POSITIVE SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATION WITH PEAKS ON A CLIFFORD TORUS

SANJIBAN SANTRA AND JUNCHENG WEI

Abstract. We prove the existence of large energy positive solutions for a stationary nonlinear Schrödinger equation

$$\Delta u - V(x)u + u^p = 0 \text{ in } \mathbb{R}^N$$

with peaks on a Clifford type torus. Here

$$\begin{aligned} V(x) = V(r_1, r_2, \cdots, r_s) &= 1 + \frac{1}{(a_1 r_1^m + a_2 r_2^m + a_3 r_3^m + ... + a_s r_s^m)} \\ &+ \mathcal{O}\Big(\frac{1}{(a_1 r_1^m + a_2 r_2^m + a_3 r_3^m + ... + a_s r_s^m)^{1+\tau}}\Big) \end{aligned}$$

where $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \cdots \times \mathbb{R}^{N_s}$, with $N_i \geq 2$ for all $i = 1, 2, ...s$, $m > 1, \tau > 0, r_i = |x_i|$. Each r_i is a function $r, \phi_1, \cdots .\phi_{i-1}$ and is defined by the generalized notion of spherical coordinates. The solutions are obtained by a $\max\limits_{(r,\phi_1,\cdots\phi_{s-1})}$ or a $\max\limits_{r} \min\limits_{(\phi_1,\cdots\phi_{s-1})}$ process.

1. Introduction

Positive entire solution of

$$\Delta u - u + u^p = 0 \text{ on } \mathbb{R}^N \tag{1.1}$$

where $1 < p < (\frac{N+2}{N-2})_+$, vanishing at infinity have been studied in many context. This class of problems arises in plasma and condensed-matter physics. For example, if one simulates the interaction-effect among many particles by introducing a nonlinear term, we obtain a nonlinear Schrödinger equation,

$$-i\varepsilon\frac{\partial\psi}{\partial t} = \varepsilon^2\Delta_x\psi - Q(x)\psi + |\psi|^{p-1}\psi$$

where i is an imaginary unit and $p > 1$. Making an *Ansatz*

$$\psi(x,t) = exp(-\frac{i\lambda t}{\varepsilon})u(x)$$

one finds that u solves

$$\varepsilon^2\Delta u - V(x)u + u^p = 0;\ \ u \in H^1(\mathbb{R}^N) \tag{1.2}$$

where $V = Q + \lambda$ is a smooth potential. Let V be a smooth potential which is bounded below by a positive constant. A considerable attention has been paid in recent years to the problem of constructing standing waves in the so-called semi-classical limit of (1.2) $\varepsilon \to 0$. In the pioneering work [18], Floer and Weinstein constructed positive solutions to (1.2) when $p = 3, N = 1$, such that the concentration takes place near a given non-degenerate critical point x_0 of V and the

2010 *Mathematics Subject Classification.* Primary 35B08, 35J20; Secondary 35J65.
Key words and phrases. Blow-up analysis, polynomial nonlinearity, clifford torus.

solutions are exponentially small outside any neighborhood of x_0. This was later extended by Oh [20], [21] for the higher dimensional case. del Pino and Felmer [9] extended the idea for a large class of nonlinearities with V which is only locally Hölder continuous function. Byeon and Tanaka [7] proved that under the optimal conditions of Berestycki-Lions on the nonlinearity, there exists a solution concentrating around the topologically stable critical points of V, which are characterized by mini-max method. In smooth bounded domain the problem (1.2) with Dirichlet and Neumann boundary condition have been studied by many other authors some of them being [1], [3], [6], [10], [11]. Higher dimensional concentrating solutions of (1.2) was studied by Ambrosetti, Malchiodi and Ni in symmetric domain [2], [4]; they consider solutions which concentrate on spheres, i.e. on $(N-1)$- dimensional manifolds. Also see del Pino, Kowalczyk and Wei [12] in $\mathbb{R}^2$ and Esposito et. al. [17] for the Dirichlet case in an annulus. Pacella and Srikanth [22] employed the symmetry of the domain to construct solutions which concentrate on spheres for some singularly perturbed problems.

In this paper, we consider the equation

$$\Delta u - V(x)u + u^p = 0, u > 0;\ u \in H^1(\mathbb{R}^N) \tag{1.3}$$

where $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \cdots \times \mathbb{R}^{N_s}$, where $N_i \geq 2$ for all $i = 1, 2, ...s$, $m > 1, r_i = |x_i|$. Here $V(x) = V(x_1, x_2, \cdots x_s)$ with $x_i \in \mathbb{R}^{N_i}$

$$\begin{aligned} V(x) = V(r_1, r_2, \cdots, r_s) \quad &= \quad 1 + \frac{1}{(a_1 r_1^m + a_2 r_2^m + a_3 r_3^m + ... + a_s r_s^m)} \\ &+ \quad \mathcal{O}\Big(\frac{1}{(a_1 r_1^m + a_2 r_2^m + a_3 r_3^m + ... + a_s r_s^m)^{1+\tau}}\Big) \end{aligned} \tag{1.4}$$

where $\tau > 0$, $a_i > 0$ and $a_i \neq a_j$ for some $i \neq j$. Moreover, r_i are given by the generalization of spherical coordinates and defined by

$$\begin{cases} r_1 = r \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{s-1} \\ r_2 = r \sin\phi_1 \sin\phi_2 \cdots \cos\phi_{s-1} \\ \quad \cdots\cdots \\ r_{s-1} = r \sin\phi_1 \cos\phi_2 \\ r_s = r\cos\phi_1; \end{cases} \tag{1.5}$$

where $\phi_i \in [0, \pi], i = 1, 2 \cdots s-2; \phi_{s-1} = [0, 2\pi]$. Define the point

$$P_{j_1 j_2 \cdots j_s} = (P_{j_1}, P_{j_2}, \cdots P_{j_s}) = (r_1 e^{\frac{i(j_1-1)\pi}{k}}, r_2 e^{\frac{i(j_2-1)\pi}{k}}, \cdots, r_s e^{\frac{i(j_s-1)\pi}{k}}); \tag{1.6}$$

where i denotes the square root of -1. Hence any point defined by (1.6) is a function of r and ϕ_i where $i = 1, 2, \cdots, s-1$. We are going to construct solutions which has peak at the point $P_{j_1 j_2 \cdots j_s}$.

We define the approximate solution as:

$$W_{j_1 j_2 \cdots j_s}(x) = w(x - P_{j_1 j_2 \cdots j_s}) \tag{1.7}$$

where $1 \leq j_i \leq k$ for all $1 \leq i \leq s$. Here we identify the Euclidean space $\mathbb{R}^{N_i}$ with $\mathbb{C} \times \mathbb{R}^{N_i - 2}$, and the coordinates of a point $\mathbb{R}^{N_i}$ are given by $(z, \vec{0})$ where $z \in \mathbb{C}$ and $\vec{0} \in \mathbb{R}^{N_i-2}$. Moreover, w is the unique positive entire solution of

$$\Delta w - w + w^p = 0;\ w \in H^1(\mathbb{R}^N). \tag{1.8}$$

It is well known by [19] that $w(x) = w(|x|)$ and the asymptotic behavior of w at infinity is given by

$$\begin{cases} w(x) = A|x|^{-\frac{N-1}{2}} e^{-|x|}\left(1 + \mathcal{O}\left(\frac{1}{|x|}\right)\right) \\ w'(x) = -A|x|^{-\frac{N-1}{2}} e^{-|x|}\left(1 + \mathcal{O}\left(\frac{1}{|x|}\right)\right) \end{cases} \tag{1.9}$$

for some constant $A > 0$. Moreover, w is non-degenerate, that is

$$Ker_{H^1(\mathbb{R}^N)}(\Delta - 1 + pw^{p-1}) = \left\{\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \cdots \frac{\partial w}{\partial x_N}\right\}. \tag{1.10}$$

Theorem 1.1. *There exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, there exists $r \in [\gamma_1 k \ln k, \gamma_2 k \ln k]$ and $\phi_i \in \mathcal{R}_i$ (for the definition of $\mathcal{R}_i$ $i = 1, \cdots, s-1$ see Lemma 2.1), with*

$$u_k(x) = \sum_{j_1, j_2, \cdots j_s = 1}^{k} W_{j_1, j_2, \cdots j_s}(x) + \varphi_k(x) \tag{1.11}$$

being a solution u_k of (1.3) and $\varphi_k(x) \to 0$ as $k \to \infty$ locally uniformly where $\gamma_1 > 0$ and $\gamma_2 > 0$ are positive constants independent of k.

We recall some previous results. Wei and Yan [23] considered the problem

$$\Delta u - V(x)u + u^p = 0, u > 0;\ u \in H^1(\mathbb{R}^N) \tag{1.12}$$

with symmetric potential

$$V(x) = V(r) = V_0 + \frac{a}{r^m} + \mathcal{O}\left(\frac{1}{r^{m+\sigma}}\right) \tag{1.13}$$

for some $V_0 > 0$, $a > 0, \sigma > 0$ and $m > 1$, and proved that (1.12) has infinitely many non-radial solutions. In fact, they proved that (1.12) admits solutions with large number of bumps on a large circle near the infinity. They conjectured that similar result holds for non-symmetric potentials. In this regard, there are two recent papers with different approaches. In [13], del Pino, the second author and Yao used the intermediate Lyapunov-Schmidt reduction method to prove the existence of infinitely many positive solutions to (1.12) for non-symmetric potentials, when $N = 2$, and (m, p, σ) satisfies

$$\min\left\{1, \frac{p-1}{2}\right\} m > 2, \sigma > 2. \tag{1.14}$$

On the other hand, Devillanova and Solimini [14] used variational methods to show that there are infinitely many positive solutions to (1.12) for non-symmetric potentials, when $N = 2$, and $V(x)$ satisfies

$$\frac{A_1}{|x|^s} \leq V(x) - V_\infty \leq \frac{A_2}{|x|}, \text{ for } x \text{ large and } \ s < 4 \tag{1.15}$$

Moreover, if $V(x)$ tends to V_∞ from above with a suitable

$$V(x) \geq V_\infty, \lim_{|x| \to \infty} (V(x) - V_\infty) e^{\eta |x|} = +\infty \text{ for some } \eta \in (0, \sqrt{V_\infty}) \tag{1.16}$$

and V satisfies a global condition:

$$\sup_{x \in \mathbb{R}^N} \|V(x) - V_\infty\|_{L^{\frac{N}{2}} B_1(x)} < \nu \tag{1.17}$$

where ν is a small positive constant, Cerami, Passaseo and Solimini [8]; Ao and Wei [5] proved that (1.12) admits infinitely many positive solutions by purely variational methods.

Remark 1.1. *Theorem 1.1 deals with the anisotropic case. Here we have the following asymptotic expansion* $V = V_\infty + \frac{a(\theta)}{r^m} + \mathcal{O}\left(\frac{1}{r^{m+\tau}}\right)$ *where* $a(\theta)$ *is anisotropic. In this case, even the distribution of spikes is not known.*

Here we allow $N \geq 3$ and $m \geq 4$ (comparing with [14]). Our result suggests that the following conjecture should be true:

Conjecture: *There are infinitely many positive solutions to (1.12) provided* V *satisfies*

$$\frac{A_1}{|x|^{m_1}} \leq V(x) - V_\infty \leq \frac{A_2}{|x|^{m_2}}, \text{ for } x \text{ large and } m_1 \geq m_2 > 0 \tag{1.18}$$

Finally, we mention several results on concentrations on spheres. M. del Pino et. al. [15] considered the Yamabe problem

$$\Delta u + \frac{N(N-2)}{2}|u|^{2^\star - 2}u = 0; \ u \in D^{1,2}(\mathbb{R}^N). \tag{1.19}$$

They construct infinitely many sign-changing solutions for (1.19). The idea of the proof is as follows. Decompose $\mathbb{R}^N = \mathbb{C} \times \mathbb{R}^{N-2}$. Then they produce solution of the form

$$u_k(x) = U(x) - \sum_{j=1}^{k} \mu_k^{-\frac{N-2}{2}} U\left(\frac{x - \xi_j}{\mu_k}\right) + o(1) \tag{1.20}$$

where $U(x) = c_N(\frac{2}{1+|x|^2})^{\frac{N-2}{2}}$, $\mu_k = \frac{c_N}{k^2}$ when $N \geq 4$; $\mu_k = \frac{c_N}{k^2(\log k)^2}$ when $N = 3$ and $\xi_j(k) = (e^{\frac{2j\pi i}{k}}, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$.

In dimension $N \geq 5$, del Pino et. al. [16] obtained a sequences of solutions whose energy concentrates along a two dimensional Clifford torus for the problem

$$\Delta_{\mathbb{S}^3} u + \frac{N(N-2)}{4}(1 - |u|^{2^\star - 2})u = 0 \ \text{ on } \mathbb{S}^N. \tag{1.21}$$

2. PRELIMINARIES

We are given that V satisfies (1.4) and r_i satisfies (1.5). Using (1.5) we obtain

$$\begin{aligned} a_1 r_1^m + a_2 r_2^m + a_3 r_3^m + ... + a_s r_s^m = r^m \Bigg[\sin^m \phi_1 \Bigg[& a_1 \sin^m \phi_2 \cdots \sin^m \phi_{s-1} \\ + \quad a_2 \sin^m \phi_2 \cdots \cos^m \phi_{s-1} + \cdots a_{s-1} \cos^m \phi_2 \Bigg] & + a_s \cos^m \phi_1 \Bigg] \end{aligned}$$

Let

$$\mathcal{S}(\phi_1, \phi_1, \cdots \phi_{s-1}) = \sin^m \phi_1 H_1(\phi_2, \phi_3, \cdots \phi_{s-1}) + a_s \cos^m \phi_1 \tag{2.1}$$

where

$$\begin{aligned} H_1(\phi_2, \phi_3, \cdots \phi_{s-1}) &= a_1 \sin^m \phi_2 \cdots \sin^m \phi_{s-1} \\ &+ a_2 \sin^m \phi_2 \cdots \cos^m \phi_{s-1} + \cdots a_{s-1} \cos^m \phi_2 \\ &= \sin^m \phi_2 H_2(\phi_3, \cdots \phi_{s-1}) + a_{s-1} \cos^m \phi_2. \end{aligned} \tag{2.2}$$

For if $i = 1, 2, \cdots, s-1$; $0 < \phi_i < \frac{\pi}{2}$, then $\mathcal{S}(\phi_1, \cdots \phi_{s-1})$ and $H_i(\phi_{i+1}, \cdots \phi_{s-1})$ are positive functions.

Now we describe two lemmas which will be crucial for the proof of the main theorem.

Lemma 2.1. *Let $g_0(\phi_1) = [H_1 \sin^m \phi_1 + a_s \cos^m \phi_1]$. Then g_0 attains a maximum at a point $\phi_1 = \phi_{1,0} = \tan^{-1}\left(\frac{a_s}{H_1}\right)^{\frac{1}{m-2}}$ whenever $m < 2$ and g_0 attains a minimum at $\phi_{1,0} = \tan^{-1}\left(\frac{a_s}{H_1}\right)^{\frac{1}{m-2}}$ whenever $m > 2$.*

Proof. Differentiating we obtain $g_0'(\phi_1) = \frac{1}{2}(H_1 \sin^{m-2}\phi_1 - a_s \cos^{m-2}\phi_1)\sin 2\phi_1$. Hence $g_0'(\phi_1) = 0$ implies that $\phi_{1,0} = \tan^{-1}\left(\frac{a_s}{H_1}\right)^{\frac{1}{m-2}}$. Moreover,

$$g_0''(\phi_{1,0}) = \frac{(m-2)}{4}(H_1 \sin^{m-4}\phi_{1,0} + a_s \cos^{m-4}\phi_{1,0})\sin^2 2\phi_{1,0}.$$

As a result, $g_0''(\phi_{1,0}) < 0$ when $m < 2$ and $g_0''(\phi_{1,0}) > 0$ when $m > 2$ which implies that g_0 achieves its maximum at a point $\phi_{1,0}$ and g_0 achieves its minimum at $\phi_{1,0}$ when $m > 2$. □

Remark 2.1. *Similarly for $i = 1, 2, \cdots, s-2$; $g_i(\phi_{i+1}) = [H_{i+1}\sin^m \phi_{i+1} + a_{s-i}\cos^m \phi_{i+1}]$ attains a maximum at $\phi_{i,0} = \tan^{-1}\left(\frac{a_{s-i}}{H_{i+1}}\right)^{\frac{1}{m-2}}$ whenever $m < 2$ and g_i attains a minimum at $\phi_{i,0} = \tan^{-1}\left(\frac{a_{s-i}}{H_{i+1}}\right)^{\frac{1}{m-2}}$ whenever $m > 2$.*

Remark 2.2. *Note that when $m = 2$, $g_0(\phi_1) = [H_1 \sin^2 \phi_1 + a_s \cos^2 \phi_1]$ has a critical point at $\phi_1 = \frac{\pi}{2}$. But*

$$g_0''(\phi_1) = 2[H_1 - a_s]\cos 2\phi_1$$

which implies that g_0 has a maximum if $H_1 > a_s$ and g_0 has a minimum if $H_1 < a_s$ at $\phi_1 = \frac{\pi}{2}$. But $r_s = r\cos\phi_1$ can be very small when ϕ is close to $\phi_1 = \pi/2$. Then the distance between the spikes and the location of the spikes may become $\mathcal{O}(1)$ which in our case breaks down the linear theory. As a result, in the case $m = 2$, we cannot use the method in Theorem 1.1.

Lemma 2.2. *Let $F(r) = r^{-m} - e^{-\frac{\pi r}{k}}$ where $0 < r < +\infty$. Then F attains its maximum at a point $r = (\frac{m+1}{\pi} + o(1))k\ln k$.*

Proof. In fact, it is easy to check that F has a critical point at $r = (\frac{m+1}{\pi} + o(1))k\ln k$. □

Choose a $\delta > 0$ small such that $\mathcal{R}_i = [\phi_{i,0} - \delta, \phi_{i,0} + \delta]$ with $\phi_{i,0} - \delta > 0$ and $\phi_{i,0} + \delta < \frac{\pi}{2}$ where $i = 1, 2 \cdots, s-1$. Let $M > 0$ be large and $\chi_{j_1 j_2 \cdots j_s}$ be a smooth function with compact support such that

$$\chi_{j_1 j_2 \cdots j_s}(x) = \begin{cases} 1 & \text{if } |x - P_{j_1 j_2 \cdots j_s}| < \dfrac{r}{2M} \\ 0 & \text{if } |x - P_{j_1 j_2 \cdots j_s}| > \dfrac{3r}{4M} \end{cases} \tag{2.3}$$

and $\text{supp}\chi_{j_1j_2\cdots j_s} \cap \text{supp}\chi_{k_1k_2\cdots k_s} = \emptyset$ whenever $(j_1, j_2, \cdots j_s) \neq (k_1, k_2 \cdots k_s)$. Now define

$$Z_{j_1j_2\cdots j_s n} = \chi_{j_1j_2\cdots j_s}(x)\frac{\partial W_{j_1j_2\cdots j_s}}{\partial x_n}; 1 \leq j_1, j_2 \cdots j_s \leq k \text{ and } 1 \leq n \leq N.$$

Furthermore, define

$$D = \{r : r \in [\gamma_1 k \ln k, \gamma_2 k \ln k]\}. \tag{2.4}$$

We are going to construct solutions of (1.3) using the

$$\max_{(r,\phi_1,\cdots\phi_{s-1})\in D\times\mathcal{R}_1\cdots\times\mathcal{R}_{s-1}} \Psi(r, \phi_1, \phi_2, \cdots \phi_{s-1})$$

or

$$\max_{r\in D}\ \min_{(\phi_1,\cdots\phi_{s-1})\in\mathcal{R}_1\cdots\times\mathcal{R}_{s-1}} \Psi(r, \phi_1, \phi_2, \cdots \phi_{s-1})$$

where Ψ will be defined in (6.1). If we substitute

$$u_k(x) = \sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}(x) + \varphi_k(x)$$

in (1.3), then we can write (1.3) as

$$S[u_k] = L(\varphi) + E + N(\varphi) = 0; \tag{2.5}$$

where

$$L(\varphi) = \Delta\varphi - \varphi + p\Big(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}\Big)^{p-1}\varphi \tag{2.6}$$

the error due to the approximation

$$\begin{aligned} E &= \Big(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}\Big)^{p} - \Big(\sum_{j_1j_2\cdots j_s=1}^{k} W^{p}_{j_1j_2\cdots j_s}\Big) \\ &\quad - \sum_{j_1j_2\cdots j_s=1}^{k} (V(x) - 1)W_{j_1j_2\cdots j_s} \end{aligned} \tag{2.7}$$

and the remainder

$$\begin{aligned} N(\varphi) &= \Big(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s} + \varphi\Big)^{p} - \Big(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}\Big)^{p} \\ &\quad - p\Big(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}\Big)^{p-1}\varphi + (1 - V(x))\varphi. \end{aligned} \tag{2.8}$$

Define the norm by

$$\|\varphi\|_\star = \sup_{\mathbb{R}^N}\Big(\sum_{j_1j_2\cdots j_s=1}^{k} e^{\eta|x-P_{j_1,j_2,\cdots j_s}|}\Big)|\varphi(x)|.$$

for some $0 < \eta < 1$.

3. Linear Theory

We first study the model problem

$$
(3.1) \qquad \begin{cases} L(\varphi) = h + \displaystyle\sum_{n=1}^{N} \sum_{j_i, j_2, \cdots j_s = 1}^{k} c_{j_1 j_2 \cdots j_s n} Z_{j_1 j_2 \cdots j_s n} \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} \varphi Z_{j_1 j_2 \cdots j_s n} dx = 0 \text{ for } n = 1, \cdots N; 1 \leq j_i, j_2, \cdots j_s \leq k \end{cases}
$$

where h lies in some space. In some sense L is made up of operators $L_{j_1 j_2 \cdots j_s}$ where

$$
(3.2) \qquad L_{j_1 j_2 \cdots j_s}(\varphi) = \Delta\varphi - \varphi + pW^{p-1}_{j_1 j_2 \cdots j_s}\varphi.
$$

Lemma 3.1. *Let h be a function with $\|h\|_\star < +\infty$ and assume $\phi_i \in \mathcal{R}_i$; $(c_{j_1 j_2 \cdots j_s n}, \varphi)$ is a solution to (3.1). There exists $\eta \in (0,1)$, $C > 0$ and $r_0 > 0$ such that for all $r \geq r_0$ satisfying (3.1), we have*

$$
(3.3) \qquad \|\varphi\|_\star \leq C\|h\|_\star.
$$

Proof. If possible, let there exists a solution to (3.1) with

$$
\|h\|_\star \to 0, \|\varphi\|_\star = 1.
$$

We claim, that

$$
c_{j_1 j_2 \cdots j_s n} \to 0
$$

for all n and $1 \leq j_i \leq k, i = 1, \cdots s$. First note that

$$
(3.4) \qquad \int_{\mathbb{R}^N} Z_{j_1 j_2 \cdots j_s p} Z_{k_1 k_2 \cdots k_s q} dx = 0
$$

if $p \neq q$ or $(j_1, j_2 \cdots, j_s) \neq (k_1, k_2 \cdots, k_s)$. Multiplying (3.1) by $Z_{j_1 j_2 \cdots j_s n}$ we obtain

$$
(3.5) \qquad \int_{\mathbb{R}^N} L(\varphi) Z_{j_1 j_2 \cdots j_s n} = \int_{\mathbb{R}^N} h Z_{j_1 j_2 \cdots j_s n} + c_{j_1 j_2 \cdots j_s n} \int_{\mathbb{R}^N} Z^2_{j_1 j_2 \cdots j_s n}.
$$

Moreover, there exists a small $\eta > 0$ such that

$$
\int_{\mathbb{R}^N} Z^2_{j_1 j_2 \cdots j_s n} dx = \int_{\mathbb{R}^N} \left(\frac{\partial w}{\partial x_n}\right)^2 dx + \mathcal{O}(e^{-(1-\eta)r}).
$$

On the other hand

$$
\int_{\mathbb{R}^N} h Z_{j_1 j_2 \cdots j_s n} dx \leq C\|h\|_\star.
$$

When $p > 2$, by the integrating by parts, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} L(\varphi) Z_{j_1 j_2 \cdots j_s n} dx &= \int_{\mathbb{R}^N} \Big[\Delta\varphi - \varphi + p\Big(\sum_{j_1, \cdots j_s = 1}^{k} W_{j_1 j_2 \cdots j_s} \Big)^{p-1} \varphi \Big] Z_{j_1 j_2 \cdots j_s n} \\
&= \int_{\mathbb{R}^N} [\Delta Z_{j_1 j_2 \cdots j_s n}\varphi - Z_{j_1 j_2 \cdots j_s n}\varphi + p\Big(\sum_{j_1 j_2 \cdots j_s = 1}^{k} W_{j_1 j_2 \cdots j_s} \Big)^{p-1} Z_{j_1 j_2 \cdots j_s n}\varphi] \\
&= p\int_{\mathbb{R}^N} \Big[\Big(\sum_{j_1, j_2, \cdots, j_s = 1}^{k} W_{j_1 j_2 \cdots j_s} \Big)^{p-1} - W^{p-1}_{j_1 j_2 \cdots j_s} \Big] Z_{j_1 j_2 \cdots j_s n}\varphi \\
&+ 2\int_{\mathbb{R}^N} \nabla\chi_{j_1 j_2 \cdots j_s} \nabla \frac{\partial W_{j_1 j_2 \cdots j_s}}{\partial x_n}\varphi dx + \int_{\mathbb{R}^N} \Delta\chi_{j_1 j_2 \cdots j_s} \frac{\partial W_{j_1 j_2 \cdots j_s}}{\partial x_n}\varphi dx \\
&= \mathcal{O}\Big(\sum_{(j_1 j_2 \cdots j_s) \neq (k_1 k_2 \cdots k_s)} e^{-(p-2)|P_{j_1 j_2 \cdots j_s} - P_{k_1 k_2 \cdots k_s}|} \Big) \int_{\mathbb{R}^N} Z_{j_1 j_2 \cdots j_s n}\varphi \\
&+ 2\int_{\mathbb{R}^N} \nabla\chi_{j_1 j_2 \cdots j_s} \nabla \frac{\partial W_{j_1 j_2 \cdots j_s}}{\partial x_n}\varphi dx + \int_{\mathbb{R}^N} \Delta\chi_{j_1 j_2 \cdots j_s} \frac{\partial W_{j_1 j_2 \cdots j_s}}{\partial x_n}\varphi dx \\
&= \mathcal{O}\Big(\sum_{(j_1 j_2 \cdots j_s) \neq (k_1 k_2 \cdots k_s)} e^{-(p-2)|P_{j_1 j_2 \cdots j_s} - P_{k_1 k_2 \cdots k_s}|} \Big) \|\varphi\|_\star \\
&+ \mathcal{O}(e^{-(1-\eta)r}) \|\varphi\|_\star. \qquad (3.6)
\end{aligned}$$

When $1 < p \leq 2$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} L(\varphi) Z_{j_1 j_2 \cdots j_s n} dx &= \int_{\mathbb{R}^N} [\Delta\varphi - \varphi + p\Big(\sum_{j_1, \cdots j_s = 1}^{k} W_{j_1 j_2 \cdots j_s} \Big)^{p-1} \varphi] Z_{j_1 j_2 \cdots j_s n} \\
&= \int_{\mathbb{R}^N} [\Delta Z_{j_1 j_2 \cdots j_s n}\varphi - Z_{j_1 j_2 \cdots j_s n}\varphi + p\Big(\sum_{j_1 j_2 \cdots j_s}^{k} W_{j_1 j_2 \cdots j_s} \Big)^{p-1} \varphi Z_{j_1 j_2 \cdots j_s n}] \\
&= p\int_{\mathbb{R}^N} \Big[\Big(\sum_{j_1 j_2 \cdots j_s}^{k} W_{j_1 j_2 \cdots j_s} \Big)^{p-1} - W^{p-1}_{j_1 j_2 \cdots j_s} \Big] Z_{j_1 j_2 \cdots j_s n}\varphi \\
&+ 2\int_{\mathbb{R}^N} \nabla\chi_{j_1 j_2 \cdots j_s} \nabla \frac{\partial W_{j_1 j_2 \cdots j_s}}{\partial x_n}\varphi dx + \int_{\mathbb{R}^N} \Delta\chi_{j_1 j_2 \cdots j_s} \frac{\partial W_{j_1 j_2 \cdots j_s}}{\partial x_n}\varphi dx \\
&= \mathcal{O}\Big(\Big(\sum_{(j_1 j_2 \cdots j_s) \neq (k_1 k_2 \cdots k_s)} W_{j_1 j_2 \cdots j_s} W_{k_1 k_2 \cdots k_s} \Big)^{\frac{p-1}{2}} \Big) \int_{\mathbb{R}^N} Z_{j_1 j_2 \cdots j_s n}\varphi \\
&+ \mathcal{O}(e^{-(1-\eta)r}) \|\varphi\|_\star \\
&= \mathcal{O}\Big(\sum_{(j_1 j_2 \cdots j_s) \neq (k_1 k_2 \cdots k_s)} e^{-\frac{p-1}{2}|P_{j_1 j_2 \cdots j_s} - P_{k_1 k_2 \cdots k_s}|} \Big) \|\varphi\|_\star \\
&+ \mathcal{O}(e^{-(1-\eta)r}) \|\varphi\|_\star. \qquad (3.7)
\end{aligned}$$

Hence from (3.5) we have

$$|c_{j_1 j_2 \cdots j_s n}| \leq C[\|h\|_\star + \mathcal{O}(e^{-(1-\eta)r})\|\varphi\|_\star]; \qquad (3.8)$$

and as a result we obtain

$$|c_{j_1 j_2 \cdots j_s n}| \to 0 \text{ as } r \to \infty.$$

Now define

$$R(x)=\sum_{j_1j_2\cdots j_s=1}^{k} e^{-\eta|x-P_{j_1j_2\cdots j_s}|} \tag{3.9}$$

for some $\eta\in(0,1)$. Then we have

$$L(R)\geq \frac{1}{2}(1-\eta^2)R; x\in\mathbb{R}^N\setminus\cup_{j_1j_2\cdots j_s}^k B_\delta(P_{j_1j_2\cdots j_s})$$

for some $\delta>0$ independent k. Hence we can use the barrier as R to obtain

$$|\varphi(x)|\leq C\bigg(\|h\|_\star+\sum_{j_1j_2\cdots j_s=1}^{k}\|\varphi\|_{L^\infty(\partial B_\delta(P_{j_1j_2\cdots j_s}))}\bigg)R(x) \tag{3.10}$$

in $\mathbb{R}^N\setminus\cup_{j_1j_2\cdots j_s}^k B_\delta(P_{j_1j_2\cdots j_s})$. Now we prove the main part. If possible, let there be a sequence of $r_\alpha\to+\infty$ with h_α and φ_α such that

$$\|h_\alpha\|_\star\to 0, \|\varphi_\alpha\|_\star=1$$

as $\alpha\to+\infty$. But by (3.8)

$$|c^{(\alpha)}_{j_1j_2\cdots j_sn}|\to 0 \text{ as } \alpha\to\infty$$

and due to the exponentially decay of $Z_{j_1j_2\cdots j_sn}$ we have

$$\bigg\|\sum_{n=1}^{N}\sum_{j_1j_2\cdots j_s=1}^{k} c^{(\alpha)}_{j_1j_2\cdots j_sn}Z_{j_1j_2\cdots j_sn}\bigg\|_\star\to 0. \tag{3.11}$$

Hence there exists a point of $P^{(\alpha)}_{j_1j_2\cdots j_s}$ where $P^{(\alpha)}_{j_1j_2\cdots j_s}$ is a function of $r_\alpha\in D$ and $\phi_{i,\alpha}\in\mathcal{R}_i$ such that

$$\|\varphi_\alpha\|_{L^\infty(B_r(P^{(\alpha)}_{j_1j_2\cdots j_s}))}\geq c>0.$$

By the standard elliptic estimate and the Arzela–Ascoli's theorem, φ_α converges locally uniformly to φ as $\alpha\to\infty$ where φ satisfies

$$(\Delta-1+pw^{p-1})\varphi=0 \text{ in } \mathbb{R}^N$$

with $|\varphi(x)|\leq ce^{-\eta|x|}$ for some $\eta>0$ and $c>0$. Moreover, note that φ_α satisfies the orthogonality condition. Hence we must have

$$\int_{\mathbb{R}^N}\varphi\nabla w dx=0. \tag{3.12}$$

This implies $\varphi\equiv 0$ as w is non-degenerate, a contradiction. □

Lemma 3.2. *There exists $\eta\in(0,1)$, $C>0$ such that for all $r\geq r_0$ and $\phi_i\in\mathcal{R}_i$, there exists a unique solution $(c_{j_1j_2\cdots j_sn},\varphi)$ satisfying* (3.1)*. Furthermore,*

$$\|\varphi\|_\star\leq C\|h\|_\star. \tag{3.13}$$

Proof. Define the Sobolev space

$$\mathcal{H}=\bigg\{\varphi\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}\varphi Z_{j_1j_2\cdots j_sn}dx=0; n=1,\cdots,N; 1\leq j_i\leq k, i=1,2,\cdots s\bigg\}.$$

Then (3.1) is expressible as

$$\varphi+K(\varphi)=\tilde{h}. \tag{3.14}$$

where $\tilde{h}$ is defined by duality and $K:\mathcal{H}\to\mathcal{H}$ is a linear compact operator. Using the Fredholm's alternative, (3.1) has a unique solution for each $\tilde{h}$ which is equivalent to showing that the equation admit a unique solution for $\tilde{h}=0$ which in turn follows from Lemma 3.1. The estimate (3.13) follows directly from Lemma 3.1. Moreover, if φ is a unique solution of (3.1), we can write $\varphi = A(h)$ and hence from (3.13) we have

$$\|A(h)\|_\star \le C\|h\|_\star. \tag{3.15}$$

□

4. The non-linear problem

Now we consider a nonlinear projected problem

$$\begin{cases} L(\varphi)+E+N(\varphi) = \displaystyle\sum_{n=1}^{N}\sum_{j_1j_2\cdots j_s=1}^{k} c_{j_1j_2\cdots j_s n} Z_{j_1j_2\cdots j_s n} \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} \varphi Z_{j_1j_2\cdots j_s n}dx = 0 \text{ for } n=1,\cdots N; 1\le j_1,j_2,\cdots j_s \le k. \end{cases} \tag{4.1}$$

We are going to show the solvability of (4.1) in $(c_{j_1j_2\cdots j_s n},\varphi)$ whenever $r\in D$ and $\phi_i \in \mathcal{R}_i$ with $i=1,2\cdots,s-1$.

Lemma 4.1. *There exist $r_0>0$ large and $C>0$ such that for all $r\ge r_0$ and for any $r\in D$, $\phi_i\in\mathcal{R}_i$, there exists a unique solution $(c_{j_1j_2\cdots j_s n},\varphi)$ of (4.1). Furthermore,*

$$\|\varphi\|_\star \le Cr^{-m}. \tag{4.2}$$

Proof. Note that φ solves (4.1) if and only if

$$\varphi = A(-E-N(\varphi)) \tag{4.3}$$

where A is the linear operator introduced in Lemma 3.2. If we define

$$F(\varphi) = A(-E-N(\varphi)); \tag{4.4}$$

then we are reduced to studying the fixed points of the map F. Define a ball

$$\mathcal{B} = \{\varphi\in\mathcal{H} : \|\varphi\|_\star \le \eta r^{-m}\}. \tag{4.5}$$

for some $\eta>0$. Now we claim that

$$\|E\|_\star \le Cr^{-m}. \tag{4.6}$$

Fix a point $P_{j_1j_2\cdots j_s}$ with $|x-P_{j_1j_2\cdots j_s}|\le \frac{r}{2+\sigma}$ where $\sigma>0$ is small number. Then we have

$$|x-P_{k_1k_2\cdots k_s}| \ge |P_{k_1k_2\cdots k_s}-P_{j_1j_2\cdots j_s}| - \frac{r}{2+\sigma} \ge \frac{r}{2}+\frac{r\sigma}{2(2+\sigma)}$$

whenever $|P_{j_1j_2\cdots j_s}-P_{k_1k_2\cdots k_s}|\ge r$.

Hence we obtain,

$$\begin{aligned} |E| &\le CW_{j_1j_2\cdots j_s}^{p-1}\sum_{(k_1k_2\cdots k_s)\ne(j_1j_2\cdots j_s)} W_{k_1k_2\cdots k_s} + \frac{C}{r^m\mathcal{S}(\phi_1,\phi_2,\cdots\phi_{s-1})}\sum_{j_1j_2\cdots j_s=1} W_{j_1j_2\cdots j_s} \\ &\le CW_{j_1j_2\cdots j_s}^{p-1}\sum_{(k_1k_2\cdots k_s)\ne(j_1j_2\cdots j_s)} w(x-P_{k_1k_2\cdots k_s}) + \frac{C}{r^m\mathcal{S}(\phi_1,\phi_2,\cdots\phi_{s-1})}\sum_{j_1j_2\cdots j_s=1} W_{j_1j_2\cdots j_s} \\ &\le CW_{j_1j_2\cdots j_s}^{p-1}\sum_{(k_1k_2\cdots k_s)\ne(j_1j_2\cdots j_s)} e^{-\frac{r}{2}-\frac{r\sigma}{2(2+\sigma)}} + \frac{C}{r^m\mathcal{S}(\phi_1,\phi_2,\cdots\phi_{s-1})}\sum_{j_1j_2\cdots j_s=1} W_{j_1j_2\cdots j_s}. \end{aligned}$$

In the region $|x - P_{j_1j_2\cdots j_s}| > \frac{r}{2+\sigma}$, choosing $0 < \mu < 1$

$$\begin{aligned} |E| &\leq C \sum_{j_1j_2\cdots j_s=1} W^p_{j_1j_2\cdots j_s} + C \sum_{j_1j_2\cdots j_s=1} W_{j_1j_2\cdots j_s} \\ &\leq C\Big(\sum_{j_1j_2\cdots j_s=1} e^{-\mu|x-P_{j_1j_2\cdots j_s}|}\Big) e^{-\frac{(p-\mu)r}{2+\sigma}} + C \sum_{j_1j_2\cdots j_s=1} W_{j_1j_2\cdots j_s} \\ &\leq C\Big(\sum_{j_1j_2\cdots j_s=1} e^{-\mu|x-P_{j_1j_2\cdots j_s}|}\Big) e^{-\frac{(p-\mu)r}{2+\sigma}} + C\Big(\sum_{j_1j_2\cdots j_s=1}^{k} e^{-\mu|x-P_{j_1j_2\cdots j_s}|}\Big) e^{-\frac{(1-\mu)r}{2+\sigma}}. \end{aligned}$$

Hence the result follows. Moreover, for any $\varphi \in \mathcal{B}$ we have

$$|N(\varphi)| \leq C(|\varphi|^2 + |\varphi|^p + r^{-m}|\varphi|).$$

Hence

$$\|N(\varphi)\|_\star \leq C(\|\varphi\|^2_\star + \|\varphi\|^p_\star + r^{-m}\|\varphi\|_\star). \tag{4.7}$$

Now we need to check whether the map (4.4) is in fact a contraction from $\mathcal{B}$ to $\mathcal{B}$. We have

$$\|F(\varphi)\|_\star = \|A(E + N(\varphi))\|_\star \leq C\|E\|_\star + C\|N(\varphi)\|_\star \leq \eta r^{-m} \tag{4.8}$$

Moreover, for any $\varphi_1, \varphi_2 \in \mathcal{B}$

$$\|F(\varphi_1) - F(\varphi_2)\|_\star \leq C\|N(\varphi_1) - N(\varphi_2)\|_\star = o(1)\|\varphi_1 - \varphi_2\|_\star. \tag{4.9}$$

As a consequence of the contraction mapping principle, we obtain the required result. □

5. The Reduced Problem

Denote the functional associated to (1.3) by

$$I(u) = \int_{\mathbb{R}^N} \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}V(x)u^2 - \frac{1}{p+1}u^{p+1}\right] dx.$$

Lemma 5.1. *Then we have*

$$k^{-s} I\Big(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}\Big) = I_0 + \frac{A}{2r^m \mathcal{S}(\phi_1, \phi_2, \cdots \phi_{s-1})} - \frac{B}{2} e^{-\frac{2\pi r}{k}} + \mathcal{O}\Big(\frac{1}{r^{m+\tau}}\Big)$$

where $I_0 = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^N} w^{p+1} dx$; $A = \int_{\mathbb{R}^N} w^2 dx$ *and some constant* $B > 0$.

Proof. We write

$$\begin{aligned} I(u) &= \int_{\mathbb{R}^N} \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{2}V(x)u^2 - \frac{1}{p+1}u^{p+1}\right] dx \\ &= \int_{\mathbb{R}^N} \left[\frac{1}{2}(|\nabla u|^2 + u^2) + \frac{1}{2}(V(x) - 1)u^2 - \frac{1}{p+1}u^{p+1}\right] dx \\ &= \frac{1}{2}\mathcal{A} + \frac{1}{2}\mathcal{B} - \frac{1}{p+1}\mathcal{C} \end{aligned} \tag{5.1}$$

where $\mathcal{A} = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx, \mathcal{B} = \int_{\mathbb{R}^N} (V(x) - 1) u^2 dx$ and $\mathcal{C} = \int_{\mathbb{R}^N} u^{p+1} dx$. Hence we obtain

$$\begin{aligned}
\mathcal{A} &= \int_{\mathbb{R}^N} \Big(\Big| \nabla \Big(\sum_{j_1 j_2 \cdots j_s = 1}^{k} W_{j_1 j_2 \cdots j_s} \Big) \Big|^2 + \Big(\sum_{j_1 j_2 \cdots j_s}^{k} W_{j_1 j_2 \cdots j_s} \Big)^2 \Big) dx \\
&= \sum_{j_1 j_2 \cdots j_s = 1}^{k} \sum_{k_1 k_2 \cdots k_s = 1}^{k} \int_{\mathbb{R}^N} \Big(W^p_{j_1 j_2 \cdots j_s} \Big) W_{k_1 k_2 \cdots k_s} dx \\
&= k^s \int_{\mathbb{R}^N} w^{p+1} dx + \sum_{(j_1 j_2 \cdots j_s) \neq (k_1 k_2 \cdots k_s)} \int_{\mathbb{R}^N} w(x - (P_{j_1 j_2 \cdots j_s} - P_{k_1 k_2 \cdots k_s})) w^p(x) dx.
\end{aligned}$$

Using (1.4) we obtain

$$\begin{aligned}
\mathcal{B} &= \int_{\mathbb{R}^N} \Big(V(x) - 1 \Big) \Big(\sum_{j_1 j_2 \cdots j_s = 1}^{k} W_{j_1 j_2 \cdots j_s} \Big)^2 dx \\
&= \int_{\mathbb{R}^N} \Big(V(x) - 1 \Big) W^2_{j_1 j_2 \cdots j_s} dx + \sum_{(j_1 j_2 \cdots j_s) \neq (k_1 k_2 \cdots k_s)} \int_{\mathbb{R}^N} \Big(V(x) - 1 \Big) W_{k_1 k_2 \cdots k_s} W_{j_1 j_2 \cdots j_s} \\
&= \int_{\mathbb{R}^N} \Big(V(x + P_{j_1 j_2 \cdots j_s}) - 1 \Big) w^2(x) dx + o(e^{-\frac{\pi r}{k}}) \\
&= \int_{\mathbb{R}^N} \Big(V(x_1 + P_{j_1}, x_2 + P_{j_2}, \cdots, x_s + P_{j_s}) - 1 \Big) w^2(x) dx \\
&= \int_{B_{r/2}(0)} \Big(\frac{1}{a_1 |x_1 + P_{j_1}|^m + a_2 |x_2 + P_{j_2}|^m + \cdots + a_s |x_s + P_{j_s}|^m} \Big) w^2(x) dx \\
&+ \mathcal{O} \Big(\int_{B_{r/2}(0)} \Big(\frac{1}{a_1 |x_1 + P_{j_1}|^m + a_2 |x_2 + P_{j_2}|^m + \cdots + a_s |x_s + P_{j_s}|^m} \Big)^{1+\tau} w^2(x) \Big) dx \\
&+ \mathcal{O}(e^{-r(1-\eta)})
\end{aligned}$$

for some $\eta > 0$ small. Moreover, for any $(x_1, x_2, \cdots x_s) \in B_{r/2}(0)$

$$\begin{aligned}
|x_1 + P_{j_1}|^m &= |P_{j_1}|^m \Big(1 + \mathcal{O} \Big(\frac{|x_1|}{|P_{j_1}|} \Big) \Big); \\
|x_2 + P_{j_2}|^m &= |P_{j_2}|^m \Big(1 + \mathcal{O} \Big(\frac{|x_2|}{|P_{j_2}|} \Big) \Big) \\
&\cdots \\
|x_s + P_{j_s}|^m &= |P_{j_s}|^m \Big(1 + \mathcal{O} \Big(\frac{|x_s|}{|P_{j_s}|} \Big) \Big)
\end{aligned} \tag{5.2}$$

and hence

$$\begin{aligned}
& a_1 |x_s + P_{j_1}|^m + a_2 |x_2 + P_{j_2}|^m + \cdots + a_s |x_s + P_{j_s}|^m \\
=\ & a_1 |P_{j_1}|^m \Big(1 + \mathcal{O} \Big(\frac{|x_1|}{|P_{j_1}|} \Big) \Big) + a_2 |P_{j_2}|^m \Big(1 + \mathcal{O} \Big(\frac{|x_2|}{|P_{j_2}|} \Big) \Big) + \cdots a_s |P_{j_s}|^m \Big(1 + \mathcal{O} \Big(\frac{|x_s|}{|P_{j_s}|} \Big) \Big) \\
=\ & a_1 |P_{j_1}|^m + a_2 |P_{j_2}|^m + \cdots a_s |P_{j_s}|^m \\
+\ & a_1 |P_{j_1}|^m \mathcal{O} \Big(\frac{|x_1|}{|P_{j_1}|} \Big) + a_2 |P_{j_2}|^m \mathcal{O} \Big(\frac{|x_2|}{|P_{j_2}|^m} \Big) \cdots + a_s |P_{j_s}|^m \mathcal{O} \Big(\frac{|x_s|}{|P_{j_s}|} \Big).
\end{aligned}$$

As a result we have,

$$
\begin{aligned}
& (a_1|x_s+P_{j_1}|^m+a_2|x_2+P_{j_2}|^m+\cdots+a_s|x_s+P_{j_s}|^m)^{-1}\\
= & \frac{1}{a_1|P_{j_1}|^m+a_2|P_{j_2}|^m\cdots+a_s|P_{j_s}|^m}\\
\times & \left(1+\mathcal{O}\left(\frac{a_1|P_{j_1}|^{m-1}|x_1|+a_2|P_{j_2}|^{m-1}|x_2|+\cdots+a_s|P_{j_s}|^{m-1}|x_s|}{a_1|P_{j_1}|^m+a_2|P_{j_2}|^m+\cdots+a_s|P_{j_s}|^m}\right)\right).
\end{aligned}
$$

Hence we have,

$$
\begin{aligned}
& \int_{B_{r/2}(0)}\left(\frac{1}{a_1|x_1+P_{j_1}|^m+a_2|x_2+P_{j_2}|^m+\cdots a_s|x_s+P_{j_s}|^m}\right)w^2(x)dx\\
= & \left(\frac{1}{a_1|P_{j_1}|^m+a_2|P_{j_2}|^m+\cdots a_s|P_{j_s}|^m}\right)\int_{\mathbb{R}^N}w^2dx\\
+ & \left(\frac{1}{a_1|P_{j_1}|^m+a_2|P_{j_2}|^m+\cdots a_s|P_{j_s}|^m}\right)^2\\
\times & \mathcal{O}\left(\int_{\mathbb{R}^N}(a_1|P_{j_1}|^{m-1}|x_1|+a_2|P_{j_2}|^{m-1}|x_2|\cdots+a_s|P_{j_s}|^{m-1}|x_s|)w^2dx\right)\\
= & \left(\frac{1}{\mathcal{S}(\phi_1,\phi_2,\cdots\phi_{s-1})r^m}\right)\int_{\mathbb{R}^N}w^2dx+\mathcal{O}\left(\frac{1}{(\mathcal{S}(\phi_1,\cdots,\phi_{s-1}))^2r^{m+1}}\right).
\end{aligned}
$$

Moreover, as $p>1$ using the Taylor expansion we obtain,

$$
\begin{aligned}
\mathcal{C} = & \int_{\mathbb{R}^N}\left(\sum_{j_1j_2\cdots j_s=1}^{k}W_{j_1j_2\cdots j_s}\right)^{p+1}dx\\
= & \sum_{j_1j_2\cdots j_s=1}^{k}\int_{\mathbb{R}^N}W_{j_1j_2\cdots j_s}^{p+1}dx+(p+1)\sum_{(k_1k_2\cdots k_s)\neq(j_1j_2\cdots j_s)}\int_{\mathbb{R}^N}W_{j_1j_2\cdots j_s}^{p}W_{k_1k_2\cdots k_s}dx\\
+ & \mathcal{O}\left(\sum_{(j_1j_2\cdots j_s)\neq(k_1k_2\cdots k_s)}\int_{\mathbb{R}^N}W_{j_1j_2\cdots j_s}^{p-1}W_{k_1k_2\cdots k_s}^2dx\right)\\
= & k^s\int_{\mathbb{R}^N}w^{p+1}dx+(p+1)\sum_{(j_1j_2\cdots j_s)\neq(k_1k_2\cdots k_s)}\int_{\mathbb{R}^N}w^p(x)w(x-(P_{j_1j_2\cdots j_s}-P_{k_1k_2\cdots k_s}))dx\\
+ & \mathcal{O}\left(\sum_{(j_1j_2\cdots j_s)\neq(k_1k_2\cdots k_s)}\int_{\mathbb{R}^N}w^{p-1}(x)w^2(x-P_{j_1j_2\cdots j_s}+P_{k_1k_2\cdots k_s})dx\right).
\end{aligned}
$$

Hence from (5.1) we obtain

$$
\begin{aligned}
I(u) = & \frac{(p-1)k^s}{2(p+1)}\int_{\mathbb{R}^N}w^{p+1}dx+\left(\frac{k^s}{2\mathcal{S}(\phi_1,\phi_2,\cdots\phi_{s-1})r^m}\right)\int_{\mathbb{R}^N}w^2dx\\
- & \frac{1}{2}\sum_{(j_1j_2\cdots j_s)\neq(k_1k_2\cdots k_s)}\int_{\mathbb{R}^N}w^p(x)w(x-(P_{j_1j_2\cdots j_s}-P_{k_1k_2\cdots k_s}))dx\\
+ & \mathcal{O}\left(\sum_{(j_1j_2\cdots j_s)\neq(k_1k_2\cdots k_s)}\int_{\mathbb{R}^N}w^{p-1}(x)w^2(x-P_{j_1j_2\cdots j_s}+P_{k_1k_2\cdots k_s})dx\right)\\
+ & \mathcal{O}\left(\frac{k^s}{r^{m+\tau}}\right). \qquad (5.3)
\end{aligned}
$$

Moreover, for $(j_1, j_2\cdots, j_s) \neq (k_1, k_2, \cdots k_s)$

$$\begin{aligned}
&|P_{j_1j_2\cdots j_s} - P_{k_1k_2\cdots k_s}|^2\\
= \;& 4r_1^2\Big[\sin^2\frac{(j_1-k_1)\pi}{2k}\Big] + 4r_2^2\Big[\sin^2\frac{(j_2-k_2)\pi}{2k}\Big] + \cdots + 4r_s^2\Big[\sin^2\frac{(j_s-k_s)\pi}{2k}\Big]\\
= \;& 4r^2\Big[\sin^2\frac{(j_1-k_1)\pi}{2k}\sin^2\phi_1\sin^2\phi_2\cdots\sin^2\phi_{s-1}\\
+ \;& \sin^2\frac{(j_2-k_2)\pi}{2k}\sin^2\phi_1\sin^2\phi_2\cdots\cos^2\phi_{s-1} + \cdots + \sin^2\frac{(j_s-k_s)\pi}{2k}\cos^2\phi_1\Big].
\end{aligned}$$

Hence if $|P_{j_1j_2\cdots j_s} - P_{k_1k_2\cdots k_s}|$ is finite

$$|P_{j_1j_2\cdots j_s} - P_{k_1k_2\cdots k_s}| \sim \frac{\pi r}{k} \tag{5.4}$$

as $k \to \infty$. Moreover, if $|P_{j_1j_2\cdots j_s} - P_{k_1k_2\cdots k_s}|$ is large, then

$$|P_{j_1j_2\cdots j_s} - P_{k_1k_2\cdots k_s}| \sim r^2 \tag{5.5}$$

and by the exponential decay of w, the contribution due to $\exp(-|P_{j_1j_2\cdots j_s} - P_{k_1k_2\cdots k_s}|)$ is a very small term. Furthermore, there exist $B'(N,p) > 0$ and $\delta > 1$ such that

$$\int_{\mathbb{R}^N} w^p(x)w(x-a)dx = B'\psi(|a|)a.e_n + \mathcal{O}(e^{-\delta|a|}) \tag{5.6}$$

where $\psi(s) = e^{-s}s^{-\frac{N+1}{2}}$ and e_n is unit vector with n-th coordinate 1 and the other entries 0. Hence

$$\begin{aligned}
&\sum_{(j_1j_2\cdots j_s)\neq(k_1k_2\cdots k_s)} \int_{\mathbb{R}^N} w^p(x)w(x-(P_{j_1j_2\cdots j_s} - P_{k_1k_2\cdots k_s}))dx\\
= \;& k^s e^{-\frac{\pi r}{k}}(B + o(1)).
\end{aligned} \tag{5.7}$$

where B is some positive constant. As a result, we obtain

$$\begin{aligned}
k^{-s}I(u) \;=\;& \frac{(p-1)}{2(p+1)}\int_{\mathbb{R}^N} w^{p+1}dx + \bigg(\frac{A}{2\mathcal{S}(\phi_1,\cdots\phi_{s-1})r^m}\bigg)\\
-\;& \frac{B}{2}e^{-\frac{\pi r}{k}} + \mathcal{O}\bigg(\frac{1}{r^{m+\tau}}\bigg) + \mathcal{O}\bigg(e^{-\frac{2\pi r}{k}}\bigg)
\end{aligned}$$

where $A = \int_{\mathbb{R}^N} w^2 dx$. □

6. Max-Procedure or Max-Min Procedure

Define

$$I\bigg(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s} + \varphi_k\bigg) = \Psi(r, \phi_1, \cdots \phi_{s-1}). \tag{6.1}$$

Now we are going to maximize $\Psi(r, \phi_1, \cdots \phi_{s-1})$ with respect to $r \in D$ and $\phi_i \in \mathcal{R}_i$. Define the norm on $H^1(\mathbb{R}^N)$ as

$$\|\varphi\|_{H^1(\mathbb{R}^N)} = \bigg(\int_{\mathbb{R}^N}[|\nabla\varphi|^2 + V(x)\varphi^2]dx\bigg)^{\frac{1}{2}}.$$

First we write

$$I\left(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}+\varphi_k\right) = I\left(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}\right)+\int_{\mathbb{R}^N} E(u)\varphi_k+\mathcal{O}(\|\varphi_k\|^2_{H^1(\mathbb{R}^N)}).$$

Using (4.6) and (3.13) we have

$$\|E\|_\star \le \eta r^{-m} \text{ and } \|\varphi_k\|_\star \le \eta r^{-m}; \tag{6.2}$$

which implies

$$\|E\|_{H^1(\mathbb{R}^N)} \le \eta k^{\frac{s}{2}} r^{-m} \text{ and } \|\varphi_k\|_{H^1(\mathbb{R}^N)} \le \eta k^{\frac{s}{2}} r^{-m}. \tag{6.3}$$

Hence we have

$$I\left(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}+\varphi_k\right) = I\left(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}\right)+\mathcal{O}(k^s r^{-2m}).$$

So we can use Lemma 5.1 to obtain

$$\Psi(r,\phi_1,\cdots\phi_{s-1}) = k^s\left[I_0+\frac{A}{2r^m\mathcal{S}(\phi_1,\cdots\phi_{s-1})}-\frac{B}{2}e^{-\frac{\pi r}{k}}+\mathcal{O}\left(\frac{1}{r^{m+\tau}}\right)\right].$$

Note that if

$$Z(r,\phi_1,\cdots\phi_{s-1})=\frac{A}{2r^m\mathcal{S}(\phi_1,\cdots\phi_{s-1})}-\frac{B}{2}e^{-\frac{\pi r}{k}} \tag{6.4}$$

Using Lemma 2.2, there exists $(r_0,\phi_{1,0},\cdots\phi_{s-1,0})$ such that $Z_r=Z_{\phi_1}=\cdots=Z_{\phi_{s-1}}=0$ and $\max\{Z_{rr},Z_{\phi_1,\phi_1},\cdots,Z_{\phi_{s-1},\phi_{s-1}}\}<0$ and all the mixed derivatives are zero at the point $(r_0,\phi_{1,0},\cdots\phi_{s-1,0})$. Which implies the Hessian associated to Z is positive definite. Hence $\Psi(r,\phi_1,\cdots\phi_{s-1})$ attains a maximum at an interior point $(r_0,\phi_{1,0},\cdots\phi_{s-1,0})\in D\times\mathcal{R}_1\times\mathcal{R}_2\cdots\mathcal{R}_{s-1}$.
Furthermore, there exists $(r_0,\phi_{1,0},\cdots\phi_{s-1,0})$ such that $Z_r=Z_{\phi_1}=\cdots=Z_{\phi_{s-1}}=0$, $Z_{rr}<0$ and $\min\{Z_{\phi_1,\phi_1},\cdots,Z_{\phi_{s-1},\phi_{s-1}}\}>0$ and all the mixed derivatives are zero at the point $(r_0,\phi_{1,0},\cdots\phi_{s-1,0})$. Which implies the Hessian associated to Z has both positive and negative eigenvalues. Hence $(r_0,\phi_{1,0},\cdots\phi_{s-1,0})\in D\times\mathcal{R}_1\times\mathcal{R}_2\cdots\mathcal{R}_{s-1}$ is a saddle point of Ψ. This point is actually a $\max-\min$ saddle point.

7. Proof of Theorem 1.1

By section 6, there exists $k_0\in\mathbb{N}$ such that for all $k\ge k_0$ there exists a C^1 map such that for any $r\in D,\phi_i\in\mathcal{R}_i$ there associates φ_k with

$$\begin{cases} S\left[\left(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}+\varphi_k\right)\right]=\sum_{n=1}^{N}\sum_{j_1j_2\cdots j_s=1}^{k} c_{j_1j_2\cdots j_s n}Z_{j_1j_2\cdots j_s n}; \\ \int_{\mathbb{R}^N}\varphi Z_{j_1j_2\cdots j_s n}dx=0 \end{cases} \tag{7.1}$$

for some constant $c_{j_1j_2\cdots j_s n}\in\mathbb{R}^{k^sN}$. Here S is defined in (2.5). We are going to prove Theorem 1.1 by showing that $c_{j_1j_2\cdots j_s n}=0$ for all $1\le j_1,j_2,\cdots,j_s\le k$ and $1\le n\le N$. This will imply

$$S\left[\left(\sum_{j_1j_2\cdots j_s=1}^{k} W_{j_1j_2\cdots j_s}+\varphi_k\right)\right]=0$$

which will in fact prove that

$$\sum_{j_1 j_2 \cdots j_s = 1}^{k} W_{j_1 j_2 \cdots j_s} + \varphi_k$$

is a solution of (1.3). By the previous section, we know that there exists a critical point $(r_0, \phi_{1,0}, \cdots \phi_{s-1,0})$ of Ψ in $D \times \mathcal{R}_1 \times \mathcal{R}_2 \cdots \mathcal{R}_{s-1}$ such that

$$\Psi(r_0, \phi_{1,0}, \cdots \phi_{s-1,0}) = \max_{(r, \phi_1, \cdots \phi_{s-1})} \Psi(r, \phi_{1,0}, \cdots \phi_{s-1,0})$$

or

$$\Psi(r_0, \phi_{1,0}, \cdots \phi_{s-1,0}) = \max_{r} \min_{(\phi_1, \cdots \phi_{s-1})} \Psi(r, \phi_{1,0}, \cdots \phi_{s-1,0}).$$

Let that point be $P_{j_1 j_2 \cdots j_s}$ where the maximum or the max − min is attained. Then we must have

$$D_{P_{j_1 j_2 \cdots j_s n}} \Big|_{P = P_{j_1 j_2 \cdots j_s}} \Psi = 0. \tag{7.2}$$

Choose

$$\eta_{j_1 j_2 \cdots j_s n}(x) = \frac{\partial}{\partial P_{j_1 j_2 \cdots j_s n}} \Big(\sum_{j_1 j_2 \cdots j_s = 1}^{k} W_{j_1 j_2 \cdots j_s} + \varphi_k \Big) \Big|_{P = P_{j_1 j_2 \cdots j_s}},$$

then (7.2) reduces to

$$\int_{\mathbb{R}^N} \nabla u_k \nabla \eta_{j_1 j_2 \cdots j_s n}(x) \Big|_{P = P_{j_1 j_2 \cdots j_s}} + \int_{\mathbb{R}^N} V(x) u_k \eta_{j_1 j_2 \cdots j_s n}(x) \Big|_{P = P_{j_1 j_2 \cdots j_s}}$$
$$- \int_{\mathbb{R}^N} u_k^p \eta_{j_1 j_2 \cdots j_s n}(x) \Big|_{P = P_{j_1 j_2 \cdots j_s}} = 0.$$

As a result, we must have

$$\sum_{n=1}^{N} \sum_{j_1 j_2 \cdots j_s = 1}^{k} c_{j_1 j_2 \cdots j_s n} \int_{\mathbb{R}^N} Z_{j_1 j_2 \cdots j_s n} \eta_{k_1 k_2 \cdots k_s q} \Big|_{P = P_{j_1 j_2 \cdots j_s}} = 0 \tag{7.3}$$

where $1 \leq k_1, k_2 \cdots k_s \leq k$ and $1 \leq q \leq N$. Note that (7.3) is a homogeneous system of equations. Now we are going to show that (7.3) is a diagonally dominant system. This will allow us to invert the matrix system. Then we can prove that $c_{j_1 j_2 \cdots j_s n} = 0$ for all $1 \leq j_1, j_2, \cdots, j_s \leq k$ and $1 \leq n \leq N$.
From the orthogonality assumption, we have

$$\int_{\mathbb{R}^N} \varphi_k Z_{j_1 j_2 \cdots j_s n} dx = 0. \tag{7.4}$$

But this implies that

$$\int_{\mathbb{R}^N} \frac{\partial \varphi_k}{\partial P_{k_1 k_2 \cdots k_s q}} Z_{j_1 j_2 \cdots j_s n} dx \Big|_{P = P_{j_1 j_2 \cdots j_s}}$$
$$= -\int_{\mathbb{R}^N} \varphi_k \frac{\partial Z_{j_1 j_2 \cdots j_s n}}{\partial P_{k_1 k_2 \cdots k_s q}} dx \Big|_{P = P_{j_1 j_2 \cdots j_s}} = 0 \tag{7.5}$$

whenever $(j_1, j_2 \cdots, j_s) \neq (k_1, k_2 \cdots k_s)$.
Furthermore, when $(j_1, j_2 \cdots, j_s) = (k_1, k_2 \cdots k_s)$

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{\partial \varphi_k}{\partial P_{j_1, j_2 \cdots, j_s q}} Z_{j_1, j_2 \cdots, j_s n} dx \Big|_{P = P_{j_1 j_2 \cdots j_s = 1}} &= -\int_{\mathbb{R}^N} \varphi_k \frac{\partial Z_{j_1 j_2 \cdots j_s n}}{\partial P_{j_1 j_2 \cdots j_s q}} dx \Big|_{P = P_{j_1 j_2 \cdots j_s}} \\ (7.6) \qquad\qquad &\leq C \|\varphi_k\|_\star = \mathcal{O}(r^{-m}). \end{aligned}$$

Whenever $(j_1, j_2 \cdots, j_s) \neq (k_1, k_2 \cdots, k_s)$ we obtain

$$(7.7) \qquad \int_{\mathbb{R}^N} \frac{\partial W_{j_1 j_2 \cdots j_s}}{\partial P_{k_1 k_2 \cdots k_s q}} Z_{j_1 j_2 \cdots j_s n} dx \Big|_{P = P_{j_1 j_2 \cdots j_s}} = \mathcal{O}(e^{-\eta |P_{j_1 j_2 \cdots j_s} - P_{k_1 k_2 \cdots k_s}|}).$$

But for $(j_1, j_2 \cdots, j_s) = (k_1, k_2 \cdots, k_s)$, we have

$$(7.8) \qquad \int_{\mathbb{R}^N} \frac{\partial W_{j_1 j_2 \cdots j_s}}{\partial P_{j_1 j_2 \cdots j_s q}} Z_{j_1 j_2 \cdots j_s n} dx \Big|_{P = P_{j_1 j_2 \cdots j_s = 1}} = \delta_{nq} \int_{\mathbb{R}^N} w_{x_q}^2 dx + \mathcal{O}(e^{-r})$$

where δ_{nq} is the Kronecker delta function. As a result, the off-diagonal term $(j_1, j_2, \cdots j_s, n)$ of (7.3) can be written as

$$\begin{aligned} \sum_{(j_1, j_2 \cdots, j_s) \neq (k_1, k_2 \cdots, k_s)} \int_{\mathbb{R}^N} Z_{j_1 j_2 \cdots j_s n} \eta_{k_1 k_2 \cdots k_s q} &+ \sum_{(j_1, j_2 \cdots, j_s), n \neq q} \int_{\mathbb{R}^N} Z_{j_1 j_2 \cdots j_s n} \eta_{j_1 j_2 \cdots j_s q} \\ &= \mathcal{O}(r^{-m}) = o(1) \end{aligned}$$

which is obtained by using (7.5), (7.6) and (7.7).

Acknowledgements: The first author was supported by CONACYT. The second author was supported by NSERC of Canada.

References

[1] A. Ambrosetti, M. Badiale, S. Cingolani; Semiclassical states of nonlinear Schrödinger equations. *Arch. Rational Mech. Anal.* 140 (1997), no. 3, 285–300.

[2] A. Ambrosetti, A. Malchiodi, W. Ni; Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. I. *Comm. Math. Phys.* 235 (2003), no. 3, 427–466.

[3] A. Ambrosetti, A. Malchiodi, S. Secchi; Multiplicity results for some nonlinear Schrödinger equations with potentials. *Arch. Ration. Mech. Anal.* 159 (2001), no. 3, 253–271.

[4] A. Ambrosetti, A. Malchiodi, W. Ni; Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. II. *Indiana Univ. Math. J.* 53 (2004), no. 2, 297–329.

[5] W. Ao, J. Wei; Infinitely many positive solutions for nonlinear equations with non-symmetric potential, *Cal. Var. PDE.* to appear.

[6] J. Byeon, Z. Wang; Standing waves with a critical frequency for nonlinear Schrödinger equations. *Arch. Ration. Mech. Anal.* 165 (2002), no. 4, 295–316.

[7] J. Byeon, K. Tanaka; Semi-classical standing waves for nonlinear Schrödinger equations at structurally stable critical points of the potential. *J. Eur. Math. Soc.* 15 (2013), no. 5, 1859–1899.

[8] G. Cerami, D. Passaseo, S. Solimini; Infinitely many positive solutions to some scalar field equations with nonsymmetric coefficients. *Comm. Pure Appl. Math.* 66 (2013), no. 3, 372–413.

[9] M. del Pino, P. Felmer; Local mountain passes for semilinear elliptic problems in unbounded domains. *Calc. Var. PDE* 4 (1996), no. 2, 121–137.

[10] M. del Pino, P. Felmer; Semi-Classical States for Nonlinear Schrödinger Equations *J. Funct. Anal.*, Vol. 149, no. 01 (1997), 245–265.

[11] M. del Pino, P. Felmer; Multi-peak bound states for nonlinear Schrödinger equations. *Ann. Inst. H. Poincare Anal. Non Lineaire* 15 (1998), no. 2, 127–149.

[12] M. del Pino, M. Kowalcyzk, J. Wei; Concentration on curves for nonlinear Schrödinger equations. *Comm. Pure Appl. Math.* 60 (2007), no. 1, 113–146.

[13] M. del Pino, J. Wei, W. Yao; Infinitely many positive solutions for nonlinear Schrodinger equations with non-symmetric potentials. *Cal. Var. PDE.* 53(2015), no.1-2, 473-523.

[14] G. Devillanova, S. Solimini; Infinitely many positive solutions to some nonsymmetric scalar field equations: the planar case. *Cal. Var. PDE.* to appear.

[15] M. del Pino, M. Musso, F. Pacard, A. Pistoia; Large energy entire solutions for the Yamabe equation. *J. Diff. Equations.* 251 (2011), no. 9, 2568–2597.

[16] M. del Pino, M. Musso, F. Pacard, A. Pistoia; Torus action on $\mathbb{S}^N$ and sign changing solutions for conformally invariant equations. *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* (5) 12, (2013), 209–237.

[17] P. Esposito, G.Mancini, S.Santra, P. Srikanth; Asymptotic behavior of radial solutions for a semilinear elliptic problem on an annulus through Morse index. *J. Diff. Equations.* 239 (2007), no. 1, 1–15.

[18] A. Floer, A. Weinstein; Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. *J. Funct. Anal.* **69** (1986), no. 3, 397–408.

[19] Man Kam Kwong; Uniqueness of positive solutions of $\Delta u - u + u^p = 0$. *Arch. Ration. Mech. Anal.* 105 (1989), no. 5, 243–266.

[20] Yong-Geun Oh; On positive multi-bump bound states of nonlinear Schrödinger equations under multiple well potential. *Comm. Math. Phys.* **131** (1990), no. 2, 223–253.

[21] Yong-Geun Oh; Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$. *Comm. PDE* **13** (1988), no. 12, 1499–1519.

[22] F. Pacella, P. Srikanth; A reduction method for semilinear elliptic equations and solutions concentrating on spheres. *J. Funct. Anal.* 266 (2014), no. 11, 6456–6472.

[23] J. Wei, S. Yan; Infinitely many solutions for the nonlinear Schrödinger equations in $\mathbb{R}^N$. *Calc. Var. PDE.* 37 (2010), no. 3-4, 423–439.

S. Santra, Department of Mathematics, Centro de Investigacióne en Mathematicás, Guanajuato, México

E-mail address: sanjiban@cimat.mx

J. Wei, Department of Mathematics, The University of British Columbia, Vancouver, BC V6T1Z2, Canada

E-mail address: jcwei@math.ubc.ca