begin{cases} \Delta w_1 + 2e^{w_1} - e^{w_2} = 0 \text{ in } \mathbb{R}^2, \int_{\mathbb{R}^2} e^{w_1} < +\infty, \\ \Delta w_2 + 2e^{w_2} - e^{w_1} = 0 \text{ in } \mathbb{R}^2, \int_{\mathbb{R}^2} e^{w_2} < +\infty \end{cases} \quad (1.6)$$

whose solutions are completely characterized in Jost and Wang [20] and Lin, Wei and Ye [26]. It is known that the masses are given by

$$\int_{\mathbb{R}^2} e^{w_1} = \int_{\mathbb{R}^2} e^{w_2} = 8\pi.$$

The purpose of this paper is to show that the intermediate blow-up scenario does indeed occur. Namely for $SU(3)$ Toda system (1.3) in a symmetric domain (see definition below), we shall prove the existence of blowing-up solutions with local masses $(8\pi, 4\pi)$ and $(4\pi, 8\pi)$. Note that there is no uniform limiting profile as in (1.6). Instead, both u_1 and u_2 have bubbles at the same place but with different blowing up rates and different limiting profiles (see remarks below).

In fact, more generally, we consider the $SU(n+1)$ Toda system

$$\begin{cases} \Delta u_1 + 2\lambda e^{u_1} - \lambda e^{u_2} - \dots - \lambda e^{u_k} = 0 & \text{in } \Omega, \\ \Delta u_2 - \lambda e^{u_1} + 2\lambda e^{u_2} - \dots - \lambda e^{u_k} = 0 & \text{in } \Omega, \\ \vdots & \ddots & \vdots \\ \Delta u_k - \lambda e^{u_1} - \lambda e^{u_2} - \dots + 2\lambda e^{u_k} = 0 & \text{in } \Omega, \\ u_1 = u_2 = \dots = u_k = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.7)$$

where Ω is a smooth bounded domain in \mathbb{R}^2 and λ is a small positive parameter.

We will assume that Ω is k -symmetric, i.e.

$$x \in \Omega \quad \text{if and only if} \quad \mathfrak{R}_k x \in \Omega, \quad \text{where} \quad \mathfrak{R}_k := \begin{pmatrix} \cos \frac{\pi}{k} & \sin \frac{\pi}{k} \\ -\sin \frac{\pi}{k} & \cos \frac{\pi}{k} \end{pmatrix}. \quad (1.8)$$

The following is the main result of this paper.

Theorem 1.2. *Assume that Ω is a k -symmetric domain (see (1.8)). If λ is small enough problem (1.7) has a solution $(u_\lambda^1, \dots, u_\lambda^k)$ such that $u_\lambda^i(x) = u_\lambda^i(\mathfrak{R}_k x)$ for any $x \in \Omega$. Moreover, it satisfies*

$$\lim_{\lambda \rightarrow 0} \lambda \int_{\Omega} e^{u_\lambda^i(x)} dx = 2^{i+1} \pi, \quad i = 1, \dots, k. \quad (1.9)$$

Remark 1.1. *The symmetry condition (1.8) is a technical condition. In the case of a general domain Ω with no symmetry, for $SU(3)$ Toda system with blow-up mass $(8\pi, 4\pi)$, $(4\pi, 8\pi)$, $(8\pi, 8\pi)$, some necessary conditions are needed. For example, in the fully blowing-up case, there are six necessary conditions (see Lin, Wei and Zhao [24, 25]). For our problem, in the case of a general domain with no symmetry, we expect that there should be at least four necessary conditions.*

Remark 1.2. *As remarked earlier, there are no fully coupled limiting profile. For each $i = 1, \dots, k$, after some scaling, u_i has the following limiting profile*

$$-\Delta w = |x|^{\alpha_i-2} e^w, \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{\alpha_i-2} e^w < \infty \quad (1.10)$$

where $\alpha_i = 2^i$. Equation (1.10) plays an important role in our construction. It is known that all solutions to (1.10) have been classified by Prajapat-Tarantello [38]. In fact solutions to (1.10) are also nondegenerate—a key property that we shall use later (see del Pino, Esposito and Musso [12] and Lin, Wei and Ye [26]).

Remark 1.3. *The construction we perform here is inspired by a recent result obtained by Grossi and Pistoia [16], where they consider the sinh-Poisson equation*

$$-\Delta u = \lambda \sinh u \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \quad (1.11)$$

Ω being a smooth bounded domain in \mathbb{R}^2 and λ being a small positive parameter. For any integer k , Grossi and Pistoia [16] construct a family of solutions to (1.11) which blows-up at the origin as $\lambda \rightarrow 0$ with positive and negative masses $4\pi k(k-1)$ and $4\pi k(k+1)$, respectively, provided $0 \in \Omega$ and Ω is symmetric with respect to the origin. In particular, their result gives a complete answer to an open problem formulated by Jost, Wang, Ye and Zhou in [21] similar to the one claimed in Theorem 1.1.

Remark 1.4. *In the case of $SU(3)$ Toda system, according to Lin, Wei and Zhang [27], there are two possible scenarios for the bubbling behavior $(8\pi, 4\pi)$. Theorem 1.2 exhibits the first type. The second type is such that both u_1 and u_2 have the limiting profile (1.1). u_1 is the sum of two bubbles and u_2 has only one bubble. An open question is if the second type bubbling exists.*

Let us comment on some recent related works. In [29, 30, 31], Lin and Zhang studied general Liouville type systems with nonnegative coefficients. For Toda systems with singularities, the classification of local masses is given in Lin, Wei and Zhang [27]. Sharp estimates for fully blow-up solutions for $SU(3)$ Toda system are given in Lin, Wei and Zhao [24, 25]. See also related studies by Malchiodi-Ndiaye [32], Ohtsuka and Suzuki [37]. As far as we know, Theorem 1.2 seems to be the first existence result on bubbling solutions to the $SU(3)$ Toda system.

Acknowledgment. Monica Musso has been partly supported by Fondecyt Grant 1120151 and CAPDE-Anillo ACT-125, Chile. Angela Pistoia has been supported by “Accordi Interuniversitari di Collaborazione Culturale e Scientifica Internazionale, A.F. 2012 between Università La Sapienza Roma and Pontificia Universidad Católica de Chile”. Juncheng Wei was supported by a GRF grant from RGC of Hong Kong. We thank Professors Chang-Shou Lin and Lei Zhang for their interests in this work.

2. THE ANSATZ

Let $\alpha \geq 2$. Let us introduce the functions

$$w_\delta^\alpha(x) := \ln 2\alpha^2 \frac{\delta^\alpha}{(\delta^\alpha + |x|^\alpha)^2} \quad x \in \mathbb{R}^2, \quad \delta > 0 \quad (2.1)$$

which solve the singular Liouville problem

$$-\Delta w = |x|^{\alpha-2} e^w \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{\alpha-2} e^{w(x)} dx < +\infty. \quad (2.2)$$

Functions w_δ^α with suitable choices of (α, δ) constitute the main terms in the bubbling profiles of u_i . Let us introduce the projection Pu of a function u into $H_0^1(\Omega)$, i.e.

$$\Delta Pu = \Delta u \quad \text{in } \Omega, \quad Pu = 0 \quad \text{on } \partial\Omega. \quad (2.3)$$

We look for a solution to (1.7) as

$$\mathbf{u}_\lambda := (u_{1\lambda}, \dots, u_{k\lambda}) = \mathbf{W}_\lambda + \boldsymbol{\phi}_\lambda, \quad (2.4)$$

with $\mathbf{W}_\lambda(x) := (W_\lambda^1, \dots, W_\lambda^k)$ and $\boldsymbol{\phi} := (\phi_{1\lambda}, \dots, \phi_{k\lambda})$.

Here for any $i = 1, \dots, k$

$$W_\lambda^i(x) := Pw_i(x) - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k Pw_j(x), \quad w_i(x) := w_{\delta_i^{\alpha_i}}^\alpha(x) \quad (2.5)$$

where

$$\alpha_i = 2^i \quad (2.6)$$

and the concentration parameters satisfy

$$\delta_i := d_i \lambda^{\frac{2^{k-i}}{\alpha_i}} = d_i \lambda^{2^{k-2i}} \quad \text{for some } d_i > 0. \quad (2.7)$$

Let us point out that from (2.7) the following relations hold

$$\frac{\delta_i}{\delta_{i+1}} = \frac{d_i}{d_{i+1}} \lambda^{\frac{3}{4} 2^{k-2i}} \quad \text{for } i = 1, \dots, k-1. \quad (2.8)$$

The rest term $\boldsymbol{\phi} \in \mathbf{H}_k$ where (see (1.8))

$$\mathbf{H}_k := H_e^k \quad \text{with } H_e := \{ \boldsymbol{\phi} \in H_0^1(\Omega) : \phi(x) = \phi(\mathfrak{R}_k x) \text{ for any } x \in \Omega \}. \quad (2.9)$$

The choice of δ_i 's and α_i 's is motivated by the need for the interaction among bubbles to be small. Indeed, an important feature is that each bubble interacts with the other one and in general the interaction is not negligible! The interaction will be measured in Lemma 3.1 using the function $\Theta := (\Theta_1, \dots, \Theta_k)$ defined as

$$\Theta_j(y) := \left(Pw_j - w_j - \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^k Pw_i \right) (\delta_j y) - (\alpha_j - 2) \ln |\delta_j y| + \ln(2\lambda), \quad j = 1, \dots, k. \quad (2.10)$$

The choice of parameters α_j and δ_j made in (2.6) and (2.7) ensures that Θ_j is small. In order to estimate Θ_j we need to introduce the sets

$$A_i := \left\{ x \in \Omega : \sqrt{\delta_i \delta_{i-1}} \leq |x| \leq \sqrt{\delta_i \delta_{i+1}} \right\}, \quad i = 1, \dots, k, \quad (2.11)$$

where we set $\delta_0 := 0$ and $\delta_{k+1} := +\infty$.

We point out that if $j, \ell = 1, \dots, k$

$$\frac{A_j}{\delta_\ell} = \left\{ y \in \frac{\Omega}{\delta_\ell} : \frac{\sqrt{\delta_{j-1} \delta_j}}{\delta_\ell} \leq |y| \leq \frac{\sqrt{\delta_j \delta_{j+1}}}{\delta_\ell} \right\}$$

and so *roughly speaking* $\frac{A_j}{\delta_j^\ell}$ shrinks to the origin if $\ell > j$, $\frac{A_j}{\delta_j^\ell}$ invades the whole space \mathbb{R}^2 and $\frac{A_j}{\delta_j^\ell}$ runs off to infinity if $\ell < j$.

More precisely, in order to have Θ_j small in Lemma 2.2 we will need to choose δ_j 's and α_j 's so that

$$-(\alpha_j - 2) + \sum_{\substack{i=1 \\ i < j}}^k \alpha_i = 0 \quad (2.12)$$

and

$$-\alpha_j \ln \delta_j + \sum_{\substack{i=1 \\ i > j}}^k \alpha_i \ln \delta_i - \ln(2\alpha_j^2) + h_j(0) - \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^k h_i(0) + \ln 2\lambda = 0, \quad (2.13)$$

where we agree that if $j = 1$ or $j = k$ the sum over the indices $i < j$ or $i > j$ is zero, respectively. Here $h_i(x) := 4\pi\alpha_i H(x, 0)$. Moreover,

$$G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} + H(x, y), \quad x, y \in \Omega \quad (2.14)$$

is the Green's function of the Dirichlet Laplacian in Ω and $H(x, y)$ is its regular part.

By (2.12) we immediately deduce

$$\alpha_1 = 2 \quad \text{and} \quad \alpha_{j+1} = 2\alpha_j \quad \text{if } j = 1, \dots, k-1 \quad (2.15)$$

and therefore (2.6). Moreover, by (2.13) we immediately deduce that

$$\delta_k^{\alpha_k} = \frac{e^{h_k(0) - \frac{1}{2} \sum_{\substack{i=1 \\ i \neq k}}^k h_i(0)}}{\alpha_k^2} \lambda \quad (2.16)$$

and

$$\delta_j^{\alpha_j} = \frac{e^{h_j(0) - \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^k h_i(0)}}{\alpha_j^2} \delta_{j+1}^{\alpha_{j+1}} \dots \delta_k^{\alpha_k} \lambda \quad \text{for } j = 1, \dots, k-1, \quad (2.17)$$

which implies (2.7).

By the maximum principle we easily deduce that

Lemma 2.1.

$$\begin{aligned} Pw_i(x) &= w_i(x) - \ln(2\alpha_i^2 \delta_i^{\alpha_i}) + h_i(x) + O(\delta_i^{\alpha_i}) \\ &= -2 \ln(\delta_i^{\alpha_i} + |x|^{\alpha_i}) + h_i(x) + O(\delta_i^{\alpha_i}) \end{aligned} \quad (2.18)$$

and for any $i, j = 1, \dots, k$

$$Pw_i(\delta_j y) = \begin{cases} -2\alpha_i \ln(\delta_j |y|) + h_i(0) \\ \quad + O\left(\frac{1}{|y|^{\alpha_i}} \left(\frac{\delta_i}{\delta_j}\right)^{\alpha_i}\right) + O(\delta_j |y|) + O(\delta_i^{\alpha_i}) & \text{if } i < j, \\ -2\alpha_i \ln \delta_i - 2 \ln(1 + |y|^{\alpha_i}) + h_i(0) \\ \quad + O(\delta_i |y|) + O(\delta_i^{\alpha_i}) & \text{if } i = j, \\ -2\alpha_i \ln \delta_i + h_i(0) \\ \quad + O\left(|y|^{\alpha_i} \left(\frac{\delta_j}{\delta_i}\right)^{\alpha_i}\right) + O(\delta_j |y|) + O(\delta_i^{\alpha_i}) & \text{if } i > j. \end{cases} \quad (2.19)$$

Here $h_i(x) := 4\pi\alpha_i H(x, 0)$.

Now, we are in position to prove the following crucial estimates.

Lemma 2.2. Assume (2.6) and (2.7). If $j = 1, \dots, k$ we have

$$|\Theta_j(y)| = O\left(\delta_j|y| + \lambda^{\frac{3}{2k}}\right) \quad \text{for any } y \in \frac{A_j}{\delta_j} \quad (2.20)$$

and in particular

$$\sup_{y \in \frac{A_j}{\delta_j}} |\Theta_j(y)| = O(1). \quad (2.21)$$

Proof. By Lemma 2.1 (also using the mean value theorem $h_j(\delta_j|y|) = h_j(0) + O(\delta_j|y|)$), by (2.12) and by (2.13) we deduce

$$\begin{aligned} \Theta_j(y) &= \left[-\alpha_j \ln \delta_j - \ln(2\alpha_j^2) + h_j(0) + O(\delta_j|y|) + O\left(\delta_j^{\alpha_j}\right) \right] - (\alpha_j - 2) \ln |\delta_j y| \\ &\quad - \frac{1}{2} \sum_{i < j} \left[-2\alpha_i \ln(\delta_j|y|) + h_i(0) + O\left(\frac{1}{|y|^{\alpha_i}} \left(\frac{\delta_i}{\delta_j}\right)^{\alpha_i}\right) + O(\delta_j|y|) + O(\delta_i^{\alpha_i}) \right] \\ &\quad - \frac{1}{2} \sum_{i > j} \left[-2\alpha_i \ln \delta_i + h_i(0) + O\left(|y|^{\alpha_i} \left(\frac{\delta_j}{\delta_i}\right)^{\alpha_i}\right) + O(\delta_j|y|) + O(\delta_i^{\alpha_i}) \right] \\ &\quad + \ln 2\lambda \\ &= \underbrace{\left[-\alpha_j \ln \delta_j + \sum_{i > j} \alpha_i \ln \delta_i - \ln(2\alpha_j^2) + h_j(0) - \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^k h_i(0) + \ln 2\lambda \right]}_{= 0 \text{ because of (2.13)}} \\ &\quad + \underbrace{\left[-(\alpha_j - 2) + \sum_{i < j} \alpha_i \right]}_{= 0 \text{ because of (2.12)}} \ln(\delta_j|y|) \\ &\quad + O(\delta_j|y|) + \sum_{i=1}^k O(\delta_i^{\alpha_i}) + \sum_{i < j} O\left(\frac{1}{|y|^{\alpha_i}} \left(\frac{\delta_i}{\delta_j}\right)^{\alpha_i}\right) + \sum_{i > j} O\left(|y|^{\alpha_i} \left(\frac{\delta_j}{\delta_i}\right)^{\alpha_i}\right) \\ &= O(\delta_j|y|) + \sum_{i=1}^k O(\delta_i^{\alpha_i}) + \sum_{i < j} O\left(\frac{1}{|y|^{\alpha_i}} \left(\frac{\delta_i}{\delta_j}\right)^{\alpha_i}\right) + \sum_{i > j} O\left(|y|^{\alpha_i} \left(\frac{\delta_j}{\delta_i}\right)^{\alpha_i}\right). \end{aligned}$$

By (2.17) we deduce that

$$O(\delta_i^{\alpha_i}) = O(\lambda) \quad \text{because } 1 \leq i \leq k.$$

Moreover, if $y \in \frac{A_j}{\delta_j}$ then $\sqrt{\frac{\delta_{j-1}}{\delta_j}} \leq |y| \leq \sqrt{\frac{\delta_{j+1}}{\delta_j}}$ and so if $j = 2, \dots, k$ and $i < j$ we have

$$O\left(\frac{1}{|y|^{\alpha_i}} \left(\frac{\delta_i}{\delta_j}\right)^{\alpha_i}\right) = O\left(\left(\frac{\delta_i^2}{\delta_{j-1}\delta_j}\right)^{\frac{\alpha_i}{2}}\right) = O\left(\left(\frac{\delta_{j-1}}{\delta_j}\right)^{\frac{\alpha_i}{2}}\right) = O\left(\lambda^{\frac{3}{2}2^{k-2j+i}}\right) = O\left(\lambda^{\frac{3}{2k}}\right),$$

and if $j = 1, \dots, k-1$ and $i > j$ we have

$$O\left(|y|^{\alpha_i} \left(\frac{\delta_j}{\delta_i}\right)^{\alpha_i}\right) = O\left(\left(\frac{\delta_{j+1}\delta_j}{\delta_i^2}\right)^{\frac{\alpha_i}{2}}\right) = O\left(\left(\frac{\delta_j}{\delta_{j+1}}\right)^{\frac{\alpha_i}{2}}\right) = O\left(\lambda^{\frac{3}{8}2^{k-2j+i}}\right) = O\left(\lambda^{\frac{3}{2}}\right).$$

We used (2.8) and (2.6). Collecting all the previous estimates, we get (2.20).

Estimate (2.21) follows immediately by (2.20), because if $y \in \frac{A_j}{\delta_j}$ then $\delta_j|y| = O(1)$. \square

In the following, we will denote by

$$\|u\|_p := \left(\int_{\Omega} |u(x)|^p dx \right)^{\frac{1}{p}} \quad \text{and} \quad \|u\| := \left(\int_{\Omega} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}}$$

the usual norms in the Banach spaces $L^p(\Omega)$ and $H_0^1(\Omega)$, respectively. We also denote by $\mathbf{u} := (u_1, \dots, u_k) \in (H_0^1(\Omega))^k$ and we set

$$\|\mathbf{u}\|_p = \sum_{i=1}^k \|u_i\|_p \quad \text{and} \quad \|\mathbf{u}\| = \sum_{i=1}^k \|u_i\|.$$

3. ESTIMATE OF THE ERROR TERM

In this section we will estimate the following error term

$$\begin{aligned} \mathcal{R}_\lambda(x) &:= \left(R_\lambda^1(x), \dots, R_\lambda^k(x) \right), \quad x \in \Omega, \quad \text{where} \\ R_\lambda^j(x) &:= -\Delta W_\lambda^j(x) - 2\lambda e^{W_\lambda^j(x)} + \lambda \sum_{\substack{i=1 \\ i \neq j}}^k e^{W_\lambda^i(x)}, \quad j = 1, \dots, k. \end{aligned} \quad (3.1)$$

Lemma 3.1. *Let \mathcal{R}_λ as in (3.1). There exists $p_0 > 1$ and $\lambda_0 > 0$ such that for any $p \in (1, p_0)$ and $\lambda \in (0, \lambda_0)$ we have*

$$\|\mathcal{R}_\lambda\|_p = O\left(\lambda^{\frac{1}{2k} \frac{2-p}{p}}\right).$$

Proof. We will show that if p is close enough to 1

$$\|R_\lambda^i\|_p = O\left(\lambda^{\frac{1}{2k} \frac{2-p}{p}}\right), \quad i = 1, \dots, k. \quad (3.2)$$

The claim will follow. By (2.5) we have

$$\begin{aligned} R_\lambda^i &= -\Delta W_\lambda^i - 2\lambda e^{W_\lambda^i} + \lambda \sum_{\substack{j=1 \\ j \neq i}}^k e^{W_\lambda^j} \\ &= -\Delta \left(Pw_i - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k Pw_j \right) - 2\lambda e^{Pw_i - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k Pw_j} + \lambda \sum_{\substack{j=1 \\ j \neq i}}^k e^{Pw_j - \frac{1}{2} \sum_{\substack{\ell=1 \\ \ell \neq j}}^k Pw_\ell} \\ &= |x|^{\alpha_i - 2} e^{w_i(x)} - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k |x|^{\alpha_j - 2} e^{w_j(x)} - 2\lambda e^{Pw_i - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k Pw_j} + \lambda \sum_{\substack{j=1 \\ j \neq i}}^k e^{Pw_j - \frac{1}{2} \sum_{\substack{\ell=1 \\ \ell \neq j}}^k Pw_\ell} \end{aligned} \quad (3.3)$$

Therefore, by (2.10) we get

$$\begin{aligned}
\int_{\Omega} |R_{i\lambda}(x)|^p dx &= O \left(\sum_{j=1}^k \int_{\Omega} \left| |x|^{\alpha_j-2} e^{w_j(x)} - 2\lambda e^{Pw_j - \frac{1}{2} \sum_{\ell \neq j} Pw_{\ell}} \right|^p dx \right) \\
&= O \left(\sum_{j=1}^k \int_{A_j} \left| |x|^{\alpha_j-2} e^{w_j(x)} - 2\lambda e^{Pw_j - \frac{1}{2} \sum_{\ell \neq j} Pw_{\ell}} \right|^p dx \right) \\
&+ O \left(\sum_{j=1}^k \sum_{\substack{r=1 \\ r \neq j}}^k \int_{A_r} \left| |x|^{\alpha_j-2} e^{w_j(x)} \right|^p dx \right) + O \left(\sum_{j=1}^k \sum_{\substack{r=1 \\ r \neq j}}^k \int_{A_r} \left| \lambda e^{Pw_j(x) - \frac{1}{2} \sum_{\ell \neq j} Pw_{\ell}(x)} \right|^p dx \right) \quad (3.4)
\end{aligned}$$

Let us estimate the first term in (3.4), which gives the rate of $\|R_{\lambda}^i\|_p$. For any $j = 1, \dots, k$ we have

$$\begin{aligned}
&\int_{A_j} \left| |x|^{\alpha_j-2} e^{w_j(x)} - 2\lambda e^{Pw_j - \frac{1}{2} \sum_{\ell \neq j} Pw_{\ell}} \right|^p dx \quad (\text{we use (2.10)}) \\
&= \int_{A_j} \left| |x|^{\alpha_j-2} e^{w_j(x)} \left[1 - e^{\Theta_j(x/\delta_j)} \right] \right|^p dx \quad (\text{we set } x = \delta_j y) \\
&= \delta_j^{2-2p} \int_{\frac{A_j}{\delta_j}} \frac{|y|^{(\alpha_j-2)p}}{(1+|y|^{\alpha_j})^{2p}} \left| 1 - e^{\Theta_j(y)} \right|^p dy \quad (\text{we use that } e^t - 1 = e^{\eta t} \text{ for some } \eta \in (0, 1) \text{ and Lemma 2.2}) \\
&= O \left(\delta_j^{2-2p} \int_{\frac{A_j}{\delta_j}} \frac{|y|^{(\alpha_j-2)p}}{(1+|y|^{\alpha_j})^{2p}} |\Theta_j(y)|^p dy \right) = \\
&= O \left(\delta_j^{2-2p} \int_{\frac{A_j}{\delta_j}} \frac{|y|^{(\alpha_j-2)p}}{(1+|y|^{\alpha_j})^{2p}} \left| \delta_j |y| + \lambda \frac{3}{2^k} \right|^p dy \right) \quad (\text{we use that (2.7)}) \\
&= O \left(\delta_j^{2-2p} \lambda \frac{3}{2^k} p \right) + O \left(\delta_j^{2-p} \right) \quad (\text{we use that } \delta_1 \leq \delta_j \leq \delta_k) \\
&= O \left(\lambda^{2^{k-2}(1-p) + \frac{3}{2^k} p} \right) + O \left(\lambda^{\frac{1}{2^k}(2-p)} \right) = O \left(\lambda^{\frac{1}{2^k}(2-p)} \right). \quad (3.5)
\end{aligned}$$

Let us estimate the second term in (3.4). For any $j = 1, \dots, k$ and $r \neq j$ we have

$$\begin{aligned}
&\int_{A_r} \left| \frac{|x|^{\alpha_j-2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \right|^p dx \quad (\text{we scale } x = \delta_j y) \\
&= C \delta_j^{2-2p} \int_{\frac{\sqrt{\delta_r-1}\delta_r}{\delta_j} \leq |y| \leq \frac{\sqrt{\delta_r}\delta_r+1}{\delta_j}} \frac{|y|^{(\alpha_j-2)p}}{(1+|y|^{\alpha_j})^{2p}} dy
\end{aligned}$$

$$\begin{aligned}
& \left\{ \begin{aligned} & O \left(\delta_j^{2-2p} \left(\frac{\sqrt{\delta_r \delta_{r+1}}}{\delta_j^2} \right)^{(\alpha_j-2)p+2} \right) = O \left(\delta_j^{2-2p} \left(\frac{\delta_r}{\delta_{r+1}} \right)^{\frac{(\alpha_j-2)p+2}{2}} \right) \\ & \text{if } r = 1, \dots, k-1 \text{ and } j > r, \end{aligned} \right. \\
= & \left\{ \begin{aligned} & O \left(\delta_j^{2-2p} \left(\frac{\delta_j}{\sqrt{\delta_{r-1} \delta_r}} \right)^{-(\alpha_j+2)p+2} \right) = O \left(\delta_j^{2-2p} \left(\frac{\delta_{r-1}}{\delta_r} \right)^{\frac{(\alpha_j+2)p-2}{2}} \right) \\ & \text{if } r = 2, \dots, k \text{ and } j < r. \end{aligned} \right. \\
& \text{(we use (2.7) and (2.8))} \\
& \left\{ \begin{aligned} & O \left(\delta_1^{2-2p} \left(\frac{\delta_{j-1}}{\delta_j} \right)^{\frac{\alpha_j}{2}-(p-1)} \right) = O \left(\lambda^{-2k-1(p-1)} \lambda^{\frac{3}{4}2^{k-2j+2}} (2^{j-1}-(p-1)) \right) \\ & \text{if } r = 1, \dots, k-1 \text{ and } j > r, \end{aligned} \right. \\
= & \left\{ \begin{aligned} & O \left(\delta_1^{2-2p} \left(\frac{\delta_{k-1}}{\delta_k} \right)^{\frac{\alpha_j}{2}+(p-1)} \right) = O \left(\lambda^{-2k-1(p-1)} \lambda^{\frac{3}{2k}(2^{j-1}+(p-1))} \right) \\ & \text{if } r = 2, \dots, k \text{ and } j < r. \end{aligned} \right. \\
& \text{(we compare with (3.5))} \\
& = o \left(\lambda^{\frac{1}{2k}(2-p)} \right), \tag{3.6}
\end{aligned}$$

for some provided p is close enough to 1.

Let us estimate the third term in (3.4). For any $j = 1, \dots, k$ and $r \neq j$,

$$\begin{aligned}
& \int_{A_r} \left| \lambda e^{Pw_j(x) - \frac{1}{2} \sum_{\substack{\ell=1 \\ \ell \neq j}}^k Pw_\ell(x)} \right|^p dx \quad \text{(we apply (2.18))} \\
& = O \left(\lambda^p \int_{A_r} \left| \frac{1}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \prod_{\ell \neq j} (\delta_\ell^{\alpha_\ell} + |x|^{\alpha_\ell}) \right|^p dx \right) \tag{3.7}
\end{aligned}$$

Firstly, we consider the case $k = 2$. We have only to estimate

$$\int_{A_1} \left| \frac{\delta_1^{\alpha_1} + |x|^{\alpha_1}}{(\delta_2^{\alpha_2} + |x|^{\alpha_2})^2} \right|^p dx \quad \text{and} \quad \int_{A_2} \left| \frac{\delta_2^{\alpha_2} + |x|^{\alpha_2}}{(\delta_1^{\alpha_1} + |x|^{\alpha_1})^2} \right|^p dx \tag{3.8}$$

with $\alpha_1 = 2$, $\alpha_2 = 4$ and $\delta_1 \sim \lambda$, $\delta_2 \sim \lambda^{\frac{1}{4}}$. Therefore, we have

$$\begin{aligned}
& \int_{A_1} \left| \frac{\delta_1^{\alpha_1} + |x|^{\alpha_1}}{(\delta_2^{\alpha_2} + |x|^{\alpha_2})^2} \right|^p dx = \int_{\{x \in \Omega : |x| \leq \sqrt{\delta_1 \delta_2}\}} \left| \frac{\delta_1^2 + |x|^2}{(\delta_2^4 + |x|^4)^2} \right|^p dx \\
& = O \left(\int_{\{|y| \leq \sqrt{\frac{\delta_1}{\delta_2}}\}} \delta_2^{2-8p} \delta_1^{2p} \frac{1}{(1+|y|^4)^{2p}} dy \right) + O \left(\int_{\{|y| \leq \sqrt{\frac{\delta_1}{\delta_2}}\}} \delta_2^{2-6p} \frac{|y|^{2p}}{(1+|y|^4)^2} dy \right) \\
& = O \left(\delta_2^{1-8p} \delta_1^{2p+1} \right) + O \left(\delta_2^{1-7p} \delta_1^{p+1} \right) = O \left(\lambda^{\frac{5}{4}} \right) + O \left(\lambda^{\frac{5}{4} - \frac{3}{4}p} \right) \tag{3.9}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{A_2} \left| \frac{\delta_2^{\alpha_2} + |x|^{\alpha_2}}{(\delta_1^{\alpha_1} + |x|^{\alpha_1})^2} \right|^p dx = \int_{\{x \in \Omega : |x| \geq \sqrt{\delta_1 \delta_2}\}} \left| \frac{\delta_2^4 + |x|^4}{(\delta_1^2 + |x|^2)^2} \right|^p dx \\
& = O \left(\int_{\{|y| \geq \sqrt{\frac{\delta_2}{\delta_1}}\}} \delta_1^{2-4p} \delta_2^{4p} \frac{1}{(1+|y|^2)^{2p}} dy \right) + O \left(\int_{\Omega} dy \right) \\
& = O \left(\delta_2^{1+2p} \delta_1^{1-2p} \right) + O(1) = O \left(\lambda^{\frac{5}{4}} \right) + O(1)
\end{aligned} \tag{3.10}$$

By (3.9) and (3.10), we can compare (3.7) with (3.5) and we get

$$\int_{A_r} \left| \lambda e^{Pw_r(x) - \frac{1}{2} \sum_{\substack{\ell=1 \\ \ell \neq j}}^k Pw_\ell(x)} \right|^p dx = o \left(\lambda^{\frac{1}{4}(2-p)} \right) \tag{3.11}$$

provided p is close enough to 1.

Now, let us consider the general case. We estimate (3.7) when $p = 1$. We have to estimate the following terms

$$\lambda \delta_1^{\alpha_1} \dots \delta_{j-1}^{\alpha_{j-1}} \delta_{j+1}^{\alpha_{j+1}} \dots \delta_k^{\alpha_k} \int_{A_r} \frac{1}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} dx = O(\lambda) \tag{3.12}$$

$$\lambda \int_{A_r} \frac{|x|^{\alpha_1 + \dots + \alpha_{j-1} + \alpha_{j+1} + \dots + \alpha_k}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} dx = O \left(\lambda^{\frac{3}{2^k}(2^{k-1}-1)} \right) \tag{3.13}$$

$$\lambda \delta_{\sigma_{h+1}}^{\alpha_{\sigma_{h+1}}} \dots \delta_{\sigma_{k-1}}^{\alpha_{\sigma_{k-1}}} \int_{A_r} \frac{|x|^{\alpha_{\sigma_1} + \dots + \alpha_{\sigma_h}}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} dx = o(\lambda) \tag{3.14}$$

where $\{\sigma_1, \dots, \sigma_{k-1}\}$ is a permutation of the indices $\{1, \dots, j-1, j+1, \dots, k\}$.

Let us estimate (3.12).

$$\begin{aligned}
& \delta_1^{\alpha_1} \dots \delta_{j-1}^{\alpha_{j-1}} \delta_{j+1}^{\alpha_{j+1}} \dots \delta_k^{\alpha_k} \int_{A_r} \frac{1}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} dx \quad (\text{we scale } x = \delta_j y) \\
& = \delta_1^{\alpha_1} \dots \delta_{j-1}^{\alpha_{j-1}} \delta_{j+1}^{\alpha_{j+1}} \dots \delta_k^{\alpha_k} \delta_j^{2-2\alpha_j} \int_{\left\{ \frac{\sqrt{\delta_{r-1}\delta_r}}{\delta_j} \leq |y| \leq \frac{\sqrt{\delta_r\delta_{r+1}}}{\delta_j} \right\}} \frac{1}{(1+|y|^{\alpha_j})^2} dy \\
& = \begin{cases} O \left(\delta_1^{\alpha_1} \dots \delta_{j-1}^{\alpha_{j-1}} \delta_j^{2-\alpha_j} \lambda \right) & \text{if } j < k \text{ (because of (2.17))} \\ O \left(\delta_1^{\alpha_1} \delta_2^{\alpha_2} \dots \delta_{j-1}^{\alpha_{j-1}} \delta_{j+1}^{\alpha_{j+1}} \dots \delta_{k-1}^{\alpha_{k-1}} \delta_k^{2-\alpha_k} \right) & \text{if } j = k \end{cases} \\
& = \begin{cases} O \left(\frac{\delta_1^{\alpha_1}}{\delta_j^{\alpha_j}} \dots \delta_{j-1}^{\alpha_{j-1}} \delta_j^2 \lambda \right) & \text{if } j < k \\ O \left(\frac{\delta_1^{\alpha_1}}{\delta_k^{\alpha_k}} \dots \delta_{j-1}^{\alpha_{j-1}} \delta_{j+1}^{\alpha_{j+1}} \dots \delta_{k-1}^{\alpha_{k-1}} \delta_k^2 \right) & \text{if } j = k \end{cases} \\
& = O(1).
\end{aligned}$$

Therefore (3.12) follows.

Let us estimate (3.13). We immediately get

$$\int_{A_r} \frac{|x|^{\alpha_1+\dots+\alpha_{j-1}+\alpha_{j+1}+\dots+\alpha_k}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} dx = O(1) \quad \text{if } j < k, \quad (3.15)$$

because the function is integrable at the origin, since

$$\alpha_1 + \dots + \alpha_k = \sum_{i=1}^k 2^i = 2(2^k - 1) \quad (3.16)$$

which implies

$$\alpha_1 + \dots + \alpha_{j-1} + \alpha_{j+1} + \dots + \alpha_k + 2 - 2\alpha_j = 2^j(2^{k+1-j} - 3) > 0.$$

If $j = k$ we scale $x = \delta_k y$ and we get

$$\begin{aligned} & \int_{A_r} \frac{|x|^{\alpha_1+\dots+\alpha_{j-1}+\alpha_{j+1}+\dots+\alpha_k}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} dx \\ &= O \left(\delta_k^{\alpha_1+\dots+\alpha_{k-1}+2-2\alpha_k} \int_{\left\{ \frac{\sqrt{\delta_{r-1}\delta_r}}{\delta_k} \leq |y| \leq \frac{\sqrt{\delta_r\delta_{r+1}}}{\delta_k} \right\}} \frac{|y|^{\alpha_1+\dots+\alpha_{k-1}}}{(1+|y|^{\alpha_k})^2} dy \right) \\ &= \left(\delta_k^{-2k} \left(\frac{\delta_{k-1}}{\delta_k} \right)^{\alpha_1+\dots+\alpha_{k-1}} \right) \\ &= \left(\frac{1}{\lambda} \lambda^{\frac{3}{2^k}} (2^{k-1}-1) \right), \text{ because of (2.7) and (2.8).} \end{aligned} \quad (3.17)$$

By (3.15) and (3.17) we deduce (3.13)

Let us estimate (3.14). It is useful to point out that $2\alpha_j = \alpha_{j+1}$. Therefore, it is clear that

$$\delta_{\sigma_{h+1}}^{\alpha_{\sigma_{h+1}}} \dots \delta_{\sigma_{k-1}}^{\alpha_{\sigma_{k-1}}} \int_{A_r} \frac{|x|^{\alpha_{\sigma_1}+\dots+\alpha_{\sigma_h}}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} dx = O(1) \quad (3.18)$$

if $\sigma_i \geq j+1$ for some $i = 1, \dots, h$ or $\alpha_{\sigma_1} + \dots + \alpha_{\sigma_h} + 2 - 2\alpha_j \geq 0$.

It remains to consider the case $\sigma_i \leq j-1$ for any $i = 1, \dots, h$ and $\alpha_{\sigma_1} + \dots + \alpha_{\sigma_h} + 2 - 2\alpha_j < 0$.

In particular, it means that the set of indices $\{\sigma_{h+1}, \dots, \sigma_{k-1}\}$ has to contain a permutation of the indices $\{j+1, \dots, k\}$. Then, we can write (3.14) as

$$\begin{aligned} & \delta_{j+1}^{\alpha_{j+1}} \dots \delta_k^{\alpha_k} \delta_{\sigma_{h+1}}^{\alpha_{\sigma_{h+1}}} \dots \delta_{\sigma_{j-1}}^{\alpha_{\sigma_{j-1}}} \int_{A_r} \frac{|x|^{\alpha_{\sigma_1}+\dots+\alpha_{\sigma_h}}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} dx \quad (\text{we scale } x = \delta_j y) \\ &= \delta_{j+1}^{\alpha_{j+1}} \dots \delta_k^{\alpha_k} \delta_{\sigma_{h+1}}^{\alpha_{\sigma_{h+1}}} \dots \delta_{\sigma_{j-1}}^{\alpha_{\sigma_{j-1}}} \delta_j^{2-2\alpha_j+\alpha_{\sigma_1}+\dots+\alpha_{\sigma_h}} \int_{\left\{ \frac{\sqrt{\delta_{r-1}\delta_r}}{\delta_j} \leq |y| \leq \frac{\sqrt{\delta_r\delta_{r+1}}}{\delta_j} \right\}} \frac{|y|^{\alpha_{\sigma_1}+\dots+\alpha_{\sigma_h}}}{(1+|y|^{\alpha_j})^2} dy \\ &= \delta_{\sigma_{h+1}}^{\alpha_{\sigma_{h+1}}} \dots \delta_{\sigma_{j-1}}^{\alpha_{\sigma_{j-1}}} \delta_j^{2-\alpha_j+\alpha_{\sigma_1}+\dots+\alpha_{\sigma_h}} \int_{\left\{ \frac{\sqrt{\delta_{r-1}\delta_r}}{\delta_j} \leq |y| \leq \frac{\sqrt{\delta_r\delta_{r+1}}}{\delta_j} \right\}} \frac{|y|^{\alpha_{\sigma_1}+\dots+\alpha_{\sigma_h}}}{(1+|y|^{\alpha_j})^2} dy \quad (\text{because of (2.17)}) \\ &= O \left(\delta_j^{2-\alpha_j+\alpha_{\sigma_1}+\dots+\alpha_{\sigma_{j-1}}} \int_{\left\{ \frac{\sqrt{\delta_{r-1}\delta_r}}{\delta_j} \leq |y| \leq \frac{\sqrt{\delta_r\delta_{r+1}}}{\delta_j} \right\}} \frac{|y|^{\alpha_{\sigma_1}+\dots+\alpha_{\sigma_h}}}{(1+|y|^{\alpha_j})^2} dy \right) \quad (\text{because } \delta_{\sigma_i} \leq \delta_j) \end{aligned}$$

$$\begin{aligned}
&= O \left(\int_{\left\{ \frac{\sqrt{\delta_{r-1}\delta_r}}{\delta_j} \leq |y| \leq \frac{\sqrt{\delta_r\delta_{r+1}}}{\delta_j} \right\}} \frac{|y|^{\alpha_{\sigma_1} + \dots + \alpha_{\sigma_h}}}{(1 + |y|^{\alpha_j})^2} dy \right) \\
&\quad (\text{because } \alpha_{\sigma_1} + \dots + \alpha_{\sigma_{j-1}} = \alpha_1 + \dots + \alpha_{j-1} = 2(2^{j-1} - 1) = \alpha_j - 2) \\
&= \begin{cases} O \left(\left(\frac{\delta_{j-1}}{\delta_j} \right)^{\frac{\alpha_{\sigma_1} + \dots + \alpha_{\sigma_h} + 2}{2}} \right) & \text{if } r < j \\ O \left(\left(\frac{\delta_j}{\delta_{j+1}} \right)^{\frac{-(\alpha_{\sigma_1} + \dots + \alpha_{\sigma_h}) - 2 + 2\alpha_j}{2}} \right) & \text{if } r > j \end{cases} \\
&= o(1). \tag{3.19}
\end{aligned}$$

By (3.18) and (3.19) we deduce (3.14).

It is clear that if cp is close enough to 1, by (3.7), (3.12), (3.13) and (3.14) we can compare (3.7) with (3.5) and we get

$$\int_{A_r} \left| \lambda e^{Pw_j(x) - \frac{1}{2} \sum_{\substack{\ell=1 \\ \ell \neq j}}^k Pw_\ell(x)} \right|^p dx = o \left(\lambda^{\frac{1}{2^k}(2-p)} \right) \quad \text{if } r \neq j. \tag{3.20}$$

Finally, (3.2) follows by (3.4), (3.5), (3.6) and (3.20). That concludes the proof. \square

4. THE LINEAR THEORY

Let us consider the linear operator

$$\begin{aligned}
\mathcal{L}_\lambda(\phi) &:= \left(L_\lambda^1(\phi), \dots, L_\lambda^k(\phi) \right), \text{ where} \\
L_\lambda^i(\phi) &:= -\Delta\phi^i - |x|^{\alpha_i-2} e^{w_i(x)} \phi^i + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k |x|^{\alpha_j-2} e^{w_j(x)} \phi^j, \quad j = 1, \dots, k. \tag{4.1}
\end{aligned}$$

Let us study the invertibility of the linearized operator \mathcal{L}_λ .

Proposition 4.1. *For any $p > 1$ there exists $\lambda_0 > 0$ and $c > 0$ such that for any $\lambda \in (0, \lambda_0)$ and for any $\psi \in (L^p(\Omega))^k$ there exists a unique $\phi \in (W^{2,2}(\Omega))^k \cap \mathbf{H}_k$ solution of*

$$\mathcal{L}_\lambda(\phi) = \psi \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega,$$

which satisfies

$$\|\phi\| \leq c |\ln \lambda| \|\psi\|_p.$$

Proof. We argue by contradiction. Assume there exist $p > 1$, sequences $\lambda_n \rightarrow 0$, $\psi_n := (\psi_n^1, \dots, \psi_n^k) \in (L^\infty(\Omega))^k$ and $\phi_n := (\phi_n^1, \dots, \phi_n^k) \in (W^{2,2}(\Omega))^k \cap \mathbf{H}_k$ such that for any $i = 1, \dots, k$

$$-\Delta\phi_n^i - |x|^{\alpha_i-2} e^{w_i(x)} \phi_n^i + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k |x|^{\alpha_j-2} e^{w_j(x)} \phi_n^j = \psi_n^i, \text{ in } \Omega, \quad \phi_n^i = 0 \text{ on } \partial\Omega, \tag{4.2}$$

with $\delta_{1n}, \dots, \delta_{kn}$ defined as in (2.7) and

$$\|\phi_n\| = \sum_{i=1}^k \|\phi_n^i\| = 1 \quad \text{and} \quad |\ln \lambda_n| \|\psi_n\|_p = |\ln \lambda_n| \sum_{i=1}^k \|\psi_n^i\|_p \rightarrow 0. \tag{4.3}$$

For sake of simplicity, in the following we will omit the index n in all the sequences and we write $\phi_i = \phi_n^i$, $\psi_i = \psi_n^i$. For any $i = 1, \dots, k$ we define $\tilde{\phi}_i(y) := \phi_i(\delta_i y)$ with $y \in \Omega_i := \frac{\Omega}{\delta_i}$.

Step 1: we will show that

$$\tilde{\phi}_i(y) \rightarrow \gamma_i \frac{1 - |y|^{\alpha_i}}{1 + |y|^{\alpha_i}} \text{ for some } \gamma_i \in \mathbb{R} \quad (4.4)$$

weakly in $H_{\alpha_i}(\mathbb{R}^2)$ and strongly in $L_{\alpha_i}(\mathbb{R}^2)$ (see (6.4) and (6.4)).

First of all we claim that

$$\int_{\Omega} |x|^{\alpha_i-2} e^{w_i(x)} \phi_i^2(x) dx = O(1) \text{ for any } i = 1, \dots, k. \quad (4.5)$$

Indeed, we write (4.2) for two functions ϕ_i and ϕ_ℓ with $i \neq \ell$

$$-\Delta \phi_i - |x|^{\alpha_i-2} e^{w_i(x)} \phi_i + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k |x|^{\alpha_j-2} e^{w_j(x)} \phi_j = \psi_i, \text{ in } \Omega, \phi_i = 0 \text{ on } \partial\Omega, \quad (4.6)$$

$$-\Delta \phi_\ell - |x|^{\alpha_\ell-2} e^{w_\ell(x)} \phi_\ell + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq \ell}}^k |x|^{\alpha_j-2} e^{w_j(x)} \phi_j = \psi_\ell, \text{ in } \Omega, \phi_\ell = 0 \text{ on } \partial\Omega, \quad (4.7)$$

$$(4.8)$$

then we subtract the two equations

$$-\Delta(\phi_i - \phi_\ell) - \frac{3}{2} \left(|x|^{\alpha_i-2} e^{w_i(x)} \phi_i - |x|^{\alpha_\ell-2} e^{w_\ell(x)} \phi_\ell \right) = \psi_i - \psi_\ell, \text{ in } \Omega, \phi_i - \phi_\ell = 0 \text{ on } \partial\Omega,$$

we multiply by ϕ_i , we use (4.3) and we deduce

$$\int_{\Omega} |x|^{\alpha_i-2} e^{w_i(x)} \phi_i^2(x) dx = \int_{\Omega} |x|^{\alpha_\ell-2} e^{w_\ell(x)} \phi_\ell(x) \phi_i(x) dx + O(1),$$

which implies (by summing over the index ℓ)

$$(k-1) \int_{\Omega} |x|^{\alpha_i-2} e^{w_i(x)} \phi_i^2(x) dx = \sum_{\substack{\ell=1 \\ \ell \neq i}}^k \int_{\Omega} |x|^{\alpha_\ell-2} e^{w_\ell(x)} \phi_\ell(x) \phi_i(x) dx + O(1). \quad (4.9)$$

On the other hand, if we multiply the first equation (4.6) by ϕ_i and we use (4.3), we get

$$\int_{\Omega} |x|^{\alpha_i-2} e^{w_i(x)} \phi_i^2(x) dx = \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k \int_{\Omega} |x|^{\alpha_j-2} e^{w_j(x)} \phi_j(x) \phi_i(x) dx + O(1) \quad (4.10)$$

$$= \frac{k-1}{2} \int_{\Omega} |x|^{\alpha_i-2} e^{w_i(x)} \phi_i^2(x) dx + O(1), \quad (4.11)$$

where the last equality follows by (4.9). By (4.10) we immediately deduce (4.5) when $k \neq 3$. When $k = 3$ we have to argue in a different way. For any index i , we write the equation (4.2) as

$$-\Delta \phi_i - \frac{3}{2} |x|^{\alpha_i-2} e^{w_i(x)} \phi_i + \frac{1}{2} \sum_{j=1}^3 |x|^{\alpha_j-2} e^{w_j(x)} \phi_j = \psi_i, \text{ in } \Omega, \phi_i = 0 \text{ on } \partial\Omega, \quad (4.12)$$

and we sum over the index $i = 1, 2, 3$, so we get

$$-\Delta(\phi_1 + \phi_2 + \phi_3) = \psi_1 + \psi_2 + \psi_3, \text{ in } \Omega, \phi_1 + \phi_2 + \phi_3 = 0 \text{ on } \partial\Omega.$$

Since $\|\psi_1 + \psi_2 + \psi_3\|_p = o(1)$ (because of (4.3)), the standard regularity implies that $\|\phi_1 + \phi_2 + \phi_3\|_\infty = o(1)$. Now, we multiply equation in (4.12) by ϕ_i and we sum over the index $i = 1, 2, 3$, so we obtain

(using (4.3))

$$\begin{aligned}
\frac{3}{2} \sum_{i=1}^3 \int_{\Omega} |x|^{\alpha_i-2} e^{w_i(x)} \phi_i^2(x) dx &= 1 + \frac{1}{2} \sum_{j=1}^3 \int_{\Omega} |x|^{\alpha_j-j} e^{w_j(x)} \phi_j(x) (\phi_1 + \phi_2 + \phi_3)(x) dx \\
&\leq 1 + \frac{1}{2} \|\phi_1 + \phi_2 + \phi_3\|_{\infty} \sum_{j=1}^3 \int_{\Omega} |x|^{\alpha_j-j} e^{w_j(x)} |\phi_j(x)| dx \\
&\leq 1 + \frac{1}{2} \|\phi_1 + \phi_2 + \phi_3\|_{\infty} \sum_{j=1}^3 \left(\int_{\Omega} |x|^{\alpha_j-j} e^{w_j(x)} |\phi_j(x)|^2 \right)^{1/2} dx
\end{aligned}$$

which implies

$$\sum_{i=1}^3 \int_{\Omega} |x|^{\alpha_i-2} e^{w_i(x)} \phi_i^2(x) dx = O(1),$$

because $\|\phi_1 + \phi_2 + \phi_3\|_{\infty} = o(1)$. That proves (4.5) when $k = 3$.

Now, by (4.5) we deduce that each ϕ_i is bounded in the space $H_{\alpha_i}(\mathbb{R}^2)$ defined in (6.4). Indeed, if we scale we get

$$\int_{\Omega_j} |\nabla \tilde{\phi}_j(y)|^2 dy = \delta_j^2 \int_{\Omega_j} |\nabla \tilde{\phi}_j(\delta_j y)|^2 dy = \int_{\Omega} |\nabla \phi_j(x)|^2 dx = 1$$

and

$$\int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} (\tilde{\phi}_j(y))^2 dy = \int_{\Omega} 2\alpha_j^2 \frac{\delta_j^{\alpha_j} |x|^{\alpha_j-2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \phi_j^2(x) dx.$$

Finally, by Proposition (6.1) we can assume that (up to a subsequence) $\tilde{\phi}_j \rightharpoonup \tilde{\phi}_j^*$ weakly in $H_{\alpha_j}(\mathbb{R}^2)$ and strongly in $L_{\alpha_j}(\mathbb{R}^2)$.

Now, by (4.2) we deduce that each function $\tilde{\phi}_j$ solves the problem

$$-\Delta \tilde{\phi}_j = 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j + \rho_j(y) \text{ in } \Omega^j, \quad \tilde{\phi}_j = 0 \text{ on } \partial\Omega_j, \quad (4.13)$$

where

$$\rho_j(y) := \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^k 2\alpha_i^2 \frac{\delta_i^{\alpha_i} \delta_j^{\alpha_j} |y|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + \delta_j^{\alpha_j} |y|^{\alpha_i})^2} \phi_i(\delta_j y) + \delta_j^2 \psi_j(\delta_j y).$$

Now, let $\varphi \in C_0^{\infty}(\mathbb{R}^2)$ be a given function and let \mathcal{K} its support. It is clear that if n is large enough

$$\mathcal{K} \subset \frac{A_j}{\delta_j} = \left\{ y \in \Omega^j : \sqrt{\frac{\delta_{j-1}}{\delta_j}} \leq |y| \leq \sqrt{\frac{\delta_{j+1}}{\delta_j}} \right\},$$

where A_j is the annulus defined in (2.11). We multiply equation (4.13) by φ and we get

$$\int_{\mathcal{K}} \nabla \tilde{\phi}_j(y) \nabla \varphi(y) dy - \int_{\mathcal{K}} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j(y) \varphi(y) dy = \int_{\mathcal{K}} \rho_j(y) \varphi(y) dy.$$

Therefore, passing to the limit we get

$$\int_{\mathcal{K}} \nabla \tilde{\phi}_j^*(y) \nabla \varphi(y) dy - \int_{\mathcal{K}} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j^*(y) \varphi(y) dy = 0 \quad \forall \varphi \in C_0^{\infty}(\mathbb{R}^2), \quad (4.14)$$

because

$$\begin{aligned}
& \sum_{\substack{i=1 \\ i \neq j}}^k \int_{\mathcal{K}} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} \delta_j^{\alpha_j} |y|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + \delta_j^{\alpha_j} |y|^{\alpha_i})^2} \phi_i(\delta_j y) \varphi(y) dy \\
&= O \left(\sum_{\substack{i=1 \\ i \neq j}}^k \int_{\frac{A_j}{\delta_j}} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} \delta_j^{\alpha_j} |y|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + \delta_j^{\alpha_j} |y|^{\alpha_i})^2} |\phi_i(\delta_j y)| dy \right) \quad (\text{because } \mathcal{K} \subset \frac{A_j}{\delta_j}) \\
&= O \left(\sum_{\substack{i=1 \\ i \neq j}}^k \int_{A_j} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} |\phi_i(x)| dx \right) \quad (\text{we scale } x = \delta_j y) \\
&= O \left(\sum_{\substack{i=1 \\ i \neq j}}^k \left(\int_{A_j} \left| 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} dx \right|^p \right)^{1/p} \|\phi_i\|_q \right) \quad (\text{we use Hölder's estimate}) \\
&= o(1) \quad (\text{we use estimate (3.6) and the fact that } |\phi_i|_q \leq 1)
\end{aligned}$$

and

$$\int_{\mathcal{K}} \delta_j^2 \psi_j(\delta_j y) \varphi(y) dy = O \left(\int_{\Omega_j} \delta_j^2 |\psi_j(\delta_j y)| dy \right) = O \left(\int_{\Omega} |\psi_j(x)| dx \right) = O(\|\psi_j\|_p) = o(1).$$

By (4.14) we deduce that $\tilde{\phi}_j^*$ is a solution to the equation

$$-\Delta \tilde{\phi}_j^* = 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j^* \text{ in } \mathbb{R}^2 \setminus \{0\}.$$

Finally, since $\int_{\mathbb{R}^2} |\nabla \phi_0^j(y)|^2 dy \leq 1$ it is standard to see that $\tilde{\phi}_j^*$ is a solution in the whole space \mathbb{R}^2 . By Theorem 6.1 we get the claim.

Step 2: we will show that $\gamma_j = 0$ for any $j = 1, \dots, k$.

Here we are inspired by some ideas used by Gladiali-Grossi [15].

We set

$$\sigma_i(\lambda) := \ln \lambda \int_{\Omega_i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \tilde{\phi}_i(y) dy. \quad (4.15)$$

We will show that

$$\sigma_i := \lim_{\lambda \rightarrow 0} \sigma_i(\lambda) = 0 \text{ for any } i = 1, \dots, k. \quad (4.16)$$

We know that ϕ_i solves the problem (see (4.13))

$$-\Delta \phi_i = 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k 2\alpha_j^2 \frac{\delta_j^{\alpha_j} |x|^{\alpha_j-2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \phi_j + \psi_i \text{ in } \Omega, \quad \phi_i = 0 \text{ on } \partial\Omega. \quad (4.17)$$

Set $Z_i(x) := \frac{\delta_i^{\alpha_i} - |x|^{\alpha_i}}{\delta_i^{\alpha_i} + |x|^{\alpha_i}}$. We know that Z_i solves (see Theorem 6.1)

$$-\Delta Z_i = 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} Z_i \text{ in } \mathbb{R}^2.$$

Let PZ_i be its projection onto $H_0^1(\Omega)$ (see (2.3)), i.e.

$$-\Delta PZ_i = 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} Z_i \text{ in } \Omega, \quad PZ_i = 0 \text{ on } \partial\Omega. \quad (4.18)$$

By maximum principle (see also Lemma 2.1) we deduce that

$$PZ_i(x) = Z_i(x) + 1 + O(\delta_i^{\alpha_i}) = \frac{2\delta_i^{\alpha_i}}{\delta_i^{\alpha_i} + |x|^{\alpha_i}} + O(\delta_i^{\alpha_i}) \quad (4.19)$$

from which we get

$$PZ_i(\delta_j y) = \begin{cases} O\left(\frac{1}{|y|^{\alpha_i}} \left(\frac{\delta_i}{\delta_j}\right)^{\alpha_i}\right) + O(\delta_i^{\alpha_i}) & \text{if } i < j, \\ \frac{2}{1 + |y|^{\alpha_i}} + O(\delta_i^{\alpha_i}) & \text{if } i = j, \\ 2 + O\left(|y|^{\alpha_i} \left(\frac{\delta_j}{\delta_i}\right)^{\alpha_i}\right) + O(\delta_i^{\alpha_i}) & \text{if } i > j. \end{cases} \quad (4.20)$$

Now, we multiply (4.17) by $(\ln \lambda)PZ_i$ and (4.18) by $(\ln \lambda)\phi$. If we subtract the two equations obtained, we get

$$\begin{aligned} & \ln \lambda \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i(x) PZ_i(x) dx - \frac{1}{2} \ln \lambda \sum_{\substack{j=1 \\ j \neq i}}^k \int_{\Omega} 2\alpha_j^2 \frac{\delta_j^{\alpha_j} |x|^{\alpha_j-2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \phi_j(x) PZ_i(x) dx \\ & + \ln \lambda \int_{\Omega} \psi_i(x) PZ_i(x) dx = \ln \lambda \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i(x) Z_i(x) dx \end{aligned}$$

and so

$$\begin{aligned} & \ln \lambda \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i(x) (PZ_i(x) - Z_i(x)) dx \\ & - \frac{1}{2} \ln \lambda \sum_{\substack{j=1 \\ j \neq i}}^k \int_{\Omega} 2\alpha_j^2 \frac{\delta_j^{\alpha_j} |x|^{\alpha_j-2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \phi_j(x) PZ_i(x) dx \\ & + \ln \lambda \int_{\Omega} \psi_i(x) PZ_i(x) dx = 0. \end{aligned} \quad (4.21)$$

We are going to pass to the limit in (4.21).

The last term is

$$\ln \lambda \int_{\Omega} \psi_i(x) PZ_i(x) dx = O(|\ln \lambda| \|\psi_i\|_p) = o(1), \quad (4.22)$$

because of (4.3) and since by (4.19) we get $\|PZ_i\|_{\infty} = O(1)$.

The first term is

$$\begin{aligned} & \ln \lambda \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i(x) (PZ_i(x) - Z_i(x)) dx \\ & \quad (\text{we scale } x = \delta_i y \text{ and we apply (4.19)}) \\ & = \ln \lambda \int_{\Omega_i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1 + |y|^{\alpha_i})^2} \tilde{\phi}_i(y) dy + O\left(\delta_i^{\alpha_i} |\ln \lambda| \int_{\Omega_i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1 + |y|^{\alpha_i})^2} |\tilde{\phi}_i(y)| dy\right) \\ & \quad (\text{we use (4.15) and (4.4)}) \\ & = \sigma_i(\lambda) + o(1). \end{aligned} \quad (4.23)$$

We estimate the second term. If $j \neq i$ we get

$$\begin{aligned}
& \ln \lambda \int_{\Omega} 2\alpha_j^2 \frac{\delta_j^{\alpha_j} |x|^{\alpha_j-2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \phi_j(x) PZ_i(x) dx \quad (\text{we scale } x = \delta_j y) \\
&= \ln \lambda \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \tilde{\phi}_j(y) PZ_i(\delta_j y) dy \quad (\text{we use (4.20)}) \\
&= \begin{cases} \left(2 \ln \lambda \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \tilde{\phi}_j(y) dy + \right. \\ \quad \left. + O \left(\left| \ln \lambda \right| \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} |\tilde{\phi}_j(y)| \left(|y|^{\alpha_i} \left(\frac{\delta_j}{\delta_i} \right)^{\alpha_i} + \delta_i^{\alpha_i} \right) dy \right) \right) & \text{if } j < i \\ \\ \left(O \left(\left| \ln \lambda \right| \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} |\tilde{\phi}_j(y)| \left(\frac{1}{|y|^{\alpha_i}} \left(\frac{\delta_i}{\delta_j} \right)^{\alpha_i} + \delta_i^{\alpha_i} \right) dy \right) \right) & \text{if } j > i. \end{cases} \\
& \quad (\text{we use (4.15), (4.25), (4.26) and (4.27)}) \\
&= \begin{cases} 2\sigma_j(\lambda) + o(1) & \text{if } j < i \\ o(1) & \text{if } j > i. \end{cases} \tag{4.24}
\end{aligned}$$

By (4.21), (4.22), (4.23) and (4.24) we get

$$\sigma_1(\lambda) = o(1) \text{ and } \sigma_i(\lambda) - \sum_{j=1}^{i-1} \sigma_j(\lambda) = o(1) \text{ for any } i = 2, \dots, k,$$

which implies passing to the limit and using the definition of σ_i given in (4.16),

$$\sigma_1 = 0 \text{ and } \sigma_i - \sum_{j=1}^{i-1} \sigma_j = 0 \text{ for any } i = 2, \dots, k.$$

Therefore, (4.16) immediately follows.

We used the following three estimates. If $j < i$ we have

$$\begin{aligned}
& \left(\left| \ln \lambda \right| \frac{\delta_j}{\delta_i} \right)^{\alpha_i} \int_{\Omega_j} \frac{|y|^{\alpha_j+\alpha_i-2}}{(1 + |y|^{\alpha_j})^2} |\tilde{\phi}_j(y)| dy \quad (\text{by Hölder's inequality}) \\
&= O \left(\left| \ln \lambda \right| \left(\frac{\delta_j}{\delta_i} \right)^{\alpha_i} \delta_j^{\frac{2(1-p)}{p}} \|\phi_j\| \left(\int_{\mathbb{R}^2} \left(\frac{|y|^{\alpha_j+\alpha_i-2}}{(1 + |y|^{\alpha_j})^2} \right)^p dy \right)^{1/p} \right) \\
& \quad (\text{we use } \alpha_j > \alpha_i \text{ and we choose } p \text{ close to } 1) \\
&= O \left(\left| \ln \lambda \right| \left(\frac{\delta_j}{\delta_i} \right)^{\alpha_i} \delta_j^{\frac{2(1-p)}{p}} \right) = o(1) \tag{4.25}
\end{aligned}$$

and if $j > i$ we have

$$\begin{aligned}
& \left(|\ln \lambda| \frac{\delta_i}{\delta_j} \right)^{\alpha_i} \int_{\Omega_j} \frac{1}{|y|^{\alpha_i - \alpha_j + 2} (1 + |y|^{\alpha_j})^2} |\tilde{\phi}_j(y)| dy \quad (\text{by Hölder's inequality}) \\
&= O \left(|\ln \lambda| \left(\frac{\delta_j}{\delta_i} \right)^{\alpha_i} \delta_j^{\frac{2(1-p)}{p}} \|\phi_j\| \left(\int_{\mathbb{R}^2} \left(\frac{1}{|y|^{\alpha_i - \alpha_j + 2} (1 + |y|^{\alpha_j})^2} \right)^p dy \right)^{1/p} \right) \\
&\quad (\text{we use } \alpha_i > \alpha_j \text{ and we choose } p \text{ close to } 1) \\
&= O \left(|\ln \lambda| \left(\frac{\delta_j}{\delta_i} \right)^{\alpha_i} \delta_j^{\frac{2(1-p)}{p}} \right) = o(1); \tag{4.26}
\end{aligned}$$

moreover for any i and j we have

$$\begin{aligned}
& |\ln \lambda| \delta_i^{\alpha_i} \int_{\Omega_j} \frac{|y|^{\alpha_j - 2}}{(1 + |y|^{\alpha_j})^2} |\tilde{\phi}_j(y)| dy \quad (\text{by Hölder's inequality}) \\
&= O \left(|\ln \lambda| \delta_i^{\alpha_i} \delta_j^{\frac{2(1-p)}{p}} \|\phi_j\| \left(\int_{\mathbb{R}^2} \left(\frac{|y|^{\alpha_j - 2}}{(1 + |y|^{\alpha_j})^2} \right)^p dy \right)^{1/p} \right) \\
&\quad (\text{we choose } p \text{ close to } 1) \\
&= O \left(|\ln \lambda| \delta_i^{\alpha_i} \delta_j^{\frac{2(1-p)}{p}} \right) = o(1). \tag{4.27}
\end{aligned}$$

Finally, we have all the ingredients to show that

$$\gamma_i = 0 \text{ for any } i = 1, \dots, k. \tag{4.28}$$

We know that Pw_i solves the problem

$$-\Delta Pw_i = 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i - 2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \text{ in } \Omega, \quad Pw_i = 0 \text{ on } \partial\Omega. \tag{4.29}$$

Now, we multiply (4.29) by ϕ_i and (4.17) by Pw_i , we subtract the two equations and we get

$$\begin{aligned}
& \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i - 2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i(x) dx \\
&= \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i - 2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i(x) Pw_i dx - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k \int_{\Omega} 2\alpha_j^2 \frac{\delta_j^{\alpha_j} |x|^{\alpha_j - 2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \phi_j(x) Pw_i(x) dx \\
&+ \int_{\Omega} \psi_i(x) Pw_i(x) dx. \tag{4.30}
\end{aligned}$$

We want to pass to the limit in (4.30).

The L.H.S. of (4.30) reduces to

$$\begin{aligned}
& \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i - 2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i(x) dx \quad (\text{we scale } x = \delta_i y) \\
&= \int_{\Omega_i} 2\alpha_i^2 \frac{|y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} \tilde{\phi}_i(y) dy = o(1) \quad (\text{because of (4.36) and (4.4)}). \tag{4.31}
\end{aligned}$$

The last term of the R.H.S. of (4.30) gives

$$\int_{\Omega} \psi_i(x) Pw_i(x) dx = O(|\ln \lambda| \|\psi_i\|_p) o(1), \quad (4.32)$$

because of (4.3) and since by (2.18) we get $\|Pw_i\|_{\infty} = O(|\ln \lambda|)$.

Finally, we claim that the first term of the R.H.S. of (4.30) is

$$\begin{aligned} & \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i(x) Pw_i dx - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k \int_{\Omega} 2\alpha_j^2 \frac{\delta_j^{\alpha_j} |x|^{\alpha_j-2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \phi_j(x) Pw_i(x) dx \\ &= \begin{cases} 4\pi\alpha_i \left(\gamma_i - \sum_{j=i+1}^k \gamma_j \right) + o(1) & \text{if } i = 1, \dots, k-1, \\ 4\pi\alpha_k \gamma_k + o(1) & \text{if } i = k. \end{cases} \end{aligned} \quad (4.33)$$

Therefore, passing to the limit, by (4.30), (4.31), (4.32) and (4.33) we immediately get

$$\gamma_k = 0 \text{ and } \gamma_i - \sum_{j=i+1}^k \gamma_j = 0 \text{ for any } i = 1, \dots, k-1,$$

which implies (4.28).

It only remains to prove (4.33). We have

$$\begin{aligned} & \int_{\Omega} 2\alpha_j^2 \frac{\delta_j^{\alpha_j} |x|^{\alpha_j-2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \phi_j(x) Pw_i(x) dx \text{ (we scale } x = \delta_j y) \\ &= \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1 + |y|^{\alpha_j})^2} \tilde{\phi}_j(y) Pw_i(\delta_j y) dy \text{ (we use (2.19))} \end{aligned}$$

$$= \left\{ \begin{array}{l} \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j(y) (-2\alpha_i \ln \delta_i + h_i(0)) dy + \\ \quad + O \left(\int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} |\tilde{\phi}_j(y)| \left(|y|^{\alpha_i} \left(\frac{\delta_j}{\delta_i} \right)^{\alpha_i} + \delta_j |y| + \delta_i^{\alpha_i} \right) dy \right) \quad \text{if } j < i \\ \\ \int_{\Omega_i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \tilde{\phi}_i(y) (-2\alpha_i \ln \delta_i - 2 \ln(1+|y|^{\alpha_i}) + h_i(0)) dy + \\ \quad + O \left(\int_{\Omega_i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} |\tilde{\phi}_i(y)| (\delta_i |y| + \delta_i^{\alpha_i}) dy \right) \quad \text{if } j = i \\ \\ \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j(y) (-2\alpha_i \ln(\delta_j |y|) + h_i(0)) dy + \\ \quad + O \left(\int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} |\tilde{\phi}_j(y)| \left(\frac{1}{|y|^{\alpha_i}} \left(\frac{\delta_i}{\delta_j} \right)^{\alpha_i} + \delta_j |y| + \delta_i^{\alpha_i} \right) dy \right) \quad \text{if } j > i \end{array} \right.$$

(we use the relation between δ_i and λ in (2.7) and we use (4.25), (4.26), (4.27) and (4.35))

$$= \left\{ \begin{array}{l} \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j(y) [-2\alpha_i \ln d_i - 2(2(k-i)+1) \ln \lambda + h_i(0)] dy \\ \quad + o(1) \text{ if } j < i \\ \\ \int_{\Omega_i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \tilde{\phi}_i(y) [-2\alpha_i \ln d_i - 2(2(k-i)+1) \ln \lambda - 2 \ln(1+|y|^{\alpha_i}) + h_i(0)] dy \\ \quad + o(1) \text{ if } j = i \\ \\ \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j(y) [-2\alpha_i \ln d_j - 2(2(k-j)+1) \ln \lambda - 2\alpha_i \ln |y| + h_i(0)] dy \\ \quad + o(1) \text{ if } j > i \end{array} \right.$$

(we use the definition of σ_i in (4.15) and we use (4.4) and (4.36))

$$= \begin{cases} -2(2(k-i)+1)\sigma_j(\lambda) + o(1) & \text{if } j < i \\ -2(2(k-i)+1)\sigma_i(\lambda) + \int_{\Omega_i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \tilde{\phi}_i(y) [-2\ln(1+|y|^{\alpha_i})] dy + o(1) & \text{if } j = i \\ -2(2(k-j)+1)\sigma_j(\lambda) + \int_{\Omega_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j(y) [-2\alpha_i \ln|y|] dy + o(1) & \text{if } j > i \end{cases}$$

(we use (4.16) and (4.4) because $\ln(1+|y|^{\alpha_j}), \ln|y| \in L_{\alpha_j}(\mathbb{R}^2)$)

$$= \begin{cases} o(1) & \text{if } j < i \\ \gamma_i \int_{\mathbb{R}^2} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}} [-2\ln(1+|y|^{\alpha_i})] dy + o(1) & \text{if } j = i \\ \gamma_j \int_{\mathbb{R}^2} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \frac{1-|y|^{\alpha_j}}{1+|y|^{\alpha_j}} [-2\alpha_i \ln|y|] dy + o(1) & \text{if } j > i \end{cases}$$

(we use (4.37) and (4.38))

$$= \begin{cases} o(1) & \text{if } j < i \\ 4\pi\alpha_i\gamma_i + o(1) & \text{if } j = i \\ 8\pi\alpha_i\gamma_j + o(1) & \text{if } j > i \end{cases} \quad (4.34)$$

If we sum (4.34) over the index j we get (4.33).

We used the following estimate. For any j we have

$$\begin{aligned} & \delta_j \int_{\Omega_j} \frac{|y|^{\alpha_j-1}}{(1+|y|^{\alpha_j})^2} |\tilde{\phi}_j(y)| dy \quad (\text{by Hölder's inequality}) \\ &= O \left(\delta_j \delta_j^{\frac{2(1-p)}{p}} \|\phi_j\| \left(\int_{\mathbb{R}^2} \left(\frac{|y|^{\alpha_j-1}}{(1+|y|^{\alpha_j})^2} \right)^p dy \right)^{1/p} \right) \\ & \quad (\text{we choose } p \text{ close to } 1) \\ &= O \left(\delta_j^{\frac{2-p}{p}} \right) = o(1). \end{aligned} \quad (4.35)$$

A straightforward computation leads to

$$\int_{\Omega} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}} dy = 0, \quad (4.36)$$

$$\int_{\Omega} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}} \ln(1+|y|^{\alpha_i})^2 dy = -4\pi\alpha_i, \quad (4.37)$$

$$\int_{\Omega} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}} \ln|y| dy = -4\pi. \quad (4.38)$$

Step 3: we will show that a contradiction arises! We multiply equation (4.2) by ϕ and we get

$$\begin{aligned} 1 &= \sum_{i=1}^k \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi^2(x) dx + \int_{\Omega} \psi(x) \phi(x) dx \\ &= \sum_{i=1}^k \int_{\Omega^i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} (\phi^i(y))^2 dy + O(\|\psi\|_p \|\phi\|) \quad (\text{we use (4.3)}) \\ &= \sum_{i=1}^k \int_{\Omega^i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} (\phi^i(y))^2 dy + o(1) \\ &= o(1) \quad (\text{because } \phi^i \rightarrow 0 \text{ strongly in } L_{\alpha_i}(\mathbb{R}^2)) \end{aligned}$$

and a contradiction arises!

Step 3: we will show that a contradiction arises!

We multiply each equation (4.2) by ϕ_i , we sum over the indices i 's and we get

$$\begin{aligned} 1 &= \sum_{i=1}^k \int_{\Omega} 2\alpha_i^2 \frac{\delta_i^{\alpha_i} |x|^{\alpha_i-2}}{(\delta_i^{\alpha_i} + |x|^{\alpha_i})^2} \phi_i^2(x) dx - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^k \int_{\Omega} 2\alpha_j^2 \frac{\delta_j^{\alpha_j} |x|^{\alpha_j-2}}{(\delta_j^{\alpha_j} + |x|^{\alpha_j})^2} \phi_j(x) \phi_i(x) dx \\ &+ \sum_{i=1}^k \int_{\Omega} \psi_i(x) \phi_i(x) dx \\ &= \sum_{i=1}^k \int_{\Omega/\delta_i} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \tilde{\phi}_i^2(y) dy - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^k \int_{\Omega/\delta_j} 2\alpha_j^2 \frac{|y|^{\alpha_j-2}}{(1+|y|^{\alpha_j})^2} \tilde{\phi}_j(y) \phi_i(\delta_j y) dy \\ &+ O\left(\sum_{i=1}^k \|\psi_i\|_p \|\phi_i\|\right) \quad (\text{we use (4.3)}) \\ &= o(1) \quad (\text{because } \tilde{\phi}_i \rightarrow 0 \text{ strongly in } L_{\alpha_i}(\mathbb{R}^2) \text{ for any } i = 1, \dots, k) \end{aligned}$$

and a contradiction arises!

□

5. A CONTRACTION MAPPING ARGUMENT AND THE PROOF OF THE MAIN THEOREM

First of all we point out that $\mathbf{W}_\lambda + \phi_\lambda$ is a solution to (1.7) if and only if ϕ_λ is a solution of the problem

$$\mathcal{L}_\lambda(\phi) = \mathcal{N}_\lambda(\phi) + \mathcal{S}_\lambda(\phi) + \mathcal{R}_\lambda \text{ in } \Omega \quad (5.1)$$

where the error term \mathcal{R}_λ is defined in (3.1), the linear operator \mathcal{L}_λ is defined in (4.1), the higher order linear operator $\mathcal{S}_\lambda(\boldsymbol{\phi})$ is defined as

$$\begin{aligned} \mathcal{S}_\lambda(\boldsymbol{\phi}) &:= \left(S_\lambda^1(\boldsymbol{\phi}), \dots, S_\lambda^k(\boldsymbol{\phi}) \right), \text{ where} \\ S_\lambda^i(\boldsymbol{\phi}) &:= \left(|x|^{\alpha_i-2} e^{w_i} - 2\lambda e^{W_\lambda^i} \right) \phi^i + \sum_{\substack{j=1 \\ j \neq i}}^k \left(-\frac{1}{2} |x|^{\alpha_j-2} e^{w_j} + \lambda e^{W_\lambda^j} \right) \phi^j, \quad i = 1, \dots, k \end{aligned} \quad (5.2)$$

and the higher order term \mathcal{N}_λ is defined as

$$\begin{aligned} \mathcal{N}_\lambda(\boldsymbol{\phi}) &:= \left(N_\lambda^1(\boldsymbol{\phi}), \dots, N_\lambda^k(\boldsymbol{\phi}) \right), \text{ where} \\ N_\lambda^i(\boldsymbol{\phi}) &:= -2\lambda e^{W_\lambda^i} \left(e^{\phi^i} + 1 + \phi^i \right) + \lambda \sum_{\substack{j=1 \\ j \neq i}}^k e^{W_\lambda^j} \left(e^{\phi^j} + 1 + \phi^j \right), \quad i = 1, \dots, k. \end{aligned} \quad (5.3)$$

Proposition 5.1. *There exists $p_0 > 1$, $\lambda_0 > 0$ and $R_0 > 0$ such that for any $p \in (1, p_0)$, $\lambda \in (0, \lambda_0)$ and $R \geq R_0$ we have such that for any $\lambda \in (0, \lambda_0)$ there exists a unique solution $\boldsymbol{\phi}_\lambda = (\phi_\lambda^1, \dots, \phi_\lambda^k) \in H_e^k$ (see (2.9)) to the system*

$$\Delta(W_\lambda^i + \phi_\lambda^i) + 2\lambda e^{W_\lambda^i + \phi_\lambda^i} - \lambda \sum_{\substack{j=1 \\ j \neq i}}^k e^{W_\lambda^j + \phi_\lambda^j} = 0 \text{ in } \Omega, \phi^i = 0 \text{ on } \partial\Omega \text{ for } i = 1, \dots, k. \quad (5.4)$$

and

$$\|\boldsymbol{\phi}_\lambda\| \leq R \lambda^{\frac{1}{2^k} \frac{2-p}{p}} |\ln \lambda|$$

for some $\epsilon > 0$.

Proof. As a consequence of Proposition 4.1, we conclude that $\boldsymbol{\phi}$ is a solution to (5.4) if and only if it is a fixed point for the operator $\mathcal{T}_\lambda : \mathbf{H}_k \rightarrow \mathbf{H}_k$, defined by

$$T_\lambda(\boldsymbol{\phi}) = (\mathcal{L}_\lambda)^{-1} (\mathcal{N}_\lambda(\boldsymbol{\phi}) + \mathcal{S}_\lambda(\boldsymbol{\phi}) + \mathcal{R}_\lambda),$$

where \mathcal{L}_λ , \mathcal{S}_λ , \mathcal{N}_λ and \mathcal{R}_λ are defined in (3.1), (5.2), (5.3) and (3.1), respectively.

Let us introduce the ball $B_{\lambda,R} := \left\{ \boldsymbol{\phi} \in \mathbf{H}_k : \|\boldsymbol{\phi}\| \leq R \lambda^{\frac{1}{2^k} \frac{2-p}{p}} \right\}$. We will show that $T_\lambda : B_{\lambda,R} \rightarrow B_{\lambda,R}$ is a contraction mapping provided λ is small enough and R is large enough.

Let us prove that T_λ maps the ball $B_{\lambda,r}$ into itself, i.e.

$$\|\boldsymbol{\phi}\| \leq R \lambda^{\frac{1}{2^k} \frac{2-p}{p}} |\ln \lambda| \implies \|\mathcal{T}_\lambda(\boldsymbol{\phi})\| \leq R \lambda^{\frac{1}{2^k} \frac{2-p}{p}} |\ln \lambda|. \quad (5.5)$$

By Lemma 5.2 (where we take $h = \mathcal{N}_\lambda(\boldsymbol{\phi}) + \mathcal{R}_\lambda$), by (5.7), by Lemma 5.1 and by Lemma 3.1 we deduce that:

$$\begin{aligned} \|\mathcal{T}_\lambda(\boldsymbol{\phi})\| &\leq c |\ln \lambda| \left(\|\mathcal{N}_\lambda(\boldsymbol{\phi})\|_p + \|\mathcal{S}_\lambda(\boldsymbol{\phi})\|_p + \|\mathcal{R}_\lambda\|_p \right) \\ &\leq c |\ln \lambda| \left(\lambda^{2^k-1 \frac{1-pr}{pr}} \|\boldsymbol{\phi}\|^2 + \lambda^{\frac{1}{2^k} \frac{2-p}{p}} \|\boldsymbol{\phi}\| + \lambda^{\frac{1}{2^k} \frac{2-p}{p}} \right) \\ &\leq R \lambda^{\frac{1}{2^k} \frac{2-p}{p}} \end{aligned}$$

provided r and p are close enough to 1, R is suitable large and λ is small enough. That proves (5.5).

Let us prove that T_λ is a contraction mapping, i.e. there exists $L > 1$ such that

$$\|\boldsymbol{\phi}\| \leq R \lambda^{\frac{1}{2^k} \frac{2-p}{p}} |\log \rho| \implies \|\mathcal{T}_\lambda(\boldsymbol{\phi}_1) - \mathcal{T}_\lambda(\boldsymbol{\phi}_2)\| \leq L \|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2\|. \quad (5.6)$$

By Lemma 5.2 (where we take $\psi = \mathcal{N}_\lambda(\phi_1) - \mathcal{N}_\lambda(\phi_2)$) and by (5.8), we deduce that:

$$\begin{aligned} \|\mathcal{T}_\lambda(\phi)\| &\leq c|\ln \lambda| \left(\|\mathcal{N}_\lambda(\phi_1) - \mathcal{N}_\lambda(\phi_2)\|_p + \|\mathcal{S}_\lambda(\phi_1 - \phi_2)\|_p \right) \leq \\ &\leq c|\ln \lambda| \left(\lambda^{2^{k-1} \frac{1-pr}{pr}} \|\phi_1 - \phi_2\| (\|\phi_1\| + \|\phi_2\|) + \lambda^{\frac{1}{2^k} \frac{2-p}{p}} \|\phi_1 - \phi_2\| \right) \\ &\leq L \sum_{i=1}^k \|\phi_1^i - \phi_2^i\| = L \|\phi_1 - \phi_2\| \text{ for some } L < 1, \end{aligned}$$

provided r and p are close enough to 1, R is suitable large and λ is small enough. That proves (5.6). \square

Lemma 5.1. *Let \mathcal{S}_λ as in (5.2). There exists $p_0 > 1$ and $\lambda_0 > 0$ such that for any $p \in (1, p_0)$ and $\lambda \in (0, \lambda_0)$ we have*

$$\|\mathcal{S}_\lambda(\phi)\|_p = O\left(\lambda^{\frac{1}{2^k} \frac{2-p}{p}} \|\phi\|\right)$$

Proof. We have that

$$\begin{aligned} \|\mathcal{S}_\lambda(\phi)\|_p &= \sum_{i=1}^k \|S_\lambda^i(\phi)\|_p \\ &= O\left(\sum_{i=1}^k \left\| \left(|x|^{\alpha_i-2} e^{w_i} - 2\lambda e^{W_\lambda^i} \right) \phi^i \right\|_p \right) \text{ (we use Hölder's inequality with } \frac{1}{q} + \frac{1}{s} = 1) \\ &= O\left(\sum_{i=1}^k \left\| \left(|x|^{\alpha_i-2} e^{w_i} - 2\lambda e^{W_\lambda^i} \right) \right\|_{pq} \|\phi^i\|_{ps} \right) \text{ (we use estimate (3.4))} \\ &= O\left(\lambda^{\frac{1}{2^k} \frac{2-p}{p}} \sum_{i=1}^k \|\phi^i\|\right) = O\left(\lambda^{\frac{1}{2^k} \frac{2-p}{p}} \|\phi\|\right), \end{aligned}$$

which proves the claim. \square

Lemma 5.2. *There exists $s_0 > 1$ and $\lambda_0 > 0$ such that for any $p > 1$, $r > 1$ with $pr \in (1, s_0)$ and $\lambda \in (0, \lambda_0)$ we have for any $\phi, \phi_1, \phi_2 \in \{\phi \in H_0^1(\Omega) : \|\phi\| \leq 1\}$*

$$\|\mathcal{N}_\lambda(\phi)\|_p = O\left(\lambda^{2^{k-1} \frac{1-pr}{pr}} \|\phi\|^2\right) \quad (5.7)$$

and

$$\|\mathcal{N}_\lambda(\phi_1) - \mathcal{N}_\lambda(\phi_2)\|_p = O\left(\lambda^{2^{k-1} \frac{1-pr}{pr}} \|\phi_1 - \phi_2\| (\|\phi_1\| + \|\phi_2\|)\right). \quad (5.8)$$

Proof. For any $i = 1, \dots, k$ set

$$\mathbf{N}_\lambda^i(\phi) := \lambda e^{W_\lambda^i} \left(e^\phi + 1 + \phi \right).$$

By the definition of \mathcal{N}_λ in (5.3) we immediately deduce that

$$\|\mathcal{N}_\lambda(\phi)\|_p = \sum_{i=1}^k \|\mathbf{N}_\lambda^i(\phi)\|_p = O\left(\sum_{i=1}^k \|\mathbf{N}_\lambda^i(\phi)\|_p\right) \quad (5.9)$$

and

$$\|\mathcal{N}_\lambda(\phi_1) - \mathcal{N}_\lambda(\phi_2)\|_p = \sum_{i=1}^k \|\mathbf{N}_\lambda^i(\phi_1) - \mathbf{N}_\lambda^i(\phi_2)\|_p = O\left(\sum_{i=1}^k \|\mathbf{N}_\lambda^i(\phi_1) - \mathbf{N}_\lambda^i(\phi_2)\|_p\right). \quad (5.10)$$

We are going to prove that there exist some positive constants c_i such that

$$\|\mathbf{N}_\lambda^i(\phi)\|_p = O\left(e^{c_i \|\phi\|^2} \lambda^{2^{k-1} \frac{1-pr}{pr}} \|\phi\|^2\right) \quad (5.11)$$

and

$$\|\mathbf{N}_\lambda^i(\phi_1) - \mathbf{N}_\lambda^i(\phi_2)\|_p = O\left(e^{c_i (\|\phi_1\|^2 + \|\phi_2\|^2)} \lambda^{2^{k-1} \frac{1-pr}{pr}} \|\phi_1 - \phi_2\| (\|\phi_1\| + \|\phi_2\|)\right). \quad (5.12)$$

Estimate (5.7) follows by (5.9) and (5.11) since $\|\phi\| \leq 1$ and estimate (5.8) follows by (5.10) and (5.12), since $\|\phi_1\|, \|\phi_2\| \leq 1$.

Let us prove (5.11) and (5.12). Since (5.11) follows by (5.12) choosing $\phi_2 = 0$, we only prove (5.12). We point out that

$$\mathbf{N}_\lambda^i(\phi_1) - \mathbf{N}_\lambda^i(\phi_2) = \lambda e^{W_\lambda^i} (e^{\phi_1} - e^{\phi_2} - \phi_1 + \phi_2)$$

By the mean value theorem, we easily deduce that

$$|e^a - e^b - a + b| \leq e^{|a|+|b|} |a - b| (|a| + |b|) \text{ for any } a, b \in \mathbb{R}.$$

Therefore, we have

$$\begin{aligned} \|\mathbf{N}_\lambda^i(\phi_1) - \mathbf{N}_\lambda^i(\phi_2)\|_p &= \left(\int_{\Omega} \lambda^p e^{pW_\lambda^i} |e^{\phi_1} - e^{\phi_2} - \phi_1 + \phi_2|^p dx \right)^{1/p} \\ &\leq c \sum_{j=1}^2 \left(\int_{\Omega} \lambda^p e^{pW_\lambda^i} e^{p|\phi_1|+p|\phi_2|} |\phi_1 - \phi_2|^p |\phi_j|^p dx \right)^{1/p} \\ &\quad \text{(we use Hölder's inequality with } \frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1 \text{)} \\ &\leq c \sum_{j=1}^2 \left(\int_{\Omega} \lambda^{pr} e^{prW_\lambda^i} dx \right)^{1/(pr)} \left(\int_{\Omega} e^{ps|\phi_1|+ps|\phi_2|} dx \right)^{1/(ps)} \left(\int_{\Omega} |\phi_1 - \phi_2|^{pt} |\phi_j|^{pt} dx \right)^{1/(pt)} \\ &\quad \text{(we use Lemma 5.3)} \\ &\leq c \sum_{j=1}^2 \left(\int_{\Omega} \lambda^{pr} e^{prW_\lambda^i} dx \right)^{1/(pr)} e^{(ps)/(8\pi)(|\phi_1|^2+|\phi_2|^2)} \|\phi_1 - \phi_2\| \|\phi_j\|. \end{aligned} \quad (5.13)$$

We have to estimate

$$\int_{\Omega} \lambda^{pr} e^{prW_\lambda^i(x)} dx = \sum_{j=1}^k \int_{A_j} \lambda^{pr} e^{prW_\lambda^i(x)} dx,$$

where A_j is the annulus defined in (2.11).

If $j = i$ we get

$$\begin{aligned} &\int_{A_i} \lambda^{pr} e^{prW_\lambda^i(x)} dx \text{ (we use (2.10))} \\ &= \delta_i^2 \lambda^{pr} \int_{\frac{A_i}{\delta_i}} e^{pr[w_i(\delta_i y) + (\alpha_i - 2) \ln |\delta_j y| - \ln 2\lambda + \Theta_i(y)]} dy \\ &= \delta_i^{2-2pr} \int_{\frac{A_i}{\delta_i}} \left(2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} \right)^{pr} e^{pr\Theta_i(y)} dy \text{ (we use Lemma (2.2))} \\ &= O\left(\delta_i^{2-2pr}\right) = O\left(\lambda^{2k-1(1-pr)}\right) \text{ (because } \delta_j \geq \delta_1 = O\left(\lambda^{2k-2}\right) \text{ and } pr > 1\text{)}. \end{aligned}$$

If $j \neq i$ by (3.20) we deduce

$$\int_{A_j} \lambda^{pr} e^{prW_\lambda^i(x)} dx = o\left(\lambda^{\frac{1}{2k}(2-pr)}\right).$$

Therefore, estimate (5.12) follows. □

We recall the following Moser-Trudinger inequality [33, 39],

Lemma 5.3. *There exists $c > 0$ such that for any bounded domain Ω in \mathbb{R}^2*

$$\int_{\Omega} e^{4\pi u^2/\|u\|^2} dx \leq c|\Omega|, \text{ for any } u \in H_0^1(\Omega).$$

In particular, there exists $c > 0$ such that for any $\eta \in \mathbb{R}$

$$\int_{\Omega} e^{\eta u} \leq c|\Omega|e^{\frac{\eta^2}{16\pi}\|u\|^2}, \text{ for any } u \in H_0^1(\Omega).$$

Proof of Theorem 1.2. By Proposition 5.1 we have that $\mathbf{u}_\lambda = \mathbf{W}_\lambda + \phi_\lambda$ is a solution to (1.7).

Let us prove (1.9). Let $i = 1, \dots, k$ be fixed. By the mean value theorem we deduce

$$\int_{\Omega} \lambda e^{u_\lambda^i(x)} dx = \int_{\Omega} \lambda e^{W_\lambda^i(x) + \phi_\lambda^i(x)} dx = \int_{\Omega} \lambda e^{W_\lambda^i} \left(1 + e^{t\phi_\lambda^i} \phi_\lambda^i\right) dx = \int_{\Omega} \lambda e^{W_\lambda^i} dx + o(1),$$

since, arguing exactly as in the proof of Lemma 5.2, we get (for some $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$)

$$\int_{\Omega} \lambda e^{W_\lambda^i} e^{t\phi_\lambda^i} \phi_\lambda^i dx = O\left(\|\lambda e^{W_\lambda^i}\|_p \|\lambda e^{t\phi_\lambda^i}\|_q \|\phi_\lambda^i\|_s\right) = o(1).$$

So we only have to estimate

$$\begin{aligned} \int_{\Omega} \lambda e^{W_\lambda^i} dx &= \int_{\Omega} \lambda e^{Pw_i(x) - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k Pw_j(x)} dx \text{ (we use (2.11))} \\ &= \sum_{r=1}^k \int_{A_r} \lambda e^{Pw_i(x) - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k Pw_j(x)} dx \\ &= \int_{A_i} |x|^{\alpha_i - 2} e^{w_i(x)} dx + \int_{A_i} \left(\lambda e^{Pw_i(x) - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k Pw_j(x)} - |x|^{\alpha_i - 2} e^{w_i(x)} \right) dx \\ &+ \sum_{\substack{r=1 \\ r \neq j}}^k \int_{A_r} \lambda e^{Pw_i(x) - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^k Pw_j(x)} dx \text{ (we apply (3.5) and (3.11))} \\ &= \int_{A_i} |x|^{\alpha_i - 2} e^{w_i(x)} dx + o(1) \text{ (we scale } x = \delta_i y) \\ &= \int_{\frac{A_i}{\delta_i}} |y|^{\alpha_i - 2} e^{w^{\alpha_i}(y)} dy + o(1) \text{ (because of (5.14))} \\ &= 4\pi\alpha_i + o(1). \end{aligned}$$

Here we used the following result of Chen-Li [5]

$$\int_{\mathbb{R}^2} |y|^{\alpha - 2} e^{w^\alpha(y)} dy = 2\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{\alpha - 2}}{(1 + |y|^\alpha)^2} dy = 4\pi\alpha. \quad (5.14)$$

That concludes the proof. □

We have the following result.

Theorem 6.1. *Assume $\alpha = 2^i$ for some integer $i \geq 1$. If ϕ satisfies*

$$\phi(y) = \phi(\mathfrak{R}_k y) \text{ for any } y \in \mathbb{R}^2, \quad \text{where } \mathfrak{R}_k := \begin{pmatrix} \cos \frac{\pi}{k} & \sin \frac{\pi}{k} \\ -\sin \frac{\pi}{k} & \cos \frac{\pi}{k} \end{pmatrix} \quad (6.1)$$

and solves the equation

$$-\Delta \phi = 2\alpha^2 \frac{|y|^{\alpha-2}}{(1+|y|^\alpha)^2} \phi \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla \phi(y)|^2 dy < +\infty, \quad (6.2)$$

then there exists $\gamma \in \mathbb{R}$ such that

$$\phi(y) = \gamma \frac{1 - |y|^\alpha}{1 + |y|^\alpha}.$$

Proof. Del Pino-Esposito-Musso in [12] proved that all the bounded solutions to (6.2) are a linear combination of the following functions (which are written in polar coordinates)

$$\phi_0(y) := \frac{1 - |y|^\alpha}{1 + |y|^\alpha}, \quad \phi_1(y) := \frac{|y|^{\frac{\alpha}{2}}}{1 + |y|^\alpha} \cos \frac{\alpha}{2} \theta, \quad \phi_2(y) := \frac{|y|^{\frac{\alpha}{2}}}{1 + |y|^\alpha} \sin \frac{\alpha}{2} \theta.$$

We observe that ϕ_0 always satisfies (6.1), while if $\alpha = 2^i$ for some integer $i \geq 1$ the functions ϕ_1 and ϕ_2 do not satisfy (6.1). In [16] it was proved that any solution ϕ of (6.2) is actually a bounded solution. That concludes the proof. \square

For any $\alpha \geq 2$ let us consider the Banach spaces

$$L_\alpha(\mathbb{R}^2) := \left\{ u \in W_{loc}^{1,2}(\mathbb{R}^2) : \left\| \frac{|y|^{\frac{\alpha-2}{2}}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\} \quad (6.3)$$

and

$$H_\alpha(\mathbb{R}^2) := \left\{ u \in W_{loc}^{1,2}(\mathbb{R}^2) : \|\nabla u\|_{L^2(\mathbb{R}^2)} + \left\| \frac{|y|^{\frac{\alpha-2}{2}}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\}, \quad (6.4)$$

endowed with the norms

$$\|u\|_{L_\alpha} := \left\| \frac{|y|^{\frac{\alpha-2}{2}}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)} \quad \text{and} \quad \|u\|_{H_\alpha} := \left(\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \left\| \frac{|y|^{\frac{\alpha-2}{2}}}{1 + |y|^\alpha} u \right\|_{L^2(\mathbb{R}^2)}^2 \right)^{1/2}.$$

Proposition 6.1. *The embedding $i_\alpha : H_\alpha(\mathbb{R}^2) \hookrightarrow L_\alpha(\mathbb{R}^2)$ is compact.*

Proof. See [16]. \square

REFERENCES

- [1] S. Baraket and F. Pacard, Construction of singular limits for a semilinear elliptic equation in dimension 2, *Cal. Var. PDE*, **6**, (1998), 1-38.
- [2] D. Bartolucci, C.-C. Chen, C.-S. Lin and G. Tarantello, Profile of blow-up solutions to mean field equations with singular data, *Comm. Partial Differential Equations* 29 (2004), no. 7-8, 1241–1265.
- [3] H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equation* 16 (1991), 1223-1254.
- [4] J. Bolton, G.R. Jensen, M. Rigoli, and L.M. Woodward, On conformal minimal immersions of S^2 into $\mathbb{C}P^n$, *Math. Ann.* **279**(4) (1988), 599-620.
- [5] W. Chen, C. Li, Qualitative properties of solutions to some nonlinear elliptic equations in \mathbb{R}^2 . *Duke Math. J.* **71** (1993) 427–439.
- [6] C.-C. Chen, C.-S. Lin, Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. *Comm. Pure Appl. Math.* 55 (2002), no. 6, 728–771.
- [7] C.C. Chen, C.S. Lin, Topological degree for a mean field equation on Riemann surfaces, *Comm. Pure Appl. Math.* 56 (2003), no. 12, 1667-1727.

- [8] S. Chanillo, M. K-H Kiessling, Conformally invariant systems of nonlinear PDE of Liouville type, *Geom. Funct. Anal.* 5 (1995), no. 6, 924–947.
- [9] S.S. Chern, J.G. Wolfson, Harmonic maps of the two-sphere into a complex Grassmann manifold. II, *Ann. of Math.* **125(2)** (1987), 301-335.
- [10] M. Chipot, I. Shafrir, G. Wolansky, On the solutions of Liouville systems, *J. Differential Equations* 140 (1997), no. 1, 59–105.
- [11] G. Dunne, *Self-dual Chern-Simons theories*. Lecture Notes in Physics, Springer, Berlin, 1995.
- [12] M. Del Pino, P. Esposito, M. Musso, Nondegeneracy of entire solutions of a singular Liouville equation. *Proc. Amer. Math. Soc.* **140** (2012), no. 2, 581–588.
- [13] M. Del Pino, M. Kowalczyk, M. Musso, Singular limits in Liouville-type equations. *Calc. Var. Partial Differential Equations*, **24**, (2005), 47-81.
- [14] P. Esposito, M. Grossi, A. Pistoia, On the existence of blowing-up solutions for a mean field equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22 (2005), no. 2, 227–257.
- [15] F. Gladiali, M. Grossi, On the spectrum of a nonlinear planar problem. *Ann. IHP Anal. Non Lineaire* **26** (2009), 191–222.
- [16] M. Grossi, A. Pistoia, Multiple blow-up phenomena for the sinh-Poisson equation *Archive for Rational Mech. and Anal.* (to appear) arXiv:1210.5719
- [17] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.
- [18] M.A. Guest, *Harmonic Maps, Loop Groups, and Integrable Systems*, London Mathematical Society Student Texts **38**, Cambridge University Press, Cambridge, 1997.
- [19] J. Jost, C.-S. Lin, G.F. Wang, Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions. *Comm. Pure Appl. Math.* 59 (2006), no. 4, 526–558.
- [20] J. Jost, G.F. Wang, Classification of solutions of a Toda system in \mathbb{R}^2 . *Int. Math. Res. Not.* 2002, no. 6, 277–290.
- [21] J. Jost, G.F. Wang, D. Ye, C. Zhou, C. The blowup analysis of solutions to the elliptic sinh-Gordon equation. *Calc. Var. Partial Diff. Equ.* **31** (2008) no.2, 263–276.
- [22] Y.Y. Li, Harnack type inequality: the method of moving planes. *Comm. Math. Phys.* 200 (1999), no. 2, 421–444.
- [23] Y.Y. Li, I. Shafrir, Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. *Indiana Univ. Math. J.* 43 (1994), no. 4, 1255–1270.
- [24] C.-S. Lin, J.C. Wei, C.Y. Zhao, Asymptotic Behavior of $SU(3)$ Toda System in a bounded domain. *Manuscripta Math.* 137(2012), No. 1-2, 1-18.
- [25] C.-S. Lin, J. C. Wei, C.Y. Zhao, Sharp estimates for fully bubbling solutions of a $SU(3)$ Toda system, *Gem. Funct. Anal.* 22(2012), no.6, 1591-1635.
- [26] C.S. Lin, J.C. Wei, D. Ye, Classification and Nondegeneracy of $SU(n+1)$ Toda System with singular sources *Invent. Math.* 190(2012), no.1, 169-207.
- [27] C.S. Lin, J.C. Wei, L. Zhang, Classification of blowup limits for $SU(3)$ singular Toda systems, preprint 2013.
- [28] C.S. Lin, S. Yan, Bubbling solutions for relativistic Abelian Chern-Simons model on a torus, *Comm. Math. Phys.* **297** (2010), 733-758.
- [29] C.-S. Lin, L. Zhang, Profile of bubbling solutions to a Liouville system. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 27 (2010), no. 1, 117–143.
- [30] C.-S. Lin, L. Zhang, A Topological Degree Counting for some Liouville Systems of Mean Field Equations. *Comm. Pure Appl. Math.* 64 (2011), no. 4, 556–590.
- [31] C.-S. Lin, L. Zhang, On Liouville systems at critical parameters, part 1: One bubble, (2011), preprint.
- [32] A. Malchiodi, C.B. Ndiaye, Some existence results for the Toda system on closed surfaces, *Att. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 18(2007), no.4, 391-412.
- [33] J. Moser, *A sharp form of an inequality by N. Trudinger*. Indiana Univ. Math. J. **20** (1970/71), 1077-1092.
- [34] K. Nagasaki, T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearity, *Asymptotic Analysis* 3(1990), 173–188.
- [35] M. Nolasco, G. Tarantello, Double vortex condensates in the Chern-Simons theory, *Calc. Var. and P.D.E.* **9** (1999), 31-94.
- [36] M. Nolasco, G. Tarantello, Vortex condensates for the $SU(3)$ Chern-Simons theory, *Comm. Math. Phys.* **213(3)** (2000), 599-639.
- [37] H. Ohtsuka, T. Suzuki, Blow-up analysis for $SU(3)$ Toda system, *J. Diff. Eqns.* 232(2007), no.2, 419-440.
- [38] J. Prajapat, G. Tarantello, On a class of elliptic problems in \mathbb{R}^2 : symmetry and uniqueness results, *Proc. Royal Society Edin.* **131A** (2001), 967-985.
- [39] N.S. Trudinger, On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.* **17** (1967), 473–483.
- [40] Y. Yang, The relativistic non-abelian Chern-Simons equation, *Comm. Phys.* **186(1)** (1999), 199-218.
- [41] Y.S. Yang, *Solitons in field theory and nonlinear analysis*. Springer Monographs in Mathematics. Springer, New York, 2001.

(Monica Musso) DEPARTAMENTO DE MATEMATICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE CASILLA 306, CORREO 22 SANTIAGO, CHILE.

E-mail address: `mmusso@mat.puc.cl`

(Angela Pistoia) DIPARTIMENTO SBAI, UNIVERSITÀ DI ROMA "LA SAPIENZA", VIA ANTONIO SCARPA 16, 00161 ROMA, ITALY

E-mail address: `pistoia@dmmm.uniroma1.it`

(Juncheng Wei) DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, N.T., HONG KONG AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B.C., CANADA, V6T 1Z2.

E-mail address: `wei@math.cuhk.edu.hk`