SIGN-CHANGING BLOWING-UP SOLUTIONS FOR SUPERCRITICAL BAHRI-CORON'S PROBLEM

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Abstract: Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ with smooth boundary $\partial\Omega$ and a small hole. We give the first example of sign-changing *bubbling* solutions to the nonlinear elliptic problem

$$-\Delta u = |u|^{\frac{n+2}{n-2}+\varepsilon-1}u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

where ε is a small positive parameter. The basic cell in the construction is the sign-changing nodal solution to the critical Yamabe problem

$$-\Delta w = |w|^{\frac{4}{n-2}}w, \quad w \in \mathcal{D}^{1,2}(\mathbb{R}^n)$$

which has large number $(3n)$ of kernels.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$ with smooth boundary $\partial\Omega$. In this paper we establish existence of a new type of *bubbling* solutions to the nonlinear elliptic problem

$$-\Delta u = |u|^{\frac{n+2}{n-2}+\varepsilon-1}u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where ε is a small positive parameter.

It is known that solvability for Problem

$$-\Delta u = |u|^{q-1}u \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega, \tag{2}$$

is an elementary fact when $1 < q < \frac{n+2}{n-2}$. This is no longer the case for $q \geq \frac{n+2}{n-2}$ due to the loss of compactness of Sobolev embeddings. Our aim is to analyze solutions exhibiting *bubbling behavior* to the above problem when one lets the exponent q approach $\frac{n+2}{n-2}$ from above.

Pohozaev [27] showed that if Ω is strictly star-shaped then no solution of (2) exists if $q \geq \frac{n+2}{n-2}$. In contrast Kazdan and Warner [19] showed that, if Ω is a radially symmetric annulus, $\Omega = \{a < |x| < b\}$, there exists a radial positive solution to Problem (2) for any exponent $q > 1$. Without symmetry the question is harder. This issue was first considered by Coron [9] who found that (2) has a positive solution when $q = \frac{n+2}{n-2}$ in any domain exhibiting a small hole. Also a second solution exists in Coron's setting, as shown in [7], see also the results in [6, 8] and reference therein. The most general result concerning existence of positive solutions to (2) for $q = \frac{n+2}{n-2}$ is obtained by Bahri and Coron [2]: if some homology group of Ω with coefficients in $\mathbf{Z}_2$ is not trivial, then (2) has at least one positive solution, in particular in any three-dimensional domain which is not contractible to a point. Examples showing that this condition is actually not necessary for solvability were found by Dancer [10], Ding [12] and Passaseo [23], for $q = \frac{n+2}{n-2}$ and also for very super critical powers $q \geq \frac{n+1}{n-3} > \frac{n+2}{n-2}$, see [24]. The question of existence for super-critical powers close to critical has been addressed in [13, 14, 15, 25, 4], where existence of *positive*

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solutions to (2) is established. These solutions become *unbounded* as the exponent $q \downarrow \frac{n+2}{n-2}$ and they develop a *blowing-up* profile.

By a *blowing-up solution* for (2) near the critical exponent we mean an unbounded sequence of solutions u_j of (2) for $q = q_j \to \frac{n+2}{n-2}$. Setting

$$M_j = \max_{\Omega} |u_j| = |u_j(x_j)| \to +\infty$$

we see then that the scaled function

$$v_j(y) = M_j\, u_j(x_j + M_j^{(q_j-1)/2}\, y),$$

satisfies

$$\Delta v_j + |v_j|^{q_j-1} v_j = 0$$

in the expanding domain $\Omega_j = M_j^{(q_j-1)/2}(\Omega - x_j)$. Assuming for instance that x_j stays away from the boundary of Ω, elliptic regularity implies that locally over compacts around the origin, v_j converges up to subsequences to a solution of

$$\Delta w + |w|^{\frac{4}{n-2}} w = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{3}$$

Back to the original variable, "near x_j" the behavior of $u_j(y)$ can be approximated as

$$u_j(x) \sim \frac{1}{M_j}\, w\left(\frac{x - x_j}{M_j^{\frac{q_j-1}{2}}}\right) \tag{4}$$

If the solution u_j develops a *positive* bubbling around x_j, then the limit profile (3) is necessarily positive. It is known, see [5, 22], that for the convenient choice $\alpha_n = (n(n-2))^{\frac{n-2}{4}}$, this solution is explicitly given by

$$\bar{w}(z) = \alpha_n \left(\frac{1}{1+|z|^2}\right)^{\frac{n-2}{2}}. \tag{5}$$

which corresponds precisely to an extremal of σ_n, the best constant in the critical Sobolev embedding,

$$\sigma_n = \inf_{u \in C_0^1(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2}{\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}, \tag{6}$$

see [1, 30]. Thus, a solution blowing-up positively near x_j looks at main order as

$$u_j(x) \sim \alpha_n \left(\frac{1}{1 + M_j^{\frac{4}{n-2}}\, |x - x_j|^2}\right)^{\frac{n-2}{2}} M_j. \tag{7}$$

In [13, 14] this issue has been addressed for a class of domains which includes that considered by Coron in [9]. It is established that a *positive* solution to (2) exists for $q = \frac{n+2}{n-2} + \varepsilon$ with any small $\varepsilon > 0$, or equivalently to (1), if for instance Ω is a smooth domain exhibiting a sufficiently small hole: considering ε as a small parameter, the solution exhibits single-bubbling around exactly two points and ceases to exist when $\varepsilon = 0$. More precisely, let $\mathcal{D}$ be a bounded, smooth domain in $\mathbb{R}^n$, $n \geq 3$, and P a point of $\mathcal{D}$. Let us consider the domain

$$\Omega = \mathcal{D} \setminus \overline{B}(P, \delta) \tag{8}$$

where $\delta > 0$ is a small number. Then there exists a $\delta_0 > 0$, which depends on $\mathcal{D}$ and the point P such that if $0 < \delta < \delta_0$ is fixed and Ω is the domain given by (8), then the following holds: There exists $\varepsilon_0 > 0$ and a solution u_ε, $0 < \varepsilon < \varepsilon_0$ of (1) of the form

$$u_\varepsilon(x) = \sum_{j=1}^{2} \alpha_n \left(\frac{1}{1 + \varepsilon^{-\frac{2}{n-2}} \Lambda_{j\varepsilon}^{-2} |x - \xi_j^\varepsilon|^2} \right)^{\frac{n-2}{2}} \Lambda_{j\varepsilon}^{\frac{n-2}{2}} \varepsilon^{\frac{1}{2}} \,(\,1 \,+\, o(1)\,), \tag{9}$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$. The numbers $\Lambda_{j\varepsilon}$ and the points ξ_j^ε converge (up to subsequences) to a critical point of certain function built upon the Green's function of Ω.

Another kind of construction for positive solutions to (1) has been recently proposed by Vaira [31]: if Ω is such that Problem (1) at $\varepsilon = 0$ admits a positive non-degenerate solution u_0, then Problem (1) has a solution that at main order looks like the sum of u_0 and a blowing-up profile as the one described in (7).

Not much is known about sign-changing solutions to (1), in fact as far as we know no existence results are available in the literature. One may ask for existence of sign-changing solutions with a blowing-up profile like the one described in (7), with a minus sign in front, namely sign-changing solution blowing-up *negatively* at one or more points in Ω. Unfortunately, sign-changing solutions blowing-up *negatively* at one point or at two points do not exist, as shown in [3].

The purpose of this work is to give the first construction of blowing-up sign-changing solutions for (1). In fact, we show the existence of solutions with the shape described in (4), where w is a sign-changing solution to the limit profile (3). Not much is known about sign-changing solutions to (3), being the only available results [11, 16, 17]. Furthermore, in order to perform a gluing construction as the one described for positive solutions, an important property of the solution w to the limit problem (3) is needed: its non-degeneracy.

In [16] it is proven that there exists an integer k_0 such that for any integer $k \geq k_0$, a solution solution $Q = Q_k$ to Problem

$$\Delta u + |u|^{p-1} u = 0 \quad \text{in} \quad \mathbb{R}^n, \quad p = \frac{n+2}{n-2}, \tag{10}$$

exists. Furthermore, if we define the energy by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} \, dx, \tag{11}$$

we have

$$E(Q_k) = \begin{cases} (k+1)\, S_n \, \left(1 + O(k^{2-n})\right) & \text{if } n \geq 4\,, \\ (k+1)\, S_3 \, \left(1 + O(k^{-1} |\log k|^{-1}\right) & \text{if } n = 3 \end{cases}$$

as $k \to \infty$, where S_n is a positive constant, depending on n. For later purpose, we mention that we also have

$$\int_{\mathbb{R}^n} |Q_k|^{p+1} = \begin{cases} (k+1) \int_{\mathbb{R}^n} \bar{w}^{p+1} \, \left(1 + O(k^{2-n})\right) & \text{if } n \geq 4\,, \\ (k+1) \int_{\mathbb{R}^3} \bar{w}^6 \, \left(1 + O(k^{-2} |\log k|^{-1}\right) & \text{if } n = 3 \end{cases} \tag{12}$$

and

$$\int_{\mathbb{R}^n} \frac{|Q_k|^{p+1}}{|y|^2} \, dy = \begin{cases} \int_{\mathbb{R}^n} \frac{\bar{w}^{p+1}}{|y|^2} + k \int_{\mathbb{R}^n} \bar{w}^{p+1} + O(k^{3-n}) & \text{if } n \geq 4\,, \\ \int_{\mathbb{R}^3} \frac{\bar{w}^6}{|y|^2} + k \int_{\mathbb{R}^3} \bar{w}^6 + O(k^{-1} |\log k|^{-1}) & \text{if } n = 3 \end{cases} \tag{13}$$

as $k \to \infty$, where $\bar{w}$ is given in (5).

The solution $Q = Q_k$ decays at infinity like the fundamental solution, namely

$$\lim_{|x| \to \infty} |x|^{n-2} \, Q_k(x) = \left[\frac{4}{n(n-2)} \right]^{\frac{n-2}{4}} \, 2^{\frac{n-2}{2}} \, (1 + c_k) \tag{14}$$

where

$$c_k = \begin{cases} O(k^{-1}) & \text{if } n \geq 4\,, \\ O(k^{-1}|\log k|^2) & \text{if } n = 3 \end{cases} \quad \text{as} \quad k \to \infty.$$

Furthermore, the solution $Q = Q_k$ has a positive global non degenerate maximum at $x = 0$. To be more precisely we have

$$Q(x) = [n(n-2)]^{\frac{n-2}{4}} \left(1 - \frac{n-2}{2}|x|^2 + O(|x|^3)\right) \quad \text{as} \quad |x| \to 0, \tag{15}$$

and also there exists $\eta > 0$, depending on k_0, but independent of k, so that

$$\eta \leq Q(x) \leq Q(0) \quad \text{for all} \quad |x| \leq \frac{1}{2}, \tag{16}$$

for any k. Another property for the solution $Q = Q_k$ is that it is invariant under rotation of angle $\frac{2\pi}{k}$ in the x_1, x_2 plane, namely

$$Q(e^{\frac{2\pi}{k}}\bar{x}, x') = Q(\bar{x}, x'), \quad \bar{x} = (x_1, x_3), \quad x' = (x_3, \ldots, x_n). \tag{17}$$

It is even in the x_j-coordinates, for any $j = 2, \ldots, n$

$$Q(x_1, \ldots, x_j, \ldots, x_n) = Q(x_1, \ldots, -x_j, \ldots, x_n), \quad j = 2, \ldots, n. \tag{18}$$

It respects invariance under Kelvin's transform:

$$Q(x) \ = \ |x|^{2-n} Q(|x|^{-2}x)\ . \tag{19}$$

A detailed description of these solutions is given in Appendix 8. These solutions are non-degenerate, as proved in [21], in the sense precised in Section 2. More precisely the dimension of the kernels of the linearized operator at Q

$$-\Delta\phi = p|Q|^{p-1}\phi$$

is shown to be $3n$.

In this paper we show that, if the domain Ω has a small hole, like in [13], then a large number of sign-changing solutions to (1) exist: they blow-up with a profile Q near two points of the domain, and they converges to 0, as $\varepsilon \to 0$, far from these two points.

We have the validity of the following result

Theorem 1.1. *Let $\mathcal{D}$ be a smooth bounded connected domain in $\mathbb{R}^n$ containing the origin 0. There exists δ_0 such that, if $\delta \in (0, \delta_0)$ is fixed and Ω is the set defined by $\Omega = \mathcal{D} \setminus \omega$, for any smooth domain $\omega \subset \bar{B}(0,\delta) \subset \mathcal{D}$, then there exists a sequence of ε_k such that, for any $\varepsilon \in (0, \varepsilon_k)$ there exists a sign changing solution u_ε to (1), with*

$$u_\varepsilon(x) = \sum_{j=1}^{2} Q\left(\frac{x - \xi_{j\varepsilon}}{\Lambda_{j\varepsilon}^2 \varepsilon^{\frac{2}{n-2}}}\right) \Lambda_{j\varepsilon}^{\frac{n-2}{2}} \varepsilon^{\frac{1}{2}}\ (1 + o(1))$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$. Up to subsequence,

$$\lim_{\varepsilon \to 0} \Lambda_{j\varepsilon} = \Lambda_j \in (0, \infty), \quad \lim_{\varepsilon \to 0} \xi_{j\varepsilon} = \xi_j \in \Omega, \quad \textit{with} \quad \xi_1 \neq \xi_2.$$

Furthermore

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2\, dx - \frac{n-2}{2n + \varepsilon(n-2)}\int_\Omega |u_\varepsilon|^{\frac{2n}{n-2}+\varepsilon} = 2\,(k+1)\, S_n + O(\varepsilon|\ln \varepsilon|).$$

Theorem 1.1 exhibits new concentration phenomena in which the basic cell of bubbling is not the positive solution. In the positive bubbling case, the kernel at the basic cell always contains $n+1$ dimensional kernels which corresponds to precisely the translation and scaling. For sign-changing bubbling solution, which is non-radial, the kernel at the basic cell contains not only the translations and scaling but also effects of rotation and Kelvin transform. The main difficulty is to find enough parameters to adjust. This is the main contributions of this paper. As we discover, the dominating role played is still the translation and scaling.

We mention related results on sign-changing bubbling solutions. For scalar curvature type equations, examples of sign-changing blowing-up solutions are constructed by Robert-Vetois in [28]-[29]. There negative bubbles are added to a positive solution to the Yamabe problem. The basic cell is still the single positive solution. Sign-changing bubbling solutions are constructed in the slightly subcritical problems ($\epsilon < 0$) by Pistoia and Vetois [26]. The basic cell is a combination of positive and negative solution.

In Section 2 we precise the notion of *non-degeneracy* for Q. In Section 3 we describe the projection of the function Q into $H_0^1(\Omega)$ and we give the expansion of the energy associated to the sum of two projected copies of Q. Section 4 is devoted to explain the construction of our solution and the scheme of the proof.

2. About the non-degeneracy of the basic cell

In [21], we proved that these solutions are *non degenerate.* To explain this, let us fix one solution $Q = Q_k$ of the family and define the linearized equation around Q for Problem (10) as follows

$$L(\phi) = \Delta\phi + p|Q|^{p-1}\phi. \tag{20}$$

The invariances (17), (18), (19), together with the natural invariance of any solution to (10) under translation (if u solves (10) then also $u(x+\xi)$ solves (10) for any $\xi \in \mathbb{R}^n$) and under dilation (if u solves (10) then $\lambda^{-\frac{n-2}{2}}u(\lambda^{-1}x)$ solves (10) for any $\lambda > 0$) produce some *natural* functions φ in the kernel of L, namely

$$L(\varphi) = 0.$$

These are the $3n$ linearly independent functions we introduce next:

$$z_0(x) = \frac{n-2}{2}Q(x) + \nabla Q(x)\cdot x, \tag{21}$$

$$z_\alpha(x) = \frac{\partial}{\partial x_\alpha}Q(x), \quad \text{for} \quad \alpha = 1,\dots,n, \tag{22}$$

and

$$z_{n+1}(x) = -x_2\frac{\partial}{\partial x_1}Q(x) + x_1\frac{\partial}{\partial x_2}Q(x), \tag{23}$$

$$z_{n+2}(x) = -2x_1 z_0(x) + |x|^2 z_1(x), \quad z_{n+3}(x) = -2x_2 z_0(x) + |x|^2 z_2(x) \tag{24}$$

and, for $l = 3,\dots,n$

$$z_{n+l+1}(x) = -x_l z_1(x) + x_1 z_l(x), \quad z_{2n+l-1}(x) = -x_l z_2(x) + x_2 z_l(x). \tag{25}$$

Indeed, a direct computation gives that

$$L(z_\alpha) = 0, \quad \text{for all} \quad \alpha = 0,1,\dots,3n-1.$$

The function z_0 defined in (21) is related to the invariance of Problem (10) with respect to dilation $\lambda^{-\frac{n-2}{2}}Q(\lambda^{-1}x)$. The functions z_i, $i = 1,\dots,n$, defined in (22) are related to the invariance of Problem (10) with respect to translation $Q(x+\xi)$. The function

z_{n+1} defined in (23) is related to the invariance of Q under rotation in the (x_1, x_2) plane. The two functions z_{n+2} and z_{n+3} defined in (24) are related to the invariance of Problem (10) under Kelvin transformation (19). The functions defined in (25) are related to the invariance under rotation in the (x_1, x_l) plane and in the (x_2, x_l) plane respectively.

Let us be more precise. Denote by $O(n)$ the orthogonal group of $n \times n$ matrices M with real coefficients, so that $M^T M = I$, and by $SO(n) \subset O(n)$ the special orthogonal group of all matrices in $O(n)$ with $detM = 1$. $SO(n)$ is the group of all rotations in $\mathbb{R}^n$, it is a compact group, which can be identified with a compact set in $\mathbb{R}^{\frac{n(n-1)}{2}}$. Consider the sub group $\hat{S}$ of $SO(n)$ generated by rotations in the (x_1, x_2)-plane, in the (x_j, x_α)-plane, for any $j = 1, 2$ and $\alpha = 3, \ldots, n$. We have that $\hat{S}$ is compact and can be identified with a compact manifold of dimension $2n - 3$, with no boundary. In other words, there exists a smooth injective map $\chi : \hat{S} \to \mathbb{R}^{\frac{n(n-1)}{2}}$ so that $\chi(\hat{S})$ is a compact manifold of dimension $2n - 3$ with no boundary and $\chi^{-1} : \chi(\hat{S}) \to \hat{S}$ is a smooth parametrization of $\hat{S}$ in a neighborhood of the Identity. Thus we write

$$\theta \in K = \chi(\hat{S}), \quad R_\theta = \chi^{-1}(\theta)$$

where K a compact manifold of dimension $2n - 3$ with no boundary and R_θ denotes a rotation in $\hat{S}$.

Let $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$, and define

$$\Theta_A(x) = \lambda^{-\frac{n-2}{2}} |\eta_{\lambda,\xi,a}(x)|^{2-n} \, Q \left(\frac{R_\theta \left(\frac{x-\xi}{\lambda} - a |\frac{x-\xi}{\lambda}|^2 \right)}{|\eta_{\lambda,\xi,a}(x)|^2} \right), \tag{26}$$

where

$$\eta_{\lambda,\xi,a}(x) = \frac{x - \xi}{|x - \xi|} - a \frac{|x - \xi|}{\lambda} \tag{27}$$

and Q is our fixed non degenerate solution to Problem (10) described above. Here and in what follows in the paper, with abuse of notation but with no ambiguity, we use the notation a to denotes a vector in $\mathbb{R}^2$, $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$, as well as the vector in $\mathbb{R}^n$, whose first two components are a_1, a_2, and all the other components are zero, namely

$$a = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \in \mathbb{R}^n.$$

In [18] it is proven that for any choice of A, the function Θ_A is still a solution of (10), namely

$$\Delta \Theta_A + |\Theta_A|^{p-1} \Theta_A = 0, \quad \text{in} \quad \mathbb{R}^n.$$

For any set of parameters $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$, we introduce the function

$$\begin{aligned} Q_A(x) &= \Theta_A(R_\theta^{-1} x) \\ &= \lambda^{-\frac{n-2}{2}} \left| \eta_{\lambda,\xi,a}(R_\theta^{-1} x) \right|^{2-n} Q \left(\frac{R_\theta \left(\frac{R_\theta^{-1} x - \xi}{\lambda} - a |\frac{R_\theta^{-1} x - \xi}{\lambda}|^2 \right)}{|\eta_{\lambda,\xi,a}(R_\theta^{-1} x)|^2} \right) \end{aligned} \tag{28}$$

More explicitly

$$Q_A(x) = \lambda^{-\frac{n-2}{2}} \left| \frac{x - R_\theta \xi}{|x - R_\theta \xi|} - R_\theta a \frac{|x - R_\theta \xi|}{\lambda} \right|^{2-n} \times$$

$$\times Q\left(\frac{\frac{x-R_\theta\xi}{\lambda}-R_\theta a|\frac{x-R_\theta\xi}{\lambda}|^2}{1-2R_\theta a\cdot(\frac{x-R_\theta\xi}{\lambda})+|a|^2|\frac{x-R_\theta\xi}{\lambda}|^2}\right).$$

Easy but long computations give the following *natural* relations between z_α and differentiation of Q_A with respect to each component of A. More precisely, one has

$$z_0(y)=-\frac{\partial}{\partial\lambda}\left[Q_A(x)\right]_{|\lambda=1,\xi=0,a=0,\theta=0} \tag{29}$$

$$z_\alpha(y)=-\frac{\partial}{\partial\xi_\alpha}\left[Q_A(x)\right]_{|\lambda=1,\xi=0,a=0,\theta=0}, \quad \alpha=1,\dots,n, \tag{30}$$

$$z_{n+2}(y)=\frac{\partial}{\partial a_1}\left[Q_A(x)\right]_{|\lambda=1,\xi=0,a=0,\theta=0} \tag{31}$$

$$z_{n+3}(y)=\frac{\partial}{\partial a_2}\left[Q_A(x)\right]_{|\lambda=1,\xi=0,a=0,\theta=0}. \tag{32}$$

Now, let $\theta=(\theta_{12},\theta_{13},\dots,\theta_{1n},\theta_{23},\dots,\theta_{2n})$, where θ_{ij} represents the rotation in the (i,j)-plane. Then

$$z_{n+1}(y)=\frac{\partial}{\partial\theta_{12}}\left[Q_A(x)\right]_{|\lambda=1,\xi=0,a=0,\theta=0} \tag{33}$$

and, for any $l=3,\dots,n$,

$$z_{n+l+1}(y)=\frac{\partial}{\partial\theta_{1l}}\left[Q_A(x)\right]_{|\lambda=1,\xi=0,a=0,\theta=0} \tag{34}$$

$$z_{2n+l-1}(y)=\frac{\partial}{\partial\theta_{2l}}\left[Q_A(x)\right]_{|\lambda=1,\xi=0,a=0,\theta=0}. \tag{35}$$

Following [18], a solution Q is said to be non degenerate if

$$\mathrm{Kernel}(L)=\mathrm{Span}\{z_\alpha\ :\ \alpha=0,1,2,\dots,3n-1\}, \tag{36}$$

or equivalently, any bounded (or any solution in $\mathcal{D}^{1,2}$) of $L(\varphi)=0$ is a linear combination of the functions z_α, $\alpha=0,\dots,3n-1$.

In [21] we proved that, under certain condition on the dimension n, the solution Q is non-degenerate. Indeed, in [21] we showed that in all dimensions $n\le 48$, any solution $Q=Q_k$ is non degenerate in the sense defined above. If dimension $n\ge 49$, our result [21] guarantees the existence of a subsequence of solutions Q_{k_j} each one of which is non degenerate.

3. First approximation and expansion of the energy

The existence of a non degenerate solution in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ of (10) is the basic element for our construction. In fact, we can perform our construction starting from any one of the infinitely many solutions $Q=Q_k$ with the property of being non degenerate. Thus, from now on, we fix a function $Q=Q_k$, and for simplicity of notation we drop the index k.

The function Q_A defined in (28) will be the building block of our construction. We first correct it so that it satisfies zero boundary condition on $\partial\Omega$. This is done defining PQ_A to be the projection of Q_A onto $H_0^1(\Omega)$, namely the unique solution to

$$\Delta u=\Delta Q_A \quad \text{in} \quad \Omega, \quad u=0 \quad \text{on} \quad \partial\Omega. \tag{37}$$

In other words, $PQ_A=Q_A-\varphi_A$ where φ_A solves

$$\Delta u=0 \quad \text{in} \quad \Omega, \quad u=Q_A \quad \text{on} \quad \partial\Omega. \tag{38}$$

Next Lemma provides a precise description of PQ_A and φ_A, when $\lambda \to 0$. To state the result we need to recall the following. Let us denote by $G(x,y)$ the Green's function of the domain, namely G satisfies

$$\Delta_x G(x,y) = \delta(x-y), \quad x \in \Omega, \quad G(x,y) = 0, \quad x \in \partial\Omega, \tag{39}$$

where $\delta(x)$ denotes the Dirac mass at the origin. We denote by $H(x,y)$ its regular part, namely

$$H(x,y) = \Gamma(x-y) - G(x,y) \tag{40}$$

where Γ denotes the fundamental solution of the Laplacian,

$$\Gamma(x) = b_n |x|^{2-n}, \tag{41}$$

so that H satisfies

$$\Delta_x H(x,y) = 0, \quad x \in \Omega, \quad H(x,y) = \Gamma(x-y), \quad x \in \partial\Omega.$$

Its *diagonal* $H(x,x)$ is usually called Robin function.

Lemma 3.1. *Assume that $a \in \mathbb{R}^2$, $\xi \in \mathbb{R}^n$, and $\theta \in K$ are fixed so that $R_\theta \xi \in \Omega$. We have the validity of the following estimates:*

$$\varphi_A(x) = b_n^{-1} \lambda^{\frac{n-2}{2}} Q\left(R_\theta a\right) H(x, R_\theta \xi) + O(\lambda^{\frac{n}{2}}) \tag{42}$$

uniformly for $x \in \Omega$, as $\lambda \to 0$. Furthermore,

$$PQ_A(x) = b_n^{-1} \lambda^{\frac{n-2}{2}} Q\left(R_\theta a\right) G(x, R_\theta \xi) + O(\lambda^{\frac{n}{2}}) \tag{43}$$

uniformly for x in compact sets of $\Omega \setminus \{R_\theta \xi\}$, as $\lambda \to 0$. In (42) and (43), H and G are the functions defined respectively in (40) and (39), b_n is a positive constant defined in (41), and $O(1)$ denotes a smooth function of x which is uniformly bounded as $\lambda \to 0$.

Proof. Let $x \in \partial\Omega$ and let $\delta > 0$ so that dist $(x, R_\theta \xi) > \delta$. Define $X = \frac{x - R_\theta \xi}{\lambda}$. The argument of the function Q in (28) gets re-written as

$$\frac{\frac{x-R_\theta\xi}{\lambda} - R_\theta a \left|\frac{x-R_\theta\xi}{\lambda}\right|^2}{1 - 2R_\theta a \cdot \left(\frac{x-R_\theta\xi}{\lambda}\right) + |a|^2 \left|\frac{x-R_\theta\xi}{\lambda}\right|^2} = \frac{X - R_\theta a |X|^2}{1 - 2R_\theta a \cdot X + |a|^2 |X|^2}$$

Observe that $X = \frac{x-R_\theta\xi}{\lambda} \to \infty$ uniformly on $x \in \partial\Omega$, as $\lambda \to 0$. Thus we have

$$\begin{aligned} \frac{X - R_\theta a |X|^2}{1 - 2R_\theta a \cdot (X) + |a|^2 |X|^2} &= \frac{X - R_\theta a |X|^2}{|a|^2 |X|^2 \left[1 - 2\frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + \frac{1}{|a|^2 |X|^2}\right]^2} \\ &= -\frac{R_\theta a}{|a|^2} \left[1 - 2\frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + \frac{1}{|a|^2 |X|^2}\right]^{-2} \\ &\quad + \frac{X}{|a|^2 |X|^2} \left[1 - 2\frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + \frac{1}{|a|^2 |X|^2}\right]^{-2} \end{aligned} \tag{44}$$

A Taylor expansion gives that, uniformly for $x \in \partial\Omega$, one has

$$\begin{aligned}\left[1-2\frac{R_\theta a}{|a|^2}\cdot\frac{X}{|X|^2}+\frac{1}{|a|^2|X|^2}\right]^{-2} &= 1-2\left(-2\frac{R_\theta a}{|a|^2}\cdot\frac{X}{|X|^2}+\frac{1}{|a|^2|X|^2}\right)\\ &+3\left(-2\frac{R_\theta a}{|a|^2}\cdot\frac{X}{|X|^2}+\frac{1}{|a|^2|X|^2}\right)^2+O(\lambda^2)\\ &=1+4\frac{R_\theta a}{|a|^2}\cdot\frac{X}{|X|^2}\\ &-\frac{2}{|a|^2|X|^2}+12\left|\frac{R_\theta a}{|a|^2}\cdot\frac{X}{|X|^2}\right|^2+O(\lambda^3)\\ &=1+4\frac{R_\theta a}{|a|^2}\cdot\frac{X}{|X|^2}+O(\lambda^2)\end{aligned}$$

as $\lambda \to 0$. Inserting this information in (44), we get that

$$\begin{aligned}\frac{X-R_\theta a|X|^2}{1-2R_\theta a\cdot(X)+|a|^2|X|^2} &= -\frac{R_\theta a}{|a|^2}+\frac{X}{|a|^2|X|^2}\\ &-4\frac{R_\theta a}{|a|^2}\cdot\frac{X}{|X|^2}+O(\lambda^2)\\ &=-\frac{R_\theta a}{|a|^2}+\frac{x-R_\theta\xi}{|a|^2|x-R_\theta\xi|^2}\lambda\\ &-4\frac{R_\theta a}{|a|^2}\frac{R_\theta a}{|a|^2}\cdot\frac{x-R_\theta\xi}{|x-R_\theta\xi|^2}\lambda+O(\lambda^2).\end{aligned}\tag{45}$$

Thanks to (45), we get that

$$\begin{aligned}&Q\left(\frac{X-R_\theta a|X|^2}{1-2a\cdot(X)+|a|^2|X|^2}\right)=Q(-\frac{R_\theta a}{|a|^2})\\ &+\nabla Q(-\frac{R_\theta a}{|a|^2})[\frac{x-R_\theta\xi}{|a|^2|x-R_\theta\xi|^2}-4\frac{R_\theta a}{|a|^2}\frac{R_\theta a}{|a|^2}\cdot\frac{x-R_\theta\xi}{|x-R_\theta\xi|^2}]\lambda+O(\lambda^2)\end{aligned}\tag{46}$$

uniformly for $x \in \partial\Omega$, as $\lambda \to 0$.

On the other hand, in the same region $x \in \partial\Omega$, we have that

$$\begin{aligned}\left|\frac{x-R_\theta\xi}{|x-R_\theta\xi|}-R_\theta a\frac{|x-R-\theta\xi|}{\lambda}\right|^{2-n} &= \frac{\lambda^{n-2}}{|a|^{n-2}|x-R_\theta\xi|^{n-2}}\\ &+\frac{n-2}{|a|^{n-2}|x-R_\theta\xi|^{n-2}}\frac{R_\theta a}{|a|^2}\cdot\frac{x-R_\theta\xi}{|x-R_\theta\xi|^2}\lambda^{n-1}\\ &+O(\lambda^n).\end{aligned}\tag{47}$$

We conclude from (46) and (47) that, uniformly for $x \in \partial\Omega$,

$$\begin{aligned}\varphi_A(x) &= \frac{\lambda^{\frac{n-2}{2}}}{|x-R_\theta\xi|^{n-2}}\frac{1}{|a|^{n-2}}Q(-\frac{R_\theta a}{|a|^2})\\ &+\frac{\lambda^{\frac{n}{2}}}{|x-R_\theta\xi|^{n-2}}\frac{1}{|a|^{n-2}}Q(-\frac{R_\theta a}{|a|^2})\,(n-2)\,\frac{R_\theta a}{|a|^2}\cdot\frac{x-R_\theta\xi}{|x-R_\theta\xi|^2}\\ &+\frac{\lambda^{\frac{n}{2}}}{|x-R_\theta\xi|^{n-2}}\frac{1}{|a|^{n-2}}[\frac{x-R_\theta\xi}{|a|^2|x-R_\theta\xi|^2}-4\frac{R_\theta a}{|a|^2}\frac{R_\theta a}{|a|^2}\cdot\frac{x-R_\theta\xi}{|x-R_\theta\xi|^2}]\\ &+O(\lambda^{\frac{n+2}{2}})\\ &=\frac{\lambda^{\frac{n-2}{2}}}{|x-R_\theta\xi|^{n-2}}Q(R_\theta a)+O(\lambda^{\frac{n}{2}})\end{aligned}\tag{48}$$

as $\lambda \to 0$. Observe that we have used the fact that $\frac{1}{|a|^{n-2}} Q(-\frac{R_\theta a}{|a|^2}) = Q(R_\theta a)$. A direct application of the maximum principle guarantees the validity of (42). To prove (43), it is enough to observe that, for any x in a compact set of $\Omega \setminus \{R_\theta \xi\}$,

$$Q_A(x) = \frac{\lambda^{\frac{n-2}{2}}}{|x - R_\theta \xi|^{n-2}} Q(R_\theta a) + O(\lambda^{\frac{n}{2}}) \tag{49}$$

as $\lambda \to 0$. This concludes the proof of the Lemma. □

We consider now two sets $A_1 = (\lambda_1, \xi_1, a_1, \theta_1)$ and $A_2 = (\lambda_2, \xi_2, a_2, \theta_2)$ and the functions

$$Q_i = Q_{A_i}, \quad PQ_i = PQ_{A_i}, \quad i = 1, 2.$$

Our purpose is to estimate the following quantity

$$J_0(PQ_1 + PQ_2) = \frac{1}{2} \int_\Omega |\nabla(PQ_1 + PQ_2)|^2 - \frac{1}{p+1} \int_\Omega (PQ_1 + PQ_2)^{p+1}.$$

We use the notations

$$\hat{\xi}_i = R_{\theta_i} \xi_i, \quad \hat{a}_i = R_{\theta_i} a_i, \quad i = 1, 2. \tag{50}$$

Let us now fix a number $\delta > 0$ and consider the following constraints

$$\begin{aligned} &(\hat{\xi}_1, \hat{\xi}_2) \in \Omega \times \Omega : \operatorname{dist}(\hat{\xi}_i, \partial\Omega) > \delta,\ |\hat{\xi}_1 - \hat{\xi}_2| > \delta \\ &\theta_i \in K, \ \ |a_i| \le \frac{1}{2}, \quad i = 1, 2. \end{aligned} \tag{51}$$

Let us set

$$\gamma_n = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla Q|^2 - \frac{1}{p+1} \int_{\mathbb{R}^n} |Q|^{p+1} \tag{52}$$

Lemma 3.2. *Given $\delta > 0$ we have the validity of the expansion*

$$\begin{aligned} J_0(PQ_1 + PQ_2) = &\ 2\gamma_n + \alpha_n H(\hat{\xi}_1, \hat{\xi}_1)\, Q(\hat{a}_1)^2\, \lambda_1^{n-2} \\ &+ \alpha_n H(\hat{\xi}_2, \hat{\xi}_2)\, Q(\hat{a}_2)^2\, \lambda_2^{n-2} \\ &- \alpha_n 2G(\hat{\xi}_1, \hat{\xi}_2)\, Q(\hat{a}_1) Q(\hat{a}_2)\, \lambda_1^{\frac{n-2}{2}} \lambda_2^{\frac{n-2}{2}} \\ &+ o(\max\{\lambda_1, \lambda_2\}^{n-2}) \end{aligned} \tag{53}$$

as $\lambda_1, \lambda_2 \to 0$, uniformly in the set satisfying constraints (51). In (53), $\alpha_n = \frac{1}{2b_n^2}$ with b_n the positive constant defined in (41).

Proof. The full expansion (53) is consequence of the following formulas:

$$\int_\Omega |\nabla PQ_i|^2 = \int_{\mathbb{R}^n} |\nabla Q|^2 - 2\alpha_n H(\hat{\xi}_i, \hat{\xi}_i)\, Q^2(\hat{a}_i)\, \lambda_i^{n-2} + o(\lambda_i^{n-2}), \tag{54}$$

$$\int_\Omega \nabla PQ_1 \nabla PQ_2 = 2\alpha_n G(\hat{\xi}_1, \hat{\xi}_2)\, Q(\hat{a}_1) Q(\hat{a}_2)\, \lambda_1^{\frac{n-2}{2}} \lambda_2^{\frac{n-2}{2}} + o(\max\{\lambda_1, \lambda_2\}^{n-2}), \tag{55}$$

$$\begin{aligned} &\frac{1}{p+1} \int_\Omega \left[|PQ_1 + PQ_2|^{p+1} - |PQ_1|^{p+1} - |PQ_2|^{p+1} \right] \\ &\qquad = 4\alpha_n G(\hat{\xi}_1, \hat{\xi}_2)\, Q(\hat{a}_1) Q(\hat{a}_2)\, \lambda_1^{\frac{n-2}{2}} \lambda_2^{\frac{n-2}{2}} + o(\max\{\lambda_1, \lambda_2\}^{n-2}) \end{aligned} \tag{56}$$

and

$$\frac{1}{p+1} \int_\Omega |PQ_i|^{p+1} = \frac{1}{p+1} \int_{\mathbb{R}^n} |Q|^{p+1} - 2\alpha_n H(\hat{\xi}_i, \hat{\xi}_i)\, Q^2(\hat{a}_i)\, \lambda_i^{n-2} + o(\lambda_i^{n-2}). \tag{57}$$

Indeed, we decompose

$$J_0(PQ_1+PQ_2)=\sum_{i=1,2}\frac{1}{2}\left[\int_\Omega|\nabla PQ_i|^2-\frac{1}{p+1}\int_\Omega|PQ_i|^{p+1}\right]+\int_\Omega\nabla PQ_1\nabla PQ_2$$
$$-\frac{1}{p+1}\int_\Omega\left[|PQ_1+PQ_2|^{p+1}-|PQ_1|^{p+1}-|PQ_2|^{p+1}\right].$$

Thus substituting estimates (54), (55), (56) and (57) in this relation we obtain the thesis.

In what is left of this proof, we shall show the validity of (54), (55), (56) and (57).

Proof of (54). For simplicity, we write $\varphi_i(x)=\varphi_{A_i}(x)$. An integration by parts gives that

$$\int_\Omega|\nabla PQ_i|^2\,dx=\int_\Omega|Q_i|^{p+1}\,dx-\int_\Omega|Q_i|^{p-1}Q_i\varphi_i\,dx \tag{58}$$

taking into account that $-\Delta PQ_i=|Q_i|^{p-1}Q_i$. Let us first compute $\int_\Omega|Q_i|^{p+1}\,dx$, writing

$$\int_\Omega|Q_i|^{p+1}\,dx=\int_{|x-\hat\xi_i|<\delta}|Q_i|^{p+1}\,dx+\int_{|x-\hat\xi_i|>\delta}|Q_i|^{p+1}\,dx$$

for some $\delta>0$ fixed and small. In the ball $|x-\hat\xi_i|<\delta$ we introduce the change of variables $y=\frac{x-\hat\xi_i}{\lambda_i}$, so that

$$\begin{aligned}\int_{|x-\hat\xi_i|<\delta}|Q_i|^{p+1}\,dx&=\int_{|y|<\frac{\delta}{\lambda_i}}\left[\left|\frac{y}{|y|}-\hat a_i|y|\right|^{2-n}|Q|\left(\frac{\frac{y}{|y|^2}-\hat a_i}{|\frac{y}{|y|^2}-\hat a_i|^2}\right)\right]^{p+1}dy\\
&=\int_{|y|<\frac{\delta}{\lambda_i}}\left[\left|\frac{y}{|y|^2}-\hat a_i\right|^{2-n}|y|^{2-n}|Q|\left(\frac{\frac{y}{|y|^2}-\hat a_i}{|\frac{y}{|y|^2}-\hat a_i|^2}\right)\right]^{p+1}dy\\
&\quad\text{since}\quad |w|^{2-n}Q(\frac{w}{|w|^2})=Q(w)\\
&=\int_{|y|<\frac{\delta}{\lambda_i}}\left[|y|^{2-n}|Q|\left(\frac{y}{|y|^2}-\hat a_i\right)\right]^{p+1}dy\\
&=\int_{|y|<\frac{\delta}{\lambda_i}}|y|^{2n}\left[|Q|\left(\frac{y}{|y|^2}-\hat a_i\right)\right]^{p+1}dy\\
&\quad\text{using the change of variables}\quad z=\frac{y}{|y|^2}\\
&=\int_{|z|>\frac{\lambda_i}{\delta}}|Q\left(z-\hat a_i\right)|^{p+1}\,dz=\int_{\mathbb{R}^n}|Q|^{p+1}\,dz+O(\lambda_i^n),\end{aligned}$$

where $O(1)$ denotes a generic smooth function of the parameters that is uniformly bounded as $\lambda_i\to0$. Observe that the last expansion is consequence of (15), (16) and (51). On the other hand, in the set $|x-\hat\xi_i|>\delta$ we have the validity of the expansion (49), so that we conclude that

$$\left|\int_{|x-\hat\xi_i|>\delta}|Q_i|^{p+1}\,dx\right|\le C\lambda_i^{\frac{n+2}{2}}\,|Q(\hat a_i)|\le C\lambda_i^{\frac{n+2}{2}}$$

where again we use the assumption in (51) that $|a_i|\le\frac12$, and also (15)-(16).

We thus conclude that

$$\int_\Omega|PQ_i|^{p+1}\,dz=\int_{\mathbb{R}^n}|Q|^{p+1}\,dz+O(\lambda_i^{\frac{n+2}{2}}). \tag{59}$$

We turn now to $\int_\Omega |Q_i|^{p-1}Q_i\varphi_i\,dx$. We claim that

$$\int_\Omega |Q_i|^{p-1}Q_i\varphi_i\,dx = b_n^{-2}\lambda_i^{n-2}Q^2(\hat a_i)H(\hat\xi_i,\hat\xi_i)+O(\lambda_i^{n-1}), \tag{60}$$

where $O(1)$ is uniformly bounded, as $\lambda_i\to 0$, in the set of parameters satisfying (51). We decompose

$$\int_\Omega |Q_i|^{p-1}Q_i\varphi_i\,dx = \int_{|x-\hat\xi_i|<\delta}|Q_i|^{p-1}Q_i\varphi_i\,dx+\int_{\Omega\cap|x-\hat\xi_i|>\delta}|Q_i|^{p-1}Q_i\varphi_i\,dx$$

Recalling the validity of the expansion (42) for φ_i, we get that

$$\begin{aligned}\int_{|x-R_{\theta_i}\xi_i|<\delta}|Q_i|^{p-1}Q_i\varphi_i\,dx &= b_n^{-1}\lambda_i^{\frac{n-2}{2}}Q(\hat a_i)H(\hat\xi_i,\hat\xi_i)\left(\int_{|x-\hat\xi_i|<\delta}|Q_i|^{p-1}Q_i\,dx\right)\\ &+\left(\int_{|x-\hat\xi_i|<\delta}|Q_i|^{p-1}Q_i\,dx\right)O(\lambda_i^{n-1})\\ &+\left(\int_{|x-\hat\xi_i|<\delta}|Q_i|^{p-1}Q_i\,|x-\hat\xi_i|dx\right)O(\lambda_i^{\frac{n-2}{2}}).\end{aligned}$$

In the ball $|x-\hat\xi_i|<\delta$ we introduce the change of variables $y=\frac{x-\hat\xi_i}{\lambda_i}$, and using the invariance of Q under Kelvin transform,

$$\begin{aligned}\int_{|x-\hat\xi_i|<\delta}|Q_i|^{p-1}Q_i\,dx &= \lambda_i^{\frac{n-2}{2}}\int_{|y|<\frac{\delta}{\lambda}}\left[|y|^{2-n}|Q|\left(\frac{y}{|y|^2}-\hat a_i\right)\right]^p dy\\ &= \lambda_i^{\frac{n-2}{2}}\int_{|y|<\frac{\delta}{\lambda}}|y|^{n+2}\left[|Q|\left(\frac{y}{|y|^2}-\hat a_i\right)\right]^p dy\\ &\qquad\text{using the change of variables}\quad z=\frac{y}{|y|^2}\\ &= \lambda_i^{\frac{n-2}{2}}\int_{|z|>\frac{\lambda}{\delta}}\frac{1}{|z|^{n-2}}\,|Q\,(z-\hat a_i)|^p\,dz\\ &= \lambda_i^{\frac{n-2}{2}}\int_{\mathbb{R}^n}\frac{1}{|z+\hat a_i|^{n-2}}\,|Q|^p\,dz+o(\lambda_i^{\frac{n-2}{2}}),\end{aligned}$$

as $\lambda_i\to 0$. Recall now that

$$Q(\hat a_i)=b_n\int_{\mathbb{R}^n}\frac{1}{|z+\hat a_i|^{n-2}}\,|Q|^p\,dz$$

Thus we get

$$\int_{|x-\hat\xi_i|<\delta}|Q_i|^{p-1}Q_i\,dx = b_n^{-1}\lambda_i^{\frac{n-2}{2}}Q(\hat a_i)+o(\lambda_i^{\frac{n-2}{2}}). \tag{61}$$

On the other hand, using again the change of variables $y=\frac{x-\hat\xi_i}{\lambda_i}$ one finds directly that

$$\int_{|x-\hat\xi_i|<\delta}|Q_i|^{p-1}Q_i\,|x-\hat\xi_i|dx=O(\lambda_i^{\frac{n}{2}}).$$

On the other hand, we have

$$\int_{\Omega\cap|x-\hat\xi_i|>\delta}|Q_i|^{p-1}Q_i\varphi_i\,dx=O(\lambda_i^{n-1})$$

as direct consequence of (42) and (49). Collecting the above estimates we get the validity of (60). Expansion (54) follows directly from (59) and (60).

Proof of (55). Arguing as in the proof of (54), it holds

$$\int_\Omega \nabla PQ_1 \nabla PQ_2 = \int_\Omega |Q_1|^{p-1}Q_1PQ_2 = \left(\int_{|x-\hat{\xi}_1|<\delta} |Q_1|^{p-1}Q_1PQ_2\,dx\right)(1+o(1)),$$

as $\lambda_i \to 0$, $i=1,2$. Now, using (43), we get

$$\int_{|x-\hat{\xi}_1|<\delta} |Q_1|^{p-1}Q_1PQ_2\,dx = b_n^{-1}\lambda_2^{\frac{n-2}{2}} Q(\hat{a}_2)G(\hat{\xi}_1,\hat{\xi}_2)\times \left(\int_{|x-\hat{\xi}_1|<\delta} |Q_1|^{p-1}Q_1\,dx\right)(1+O(\lambda_2))$$

Thanks to (61), we conclude that

$$\int_{|x-\hat{\xi}_1|<\delta} |Q_1|^{p-1}Q_1PQ_2\,dx = b_n^{-2}\lambda_1^{\frac{n-2}{2}}\lambda_2^{\frac{n-2}{2}} Q(\hat{a}_1)Q(\hat{a}_2)G(\hat{\xi}_1,\hat{\xi}_2) + o(\max\{\lambda_1,\lambda_2\}^{n-2}). \tag{62}$$

This gives the validity of (55).

Proof of (56). We write

$$\begin{aligned}\frac{1}{p+1}\int_\Omega & \left[|PQ_1+PQ_2|^{p+1} - |PQ_1|^{p+1} - |PQ_2|^{p+1}\right]\\ &= \int_{|x-\hat{\xi}_1|<\delta} |PQ_1|^{p-1}PQ_1PQ_2\\ &+ \int_{|x-\hat{\xi}_2|<\delta} |PQ_2|^{p-1}PQ_2PQ_1 + O((\lambda_1\lambda_2)^{\frac{n+2}{2}})\\ &= \int_{|x-\hat{\xi}_1|<\delta} |Q_1|^{p-1}Q_1PQ_2\\ &+ \int_{|x-\hat{\xi}_2|<\delta} |Q_2|^{p-1}Q_2PQ_1 + O((\lambda_1\lambda_2)^{\frac{n+2}{2}})\end{aligned}$$

At this point, (56) follows directly from (62).

Proof of (57). Using the estimate (49), we see that

$$\frac{1}{p+1}\int_\Omega |PQ_i|^{p+1}\,dx = \frac{1}{p+1}\int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p+1}\,dx + O(\lambda_i^{n+2}).$$

On the other hand, a Taylor expansion gives

$$\begin{aligned}\frac{1}{p+1}\int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p+1}\,dx &= \frac{1}{p+1}\int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p+1}\,dx\\ &+ \left(\int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p-1}PQ_i\varphi_i\,dx\right)(1+o(1))\\ &= \frac{1}{p+1}\int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p+1}\,dx\\ &+ \left(\int_{|x-\hat{\xi}_i|<\delta} |Q_i|^{p-1}Q_i\varphi_i\,dx\right)(1+o(1))\end{aligned}$$

We now apply (59) and (60) to get (57). □

Let us consider the energy functional, associated to problem (1),

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1+\varepsilon}\int_\Omega u^{p+1+\varepsilon}. \tag{63}$$

Next Lemma provides the expansion of $J_\varepsilon(PQ_1+PQ_2)$ in terms of $J_0(PQ_1+PQ_2)$, as $\varepsilon \to 0$.

Lemma 3.3. *Let $\delta > 0$ and assume that*

$$\delta < \frac{\lambda_i^{n-2}}{\varepsilon} < \delta^{-1}, \quad i = 1,2 \tag{64}$$

as $\varepsilon \to 0$. We have the validity of the expansion

$$\begin{aligned} J_\varepsilon(PQ_1+PQ_2) = J_0(PQ_1+PQ_2) + 2\varepsilon A_n + \frac{(n-2)^2}{4n}\varepsilon \log(\lambda_1\lambda_2)\int_{\mathbb{R}^n} |Q|^{p+1} \\ - \frac{(n-2)^2}{2n}\,\varepsilon\left[\sum_{j=1}^{2}\int |Q|^{p+1}\log|y+\hat{a}_j|\,dy\right] + o(\varepsilon). \end{aligned}$$

as $\varepsilon \to 0$, uniformly in the set satisfying constraints (51). In the above formula

$$A_n = \left[\frac{1}{(p+1)^2}\int_{\mathbb{R}^n}|Q|^{p+1} - \frac{1}{p+1}\int_{\mathbb{R}^n}|Q|^{p+1}\log|U|\right].$$

Proof. We first write

$$\begin{aligned} &J_\varepsilon(PQ_1+PQ_2) - J_0(PQ_1+PQ_2) \\ &\quad = \left(\frac{1}{p+1} - \frac{1}{p+1+\varepsilon}\right)\int |PQ_1+PQ_2|^{p+1} \\ &\quad + \frac{1}{p+1+\varepsilon}\int \left[|PQ_1+PQ_2|^{p+1} - |PQ_1+PQ_2|^{p+1+\varepsilon}\right] \\ &\quad = 2\frac{\varepsilon}{(p+1)^2}\int |Q|^{p+1} - \frac{\varepsilon}{p+1}\int |PQ_1+PQ_2|^{p+1}\log|PQ_1+PQ_2| \\ &\quad + O(\varepsilon^2) \end{aligned}$$

Arguing as in the proof of Lemma 3.2, we find that

$$\int |PQ_1+PQ_2|^{p+1}\log|PQ_1+PQ_2| = \sum_{j=1}^{2}\int_{|x-\hat{\xi}_j|<\delta} |PQ_j|^{p+1}\log|PQ_j| + O(\varepsilon).$$

Let us fix $j = 1$. After the change of variables $z = \frac{x-\hat{\xi}_1}{\lambda_1}$, we get that

$$\begin{aligned}
\int_{|x-\hat{\xi}_1|<\delta} |PQ_1|^{p+1} \log |PQ_1| &= -\frac{n-2}{2} \log \lambda_1 \int_{|z|<\frac{\delta}{\lambda_1}} |z|^{-2n} |Q|^{p+1}\left(\frac{z}{|z|^2} - \hat{a}_1\right) \\
&+ \int_{|z|<\frac{\delta}{\lambda_1}} |z|^{-2n} |Q|^{p+1}\left(\frac{z}{|z|^2} - \hat{a}_1\right) \log \left|\frac{z}{|z|^2} - \hat{a}_1\right| \\
&+ \int_{|z|<\frac{\delta}{\lambda_1}} |z|^{-2n} |Q|^{p+1}\left(\frac{z}{|z|^2} - \hat{a}_1\right) \log |z|^{2-n} \\
\text{(using the change of variables} \quad & y = \frac{z}{|z|^2}) \\
&= -\frac{n-2}{2} \log \lambda_1 \int_{|y|>\frac{\lambda_1}{\delta}} |Q|^{p+1}(y - \hat{a}_1) \\
&+ \int_{|y|>\frac{\lambda_1}{\delta}} |Q|^{p+1}(y - \hat{a}_1) \log |y - \hat{a}_1| \\
&+ (n-2) \int_{|y|>\frac{\lambda_1}{\delta}} |Q|^{p+1}(y - \hat{a}_1) \log |y| \\
&= -\frac{n-2}{2} \log \lambda_1 \int |Q|^{p+1} + \int |Q|^{p+1}(y) \log |y| \\
&+ (n-2) \int |Q|^{p+1}(y) \log |y + a_1| + O(\varepsilon).
\end{aligned}$$

Collecting together all the previous expansions, we get the validity of (65). □

We shall now choose the numbers λ_i in terms of ε in a more specific way: we will assume

$$\alpha_n \lambda_i^{n-2} = \frac{n-2}{2n} \left(\int_{\mathbb{R}^n} |Q|^{p+1} \right) \Lambda_i^2 \varepsilon. \tag{65}$$

Combining the results of the previous two lemmas, and our choice (65) for λ_i, we get the following result.

Lemma 3.4. *Let $\delta > 0$ and assume that*

$$\begin{aligned}
&(\hat{\xi}_1, \hat{\xi}_2) \in \Omega \times \Omega : dist(\hat{\xi}_i, \partial\Omega) > \delta, \ |\hat{\xi}_1 - \hat{\xi}_2| > \delta \\
&\delta < \Lambda_i < \delta^{-1}, \qquad \theta_i \in K, \quad |a_i| \le \frac{1}{2}, \quad i = 1, 2,
\end{aligned}$$

where

$$\alpha_n \lambda_i^{n-2} = \frac{n-2}{2n} \left(\int_{\mathbb{R}^n} |Q|^{p+1} \right) \Lambda_i^2 \varepsilon, \quad \hat{\xi}_i = R_{\theta_i} \xi_i, \quad \hat{a}_i = R_{\theta_i} a_i, \quad i = 1, 2.$$

Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ we have

$$\begin{aligned}
J_\varepsilon(U_1 + U_2) &= 2\gamma_n + \gamma_{1,n} \varepsilon \log \varepsilon + \gamma_{2,n} \varepsilon \\
&+ \varepsilon \frac{n-2}{2n} \left(\int_{\mathbb{R}^n} |Q|^{p+1} \right) \Psi(A_1, A_2) + o(\varepsilon),
\end{aligned} \tag{66}$$

uniformly with respect to sets of parameters A_1 and A_2 satisfying (74). Here

$$\begin{aligned}
\Psi(A_1, A_2) &= H(\hat{\xi}_1, \hat{\xi}_1) Q^2(\hat{a}_1) \Lambda_1^2 + H(\hat{\xi}_2, \hat{\xi}_2) Q^2(\hat{a}_2) \Lambda_2^2 \\
&- 2G(\hat{\xi}_1, \hat{\xi}_2) Q(\hat{a}_1) Q(\hat{a}_2) \Lambda_1 \Lambda_2 + \log \Lambda_1 \Lambda_2 \\
&+ \frac{n-2}{\int_{\mathbb{R}^n} |Q|^{p+1}} \left[\sum_{j=1}^{2} \int |Q|^{p+1} \log \frac{1}{|y + \hat{a}_j|} \right],
\end{aligned} \tag{67}$$

while $\gamma_{1,n}$ and $\gamma_{2,n}$ are constants whose definition depends only on n, and it is independent of ε.

4. Scheme of the proof

The solution predicted by Theorem 1.1 will have the form

$$u(x) = (1+\zeta)\left(u_0(x) + \tilde{\phi}(x)\right), \tag{68}$$

where ζ is the number defined by

$$\zeta = \varepsilon^{\varepsilon \frac{p}{2(p-1+\varepsilon)}} - 1 \tag{69}$$

and u_0 is the function defined by

$$u_0(x) = PQ_1(x) + PQ_2(x) \tag{70}$$

where for $j = 1, 2$, PQ_j is the $H_0^1(\Omega)$-projection of the function Q_{A_j} defined in (28), for some parameter $A_j = (\lambda_j, \xi_j, a_j, \theta_j)$. In (68), the function $\tilde{\phi}$ has to be determined to have that u is a solution to (1).

It is useful to rephrase Problem (1) in some expanded domain $\Omega_\varepsilon = \varepsilon^{-\frac{1}{n-2}}\,\Omega$. After the change of variables

$$v(y) = \varepsilon^{\frac{1}{2}} u(\varepsilon^{\frac{1}{n-2}} y), \quad y = \varepsilon^{-\frac{1}{n-2}}\, x \in \Omega_\varepsilon, \tag{71}$$

Problem (1) gets re-written as

$$\Delta v + v^{p+\varepsilon} = 0, \quad v > 0, \quad \text{in} \quad \Omega_\varepsilon \quad v = 0, \quad \text{on} \quad \partial\Omega_\varepsilon, \tag{72}$$

In the expanded variables, the solution in (68) gets into the form

$$v(y) = v_0(y) + \phi(y), \quad \text{where} \quad v_0(y) = \varepsilon^{\frac{1}{2}} u_0(\varepsilon^{\frac{1}{n-2}} y). \tag{73}$$

In order to determine the unknown function ϕ, we proceed in two steps. In the first step, we fix the parameters A_1 and A_2, and we find ϕ as solution of a proper non linear projected problem. With abuse of notation, we denote

$$A_j = (\Lambda_j, \xi_j, a_j, \theta_j) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2 \times K$$

and we assume the following constraints on A_j, $j = 1, 2$:

$$\begin{gathered}(\hat{\xi}_1, \hat{\xi}_2) \in \Omega \times \Omega : \operatorname{dist}(\hat{\xi}_i, \partial\Omega) > \delta,\ |\hat{\xi}_1 - \hat{\xi}_2| > \delta \\ \delta < \Lambda_i < \delta^{-1}, \qquad \theta_i \in \chi(\hat{S}), \quad |a_i| \le \frac{1}{2}, \quad i = 1, 2,\end{gathered} \tag{74}$$

for some fixed $\delta > 0$, where

$$\lambda_i^{n-2} = \beta_n \Lambda_i^2 \varepsilon, \quad \hat{\xi}_i = R_{\theta_i}\xi_i, \quad \hat{a}_i = R_{\theta_i} a_i, \quad i = 1, 2,$$

with β_n the positive number defined by $\beta_n = \frac{(n-2)\int_{\mathbb{R}^n} |Q|^{p+1}}{2n\alpha_n}$. Observe that a direct computation gives that, as $\varepsilon \to 0$,

$$\begin{aligned} v_0(y) = \sum_{j=1}^{2} (\Lambda_j^*)^{-\frac{n-2}{2}} \left|\eta_{\Lambda_j^*, R_{\theta_j}\xi_j', a_j}(y)\right|^{2-n} Q\left(\frac{\frac{y - R_{\theta_j}\xi_j'}{\Lambda_j^*} - R_{\theta_j} a_j \left|\frac{y - R_{\theta_j}\xi_j}{\Lambda_j^*}\right|^2}{\eta_{\Lambda_j^*, R_{\theta_j}\xi_j' a_j}(y)}\right) \\ + o(1) \end{aligned} \tag{75}$$

where the function η is defined in (27), $\Lambda_i^* = (\beta_n \Lambda_i^2)^{\frac{1}{n-2}}$, see (65), and $\xi_j' = \varepsilon^{-\frac{1}{n-2}}\xi_j$.

Consider the following functions, for any $\alpha = 0, 1, \ldots, 3n-1$, and $j = 1, 2$

$$\bar{Z}_{\alpha j} = (\Lambda_j^*)^{-\frac{n-2}{2}} \left|\eta_{\Lambda_j^*, R_{\theta_j}\xi_j', a_j}(y)\right|^{2-n} z_\alpha \left(\frac{\frac{y - R_{\theta_j}\xi_j'}{\Lambda_j^*} - R_{\theta_j} a_j \left|\frac{y - R_{\theta_j}\xi_j'}{\Lambda_j^*}\right|^2}{\eta_{\Lambda_j^*, R_{\theta_j}\xi_j', a_j}(y)} \right)$$

where the functions z_α are defined in (21), (22), (23), (24), (25), and the function η is defined in (27). Consider furthermore their $H_0^1(\Omega_\varepsilon)$-projections $Z_{\alpha j}$, namely the unique solutions of

$$\Delta Z_{\alpha j} = \Delta \bar{Z}_{\alpha j} \quad \text{in } \Omega_\varepsilon, \quad Z_{\alpha j} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

The nonlinear projected problem that we first solve consists in finding a function ϕ such that the following equation holds

$$\begin{cases} \Delta(v_0 + \phi) + (v_0 + \phi)_+^{p+\varepsilon} = \sum_{\alpha, j} c_{\alpha j} v_j^{p-1} Z_{\alpha j} & \text{in } \Omega_\varepsilon, \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi v_j^{p-1} Z_{\alpha j} = 0 & \text{for all } \alpha, j, \end{cases} \tag{76}$$

for some constants $c_{\alpha j}$, where

$$v_i(y) = \varepsilon^{\frac{1}{2}} P Q_i(\varepsilon^{\frac{1}{n-1}} y), \quad i = 1, 2.$$

A fundamental tool to solve properly Problem (76) consists in developing an invertibility theory for the following linear problem. Given $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, find a function ϕ such that for certain constants $c_{\alpha j}$, $j = 1, 2$, $\alpha = 0, \ldots, 3n-1$ one has

$$\begin{cases} \Delta\phi + (p+\varepsilon) v_0^{p+\varepsilon-1} \phi = h + \sum_{\alpha, j} c_{\alpha j} v_i^{p-1} Z_{\alpha j} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} v_i^{p-1} Z_{\alpha j} \phi \, dy = 0 & \text{for all } i, j. \end{cases} \tag{77}$$

We do solve (77) in proper weighted L^∞-norms: for a function ψ defined on Ω_ε, we define

$$\begin{aligned} \|\psi\|_* = \sup_{x \in \Omega_\varepsilon} | \left(\sum_{j=1}^2 (1 + |x - \xi_j'|^2)^{-\frac{n-2}{2}} \right)^{-\beta} \psi(x)| \\ + \sup_{x \in \Omega_\varepsilon} | \left(\sum_{j=1}^2 (1 + |x - \xi_j'|^2)^{-\frac{n-2}{2}} \right)^{-\beta - \frac{1}{n-2}} D\psi(x)|, \end{aligned}$$

where $\beta = 1$ if $n = 3$ and $\beta = \frac{2}{n-2}$ otherwise, and

$$\|\psi\|_{**} = \sup_{x \in \Omega_\varepsilon} | \left(\sum_{j=1}^2 (1 + |x - \xi_j'|^2)^{-\frac{n-2}{2}} \right)^{-\frac{4}{n-2}} \psi(x)|.$$

In Section 5, we shall establish

Proposition 4.1. *Assume constraints (74) hold on the parameter sets A_1 and A_2. Then there are numbers $\varepsilon_0 > 0$, $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, problem (77) admits a unique solution $\phi \equiv L_\varepsilon(h)$. Besides,*

$$\|L_\varepsilon(h)\|_* \leq C\|h\|_{**}, \quad |c_{\alpha j}| \leq C\|h\|_{**} \tag{78}$$

and

$$\|\nabla_{\Lambda, \xi', a, \theta} \phi\|_* \leq C\|h\|_{**}. \tag{79}$$

A fixed point argument using contraction mapping Theorem gives as a direct byproduct of the previous Proposition the following result that states unique solvability of the non linear Problem (76), for any given sets of parameters A_1 and A_2.

Proposition 4.2. *Assume constraints (74) hold on the parameter sets A_1 and A_2. Then there is a constant $C > 0$, such that for all small ε there exists a unique solution $\phi = \phi(\xi', \Lambda)$ to Problem (76) with*

$$||\phi||_* \leq C\varepsilon, \quad \|\nabla_{\Lambda,\xi',a,\theta}\phi\|_* \leq C\varepsilon \tag{80}$$

The proof of Proposition 4.2 is postponed to Section 6.

Looking back at Problem (76), it is immediate to observe that $v_0 + \phi$ is a solution to the scaled Problem (72) if the constants $c_{\alpha j}$ appearing in (76) are all zero. The second step in our argument consists in showing that the constants $c_{\alpha j}$ can be made all equal to zero provided the parameter sets A_1 and A_2 are properly chosen. Let us explain this second part of our argument.

Consider the function of A_1 and A_2 defined by

$$I(A_1, A_2) \equiv J_\varepsilon((1+\zeta)\left(u_0(x) + \tilde{\phi}(x)\right)), \tag{81}$$

where J_ε is defined in (63), u_0 is given by (70) and $\tilde{\phi}(x) = \varepsilon^{-\frac{1}{2}}\phi(\varepsilon^{-\frac{1}{n-2}}x)$, where ϕ is the unique solution to Problem (76) as predicted by Proposition 4.2. Recall that

$$A_j = (\Lambda_j, \xi_j, a_j, \theta_j) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2 \times K, \quad j = 1, 2.$$

A consequence of Proposition 4.2, and estimate (80), is that the function I depends in a C^1 sense on the parameters A_1 and A_2. Furthermore, we see that

$$I(A_1, A_2) = (1+\zeta)^2 F_\varepsilon(v_0 + \phi),$$

where ζ is defined in (69) and

$$F_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |Dv|^2 - \frac{1}{p+\varepsilon+1}\int_{\Omega_\varepsilon} v^{p+\varepsilon+1}.$$

The key observation of our argument is the following

Lemma 4.3. *$u = (1+\zeta)(u_0 + \tilde{\phi})$ is a solution of problem (1) if and only if (A_1, A_2) is a critical point of I.*

Before giving the proof of Lemma 4.3, an observation is in order. Let us recall that, for $A = (\lambda, \xi, a, \theta) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$, the function Q_A is defined as

$$Q_A(x) = \lambda^{-\frac{n-2}{2}} \left| \frac{x-\hat{\xi}}{|x-\hat{\xi}|} - \hat{a}\frac{|x-\hat{\xi}|}{\lambda} \right|^{2-n} Q\left(\frac{\frac{x-\hat{\xi}}{\lambda} - \hat{a}|\frac{x-\hat{\xi}}{\lambda}|^2}{1 - 2\hat{a}\cdot(\frac{x-\hat{\xi}}{\lambda}) + |a|^2|\frac{x-\hat{\xi}}{\lambda}|^2} \right),$$

with

$$\hat{\xi} = R_\theta \xi, \quad \hat{a} = R_\theta a.$$

Recall the functions z_α, $\alpha = 0, \ldots, 3n-1$, defined in (21), (22), (23), (24), (25). These are the only elements in the kernel of the linear operator $L(\varphi) = \Delta\varphi + p|Q|^{p-2}Q\varphi$ (see [21]). We are in a position to prove

Proof of Lemma 4.3. Consider, for example, the derivative of I with respect to a_{11}, the first component of $a_1 = (a_{11}, a_{12}) \in \mathbb{R}^2$. We start observing that that $\frac{\partial}{\partial a_{11}}I = 0$ is equivalent to say that $\frac{\partial}{\partial a_{11}}F_\varepsilon(v_0 + \phi) = 0$. Next we compute $\frac{\partial}{\partial a_{11}}F_\varepsilon(v_0 + \phi) = DF_\varepsilon(v_0 + \phi)[\frac{\partial}{\partial a_{11}}v_0 + \frac{\partial}{\partial a_{11}}\phi]$. On the other hand, using (31), one sees that

$$\frac{\partial}{\partial a_{11}}v_0 = \bar{Z}_{n+2,1} = Z_{n+2,1} + o(1)$$

where

$$\bar{Z}_{\alpha j} = (\Lambda_j^*)^{-\frac{n-2}{2}} \left|\eta_{\Lambda_j^*, R_{\theta_j}\xi_j', a_j}(y)\right|^{2-n} z_\alpha \left(\frac{\frac{y - R_{\theta_j}\xi_j'}{\Lambda_j^*} - R_{\theta_j} a_j |\frac{y-R_{\theta_j}\xi_j'}{\Lambda_j^*}|^2}{\eta_{\Lambda_j^*, R_{\theta_j}\xi_j', a_j}(y)} \right)$$

and $Z_{\alpha j}$ is the $H_0^1(\Omega_\varepsilon)$-projection of $\bar{Z}_{\alpha j}$. Taking into account that $\|\frac{\partial}{\partial a_{11}}\phi\|_* = o(1)$, as $\varepsilon \to 0$, we get that $\frac{\partial}{\partial a_{11}} I = 0$ is equivalent to say $DF_\varepsilon(v_0 + \phi)[Z_{n+2,1} + o(1)] = 0$. Based on this argument, we can say that the conditions $\nabla I(A_1, A_2) = 0$ are equivalent to say that

$$DF_\varepsilon(v_0 + \phi)[Z_{\alpha j} + o(1)] = 0 \tag{82}$$

for all α, for all j. Digging further in the above set of equalities, and using the fact that, by definition, $DF_\varepsilon(v_0 + \phi)[g] = 0$ for all functions such that $\int_{\Omega_\varepsilon} v_0^{p-1} Z_{\beta k} g = 0$, we can be more precise in (82): indeed, one has that

$$DF_\varepsilon(v_0 + \phi)[Z_{\alpha j} + o(1)\Theta] = 0 \quad \text{for all} \quad \alpha, j$$

where Θ is a uniformly bounded function, that belongs to the vector space generated by the functions $Z_{\beta,i}$. From the above relation we thus conclude that the $6n$ conditions $\nabla I(A_1, A_2) = 0$ are equivalent to the $6n$ conditions

$$DF_\varepsilon(v_0 + \phi)[Z_{\alpha j}] = 0$$

for all α, j. By definition of the $c_{\alpha j}$, it is easily seen that this is indeed equivalent to $c_{\alpha j} = 0$ for all α, j. This concludes the proof of the Lemma.

□

The result of Lemma 4.3 says that the function defined in (68)

$$u(x) = (1 + \zeta)\left(u_0(x) + \tilde{\phi}(x)\right),$$

where $\tilde{\phi}(x) = \varepsilon^{-\frac{1}{2}}\phi(\varepsilon^{-\frac{1}{n-2}}x)$ and ϕ is the unique solution to Problem (76) as predicted by Proposition 4.2, is a solution to (1) if (A_1, A_2) is a critical point for I defined in (81).

Our purpose is thus to establish the existence of a critical point for $I(A_1, A_2)$. To this purpose, we first give an asymptotic estimate for the function $I(A_1, A_2)$. We prove

Proposition 4.4. *Let ζ be given by (69). Then we have the expansion,*

$$\begin{aligned} \varepsilon^{2\zeta - 1} I(A_1, A_2) &= 2\gamma_n + \gamma_{1,n}\varepsilon \log\varepsilon + \gamma_{2,n}\varepsilon \\ &\quad + w_n \varepsilon\, \Psi(A_1, A_2) + o(\varepsilon)\theta(A_1, A_2), \end{aligned} \tag{83}$$

uniformly with respect to (A_1, A_2) satisfying constaint (74), where θ and its derivatived $D\theta$ are smooth functions that are uniformly bounded, independently of ε. We refer to the statement of Lemma 3.4 for the definition of the function Ψ and the constants $\gamma_{1,n}$, $\gamma_{2,n}$.

The proof of this result is postponed to the end of Section 6.

The final argument to get our Theorem 1.1 is to show that the function Ψ in (83) defined also in (67) has a critical point, in fact a robust critical point of min max type, that persists under small C^1 perturbation. This is where we need that our domain Ω has the shape of a smooth bounded connected domain with a small removed hole. We show the existence of a min max structure for Ψ in Section 7. This completes the proof of Theorem 1.1.

The rest of the paper is devoted to give detailed proofs of all our previous statements.

5. The linear problem: proof of Proposition 4.1

Proof of Proposition 4.1. The proof of this result is divided into two steps: we first assume the existence of a solution, and we prove the estimates (78), then we show existence of ϕ.

To prove (78), assume that there exists sequence $\varepsilon = \varepsilon_n \to 0$ such that there are functions ϕ_ε and h_ε with $\|h_\varepsilon\|_{**} = o(1)$ such that

$$\Delta\phi_\varepsilon + (p+\varepsilon)v_0^{p-1+\varepsilon}\phi_\varepsilon = h_\varepsilon + \sum_{\alpha,j} c_{\alpha j} v_j^{p-1} Z_{\alpha j} \quad \text{in } \Omega_\varepsilon$$

$$\phi_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

$$\int_{\Omega_\varepsilon} v_j^{p-1} Z_{\alpha j}\phi_\varepsilon \, dx = 0 \text{ for all } \alpha, j,$$

for certain constants $c_{\alpha j}$, depending on ε. We shall show that $\|\phi_\varepsilon\|_* \to 0$.

We first establish that

$$\|\phi_\varepsilon\|_\rho = \sup_{x\in\Omega_\varepsilon} \left| \left(\sum_{j=1}^{2} (1+|x-\xi_j'|^2)^{-\frac{n-2}{2}} \right)^{-(\beta-\rho)} \phi_\varepsilon(x) \right| + \sup_{x\in\Omega_\varepsilon} \left| \left(\sum_{j=1}^{2} (1+|x-\xi_j'|^2)^{-\frac{n-2}{2}} \right)^{-(\beta-\rho-\frac{1}{n-2})} D\phi_\varepsilon(x) \right| \to 0$$

with $\rho > 0$ a small fixed number. To do this, we assume the opposite, so that with no loss of generality we may take $\|\phi_\varepsilon\|_\rho = 1$. Testing the above equation against $Z_{\beta k}$, integrating by parts twice we get that

$$\sum c_{\alpha j} \int_{\Omega_\varepsilon} v_j^{p-1} Z_{\alpha j} Z_{\beta k} = \int_{\Omega_\varepsilon} [\Delta Z_{\beta k} + (p+\varepsilon) v_0^{p-1+\varepsilon} Z_{\beta k}\phi] - \int_{\Omega_\varepsilon} h_\varepsilon Z_{\beta,k}. \tag{84}$$

Formula (84) defines a linear system in the $6n$ variables $c_{\alpha,j}$, which is uniquely solvable, with bounded inverse. This is due to the following facts: fix α and j. If $k \neq j$, then

$$\int_{\Omega_\varepsilon} v_j^{p-1} Z_{\alpha j} Z_{\beta k} = O(\varepsilon^{\frac{n}{n-2}}), \quad \text{for any} \quad \beta.$$

If $k = j$, then we have

$$\int_{\Omega_\varepsilon} v_j^{p-1} Z_{\alpha j} Z_{\beta j} = \begin{cases} \int_{\mathbb{R}^n} |Q|^{p-1} z_\alpha^2 + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } \alpha = \beta\,, \\ \int_{\mathbb{R}^n} |Q|^{p-1} z_1 z_{n+2} + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } \alpha = 1, \beta = n+2\,, \\ \int_{\mathbb{R}^n} |Q|^{p-1} z_2 z_{n+3} + O(\varepsilon^{\frac{n}{n-2}}) & \text{if } \alpha = 2, \beta = n+3\,, \\ O(\varepsilon^{\frac{n}{n-2}}) & \text{otherwise.} \end{cases}$$

Moreover the numbers $\int_{\mathbb{R}^n} |Q|^{p-1} z_\alpha^2$, $\alpha = 0, 1, \ldots, 3n-1$, and $\int_{\mathbb{R}^n} |Q|^{p-1} z_1 z_{n+2}$, $\int_{\mathbb{R}^n} |Q|^{p-1} z_2 z_{n+3}$ are fixed numbers, different from zero, that are independent of ε. The above computations tell us several facts: first of all, the linear system (84) of $6n$ equations in the $6n$ variables $c_{\alpha j}$ at main order decouples in two systems, the first a $3n \times 3n$ system in the variables $c_{\alpha,1}$ and the second in a $3n \times 3n$ system in the variables $c_{\alpha 2}$. Second, if we analyze for instant the system in the $c_{\alpha 1}$, we see that the coefficients $c_{01}, \ldots c_{(3n-1)1}$ are coupled, at main order, but the coupling is very clear, in fact only two coupling occurs: the variable c_{11} with the variable $c_{n+2,1}$ and the variable c_{21} with the variable $c_{n+3,1}$. Except for these coupling, the system in the $c_{\alpha 1}$ decouples at main order, as $\varepsilon \to 0$. We finally observe that the matrices

$$\begin{bmatrix} \int_{\mathbb{R}^n} |Q|^{p-1} z_1^2 & \int_{\mathbb{R}^n} |Q|^{p-1} z_1 z_{n+2} \\ \int_{\mathbb{R}^n} |Q|^{p-1} z_1 z_{n+2} & \int_{\mathbb{R}^n} |Q|^{p-1} z_{n+2}^2 \end{bmatrix},$$

$$\begin{bmatrix} \int_{\mathbb{R}^n} |Q|^{p-1} z_2^2 & \int_{\mathbb{R}^n} |Q|^{p-1} z_2 z_{n+3} \\ \int_{\mathbb{R}^n} |Q|^{p-1} z_2 z_{n+3} & \int_{\mathbb{R}^n} |Q|^{p-1} z_{n+3}^2 \end{bmatrix} \tag{85}$$

are invertible. Thus we conclude that (84) defines a linear system in the $6n$ variables $c_{\alpha,j}$, which is uniquely solvable, with bounded inverse.

On the other hand, it is easy to see that we have, for $l = 1, 2$,

$$\int_{\Omega_\varepsilon} \Delta Z_{\alpha k} + (p+\varepsilon) v_0^{p+\varepsilon-1} Z_{\alpha k} \phi = o(1) \|\phi\|_\rho, \quad \text{and} \quad |\int_{\Omega_\varepsilon} h_\varepsilon, Z_{\beta k}| \le C \|h_\varepsilon\|_{**}.$$

Thus, we conclude that

$$|c_{\alpha j}| \le C \|h_\varepsilon\|_{**} + o(1) \|\phi_\varepsilon\|_\rho \tag{86}$$

so that $c_{ij} = o(1)$. Let G_ε denotes the Green's function of Ω_ε. We have for $x \in \Omega_\varepsilon$

$$\begin{aligned} \phi_\varepsilon(x) = (p+\varepsilon) \int_{\Omega_\varepsilon} G_\varepsilon(x,y) v_0^{p+\varepsilon-1} \phi_\varepsilon dy \\ - \int_{\Omega_\varepsilon} G_\varepsilon(x,y) h_\varepsilon \, dy - \sum c_{\alpha j} \int_{\Omega_\varepsilon} v_j^{p-1} Z_{\alpha j} G_\varepsilon(x,y) \, dy \end{aligned} \tag{87}$$

Furthermore, the function ϕ_ε is of class C^1 and

$$\partial_{x_j} \phi_\varepsilon(x) = p \int_{\Omega_\varepsilon} \partial_{x_j} G_\varepsilon(x,y) v_0^{p+\varepsilon-1} \phi_\varepsilon dy =$$

$$- \int_{\Omega_\varepsilon} \partial_{x_j} G_\varepsilon(x,y) h_\varepsilon \, dy - \sum c_{\alpha j} \int_{\Omega_\varepsilon} v_0^{p-1} Z_{\alpha j} \partial_{x_j} G_\varepsilon(x,y) \, dy \quad x \in \Omega_\varepsilon. \tag{88}$$

Direct estimates give

$$\begin{aligned} \int_{\Omega_\varepsilon} G_\varepsilon(x,y) |h_\varepsilon| \, dy &\le \|h_\varepsilon\|_{**} C \int_{\mathbb{R}^n} \Gamma(x-y) \sum_{j=1}^2 \frac{1}{(1+|y-\xi_j'|^2)^2} \, dy \\ &\le C \|h_\varepsilon\|_{**} \sum_{j=1}^2 \left(\frac{1}{(1+|x-\xi_1'|^2)^{\frac{n-2}{2}}} \right)^\beta, \end{aligned}$$

$$\begin{aligned} |\int_{\Omega_\varepsilon} v_j^{p-1} Z_{\alpha j} G_\varepsilon(x,y) \, dy| &\le C \int_{\mathbb{R}^n} \Gamma(x-y) \sum_{j=1}^2 \frac{1}{(1+|y-\xi_i'|^2)^{\frac{n+3}{2}}} \\ &\le C \sum_{j=1}^2 \frac{1}{(1+|x-\xi_1'|^2)^{\frac{n-2}{2}}} \end{aligned}$$

and

$$\int_{\Omega_\varepsilon} G_\varepsilon(x,y) v_0^{p+\varepsilon-1} |\phi_\varepsilon| dy \le C \|\phi_\varepsilon\|_\rho \sum_{j=1}^2 \left(\frac{1}{(1+|x-\xi_j'|^2)^{\frac{n-2}{2}}} \right)^\beta.$$

Analogously we get

$$\int_{\Omega_\varepsilon} |\partial_{x_j} G_\varepsilon(x,y) h|\, dy \le \|h\|_{**} C \sum_j \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} (1+|y-\xi_j'|^2)^{-2}\, dy$$

$$\le C\|h\|_{**} \sum_{j=1}^{2} \left(\frac{1}{(1+|x-\xi_1'|^2)^{\frac{n-2}{2}}} \right)^{\beta + \frac{1}{n-2}},$$

$$|\int_{\Omega_\varepsilon} v_0^{p-1} Z_{\alpha j} \partial_{x_j} G_\varepsilon(x,y)\, dy| \le C(\|\phi_\varepsilon\|_\rho + \|h\|_{**}) \sum \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} \left((1+|y-\xi_i'|^2)^{-\frac{n+3}{2}} \right)$$

$$\le C(\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}) \sum_{j=1}^{2} \left(\frac{1}{(1+|x-\xi_1'|^2)^{\frac{n-2}{2}}} \right)^{\beta + \frac{1}{n-2}}$$

and

$$\int_{\Omega_\varepsilon} |\partial_{x_j} G_\varepsilon(x,y) v_0^{p+\varepsilon-1} \phi_\varepsilon| dy \le C\|\phi_\varepsilon\|_\rho \sum_{j=1}^{2} \left(\frac{1}{(1+|x-\xi_1'|^2)^{\frac{n-2}{2}}} \right)^{\beta + \frac{1}{n-2}}.$$

Equation (87) and the above estimates imply that

$$|\phi_\varepsilon(x)| \le C(\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}) \left(\sum_{j=1}^{2} \frac{1}{(1+|x-\xi_j'|^2)^{\frac{n-2}{2}}} \right)^{\beta}$$

and

$$|D\phi_\varepsilon(x)| \le C(\|\phi_\varepsilon\|_\rho + \|h_\varepsilon\|_{**}) \left(\sum_{j=1}^{2} \frac{1}{(1+|x-\xi_j'|^2)^{\frac{n-2}{2}}} \right)^{\beta + \frac{1}{n-2}}.$$

In particular

$$\left(\sum_{j=1}^{2} \frac{1}{(1+|x-\xi_j'|^2)^{\frac{n-2}{2}}} \right)^{-(\beta-\rho)} |\phi_\varepsilon(x)| \le C \left(\sum_{j=1}^{2} \frac{1}{(1+|x-\xi_j'|^2)^{\frac{n-2}{2}}} \right)^{\rho}.$$

Since $\|\phi_\varepsilon\|_\rho = 1$, we assume that $\|\phi_\varepsilon\|_{L^\infty(B_R(\xi_1'))} > \gamma$ for certain $R > 0$ and $\gamma > 0$ independent of ε. for either $i = 1$ or $i = 2$. Then local elliptic estimates and the bounds above yield that, up to a subsequence, $\tilde\phi_\varepsilon(x) = \phi_\varepsilon(x - \xi_1')$ converges uniformly over compacts of $\mathbb{R}^N$ to a nontrivial solution $\tilde\phi$ of

$$\Delta\tilde\phi + p|Q|^{p-1}\tilde\phi = 0, \tag{89}$$

which besides satisfies

$$|\tilde\phi(x)| \le C|x|^{(2-n)\beta}. \tag{90}$$

In dimension $n = 3$ this means $|\tilde\phi(x)| \le C|x|^{2-n}$. In higher dimension, a bootstrap argument of $\tilde\phi$ solution of (89), using estimate (90), gives $|\tilde\phi(x)| \le C|x|^{2-n}$. Thanks to non degenerate result in [21], this implies that $\tilde\phi$ is a linear combination of the functions z_α, defined in (21), (22), (23), (24) and (25). On the other hand, dominated convergence Theorem gives that the orthogonality conditions $\int_{\Omega_\varepsilon} \phi_\varepsilon v_j^{p-1} Z_{\alpha j} = 0$ pass to the limit, thus getting

$$\int_{\mathbb{R}^n} |Q|^{p-1} z_\alpha \tilde\phi = 0 \quad \text{for all} \quad \alpha = 0, \dots, 3n-1.$$

Hence the only possibility is that $\tilde{\phi} \equiv 0$, which is a contradiction which yields the proof of $\|\phi_\varepsilon\|_\rho \to 0$. Moreover, we observe that

$$\|\phi_\varepsilon\|_* \le C(\|h_\varepsilon\|_{**} + \|\phi_\varepsilon\|_\rho),$$

hence $\|\phi_\varepsilon\|_* \to 0$.

Now we are in a position to prove the existence of ϕ solution to (77). To do this, let us consider the space

$$H = \{\phi \in H_0^1(\Omega_\varepsilon) \mid \int_{\Omega_\varepsilon} v_j^{p-1} Z_{\alpha j}, \phi >= 0 \ \forall\, \alpha, j \ \}$$

endowed with the usual inner product $[\phi,\psi] = \int_{\Omega_\varepsilon} \nabla\phi\nabla\psi$. Problem (77) expressed in weak form is equivalent to that of finding a $\phi \in H$ such that

$$[\phi,\psi] = \int_{\Omega_\varepsilon} \left((p+\varepsilon)v_0^{p+\varepsilon-1}\phi - h\right)\psi \qquad \forall\psi \quad \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operational form

$$\phi = T_\varepsilon(\phi) + \tilde{h} \tag{91}$$

with certain $\tilde{h} \in H$ which depends linearly in h and where T_ε is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = T_\varepsilon(\phi)$ has only the zero solution in H. Assume it has a nontrivial solution $\phi = \phi_\varepsilon$, which with no loss of generality may be taken so that $\|\phi_\varepsilon\|_* = 1$. But for what we proved before, necessarily $\|\phi_\varepsilon\|_* \to 0$. This is certainly a contradiction that proves that this equation only has the trivial solution in H. We conclude then that for each h, problem (77) admits a unique solution. Standard arguments give then the validity of (78).

We go now to the issue of the dependence of the solution ϕ to (77) on the parameters $A_1' = (\Lambda_1, \xi_1', a_1, \theta_1)$ and $A_2' = (\Lambda_2, \xi_2', a_2, \theta_2)$. Let us fix $j = 1$ and define $A_1' = (A_{11}, A_{12}, \dots, A_{13n})$ the components of the vector A_1'. Let us differential ϕ with respect to A_{1l}, for some $l = 1, \dots, 3n$. We set formally $Z = \frac{\partial}{\partial A_{1l}}\phi$.

We define the number $b_{\alpha j}$ so that

$$\int_{\Omega_\varepsilon} v_i^{p-1} Z_{\beta i}[Z - \sum_{\alpha j} b_{\alpha j} Z_{\alpha j}] = 0, \quad \text{for all} \quad \beta, i.$$

This amounts to solving a linear system in the constants $b_{\alpha j}$,

$$\sum_{\beta j} b_{\beta j} \int_{\Omega_\varepsilon} v_i^{p-1} Z_{\alpha i} Z_{\beta j} = \int_{\Omega_\varepsilon} \frac{\partial}{\partial A_{1l}}(v_i^{p-1} Z_{\alpha i})\phi, \tag{92}$$

as a direct differentiation with respect to A_{1l} of the orthogonal conditions $\int_{\Omega_\varepsilon} v_i^{p-1} Z_{\alpha i}\phi = 0$ directly shows. Arguing as in (84), we see that (92) is uniquely solvable and that

$$b_{\beta i} = O(\|\phi\|_*)$$

uniformly for parameters A_1' and A_2' in the considered region. Thus $\eta \in H_0^1(\Omega_\varepsilon)$ and

$$\int_{\Omega_\varepsilon} v_i^{p-1} Z_{\alpha i}\eta = 0 \quad \text{ for all } \alpha, i. \tag{93}$$

On the other hand, a direct but long computation shows that

$$\Delta\eta + (p+\varepsilon)v_0^{p-1+\varepsilon}\eta = f + \sum_{\alpha,j} d_{\alpha j} v_j^{p-1} Z_{\alpha j} \quad \text{in } \Omega_\varepsilon, \tag{94}$$

where $d_{\alpha j} = \frac{\partial}{\partial A_{1l}} c_{\alpha j}$ and

$$f = \sum_{\alpha,j} b_{\alpha j}(-(\Delta + (p+\varepsilon)v_0^{p-1+\varepsilon})Z_{\alpha j} + c_{\alpha j}\partial_{A_{1l}}(v_j^{p-1}Z_{\alpha j}) -$$

$$(p+\varepsilon)\partial_{A_{1l}}(v_0^{p-1+\varepsilon}\phi), \tag{95}$$

Thus we have that $\eta = L_\varepsilon(f)$. Moreover, we easily see that

$$\|\phi\partial_{A_{1l}}(v_0^{p-1+\varepsilon})\|_{**} \le C\|\phi\|_*.$$

On the other hand

$$|\partial_{A_{1l}}(v_i^{p-1}Z_{\alpha i}(x))| \le C|x - \xi_i'|^{-n-4},$$

hence

$$\|c_{\alpha i}\partial_{A_{1l}} v_i^{p-1} Z_{\alpha i}\|_{**} \le C\|h\|_{**}$$

since we have that $c_{\alpha i} = O(\|h\|_{**})$. We conclude that

$$\|f\|_{**} \le C\|h\|_{**}.$$

Reciprocally, if we define

$$Z = L_\varepsilon(f) + \sum_{\alpha,j} b_{\alpha j} v_j^{p-1} Z_{\alpha j},$$

with $b_{\alpha j}$ given by relations (92) and f by (95), we check that indeed $Z = \partial_{A_{1l}}\phi$. In fact Z depends continuously on the parameters A_1', A_2' and h for the norm $\|\ \|_*$, and $\|Z\|_* \le C\|h\|_{**}$ for parameters in the considered region. The corresponding result for differentiation with respect to the A_2' follow similarly.

In other words, we proved that $(A_1', A_2') \mapsto L_\varepsilon$ is of class C^1 in $\mathcal{L}(L^\infty_{**}, L^\infty_*)$ and, for instance,

$$(D_{A_{1l}}L_\varepsilon)(h) = L_\varepsilon(f) + \sum_{\alpha,j} b_{\alpha j} Z_{\alpha j}, \tag{96}$$

where f is given by (95) and $b_{\alpha j}$ by (92) . This concludes the proof.

□

6. The non-linear Problem: proof of Proposition 4.2

Proof of Proposition 4.2. We write the equation in (76) as

$$\Delta\phi + (p+\varepsilon)v_0^{p+\varepsilon-1}\phi = E - N_\varepsilon(\phi) + \sum_{\alpha,j} c_{\alpha j} v_j^{p-1} Z_{\alpha j} \quad \text{in} \quad \Omega_\varepsilon$$

where

$$N_\varepsilon(\phi) = (v_0 + \phi)_+^{p+\varepsilon} - v_0^{p+\varepsilon} - (p+\varepsilon)v_0^{p+\varepsilon-1}\phi, \quad E = v_0^{p+\varepsilon} - Q_1^p - Q_2^p. \tag{97}$$

Observe that

$$|E| \le C\left(|v_0^{p+\varepsilon} - v_0^p| + |v_0^p - Q_1^p - Q_2^p|\right) \le C\varepsilon\left(|Q_i|^p|\log|Q_i||(x) + \frac{1}{1+|x-\xi_i'|^4}\right)$$

in the regions where $|x - \xi_i'| \le \bar\delta\varepsilon^{-\frac{1}{N-2}}$, for small $\bar\delta > 0$. Taking into account that $|E| \le C\varepsilon^{\frac{n+2}{n-2}}$ in the complement of these two regions, we get

$$\|E\|_{**} \le C\varepsilon.$$

To estimate $N_\varepsilon(\phi)$, it is convenient, and sufficient for our purposes, to assume $\|\phi\|_* < 1$. Note that, if $n \le 6$, then $p \ge 2$ and we can estimate

$$|N_\varepsilon(\phi)| \le C|v_1 + v_2|^{p-2}|\phi|^2$$

and hence

$$\|N_\varepsilon(\phi)\|_{**} \le C\|\phi\|_*^2.$$

Assume now that $n > 6$. If $|\phi| \geq \frac{1}{2}$ we have

$$|N_\varepsilon(\phi)| \leq C|\phi|^p, \quad \|N(\phi)\|_{**} \leq C\varepsilon^{-\frac{n-6}{2}}\|\phi\|_*^p.$$

Let us consider now the case $|\phi| \leq \frac{1}{2}v_0$. In the region where $dist(y, \partial\Omega_\varepsilon) \geq \delta\varepsilon^{-\frac{1}{n-2}}$, for some $\delta > 0$, then $v_0(y) \geq \alpha_\delta U(y)$ for some $\alpha_\delta > 0$; hence in this region, we have

$$|N(\phi)| \leq CU^{2\beta-1}\|\phi\|_*^2 \leq C\varepsilon^{(2\beta-1)}\|\phi\|_*^2.$$

On the other hand, when $dist(y, \partial\Omega_\varepsilon) \leq \delta\varepsilon^{-\frac{1}{n-2}}$, the following facts occur: $U(y)$, $v_0(y) = O(\varepsilon)$ and, as $y \to \partial\Omega_\varepsilon$, $v_0(y) = C\varepsilon^{\frac{n-1}{n-2}}dist(y, \partial\Omega_\varepsilon)(1 + o(1))$. This second assertion is a consequence of the fact that the Green function of the domain Ω vanishes linearly with respect to $dist(x, \partial\Omega)$ as $x \to \partial\Omega$. These two facts imply that, if $dist(y, \partial\Omega_\varepsilon) \leq \delta\varepsilon^{-\frac{1}{n-2}}$ and $\phi(y) \neq 0$ (otherwise $N(\phi)(y) = 0$), then

$$\begin{aligned}
\|N(\phi)\|_{**} &\leq U^{-\frac{4}{n-2}}v_0^{p-2}|\phi|^2 \\
&\leq CU^{-\frac{4}{n-2}}\left(\varepsilon^{\frac{n-1}{n-2}}dist(y, \partial\Omega_\varepsilon)\right)^{p-2}dist(y, \partial\Omega_\varepsilon)^2|D\phi(\bar{y})|^2 \\
&\leq CU^{-\frac{4}{n-2}+2\beta+\frac{2}{n-2}} + \varepsilon^{\frac{n-1}{n-2}(p-2)-\frac{p}{2}}\|\phi\|_*^2 \leq C\varepsilon^{-\frac{n-6}{n-2}}\|\phi\|_*^2.
\end{aligned}$$

Combining these relations we get

$$\|N(\phi)\|_{**} \leq \begin{cases} C\|\phi\|_*^2 & \text{if } n \leq 6 \\ C\varepsilon^{-\frac{n-6}{n-2}}\|\phi\|_*^2 & \text{if } n > 6. \end{cases} \tag{98}$$

Now, we are in position to prove that problem (76) has a unique solution $\phi = \widetilde{\phi} + \widetilde{\psi}$, with

$$\widetilde{\psi} := -T_\varepsilon(E), \tag{99}$$

with the required properties. Here T_ε denotes the linear operator defined by Proposition 4.1, namely $T_\varepsilon(h) = \phi$ is $L_\varepsilon\phi = h$. We see that problem (76) is equivalent to solving a fixed point problem. Indeed $\phi = \tilde{\phi} + \tilde{\psi}$ is a solution of (76) if and only if

$$\tilde{\phi} = -T_\varepsilon(N(\tilde{\phi} + \tilde{\psi})) \equiv A_\varepsilon(\tilde{\phi}).$$

We proceed to prove that the operator A_ε defined above is a contraction inside a properly chosen region. Since $\|E\|_* \leq C\varepsilon$, the result of Proposition 4.1 gives that

$$\|\tilde{\psi}\|_{**} \leq C\varepsilon$$

and

$$\|N(\tilde{\psi} + \eta)\|_{**} \leq \begin{cases} C(\varepsilon^2 + \varepsilon\|\eta\|_* + \|\eta\|_*^2) & \text{if } n \leq 6 \\ C(\varepsilon^{1+\frac{4}{n-2}} + \varepsilon^{\frac{4}{n-2}}\|\eta\|_* + \varepsilon^{-\frac{n-6}{n-2}}\|\eta\|_*^2) & \text{if } n > 6. \end{cases} \tag{100}$$

Call

$$F = \{\eta \in H_0^1 \, : \, ||\eta||_* \leq R\varepsilon\}.$$

¿From Proposition 4.1 and (100) we conclude that, for ε sufficiently small and any $\eta \in lF$ we have

$$\|A_\varepsilon(\eta)\|_* \leq C\varepsilon.$$

If we choose R big enough in the definition of F, we get then that A_ε maps F in itself. Now we will show that the map A_ε is a contraction, for any ε small enough. That will imply that A_ε has a unique fixed point in F and hence problem (76) has a unique solution. For any η_1, η_2 in F we have

$$\|A_\varepsilon(\eta_1) - A_\varepsilon(\eta_2)\|_* \leq C\|N_\varepsilon(\tilde{\psi} + \eta_1) - N_\varepsilon(\tilde{\psi} + \eta_2)\|_{**},$$

hence we just need to check that N is a contraction in its corresponding norms. By definition of N

$$D_{\bar\eta}N_\varepsilon(\bar\eta) = (p+\varepsilon)[(v_0+\bar\eta)_+^{p+\varepsilon-1} - v_0^{p+\varepsilon-1}].$$

Hence we get

$$|N_\varepsilon(\tilde\psi+\eta_1) - N_\varepsilon(\tilde\psi+\eta_2)| \le C\bar v_0^{p-2}|\bar\eta||\eta_1-\eta_2|.$$

for some $\bar\eta$ in the segment joining $\tilde\psi+\eta_1$ and $\tilde\psi+\eta_2$. Hence, we get for small enough $\|\bar\eta\|_*$,

$$\|N(\tilde\psi+\eta_1) - N(\tilde\psi+\eta_2)\|_{**} \le C\varepsilon^{p-2+2\beta}\|\bar\eta\|_*\|\eta_1-\eta_2\|_*.$$

We conclude that there exists $c\in(0,1)$ such that

$$\|N(\tilde\psi+\eta_1) - N(\tilde\psi+\eta_2)\|_{**} \le c\|\eta_1-\eta_2\|_*.$$

This concludes the proof of existence of ϕ solution to (76), and the first estimate in (80). We devote the rest to prove the second estimate in (80).

We recall that ϕ is defined through the relation

$$B(A_1,A_2,\phi) \equiv \phi + L_\varepsilon(N_\varepsilon(\phi+\psi)) = 0.$$

We have that

$$D_\phi B(A_1,A_2,\phi)[\theta] = \theta + L_\varepsilon(\theta D_{\bar\phi}N(\phi+\psi)) \equiv \theta + M(\theta)$$

where

$$D_{\bar\phi}N(A_1,A_2,\bar\phi) = (p+\varepsilon)[(v_0+\bar\phi)_+^{p+\varepsilon-1} - v_0^{p+\varepsilon-1}].$$

Now,

$$\|M(\theta)\|_* \le C\|(\theta D_{\bar\phi}N(\phi+\psi))\|_{**} \le C\|v_0^{-\frac{4}{n-2}+\beta}D_{\bar\phi}N(\phi+\psi))\|_\infty\|\theta\|_*,$$

and

$$\bar v_0^{-\frac{4}{N-2}+\beta}|D_{\bar\phi}N_\varepsilon(\phi+\psi))| \le v_0^{2\beta-1}\|\phi+\psi\|_* \le C\varepsilon^{\min\{2\beta,1\}}.$$

It follows that for small ε, the linear operator $D_\phi B(A_1,A_2,\phi)$ is invertible in L^∞_*, with uniformly bounded inverse. It also depends continuously on its parameters. Define again $A_1' = (\Lambda_1,\xi_1',a_1,\theta_1)$ and $A_2' = (\Lambda_2,\xi_2',a_2,\theta_2)$. Let us fix $j=1$ and define $A_1' = (A_{11},A_{12},\dots,A_{13n})$ the components of the vector A_1'. Let us differential ϕ with respect to A_{1l}, for some $l=1,\dots,3n$. We have

$$D_{A_{1l}}N(A_1,A_2,\bar\phi) = (p+\varepsilon)[(v_0+\bar\phi)_+^{p+\varepsilon-1} - v_0^{p+\varepsilon-1} - (p+\varepsilon-1)v_0^{p+\varepsilon-2}\bar\phi]D_{A_{1l}}v_0. \tag{101}$$

and

$$D_{A_{1l}}B(A_1,A_2,\phi) = (D_{A_{1l}}L_\varepsilon)(N(\phi+\psi)) + \big[L_\varepsilon((D_{A_{1l}}N)(A_1,A_2,\phi+\psi)) + L_\varepsilon((D_{\bar\phi}N)(A_1,A_2,\phi+\psi)D_{A_{1l}}\psi)\big].$$

Here $D_{A_{1l}}L_\varepsilon$ is the operator defined by the expression (96) and the second quantity by (101). Observe also that

$$D_{A_{1l}}\psi = (D_{A_{1l}}L_\varepsilon)(E) + L_\varepsilon(D_{A_{1l}}E). \tag{102}$$

see (99). Also,

$$D_{A_{1l}}E = (p+\varepsilon)v_0^{p+\varepsilon-1}D_{A_{1l}}v_1 - pv_1^{p-1}D_{A_{1l}}v_1. \tag{103}$$

These expressions also depend continuously on their parameters.

The implicit function theorem then applies to yield that $\phi(A_1,A_2)$ indeed defines a C^1 function into L^∞_*. Moreover, we have for instance

$$D_{A_{1l}}\phi = -(D_\phi B(A_1,A_2,\phi))^{-1}\big[(D_{A_{1l}}L_\varepsilon)(N(\phi+\psi)) + [L_\varepsilon(D_{A_{1l}}[N(A_1,A_2,\phi+\psi)]) + L_\varepsilon((D_{\bar\phi}N)(A_1,A_2,\phi+\psi)D_{A_{1l}}\psi)]\big].$$

Hence,

$$\|D_{A_{1l}}\phi\|_* \le C(\|N(\phi+\psi)\|_{**} + \|D_{A_{1l}}N(A_1,A_2,\phi+\psi)\|_{**} + \|D_{\bar\phi}N(A_1,A_2,\psi+\phi)D_{A_{1l}}\psi\|_{**}),$$

thanks to (96). On the other hand, we get

$$\|N_\varepsilon(\phi+\psi)\|_{**} \le \begin{cases} C\varepsilon^2 & \text{if } n\le 6\\ C\varepsilon^{p\beta+1} & \text{if } n>6. \end{cases} \tag{104}$$

Thus, from (101) we have

$$\begin{aligned}|(D_{A_{1l}}N)(A_1,A_2,\bar\phi)| &\le C\bar v_0^{\frac{n-1}{n-2}}|(v_0+\bar\phi)_+^{p+\varepsilon-1} - v_0^{p+\varepsilon-1} - (p+\varepsilon-1)v_0^{p+\varepsilon-2}\bar\phi|\\ &\le C\bar v_0^{\frac{5}{n-2}+\varepsilon+\beta}\|\bar\phi\|_*,\end{aligned}$$

hence

$$\|(D_{A_{1l}}N)(A_1,A_2,\psi+\phi)\|_{**} \le C\|\phi+\psi\|_* \le C\varepsilon.$$

In similar way we get that

$$\|D_{\bar\phi}N(A_1,A_2,\psi+\phi)D_{A_{1l}}\psi\|_{**} \le C\varepsilon.$$

Hence, we finally get

$$\|D_{A_{1l}}\phi\|_* \le C\varepsilon,$$

as desired. A similar estimate holds for differentiation with respect to the other variables. This concludes the proof.

□

Proof of Proposition 4.4. We write

$$\begin{aligned}I(A_1,A_2) = J_\varepsilon((1+\zeta)(u_0+\tilde\phi)) - J_\varepsilon((u_0+\tilde\phi))\\ + J_\varepsilon((u_0+\tilde\phi)) - J_\varepsilon(PQ_1+PQ_2).\end{aligned}$$

Since $\tilde\phi(x) = \varepsilon^{-\frac12}\phi(\varepsilon^{-\frac{1}{n-2}}x)$, and $\|\phi\|_* \le C\varepsilon$, we have that

$$J_\varepsilon((1+\zeta)(u_0+\tilde\phi)) - J_\varepsilon((u_0+\tilde\phi)) = J_\varepsilon((1+\zeta)(u_0)) - J_\varepsilon((u_0)) + o(\varepsilon)$$

At this point, arguing like in the proof of Lemma 3.4 we are able to show that

$$J_\varepsilon((1+\zeta)(u_0)) - J_\varepsilon((u_0)) = 2\tilde\gamma_n\varepsilon\log\varepsilon + O(\varepsilon^2|\log\varepsilon|),$$

where $\tilde\gamma_n$ is a fixed constant, independent of ε. Observe also that

$$\nabla_{A_1,A_2}\left[J_\varepsilon((1+\zeta)(u_0)) - J_\varepsilon((u_0))\right] = O(\varepsilon^2|\log\varepsilon|)$$

uniformly for parameters A_1 and A_2 in the considered region. Given the result of Lemma 3.4, we need to show that

$$I(A_1,A_2) - J_\varepsilon(PQ_1+PQ_2) = o(\varepsilon) \tag{105}$$

and

$$\nabla_{A_1,A_2}[I(A_1,A_2) - J_\varepsilon(PQ_1+PQ_2)] = o(\varepsilon). \tag{106}$$

Recall now that $u_0 = PQ_1+PQ_2$. Let us apply a Taylor expansion

$$J_\varepsilon((u_0+\tilde\phi)) - J_\varepsilon(PQ_1+PQ_2) = \int_0^1 tdtD^2J_\varepsilon(u_0+t\tilde\phi)[\tilde\phi,\tilde\phi], \tag{107}$$

since $0 = DF_\varepsilon(v_0+\phi)[\phi] = (1+\zeta)^2DJ_\varepsilon(u_0\tilde\phi)[\tilde\phi]$. Now, from the definition of ϕ, we see that

$$\int_0^1 tdtD^2J_\varepsilon(u_0+t\tilde\phi)[\tilde\phi,\tilde\phi] = (1+\zeta)^2\int_0^1 tdtD^2F_\varepsilon(v_0+t\phi)[\phi,\phi] =$$

$$(1+\zeta)^2\int_0^1 tdt[\int_{\Omega_\varepsilon}|\nabla\phi|^2 - (p+\varepsilon)(v_0+t\phi)^{p+\varepsilon-1}\phi^2] =$$

$$(1+\zeta)^2\int_0^1 tdt(\int_{\Omega_\varepsilon}N(\phi+\psi)\phi + \int_{\Omega_\varepsilon}(p+\varepsilon)[v_0^{p+\varepsilon-1} - (v_0+\psi+t\phi)^{p+\varepsilon-1}]\phi^2). \tag{108}$$

Since, we recall $\|\phi\|_* + \|\psi\|_* = O(\varepsilon)$, the above relation together with (104) yield in particular,

$$I(A_1, A_2) - J_\varepsilon(u_0 + \tilde{\phi}) = \begin{cases} O(\varepsilon^2) & \text{if } n < 6 \\ O(\varepsilon^2 |\log \varepsilon|) & \text{if } n = 6 \\ O(\varepsilon^{1+\frac{4}{N-2}}) & \text{if } n \geq 7, \end{cases} \tag{109}$$

uniformly on A_1, A_2 in the considered region. Let us estimate now difference in derivatives. Define again $A_1' = (\Lambda_1, \xi_1', a_1, \theta_1)$ and $A_2' = (\Lambda_2, \xi_2', a_2, \theta_2)$. Let us fix $j = 1$ and define $A_1' = (A_{11}, A_{12}, \dots, A_{13n})$ the components of the vector A_1'. Let us differential ϕ with respect to A_{1l}, for some $l = 1, \dots, 3n$. Differentiating with respect to A_{1l} variables we get form (108) that

$$D_{A_{1l}}[I(A_1, A_2) - J_\varepsilon(u_0 + \tilde{\phi})] = (1-\zeta)^2 \int_0^1 tdt(\int_{\Omega_\varepsilon} D_{A_{1l}}[(N(\phi + \psi))\phi]$$
$$+ (p+\varepsilon) \int_{\Omega_\varepsilon} \nabla_{A_{1l}}[((v_0 + \psi + t\phi)^{p+\varepsilon-1} - v_0^{p+\varepsilon-1})\phi^2]).$$

Using the computations in the proof of Proposition 4.2 we get that

$$D_{A_{1l}}[I(A_1, A_2) - J_\varepsilon(u_0 + \tilde{\psi})] = o(\varepsilon).$$

Now,

$$J_\varepsilon(u_0 + \hat{\psi}) - J_\varepsilon(u_0) = (1-\zeta)^2 [F_\varepsilon(v_0 + \psi) - F_\varepsilon(v_0)] =$$

$$(1-\zeta)^2 \{ \int_0^1 (1-t)dt[(p+\varepsilon) \int_{\Omega_\varepsilon} ((v_0 + t\psi)^{p+\varepsilon-1} - v_0^{p+\varepsilon-1})\psi^2] - 2 \int_{\Omega_\varepsilon} E\psi \} \tag{110}$$

where we have used that

$$DF_\varepsilon(v_0)[\psi] = -\int_{\Omega_\varepsilon} E\psi.$$

Arguing as before and taking into account that (109) holds, we get (105). On the other hand, using (110), we see that

$$D_{A_{1l}}[J_\varepsilon(u_0 + \hat{\psi}) - J_\varepsilon(u_0)] = (1-\zeta)^2 D_{A_{1l}} \{ \int_0^1 (1-t)dt \times$$
$$[(p+\varepsilon) \int_{\Omega_\varepsilon} ((v_0 + t\psi)^{p+\varepsilon-1} - v_0^{p+\varepsilon-1})\psi^2] - 2 \int_{\Omega_\varepsilon} E\psi \}$$
$$= o(\varepsilon) - 2D_{A_{1l}}(\int_{\Omega_\varepsilon} E\psi).$$

On the other hand, we have that

$$D_{A_{1l}}(\int_{\Omega_\varepsilon} R^\varepsilon \psi) = \begin{cases} O(\varepsilon^{2-\frac{1}{n-2}}) & \text{if } n \leq 5 \\ O(\varepsilon^{\frac{7}{4}} |\log \varepsilon|) & \text{if } n = 6 \\ O(\varepsilon^{1+\frac{4}{n-2}-\frac{1}{n-2}}) & \text{if } n \geq 7. \end{cases}$$

This concludes the proof of the Proposition.

□

7. The min-max

In this section we set up a min-max scheme to find a critical point of the function Ψ, defined in (67). We write

$$\Psi(\Lambda, \xi, a, \theta) = \Psi(\Lambda_1, \Lambda_2, \xi_1, \xi_2, a_1, a_2, \theta_1, \theta_2),$$

where

$$(\Lambda, \xi, a, \theta) \in \mathbb{R}_+^2 \times (\Omega \times \Omega \setminus \{\xi_1 = \xi_2\}) \times B^2 \times K^2$$

where

$$B = \{(x_1, x_2) \,:\, \sqrt{x_1^2 + x_2^2} \le \frac{1}{2}\},$$

and K is a compact manifold of dimension $2n-3$, without boundary.

For $t = (t_1, t_2)$, define

$$\Psi_1(t, \xi, \theta) = H(\hat{\xi}_1, \hat{\xi}_1) t_1^2 + H(\hat{\xi}_2, \hat{\xi}_2) t_2^2 - 2G(\hat{\xi}_1, \hat{\xi}_2) t_1 t_2 + \log (t_1 t_2) . \tag{111}$$

We observe that

$$\Psi(\Lambda, \xi, a, \theta) = \Psi_1(\Lambda Q(a), \xi, \theta) + \Psi_2(a), \tag{112}$$

where $\Lambda Q(a) = (\Lambda_1 Q(\hat{a}_1), \Lambda_2 Q(\hat{a}_2))$ and

$$\Psi_2(a) = -\log (Q(a_1) Q(a_2)) + \frac{n-2}{\int_{\mathbb{R}^n} |Q|^{p+1}} \left[\sum_{j=1}^{2} \int |Q|^{p+1} \log \frac{1}{|y + a_j|} \right] . \tag{113}$$

Thus, we shall define a min-max scheme to find a critical point of the function $\hat{\Psi}$ given by

$$\hat{\Psi}(t, \xi, a, \theta) = \Psi_1(t, \xi, \theta) + \Psi_2(a) \tag{114}$$

for

$$(t, \xi, a, \theta) \in \mathbb{R}_+^2 \times (\Omega \times \Omega \setminus \{\xi_1 = \xi_2\}) \times B^2 \times K^2.$$

If we show that $\hat{\Psi}$ has a critical point of min-max type, we authomatically have that our original function Ψ has a critical point which is robust under small C^0 perturbation. As we explained in Section 4, this fact will conclude the proof of our result.

We start with the observation that the function Ψ_2 has a non degenerate minimum at $a_1 = 0$, $a_2 = 0$. Indeed, a Taylor expansion gives that

$$\begin{aligned}
\Psi_2(a) &= -2 \log Q(0) + 2 \frac{n-2}{\int_{\mathbb{R}^n} |Q|^{p+1}} \int |Q|^{p+1} \log \frac{1}{|y|} \, dy \\
&\quad + \frac{n-2}{4n} \left[2n \int |Q|^{p+1} - (2n-1) \int \frac{|Q|^{p+1}}{|y|^2} \, dy \right] \left(|a_1|^2 + |a_2|^2 \right) \\
&\quad + o\left(|a_1|^2 + |a_2|^2 \right)
\end{aligned}$$

as $\left(|a_1|^2 + |a_2|^2 \right) \to 0$. On the other hand, we have

$$\left[2n \int |Q|^{p+1} - (2n-1) \int \frac{|Q|^{p+1}}{|y|^2} \, dy \right] > 0.$$

The above inequality is consequence of (12)–(13) and of a direct computations for

$$\int_{\mathbb{R}^n} \frac{\bar{w}^{p+1}}{|y|^2} \, dy = \frac{3}{(n+1)(n-2)} \int_{\mathbb{R}^n} \bar{w}^{p+1} \tag{115}$$

where $\bar{w}$ is given by (5). Formula (115) follows from

$$\int_0^\infty \left(\frac{r}{1+r^2} \right)^2 \frac{1}{r^{\alpha+1}} \, dr = \frac{\Gamma(\frac{q-\alpha}{2}) \, \Gamma(\frac{q+\alpha}{2})}{2\Gamma(q)}.$$

We recall now that $\Omega = \mathcal{D} \setminus \omega$, where $\omega \subset \bar{B}(0, \delta) \subset \mathcal{D}$, and define

$$\varphi(\xi_1, \xi_2) = H(\xi_1, \xi_1)^{\frac{1}{2}} \, H(\xi_2, \xi_2)^{\frac{1}{2}} - G(\xi_1, \xi_2).$$

The following result holds true (see Corollary 2.1, [13])

Corollary 7.1. *For any (fixed) sufficiently small $\sigma > 0$ there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and for any smooth domain $\omega \subset B(0, \delta)$ it holds*

$$\varphi(\xi_1, \xi_2) < 0 \qquad \forall\ (\xi_1, \xi_2) \in S,$$

where the manifold S is defined by

$$S = \{(\xi_1, \xi_2) \in \Omega^2 \mid |x_1| = |x_2| = \sigma\}.$$

For any $\xi = (\xi_1, \xi_2) \in S$ we let $d(\xi) = (d_1(\xi), d_2(\xi)) \in \mathbb{R}^2_+$ be the negative direction of the quadratic form defining Ψ. We easily see that there is a constant $c > 0$ so that $c < d_1(\xi)d_2(\xi) < c^{-1}$ for all $\xi \in S$.

We have now the tools to construct a critical point of "min-max" type of the function $\hat{\Psi}$, defined in (114). This construction has similarities with the ones developed in [13] and [20]. Let us introduce for $l > 0$ and $\rho > 0$ the following manifold

$$W^l_\rho = \{\xi \in \Omega^2 \mid \varphi(\xi) < -l\} \cap V_\rho,$$

where

$$V_\rho = \{(\xi_1, \xi_2) \in \Omega^2 \mid \operatorname{dist}(\xi_1, \partial\Omega) > \rho,\ \operatorname{dist}(\xi_2, \partial\Omega) > \rho,\ |\xi_1 - \xi_2| > \rho\}.$$

If we take $\lambda_0 = -\max_{\xi \in S} \varphi(\xi)$ and $\rho_0 = \operatorname{dist}(S, \partial\Omega)$, then for any $\rho \in (0, \rho_0)$ and $l \in (0, l_0)$ we have that $S \subset W^l_\rho$. Moreover, for any $R > 1$

$$\max_{\substack{\xi \in S,\, R^{-1} \le r \le R \\ a \in B^2,\, \theta \in K^2}} \hat{\Psi}(rd(\xi), \xi, a, \theta) > \max_{\substack{\xi \in S,\, r = R^{-1}, R, \\ a \in B^2,\, \theta \in K^2}} \hat{\Psi}(rd(\xi), \xi, a, \theta), \tag{116}$$

where $d(\xi) = \big(d_1(\xi), d_2(\xi)\big) \in \mathbb{R}^2_+$ is the negative direction of the quadratic form defining Ψ. This is a direct consequence of Corollary 7.1.

Now let A and B be fixed numbers defined as follows

$$B = \max_{\substack{\xi \in S,\, R^{-1} \le r \le R \\ a \in B^2,\, \theta \in K^2}} \hat{\Psi}(rd(\xi), \xi, a, \theta) > A > \max_{\substack{\xi \in S,\, r = R^{-1}, R, \\ a \in B^2,\, \theta \in K^2}} \hat{\Psi}(rd(\xi), \xi, a, \theta). \tag{117}$$

There exists $R > 0$ large such that for any $l \in (0, l_0)$ it holds

$$\begin{aligned} B = \max_{\substack{\xi \in S,\, R^{-1} \le r \le R \\ a \in B^2,\, \theta \in K^2}} \hat{\Psi}(rd(\xi), \xi, a, \theta) &\ge \max_{\substack{\xi \in S,\, \Lambda \in I \\ a \in B^2,\, \theta \in K^2}} \hat{\Psi}(t, \xi, a, \theta) \\ \ge \max_{\substack{\xi \in W^l_\rho,\, \Lambda \in I \\ a \in B^2,\, \theta \in K^2}} \hat{\Psi}(rd(\xi), \xi, a, \theta) > A &> \max_{\substack{\xi \in S,\, r = R^{-1}, R, \\ a \in B^2,\, \theta \in K^2}} \hat{\Psi}(rd(\xi), \xi, a, \theta), \end{aligned} \tag{118}$$

where I is the hyperbola in $\mathbb{R}^2_+$ defined by $I = \{t \in \mathbb{R}^2_+ \mid t_1 t_2 = 1\}$. Indeed, for any $t \in I$, we have

$$\hat{\Psi}(t, \xi, a, \theta) \ge -G(\xi_1, \xi_2) \ge -\frac{1}{\rho^{n-2}} > A, \tag{119}$$

provided that R is choosen properly.

Lemma 7.2. *There exist $l_0 > 0$ and $\rho_0 > 0$ such that for any $l \in (0, l_0)$ and $\rho \in (0, \rho_0)$ the function Ψ satisfies the following property:*

for any sequence $(t_n, \xi_n, a_n, \theta_n)$ in $[R^{-1}, R]^2 \times W^l_\rho \times B^2_2 \times K^2$ such that

$$\lim_n (t_n, \xi_n, a_n, \theta_n) = (t, \xi, a, \theta) \in \partial([R^{-1}, R]^2 \times W^l_\rho \times B^2 \times K^2)$$

and $\hat{\Psi}(t_n, \xi_n, a_n, \theta_n) \in [A, B]$ there exists a vector T tangent to $\partial([R^{-1}, R]^2 \times W^l_\rho \times B^2 \times K^2)$ at (t, ξ, a, θ), such that

$$\nabla\hat{\Psi}(t, \xi, a, \theta) \cdot T \neq 0.$$

Proof. First of all we observe that t_n is component-wise bounded from below and from above by a positive constant. In fact, if $|t_n| \to +\infty$ and $|t_n| \to 0$ then $|\hat{\Psi}(t_n, \xi_n, a_n, \theta_n)| \to +\infty$, which is impossible. Thus we have that $t \notin \{R^{-1}, R\}$. Furthermore, since $a \to \hat{\Psi}$ has a minimum in B, if $a \in \partial B^2$, then $\nabla_a \hat{\Psi} \neq 0$, and we can choose $T = \nabla_a \hat{\Psi}$. Similarly, if $\nabla_t \hat{\Psi}(t, \xi, a, \theta) \neq 0$, then T can be chosen parallel to $\nabla_t \hat{\Psi}(t, \xi)$. Then assume that $\nabla_t \hat{\Psi}(t, \xi, a, \theta) = 0$, t satisfies

$$t_1^2 = -\frac{H(\hat{\xi}_2, \hat{\xi}_2)^{1/2}}{H(\hat{\xi}_1, \hat{\xi}_1)^{1/2} \varphi(\hat{\xi}_1, \hat{\xi}_2)}, \quad t_2^2 = -\frac{H(\hat{\xi}_1, \hat{\xi}_1)^{1/2}}{H(\hat{\xi}_2, \hat{\xi}_2)^{1/2} \varphi(\hat{\xi}_1, \hat{\xi}_2)},$$

and $\hat{\xi}$ satisfies $\varphi(\hat{\xi}) < 0$. Substituting back in $\hat{\Psi}$, we get

$$\hat{\Psi} = -\frac{1}{2} + \frac{1}{2} \log \frac{1}{|\varphi(\hat{\xi}_1, \hat{\xi}_2)|}.$$

Thus we conclude the proof of the Lemma, using the following result proved in [13].

Lemma 7.3. *Given $c < 0$ there exists a sufficiently small number $\rho > 0$ with the following property: If $(\bar{\xi}_1, \bar{\xi}_2) \in \partial(\Omega_\rho \times \Omega_\rho)$ is such that $\varphi(\bar{\xi}_1, \bar{\xi}_2) = c$, then there is a vector τ, tangent to $\partial(\Omega_\rho \times \Omega_\rho)$ at the point $(\bar{\xi}_1, \bar{\xi}_2)$, so that*

$$\nabla \varphi(\bar{\xi}_1, \bar{\xi}_2) \cdot \tau \neq 0. \tag{120}$$

The number ρ does not depend on c.

□

We now have the tools to show the validity of the following fact

Proposition 7.4. *There exists a critical level for $\hat{\Psi}$ between A and B.*

Proof. First we claim that the function $\hat{\Psi}$ constrained to $\mathbb{R}_+^2 \times W_\rho^l \times B_2^2 \times K^2$ satisfies the Palais-Smale condition in $[A, B]$. Indeed, let $(t_n, \xi_n, a_n, \theta_n)$ in $\mathbb{R}_+^2 \times W_\rho^l \times B^2 \times K^2$ be such that $\lim_n \hat{\Psi}(t_n, \xi_n, a_n, \theta_n) \in [A, B]$ and $\lim_n \nabla \hat{\Psi}(t_n, \xi_n, a_n, \theta_n) = 0$. Arguing as in the proof of Lemma 7.2 it can be shown that t_n remains bounded component-wise from above and below by a positive constant.

Assume now by contradiction that there are no critical levels in the interval $[A, B]$. We can define an appropriate negative gradient flow that will remain in $[R^{-1}, R]^2 \times W_\rho^l \times B_2^2 \times K^2$ at any level $c \in [A, B]$. Moreover the Palais-Smale condition holds in $[A, B]$. Hence there exists a continuous deformation

$$\eta : [0, 1] \times \hat{\Psi}^B \to \hat{\Psi}^B$$

such that for some $A' \in (0, A)$

$$\eta(0, u) = u \qquad \forall\, u \in \hat{\Psi}^B, \quad \eta(t, u) = u \qquad \forall\, u \in \hat{\Psi}^{A'}, \quad \eta(1, u) \in \hat{\Psi}^{A'}.$$

Let us define the sets $\mathcal{C} = I \times W_\rho^l$,

$$\mathcal{A} = \{(t, \xi, a, \theta) \in \mathbb{R}_+^2 \times W_\rho^l \times B^2 \times K^2 \mid \xi \in S,\ t = rd(\xi),\ R^{-1} \leq r \leq R\},$$

$$\partial\mathcal{A} = \{(t, \xi, a, \theta) \in \mathbb{R}_+^2 \times W_\rho^l \times B^2 \times K^2 \mid \xi \in S,\ t = R^{-1} \text{ or } t = Rd(\xi)\}.$$

From (118) we deduce that $\mathcal{A} \subset \hat{\Psi}^B$, $\partial\mathcal{A} \subset \hat{\Psi}^{A'}$ and $\hat{\Psi}^{A'} \cap \mathcal{C} = \emptyset$. Therefore

$$\eta(0, u) = u \qquad \forall\, u \in \mathcal{A}, \quad \eta(t, u) = u \qquad \forall\, u \in \partial\mathcal{A}, \quad \eta(1, \mathcal{A}) \cap \mathcal{C} = \emptyset. \tag{121}$$

For any $(t, \xi, a, \theta) \in \mathcal{A}$ and for any $s \in [0, 1]$ we denote

$$\eta\big(s, (t, \xi, a, \theta)\big) = \big(\widetilde{t}, \widetilde{\xi}, \widetilde{a}, \widetilde{\theta}\big) \in \mathbb{R}_+^2 \times W_\rho^l \times B^2 \times K^2.$$

We define the set

$$\mathcal{B} = \{(t,\xi) \in \mathcal{A} \mid \widetilde{\Lambda} \in I\}.$$

Since $\eta(1,\mathcal{A})\cap\mathcal{C} = \emptyset$ it holds $\mathcal{B} = \emptyset$. Now let $\mathcal{U}$ be a neighborhood of $\mathcal{B}$ in $\mathbb{R}^2_+ \times W^l_\rho \times B^2 \times K^2$ such that $H^*(\mathcal{U}) = H^*(\mathcal{B})$. If $\pi : \mathcal{U} \to \mathcal{S}$ denotes the projection, arguing like in Lemma 7.1 of [13] we can show that

$$\pi^* : H^*(\mathcal{S}) \to H^*(\mathcal{U}) \quad \text{is a monomorphism.}$$

This condition provides a contradiction, since $H^*(\mathcal{U}) = \{0\}$ and $H^*(\mathcal{S}) \neq \{0\}$. □

8. Appendix

To give a first description of these solutions, let us introduce some notations. Fix an integer k. For any integer $l = 1,\dots,k$, we define angles θ_l and vectors n_l, t_l by

$$\theta_l = \frac{2\pi}{k}(l-1), \quad \mathsf{n}_l = (\cos\theta_l, \sin\theta_l, \mathsf{0}), \quad \mathsf{t}_l = (-\sin\theta_l, \cos\theta_l, \mathsf{0}). \tag{122}$$

Here $\mathsf{0}$ stands for the zero vector in $\mathbb{R}^{n-2}$. Notice that $\theta_1 = 0$, $\mathsf{n}_1 = (1, 0, \mathsf{0})$, and $\mathsf{t}_1 = (0, 1, \mathsf{0})$.

In [16] it was proved that there exists k_0 such that for all integer $k > k_0$ there exists a solution $Q = Q_k$ to (10) that can be described as follows

$$Q_k(x) = U_*(x) + \tilde{\phi}(x). \tag{123}$$

where

$$U_*(x) = U(x) - \sum_{j=1}^{k} U_j(x), \tag{124}$$

while $\tilde{\phi}$ is smaller than U_*. The functions U and U_j are positive solutions to (10), respectively defined as

$$U(x) = \gamma\left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}, \quad U_j(x) = \mu_k^{-\frac{n-2}{2}} U(\mu_k^{-1}(x-\xi_j)), \tag{125}$$

where $\gamma = \left[\frac{n(n-2)}{4}\right]^{\frac{n-2}{4}}$. For any integer k large, the parameters $\mu_k > 0$ and the k points ξ_l, $l = 1,\dots,k$ are given by

$$\left[\sum_{l>1}^{k} \frac{1}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\right] \mu_k^{\frac{n-2}{2}} = \left(1 + O(\frac{1}{k})\right), \quad \text{for} \quad k \to \infty \tag{126}$$

in particular $\mu_k \sim k^{-2}$ if $n \geq 4$, and $\mu_k \sim k^{-2}|\log k|^{-2}$ if $n = 3$, as $k \to \infty$, and

$$\xi_l = \sqrt{1-\mu^2}\,(\mathsf{n}_l, 0). \tag{127}$$

The function $\tilde{\phi}$ in (123) can be further decomposed. Let us introduce some cut-off functions ζ_j to be defined as follows. Let $\zeta(s)$ be a smooth function such that $\zeta(s) = 1$ for $s < 1$ and $\zeta(s) = 0$ for $s > 2$. We also let $\zeta^-(s) = \zeta(2s)$. Then we set

$$\zeta_j(y) = \begin{cases} \zeta(\,k\eta^{-1}|y|^{-2}|(y - \xi|y|)\,|\,) & \text{if } |y| > 1\,, \\ \zeta(\,k\eta^{-1}\,|y-\xi|\,) & \text{if } |y| \leq 1\,, \end{cases}$$

in such a way that that

$$\zeta_j(y) = \zeta_j(y/|y|^2).$$

The function $\tilde{\phi}$ has the form

$$\tilde{\phi} \;=\; \sum_{j=1}^{k} \tilde{\phi}_j + \psi. \tag{128}$$

In the decomposition (128) the functions $\tilde{\phi}_j$, for $j > 1$, are defined in terms of $\tilde{\phi}_1$

$$\tilde{\phi}_j(\bar{y}, y') = \tilde{\phi}_1(e^{\frac{2\pi j}{k} i}\bar{y}, y'), \quad j = 1, \ldots, k-1. \tag{129}$$

We have that

$$\|\psi\|_{n-2} \le Ck^{1-\frac{n}{q}} \quad \text{if} \quad n \ge 4, \quad \|\psi\|_{n-2} \le \frac{C}{\log k} \quad \text{if} \quad n = 3, \tag{130}$$

where

$$\|\phi\|_{n-2} := \|\,(1+|y|^{n-2})\phi\,\|_{L^\infty(\mathbb{R}^n)}\ . \tag{131}$$

On the other hand, if we rescale and translate the function $\tilde{\phi}_1$

$$\phi_1(y) = \mu^{\frac{n-2}{2}} \tilde{\phi}_1(\xi_1 + \mu y) \tag{132}$$

we have the validity of the following estimate for ϕ_1

$$\|\phi_1\|_{n-2} \le Ck^{-\frac{n}{q}} \quad \text{if} \quad n \ge 4, \quad \|\phi_1\|_{n-2} \le \frac{C}{k \log k} \quad \text{if} \quad n = 3. \tag{133}$$

In terms of the function $\tilde{\phi}$ in the decomposition (123), equation (10) gets re-written as

$$\Delta\tilde{\phi} + p\gamma|U_*|^{p-1}\tilde{\phi} + E + \gamma N(\tilde{\phi}) = 0 \tag{134}$$

where E is defined by

$$E = \Delta U_* + f(U_*) \tag{135}$$

and

$$N(\phi) = |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\phi.$$

One has a precise control of the size of the function E when measured for instance in the following norm. Let us fix a number q, with $\frac{n}{2} < q < n$, and consider the weighted L^q norm

$$\|h\|_{**} = \|\,(1+|y|)^{n+2-\frac{2n}{q}} h\|_{L^q(\mathbb{R}^n)}. \tag{136}$$

In [16] it is proved that there exist an integer k_0 and a positive constant C such that for all $k \ge k_0$ the following estimates hold true

$$\|E\|_{**} \le Ck^{1-\frac{n}{q}} \quad \text{if} \quad n \ge 4, \quad \|E\|_{**} \le \frac{C}{\log k} \quad \text{if} \quad n = 3 \tag{137}$$

To be more precise, we have estimates for the $\|\cdot\|_{**}$-norm of the error term E in the *exterior region* $\bigcap_{j=1}^k \{|y-\xi_j| > \frac{\eta}{k}\}$, and also in the *interior regions* $\{|y-\xi_j| < \frac{\eta}{k}\}$, for any $j = 1, \ldots, k$. Here $\eta > 0$ is a positive and small constant, independent of k.

In the exterior region. We have

$$\|\,(1+|y|)^{n+2-\frac{2n}{q}} E(y)\|_{L^q(\bigcap_{j=1}^k \{|y-\xi_j|>\frac{\eta}{k}\})} \le Ck^{1-\frac{n}{q}}$$

if $n \ge 4$, while

$$\|\,(1+|y|)^{n+2-\frac{2n}{q}} E(y)\|_{L^q(\bigcap_{j=1}^k \{|y-\xi_j|>\frac{\eta}{k}\})} \le \frac{C}{\log k}$$

if $n = 3$.

In the interior regions. Now, let $|y-\xi_j|<\frac{\eta}{k}$ for some $j\in\{1,\dots,k\}$ fixed. It is convenient to measure the error after a change of scale. Define

$$\tilde E_j(y):=\mu^{\frac{n+2}{2}}E(\xi_j+\mu y),\quad |y|<\frac{\eta}{\mu k}$$

We have

$$\|\,(1+|y|)^{n+2-\frac{2n}{q}}\tilde E_j(y)\|_{L^q(|y-\xi_j|<\frac{\eta}{\mu k})}\le Ck^{-\frac{n}{q}}\quad\text{if}\quad n\ge 4$$

and

$$\|\,(1+|y|)^{n+2-\frac{2n}{q}}\tilde E_j(y)\|_{L^q(|y-\xi_j|<\frac{\eta}{\mu k})}\le \frac{C}{k\log k}\quad\text{if}\quad n=3.$$

We refer the readers to [16].

Let us now define the following functions

$$\begin{aligned}\pi_\alpha(y)&=\tfrac{\partial}{\partial y_\alpha}\tilde\phi(y),\quad\text{for}\quad \alpha=1,\dots,n;\\ \pi_0(y)&=\tfrac{n-2}{2}\tilde\phi(y)+\nabla\tilde\phi(y)\cdot y.\end{aligned}\tag{138}$$

In the above formula $\tilde\phi$ is the function defined in (123) and described in (128). Observe that the function π_0 is even in each of its variables, namely

$$\pi_0(y_1,\dots,y_j,\dots,y_n)=\pi_0(y_1,\dots,-y_j,\dots,y_n)\quad\forall j=1,\dots,n,$$

while π_α, for $\alpha=1,\dots,n$ is odd in the y_α variable, while it is even in all the other variables. Furthermore, all functions π_α are invariant under rotation of $\frac{2\pi}{k}$ in the first two coordinates, namely they satisfy (17). The functions π_α can be further described, as follows.

The functions π_α can be decomposed into

$$\pi_\alpha(y)=\sum_{j=1}^k\tilde\pi_{\alpha,j}(y)+\hat\pi_\alpha(y)\tag{139}$$

where

$$\tilde\pi_{\alpha,j}(y)=\tilde\pi_{\alpha,1}(e^{\frac{2\pi}{k}j\,i}\bar y,y').$$

Furthermore, there exists a positive constant C so that

$$\|\hat\pi_0\|_{n-2}\le Ck^{1-\frac{n}{q}},\quad \|\hat\pi_j\|_{n-1}\le Ck^{1-\frac{n}{q}},\quad j=1,\dots,k,$$

if $n\ge 4$, and

$$\|\hat\pi_0\|_{n-2}\le \frac{C}{\log k},\quad \|\hat\pi_j\|_{n-1}\le \frac{C}{\log k},\quad j=1,\dots,k,$$

if $n=3$. Furthermore, if we denote $\pi_{\alpha,1}(y)=\mu^{\frac{n-2}{2}}\tilde\pi_{\alpha,1}(\xi_1+\mu y)$, then

$$\|\pi_{0,1}\|_{n-2}\le Ck^{-\frac{n}{q}},\quad \|\pi_{\alpha,1}\|_{n-1}\le Ck^{-\frac{n}{q}},\quad \alpha=1,\dots,n$$

if $n\ge 4$, and

$$\|\pi_{0,1}\|_{n-2}\le \frac{C}{k\log k},\quad \|\pi_{\alpha,1}\|_{n-1}\le C\frac{C}{k\log k},\quad \alpha=1,\dots,3$$

if $n=3$.

For further details we refer the interested reader to [16].

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