SIGN-CHANGING BLOW-UP SOLUTIONS FOR YAMABE PROBLEM

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Abstract: Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. We are concerned with the following elliptic problem

$$\Delta_g u + hu = |u|^{\frac{4}{n-2}-\varepsilon}u, \qquad in \ \ M,$$

where $\Delta_g = -div_g(\nabla)$ is the Laplace-Beltrami operator on M, h is a $\mathcal{C}^1$ function on M, ε is a small real parameter such that ε goes to 0.

1. Introduction

Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, where g denotes the metric tensor. We are interested in the following asymptotically critical elliptic equation

$$\Delta_g u + hu = |u|^{\frac{4}{n-2}-\varepsilon}u, \qquad in \ \ M, \tag{1.1}$$

where $\Delta_g = -div_g(\nabla)$ is the Laplace-Beltrami operator on M, h is a $\mathcal{C}^1$ function on M, ε is a small real parameter such that $\varepsilon \to 0$.

If $h \equiv \frac{n-2}{4(n-1)}\mathrm{Scal}_g$, the problem

$$\Delta_g u + \frac{n-2}{4(n-1)}\mathrm{Scal}_g u = u^{2^*-1-\varepsilon} \quad \text{in} \ \ M \quad u > 0 \ \ \text{in} \ \ M, \tag{1.2}$$

is just the so called prescribed scalar curvature problem with $\varepsilon = 0$, where $2^* = \frac{2n}{n-2}$. The existence of a conformal metric with constant scalar curvature on compact Riemannian manifolds was studied by Yamabe [26], Trudinger [25], Aubin [1] and Schoen [24]. If u is a solution, then $\frac{4(n-1)}{n-2}$ is the scalar curvature of the conformal metric $\tilde{g} = u^{\frac{1}{n-2}}g$.

Recently, nonlinear elliptic equations on compact Riemannian manifold have been brought much attention. Consider the following problem

$$\varepsilon^2 \Delta_g u + u = |u|^{p-2}u \quad \text{in} \ \ M, \tag{1.3}$$

where (M,g) is a compact, connected, Riemannian manifold of class C^∞ with Riemannian metric g, $\dim M = n \geq 3$, $2 < p < \frac{2n}{n-2}$ and ε is a positive parameter. In [4], the authors proved that the problem (1.3) has a mountain pass solution u_ε which exhibits a spike layer. In particular, they proved that the maximum point of u_ε converges to a maximum point of the scalar curvature Scal_g as ε goes to zero. Multiple solutions were obtained in [2] for the problem (1.3), the authors showed that multiplicity of solutions to (1.3) depends on the topological properties of the manifold M. More precisely, they showed that problem (1.3) has at least $cat(M)+1$ nontrivial solutions provided ε is small enough. Here $cat(M)$ denotes the Lusternik-Schnirelmann category of M. In [15] the authors showed that for any stable critical point of the scalar curvature it is possible to construct a single peak solution, whose peak approaches

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such a point as ε goes to zero. In [6] the authors proved that for any fixed positive integer k, problem (1.3) has a $k-$peak solution, whose peaks collapse, as ε goes to zero, to an isolated local minimum point of the scalar curvature. Recently in [16] the authors proved that the existence of positive or sign changing multi-peak solutions of (1.3), whose both positive and negative peaks approach different stable critical points of the scalar curvature as ε goes to zero.

Regarding the asymptotically critical case (1.1) on Riemannian manifolds there are also intensive research on the existence of positive blowing-up solutions: see for instance [3] for the Yamabe equation, [9], [12], [17] for perturbations of the Yamabe equation, [5], [13] for equations on the sphere, and the references therein. In terms of sign-changing bubbling solutions, in [21–23], the authors constructed a new kind of sign-changing bubbling solution to (1.1) by imposing a negative bubble on the top of a positive solution to the Yamabe problem. In [20] the authors constructed sign-changing bubbling towers for (1.1).

In all the papers mentioned above, the canonical profile of bubbling is the positive solution to

$$\Delta u+|u|^{p-1}u=0 \quad \text{in} \quad \mathbb{R}^n, \quad p=\frac{n+2}{n-2}, \tag{1.4}$$

which can be written explicitly

$$U_{\lambda,\xi}=c_n\Big(\frac{\lambda}{\lambda^2+|x-\xi|^2}\Big)^{\frac{n-2}{2}}. \tag{1.5}$$

In this paper we are interested in gluing more complicated *sign-changing* solutions of (1.6) on Riemannian manifolds. More precisely the canonical profile is the sign-changing solution to (1.1) on the canonical sphere constructed in [7]. In [7] it is proven that there exists an integer K_0 such that for any integer $K\geq K_0$, a solution solution $Q=Q_K$ to Problem

$$\Delta u+|u|^{p-1}u=0 \quad \text{in} \quad \mathbb{R}^n, \quad p=\frac{n+2}{n-2}, \tag{1.6}$$

exists. Moreover, if we define the energy by

$$E(u)=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2\,dy-\frac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}\,dy, \tag{1.7}$$

we have

$$E(Q_K)=\begin{cases}(K+1)\,S_n\,\left(1+O(K^{2-n})\right) & \text{if } n\geq 4\,,\\ (K+1)\,S_3\,\left(1+O(K^{-1}|\log K|^{-1}\right) & \text{if } n=3\end{cases}$$

as $K\to\infty$, where S_n is a positive constant, depending on n. The solution $Q=Q_K$ decays at infinity like the fundamental solution, namely

$$\lim_{|y|\to\infty}|y|^{n-2}\,Q_K(y)=\left(\frac{4}{n(n-2)}\right)^{\frac{n-2}{4}}\,2^{\frac{n-2}{2}}\,(1+c_K) \tag{1.8}$$

where

$$c_K=\begin{cases}O(K^{-1}) & \text{if } n\geq 4\,,\\ O(K^{-1}|\log K|^2) & \text{if } n=3\end{cases} \quad \text{as} \quad K\to\infty.$$

Furthermore, the solution $Q=Q_K$ has a positive global non degenerate maximum at $y=0$. To be more precisely we have

$$Q(y)=[n(n-2)]^{\frac{n-2}{4}}\left(1-\frac{n-2}{2}|y|^2+O(|y|^3)\right) \quad \text{as} \quad |y|\to 0, \tag{1.9}$$

and also there exists $\eta > 0$, depending on K_0, but independent of K, so that

$$\eta \le Q(y) \le Q(0) \quad \text{for all} \quad |y| \le \frac{1}{2}, \tag{1.10}$$

for any K. Another property for the solution $Q = Q_K$ is that it is invariant under rotation of angle $\frac{2\pi}{K}$ in the y_1, y_2 plane, namely

$$Q(e^{\frac{2\pi}{K}} \bar{y}, y') = Q(\bar{y}, y'), \quad \bar{y} = (y_1, y_2), \quad y' = (y_3, \ldots, y_n). \tag{1.11}$$

It is even in the y_j-coordinates, for any $j = 2, \ldots, n$

$$Q(y_1, \ldots, y_j, \ldots, y_n) = Q(y_1, \ldots, -y_j, \ldots, y_n), \quad j = 2, \ldots, n. \tag{1.12}$$

It respects invariance under Kelvin's transform:

$$Q(y) = |y|^{2-n} Q(\frac{y}{|y|^2}). \tag{1.13}$$

These solutions are non-degenerate, as proved in [18], in the sense precisely in Section 6.2. More precisely, the dimensional of the kernels of the linearized operator at Q

$$-\Delta \phi = p|Q|^{p-1}\phi$$

is shown to be $3n$.

In this paper, we will use Q_K to construct sign changing solutions to problem (1.1). It was used to construct sign-changing blowing-up solutions for supercritical Bahri-Coron's problem in a bounded domain of $\mathbb{R}^n$ in the recently work [19].

For $\xi \in M$, we define the function,

$$\varphi(\xi) = h(\xi) - \frac{n-2}{4(n-1)} \left(1 + \frac{n-4}{3n} K\right) Scal_g(\xi). \tag{1.14}$$

We have the validity of the following result.

Theorem 1.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 5$. Let h be a C^1 function on M such that the operator $\Delta_g + h$ is coercive, and let ξ_0 be a C^1-stable critical point of the function $\varphi(\xi)$, and $\varphi(\xi) sign(\varepsilon) > 0$. Then there exists an integer K_0 such that for any integer $K \ge K_0$, there exists ε_K, such that for any $\varepsilon \in (0, \varepsilon_K)$, the problem (1.1) has a sign changing solution u_ε.*

This paper is organized as follows. In Section 2, we we introduce some framework and preliminary results. The proof of the main result is given in Section 3. Section 4 is devoted to perform the finite dimensional reduction. Section 5 contains the asymptotic expansion of the reduced energy. In Appendix, we will recall the construction of sign changing solution Q_K and its non-degenerate, and we also give some useful technical estimates.

2. Some preliminary results

Let M be a compact Riemannian manifold of class $\mathcal{C}^\infty$. On the tangent bundle of M it is defined the exponential map $\exp : TM \to M$ which has the following properties:

(i) exp is of class C^∞;

(ii) there exists a constant $r > 0$ such that $\exp_\xi\big|_{B(0,r)} : B(0,r) \to B_g(\xi, r)$ is a diffeomorphism for all $\xi \in M$.

where $B(0,r)$ denotes the ball in $\mathbb{R}^n$ centered at 0 with radius r and $B_g(\xi, r)$ denotes the ball in M centered at ξ with radius r with respect to the distance induced by the metric g.

Geodesic normal coordinates

$$\exp_\xi : T_\xi M \supset V \to M$$

and an isomorphism

$$E : \mathbb{R}^N \to T_\xi M$$

given by any basis of the tangent space at the fixed basepoint $\xi \in M$. If the additional structure of a Riemannian metric is imposed, then the basis defined by E may be required in addition to be orthonormal, and the resulting coordinate system is then known as a Riemannian normal coordinate system.

Normal coordinates exist on a normal neighborhood of a point ξ in M. A normal neighborhood U is a subset of M such that there is a proper neighborhood V of the origin in the tangent space $T_\xi M$ and $\exp_\xi$ acts as a diffeomorphism between U and V. Now let U be a normal neighborhood of ξ in M then the chart is given by:

$$\varphi := E^{-1} \circ \exp_\xi^{-1} : U \to \mathbb{R}^N$$

The isomorphism E can be any isomorphism between both vectorspaces, so there are as many charts as different orthonormal bases exist in the domain of E.

Fix such an r in this paper with $r < i_g/2$, where i_g denotes the injectivity radius of (M, g). Let $\mathfrak{C}$ be the atlas on M whose charts are given by the exponential map and $\mathcal{P} = \{\psi_\omega\}_{\omega \in \mathfrak{C}}$ be a partition of unity subordinate to the atlas $\mathfrak{C}$. For $u \in H_g^1(M)$, we have

$$\int_M |\nabla_g u|^2 \, dv_g = \sum_{\omega \in \mathfrak{C}} \int_\omega \psi_\omega(x) |\nabla_g u|^2 \, dv_g,$$

where $dv_g = \sqrt{\det g}\, dz$ denotes the volume form on M associated to the metric g. Moreover, if u has support inside one chart $\omega = B_g(\xi, r)$, then

$$\int_\omega |\nabla_g u|^2 \, dv_g = \int_{B(0,r)} \left(\sum_{a,b=1}^n g_\xi^{ab}(z) \frac{\partial u(\exp_\xi(z))}{\partial z_a} \frac{\partial u(\exp_\xi(z))}{\partial z_b} \right) |g_\xi(z)|^{\frac{1}{2}} \, dz,$$

where g_ξ denotes the Riemannian metric reading in $B(0, r)$ through the normal coordinates defined by the exponential map $\exp_\xi$ at ξ. We denote $|g_\xi(z)| := \det(g_\xi(z))$ and $(g_\xi^{ab})(z)$ is the inverse matrix of $g_\xi(z)$. In particular, it holds

$$g_\xi^{ab}(0) = \delta_{ab}, \qquad g_\xi(0) = Id,$$

where δ_{ab} is the Kronecker symbol and

$$\frac{\partial g_\xi^{ab}}{\partial z_c}(0) = 0 \quad \text{for any} \quad a, b, c.$$

Since M is compact, there are two strictly positive constants C and $\tilde{C}$ such that

$$\forall\ \xi \in M, \quad \forall\ \nu \in T_\xi M, \quad C\|\nu\|^2 \le g_\xi(\nu, \nu) \le \tilde{C}\|\nu\|^2.$$

Hence, we have

$$\forall\ \xi \in M, \quad C^n \le |g_\xi| \le \tilde{C}^n.$$

Let L^q be the Banach space $L^q(M)$ with the norm

$$|u|_q = \left(\int_M |u|^q \, dv_g \right)^{1/q}.$$

Since the operator $\Delta_g + h$ is coercive, the Sobolev space $H_1^2(M)$ is endowed with the scalar product $\langle \cdot, \cdot \rangle_h$ defined by

$$\langle u, v \rangle_h = \int_M \langle \nabla u, \nabla v \rangle_g dv_g + \int_M huv dv_g \tag{2.1}$$

for all $u, v \in H_1^2(M)$. We let $\|\cdot\|_h$ be the norm induced by $\langle\cdot,\cdot\rangle_h$, this norm is equivalent to the standard norm on $H_1^2(M)$.

It is clear that the embedding $i : H_1^2(M) \hookrightarrow L^{2^*}(M)$ is a continuous map. We let $i^* : L^{2n/(n+2)}(M) \hookrightarrow H_1^2(M)$ be the adjoint operator of the embedding i, the embedding i^* is a continuous map such that for any w in $L^{2n/(n+2)}(M)$, the function $u = i^*(w)$ in $H_1^2(M)$ is the unique solution of the equation $\Delta_g u + hu = w$ in M. By the continuity of the embedding $H_1^2(M)$ into $L^{2^*}(M)$, we have

$$\|i^*(w)\|_h \leq C|w|_{2n/(n+2)} \tag{2.2}$$

for some positive constant C independent of w.

In order to study the supercritical, by the standard elliptic estimates (see [11]), given a real number $s > 2n/(n-2)$, that is $ns/(n+2s) > 2n/(n+2)$, for any w in $L^{ns/(n+2s)}(M)$, the function $i^*(w)$ belongs to $L^s(M)$ and satisfies

$$|i^*(w)|_s \leq C|w|_{ns/(n+2s)} \tag{2.3}$$

for some positive constant C independent of w. For ε small, we set

$$s_\varepsilon := \begin{cases} 2^* - \frac{n}{2}\varepsilon & \text{if} \quad \varepsilon < 0; \\ 2^* & \text{if} \quad \varepsilon > 0, \end{cases}$$

and set $\mathcal{H}_\varepsilon = H_1^2(M) \cap L^{s_\varepsilon}(M)$ be the Banach space provided with the norm

$$\|u\|_{h,s_\varepsilon} = \|u\|_h + |u|_{s_\varepsilon}.$$

If $\varepsilon > 0$, the subcritical case, the space $\mathcal{H}_\varepsilon$ is the Sobolev space $H_1^2(M)$, and the norm $\|\cdot\|_{h,s_\varepsilon}$ is equivalent to the norm $\|\cdot\|_h$. And we can compute that there holds

$$\frac{ns_\varepsilon}{n+2s_\varepsilon} = \begin{cases} \frac{s_\varepsilon}{2^*-1-\varepsilon} & \text{if} \quad \varepsilon < 0; \\ \frac{2n}{n+2} & \text{if} \quad \varepsilon > 0, \end{cases} \tag{2.4}$$

Here we note that $\frac{ns_\varepsilon}{n+2s_\varepsilon} = \frac{2n}{n+2} - \frac{n(n^2+2n+2)}{n+2}\varepsilon + O(|\varepsilon|^2)$ for $\varepsilon < 0$ small.

Then by (2.2) (or (2.3) in the supercritical case), equation (1.1) can be written as

$$u = i^*(f_\varepsilon(u)), \quad u \in H_1^2(M), \tag{2.5}$$

where $f_\varepsilon(u) = |u|^{p-1-\varepsilon}u$, here and in the follows we will denote p by $p = \frac{n+2}{n-2}$.

3. The existence result

By compactness of manifold M, we have that the injectivity radius i_g of the manifold is nonzero. Fix $r > 0$ small than i_g. Let χ_r be a smooth cut-off function satisfying

$$\chi_r(z) := \begin{cases} 1 & \text{if} \quad z \in B(0, \frac{r}{2}); \\ \in (0,1) & \text{if} \quad z \in B(0,r) \backslash B(0, \frac{r}{2}); \\ 0 & \text{if} \quad z \in \mathbb{R}^n \backslash B(0,r), \end{cases} \tag{3.1}$$

and $|\nabla \chi_r(z)| \leq \frac{2}{r}$, $|\nabla^2 \chi_r(z)| \leq \frac{2}{r^2}$.

Let

$$A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times M \times \mathbb{R}^n \times \mathbb{R}^{2n-3}.$$

We will denote $A \in \mathcal{A}$ if $(\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times M \times \mathbb{R}^n \times \mathbb{R}^{2n-3}$, such that

$$\eta < t < \frac{1}{\eta}, \quad \text{for some fixed } \eta > 0, \tag{3.2}$$

$$\xi \in M, \quad a \in \mathbb{B} := \left\{ a = (a_1, a_2, 0, \ldots, 0) \in \mathbb{R}^n \ : \ |a| < \frac{1}{2} \right\}, \tag{3.3}$$

and

$$\theta = (\theta_{12}, \theta_{13}, \dots, \theta_{1n}, \theta_{23}, \dots, \theta_{2n}) \in \mathcal{O}, \tag{3.4}$$

where $\mathcal{O}$ is a compact manifold of dimension $2n-3$ with no boundary.

Now, for $A \in \mathcal{A}$, set

$$\lambda = \sqrt{t|\varepsilon|}. \tag{3.5}$$

We define the function $W_A(x) = W_{\lambda,\xi,a,\theta}(x)$ on M by

$$W_A(x) := \begin{cases} \chi_r\left(\exp_\xi^{-1}(x)\right)\widetilde{W}_A(x) & \text{if } x \in B_g(\xi, r); \\ 0 & \text{otherwise,} \end{cases} \tag{3.6}$$

where

$$\widetilde{W}_A(x) = Q\left(P_\theta \circ J \circ T_{-a} \circ J \circ D_{\lambda^{-1}} \circ P_\theta^{-1} \circ exp_\xi^{-1}(x)\right),$$

that is,

$$\widetilde{W}_A(x) = \lambda^{-\frac{n-2}{2}}\Big|\frac{exp_\xi^{-1}(x)}{d_g(x,\xi)} - P_\theta a \frac{d_g(x,\xi)}{\lambda}\Big|^{2-n} Q\left(\frac{\frac{exp_\xi^{-1}(x)}{\lambda} - P_\theta a \frac{d_g(x,\xi)^2}{\lambda^2}}{\Big|\frac{exp_\xi^{-1}(x)}{d_g(x,\xi)} - P_\theta a \frac{d_g(x,\xi)}{\lambda}\Big|^2}\right), \tag{3.7}$$

where $Q = Q_K$ is a solution of problem (1.6) for K large enough, which was proved in [7].

Moreover, let us define on M the functions

$$Z_A^i(x) := \begin{cases} \chi_r\left(\exp_\xi^{-1}(x)\right)\widetilde{Z}_A^i(x) & \text{if } x \in B_g(\xi, r); \\ 0 & \text{otherwise,} \end{cases} \tag{3.8}$$

for $i = 0, 1, 2, \cdots, 3n-1$. where

$$\widetilde{Z}_A^i(x) = \lambda^{-\frac{n-2}{2}}\Big|\frac{exp_\xi^{-1}(x)}{d_g(x,\xi)} - P_\theta a \frac{d_g(x,\xi)}{\lambda}\Big|^{2-n} z_i\left(\frac{\frac{exp_\xi^{-1}(x)}{\lambda} - P_\theta a \frac{d_g(x,\xi)^2}{\lambda^2}}{\Big|\frac{exp_\xi^{-1}(x)}{d_g(x,\xi)} - P_\theta a \frac{d_g(x,\xi)}{\lambda}\Big|^2}\right), \tag{3.9}$$

where z_i, $i = 0, 1, 2, \dots, 3n-1$, are defined in (6.5)- (6.9).

We define the projections Π_A and $\Pi_A^\perp$ of the Sobolev space $\mathcal{H}_\varepsilon$ onto the respective subspaces

$$K_A := \text{Span}\Big\{Z_A^0, Z_A^1, \cdots, Z_A^{3n-1}\Big\}, \tag{3.10}$$

$$K_A^\perp := \Big\{\phi \in \mathcal{H}_\varepsilon \ : \ \langle \phi, Z_A^i \rangle_h = 0, \forall i = 0, 1, \dots, 3n-1\Big\}, \tag{3.11}$$

where $\langle \cdot, \cdot \rangle_h$ is as in (2.1).

We will look for a solution to (2.5), or equivalently to (1.1), of the form

$$u_\varepsilon = W_A(x) + \phi_A(x), \tag{3.12}$$

where $W_A(x)$ is given by (3.6), and the rest term ϕ_A belongs to the space $K_A^\perp$. In order to solve problem (2.5) we will solve the system

$$\Pi_A^\perp\left\{W_A + \phi_A - i^*\left[f_\varepsilon\left(W_A + \phi_A\right)\right]\right\} = 0, \tag{3.13}$$

$$\Pi_A\left\{W_A + \phi_A - i^*\left[f_\varepsilon\left(W_A + \phi_A\right)\right]\right\} = 0. \tag{3.14}$$

We first give the result whose proof is postponed until Section 4 to solve equation (3.13).

Proposition 3.1. *If $n \geq 6$, for $A \in \mathcal{A}$, if ε is small enough, there exists a unique $\phi_{\varepsilon,A} = \phi(\varepsilon, A)$ which solves equation (3.13), which is continuously differential with respect to A, moreover,*

$$\|\phi_{\varepsilon,A}\|_{h,s_\varepsilon} \leq C \begin{cases} |\varepsilon| \, |\ln|\varepsilon| \, |^{2/3} & \text{if} \quad n = 6 \quad \text{and} \quad \varepsilon > 0; \\ |\varepsilon| \, |\ln|\varepsilon| \, | & \text{otherwise.} \end{cases} \tag{3.15}$$

Furthermore,

$$\|\nabla_A \phi_{\varepsilon,A}\|_{h,s_\varepsilon} \leq C \begin{cases} |\varepsilon| \, |\ln|\varepsilon| \, |^{2/3} & \text{if} \quad n = 6 \quad \text{and} \quad \varepsilon > 0; \\ |\varepsilon| \, |\ln|\varepsilon| \, | & \text{otherwise.} \end{cases} \tag{3.16}$$

where C is a positive constant.

We introduce the functional $J_\varepsilon : \mathcal{H}_\varepsilon \to \mathbb{R}$ defined by

$$J_\varepsilon(u) = \frac{1}{2}\int_M |\nabla_g u|^2 \, dv_g + \frac{1}{2}\int_M h(x) u^2 \, dv_g - \frac{1}{p+1-\varepsilon}\int_M |u|^{p+1-\varepsilon} \, dv_g,$$

It is well known that any critical point of J_ε is solution to problem (1.1). We also define the functional $\mathcal{F}_\varepsilon : \mathbb{R}^+ \times M \times \mathbb{R}^{2n-3} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{F}_\varepsilon(t, \xi, a, \theta) = J_\varepsilon\left(W_A + \phi_A\right), \tag{3.17}$$

where W_A is as (3.6) and ϕ_A is given by Proposition 3.1.

The next result, whose proof is postponed until Section 5, allows to solve equation (3.14), by reducing the problem to a finite dimensional one.

Proposition 3.2. *(i) For ε small, if (t, ξ, a, θ) is a critical point of the functional $\mathcal{F}_\varepsilon$, then $W_A + \phi_A$ is a solution of (2.5), or equivalently of problem (1.1).*

(ii) If $n \geq 6$, for $A \in \mathcal{A}$, there holds

$$J_\varepsilon(W_A(x)) = \frac{c_0}{n} - d_1 \varepsilon \log|\varepsilon| - d_2\varepsilon + \frac{\beta}{2}\Psi(t, \xi, a, \theta)\varepsilon + o(|\varepsilon|). \tag{3.18}$$

as $\varepsilon \to 0$, $\mathcal{C}^1-$uniformly with respect to A in $\mathcal{A}$, where

$$\begin{aligned} \Psi(t, \xi, a, \theta) = & -d_3 \log t + sign(\varepsilon)\varphi(\xi) t - sign(\varepsilon) d_4 a(B_{\xi,\theta}) a^T \ t \\ & + \left[sign(\varepsilon)\left(-2\varphi(\xi) + d_5 Scal_g(\xi)\right) t + d_6\right] |a|^2 + o(|a|^2) \end{aligned} \tag{3.19}$$

with

$$\varphi(\xi) = h(\xi) - \frac{n-2}{4(n-1)}\left(1 + \frac{n-4}{3n}K\right) Scal_g(\xi), \tag{3.20}$$

and

$$B_{\xi,\theta} = (P_\theta)^T (R_{ij})_{n\times n} P_\theta \tag{3.21}$$

is a $n \times n$ matrix. The constants $d_1 = \frac{n-2}{4}c_0$, $d_2 = \frac{(n-2)^2}{4n^2}c_0 - \frac{n-2}{2n}c_1 - \frac{(n-2)^2}{2}c_2$ with

$$c_0 = \int\limits_{\mathbb{R}^n} |Q(y)|^{\frac{2n}{n-2}} dy, \quad c_1 = \int\limits_{\mathbb{R}^n} |Q(y)|^{\frac{2n}{n-2}} \log|Q(y)| dy, \quad c_2 = \int\limits_{\mathbb{R}^n} |Q(y)|^{\frac{2n}{n-2}} \log|y| dy.$$

Moreover, the constants d_3, d_4, d_5, d_6 are defined by

$$d_3 = \frac{n-2}{2}c_0\beta^{-1} = \frac{n(n-2)^2(n-4)}{8(n-1)}(1+K),$$

$$d_4 = \frac{5n(n-2)(n+2)}{12(n-1)(n-6)} + \frac{(n-2)(n-4)}{12}K$$

$$d_5=\frac{(n-2)(n^2-9n-2)}{n(n-1)(n-6)}+\frac{(n-2)(n-3)(n-4)}{3n^2(n-1)}K$$
$$d_6=\frac{n^2(n-2)^2(n-4)}{8(n-1)}\left(1+\frac{n-2}{n}K\right).$$

(iii) If $n\geq 6$, there holds

$$\mathcal{F}_\varepsilon(t,\xi,a,\theta)=J_\varepsilon(W_A+\phi_A)=J_\varepsilon(W_A)+o(|\varepsilon|)$$

as $\varepsilon\to 0$, $\mathcal{C}^1$ uniformly with respect to $(t,\xi,a,\theta)\in\mathcal{A}$.

Now we are ready to prove the main result Theorem 1.1.

Proof of Theorem 1.1: From Proposition 3.2, we have that the function $u_\varepsilon=W_A+\phi_A$ is a solution of equation (2.5), or equivalently of problem (1.1) for ε small enough if we find a critical point (t,ξ,a,θ) of functional $\mathcal{F}_\varepsilon$, it is equivalent to find a critical point of the function $\Psi(t,\xi,a,\theta)$ which is given in (3.19).

Recall that $A=(t,\xi,a,\theta)\in\mathcal{A}=(\eta,\frac{1}{\eta})\times M\times\mathbb{B}\times\mathcal{O}$, where

$$\mathbb{B}:=\left\{a=(a_1,a_2,0,\dots,0)\in\mathbb{R}^n\ :\ |a|<\frac{1}{2}\right\},$$

and $\mathcal{O}$ is a compact manifold of dimension $2n-3$ with no boundary. By Proposition 3.2, we have

$$\begin{aligned}\Psi(t,\xi,a,\theta)=&-d_3\log t+\operatorname{sign}(\varepsilon)\varphi(\xi)t-\operatorname{sign}(\varepsilon)d_4a(B_{\xi,\theta})a^T\ t\\&+\left[\operatorname{sign}(\varepsilon)\left(-2\varphi(\xi)+d_5\mathrm{Scal}_g(\xi)\right)t+d_6\right]|a|^2+o(|a|^2),\end{aligned}\tag{3.22}$$

where $\varphi(\xi)$ is defined in (1.14).

Firstly, from (3.19), we have

$$\Psi(t,\xi,a,\theta)=\Phi_1(t,\xi)+O(|a|^2),\tag{3.23}$$

where

$$\Phi_1(t,\xi)=-d_3\log t+\operatorname{sign}(\varepsilon)\varphi(\xi)t,$$

with $\varphi(\xi)$ is given in (3.20). By assumption, there is a stable critical point ξ_0 of $\varphi(\xi)$, satisfying

$$\begin{cases}\varphi(\xi_0)>0, & \text{if } \varepsilon>0;\\ \varphi(\xi_0)<0, & \text{if } \varepsilon<0.\end{cases}$$

Set $t_0=\frac{d_3}{\varphi(\xi_0)}\operatorname{sign}(\varepsilon)$, we have $t_0>0$ and (t_0,ξ_0) is a critical point of $\Phi_1(t,\xi)$. Since $\deg(\nabla_g\varphi,B_g(\xi_0,\varrho),0)\neq 0$ for some $\varrho>0$, then $\deg(\nabla_g\Phi_1(t,\xi),B_g(\xi_0,\varrho),0)\neq 0$, by the continuity of the Brouwer degree via homotopy considering the function $H:[0,1]\times\mathbb{R}^+\times M\to\mathbb{R}^{n+1}$ defined by

$$\begin{aligned}H(\tau,t,\xi)=&\ \tau\left(\tfrac{\partial\Phi_1(t,\xi)}{\partial t},\left(\tfrac{\partial\Phi_1(t,\exp_\xi(y))}{\partial y_1}\right)_{|y=0},\cdots,\left(\tfrac{\partial\Phi_1(t,\exp_\xi(y))}{\partial y_n}\right)_{|y=0}\right)\\&+(1-\tau)\left(t-t_0,\left(\tfrac{\partial(\varphi\circ\exp_\xi(y))}{\partial y_1}\right)_{|y=0},\cdots,\left(\tfrac{\partial(\varphi\circ\exp_\xi(y))}{\partial y_n}\right)_{|y=0}\right).\end{aligned}$$

We get that (t_0,ξ_0) is a critical point of $\Phi_1(t,\xi)$, such that

$$\deg(\nabla_g\Phi_1,B_g(\xi_0,\varrho),0)\neq 0,$$

By Brouwer degree, we then have that (t_0,ξ_0) is a stable critical point of $\Phi_1(t,\xi)$. By Proposition 3.2, we have

$$\left|\partial_t\left(\frac{1}{\varepsilon}\mathcal{F}_\varepsilon-\Phi_1(t,\xi)\right)\right|+\left|\partial_\xi\left(\frac{1}{\varepsilon}\mathcal{F}_\varepsilon-\Phi_1(t,\xi)\right)\right|\to 0,$$

as $\varepsilon \to 0$, uniformly with respect to $A = (t, \xi, a, \xi) \in \mathcal{A}$. By the properties of the Brouwer degree, it follows that there exists a family of critical points $(t_\varepsilon, \xi_\varepsilon)$ of $\mathcal{F}_\varepsilon$ converging to (t_0, ξ_0) as $\varepsilon \to 0$.

On the other hand, we observe that the function $\theta \mapsto \Psi(t, \xi, a, \theta)$ has a maximum point $\bar{\theta}$. Because $\Psi(t, \xi, a, \theta)$ is a continuous function for θ on a compact set of R^{2n-3} without boundary. Moreover, the function $\Psi(t_0, \xi_0, a, \bar{\theta})$ has a non degenerate minimum($\varepsilon > 0$) or maximum ($\varepsilon < 0$) at $\bar{a} = (\bar{a}_1, \bar{a}_2) = (0, 0)$.

Thus, we obtain that $(t_0, \xi_0, 0, 0)$ is a stable critical point of $\Psi(t, \xi, a, \theta)$. □

4. The finite dimensional reduction

This section is devoted to the proof of Proposition 3.1. Let us introduce the linear operator $L_{\varepsilon,A} : H_1^2(M) \cap K_A \to K_A^\perp$ defined by

$$L_{\varepsilon,A}(\phi_A) := \Pi_A^\perp \left\{ \phi_A - i^* \left[f'_\varepsilon(W_A)\phi_A \right] \right\}.$$

This operator is well defined by using (2.2). Therefore equation (3.13) is equivalent to

$$L_{\varepsilon,A}(\phi_A) = N_{\varepsilon,A}(\phi_A) + R_{\varepsilon,A} \tag{4.1}$$

where

$$N_{\varepsilon,A}(\phi_A) = \Pi_A^\perp \left\{ i^* \left[f_\varepsilon(W_A + \phi_A) - f_\varepsilon(W_A) - f'_\varepsilon(W_A)\phi_A \right] \right\}, \tag{4.2}$$

and

$$R_{\varepsilon,A} = \Pi_A^\perp \left\{ i^* \left(f_\varepsilon(W_A) \right) - W_A \right\}. \tag{4.3}$$

As a first step, we want to study the invertibility of $L_{\varepsilon,A}$.

Lemma 4.1. *If $n \geq 6$ and for any $A \in \mathcal{A}$, and for any $\phi_A \in H_1^2(M) \cap K_A^\perp$, if ε is small enough, there holds*

$$\|L_{\varepsilon,A}(\phi_A)\|_{h,s_\varepsilon} \geq C\|\phi_A\|_{h,s_\varepsilon}, \tag{4.4}$$

where C is a positive constant.

Proof. We argue by contradiction. Assume there exist a sequences $\varepsilon_k \to 0$, $A_{\varepsilon_k} \in \mathcal{A}$ with $t_k \in (\eta, \frac{1}{\eta})$, $\xi_k \in M$, θ_k in a compact of $\mathbb{R}^{2n-3}$ and $a_k \in \mathbb{B} \subset \mathbb{R}^n$, and a sequences of functions $\phi_k \in H_1^2(M) \cap K_{A_k}^\perp$ such that

$$L_{\varepsilon_k,A_k}(\phi_k) = \psi_k, \quad \|\phi_k\|_{h,s_{\varepsilon_k}} = 1 \quad \text{and} \quad \|\psi_k\|_{h,s_{\varepsilon_k}} \to 0. \tag{4.5}$$

From (4.5) we get there exists $\zeta_k \in H_1^2(M) \cap K_{A_k}$ such that

$$\phi_k - i^* \left[f'_{\varepsilon_k}(W_{A_k})\phi_k \right] = \psi_k + \zeta_k. \tag{4.6}$$

Step 1, we claim that

$$\|\zeta_k\|_{h,s_\varepsilon} \to 0 \quad \text{as} \quad k \to \infty. \tag{4.7}$$

Let $\zeta_k := \sum\limits_{i=0}^{3n-1} c_k^i Z_{\lambda_{A_k}}^i$. For any $j = 0, 1, \cdots, 3n-1$, we multiply (4.6) by $Z_{A_k}^l$, and taking into account that $\phi_k, \psi_k \in K_{A_k}^\perp$, we get

$$\sum_{i=0}^{3n-1} c_k^i \left\langle Z_{A_k}^i, Z_{A_k}^j \right\rangle_h = -\left\langle i^* \left[f'_{\varepsilon_k}(W_{A_k})\phi_k \right], Z_{A_k}^j \right\rangle_h \tag{4.8}$$

By changing of variable $x = \exp_{\xi_k}(\lambda_k y)$, for $i, j = 0, 1, \cdots, 3n-1$ and for any k, we have

$$\begin{aligned}
&\left\langle Z^i_{A_k}, Z^j_{A_k} \right\rangle_h \\
&= \int_M \left\langle \nabla Z^i_{A_k}, \nabla Z^j_{A_k} \right\rangle_g \, dv_g + \int_M h(x) Z^i_{A_k} Z^j_{A_k} \, dv_g \\
&= \lambda_k^2 \int\limits_{B(0,r/\lambda_k)} \sum_{a,b=1}^n g^{ab}_{\xi_\alpha}(\lambda_k y) \left[\frac{1}{\lambda_k} \frac{\partial \Big(|y|^{2-n} z_i(\frac{y}{|y|^2} - P_\theta a) \Big)}{\partial y_a} \chi_r(\lambda_k y) + \frac{\partial \chi_r(\lambda_k y)}{\partial y_a} \Big(|y|^{2-n} z_i(\frac{y}{|y|^2} - P_\theta a) \Big) \right] \\
&\quad \times \left[\frac{1}{\lambda_k} \frac{\partial \Big(|y|^{2-n} z_j(\frac{y}{|y|^2} - P_\theta a) \Big)}{\partial y_b} \chi_r(\lambda_k y) + \frac{\partial \chi_r(\lambda_k y)}{\partial y_b} \Big(|y|^{2-n} z_j(\frac{y}{|y|^2} - P_\theta a) \Big) \right] \left| g_{\xi_{j\alpha}}(\lambda_k y) \right|^{\frac{1}{2}} \, dy \\
&\quad + \lambda_k^2 \int\limits_{B(0,r/\lambda_k)} h\left(\exp_{\xi_k}(\lambda_k y) \right) \chi_r(\lambda_k y) |y|^{2(2-n)} z_i(\frac{y}{|y|^2} - P_\theta a) z_j(\frac{y}{|y|^2} - P_\theta a) \left| g_{\xi_k}(\lambda_k y) \right|^{\frac{1}{2}} \, dz.
\end{aligned}$$

By the Taylor's expansion, from (2.1), we have

$$g^{ab}_{\xi_k}(\lambda_k y) = \delta_{ab} + O(\lambda_k^2 |y|^2) = \delta_{ab} + O(t_k |\varepsilon_k| |y|^2); \tag{4.9}$$

and

$$\left| g_{\xi_k}(\lambda_k y) \right|^{\frac{1}{2}} = 1 + O(\lambda_k^2 |z|^2) = 1 + O(t_k |\varepsilon_\alpha| |y|^2). \tag{4.10}$$

Therefore, we find

$$\begin{aligned}
\left\langle Z^i_{A_k}, Z^j_{A_k} \right\rangle_h &= -\int\limits_{\mathbb{R}^n} \Delta \Big(|y|^{2-n} z_i(\frac{y}{|y|^2} - P_\theta a) \Big) \Big(|y|^{2-n} z_j(\frac{y}{|y|^2} - P_\theta a) \Big) dy + o(\lambda_k^2) \\
&= \int_{\mathbb{R}^n} \Big| |y|^{2-n} Q(\frac{y}{|y|^2} - P_\theta a) \Big|^{p-1} |y|^{4-2n} z_i(\frac{y}{|y|^2} - P_\theta a) z_j(\frac{y}{|y|^2} - P_\theta a) dy + o(\lambda_k^2) \\
&= \int_{\mathbb{R}^n} |Q(y)|^{p-1} z_i(y) z_j(y) dy + o(\lambda_k^2) \\
&= \begin{cases} \int_{\mathbb{R}^n} |Q(y)|^{p-1} z_i^2(y) dy + o(\varepsilon) & \text{if} \quad i = j; \\ \int_{\mathbb{R}^n} |Q(y)|^{p-1} z_1(y) z_{n+2}(y) dy + o(\varepsilon) & \text{if} \quad i = 1, \ j = n+2; \\ \int_{\mathbb{R}^n} |Q(y)|^{p-1} z_2(y) z_{n+3}(y) dy + o(\varepsilon) & \text{if} \quad i = 2, \ j = n+3; \\ o(\varepsilon) & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.11}$$

Here $\int_{\mathbb{R}^n} |Q(y)|^{p-1} z_i^2(y) dy$, $\int_{\mathbb{R}^n} |Q(y)|^{p-1} z_1(y) z_{n+2}(y) dy$ and $\int_{\mathbb{R}^n} |Q(y)|^{p-1} z_2(y) z_{n+3}(y) dy$ are fixed numbers, different from zero, that are independent of ε.

Now, set

$$\phi_k(x) = \lambda_k^{-\frac{n-2}{2}} \Big| \frac{exp^{-1}_{\xi_k}(x)}{d_g(x,\xi_k)} - P_\theta a \frac{d_g(x,\xi_k)}{\lambda_k} \Big|^{2-n} \tilde{\phi}_k \left(\frac{\frac{exp^{-1}_{\xi_k}(x)}{\lambda_k} - P_\theta a \frac{d_g(x,\xi_k)^2}{\lambda_k^2}}{\Big| \frac{exp^{-1}_{\xi_k}(x)}{d_g(x,\xi_k)} - P_\theta a \frac{d_g(x,\xi_k)}{\lambda_k} \Big|^2} \right),$$

we have

$$\phi_k(exp_{\xi_k}(\lambda_k y) = \lambda_k^{-\frac{n-2}{2}} |y|^{2-n} \tilde{\phi}_k \left(\frac{y}{|y|^2} - P_\theta a \right).$$

We now have

$$\left\langle i^* \left[f'_{\varepsilon_k}(W_{A_k}) \phi_k \right], Z^j_{A_k} \right\rangle_h$$

$$
\begin{aligned}
&= \int_M f'_{\varepsilon_k}(W_{A_k}) Z^j_{A_k} \phi_k d\upsilon_g \\
&= \lambda_k^2 \int\limits_{B(0,r/\lambda_k)} f'_{\varepsilon_k}\left(\chi_r(\lambda_k y)\lambda_k^{-\frac{n-2}{2}}|y|^{2-n} Q\left(\frac{y}{|y|^2} - P_\theta a\right)\right) \times \\
&\qquad \times \chi_r(\lambda_k y)|y|^{4-2n} z_j\left(\frac{y}{|y|^2} - P_\theta a\right)\tilde{\phi}_k\left(\frac{y}{|y|^2} - P_\theta a\right))\sqrt{|g_{\xi_k}(\lambda_k y)|}dy \\
&= \lambda_k^{\frac{n-2}{2}\varepsilon} \int\limits_{\mathbb{R}^n} f'_{\varepsilon_k}\left(Q\left(\frac{y}{|y|^2} - P_\theta a\right)\right)|y|^{2n+(n-2)\varepsilon} \times \\
&\qquad \times z_j\left(\frac{y}{|y|^2} - P_\theta a\right)\tilde{\phi}_k\left(\frac{y}{|y|^2} - P_\theta a\right)\sqrt{|g_{\xi_k}(\lambda_k y)|}dy + O(\varepsilon^2) \\
&\qquad \text{set } \ \tilde{y} = \frac{y}{|y|^2} - P_\theta a \\
&= (2^* - 1 - \varepsilon_k)\lambda_k^{\frac{n-2}{2}\varepsilon} \int\limits_{\mathbb{R}^n} |Q(\tilde{y})|^{2^*-2-\varepsilon} Q(\tilde{y})|\tilde{y} + P_\theta a|^{-(n-2)\varepsilon} z_j(\tilde{y})\tilde{\phi}_k(\tilde{y})dy + O(\varepsilon^2) \\
&\to (2^* - 1)\int\limits_{\mathbb{R}^n} |Q(\tilde{y})|^{2^*-1} z_j(\tilde{y})\tilde{\phi}_k(\tilde{y})dy
\end{aligned} \tag{4.12}
$$

as $\varepsilon_k \to 0$.

Since, for any k, the function $\phi_k \in K^{\perp}_{A_k}$, by the same change of variable for $x = \exp_{\xi_k}(\lambda_k y)$, we have

$$
\begin{aligned}
0 = \left\langle Z^j_{A_k}, \phi_k \right\rangle_h &= -\int_{\mathbb{R}^n} \Delta(z_j(y))\ \tilde{\phi}_k(y) d\mu_{g_{\xi_k}} \\
&+ \lambda_k^2 \int_{\mathbb{R}^n} h\left(\exp_{\xi_k}(\lambda_k y)\right)\chi_r(\lambda_k y) z_j \tilde{\phi}_k \ d\mu_{g_{\xi_k}},
\end{aligned} \tag{4.13}
$$

where $g_{\xi_k}(y) = \exp_{\xi_k} g(\lambda_k y)$ with $d\mu_{g_{\xi_k}} = |g_{\xi_k}(\lambda_k y)|^{\frac{1}{2}}\, dz$. Then, passing the limit in (4.13), we get

$$\int_{\mathbb{R}^n} \nabla z_j \nabla \tilde{\phi}\ dy = 0.$$

Since the function z_j is a solution of equation $L(z_j) = 0$, the operator L is given in (6.4), it yields that

$$\int_{\mathbb{R}^n} \nabla z_j \nabla \tilde{\phi}\ dy = (2^* - 1)\int_{\mathbb{R}^n} |Q|^{2^*-1} z_j \tilde{\phi}\ dy = 0. \tag{4.14}$$

It follows from (4.8), (4.11), (4.12) and (4.14) that for any $i = 0, 1, \cdots, 3n - 1$, there holds $c^i_k \to 0$ as $\alpha \to \infty$, therefore our claim (4.7) is proved.

Step 2: We prove that

$$\liminf_{k\to\infty} \int_M f'_{\varepsilon_k}(W_{A_k}) u_k^2\ d\upsilon_g \to c > 0. \tag{4.15}$$

where

$$u_k = \phi_k - \psi_k - \zeta_k, \quad \text{with} \qquad \|u_k\|_h \to 1. \tag{4.16}$$

Let us write equation (4.6) as

$$\Delta_g u_k + h(x)u_k = f'_{\varepsilon_k}(W_{A_k})u_k + f'_{\varepsilon_k}(W_{A_k})(\psi_k + \zeta_k), \tag{4.17}$$

We first prove that

$$\liminf_{k\to\infty} \|u_k\|_h = c > 0. \tag{4.18}$$

In fact, by (4.17) we deduce

$$u_k = i^*\left\{f'_{\varepsilon_k}(W_{A_k})u_k + f'_{\varepsilon_k}(W_{A_k})(\psi_k + \zeta_k)\right\}, \tag{4.19}$$

and by (2.3), (4.5), (4.7) and (4.16), use the Hölder inequality, we get

$$\begin{aligned}
|u_k|_{s_{\varepsilon_k}} &\le C\left[\left|f'_{\varepsilon_k}(W_{A_k})u_k\right|_{\frac{ns_{\varepsilon_k}}{n+2s_{\varepsilon_k}}} + \left|f'_{\varepsilon_\alpha}(W_{A_k})(\psi_k+\zeta_k)\right|_{\frac{ns_{\varepsilon_k}}{n+2s_{\varepsilon_k}}}\right]\\
&\le C\left[\left|f'_{\varepsilon_k}(W_{A_k})\right|_{\frac{2ns_{\varepsilon_k}}{2n-(n-6)s_{\varepsilon_k}}}|u_k|_{2^*} + \left|f'_{\varepsilon_k}(W_{A_k})\right|_{\frac{n}{2}}|\psi_k+\zeta_k|_{s_{\varepsilon_k}}\right]\\
&\le C\left|f'_{\varepsilon_k}(W_{A_k})\right|_{\frac{2ns_{\varepsilon_k}}{2n-(n-6)s_{\varepsilon_k}}}|u_k|_{2^*} + C\left|f'_{\varepsilon_k}(W_{A_k})\right|_{\frac{n}{2}}(\|\psi_k\|_h + \|\zeta_k\|_h)\\
&\le C\left|f'_{\varepsilon_k}(W_{A_k})\right|_{\frac{2ns_{\varepsilon_k}}{2n-(n-6)s_{\varepsilon_k}}}|u_k|_{2^*} + o(1)\\
&\le C\|u_k\|_h + o(1), \qquad (4.20)
\end{aligned}$$

as $k\to\infty$. Then, if $\|u_k\|_h \to 0$, by (4.20) we deduce that also $|u_k|_{s_{\varepsilon_k}} \to 0$, this is not impossible because of (4.16). This gives the validity of (4.18).

We multiply (4.17) by u_k we deduce that

$$\|u_k\|_h^2 = \int_M f'_{\varepsilon_k}(W_{A_k})u_k^2\, d\upsilon_g + \int_M f'_{\varepsilon_k}(W_{A_k})(\psi_k+\zeta_k)u_k\, d\upsilon_g. \tag{4.21}$$

By Hölder inequality, from (4.5), (4.7), we have

$$\begin{aligned}
\left|\int_M f'_{\varepsilon_k}(W_{A_k})(\psi_k+\zeta_k)u_k\, d\upsilon_g\right| &\le \left|f'_{\varepsilon_k}(W_{A_k})\right|_{\frac{n}{2}}|\psi_k+\zeta_k|_{\frac{2n}{n-2}}|u_k|_{\frac{2n}{n-2}}\\
&\le C\|\psi_k+\zeta_k\|_h\|u_k\|_h = o(1). \qquad (4.22)
\end{aligned}$$

Then, (4.15) follows by (4.18), (4.21) and (4.22).

Step 3: Let us prove that a contradiction arises, by showing that

$$\liminf_{k\to\infty}\int_M f'_{\varepsilon_k}(W_{A_k})u_k^2\, d\upsilon_g \to 0. \tag{4.23}$$

In fact, set

$$u_k(x) = \lambda_k^{-\frac{n-2}{2}}\left|\frac{exp_{\xi_k}^{-1}(x)}{d_g(x,\xi_k)} - P_\theta a\frac{d_g(x,\xi_k)}{\lambda_k}\right|^{2-n}\tilde u_k\left(\frac{\frac{exp_{\xi_k}^{-1}(x)}{\lambda_k} - P_\theta a\frac{d_g(x,\xi_k)^2}{\lambda_k^2}}{\left|\frac{exp_{\xi_k}^{-1}(x)}{d_g(x,\xi_k)} - P_\theta a\frac{d_g(x,\xi_k)}{\lambda_k}\right|^2}\right). \tag{4.24}$$

We will show that

$$\tilde u_k \to 0 \quad \text{weakly in} \quad \mathcal{D}^{1,2}(\mathbb{R}^n) \quad \text{and strongly in} \quad L^q_{loc}(\mathbb{R}^n), \tag{4.25}$$

for any $q\in[2,2^*)$. That fact implies that

$$\int_M f'_{\varepsilon_k}(W_{A_k})u_k^2\, d\upsilon_g = \int_{B_g(\xi_{l\alpha},r)} f'_{\varepsilon_\alpha}(W_{\lambda_{l\alpha},\xi_{l\alpha}})u_\alpha^2\, d\upsilon_g$$

$$= \lambda_k^{\frac{n-2}{2}\varepsilon_k} \int\limits_{B(0,r/\lambda_k)} f'_{\varepsilon_k}\left(\chi_r(\lambda_k y)|y|^{2-n}Q(\frac{y}{|y|^2}-P_\theta a)\right)\times$$
$$\times|y|^{4-2n}\left(\tilde{u}_k(\frac{y}{|y|^2}-P_\theta a)\right)^2 |g_{\xi_k}(\lambda_{ky})|^{\frac{1}{2}}\ dy$$
$$\leq C\lambda_k^{\frac{n-2}{2}\varepsilon_k}\left||\tilde{y}+P_\theta a|^{(2-n)\varepsilon_k} f'_{\varepsilon_k}(Q(\tilde{y}))\right|_{L^{n/2}(\mathbb{R}^n)}|\tilde{u}_k(\tilde{y})|_{L^{2^*}(\mathbb{R}^n)} = o(1), \tag{4.26}$$

for $\varepsilon_k \to 0$, because $\left||\tilde{y}+P_\theta a|^{(2-n)\varepsilon_k} f'_{\varepsilon_k}(Q(\tilde{y}))\right|_{L^{n/2}(\mathbb{R}^n)} = O(1)$ holds.

Finally, we prove (4.25). By (4.17) we get

$$\int_M |\nabla_g u_k|_g\ d\upsilon_g + \int_M h(x)u_k^2\ d\upsilon_g$$
$$= \int_M f'_{\varepsilon_\alpha}(W_{A_k})u_k^2\ d\upsilon_g + \int_M f'_{\varepsilon_k}(W_{A_k})(\psi_k+\zeta_k)u_k\ d\upsilon_g$$
$$= \int_M f'_{\varepsilon_k}(W_{A_k})u_k^2\ d\upsilon_g + o(1), \tag{4.27}$$

because (4.22) holds.

By an change of variable $x = \exp_{\xi_k}(\lambda_k y)$ in (4.27), we get

$$\int_{\mathbb{R}^n} |\nabla_{g_{\xi_k}}\tilde{u}_k|_{g_{\xi_k}}\ d\mu_{\xi_k} + \lambda_k^2\int_{\mathbb{R}^n} h\left(\exp_{\xi_k}(\lambda_k y)\right)\tilde{u}_k^2\ d\mu_{g_{\xi_k}}$$
$$= \lambda_k^{\frac{n-2}{2}\varepsilon_k}\int_{\mathbb{R}^n} f'_{\varepsilon_k}\left(\chi_r(\lambda_k y)Q(y)\right)\tilde{u}_k^2\ d\mu_{g_{\xi_k}} + o(1). \tag{4.28}$$

Moreover, we observe that $\|\tilde{u}_k\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} \leq c\|u_k\|_h \leq c$, that is the sequence $\{\tilde{u}_k\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, then there exists $\tilde{u}$ such that $\tilde{u}_k(z) \to \tilde{u}$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and strongly in $L^q(\mathbb{R}^n)$ for any $q \in [2, 2^*)$ if $n \geq 3$. Then we deduce that $\tilde{u}$ solve the problem

$$\Delta\tilde{u} = (2^*-1)|Q|^{2^*-2}\tilde{u} \quad \text{in} \quad \mathbb{R}^n, \tag{4.29}$$

by (4.14), we get that the function $\tilde{u}$ is identically zero, then (4.25) holds.

Therefore from the contradiction (4.15) with (4.23), we end proof of Lemma 4.1. □

From [17], we have the following estimate of the error term $R_{\varepsilon,A}$.

Lemma 4.2. *If $n \geq 6$ and for any $A \in \mathcal{A}$, if ε is small enough, there holds*

$$\|R_{\varepsilon,A}\|_{h,s_\varepsilon} \leq C\begin{cases} |\varepsilon|\,|\ln|\varepsilon||^{2/3} & \text{if } n=6 \text{ and } \varepsilon>0; \\ |\varepsilon|\,|\ln|\varepsilon|| & \text{otherwise,}\end{cases} \tag{4.30}$$

where C is a positive constant.

Proof of Proposition 3.1: In order to solve (3.13), we need to find a fixed point for the operator $T_{\varepsilon,A} : H_1^2(M)\cap K_A^\perp \to H_1^2(M)\cap K_A^\perp$ defined

$$T_{\varepsilon,A}(\phi) = L_{\varepsilon,A}^{-1}(N_{\varepsilon,A}(\phi_A)+R_{\varepsilon,A}),$$

for ε small and for any $A \in \mathcal{A}$. We also let

$$\mathcal{B}(\beta) = \left\{\phi \in H_1^2(M)\cap K_A^\perp \ : \ \|\phi\|_{h,s_\varepsilon} \leq \beta\|R_{\varepsilon,A}\|_{h,s_\varepsilon}\right\},$$

where β is a positive constant to be chosen later on.

By Lemma 4.1, we deduce

$$\|T_{\varepsilon,A}(\phi)\|_{h,s_\varepsilon} \leq C\left(\|N_{\varepsilon,A}(\phi)\|_{h,s_\varepsilon} + \|R_{\varepsilon,A}\|_{h,s_\varepsilon}\right), \tag{4.31}$$

and

$$\|T_{\varepsilon,A}(\phi_1)-T_{\varepsilon,A}(\phi_2)\|_{h,s_\varepsilon}\le C\left(\|N_{\varepsilon,A}(\phi_1)-N_{\varepsilon,A}(\phi_2)\|_{h,s_\varepsilon}\right). \tag{4.32}$$

By (2.2) and (2.3), we deduce

$$\begin{aligned}\|N_{\varepsilon,A}(\phi)\|_{h,s_\varepsilon}\le C\left|f_\varepsilon(W_A+\phi)-f_\varepsilon(W_A)-f'_\varepsilon(W_A)\phi\right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\\ +\left|f_\varepsilon(W_A+\phi)-f_\varepsilon(W_A)-f'_\varepsilon(W_A)\phi\right|_{\frac{2n}{n+2}},\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}&\|N_{\varepsilon,A}(\phi_1)-N_{\varepsilon,A}(\phi_2)\|_{h,s_\varepsilon}\\ &\le C\left|f_\varepsilon(W_A+\phi_1)-f_\varepsilon(W_A+\phi_2)-f'_\varepsilon(W_A)(\phi_1-\phi_2)\right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\\ &\quad+\left|f_\varepsilon(W_A+\phi_1)-f_\varepsilon(W_A+\phi_2)-f'_\varepsilon(W_A)(\phi_1-\phi_2)\right|_{\frac{2n}{n+2}}.\end{aligned} \tag{4.34}$$

Then by the mean value theorem and the Hölder inequality, we obtain that if $n=6$ and $\varepsilon>0$, for any $\tau\in(0,1)$,

$$\begin{aligned}&\left|f_\varepsilon(W_A+\phi_1)-f_\varepsilon(W_A+\phi_2)-f'_\varepsilon(W_A)(\phi_1-\phi_2)\right|_{\frac{2n}{n+2}}\\ &=\left|\left(f'_\varepsilon(W_A+\phi_2+\tau(\phi_1-\phi_2))-f'_\varepsilon(W_A)\right)(\phi_1-\phi_2)\right|_{\frac{2n}{n+2}}\\ &\le C\left(|\phi_1|_{s_\varepsilon}^{\frac{2s_\varepsilon}{n}}+|\phi_2|_{s_\varepsilon}^{\frac{2s_\varepsilon}{n}}\right)|\phi_1-\phi_2|_{\frac{2n}{n-2}}\le C\left(\|\phi_1\|_{h,s_\varepsilon}^{1-\varepsilon}+\|\phi_2\|_{h,s_\varepsilon}^{1-\varepsilon}\right)\|\phi_1-\phi_2\|_{h,s_\varepsilon}.\end{aligned} \tag{4.35}$$

We note that by (2.4) we have $\frac{ns_\varepsilon}{n+2s_\varepsilon}=\frac{2n}{n+2}$ for $\varepsilon>0$.

If $n\ge 7$ or $\varepsilon<0$, there holds

$$\begin{aligned}&\left|f_\varepsilon(W_A+\phi_1)-f_\varepsilon(W_A+\phi_2)-f'_\varepsilon(W_A)(\phi_1-\phi_2)\right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\\ &\le C\left|\left(|W_A|^{2^*-3-\varepsilon}|\tau\phi_2+(1-\tau)\phi_1|+|\tau\phi_2+(1-\tau)\phi_1|^{2^*-2-\varepsilon}\right)(\phi_1-\phi_2)\right|_{\frac{ns_\varepsilon}{n+2s_\varepsilon}}\\ &\le C\left(|W_A|_{s_\varepsilon}+\|\phi_1\|_{h,s_\varepsilon}+\|\phi_2\|_{h,s_\varepsilon}\right)^{2^*-3-\varepsilon}\left(\|\phi_1\|_{h,s_\varepsilon}+\|\phi_2\|_{h,s_\varepsilon}\right)\|\phi_1-\phi_2\|_{h,s_\varepsilon}.\end{aligned} \tag{4.36}$$

Since the problem is supercritical if $\varepsilon<0$, $s>\frac{2n}{n-2}$, i.e., $\frac{ns_\varepsilon}{n+2s_\varepsilon}>\frac{2n}{n+2}$, by the embedding $L^{\frac{ns_\varepsilon}{n+2s_\varepsilon}}(M)\hookrightarrow L^{\frac{2n}{n+2}}(M)$, we get

$$\begin{aligned}&\left|f_\varepsilon(W_A+\phi_1)-f_\varepsilon(W_A+\phi_2)-f'_\varepsilon(W_A)(\phi_1-\phi_2)\right|_{\frac{2n}{n+2}}\\ &=C\left(|W_A|_{s_\varepsilon}+\|\phi_1\|_{h,s_\varepsilon}+\|\phi_2\|_{h,s_\varepsilon}\right)^{2^*-3-\varepsilon}\left(\|\phi_1\|_{h,s_\varepsilon}+\|\phi_2\|_{h,s_\varepsilon}\right)\|\phi_1-\phi_2\|_{h,s_\varepsilon}.\end{aligned} \tag{4.37}$$

Moreover, if $n\ge 7$ and $\varepsilon>0$, from (2.4), we have $\frac{ns_\varepsilon}{n+2s_\varepsilon}=\frac{2n}{n+2}$.

Taking $\phi_1=\phi,\phi_2=0$ into (4.35) or (4.36) and (4.37), from (4.33), we can get

$$\|N_{\varepsilon,\bar d,\bar\xi}(\phi)\|_{h,s_\varepsilon}\le\begin{cases}C\|\phi\|_{h,s_\varepsilon}^{2-\varepsilon} & \text{if}\quad n=6\text{ and }\varepsilon>0;\\ C\left(\|\phi\|_{h,s_\varepsilon}^{2}+\|\phi\|_{h,s_\varepsilon}^{2^*-1-\varepsilon}\right) & \text{if}\quad n\ge 7\text{ or }\varepsilon<0.\end{cases} \tag{4.38}$$

By the definition of $\mathcal{B}(\beta)$, from (4.30), (4.31) and (4.38), we can get that there exists $\beta>0$ such that

$$\phi\in\mathcal{B}(\beta)\implies T_{\varepsilon,A}(\phi)\in\mathcal{B}(\beta), \tag{4.39}$$

provided that ε is sufficiently small. Next we will show that the map $T_{\varepsilon,A}$ is a contraction map for any ε small enough.

If $n=6$ and $\varepsilon>0$, by (4.32), (4.34) and (4.35), we deduce that there exists $\vartheta\in(0,1)$ such that

$$\begin{aligned}&\|\phi_1\|_{h,s_\varepsilon},\|\phi_2\|_{h,s_\varepsilon}\le|\varepsilon|\,|\ln|\varepsilon||^{2/3}\\ &\Longrightarrow\|T_{\varepsilon,A}(\phi_1)-T_{\varepsilon,A}(\phi_2)\|_{h,s_\varepsilon}\le\vartheta\|\phi_1-\phi_2\|_{h,s_\varepsilon}.\end{aligned}\tag{4.40}$$

If $n\ge7$ or $\varepsilon<0$, by (4.32), (4.34),(4.36) and (4.37), we can deduce that there exists $\vartheta\in(0,1)$ such that

$$\begin{aligned}&\|\phi_1\|_{h,s_\varepsilon},\|\phi_2\|_{h,s_\varepsilon}\le|\varepsilon|\,|\ln|\varepsilon||\\ &\Longrightarrow\|T_{\varepsilon,A}(\phi_1)-T_{\varepsilon,A}(\phi_2)\|_{h,s_\varepsilon}\le\vartheta\|\phi_1-\phi_2\|_{h,s_\varepsilon}.\end{aligned}\tag{4.41}$$

By (4.39) and (4.40) or (4.41), we deduce that $T_{\varepsilon,A}$ is a contraction mapping from $\mathcal{B}(\beta)$ into $\mathcal{B}(\beta)$ for ε small enough, so it has a fixed point $\phi_{\varepsilon,A}$ which satisfies (3.13), and (3.15) holds from (4.30).

By the Implicit Function Theorem to prove that the map $A\to\phi_{\varepsilon,A}$ is a $\mathcal{C}^1$ map. In fact, we apply the Implicit Function Theorem to the function $G(A,\phi):A\in\mathcal{A}\times\mathcal{H}_\varepsilon\to\mathcal{H}_\varepsilon$ defined by $G(A,\phi)=\phi-L_{\varepsilon,A}^{-1}\left(N_{\varepsilon,A}(\phi)+R_{\varepsilon,A}\right)$. The proof is standard, we omit it here, see [17]. This finishes the proof. □

5. The expansion of energy

Lemma 5.1. *[14]In a normal coordinates neighborhood of $\xi\in M$, the Taylor series of g around ξ is given by*

$$g_{ij}=\delta_{ij}+\frac{1}{3}R_{kijl}z^kz^l+O(|z|^3),$$

as $|z|\to0$. Moreover, the volume element on normal coordinates has the following expansion

$$\sqrt{det(g)}=1-\frac{1}{6}R_{kl}z^kz^l+O(|z|^3),$$

where $R_{kl}=Ric(e_k,e_l)=g^{ij}R_{iklj}=g^{ij}R(e_i,e_k,e_l,e_j)$, with $\{e_i\}_1^n$ is a basic of $T_\xi(M)$.

This section is devoted to the proof of Proposition 3.2. At the first step, we have

Lemma 5.2. *For ε small, if (λ,ξ,a,θ) is a critical point of the functional $\mathcal{F}_\varepsilon$, then $W_A+\phi_A$ is a solution of (2.5), or equivalently of problem (1.1).*

Proof. Let (λ,ξ,a,θ) be a critical point of $\mathcal{F}_\varepsilon$. Let $\xi=\xi(y)=\exp_\xi(y)$, $y\in B(0,r)$. We note that $\xi(0)=\xi$. since (λ,ξ,a,θ) is a critical point of $\mathcal{F}_\varepsilon$, there holds

$$J_\varepsilon'\left(W_A+\phi_A\right)\left[\frac{\partial}{\partial t}W_A+\frac{\partial}{\partial t}\phi_A\right]=0,\tag{5.1}$$

$$J_\varepsilon'\left(W_A+\phi_A\right)\left[\frac{\partial}{\partial y_l}W_A+\frac{\partial}{\partial y_l}\phi_A\right]=0,\quad l=1,\ldots,n,\tag{5.2}$$

$$J_\varepsilon'\left(W_A+\phi_A\right)\left[\frac{\partial}{\partial\theta_{ij}}W_A+\frac{\partial}{\partial\theta_{ij}}\phi_A\right]=0,\tag{5.3}$$

and

$$J_\varepsilon'\left(W_A+\phi_A\right)\left[\frac{\partial}{\partial a_k}W_A+\frac{\partial}{\partial a_k}\phi_A\right]=0,\quad k=1,2.\tag{5.4}$$

Let ∂_m denotes ∂_t or ∂_{y_l} for $l=1,2,\cdots,n$, or ∂_{a_1}, ∂_{a_2}, or $\partial_{\theta_{ij}}$ for $\theta_{ij}\in\{\theta_{12},\theta_{13},\ldots,\theta_{1n},\theta_{23},\ldots,\theta_{2n}\}$. By (3.13) we get

$$\begin{aligned}
0=\partial_m\mathcal{F}_\varepsilon(\lambda,\xi,a,\theta)&=J_\varepsilon'\left(W_A+\phi_A\right)\left[\partial_mW_A+\partial_m\phi_A\right]\\
&=\Big\langle W_A+\phi_A-i^*\left[f_\varepsilon(W_A+\phi_A)\right],\partial_mW_A+\partial_m\phi_A\Big\rangle_h\\
&=\sum_{i=0}^{3n-1}c_\varepsilon^i\left\langle Z_A^i,\partial_mW_A+\partial_m\phi_A\right\rangle_h,
\end{aligned}$$

for some $c_\varepsilon^i\in\mathbb{R}$. Since $\partial_mW_A=Z_A^m+o(1)$ and $\partial_m\phi_A=o(1)$, thus $\partial_m\mathcal{F}_\varepsilon(\lambda,\xi,a,\theta)=0$ is equivalent to get

$$c_\varepsilon^i=0\quad\text{for any}\quad i=0,1,\cdots 3n-1.$$

for ε is small enough. □

Now we give the expansion of the energy $J_\varepsilon(W_A)$.

Let $p,q\in\mathbb{R}_+$ such that $p-q>1$ and assume that $I_p^q=\int_0^{+\infty}\frac{t^q}{(1+t)^p}dt$. An integration by parts, we have

$$I_{p+1}^q=\frac{p-q-1}{p}I_p^q,\quad\text{and}\qquad I_{p+1}^{q+1}=\frac{q+1}{p-q-1}I_{p+1}^q. \tag{5.5}$$

$$I_{n-2}^{\frac{n-8}{2}}=\frac{n}{n-6}I_{n-2}^{\frac{n-6}{2}},\qquad I_n^{\frac{n}{2}}=I_n^{\frac{n-4}{2}}=\frac{n(n-4)}{4(n-1)(n-2)}I_{n-2}^{\frac{n-6}{2}},$$

$$I_n^{\frac{n-6}{2}}=\frac{n(n+2)}{4(n-1)(n-2)}I_{n-2}^{\frac{n-6}{2}},\qquad I_n^{\frac{n-2}{2}}=\frac{(n-2)(n-4)}{n(n+2)}I_n^{\frac{n-6}{2}}=\frac{(n-4)}{4(n-1)}I_{n-2}^{\frac{n-6}{2}},$$

and

$$I_{n-1}^{\frac{n-6}{2}}=\frac{n}{2(n-2)}I_{n-2}^{\frac{n-6}{2}},\qquad I_{n-1}^{\frac{n-4}{2}}=\frac{n-4}{2(n-2)}I_{n-2}^{\frac{n-6}{2}}$$

The energy functional is

$$J_\varepsilon(W_A)=\frac{1}{2}\int_M|\nabla W_A(x)|_g^2dv_g+\frac{1}{2}\int_M h(x)|W_A(x)|^2dv_g-\frac{1}{2^*-\varepsilon}\int_M|W_A(x)|^{2^*-\varepsilon}dv_g.$$

We observe that by change of variable $x=\exp_\xi(\lambda z)$, for $z\in B(0,r)$, we have

$$\begin{aligned}
\widetilde{W}_A(x)=\widetilde{W}_A(\exp_\xi(\lambda z))&=\lambda^{-\frac{n-2}{2}}\Big|\frac{z}{|z|}-P_\theta a|z|\Big|^{2-n}Q\left(\frac{z-P_\theta a|z|^2}{\Big|\frac{z}{|z|}-P_\theta a|z|\Big|^2}\right)\\
&=\lambda^{-\frac{n-2}{2}}|z|^{2-n}\Big|\frac{z}{|z|^2}-P_\theta a\Big|^{2-n}Q\left(\frac{\frac{z}{|z|^2}-P_\theta a}{\Big|\frac{z}{|z|^2}-P_\theta a\Big|^2}\right)\\
&\qquad\text{since }\ |z|^{2-n}Q(\frac{z}{|z|^2})=Q(z)\\
&=\lambda^{-\frac{n-2}{2}}|z|^{2-n}Q\Big(\frac{z}{|z|^2}-P_\theta a\Big).
\end{aligned}$$

We set

$$\widetilde{Q}_{\tilde a}(z)=|z|^{2-n}Q\Big(\frac{z}{|z|^2}-\tilde a\Big),\quad\text{with}\quad\tilde a=P_\theta a.$$

Then we find

$$\begin{aligned} J_\varepsilon(W_A) &= \frac{1}{2}\int_M |\nabla W_A(x)|_g^2 d\upsilon_g + \frac{1}{2}\int_M h(x)|W_A(x)|^2 d\upsilon_g - \frac{1}{2^*-\varepsilon}\int_M |W_A(x)|^{2^*-\varepsilon} d\upsilon_g \\ &= \int\limits_{B(0,\frac{r}{\lambda})} \left[\frac{1}{2} g_\xi^{ij} \frac{\partial \widetilde{Q}_{\tilde a}(z)}{\partial z_i}\frac{\partial \widetilde{Q}_{\tilde a}(z)}{\partial z_j} + \frac{1}{2}\lambda^2 h(\exp_\xi(\lambda z))|\widetilde{Q}_{\tilde a}(z)|^2 - \frac{1}{2^*-\varepsilon}\lambda^{\frac{n-2}{2}\varepsilon}|\widetilde{Q}_{\tilde a}(z)|^{2^*-\varepsilon}\right] \times \\ &\qquad \times\Big(1-\frac{\lambda^2}{6}R_{kl}z^k z^l + O(\lambda^3|z|^3)\Big)dz \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{2}\int_M |\nabla W_A(x)|_g^2 d\upsilon_g &= \frac{1}{2}\int\limits_{B(0,\frac{r}{\lambda})} g_\xi^{ij} \frac{\partial \widetilde{Q}_{\tilde a}(z)}{\partial z_i}\frac{\partial \widetilde{Q}_{\tilde a}(z)}{\partial z_j}\Big(1-\frac{\lambda^2}{6}R_{kl}z^k z^l + O(\lambda^3|z|^3)\Big)dz \\ &= \frac{1}{2}\int\limits_{\mathbb{R}^n} |\nabla \widetilde{Q}_{\tilde a}(z)|^2 dz - \frac{\lambda^2}{12}R_{kl}\int\limits_{\mathbb{R}^n} |\nabla \widetilde{Q}_{\tilde a}(z)|^2 z^k z^l dz + o(\lambda^2), \end{aligned}$$

and

$$\frac{1}{2}\int_M h(x)|W_A(x)|^2 d\upsilon_g = \frac{\lambda^2}{2}h(\xi)\int\limits_{\mathbb{R}^n} |\widetilde{Q}_{\tilde a}(z)|^2 dz + o(\lambda^2).$$

On the other hand,

$$\begin{aligned} &\frac{1}{2^*-\varepsilon}\int_M |W_A(x)|^{2^*-\varepsilon} d\upsilon_g \\ &= \frac{1}{2^*-\varepsilon}\lambda^{\frac{n-2}{2}\varepsilon}\int\limits_{B(0,\frac{r}{\lambda})} |\widetilde{Q}_{\tilde a}(z)|^{2^*-\varepsilon}\Big(1-\frac{\lambda^2}{6}R_{kl}z^k z^l + O(\lambda^3|z|^3)\Big)dz \\ &= \Big(\frac{n-2}{2n} + \frac{(n-2)^2}{4n^2}\varepsilon + O(\varepsilon^2)\Big)\Big(1+\varepsilon\frac{n-2}{2}\log(\lambda) + O(\varepsilon^2)\Big) \times \\ &\quad \times \int\limits_{\mathbb{R}^n} |\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}\Big(1-\varepsilon\log|\widetilde{Q}_{\tilde a}(z)| + O(\varepsilon^2)\Big)\Big(1-\frac{\lambda^2}{6}R_{kl}z^k z^l + O(\lambda^3|z|^3)\Big)dz \\ &= \frac{n-2}{2n}\int\limits_{\mathbb{R}^n} |\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}} dz - \lambda^2\frac{n-2}{12n}R_{kl}\int\limits_{\mathbb{R}^n} |\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}} z^k z^l dz \\ &\quad +\varepsilon\Big(\frac{(n-2)^2}{4n^2}\int\limits_{\mathbb{R}^n} |\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}} dz - \frac{n-2}{2n}\int\limits_{\mathbb{R}^n} |\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}\log|\widetilde{Q}_{\tilde a}(z)| dz\Big) \\ &\quad +\varepsilon\log(\lambda)\frac{n-2}{2}\int\limits_{\mathbb{R}^n} |\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}} dz + o(\lambda^2) + o(\varepsilon). \end{aligned}$$

Therefore, for $\lambda = \sqrt{t|\varepsilon|}$, we get

$$J_\varepsilon(W_A(x)) = \frac{1}{n}\int\limits_{\mathbb{R}^n} |\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}} dz - \varepsilon\ \log(t)\frac{n-2}{4}\int\limits_{\mathbb{R}^n} |\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}} dz$$

$$+t\Big[\Big(\frac{1}{2}\int\limits_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^2dz\Big)h(\xi)-\frac{R_{kl}}{6}\Big(\frac{1}{2}\int\limits_{\mathbb{R}^n}|\nabla\widetilde{Q}_{\tilde a}(z)|^2z^kz^ldz$$

$$-\frac{n-2}{2n}\int\limits_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}z^kz^ldz\Big)\Big]|\varepsilon|$$

$$-\Big[\frac{(n-2)^2}{4n^2}\int\limits_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}dz-\frac{n-2}{2n}\int\limits_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}\log|\widetilde{Q}_{\tilde a}(z)|dz\Big]\varepsilon$$

$$-\varepsilon\log|\varepsilon|\ \frac{n-2}{4}\int\limits_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}dz\ +o(|\varepsilon|).$$

Since

$$\int\limits_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}dz=\int\limits_{\mathbb{R}^n}|z|^{-2n}\Big|Q\Big(\frac{z}{|z|^2}-\tilde a\Big)\Big|^{\frac{2n}{n-2}}dz=\int\limits_{\mathbb{R}^n}|Q(y)|^{\frac{2n}{n-2}}dy:=c_0,$$

$$\int\limits_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^2dz=\frac{1}{2}\int\limits_{\mathbb{R}^n}\frac{|Q(y)|^2}{|y+\tilde a|^4}dy,$$

and

$$\int\limits_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}\log|\widetilde{Q}_{\tilde a}(z)|dz$$

$$=\int\limits_{\mathbb{R}^n}|z|^{-2n}\Big|Q\Big(\frac{z}{|z|^2}-\tilde a\Big)\Big|^{\frac{2n}{n-2}}\log\Big||z|^{2-n}Q\Big(\frac{z}{|z|^2}-\tilde a\Big)\Big|dz$$

$$=\int\limits_{\mathbb{R}^n}\Big|Q\Big(y-\tilde a\Big)\Big|^{\frac{2n}{n-2}}\log\Big||y|^{n-2}Q\Big(y-\tilde a\Big)\Big|dy$$

$$=(n-2)\int\limits_{\mathbb{R}^n}\Big|Q(y-\tilde a)\Big|^{\frac{2n}{n-2}}\log|y|dy+\int\limits_{\mathbb{R}^n}\Big|Q\Big(y-\tilde a\Big)\Big|^{\frac{2n}{n-2}}\log\Big|Q\Big(y-\tilde a\Big)\Big|dy$$

$$=(n-2)\int\limits_{\mathbb{R}^n}|Q(y)|^{\frac{2n}{n-2}}\log\big|y+\tilde a\big|dy+\int\limits_{\mathbb{R}^n}|Q(y)|^{\frac{2n}{n-2}}\log|Q(y)|dy.$$

Then we find

$$J_\varepsilon(W_A(x))=\frac{1}{n}c_0-\frac{n-2}{4}c_0\ \varepsilon\log|\varepsilon|-\Big(\frac{(n-2)^2}{4n^2}c_0-\frac{n-2}{2n}c_1\Big)\varepsilon-\frac{n-2}{4}c_0\log(t)\varepsilon$$

$$+t\Big[\Big(\frac{1}{2}\int\limits_{\mathbb{R}^n}\frac{|Q(y)|^2}{|y+\tilde a|^4}dy\Big)h(\xi)-\frac{R_{kl}}{12}\Big(\int\limits_{\mathbb{R}^n}|\nabla\widetilde{Q}_{\tilde a}(z)|^2z^kz^ldz-\frac{n-2}{n}\int\limits_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}z^kz^ldz\Big)\Big]|\varepsilon|$$

$$+\frac{(n-2)^2}{2n}\int\limits_{\mathbb{R}^n}|Q(y)|^{\frac{2n}{n-2}}\log\big|y+\tilde a\big|dy\ \varepsilon+o(|\varepsilon|),$$

(5.6)

where

$$c_1 = \int\limits_{\mathbb{R}^n} |Q(y)|^{\frac{2n}{n-2}} \log |Q(y)| dy.$$

Now we observer that $\tilde{a} = P_\theta a$, for $|\tilde{a}|$ small, a Taylor expansion, we have

$$\begin{aligned}
|y+\tilde{a}|^{-4} &= \big(|y|^2 + 2y\tilde{a} + |\tilde{a}|^2\big)^{-2} = |y|^{-4}\Big(1 + 2\frac{y\tilde{a}}{|y|^2} + \frac{|\tilde{a}|^2}{|y|^2}\Big)^{-2} \\
&= |y|^{-4}\left(1 - 2\Big(2\frac{y\tilde{a}}{|y|^2} + \frac{|\tilde{a}|^2}{|y|^2}\Big) + 3\Big(2\frac{y\tilde{a}}{|y|^2} + \frac{|\tilde{a}|^2}{|y|^2}\Big)^2 + o(|\tilde{a}|^2)\right) \\
&= \frac{1}{|y|^4} - 4\frac{y\tilde{a}}{|y|^6} - 2\frac{|\tilde{a}|^2}{|y|^6} + 12\frac{(y\tilde{a})^2}{|y|^8} + \frac{o(|\tilde{a}|^2)}{|y|^4},
\end{aligned} \tag{5.7}$$

where $y\tilde{a} = y_1\tilde{a}_1 + y_2\tilde{a}_2$, then $\int\limits_{\mathbb{R}^n} \frac{y\tilde{a}}{|y|^6}|Q(y)|^2 dy = 0$ and $\int\limits_{\mathbb{R}^n} \frac{(y\tilde{a})^2}{|y|^8}|Q(y)|^2 dy = \frac{|\tilde{a}|^2}{n}\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^6} dy$. Thus

$$\begin{aligned}
\frac{1}{2}\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y+\tilde{a}|^4} dy &= \frac{1}{2}\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^4} dy - |\tilde{a}|^2 \int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^6} dy + \frac{6|\tilde{a}|^2}{n}\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^6} dy + o(|\tilde{a}|^2) \\
&= \frac{1}{2}\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^4} dy - |\tilde{a}|^2 \frac{n-6}{n}\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^6} dy + o(|\tilde{a}|^2) \\
&= \beta\left(\frac{1}{2} - |a|^2\right) + o(|a|^2) + o(\varepsilon),
\end{aligned} \tag{5.8}$$

where $\beta = \frac{\omega_{n-1}}{2}\alpha_n^2 I_{n-2}^{\frac{n-6}{2}}$.

Recall that $\widetilde{Q}_{\tilde{a}}(z) = |z|^{2-n} Q\Big(\frac{z}{|z|^2} - \tilde{a}\Big)$, we have

$$|\nabla \widetilde{Q}_{\tilde{a}}(z)|^2 = |z|^{-2n}|\nabla_y Q(y)|^2 + (n-2)^2|z|^{2-2n}|Q(y)|^2 + 2(n-2)|z|^{-2n}Q(y)\nabla_y Q(y) z$$

where $y = \frac{z}{|z|^2} - P_\theta a$. Thus

$$\begin{aligned}
& R_{kl}\int_{\mathbb{R}^n} |\nabla \widetilde{Q}_{\tilde{a}}(z)|^2 z^k z^l dz \\
=\ & R_{kl}\int_{\mathbb{R}^n} \Big[|\nabla_y Q(y)|^2 + (n-2)^2\frac{|Q(y)|^2}{|y+P_\theta a|^2} \\
& \qquad +2(n-2)\frac{(y+P_\theta a)\nabla_y Q(y) Q(y)}{|y+P_\theta a|^2}\Big] \frac{(y+P_\theta a)^k (y+P_\theta a)^l}{|y+P_\theta a|^4} dy \\
=\ & R_{kl}\int_{\mathbb{R}^n} \frac{|\nabla_y Q(y)|^2}{|y+P_\theta a|^4}(y+P_\theta a)^k (y+P_\theta a)^l dy \\
& +(n-2)^2 R_{kl}\int_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y+P_\theta a|^6}(y+P_\theta a)^k (y+P_\theta a)^l dy \\
& +2(n-2) R_{kl}\int_{\mathbb{R}^n} \frac{(y+P_\theta a)\nabla_y Q(y) Q(y)}{|y+P_\theta a|^6}(y+P_\theta a)^k (y+P_\theta a)^l dy \\
:=\ & I_1 + I_2 + I_3.
\end{aligned}$$

Using (5.7), we have

$$I_1 = R_{kl}\int_{\mathbb{R}^n} \left[\frac{|\nabla_y Q(y)|^2}{|y|^4} - 4\frac{|\nabla_y Q(y)|^2}{|y|^6} y P_\theta a - \frac{2(n-6)}{n}\frac{|\nabla_y Q(y)|^2}{|y|^6}|P_\theta a|^2 + o(|a|^2)\right] \times$$

$$
\begin{aligned}
&\quad\times\Big(y^ky^l+y^k(P_\theta a)^l+y^l(P_\theta a)^k+(P_\theta a)^k(P_\theta a)^l\Big)\,dy\\
&=\frac{Scal_g(\xi)}{n}\int_{\mathbb{R}^n}\left[\frac{|\nabla_yQ(y)|^2}{|y|^2}-\frac{2(n-6)}{n}\frac{|\nabla_yQ(y)|^2}{|y|^4}|a|^2\right]dy\\
&\quad+R_{kl}(P_\theta a)^k(P_\theta a)^l\int_{\mathbb{R}^n}\frac{|\nabla_yQ(y)|^2}{|y|^4}dy+o(|a|^2)+o(\varepsilon)\\
&=\frac{(n-2)^2}{n}\frac{\omega_{n-1}}{2}\alpha_n^2\left(I_n^{\frac{n-2}{2}}+KI_n^{\frac{n}{2}}-\frac{2(n-6)}{n}\Big(I_n^{\frac{n-4}{2}}+KI_n^{\frac{n}{2}}\Big)|a|^2\right)\ Scal_g(\xi)\\
&\quad+(n-2)^2\frac{\omega_{n-1}}{2}\alpha_n^2\left(I_n^{\frac{n-4}{2}}+KI_n^{\frac{n}{2}}\right)\ R_{kl}(P_\theta a)^k(P_\theta a)^l+o(|a|^2)+o(\varepsilon)\\
&=\beta\left[\frac{(n-2)^2(n-4)}{4n(n-1)}+\frac{(n-2)(n-4)}{4(n-1)}K-\frac{(n-2)(n-4)(n-6)}{2n(n-1)}(1+K)|a|^2\right]Scal_g(\xi)\\
&\quad+\beta\frac{n(n-2)(n-4)}{4(n-1)}(1+K)\ R_{kl}(P_\theta a)^k(P_\theta a)^l+o(|a|^2)+o(\varepsilon).
\end{aligned}
$$

Moreover, since

$$
|y+\tilde a|^{-6}=\frac{1}{|y|^6}-6\frac{y\tilde a}{|y|^8}-\frac{3(n-8)}{n}\frac{|\tilde a|^2}{|y|^8}+\frac{o(|\tilde a|^2)}{|y|^8},
$$

We have

$$
\begin{aligned}
I_2&=\frac{(n-2)^2}{n}Scal_g(\xi)\int_{\mathbb{R}^n}\left[\frac{|Q(y)|^2}{|y|^4}-\frac{3(n-8)}{n}\frac{|Q(y)|^2}{|y|^6}|a|^2\right]dy\\
&\quad+(n-2)^2R_{kl}(P_\theta a)^k(P_\theta a)^l\int_{\mathbb{R}^n}\frac{|Q(y)|^2}{|y|^6}dy+o(|a|^2)+o(\varepsilon)\\
&=\frac{(n-2)^2}{n}\frac{\omega_{n-1}}{2}\alpha_n^2\left(I_{n-2}^{\frac{n-6}{2}}-\frac{3(n-8)}{n}I_{n-2}^{\frac{n-8}{2}}|a|^2\right)Scal_g(\xi)\\
&\quad+(n-2)^2\frac{\omega_{n-1}}{2}\alpha_n^2I_{n-2}^{\frac{n-8}{2}}\ R_{kl}(P_\theta a)^k(P_\theta a)^l+o(|a|^2)+o(\varepsilon)\\
&=\beta\left(\frac{(n-2)^2}{n}-\frac{3(n-2)^2(n-8)}{n(n-6)}|a|^2\right)Scal_g(\xi)\\
&\quad+\beta\frac{n(n-2)^2}{n-6}\ R_{kl}(P_\theta a)^k(P_\theta a)^l+o(|a|^2)+o(\varepsilon),
\end{aligned}
$$

and

$$
\begin{aligned}
I_3&=2(n-2)R_{kl}\int_{\mathbb{R}^n}\frac{(y+P_\theta a)\nabla Q(y)Q(y)}{|y+P_\theta a|^6}(y+P_\theta a)^k(y+P_\theta a)^ldy\\
&=\frac{2(n-2)}{n}Scal_g(\xi)\int_{\mathbb{R}^n}\left[\frac{y\nabla Q(y)Q(y)}{|y|^4}-\frac{3(n-8)}{n}\frac{y\nabla Q(y)Q(y)}{|y|^4}|a|^2\right]dy\\
&\quad+2(n-2)R_{kl}(P_\theta a)^k(P_\theta a)^l\int_{\mathbb{R}^n}\frac{y\nabla Q(y)Q(y)}{|y|^6}dy+o(|a|^2)+o(\varepsilon)\\
&=-\frac{2(n-2)^2}{n}\frac{\omega_{n-1}}{2}\alpha_n^2\left(I_{n-1}^{\frac{n-4}{2}}-\frac{3(n-8)}{n}I_{n-1}^{\frac{n-6}{2}}|a|^2\right)\ Scal_g(\xi)\\
&\quad-2(n-2)^2R_{kl}\frac{\omega_{n-1}}{2}\alpha_n^2I_{n-1}^{\frac{n-6}{2}}\ (P_\theta a)^k(P_\theta a)^l+o(|a|^2)+o(\varepsilon)
\end{aligned}
$$

$$= \beta\left(-\frac{(n-2)(n-4)}{n}+\frac{3(n-2)(n-8)}{n}|a|^2\right)Scal_g(\xi)$$
$$-\beta n(n-2)\ R_{kl}(P_\theta a)^k(P_\theta a)^l+o(|a|^2)+o(\varepsilon).$$

Therefore, we obtain

$$\begin{aligned}
&I_1+I_2+I_3\\
&=\beta\left(\frac{(n-2)(n+2)}{4(n-1)}+K\frac{(n-2)(n-4)}{4(n-1)}\right)\ Scal_g(\xi)\\
&-\beta\left(\frac{(n-2)(n^3+8n^2-132n+48)}{2n(n-1)(n-6)}+K\frac{(n-2)(n-4)(n-6)}{2n(n-1)}\right)|a|^2\ Scal_g(\xi)\\
&+\beta\left(\frac{n(n-2)(n+2)(n+4)}{4(n-1)(n-6)}+\frac{n(n-2)(n-4)}{4(n-1)}K\right)R_{kl}\ (P_\theta a)^k(P_\theta a)^l\\
&+o(|a|^2)+o(\varepsilon),
\end{aligned} \tag{5.9}$$

where $\beta=\frac{\omega_{n-1}}{2}\alpha_n^2 I_{n-2}^{\frac{n-6}{2}}$.

On the other hand,

$$\begin{aligned}
&R_{kl}\int_{\mathbb{R}^n}|\widetilde{Q}_{\tilde a}(z)|^{\frac{2n}{n-2}}z^kz^l dz=R_{kl}\int_{\mathbb{R}^n}\frac{|Q(y)|^{\frac{2n}{n-2}}}{|y+P_\theta a|^4}(y+P_\theta a)^k(y+P_\theta a)^l dy\\
&=\frac{\omega_{n-1}}{2}\alpha_n^{\frac{2n}{n-2}}\frac{1}{n}\left(\left(I_n^{\frac{n-4}{2}}+KI_n^{\frac{n-2}{2}}\right)-\frac{2(n-6)}{n}\left(I_n^{\frac{n-6}{2}}+KI_n^{\frac{n-2}{2}}\right)|a|^2\right)\ Scal_g(\xi)\\
&+\frac{\omega_{n-1}}{2}\alpha_n^{\frac{2n}{n-2}}\ \left(I_n^{\frac{n-6}{2}}+KI_n^{\frac{n-2}{2}}\right)\ R_{kl}(P_\theta a)^k(P_\theta a)^l+o(|a|^2)+o(\varepsilon)\\
&=\beta\left(\frac{n(n-4)}{4(n-1)}\left(1+K\frac{n-2}{n}\right)-\frac{(n-6)(n+2)}{2(n-1)}\left(1+K\frac{(n-2)(n-4)}{n(n+2)}\right)|a|^2\right)\ Scal_g(\xi)\\
&+\beta\frac{n^2(n+2)}{4(n-1)}\left(1+K\frac{(n-2)(n-4)}{n(n+2)}\right)\ R_{kl}(P_\theta a)^k(P_\theta a)^l+o(|a|^2)+o(\varepsilon).
\end{aligned} \tag{5.10}$$

Finally, we have

$$\int_{\mathbb{R}^n}|Q(y)|^{\frac{2n}{n-2}}\log\left|y+\tilde a\right|dy=c_2+\beta\frac{n^2(n-4)}{8(n-1)}\left(1+K\frac{n-2}{n}\right)|a|^2+o(|a|^2)+o(\varepsilon), \tag{5.11}$$

where

$$c_2=\int_{\mathbb{R}^n}|Q(y)|^{\frac{2n}{n-2}}\log|y|dy.$$

From (5.6), (5.8)-(5.11), we get

$$J_\varepsilon(W_A(x))=\frac{c_0}{n}-d_1\varepsilon\log|\varepsilon|-d_2\varepsilon+\frac{\beta}{2}\Psi(t,\xi,a,\theta)\varepsilon+o(|\varepsilon|).$$

where

$$\begin{aligned}
\Psi(t,\xi,a,\theta)=&-d_3\log t+\mathrm{sign}(\varepsilon)\varphi(\xi)t-\mathrm{sign}(\varepsilon)d_4a(B_{\xi,\theta})a^T\ t\\
&+\left[\mathrm{sign}(\varepsilon)\left(-2\varphi(\xi)+d_5\mathrm{Scal}_g(\xi)\right)t+d_6\right]|a|^2+o(|a|^2)
\end{aligned}$$

where $B_{\xi,\theta} = (P_\theta)^T (R_{ij})_{n\times n} P_\theta$ is a $n \times n$ matrix, and

$$\varphi(\xi) = h(\xi) - \frac{n-2}{4(n-1)} \left(1 + \frac{n-4}{3n} K\right) Scal_g(\xi).$$

The constants c_0, c_1, c_2 and d_i, $i = 1, \dots, 6$ are given in Proposition 3.2.

6. Appendix

6.1. **The sign changing solution Q_K.** Recall that, In [7,8] it was proved that there exists k_0 such that for all integer $k > k_0$ there exists a solution $Q = Q_k$ to (1.6) that can be described as follows

$$Q_k(y) = U_*(y) + \tilde{\phi}(y). \tag{6.1}$$

where

$$U_*(y) = U(y) - \sum_{j=1}^{k} U_j(y), \tag{6.2}$$

while $\tilde{\phi}$ is smaller than U_*. The functions U and U_j are positive solutions to (1.6), respectively defined as

$$U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad U_j(x) = \mu_k^{-\frac{n-2}{2}} U(\mu_k^{-1}(y - \xi_j)), \tag{6.3}$$

where $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$. For any integer k large, the parameters $\mu_k > 0$ and the k points ξ_l, $l = 1, \dots, k$ are given by

$$\left[\sum_{l>1}^{k} \frac{1}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\right] \mu_k^{\frac{n-2}{2}} = \left(1 + O(\frac{1}{k})\right), \quad \text{for} \quad k \to \infty$$

in particular $\mu_k \sim k^{-2}$ if $n \geq 4$, and $\mu_k \sim k^{-2} |\log k|^{-2}$ if $n = 3$, as $k \to \infty$, and

$$\xi_l = \sqrt{1-\mu^2}\, (\mathsf{n}_l, 0).$$

In (6.1), $\tilde{\phi}(y)$ satisfies

$$|\tilde{\phi}(y)| = O\left(\frac{k^{-\frac{n}{q}}}{(1+|y|)^{n-2}}\right),$$

for $q > \frac{n}{2}$.

6.2. **About the non-degeneracy of the basic cell.** Let Σ be the set of nontrivial solutions of equation

$$-\Delta Q = |Q|^{\frac{4}{n-2}} Q, \quad \text{in } \mathbb{R}^n.$$

Let $\mathcal{N}$ be the group of one-to-one maps of $\mathbb{R}^n \cup \{\infty\}$ generated by

- the translations $T_a : y \mapsto y + a$, where $a \in \mathbb{R}^n$;
- the dilations $D_\lambda : y \mapsto \lambda y$, where $\lambda > 0$;
- the linear isometries P_θ;
- the inversion $J : y \mapsto \frac{y}{|y|^2}$.

From [10], for $x, \xi \in M$, we then have

$$Q\left(P_\theta \circ J \circ T_{-a} \circ J \circ D_{\lambda^{-1}} \circ P_\theta^{-1} \circ exp_\xi^{-1}(x)\right) \in \Sigma.$$

In [18], is was proved that these solutions are *non degenerate*. That is, fix one solution $Q = Q_K$ of problem (1.6) and define the linearized equation around Q as follows

$$L(\phi) = \Delta\phi + p|Q|^{p-1}\phi. \tag{6.4}$$

The invariances (1.11), (1.12), (1.13), together with the natural invariance of any solution to (1.6) under translation (if u solves (1.6) then also $u(y+\zeta)$ solves (1.6) for any $\zeta \in \mathbb{R}^n$) and under dilation (if u solves (1.6) then $\lambda^{-\frac{n-2}{2}}u(\lambda^{-1}y)$ solves (1.6) for any $\lambda > 0$) produce some *natural* functions φ in the kernel of L, namely

$$L(\varphi) = 0.$$

These are the $3n$ linearly independent functions we introduce next:

$$z_0(y) = \frac{n-2}{2}Q(y) + \nabla Q(y) \cdot y, \tag{6.5}$$

$$z_\alpha(y) = \frac{\partial}{\partial y_\alpha}Q(y), \quad \text{for} \quad \alpha = 1, \dots, n, \tag{6.6}$$

and

$$z_{n+1}(y) = -y_2\frac{\partial}{\partial y_1}Q(y) + y_1\frac{\partial}{\partial y_2}Q(y), \tag{6.7}$$

$$z_{n+2}(y) = -2y_1z_0(y) + |y|^2z_1(y), \quad z_{n+3}(y) = -2y_2z_0(y) + |y|^2z_2(y) \tag{6.8}$$

and, for $l = 3, \dots, n$

$$z_{n+l+1}(y) = -y_lz_1(y) + y_1z_l(y), \quad z_{2n+l-1}(y) = -y_lz_2(y) + y_2z_l(y). \tag{6.9}$$

Indeed, a direct computation gives that

$$L(z_\alpha) = 0, \quad \text{for all} \quad \alpha = 0, 1, \dots, 3n-1.$$

A solution Q is said to be non degenerate if

$$\text{Kernel}(L) = \text{Span}\{z_\alpha \, : \, \alpha = 0, 1, 2, \dots, 3n-1\}, \tag{6.10}$$

or equivalently, any bounded (or any solution in $\mathcal{D}^{1,2}$) of $L(\varphi) = 0$ is a linear combination of the functions z_α, $\alpha = 0, \dots, 3n-1$.

The function z_0 defined in (6.5) is related to the invariance of Problem (1.6) with respect to dilation $\lambda^{-\frac{n-2}{2}}Q(\lambda^{-1}y)$. The functions z_i, $i = 1, \dots, n$, defined in (6.6) are related to the invariance of Problem (1.6) with respect to translation $Q(y+\zeta)$. The function z_{n+1} defined in (6.7) is related to the invariance of Q under rotation in the (y_1, y_2) plane. The two functions z_{n+2} and z_{n+3} defined in (6.8) are related to the invariance of Problem (1.6) under Kelvin transformation (1.13). The functions defined in (6.9) are related to the invariance under rotation in the (y_1, y_l) plane and in the (y_2, y_l) plane respectively.

Let us be more precise. Denote by $O(n)$ the orthogonal group of $n \times n$ matrices P with real coefficients, so that $P^TP = I$, and by $SO(n) \subset O(n)$ the special orthogonal group of all matrices in $O(n)$ with $detP = 1$. $SO(n)$ is the group of all rotations in $\mathbb{R}^n$, it is a compact group, which can be identified with a compact set in $\mathbb{R}^{\frac{n(n-1)}{2}}$. Consider the sub group $\hat{S}$ of $SO(n)$ generated by rotations in the (x_1, x_2)-plane, in the (x_j, x_α)-plane, for any $j = 1, 2$ and $\alpha = 3, \dots, n$. We have that $\hat{S}$ is compact and can be identified with a compact manifold of dimension $2n-3$, with no boundary. In other words, there exists a smooth injective map $\chi : \hat{S} \to \mathbb{R}^{\frac{n(n-1)}{2}}$ so that $\chi(\hat{S})$ is a compact manifold of dimension $2n-3$ with no boundary and $\chi^{-1} : \chi(\hat{S}) \to \hat{S}$ is a smooth parametrization of $\hat{S}$ in a neighborhood of the Identity. Thus we write

$$\theta \in \mathcal{O} = \chi(\hat{S}), \quad P_\theta = \chi^{-1}(\theta)$$

where $\mathcal{O}$ is a compact manifold of dimension $2n-3$ with no boundary and P_θ denotes a rotation in $\hat{S}$. Let $\theta = (\theta_{12}, \theta_{13}, \ldots, \theta_{1n}, \theta_{23}, \ldots, \theta_{2n})$, and we write

$$P_\theta = P_{12}(\theta_{12})P_{13}(\theta_{13})P_{14}(\theta_{14}) \cdots P_{1n}(\theta_{1n})P_{23}(\theta_{23})P_{24}(\theta_{24}) \cdots P_{2n}(\theta_{2n}),$$

where $P_{ij}(\theta_{ij})$ is the Rotation in the $(i,j)-$plane,

$$P_{ij}(\theta_{ij}) = \begin{pmatrix} 1 \cdots & 0 & 0 \cdots 0 & 0 & \cdots 0 \\ \vdots \ddots & \vdots & \vdots \ddots \vdots & \vdots & \ddots \vdots \\ 0 \cdots & \cos\theta_{ij} & 0 \cdots 0 & -\sin\theta_{ij} & \cdots 0 \\ 0 \cdots & 0 & 1 \cdots 0 & 0 & \cdots 0 \\ \vdots \ddots & \vdots & \vdots \ddots \vdots & \vdots & \ddots \vdots \\ 0 \cdots & 0 & 0 \cdots 1 & 0 & \cdots 0 \\ 0 \cdots & \sin\theta_{ij} & 0 \cdots 0 & \cos\theta_{ij} & \cdots 0 \\ \vdots \ddots & \vdots & \vdots \ddots \vdots & \vdots & \ddots \vdots \\ 0 \cdots & 0 & 0 \cdots 0 & 0 & \cdots 1 \end{pmatrix}, \qquad i < j.$$

We set

$$P_\theta = (c_{ij})_{n\times n}.$$

By a direct calculation, we have

$$\begin{aligned} c_{11} &= \cos\theta_{12}\cos\theta_{13}\cos\theta_{14}\cdots\cos\theta_{1n}, \\ c_{i1} &= \sin\theta_{1i}\cos\theta_{1,i+1}\cos\theta_{1,i+2}\cdots\cos\theta_{1n}, \quad i = 2,3,\ldots,n, \end{aligned}$$

and

$$\begin{aligned} c_{12} = &-\sin\theta_{12}\cos\theta_{23}\cos\theta_{2,4}\cdots\cos\theta_{2n} \\ &-\cos\theta_{12}\sin\theta_{13}\sin\theta_{23}\cos\theta_{24}\cdots\cos\theta_{2n} \\ &-\cos\theta_{12}\cos\theta_{13}\sin\theta_{14}\sin\theta_{24}\cos\theta_{25}\cdots\cos\theta_{2n} \\ &-\cdots \\ &-\cos\theta_{12}\cos\theta_{13}\cos\theta_{14}\cdots\cos\theta_{1,n-1}\cos\theta_{1,n-1}\sin\theta_{2,n-1}\cos\theta_{2n} \\ &-\cos\theta_{12}\cos\theta_{1,3}\cos\theta_{14}\cdots\cos\theta_{1,n-1}\cos\theta_{1,n-1}\sin\theta_{1n}\sin\theta_{2n}, \end{aligned}$$

and for $i = 2,3,\ldots,n$,

$$\begin{aligned} c_{i2} = &\cos\theta_{1i}\sin\theta_{2i}\cos\theta_{2,i+1}\cos\theta_{2,i+2}\cdots\cos\theta_{2n} \\ &-\sin\theta_{1i}\sin\theta_{1,i+1}\sin\theta_{2,i+1}\cos\theta_{2,i+2}\cos\theta_{2,i+2}\cdots\cos\theta_{2n} \\ &-\sin\theta_{1i}\cos\theta_{1,i+1}\sin\theta_{1,i+2}\sin\theta_{2,i+2}\cos\theta_{2,i+3}\cos\theta_{2,i+2}\cdots\cos\theta_{2n} \\ &-\cdots \\ &-\sin\theta_{1i}\cos\theta_{1,i+1}\cos\theta_{1,i+2}\cdots\cos\theta_{1,n-2}\sin\theta_{1,n-1}\sin\theta_{2,n-1}\cos\theta_{2n} \\ &-\sin\theta_{1i}\cos\theta_{1,i+1}\cos\theta_{1,i+2}\cdots\cos\theta_{1,n-2}\cos\theta_{1,n-1}\sin\theta_{1,n}\sin\theta_{2n}. \end{aligned}$$

6.3. Some useful estimates.

Lemma 6.1. *We have*

$$\int\limits_{\mathbb{R}^n} \frac{|\nabla Q(y)|^2}{|y|^2} dy = (n-2)^2 \frac{\omega_{n-1}}{2} \alpha_n^2 I_n^{\frac{n-2}{2}} + O\left(k^{-\frac{n}{q}}\right), \tag{6.11}$$

$$\int\limits_{\mathbb{R}^n} \frac{|\nabla Q(y)|^2}{|y|^4} dy = (n-2)^2 \frac{\omega_{n-1}}{2} \alpha_n^2 I_n^{\frac{n-4}{2}} + O\left(k^{-\frac{n}{q}}\right), \tag{6.12}$$

$$\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^4} dy = \frac{\omega_{n-1}}{2} \alpha_n^2 I_{n-2}^{\frac{n-6}{2}} + O\left(k^{-\frac{n}{q}}\right), \tag{6.13}$$

$$\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^6} dy = \frac{\omega_{n-1}}{2} \alpha_n^2 I_{n-2}^{\frac{n-8}{2}} + O\left(k^{-\frac{n}{q}}\right), \tag{6.14}$$

$$\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^{\frac{2n}{n-2}}}{|y|^2} dy = \frac{\omega_{n-1}}{2} \alpha_n^{\frac{2n}{n-2}} I_n^{\frac{n-4}{2}} + \frac{\omega_{n-1}}{2} \alpha_n^{\frac{2n}{n-2}} k I_n^{\frac{n-2}{2}} + O\left(k^{-\frac{n}{q}}\right), \tag{6.15}$$

and

$$\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^{\frac{2n}{n-2}}}{|y|^4} dy = \frac{\omega_{n-1}}{2} \alpha_n^{\frac{2n}{n-2}} I_n^{\frac{n-6}{2}} + \frac{\omega_{n-1}}{2} \alpha_n^{\frac{2n}{n-2}} k I_n^{\frac{n-2}{2}} + O\left(k^{-\frac{n}{q}}\right). \tag{6.16}$$

Proof. Proof of (6.13): by the definition of Q, we have

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^4} dy &= \int\limits_{\mathbb{R}^n} \frac{\left|U(y) - \sum_{j=1}^k U_j(y) + \tilde{\phi}(y)\right|^2}{|y|^4} dy \\
&= \int\limits_{\mathbb{R}^n} \frac{|U(y)|^2}{|y|^4} dy + \sum_{j=1}^k \int\limits_{\mathbb{R}^n} \frac{|U_j(y)|^2}{|y|^4} dy + \int\limits_{\mathbb{R}^n} \frac{|\tilde{\phi}(y)|^2}{|y|^4} dy \\
&\quad + 2 \int\limits_{\mathbb{R}^n} \frac{|U(y)| \left|\sum_{j=1}^k U_j(y) + \tilde{\phi}(y)\right|}{|y|^4} dy.
\end{aligned} \tag{6.17}$$

Since

$$\int\limits_{\mathbb{R}^n} \frac{|U(y)|^2}{|y|^4} dy = \frac{\omega_{n-1}}{2} \alpha_n^2 I_{n-2}^{\frac{n-6}{2}}, \tag{6.18}$$

$$\begin{aligned}
\sum_{j=1}^k \int\limits_{\mathbb{R}^n} \frac{|U_j(y)|^2}{|y|^4} dy &= \alpha_n^2 \sum_{j=1}^k \int\limits_{\mathbb{R}^n} \frac{\mu_k^{n-2}}{(\mu_k^2 + |y-\xi_j|^2)^{n-2}} \frac{1}{|y|^4} dy \\
&= \mu_k^2 \alpha_n^2 \sum_{j=1}^k \int\limits_{\mathbb{R}^n} \frac{1}{(1+|z|^2)^{n-2}} \frac{1}{|\mu_k z + \xi_j|^4} dy
\end{aligned}$$

$$
\begin{aligned}
&= \mu_k^2\alpha_n^2 \sum_{j=1}^{k} \int\limits_{|z|\le \frac{1}{2\mu_k}} \frac{1}{(1+|z|^2)^{n-2}} \frac{1}{|\mu_k z+\xi_j|^4} dy \\
&\quad + \mu_k^2\alpha_n^2 \sum_{j=1}^{k} \int\limits_{|z|\ge \frac{1}{2\mu_k}} \frac{1}{(1+|z|^2)^{n-2}} \frac{1}{|\mu_k z+\xi_j|^4} dy \\
&= \mu_k^2\alpha_n^2 \sum_{j=1}^{k} \int\limits_{|z|\le \frac{1}{2\mu_k}} \frac{1}{(1+|z|^2)^{n-2}} \frac{1}{|\xi_j|^4} \left(1+O\big(\frac{\mu_k z}{|\xi_j|}\big)\right) dy \\
&\quad + \mu_k^2\alpha_n^2 \sum_{j=1}^{k} \int\limits_{|z|\ge \frac{1}{2\mu_k}} \frac{1}{(1+|z|^2)^{n-2}} \frac{1}{|\mu_j z|^4} \left(1+O\big(\frac{|\xi_j|}{\mu_k z}\big)\right) dy \\
&= O\big(k\mu_k^2|\log\mu_k|\big) = O\big(k^{-3}\log k\big). \qquad (6.19)
\end{aligned}
$$

Using (6.1), we have

$$
\int\limits_{\mathbb{R}^n} \frac{|\tilde{\phi}(y)|^2}{|y|^4} dy = O\Big(k^{-\frac{2n}{q}} \int\limits_{\mathbb{R}^n} \frac{1}{(1+|y|)^{2(n-2)}|y|^4} dy\Big) = O\Big(k^{-\frac{2n}{q}}\Big). \qquad (6.20)
$$

Moreover,

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^n} \frac{|U(y)||\sum_{j=1}^k U_j(y)+\tilde{\phi}(y)|}{|y|^4} dy \\
&= \int_{B(0,\delta)} \cdots + \int_{\cup_{j=1}^k B(0,\delta)} \cdots + \int_{\mathbb{R}^n\setminus(B(0,\delta)\cup\cup_{j=1}^k B(0,\delta))} \cdots \\
&= O\left(\mu_k^{\frac{n-2}{2}} + k^{-\frac{n}{q}}\right) + O\left(\mu_k^{n-\frac{n-2}{2}}\right) + O\left(k^{-\frac{n}{q}}\right) \\
&= O\left(\mu_k^{\frac{n-2}{2}}\right) + O\left(k^{-\frac{n}{q}}\right) \\
&= O\left(k^{-(n-2)}\right) + O\left(k^{-\frac{n}{q}}\right) \\
&= O\left(k^{-\frac{n}{q}}\right) = o\left(k^{-1}\right) \quad \text{since } \frac{n}{2} < q < n,\ n\ge 4. \qquad (6.21)
\end{aligned}
$$

From (6.28)-(6.21), we get (6.13).

Proof of (6.14): As the same computation as (6.13) we can get. In fact, by the definition of Q, we have

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n} \frac{|Q(y)|^2}{|y|^6} dy &= \int\limits_{\mathbb{R}^n} \frac{\big|U(y)-\sum_{j=1}^k U_j(y)+\tilde{\phi}(y)\big|^2}{|y|^4} dy \\
&= \int\limits_{\mathbb{R}^n} \frac{|U(y)|^2}{|y|^6} dy + \sum_{j=1}^{k} \int\limits_{\mathbb{R}^n} \frac{|U_j(y)|^2}{|y|^6} dy + \int\limits_{\mathbb{R}^n} \frac{|\tilde{\phi}(y)|^2}{|y|^4} dy
\end{aligned}
$$

$$+2\int\limits_{\mathbb{R}^n}\frac{|U(y)||\sum_{j=1}^k U_j(y)+\tilde{\phi}(y)|}{|y|^6}dy. \tag{6.22}$$

Since

$$\int\limits_{\mathbb{R}^n}\frac{|U(y)|^2}{|y|^6}dy=\frac{\omega_{n-1}}{2}\alpha_n^2 I_{n-2}^{\frac{n-8}{2}}, \tag{6.23}$$

$$\begin{aligned}
\sum_{j=1}^k\int\limits_{\mathbb{R}^n}\frac{|U_j(y)|^2}{|y|^6}dy &= \alpha_n^2\sum_{j=1}^k\int\limits_{\mathbb{R}^n}\frac{\mu_k^{n-2}}{(\mu_k^2+|y-\xi_j|^2)^{n-2}}\frac{1}{|y|^6}dy\\
&=\mu_k^2\alpha_n^2\sum_{j=1}^k\int\limits_{\mathbb{R}^n}\frac{1}{(1+|z|^2)^{n-2}}\frac{1}{|\mu_k z+\xi_j|^6}dy\\
&=\mu_k^2\alpha_n^2\sum_{j=1}^k\int\limits_{|z|\le\frac{1}{2\mu_k}}\frac{1}{(1+|z|^2)^{n-2}}\frac{1}{|\mu_k z+\xi_j|^6}dy\\
&\quad+\mu_k^2\alpha_n^2\sum_{j=1}^k\int\limits_{|z|\ge\frac{1}{2\mu_k}}\frac{1}{(1+|z|^2)^{n-2}}\frac{1}{|\mu_k z+\xi_j|^6}dy\\
&=\mu_k^2\alpha_n^2\sum_{j=1}^k\int\limits_{|z|\le\frac{1}{2\mu_k}}\frac{1}{(1+|z|^2)^{n-2}}\frac{1}{|\xi_j|^6}\left(1+O(\frac{\mu_k z}{|\xi_j|})\right)dy\\
&\quad+\mu_k^2\alpha_n^2\sum_{j=1}^k\int\limits_{|z|\ge\frac{1}{2\mu_k}}\frac{1}{(1+|z|^2)^{n-2}}\frac{1}{|\mu_j z|^6}\left(1+O(\frac{|\xi_j|}{\mu_k z})\right)dy\\
&=O\big(k\mu_k^2|\log\mu_k|\big)=O\big(k^{-3}\log k\big).
\end{aligned} \tag{6.24}$$

Using (6.1), we have

$$\int\limits_{\mathbb{R}^n}\frac{|\tilde{\phi}(y)|^2}{|y|^6}dy=O\Big(k^{-\frac{2n}{q}}\int\limits_{\mathbb{R}^n}\frac{1}{(1+|y|)^{2(n-2)}|y|^6}dy\Big)=O\Big(k^{-\frac{2n}{q}}\Big). \tag{6.25}$$

Moreover,

$$\int\limits_{\mathbb{R}^n}\frac{|\tilde{\phi}(y)|^2}{|y|^6}dy=O\Big(k^{-\frac{2n}{q}}\int\limits_{\mathbb{R}^n}\frac{1}{(1+|y|)^{2(n-2)}|y|^6}dy\Big)=O\Big(k^{-\frac{2n}{q}}\Big). \tag{6.26}$$

Moreover,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n}\frac{|U(y)||\sum_{j=1}^k U_j(y)+\tilde{\phi}(y)|}{|y|^6}dy\\
&=\int_{B(0,\delta)}\cdots+\int_{\cup_{j=1}^k B(0,\delta)}\cdots+\int_{\mathbb{R}^n\setminus(B(0,\delta)\cup\cup_{j=1}^k B(0,\delta))}\cdots\\
&=O\left(\mu_k^{\frac{n-2}{2}}+k^{-\frac{n}{q}}\right)+O\left(\mu_k^{n-\frac{n-2}{2}}\right)+O\left(k^{-\frac{n}{q}}\right)
\end{aligned}$$

$$= O\left(\mu_k^{\frac{n-2}{2}}\right) + O\left(k^{-\frac{n}{q}}\right)$$

$$= O\left(k^{-(n-2)}\right) + O\left(k^{-\frac{n}{q}}\right) = O\left(k^{-\frac{n}{q}}\right). \tag{6.27}$$

From (6.22)-(6.27), we get (6.14).

Proof of (6.15): we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|Q(y)|^{\frac{2n}{n-2}}}{|y|^2} dy &= \int_{\mathbb{R}^n} \frac{\left|U(y) - \sum_{j=1}^k U_j(y) + \tilde{\phi}(y)\right|^{\frac{2n}{n-2}}}{|y|^2} dy \\ &= \int_{\mathbb{R}^n} \frac{|U(y)|^{\frac{2n}{n-2}}}{|y|^2} dy + \sum_{j=1}^k \int_{\mathbb{R}^n} \frac{|U_j(y)|^{\frac{2n}{n-2}}}{|y|^2} dy + \int_{\mathbb{R}^n} \frac{|\tilde{\phi}(y)|^{\frac{2n}{n-2}}}{|y|^2} dy \\ &\quad + \sum_{\gamma=1}^{\frac{2n}{n-2}-1} \int_{\mathbb{R}^n} \frac{|U(y)|^{\gamma} | \sum_{j=1}^k U_j(y) + \tilde{\phi}(y)|^{\frac{2n}{n-2}-\gamma}}{|y|^2} dy. \end{aligned} \tag{6.28}$$

Since

$$\int_{\mathbb{R}^n} \frac{|U(y)|^{\frac{2n}{n-2}}}{|y|^2} dy = \frac{\omega_{n-1}}{2} \alpha_n^{\frac{2n}{n-2}} I_n^{\frac{n-4}{2}}, \tag{6.29}$$

$$\begin{aligned}\sum_{j=1}^k \int_{\mathbb{R}^n} \frac{|U_j(y)|^{\frac{2n}{n-2}}}{|y|^2} dy &= \alpha_n^{\frac{2n}{n-2}} \sum_{j=1}^k \int_{\mathbb{R}^n} \frac{\mu_k^n}{(\mu_k^2 + |y-\xi_j|^2)^n} \frac{1}{|y|^2} dy \\ &= \alpha_n^{\frac{2n}{n-2}} \sum_{j=1}^k \int_{\mathbb{R}^n} \frac{1}{(1+|z|^2)^n} \frac{1}{|\mu_k z + \xi_j|^2} dy \\ &= \alpha_n^{\frac{2n}{n-2}} \sum_{j=1}^k \int_{|z|\le \frac{1}{2\mu_k}} \frac{1}{(1+|z|^2)^n} \frac{1}{|\mu_k z + \xi_j|^2} dy \\ &\quad + \alpha_n^{\frac{2n}{n-2}} \sum_{j=1}^k \int_{|z|\ge \frac{1}{2\mu_k}} \frac{1}{(1+|z|^2)^n} \frac{1}{|\mu_k z + \xi_j|^2} dy \\ &= \alpha_n^{\frac{2n}{n-2}} \sum_{j=1}^k \int_{|z|\le \frac{1}{2\mu_k}} \frac{1}{(1+|z|^2)^n} \frac{1}{|\xi_j|^2} \left(1 + O\left(\frac{\mu_k z}{|\xi_j|}\right)\right) dy \\ &\quad + \alpha_n^{\frac{2n}{n-2}} \sum_{j=1}^k \int_{|z|\ge \frac{1}{2\mu_k}} \frac{1}{(1+|z|^2)^n} \frac{1}{|\mu_j z|^2} \left(1 + O\left(\frac{|\xi_j|}{\mu_k z}\right)\right) dy \\ &= \frac{\omega_{n-1}}{2} \alpha_n^{\frac{2n}{n-2}} k I_n^{\frac{n-2}{2}} + O\big(k\mu_k\big) \\ &= \frac{\omega_{n-1}}{2} \alpha_n^{\frac{2n}{n-2}} k I_n^{\frac{n-2}{2}} + O\left(k^{-\frac{n}{q}}\right), \qquad n \ge 4. \end{aligned} \tag{6.30}$$

Using (6.1), we have

$$\int_{\mathbb{R}^n} \frac{|\tilde{\phi}(y)|^{\frac{2n}{n-2}}}{|y|^2} dy = O\Big(k^{-\frac{n}{q}\frac{2n}{n-2}} \int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{2n}|y|^2} dy\Big) = O\Big(k^{-\frac{n}{q}\frac{2n}{n-2}}\Big) = O\Big(k^{-\frac{n}{q}}\Big). \tag{6.31}$$

Moreover, for $\gamma \in (1, \frac{2n}{n-2} - 1)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|U(y)|^\gamma |\sum_{j=1}^k U_j(y) + \tilde{\phi}(y)|^{\frac{2n}{n-2}-\gamma}}{|y|^2} dy \\
&= \int_{B(0,\delta)} \cdots + \int_{\cup_{j=1}^k B(0,\delta)} \cdots + \int_{\mathbb{R}^n \setminus (B(0,\delta) \cup \cup_{j=1}^k B(0,\delta))} \cdots \\
&= O\left(\mu_k^{\frac{n-2}{2}\left(\frac{2n}{n-2}-\gamma\right)} + k^{-\frac{n}{q}\left(\frac{2n}{n-2}-\gamma\right)}\right) \\
&\quad + O\left(\mu_k^{n-\frac{n-2}{2}\left(\frac{2n}{n-2}-\gamma\right)}\right) + O\left(k^{-\frac{n}{q}\left(\frac{2n}{n-2}-\gamma\right)}\right) \\
&= O\left(\mu_k^{n-\frac{n-2}{2}\gamma}\right) + O\left(\mu_k^{\frac{n-2}{2}\gamma}\right) + O\left(k^{-\frac{n}{q}\left(\frac{2n}{n-2}-\gamma\right)}\right) \\
&= O\left(\mu_k^{\frac{n-2}{2}}\right) + O\left(k^{-\frac{n}{q}}\right) \\
&= O\left(k^{-(n-2)}\right) + O\left(k^{-\frac{n}{q}}\right) = O\left(k^{-\frac{n}{q}}\right).
\end{aligned} \tag{6.32}$$

Therefore (6.15) follows from (6.28)-(6.32).

Proof of (6.16), which is the same as (6.15). □

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