The spectrum of the torus profile to a geometric variational problem with long range interaction

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Abstract

The profile problem for the Ohta-Kawasaki diblock copolymer theory is a geometric variational problem. The energy functional is defined on sets in $\mathbb{R}^3$ of prescribed volume and the energy of an admissible set is its perimeter plus a long range interaction term related to the Newtonian potential of the set. This problem admits a solution, called a torus profile, that is a set enclosed by an approximate torus of the major radius 1 and the minor radius q. The torus profile is both axially symmetric about the z axis and reflexively symmetric about the xy-plane. There is a way to set up the profile problem in a function space as a partial differential-integro equation. The linearized operator $\mathcal{L}$ of the problem at the torus profile is decomposed into a family of linear ordinary differential-integro operators $\mathcal{L}^m$ where the index $m = 0, 1, 2, ...$ is called a mode. The spectrum of $\mathcal{L}$ is the union of the spectra of the $\mathcal{L}^m$'s. It is proved that for each m, when q is sufficiently small, $\mathcal{L}^m$ is positive definite. (0 is an eigenvalue for both $\mathcal{L}^0$ and $\mathcal{L}^1$, due to the translation and rotation invariance.) As q tends to 0, more and more $\mathcal{L}^m$'s become positive definite. However no matter how small q is, there is always a mode m for which $\mathcal{L}^m$ has a negative eigenvalue. This mode grows to infinity like $(\frac{8}{q})^{3/4}$ as $q \to 0$.

1 Introduction

The Ohta-Kawasaki theory [10] for diblock copolymers is an architypical example of binary inhibitory systems. In the strong segregation limit, where the two constituents are fully separated by sharp interfaces, the free energy of the system can be written as

$$\mathcal{J}_D(\Omega) = \frac{1}{2}\mathcal{P}_D(\Omega) + \frac{\gamma}{2}\int_D \left|(-\Delta)^{-1/2}(\chi_\Omega - \omega)\right|^2 dx. \tag{1.1}$$

Here D is a bounded domain in $\mathbb{R}^3$, and there are two parameters $\gamma > 0$ and $\omega \in (0, 1)$. The input of this functional is Ω, a Lebesgue measurable subset of D, whose measure $|\Omega|$ is fixed at

$$|\Omega| = \omega|D|. \tag{1.2}$$

The first term $\mathcal{P}_D(\Omega)$ is the perimeter of Ω in D. If D is bounded by smooth surfaces, then $\mathcal{P}_D(\Omega)$ is the total area of those surfaces that are inside D. These surfaces form the set $\partial\Omega \cap D$, which is called the interface of Ω because it separates Ω from $D\backslash\Omega$ in D.

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The second term in the functional (1.1) is the most interesting. The nonlocal operator $(-\Delta)^{-1/2}$ is defined to be the positive square root of $(-\Delta)^{-1}$. For the latter operator, given $f \in L^2(D)$ and $\int_D f(x)\,dx = 0$, define $w = (-\Delta)^{-1} f$ by solving the Poisson's equation

$$-\Delta w = f \text{ in } D, \quad \partial_\nu w = 0 \text{ on } \partial D, \quad \int_D w(x)\,dx = 0. \tag{1.3}$$

In (1.3) $\partial_\nu w$ stands for the outward normal derivaitve of w on ∂D.

A stationary set of $\mathcal{J}_D$ is a solution of the equation

$$\mathcal{H}(\partial\Omega) + \gamma(-\Delta)^{-1}(\chi_\Omega - \omega) = \lambda \tag{1.4}$$

which holds on the interface $\partial\Omega \cap D$. Here $\mathcal{H}(\partial\Omega)$ is the mean curvature of $\partial\Omega$. The constant λ on the right side of (1.4) is a Lagrange multiplier corresponding to the volume constraint (1.2). If Ω shares boundary with D, then

$$\partial\Omega \cap D \perp \partial D; \tag{1.5}$$

namely the interface of Ω meets the domain boundary perpendicularly.

It is easy to show that the functional $\mathcal{J}_D$ admits a global minimizer. One can study its properties for various parameter ranges of γ and ω [1, 9, 22, 3]. One may also construct stable stationary sets of (1.1) either by finding local minimizers of $\mathcal{J}_D$ [14, 8], or by solving (1.4) as in [5, 6, 16, 15, 17, 18, 13].

Many morphological phases observed in nature are assemblies of small components with almost the same size and shape. These components arrange themselves in a very regular pattern. The most well known are the hexagonal pattern and the body centered cubic pattern. A cross section of an hexagonal pattern is a two dimensional assembly of small discs, and a body centered cubic pattern is a three dimensional assembly of small balls. These two patterns were found as stable stationary assemblies of (1.1) by the authors in [15, 17].

The starting point of these constructions is an observation that (1.4) has a counterpart on the entire space $\mathbb{R}^3$ ($\mathbb{R}^2$, resp):

$$\mathcal{H}(\partial\Omega) + \gamma\mathcal{N}(\Omega) = \lambda \ \text{ on } \partial\Omega. \tag{1.6}$$

A solution Ω to (1.6) must have a prescribed volume:

$$|\Omega| = m \tag{1.7}$$

where $m > 0$ is one of the two parameters, the other being γ; λ on the right side of (1.6) is a Lagrange multiplier corresponding to (1.7). In (1.6), $\mathcal{N}(\Omega)$ is the Newtonian potential of Ω:

$$\mathcal{N}(\Omega)(x) = \int_\Omega \frac{1}{4\pi|x-y|}\,dy. \tag{1.8}$$

The equation (1.6) has its own variational structure. A solution of (1.6) is a stationary set of the functional

$$\mathcal{J}(\Omega) = \frac{1}{2}\mathcal{P}(\Omega) + \frac{\gamma}{2}\int_\Omega \mathcal{N}(\Omega)(x)\,dx. \tag{1.9}$$

Here $\mathcal{P}(\Omega)$ is the perimeter of Ω in $\mathbb{R}^3$, i.e. the area of $\partial\Omega$.

We term (1.6) the *profile equation* of the Ohta-Kawasaki model; a solution of (1.6) is called a *profile*. A ball of volume m is a profile, because its boundary has constant mean curvature and its Newtonian potential is a radially symmetric function, hence also constant on its boundary. This ball is used as an approximation for a component in a stationary assembly constructed in [17]. In that assembly each component is close to a scaled version of the ball, and the locations of the components are determined by the geometry of D via the Green's function of Poisson's equation (1.3). Similarly in two dimensions, a disc of area m is a profile and it serves as a component for the hexagonal stationary assembly [15].

The method of building stationary assemblies from profiles is a general one, which has been lately successfully applied to other inhibitory systems [21, 2]. In addition to balls and discs, a few other profiles have been found [5, 12, 18, 19, 20].

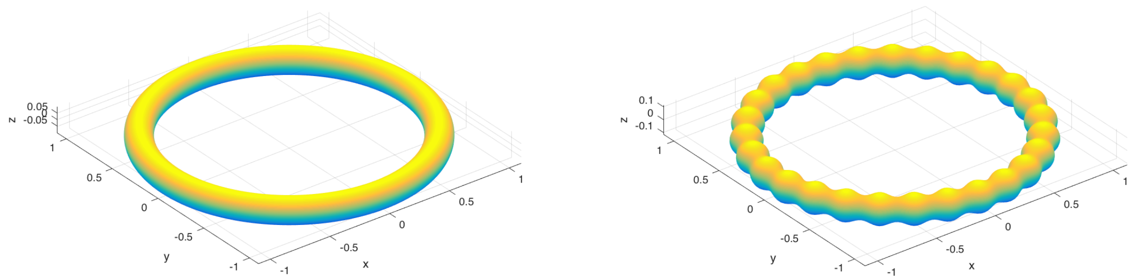

Figure 1: The left plot shows a torus profile found in [19]; the right plot shows an unstable deformation found in Part 3 of Theorem 1.1. Here $p = 1$, $q = 0.1$, and $m = 27$.

In this paper we study a profile that shapes like a solid torus in $\mathbb{R}^3$. This profile was found in [19] and is a set enclosed by a surface which is a slightly perturbed torus. Our interest in a torus shaped profile partially comes from a discovery by Pochan [11] of a block copolymer morphology phase of toroidal supramolecule assemblies. This phase was found by combining dilute solution characteristics critical for both bundling of like-charged biopolymers and block copolymer micelle formation. The key to toroid versus classic cylinder micelle formation is the interaction of the negatively charged hydrophilic block of an amphiphilic triblock copolymer with a positively charged divalent organic counterion. This produces a self-attraction of cylindrical micelles that leads to toroid formation, a mechanism akin to the toroidal bundling of semiflexible charged biopolymers such as DNA.

A perfect torus is a surface characterized by a major radius p and a minor radius q with $0 < q < p$. The profile found in [19] is bounded by an approximate torus. Nevertheless this approximate torus still has well defined major radius and minor radius, and it encloses the same volume as the perfect torus with the same radii does; see the comments after Proposition 3.1. The profile problem (1.6) has m and γ as parameters so the radii p and q of the torus profile are dependent on m and γ. In [19] we took $m = 1$. This assumption is harmless because one can always transform m to 1 by a change of space variable x and a change of γ. The two radii of the torus profile were then denoted by p_γ and q_γ, both dependent on γ. This profile exists when γ is sufficiently large. As $\gamma \to \infty$, $p_\gamma \to \infty$ and $q_\gamma \to 0$.

However it is more convenient in this paper to take the radii p and q as parameters and treat m and γ as derived quantities. Moreover, without the loss of generality we let

$$p = 1. \tag{1.10}$$

This can be achieved from the torus profile of radii p_γ and q_γ found in [19] by a change of space variable. Now q becomes the only parameter of the profile problem. A torus profile of radii 1 and q exists when q is sufficiently small; see the left plot of Figure 1. The volume of this profile is $2\pi^2 pq^2 = 2\pi^2 q^2$. Consequently under (1.10)

$$m = 2\pi^2 q^2. \tag{1.11}$$

The quantity γ is now unknown and is to be determined together with the profile as one solves (1.6).

In [19] a perturbed torus is described by a two variable function v in a function space so the energy $\mathcal{J}(\Omega)$ becomes a functional $\mathcal{J}(v)$ defined on the function space. The equation (1.6) for Ω becomes an equation

$$\mathcal{S}(v) = 0 \tag{1.12}$$

where $\mathcal{S}$ is a nonlinear partial differential-integro operator on a space of two variable functions. If the torus profile is denoted by v, then we let the Fréchet derivative of $\mathcal{S}$ at v be

$$\mathcal{L} = \mathcal{S}'(v). \tag{1.13}$$

Here $\mathcal{L}$ is a linear partial differential-integro operator. The spectrum of $\mathcal{L}$, which consists of real eigenvalues of finite multiplicity, is the focus of study in this paper. Because the profile v is an axially symmetric set about the z axis, the operator $\mathcal{L}$ inherits this symmetry and separation of variables decomposes $\mathcal{L}$ into a family of simpler operators $\mathcal{L}^m$ where the index m is called a mode and it ranges over non-negative integers. Each $\mathcal{L}^m$ is a linear ordinary differential-integro operator on a space of one variable functions. Its spectrum again consists of real eigenvalues of finite multiplicity only. The union of the spectra of these $\mathcal{L}^m$'s is the spectrum of $\mathcal{L}$. The following theorem is the main result of this paper.

Theorem 1.1 *Let $\mathcal{L}$ be the linearized operator at the torus profile, decomposed into a sequence of linear ordinary differential-integro operators $\mathcal{L}^m$, $m = 0, 1, 2, ...$*

1. *(a) There exists $\tilde{q} > 0$ such that when $q \in (0, \tilde{q})$, one of $\mathcal{L}^0$'s eigenvalues is zero with multiplicity one, and all other eigenvalues are positive.*
   (b) There exists $\widetilde{\widetilde{q}} > 0$ such that when $q \in (0, \widetilde{\widetilde{q}})$, one of $\mathcal{L}^1$'s eigenvalues is zero with multiplicity two, and all other eigenvalues are positive.
   (c) For every $M_i > 0$, there exists $\tilde{q}_i > 0$ depending on M_i such that when $q \in (0, \tilde{q}_i)$ and $m \in \{2, 3, ..., M_i\}$, all of $\mathcal{L}^m$'s eigenvalues are positive.
2. *There exist $M_{ii} > 0$ and $\tilde{q}_{ii} > 0$ such that when $q \in (0, \tilde{q}_{ii})$ and $m \geq \frac{M_{ii}}{q}$, all of $\mathcal{L}^m$'s eigenvalues are positive.*
3. *There exists $\tilde{q}_{i,ii} > 0$ such that for every $q \in (0, \tilde{q}_{i,ii})$ there is $m \in (M_i, \frac{M_{ii}}{q})$ for which $\mathcal{L}^m$ has a negative eigenvalue. Moreover, as q tends to 0, m grows to infinity like $(\frac{8}{q})^{3/4}$.*

Consequently, $\mathcal{L}$ has a negative eigenvalue when q is sufficiently small.

The presence of 0 as an eigenvalue for $\mathcal{L}^0$ and $\mathcal{L}^1$ in Part 1 of the theorem is a consequence of the translation and rotation invariance of the profile problem. For each $m \geq 2$, Part 1 asserts that the operator $\mathcal{L}^m$ becomes positive definite if q is sufficiently small.

The most interesting discovery in this paper is that the transition of $\mathcal{L}^m$ to a positive definite operator as q tends to 0 *does not occur uniformly* with respect to m. Part 3 of the theorem shows that no matter how small q is, there is always a mode m for which $\mathcal{L}^m$ has a negative eigenvalue.

In fact there are three distinct ranges for the mode m. Let M_i and M_{ii} be the two numbers in Theorem 1.1, we say that

- m is small if $0 \leq m \leq M_i$,
- m is medium if $M_i < m < \frac{M_{ii}}{q}$, and
- m is large if $\frac{M_{ii}}{q} \leq m$.

The theorem shows that when q is small, $\mathcal{L}^m$ ($m \geq 2$) is positive definite if m is small or large, but there is a medium m for which $\mathcal{L}^m$ is indefinite.

In the proof of Part 3 of the theorem an indefinite $\mathcal{L}^m$ is found when m is chosen of the order $(\frac{8}{q})^{3/4}$ in the medium range. We use the function $\varphi = 1$ as a test function for the quadratic form $\langle \mathcal{L}^m \varphi, \varphi \rangle$, and show that $\langle \mathcal{L}^m(1), 1 \rangle < 0$. The test function $\varphi = 1$ represents a particular type of perturbation, illustrated in the right plot of Figure 1, by which the rigidity of the torus profile is vulnerable.

The three parts of Theorem 1.1 are proved in Sections 5, 6, and 7 for small, large, and medium m's respectively. In Section 2 we explain our representation of perturbed tori as functions of two variables and re-formulate the profile problem (1.6) as a partial differential-integro equation, (1.12). The linearized operator $\mathcal{L}$ is derived and decomposed into a family of linear ordinary differential-integro operators $\mathcal{L}^m$, $m = 0, 1, 2, \ldots$. Before the $\mathcal{L}^m$'s can be studied, one needs a better understanding of the torus profile, and in Section 3 some fine properties of this profile are obtained and several important estimates related to the Newtonian potential operator are presented. One derives a three term expansion for each operator $\mathcal{L}^m$ in Section 4: $\mathcal{L}^m = \mathcal{L}^m_0 + \mathcal{L}^m_1 + \mathcal{L}^m_2 + \ldots$, which yields a perturbation analysis of $\mathcal{L}^m$ and ultimately leads to the proof of Part 1 of Theorem 1.1 in Section 5.

Because the torus profile is unstable when q is small, any toroidal assembly built from this profile will be unstable. We are inclined to conclude that the Ohta-Kawasaki functional (1.1) is not capable of producing the toroidal morphology phase discovered by Pochan [11].

Although it has only one intrinsic parameter, the profile problem is not simple. Take the ball profile as an example. The authors found in [18] that if $m = \frac{4\pi}{3}$, then the unit ball is a stable profile if $\gamma < 15$ and is an unstable profile if $\gamma > 15$. Knüpfer and Muratov [7] showed that if one holds $\gamma = 1$, then the ball of volume m is the global minimizer of $\mathcal{J}$ if m is sufficiently small; if m is sufficiently large, $\mathcal{J}$ does not have a global minimizer. Theorem 1.1 reveals another peculiar phenomenon in this problem.

2 The modes of $\mathcal{L}$

We recall the framework used in [19] under which the existence of a torus profile is proved. Denote the cylindrical coordinate system of $\mathbb{R}^3$ by

$$\mathbb{R}^3_c = \big\{(r, z, \sigma) : \ r \in [0, \infty), \ z \in \mathbb{R}, \ \sigma \in S^1\big\}. \tag{2.1}$$

Here S^1 denotes the unit circle, same as the interval $[0, 2\pi]$ with identified end points. The perfect torus of the radii 1 and q in a standard position is the surface $\{(1 + q\cos\theta, q\sin\theta, \sigma) : \ (\theta, \sigma) \in S^1 \times S^1\}$ in $\mathbb{R}^3_c$. To introduce a perturbed torus, replace q by a function u from $S^1 \times S^1$ to $(0, \infty)$. If u is continuous, then $\{(1 + u(\theta, \sigma)\cos\theta, u(\theta, \sigma)\sin\theta, \sigma) : \ (\theta, \sigma) \in S^1 \times S^1\}$ in $\mathbb{R}^3_c$ defines a continuous surface in $\mathbb{R}^3$. If $u(\theta, \sigma)$ is close to q for all $(\sigma, \theta) \in S^1 \times S^1$, then the surface is a perturbed torus. One denotes the region in $\mathbb{R}^3$ enclosed by this surface by Ω and the corresponding set in $\mathbb{R}^3_c$ by

$$\Omega_c = \bigcup_{(\theta,\sigma)\in S^1\times S^1} \Big\{(1 + h\cos\theta, h\sin\theta, \sigma) : \ h \in [0, u(\theta, \sigma)]\Big\}. \tag{2.2}$$

In terms of u, $\mathcal{J}(\Omega)$ is

$$\mathcal{J}(\Omega) = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \sqrt{(1 + u\cos\theta)^2(u_\theta^2 + u^2) + u^2u_\sigma^2}\, d\theta d\sigma + \frac{\gamma}{2}\int_{\Omega_c} \mathcal{N}(\Omega)(r, z, \sigma)\, r dr dz d\sigma. \tag{2.3}$$

The set Ω has the same volume as the un-perturbed solid torus, and hence the constraint

$$|\Omega| = 2\pi^2 q^2 \tag{2.4}$$

holds, which can be expressed as

$$\int_0^{2\pi}\int_0^{2\pi} \Big(\frac{u^2(\theta, \sigma)}{2} + \frac{u^3(\theta, \sigma)\cos\theta}{3}\Big)\, d\theta d\sigma = 2\pi^2 q^2, \tag{2.5}$$

since

$$|\Omega| = \int_0^{2\pi}\int_0^{2\pi}\int_0^{u(\theta,\sigma)} (1 + h\cos\theta)h\, dh d\theta d\sigma = \int_0^{2\pi}\int_0^{2\pi} \Big(\frac{u^2(\theta, \sigma)}{2} + \frac{u^3(\theta, \sigma)\cos\theta}{3}\Big)\, d\theta d\sigma.$$

Unfortunately (2.5) is a nonlinear constraint on u and is not easy to work with. One way out of this difficulty is to introduce another variable

$$v(\theta, \sigma) = \frac{u^2(\theta, \sigma)}{2} + \frac{u^3(\theta, \sigma)\cos\theta}{3}, \tag{2.6}$$

and use v to describe Ω. Then the nonlinear constraint (2.5) becomes an affine constraint

$$\int_0^{2\pi}\int_0^{2\pi} v(\theta,\sigma)\, d\theta = 2\pi^2 q^2. \tag{2.7}$$

Consequently $\mathcal{J}$ of (2.3) becomes a functional of v: $\mathcal{J} = \mathcal{J}(v)$.

For a set Ω described by an $H^2(S^1\times S^1)$ function v, let ϕ be an $H^2(S^1\times S^1)$ function of zero average, i.e.

$$\int_0^{2\pi}\int_0^{2\pi} \phi(\theta,\sigma)\, d\theta d\sigma = 0. \tag{2.8}$$

Then $v+\varepsilon\phi$ represents a volume preserving deformation of the set Ω if $|\varepsilon|$ is sufficiently small. Calculations show that the first variation of $\mathcal{J}$ at Ω in this setting is

$$\mathcal{J}'(v)(\phi) = \frac{d\mathcal{J}(v+\varepsilon\phi)}{d\varepsilon}\Big|_{\varepsilon=0} = \int_0^{2\pi}\int_0^{2\pi}\Big(\mathcal{H}(v)+\gamma\mathcal{N}(v)\Big)\phi\, d\theta d\sigma \tag{2.9}$$

where $\mathcal{H}(v)$ is the mean curvature of $\partial\Omega$. Here the mean curvature $\mathcal{H}$ and the Newtonian potential $\mathcal{N}$ are treated as operators on the function v instead of the set Ω. Let

$$\mathcal{S}(v) = \mathcal{H}(v)+\gamma\mathcal{N}(v) - \overline{\mathcal{H}(v)+\gamma\mathcal{N}(v)} \tag{2.10}$$

where $\overline{\mathcal{H}(v)+\gamma\mathcal{N}(v)}$ denotes the average of $\mathcal{H}(v)+\gamma\mathcal{N}(v)$:

$$\overline{\mathcal{H}(v)+\gamma\mathcal{N}(v)} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Big(\mathcal{H}(v)+\gamma\mathcal{N}(v)\Big)\, d\theta d\sigma. \tag{2.11}$$

Note that

$$\overline{\mathcal{S}(v)} = 0 \tag{2.12}$$

and

$$\mathcal{J}'(v)(\phi) = \int_0^{2\pi}\int_0^{2\pi} \mathcal{S}(v)\phi\, d\theta d\sigma. \tag{2.13}$$

Therefore $\mathcal{S}$ is identified as the first variation of $\mathcal{J}$. The equation (1.12), $\mathcal{S}(v)=0$, is just another way to write (1.6) for stationary sets.

The second variation of $\mathcal{J}$ is the Fréchet derivative of $\mathcal{S}$, denoted $\mathcal{S}'$. At each v, $\mathcal{S}'(v)$ is a linear operator such that

$$\mathcal{J}''(v)(\phi,\psi) = \frac{\partial^2\mathcal{J}(v+\varepsilon_1\phi+\varepsilon_2\psi)}{\partial\varepsilon_1\partial\varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0} = \int_0^{2\pi}\int_0^{2\pi}\mathcal{S}'(v)(\phi)\,\psi\, d\theta d\sigma. \tag{2.14}$$

Henceforth we denote the torus profile of radii 1 and q by Ω, represented by u or v. The existence of this profile for small q was established in [19, Theorem 1.1]. Since Ω is axially symmetric, u and v are independent of σ and we use Ω' to denote the projection of the corresponding Ω_c to the rz-plane; namely

$$\Omega' = \{(1+h\cos\theta, h\sin\theta):\ h\in[0,u(\theta)]\} \tag{2.15}$$

where u corresponds to v via the same transformation (2.6):

$$v(\theta) = \frac{u^2(\theta)}{2} + \frac{u^3(\theta)\cos\theta}{3}. \tag{2.16}$$

Let $\mathcal{L} = S'(v) : \mathcal{X}\to\mathcal{Z}$ be the second variation of $\mathcal{J}$ at the torus profile where the domain and the target are respectively

$$\begin{aligned}\mathcal{X} &= \{\phi\in H^2(S^1\times S^1):\ \int_0^{2\pi}\int_0^{2\pi}\phi(\theta,\sigma)\, d\theta d\sigma = 0\} && (2.17)\\ \mathcal{Z} &= \{g\in L^2(S^1\times S^1):\ \int_0^{2\pi}\int_0^{2\pi} g(\theta,\sigma)\, d\theta d\sigma = 0\}. && (2.18)\end{aligned}$$

By the Fredholm theory, $\mathcal{L}$ is a self-adjoint operator whose spectrum consists of eigenvalues of finite multiplicity. Denote the two terms of $\mathcal{J}$ in (2.3) by $\hat{\mathcal{J}}$ and $\check{\mathcal{J}}$ respectively so that

$$\mathcal{J}(v) = \hat{\mathcal{J}}(v) + \gamma \check{\mathcal{J}}(v). \tag{2.19}$$

Then write $\mathcal{L}$ as

$$\mathcal{L} = \hat{\mathcal{L}} + \gamma \check{\mathcal{L}} \tag{2.20}$$

where $\hat{\mathcal{L}}$ and $\check{\mathcal{L}}$ are the second variations of $\hat{\mathcal{J}}$ and $\check{\mathcal{J}}$ at v respectively.

As a second variation of the perimeter functional, $\hat{\mathcal{L}}$ is a second order linear partial differential operator followed by a projection from $L^2(S^1 \times S^1)$ to $\mathcal{Z}$. The exact expression of $\hat{\mathcal{L}}$ is fairly complex; it is better to study the quadratic form $\langle \hat{\mathcal{L}}\phi, \phi \rangle$. A deformation to v by

$$v \to v_\varepsilon = v + \varepsilon \phi \tag{2.21}$$

induces a deformation to u and we denote the latter by

$$u \to u_\varepsilon = u + \varepsilon \eta_\varepsilon. \tag{2.22}$$

Note that ϕ is independent of ε but η_ε depends on ε. Expand

$$\sqrt{(1 + u_\varepsilon \cos\theta)^2 (u_{\varepsilon,\theta}^2 + u_\varepsilon^2) + u_\varepsilon^2 u_{\varepsilon,\sigma}^2} = A + \varepsilon B_\varepsilon + \varepsilon^2 C_\varepsilon + O(\varepsilon^3), \tag{2.23}$$

where

$$A = (1 + u\cos\theta)(u_\theta^2 + u^2)^{1/2} \tag{2.24}$$

$$B_\varepsilon = \frac{(1 + u\cos\theta) u_\theta \eta_{\varepsilon,\theta} + (u + 2u^2\cos\theta + u_\theta^2 \cos\theta)\eta_\varepsilon}{(u_\theta^2 + u^2)^{1/2}} \tag{2.25}$$

$$C_\varepsilon = \frac{(1 + u\cos\theta) u^2 \eta_{\varepsilon,\theta}^2}{2(u_\theta^2 + u^2)^{3/2}} + \frac{u^2 \eta_{\varepsilon,\sigma}^2}{2(u_\theta^2 + u^2)^{1/2}(1 + u\cos\theta)} \tag{2.26}$$

$$+ \frac{(-uu_\theta + u_\theta^3 \cos\theta)\eta_{\varepsilon,\theta}\eta_\varepsilon}{(u_\theta^2 + u^2)^{3/2}} + \frac{(2u^3\cos\theta + pu_\theta^2 + 3uu_\theta^2\cos\theta)\eta_\varepsilon^2}{2(u_\theta^2 + u^2)^{3/2}}. \tag{2.27}$$

Recall that v_ε and u_ε are related by (2.6), which implies that

$$\eta_\varepsilon = \frac{\phi}{u + u^2\cos\theta} - \frac{(1 + 2u\cos\theta)\phi^2}{2(u + u^2\cos\theta)^3}\varepsilon + O(\varepsilon^2). \tag{2.28}$$

Therefore

$$\begin{aligned} B_\varepsilon &= \frac{(1 + u\cos\theta)u_\theta}{(u_\theta^2 + u^2)^{1/2}} \Big(\frac{\phi}{u + u^2\cos\theta}\Big)_\theta + \frac{(u + 2u^2\cos\theta + u_\theta^2\cos\theta)\phi}{(u_\theta^2 + u^2)^{1/2}(u + u^2\cos\theta)} \\ &+ \varepsilon\Big[\frac{(1 + u\cos\theta)u_\theta}{(u_\theta^2 + u^2)^{1/2}} \Big(\frac{-(1 + 2u\cos\theta)\phi^2}{2(u + u^2\cos\theta)^3}\Big)_\theta - \frac{(u + 2u^2\cos\theta + u_\theta^2\cos\theta)(1 + 2u\cos\theta)\phi^2}{2(u_\theta^2 + u^2)^{1/2}(u + u^2\cos\theta)^3}\Big] + O(\varepsilon^2) \end{aligned} \tag{2.29}$$

$$\begin{aligned} C_\varepsilon &= \frac{(1 + u\cos\theta)u^2}{2(u_\theta^2 + u^2)^{3/2}} \Big(\frac{\phi}{u + u^2\cos\theta}\Big)_\theta^2 + \frac{\phi_\sigma^2}{2(u_\theta^2 + u^2)^{1/2}(1 + u\cos\theta)^3} \\ &+ \frac{-uu_\theta + u_\theta^3\cos\theta}{(u_\theta^2 + u^2)^{3/2}(u + u^2\cos\theta)} \Big(\frac{\phi}{u + u^2\cos\theta}\Big)_\theta \phi + \frac{(2u^3\cos\theta + u_\theta^2 + 3uu_\theta^2\cos\theta)\phi^2}{2(u_\theta^2 + u^2)^{3/2}(u + u^2\cos\theta)^2} + O(\varepsilon). \end{aligned} \tag{2.30}$$

The first variation of $\hat{\mathcal{J}}$ is given by the leading order of B_ε, i.e.

$$\frac{d\hat{\mathcal{J}}(v + \varepsilon\phi)}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \Big[\frac{(1 + u\cos\theta)u_\theta}{(u_\theta^2 + u^2)^{1/2}} \Big(\frac{\phi}{u + u^2\cos\theta}\Big)_\theta + \frac{(u + 2u^2\cos\theta + u_\theta^2\cos\theta)\phi}{(u_\theta^2 + u^2)^{1/2}(u + u^2\cos\theta)}\Big] d\theta d\sigma. \tag{2.31}$$

The second variation of $\hat{\mathcal{J}}$, i.e. the quadratic form $\langle \hat{\mathcal{L}}\phi, \phi\rangle$, follows from the ε order term of B_ε and the leading order of C_ε:

$$\begin{aligned}
\langle \hat{\mathcal{L}}\phi, \phi\rangle &= \frac{d^2 \hat{\mathcal{J}}(v+\varepsilon\phi)}{d\varepsilon^2}\Big|_{\varepsilon=0} \\
&= \int_0^{2\pi}\int_0^{2\pi} \Big[\frac{(1+u\cos\theta)u_\theta}{(u_\theta^2+u^2)^{1/2}}\Big(\frac{-(1+2u\cos\theta)\phi^2}{2(u+u^2\cos\theta)^3}\Big)_\theta && (2.32)\\
&\quad -\frac{(u+2u^2\cos\theta+u_\theta^2\cos\theta)(1+2u\cos\theta)\phi^2}{2(u_\theta^2+u^2)^{1/2}(u+u^2\cos\theta)^3} && (2.33)\\
&\quad +\frac{(1+u\cos\theta)u^2}{2(u_\theta^2+u^2)^{3/2}}\Big(\frac{\phi}{u+u^2\cos\theta}\Big)_\theta^2 && (2.34)\\
&\quad +\frac{\phi_\sigma^2}{2(u_\theta^2+u^2)^{1/2}(1+u\cos\theta)^3} && (2.35)\\
&\quad +\frac{-uu_\theta+u_\theta^3\cos\theta}{(u_\theta^2+u^2)^{3/2}(u+u^2\cos\theta)}\Big(\frac{\phi}{u+u^2\cos\theta}\Big)_\theta\phi && (2.36)\\
&\quad +\frac{(2u^3\cos\theta+u_\theta^2+3uu_\theta^2\cos\theta)\phi^2}{2(u_\theta^2+u^2)^{3/2}(u+u^2\cos\theta)^2}\Big]\, d\theta d\sigma. && (2.37)
\end{aligned}$$

To find $\check{\mathcal{L}}$, note that the operator $\mathcal{N}$ is given by

$$\mathcal{N}(v)(\theta,\sigma) = \int_{\Omega_c} G(1+ue^{\mathrm{i}\theta},\sigma,\rho,\zeta,\tau)\, d\rho d\zeta d\tau \tag{2.38}$$

where

$$G(r,z,\sigma,\rho,\zeta,\tau) = \frac{\rho}{4\pi\sqrt{r^2+\rho^2-2r\rho\cos(\sigma-\tau)+(z-\zeta)^2}} \tag{2.39}$$

is the Green's function of $-\Delta$ on $\mathbb{R}^3$ in the cylindrical coordinates. Then

$$\begin{aligned}
\mathcal{N}'(v)\phi &= \frac{\partial \mathcal{N}(v+\varepsilon\phi)}{\partial\varepsilon}\Big|_{\varepsilon=0} \\
&= \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\int_0^{2\pi}\int_0^{2\pi}\int_0^{u_\varepsilon(\omega,\tau)} G(1+u_\varepsilon(\theta,\sigma)e^{\mathrm{i}\theta},\sigma,1+he^{\mathrm{i}\omega},\tau)\, hdhd\omega d\tau \\
&= \int_0^{2\pi}\int_0^{2\pi} G(1+ue^{\mathrm{i}\theta},\sigma,1+ue^{\mathrm{i}\omega},\tau)\frac{\phi(\omega,\tau)}{1+u(\omega,\tau)\cos\omega}\, d\omega d\tau \\
&\quad +\frac{\phi(\theta,\sigma)}{u+u^2\cos\theta}\int_{\Omega_c}\nabla G(1+ue^{\mathrm{i}\theta},\sigma,\rho,\zeta,\tau)\cdot e^{i\theta}\, d\rho d\zeta d\tau. && (2.40)
\end{aligned}$$

By (2.40),

$$\check{\mathcal{L}}\phi = \mathcal{N}'(v)\phi - \overline{\mathcal{N}'(v)\phi}. \tag{2.41}$$

For a simpler notation, we have introduces a convention to write $e^{\mathrm{i}\theta}$ for $(\cos\theta,\sin\theta)$ in the rz-plane and $e^{\mathrm{i}\omega}$ for $(\cos\omega,\sin\omega)$ in the $\rho\zeta$-plane. Also, $(\frac{\partial G}{\partial r},\frac{\partial G}{\partial z})$, the gradient of G with respect to its first two variables r and z, is denoted ∇G.

The axial symmetry of v allows us to decompose $\mathcal{L}$ into invariant subspaces. Let

$$\mathcal{Z} = \oplus_{m=0}^\infty \mathcal{Z}^m \tag{2.42}$$

where

$$\begin{aligned}
\mathcal{Z}^0 &= \Big\{\varphi(\theta):\ \varphi\in L^2(\mathbb{S}^1),\ \int_0^{2\pi}\varphi(\theta)\, d\theta = 0\Big\} && (2.43)\\
\mathcal{Z}^m &= \Big\{\varphi_1(\theta)\cos m\sigma + \varphi_2(\theta)\sin m\sigma:\ \varphi_1,\varphi_2\in L^2(\mathbb{S}^1)\Big\} && (2.44)
\end{aligned}$$

Also write

$$\mathcal{X}^m = \mathcal{X} \cap \mathcal{Z}^m. \tag{2.45}$$

For each $m = 0, 1, 2, ...$, $\mathcal{L}$ maps $\mathcal{X}^m$ to $\mathcal{Z}^m$, so each $\mathcal{Z}^m$ is an invariant subspace of $\mathcal{L}$. There exist operators $\mathcal{L}^m$ acting on $\varphi(\theta)$ so that

$$\begin{aligned} \mathcal{L}(\varphi(\theta)) &= \mathcal{L}^0\varphi, \quad \text{i.e. } \mathcal{L}^0 \text{ is } \mathcal{L} \text{ restricted to axially symmetric inputs} & (2.46)\\ \mathcal{L}(\varphi(\theta)\cos m\sigma) &= (\mathcal{L}^m\varphi)\cos m\sigma, \ \ \mathcal{L}(\varphi(\theta)\sin m\sigma) = (\mathcal{L}^m\varphi)\sin m\sigma, \ m = 1, 2, 3, ... & (2.47)\end{aligned}$$

The operator $\mathcal{L}^0$ maps from $H^2_\sharp(S^1)$ to $L^2_\sharp(S^1)$ where

$$H^2_\sharp(S^1) = \Big\{\varphi \in H^2(S^1) : \int_0^{2\pi} \varphi(\theta)\, d\theta = 0\Big\}, \quad L^2_\sharp(S^1) = \Big\{\varphi \in L^2(S^1) : \int_0^{2\pi} \varphi(\theta)\, d\theta = 0\Big\}. \tag{2.48}$$

The other $\mathcal{L}^m$'s, $m = 1, 2, 3, ...$, map from $H^2(S^1)$ to $L^2(S^1)$. We refer to m as a mode of $\mathcal{L}$. The spectrum of $\mathcal{L}$ is the union of the spectra of the $\mathcal{L}^m$'s.

Associated with $\hat{\mathcal{L}}$ (resp. $\check{\mathcal{L}}$) there also exist $\hat{\mathcal{L}}^m$ (resp. $\check{\mathcal{L}}^m$) for which analogies of (2.46) and (2.47) hold. Then each $\mathcal{L}^m$ can be written as

$$\mathcal{L}^m = \hat{\mathcal{L}}^m + \gamma\check{\mathcal{L}}^m. \tag{2.49}$$

The first operator $\hat{\mathcal{L}}^m$ is given by a quadratic form

$$\begin{aligned} \langle \hat{\mathcal{L}}^m\varphi, \varphi\rangle &= \int_0^{2\pi} \Big[\frac{(1+u\cos\theta)u_\theta}{(u_\theta^2+u^2)^{1/2}}\Big(\frac{-(1+2u\cos\theta)\varphi^2}{2(u+u^2\cos\theta)^3}\Big)_\theta & (2.50)\\ &\quad -\frac{(u+2u^2\cos\theta+u_\theta^2\cos\theta)(1+2u\cos\theta)\varphi^2}{2(u_\theta^2+u^2)^{1/2}(u+u^2\cos\theta)^3} & (2.51)\\ &\quad +\frac{(1+u\cos\theta)u^2}{2(u_\theta^2+u^2)^{3/2}}\Big(\frac{\varphi}{u+u^2\cos\theta}\Big)_\theta^2 & (2.52)\\ &\quad +\frac{m^2\varphi^2}{2(u_\theta^2+u^2)^{1/2}(1+u\cos\theta)^3} & (2.53)\\ &\quad +\frac{-uu_\theta+u_\theta^3\cos\theta}{(u_\theta^2+u^2)^{3/2}(u+u^2\cos\theta)}\Big(\frac{\varphi}{u+u^2\cos\theta}\Big)_\theta\varphi & (2.54)\\ &\quad +\frac{(2u^3\cos\theta+u_\theta^2+3uu_\theta^2\cos\theta)\varphi^2}{2(u_\theta^2+u^2)^{3/2}(u+u^2\cos\theta)^2}\Big]\, d\theta. & (2.55)\end{aligned}$$

Here we have used the same $\langle\cdot,\cdot\rangle$ to denote the inner product in $L^2(S^1)$. For the second operator $\check{\mathcal{L}}^m$, define

$$G^m(r, z, \rho, \zeta) = \frac{1}{4\pi}\int_0^{2\pi} \frac{\rho e^{\mathrm{i}m\tau}\, d\tau}{\sqrt{r^2+\rho^2-2r\rho\cos\tau+(z-\zeta)^2}}, \ m = 0, 1, 2, ... \tag{2.56}$$

One derives from (2.40)

$$\begin{aligned} \check{\mathcal{L}}^0\varphi &= \int_0^{2\pi} \frac{G^0(1+ue^{\mathrm{i}\theta}, 1+ue^{\mathrm{i}\omega})\varphi(\omega)}{1+u\cos\omega}\, d\omega + \frac{\varphi(\theta)}{u+u^2\cos\theta}\int_{\Omega'} \nabla G^0(1+ue^{\mathrm{i}\theta}, \rho, \zeta)\cdot e^{i\theta}\, d\rho d\zeta & (2.57)\\ &\quad -\mathrm{Av}\Big(\int_0^{2\pi} \frac{G^0(1+ue^{\mathrm{i}\theta}, 1+ue^{\mathrm{i}\omega})\varphi(\omega)}{1+u\cos\omega}\, d\omega + \frac{\varphi(\theta)}{u+u^2\cos\theta}\int_{\Omega'} \nabla G^0(1+ue^{\mathrm{i}\theta}, \rho, \zeta)\cdot e^{i\theta}\, d\rho d\zeta\Big) \end{aligned}$$

where Av(...) denotes the average of a function of θ over $(0, 2\pi)$, and

$$\check{\mathcal{L}}^m\varphi = \int_0^{2\pi} \frac{G^m(1+ue^{\mathrm{i}\theta}, 1+ue^{\mathrm{i}\omega})\varphi(\omega)}{1+u\cos\omega}\, d\omega + \frac{\varphi(\theta)}{u+u^2\cos\theta}\int_{\Omega'} \nabla G^0(1+ue^{\mathrm{i}\theta}, \rho, \zeta)\cdot e^{i\theta}\, d\rho d\zeta, \ m = 1, 2, 3, ... \tag{2.58}$$

3 Fine properties of Ω

We collect some properties of the torus profile.

Proposition 3.1 *A torus profile Ω, also identified by u or v, of radii 1 and q exists when q is sufficiently small. It has the following properties.*

1. *The profile Ω is an axially symmetric set.*
2. *The function v satisfies the following conditions.*

$$\int_0^{2\pi} \Big(v(\theta) - \Big(\frac{q^2}{2} + \frac{q^3}{3}\cos\theta\Big)\Big)\, d\theta = 0 \tag{3.1}$$

$$\int_0^{2\pi} \Big(v(\theta) - \Big(\frac{q^2}{2} + \frac{q^3}{3}\cos\theta\Big)\Big) \cos\theta\, d\theta = 0 \tag{3.2}$$

$$\int_0^{2\pi} \Big(v(\theta) - \Big(\frac{q^2}{2} + \frac{q^3}{3}\cos\theta\Big)\Big) \sin\theta\, d\theta = 0. \tag{3.3}$$

3. *Both u and v are even functions*

$$u(-\theta) = u(\theta),\ v(-\theta) = v(\theta),\ \forall\theta \in S^1.$$

The first two parts of the proposition are shown in the proof of [19, Theorem 1.1]. The function $\frac{q^2}{2} + \frac{q^3}{3}\cos\theta$ represents the perfect torus of radii 1 and q. Equations (3.2) and (3.3) in Part 2 give a precise meaning that the perturbed circle $1 + u(\theta)e^{\mathrm{i}\theta}$, $\theta \in S^1$, where u corresponds to v, in the rz-plane is centered at $(1, 0)$. This implies that Ω, although a perturbed torus, still has a well defined major radius equal to 1. Equation (3.1) is the same as the volume constraint (2.7). It also serves as the interpretation that the minor radius of the profile equals q.

Part 3 asserts that the torus profile has the mirror symmetry with respect to the xy-plane. This property is not claimed in [19] but can be established easily. One considers the profile equation (1.12) in the class of even functions and the same proof of [19, Theorem 1.1] still works if the function spaces used in [19] are replaced by their restrictions to even functions.

To learn more about the torus profile, one needs a better understanding of the function G^m in (2.56). Rewrite G^m as

$$G^m(r, z, \rho, \zeta) = \frac{1}{2\pi}\sqrt{\frac{\rho}{r}} \int_0^{\pi/2} \frac{\cos 2m\tau\, d\tau}{\sqrt{\beta + \sin^2\tau}}\ , \quad \text{where } \beta = \frac{(r-\rho)^2 + (z-\zeta)^2}{4r\rho}\ . \tag{3.4}$$

Lemma 3.2 below shows the asymptotic behavior of the integral in (3.4) with respect to β. Two important quantities, d_0^m and d_1^m, both dependent on m, appear in this lemma. Lemma 3.3 deals with the growth rate of the two quantities as m tends to ∞. Lemma 3.4 follows from Lemmas 3.2 and 3.3 and provides sharp estimates for G^m. The last Lemma 3.5 shows the positivity of two quantities that are closely related to d_0^m and d_m^1. We place the proofs of these lemmas in the appendix.

Lemma 3.2 *Let $\alpha \in (0, 1)$. Then for small β we have the expansion*

$$\int_0^{\pi/2} \frac{\cos 2m\tau\, d\tau}{\sqrt{\beta + \sin^2\tau}} = \Big(\frac{1}{2} + \big(\frac{m^2}{2} - \frac{1}{8}\big)\beta\Big) \log\frac{16}{\beta} + d_0^m + d_1^m\beta + R(\beta)$$

where

$$d_0^m = -2\int_0^{\pi/2} \frac{\sin^2 m\tau}{\sin\tau}\, d\tau, \quad d_1^m = \frac{1}{4} - \frac{m^2}{2} - \int_0^{\pi/2} \frac{m^2\sin^2\tau - \sin^2 m\tau}{\sin^3\tau}\, d\tau,$$

and $R(\beta)$ is a small quantitity satisfying

1. $|R(\beta)| \le C\beta^2 \log\frac{1}{\beta}$, *if* $m = 0$ *or* $m = 1$, *and*

2. $|R(\beta)| \le Cm^{2+2\alpha}\beta^{1+\alpha}$, *if* $m \ge 2$.

The constant C *in 1. does not depend on* α *or* β*; the constant* C *in 2. depends on* α *but not on* m *or* β*.*

Lemma 3.3 *For large* m,

1. $\displaystyle\int_0^{\pi/2} \frac{\sin^2 m\tau}{\sin\tau}\, d\tau = \frac{1}{2}\log m + O(1)$, *and*

2. $\displaystyle\int_0^{\pi/2} \frac{m^2\sin^2\tau - \sin^2 m\tau}{\sin^3\tau}\, d\tau = m^2\log m + O(m^2)$.

Consequently $d_0^m = -\log m + O(1)$ *and* $d_1^m = -m^2\log m + O(m^2)$.

Lemma 3.4 *Let* $r = 1 + O(q)$, $z = O(q)$, $\rho = 1 + O(q)$, *and* $\zeta = O(q)$.

1. *If* $m = 0$, *then*

$$\begin{aligned} G^0(r,z,\rho,\zeta) &= \frac{1}{2\pi}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} - \frac{r-\rho}{4\pi\rho}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} \\ &\quad + \frac{5(r-\rho)^2-(z-\zeta)^2}{32\pi\rho^2}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} + \frac{(r-\rho)^2+(z-\zeta)^2}{32\pi\rho^2} \\ &\quad + O\big(q^3\log\frac{8}{q}\big). \end{aligned}$$

2. *If* $m = 1$, *then*

$$\begin{aligned} G^1(r,z,\rho,\zeta) &= \frac{1}{2\pi}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} - \frac{r-\rho}{4\pi\rho}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} \\ &\quad + \frac{9(r-\rho)^2+3(z-\zeta)^2}{32\pi\rho^2}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} - \frac{1}{\pi}\sqrt{\frac{\rho}{r}} - \frac{(r-\rho)^2+(z-\zeta)^2}{32\pi\rho^2} \\ &\quad + O\big(q^3\log\frac{8}{q}\big). \end{aligned}$$

3. *If* $m \ge 2$, *then*

$$\begin{aligned} G^m(r,z,\rho,\zeta) &= \frac{1}{2\pi}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} - \frac{r-\rho}{4\pi\rho}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} \\ &\quad + \frac{(4m^2+5)(r-\rho)^2+(4m^2-1)(z-\zeta)^2}{32\pi\rho^2}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} \\ &\quad + \frac{d_0^m}{2\pi}\sqrt{\frac{\rho}{r}} + \Big(\frac{d_1^m}{2\pi}\Big)\frac{(r-\rho)^2+(z-\zeta)^2}{4\rho^2} \\ &\quad + O(m^2q^3\log\frac{8}{q}) + O((m^2\log m)q^3) + O\big(m^{2+2\alpha}q^{2+2\alpha}\big) \end{aligned}$$

Lemma 3.5 1. *When* $m \ge 2$, $\displaystyle\int_0^{\pi/2} \frac{m^2\sin^2\tau - \sin^2 m\tau}{\sin^3\tau}\, d\tau > 0$.

2. *When* $m \ge 2$, $\displaystyle 1 - \int_0^{\pi/2} \frac{\sin^2 m\tau}{\sin\tau}\, d\tau + \int_0^{\pi/2} \frac{m^2\sin^2\tau - \sin^2 m\tau}{\sin^3\tau}\, d\tau > 0$.

The function u that characterizes the profile Ω can be expanded into a series

$$u = u_0 + u_1 + u_2 ... \tag{3.5}$$

This is an expansion with respect to the parameter q, so each term u_k is of the order q less than the previous term u_{k-1}. However in general u_k is not just a function independent of q multiplied by a power of q; u_k may also depend on $\frac{1}{\log \frac{8}{q}}$ and each u_k can again be expanded into a series where each term is of $\frac{1}{\log \frac{8}{q}}$ order less than the previous term. This means a two parameter expansion with respect to the primary parameter q and the secondary parameter $\frac{1}{\log \frac{8}{q}}$.

The leading order term u_0 in (3.5) is

$$u_0 = q. \tag{3.6}$$

This term corresponds to the perfect torus whose two radii are 1 and q. This particular u_0 does not depend on $\frac{1}{\log \frac{8}{q}}$. Later we will see in (3.31) that

$$u_1 = 0 \text{ and } u_2 = q^3 \Big(\frac{12 \log \frac{8}{q} - 5}{288 \log \frac{8}{q} - 456} \Big) \cos 2\theta \tag{3.7}$$

In u_2 there are both q^3 and a fraction that depends on $\frac{1}{\log \frac{8}{q}}$. The fraction can itself be expended as a power series of $\frac{1}{\log \frac{8}{q}}$ so that

$$u_2 = q^3 \Big(\frac{1}{24} + \frac{7}{144} \Big(\frac{1}{\log \frac{8}{q}} \Big) + \frac{133}{1728} \Big(\frac{1}{\log \frac{8}{q}} \Big)^2 + \frac{2527}{20736} \Big(\frac{1}{\log \frac{8}{q}} \Big)^3 + ... \Big) \cos 2\theta \tag{3.8}$$

Recall that the torus profile Ω is identified by either u or v. They are related via the transformation (2.16). The function v is also expanded with respect to q so that

$$v = v_0 + v_1 + v_2 ... \tag{3.9}$$

The two leading orders of v are

$$v_0 = \frac{q^2}{2} \text{ and } v_1 = \frac{q^3}{3} \cos \theta. \tag{3.10}$$

Note that

$$v_0 + v_1 = \frac{q^2}{2} + \frac{q^3}{3} \cos \theta \tag{3.11}$$

describes the perfect torus and it is related to u_0 by (2.16).

To determine v_2 in the expansion of v, we need an estimate on $\mathcal{S}(v_0 + v_1)$. The mean curvature of $v_0 + v_1$ is

$$\mathcal{H}(v_0 + v_1)(\theta) = \frac{1}{2} \Big(\frac{1}{q} + \frac{\cos \theta}{1 + q \cos \theta} \Big) = \frac{1}{2q} + \frac{\cos \theta}{2} - \frac{q}{4} - \frac{\cos 2\theta}{4} q + O(q^2). \tag{3.12}$$

To find the Newtonian potential of $v_0 + v_1$,

$$\mathcal{N}(v_0 + v_1)(\theta) = \int_0^{2\pi} \int_0^q G^0(1 + q e^{\mathrm{i}\theta}, 1 + h e^{\mathrm{i}\omega})\, h dh d\omega = q^2 \int_0^{2\pi} \int_0^1 G^0(1 + q e^{\mathrm{i}\theta}, 1 + q h e^{\mathrm{i}\omega})\, h dh d\omega, \tag{3.13}$$

use Lemma 3.4 to break G^0 into three terms and evaluate three corresponding integrals. First with

$$r = 1 + q \cos \theta, \ z = q \sin \theta, \ \rho = 1 + q h \cos \omega, \ \zeta = q h \sin \omega,$$

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^1 \frac{1}{2\pi}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|}\,hdhd\omega \\
=\ & \frac{1}{2}\log\frac{8}{q}+\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\Big(\log\sqrt{(1+q\cos\theta)(1+qh\cos\omega)}+\log\frac{1}{|1-he^{\mathrm{i}(\theta-\omega)}|}\Big)\,hdhd\omega \\
=\ & \frac{1}{2}\log\frac{8}{q}+\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\Big(\frac{q(\cos\theta+h\cos\omega)}{2}+\frac{q^2h\cos\theta\cos\omega}{2}-\frac{q^2(\cos\theta+h\cos\omega)^2}{4}\Big)\,hdhd\omega+O(q^3) \\
=\ & \frac{1}{2}\log\frac{8}{q}+\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\Big(\frac{q\cos\theta}{2}-\frac{q^2\cos^2\theta+q^2h^2\cos^2\omega}{4}\Big)\,hdhd\omega+O(q^3) \\
=\ & \frac{1}{2}\log\frac{8}{q}+\frac{\cos\theta}{4}q-\frac{\cos^2\theta}{8}q^2-\frac{q^2}{32}+O(q^3) \\
=\ & \frac{1}{2}\log\frac{8}{q}+\frac{\cos\theta}{4}q-\frac{\cos 2\theta}{16}q^2-\frac{3}{32}q^2+O(q^3). \qquad (3.14)
\end{aligned}
$$

Here we have used the important formula

$$
\log\frac{1}{|1-he^{\mathrm{i}(\theta-\omega)}|}=\sum_{n=1}^{\infty}\frac{h^n\cos n(\theta-\omega)}{n},\ \ h\in[0,1] \qquad (3.15)
$$

which will be called upon several more times. The next term is

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^1 -\frac{r-\rho}{4\pi\rho}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|}\,hdhd\omega \\
=\ & \int_0^{2\pi}\int_0^1\frac{q(-\cos\theta+h\cos\omega)}{4\pi(1+qh\cos\omega)}\Big(\log\frac{8}{q}+\log\frac{1}{|1-he^{\mathrm{i}(\theta-\omega)}|}+\log\sqrt{(1+q\cos\theta)(1+qh\cos\omega)}\Big)\,hdhd\omega \\
=\ & \frac{q}{4\pi}\int_0^{2\pi}\int_0^1(-\cos\theta+h\cos\omega)(1-qh\cos\omega)\Big(\log\frac{8}{q}+\sum_{n=1}^{\infty}\frac{h^n\cos n(\theta-\omega)}{n}+\frac{q(\cos\theta+h\cos\omega)}{2}\Big)\,hdhd\omega \\
& +O(q^3\log\frac{8}{q}) \\
=\ & \frac{q}{4\pi}\int_0^{2\pi}\int_0^1(-\cos\theta+h\cos\omega)\Big(\log\frac{8}{q}+\sum_{n=1}^{\infty}\frac{h^n\cos n(\theta-\omega)}{n}\Big)\,hdhd\omega \\
& -\frac{q^2}{4\pi}\int_0^{2\pi}\int_0^1(-\cos\theta+h\cos\omega)h\cos\omega\Big(\log\frac{8}{q}+\sum_{n=1}^{\infty}\frac{h^n\cos n(\theta-\omega)}{n}\Big)\,hdhd\omega \\
& +\frac{q^2}{4\pi}\int_0^{2\pi}\int_0^1(-\cos\theta+h\cos\omega)\Big(\frac{\cos\theta+h\cos\omega}{2}\Big)\,hdhd\omega+O(q^3\log\frac{8}{q}) \\
=\ & \frac{q}{4\pi}\Big(-\pi\cos\theta\log\frac{8}{q}+\frac{\pi\cos\theta}{4}\Big)-\frac{q^2}{4\pi}\Big(-\frac{\pi\cos^2\theta}{4}+\frac{\pi}{4}\log\frac{8}{q}+\frac{\pi\cos 2\theta}{24}\Big)+\frac{q^2}{4\pi}\Big(-\frac{\pi\cos^2\theta}{2}+\frac{\pi}{8}\Big) \\
& +O(q^3\log\frac{8}{q}) \\
=\ & -\frac{\cos\theta}{4}q\log\frac{8}{q}+\frac{\cos\theta}{16}q-\frac{1}{16}q^2\log\frac{8}{q}-\frac{\cos 2\theta}{24}q^2+O(q^3\log\frac{8}{q}). \qquad (3.16)
\end{aligned}
$$

The third term is

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^1\frac{5(r-\rho)^2-(z-\zeta)^2}{32\pi\rho^2}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|}\,hdhd\omega \\
=\ & \frac{q^2}{32\pi}\int_0^{2\pi}\int_0^1\big(5(\cos\theta-h\cos\omega)^2-(\sin\theta-h\sin\omega)^2\big)\Big(\log\frac{8}{q}+\log\frac{1}{|1-he^{\mathrm{i}(\theta-\omega)}|}\Big)\,hdhd\omega
\end{aligned}
$$

$$
\begin{aligned}
& +O(q^3 \log \frac{8}{q}) \\
= \; & \frac{q^2}{32\pi} \int_0^{2\pi} \int_0^1 \big(6\cos^2\theta - 1 - 10h\cos\theta\cos\omega + 2h\sin\theta\sin\omega + 6h^2\cos^2\omega - h^2\big) \\
& \Big(\log\frac{8}{q} + \sum_{n=1}^{\infty} \frac{h^n \cos n(\theta-\omega)}{n}\Big) hdhd\omega + O(q^3 \log\frac{8}{q}) \\
= \; & \frac{q^2}{32\pi} \int_0^{2\pi} \int_0^1 \big(6\cos^2\theta - 1 + 6h^2\cos^2\omega - h^2\big) \log\frac{8}{q}\, hdhd\omega \\
& + \frac{q^2}{32\pi} \int_0^{2\pi} \int_0^1 \big(-10h\cos\theta\cos\omega + 2h\sin\theta\sin\omega + 6h^2\cos^2\omega\big) \sum_{n=1}^{\infty} \frac{h^n \cos n(\theta-\omega)}{n}\, hdhd\omega \\
& +O(q^3 \log\frac{8}{q}) \\
= \; & \frac{q^2}{32\pi} \log\frac{8}{q} \Big(6\pi\cos^2\theta - \pi + \frac{3\pi}{2} - \frac{\pi}{2}\Big) + \frac{q^2}{32\pi}\Big(-\frac{5\pi}{2}\cos^2\theta + \frac{\pi}{2}\sin^2\theta + \frac{\pi\cos 2\theta}{4}\Big) + O(q^3\log\frac{8}{q}) \\
= \; & \frac{3\cos 2\theta}{32} q^2 \log\frac{8}{q} + \frac{3}{32} q^2 \log\frac{8}{q} - \frac{5\cos 2\theta}{128} q^2 - \frac{1}{32} q^2 + O(q^3 \log\frac{8}{q}). \qquad (3.17)
\end{aligned}
$$

And finally the fourth term is

$$
\begin{aligned}
& \int_0^{2\pi} \int_0^1 \frac{(r-\rho)^2 + (z-\zeta)^2}{32\pi\rho^2}\, hdhd\omega \\
& = \frac{q^2}{32\pi} \int_0^{2\pi} \int_0^1 \big((\cos\theta - h\cos\omega)^2 + (\sin\theta - h\sin\omega)^2\big)\, hdhd\omega + O(q^3) \\
& = \frac{q^2}{32\pi} \int_0^{2\pi} \int_0^1 (1+h^2)\, hdhd\omega + O(q^3) \\
& = \frac{3}{64} q^2 + O(q^3). \qquad (3.18)
\end{aligned}
$$

Summing up (3.14), (3.16), (3.17), and (3.18), one deduces

$$
\begin{aligned}
\mathcal{N}(v_0+v_1) = \; & \frac{q^2}{2}\log\frac{8}{q} + \frac{\cos\theta}{4} q^3 - \frac{\cos 2\theta}{16} q^4 - \frac{3}{32} q^4 - \frac{\cos\theta}{4} q^3 \log\frac{8}{q} + \frac{\cos\theta}{16} q^3 - \frac{1}{16} q^4 \log\frac{8}{q} - \frac{\cos 2\theta}{24} q^4 \\
& + \frac{3\cos 2\theta}{32} q^4 \log\frac{8}{q} + \frac{3}{32} q^4 \log\frac{8}{q} - \frac{5\cos 2\theta}{128} q^4 - \frac{1}{32} q^4 + \frac{3}{64} q^4 + O(q^5 \log\frac{8}{q}) \\
= \; & \frac{q^2}{2}\log\frac{8}{q} - \frac{\cos\theta}{4} q^3 \log\frac{8}{q} + \frac{5\cos\theta}{16} q^3 + \frac{3\cos 2\theta}{32} q^4 \log\frac{8}{q} + \frac{1}{32} q^4 \log\frac{8}{q} - \frac{55\cos 2\theta}{384} q^4 - \frac{5}{64} q^4 \\
& + O(q^5 \log\frac{8}{q}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{S}(v_0+v_1) = \; & \frac{\cos\theta}{2} - \frac{\cos 2\theta}{4} q + O(q^2) \\
& + \gamma\Big(-\frac{\cos\theta}{4} q^3 \log\frac{8}{q} + \frac{5\cos\theta}{16} q^3 + \frac{3\cos 2\theta}{32} q^4 \log\frac{8}{q} - \frac{55\cos 2\theta}{384} q^4 + O(q^5 \log\frac{8}{q})\Big).
\end{aligned}
$$

Note that the constant terms like $\frac{1}{2q}$ in $\mathcal{H}(v_0+v_1)$ and $\frac{q^2}{2}\log\frac{8}{q}$ in $\mathcal{N}(v_0+v_1)$ vanish because the average of $\mathcal{S}(v_0+v_1)$ is 0 as in (2.12).

We expand γ into

$$
\gamma = \frac{\Gamma}{q^3} + \frac{\Gamma'}{q^2} + \frac{\Gamma''}{q} + ... \qquad (3.19)
$$

Then

$$\mathcal{S}(v_0+v_1) = \frac{\cos\theta}{2}+\Gamma\Big(-\frac{\cos\theta}{4}\log\frac{8}{q}+\frac{5\cos\theta}{16}\Big)-\frac{\cos 2\theta}{4}q$$
$$+\Gamma\Big(\frac{3\cos 2\theta}{32}q\log\frac{8}{q}-\frac{55\cos 2\theta}{384}q\Big)+\Gamma' q\Big(-\frac{\cos\theta}{4}\log\frac{8}{q}+\frac{5\cos\theta}{16}\Big)+... \tag{3.20}$$

The leading order in (3.20) vanishes so

$$\frac{\cos\theta}{2}+\Gamma\Big(-\frac{\cos\theta}{4}\log\frac{8}{q}+\frac{5\cos\theta}{16}\Big)=0. \tag{3.21}$$

Therefore

$$\Gamma=\frac{8}{4\log\frac{8}{q}-5}\ . \tag{3.22}$$

With this choice of Γ, (3.20) becomes

$$\mathcal{S}(v_0+v_1)=-\frac{\cos 2\theta}{4}q+\Gamma\Big(\frac{3\cos 2\theta}{32}q\log\frac{8}{q}-\frac{55\cos 2\theta}{384}q\Big)+\Gamma' q\Big(-\frac{\cos\theta}{4}\log\frac{8}{q}+\frac{5\cos\theta}{16}\Big)+... \tag{3.23}$$

The term v_2 in the expansion of v solves

$$\mathcal{L}_0^0 v_2-\frac{\cos 2\theta}{4}q+\Gamma\Big(\frac{3\cos 2\theta}{32}q\log\frac{8}{q}-\frac{55\cos 2\theta}{384}q\Big)+\Gamma' q\Big(-\frac{\cos\theta}{4}\log\frac{8}{q}+\frac{5\cos\theta}{16}\Big)=0. \tag{3.24}$$

Here $\mathcal{L}_0^0$ is the leading order approximation of $\mathcal{S}'(v_0)$ in the axially symmetric class given by

$$\mathcal{L}_0^0\phi=\frac{1}{2q^3}\big(-\phi_{\theta\theta}-\phi\big)+\frac{\Gamma}{q^3}\int_0^{2\pi}\frac{1}{2\pi}\log\frac{8}{q|e^{i\theta}-e^{i\omega}|}\phi(\omega)\,d\omega-\frac{\Gamma\phi(\theta)}{2q^3}. \tag{3.25}$$

Note that $\cos\theta$ and $\sin\theta$ form the kernel of $\mathcal{L}_0^0$, i.e.

$$\mathcal{L}_0^0(\cos\theta)=\mathcal{L}_0^0(\sin\theta)=0. \tag{3.26}$$

The derivation of (3.26) makes use of (3.15). We will have more to say about $\mathcal{L}_0^0$ in Sections 4 and 5.

Multiplying (3.24) by $\cos\theta$ and integrate, we deduce

$$\Gamma'=0. \tag{3.27}$$

Moreover, by Part 2 of Proposition 3.1, v_2 must be perpendicular to 1, $\cos\theta$, and $\sin\theta$, i.e.

$$\int_0^{2\pi}v_2(\theta)\,d\theta=\int_0^{2\pi}v_2(\theta)\cos\theta\,d\theta=\int_0^{2\pi}v_2(\theta)\sin\theta\,d\theta=0. \tag{3.28}$$

Then $v_2=q^4A\cos 2\theta$ by (3.24), (3.27), and (3.28) where A satisfies

$$\Big[\frac{4-1}{2q^3}+\frac{\Gamma}{q^3}\Big(\frac{1}{4}-\frac{1}{2}\Big)\Big]Aq^4-\frac{q}{4}+\frac{\Gamma}{q^3}\Big(\frac{3}{32}q^4\log\frac{8}{q}-\frac{55}{384}q^4\Big)=0$$

With Γ given by (3.22),

$$A=\frac{\frac{1}{4}-\Gamma\Big(\frac{3}{32}\log\frac{8}{q}-\frac{55}{384}\Big)}{\frac{3}{2}-\Gamma\frac{1}{4}}=\frac{12\log\frac{8}{q}-5}{288\log\frac{8}{q}-456}\ . \tag{3.29}$$

In summary we have the expansion

$$v = v_0+v_1+v_2+...$$
$$= \frac{q^2}{2}+\frac{q^3\cos\theta}{3}+q^4A\cos 2\theta+... \tag{3.30}$$

for v, where A is given in (3.29). Consequently we have an expansion for u

$$\begin{aligned} u &= u_0 + u_1 + u_2 + ... \\ &= q + q^3 A \cos 2\theta + ... \end{aligned} \tag{3.31}$$

derived from (2.16). In (3.31)

$$u_1 = 0. \tag{3.32}$$

4 Expansion of $\mathcal{L}^m$

The next objective is to study $\mathcal{L}$, the linearized operator at v, through the expansions of the $\mathcal{L}^m$'s. Recall that $\mathcal{L}$ is decomposed into the $\mathcal{L}^m$'s, $m = 0, 1, 2...$ For each m, one can expand $\mathcal{L}^m$ in terms of q:

$$\mathcal{L}^m = \mathcal{L}_0^m + \mathcal{L}_1^m + \mathcal{L}_2^m + ... \tag{4.1}$$

To this end we need expansions of $\hat{\mathcal{L}}^m$ and $\check{\mathcal{L}}^m$

$$\hat{\mathcal{L}}^m = \hat{\mathcal{L}}_0^m + \hat{\mathcal{L}}_1^m + \hat{\mathcal{L}}_2^m + ..., \quad \check{\mathcal{L}}^m = \check{\mathcal{L}}_0^m + \check{\mathcal{L}}_1^m + \check{\mathcal{L}}_2^m + ... \tag{4.2}$$

First consider the $m \geq 1$ case. The expansion (3.31) gives a very precise approximation of u. We insert (3.31) into the $\mathcal{L}^m$'s derived in Section 2 and find their expansions in terms of q. Regarding $\hat{\mathcal{L}}^m$, the leading order $\hat{\mathcal{L}}_0^m$ is derived from the leading orders of (2.51) and (2.52):

$$\langle \hat{\mathcal{L}}_0^m \varphi, \varphi \rangle = \int_0^{2\pi} \Big(\frac{\varphi_\theta^2}{2q^3} - \frac{\varphi^2}{2q^3} \Big) \, d\phi; \tag{4.3}$$

namely

$$\hat{\mathcal{L}}_0^m \varphi = -\frac{\varphi_{\theta\theta}}{2q^3} - \frac{\varphi}{2q^3}. \tag{4.4}$$

For $\hat{\mathcal{L}}_1^m$ the terms (2.50), (2.53), and (2.54) are negligible. The terms (2.51), (2.52), and (2.55) give rise to

$$\langle \hat{\mathcal{L}}_1^m \varphi, \varphi \rangle = \int_0^{2\pi} \Big[\frac{-\cos\theta \varphi^2}{2q^2} + \frac{2\sin\theta \varphi_\theta \varphi - \cos\theta \varphi_\theta^2}{2q^2} + \frac{\cos\theta \varphi^2}{q^2} \Big] \, d\theta = \int_0^{2\pi} \Big(-\frac{\cos\theta}{2q^2} \Big) \varphi_\theta^2, \tag{4.5}$$

so

$$\hat{\mathcal{L}}_1^m \varphi = \Big(\frac{\cos\theta}{2q^2} \varphi_\theta \Big)_\theta \,. \tag{4.6}$$

To find $\hat{\mathcal{L}}_2^m$, the terms (2.50)-(2.55) respectively yield

$$\begin{aligned} \langle \hat{\mathcal{L}}_2^m \varphi, \varphi \rangle &= \int_0^{2\pi} \Big[\frac{A \sin 2\theta}{q} (\varphi^2)_\theta && (4.7) \\ &\quad + \frac{2\cos^2\theta + 3A\cos 2\theta}{2q} \varphi^2 && (4.8) \\ &\quad + \frac{\cos^2\theta - 3A\cos 2\theta}{2q} \varphi_\theta^2 + \frac{2A \sin 2\theta - \sin 2\theta}{q} \varphi_\theta \varphi + \frac{\sin^2\theta}{2q} \varphi^2 && (4.9) \\ &\quad + \frac{m^2}{2q} \varphi^2 && (4.10) \\ &\quad + \frac{2A\sin 2\theta}{q} \varphi_\theta \varphi && (4.11) \\ &\quad - \frac{2\cos^2\theta}{q} \varphi^2 \Big] \, d\theta && (4.12) \\ &= \frac{1}{q} \int_0^{2\pi} \Big[\Big(\frac{1}{4} + \big(\frac{1}{4} - \frac{3A}{2} \big) \cos 2\theta \Big) \varphi_\theta^2 + \Big(\frac{m^2}{2} - \frac{1}{4} + \big(\frac{1}{4} - \frac{9A}{2} \big) \cos 2\theta \Big) \varphi^2 \Big] \, d\theta. && (4.13) \end{aligned}$$

This gives the operator

$$\hat{\mathcal{L}}_2^m\varphi = \frac{1}{q}\Big[-\Big(\frac{1}{4}+\big(\frac{1}{4}-\frac{3A}{2}\big)\cos 2\theta\Big)\varphi_\theta\Big]_\theta + \frac{1}{q}\Big[\frac{m^2}{2}-\frac{1}{4}+\big(\frac{1}{4}-\frac{9A}{2}\big)\cos 2\theta\Big]\varphi. \tag{4.14}$$

Note in (4.14) the presence of A from the expansion (3.31) of u.

Regarding $\check{\mathcal{L}}^m$, we invoke Lemma 3.4. The leading order is

$$\check{\mathcal{L}}_0^m\varphi = \int_0^{2\pi}\Big(\frac{1}{2\pi}\log\frac{8}{q|e^{\mathrm{i}\theta}-e^{\mathrm{i}\omega}|}+\frac{d_0^m}{2\pi}\Big)\varphi(\omega)\,d\omega - \frac{\varphi(\theta)}{2}\ . \tag{4.15}$$

The next order is

$$\begin{aligned}
\check{\mathcal{L}}_1^m\varphi \quad = \quad & \frac{q}{2\pi}\int_0^{2\pi}\Big[(-\cos\omega)\log\frac{8}{q|e^{\mathrm{i}\theta}-e^{\mathrm{i}\omega}|}+\frac{\cos\theta+\cos\omega}{2}\Big]\varphi(\omega)\,d\omega\\
& -\frac{q}{4\pi}\int_0^{2\pi}\Big[(\cos\theta-\cos\omega)\log\frac{8}{q|e^{\mathrm{i}\theta}-e^{\mathrm{i}\omega}|}\Big]\varphi(\omega)\,d\omega\\
& -\frac{qd_0^m}{4\pi}\int_0^{2\pi}(\cos\theta+\cos\omega)\varphi(\omega)\,d\omega\\
& +q\Big(\frac{3\cos\theta}{4}\Big)\varphi\\
& +q\Big(-\frac{\cos\theta}{4}\log\frac{8}{q}+\frac{3\cos\theta}{16}\Big)\varphi
\end{aligned}$$

which is simplified to

$$\begin{aligned}
\check{\mathcal{L}}_1^m\varphi \quad = \quad & \frac{q}{2\pi}\int_0^{2\pi}\Big[\frac{-\cos\theta-\cos\omega}{2}\log\frac{8}{q|e^{\mathrm{i}\theta}-e^{\mathrm{i}\omega}|}+\big(\frac{1}{2}-\frac{d_0^m}{2}\big)(\cos\theta+\cos\omega)\Big]\varphi(\omega)\,d\omega\\
& +q\Big(-\frac{\cos\theta}{4}\log\frac{8}{q}+\frac{15\cos\theta}{16}\Big)\varphi.
\end{aligned} \tag{4.16}$$

And

$$\begin{aligned}
\check{\mathcal{L}}_2^m\varphi \quad = \quad & \frac{q^2}{2\pi}\int_0^{2\pi}\Big(\cos^2\omega\log\frac{8}{q|e^{\mathrm{i}\theta}-e^{\mathrm{i}\omega}|}-\frac{\cos^2\theta+2\cos\theta\cos\omega+3\cos^2\omega}{4}-\frac{A(\cos 2\theta+\cos 2\omega)}{2}\Big)\varphi(\omega)\,d\omega\\
& -\frac{q^2}{4\pi}\int_0^{2\pi}\Big(-2(\cos\theta\cos\omega-\cos^2\omega)\log\frac{8}{q|e^{\mathrm{i}\theta}-e^{\mathrm{i}\omega}|}+\frac{\cos^2\theta-\cos^2\omega}{2}\Big)\varphi(\omega)\,d\omega\\
& +\frac{q^2}{32\pi}\int_0^{2\pi}\Big(\big((4m^2+5)(\cos\theta-\cos\omega)^2+(4m^2-1)(\sin\theta-\sin\omega)^2\big)\log\frac{8}{q|e^{\mathrm{i}\theta}-e^{\mathrm{i}\omega}|}\Big)\varphi(\omega)\,d\omega\\
& +\frac{d_0^m q^2}{2\pi}\int_0^{2\pi}\Big(\frac{1}{4}\cos\theta\cos\omega+\frac{3}{8}(\cos^2\theta+\cos^2\omega)\Big)\varphi(\omega)\,d\omega\\
& +\frac{d_1^m q^2}{4\pi}\int_0^{2\pi}\big(1-\cos(\theta-\omega)\big)\varphi(\omega)\,d\omega\\
& +q^2\Big(-\cos^2\theta+\frac{A\cos 2\theta}{2}\Big)\varphi(\theta)\\
& +q^2\Big(\frac{1}{8}\log\frac{8}{q}+\frac{\cos 2\theta}{8}\log\frac{8}{q}-\frac{5}{32}-\frac{19\cos 2\theta}{96}\Big)\varphi(\theta)\\
& +\frac{q^2}{32}\Big(4\log\frac{8}{q}+6\cos 2\theta\log\frac{8}{q}-3-\frac{7\cos 2\theta}{2}\Big)\varphi(\theta)\\
& +\frac{q^2}{16}\varphi(\theta)
\end{aligned}$$

which is simplified to

$$
\begin{aligned}
\check{\mathcal{L}}_2^m\varphi &= \frac{q^2}{2\pi}\int_0^{2\pi}\Big[\Big(\frac{2m^2+1}{4}-\frac{m^2\cos(\theta-\omega)}{2}+\frac{3(\cos 2\theta+\cos 2\omega)}{16}+\frac{3\cos\theta\cos\omega+\sin\theta\sin\omega}{8}\Big) \\
&\quad \log\frac{8}{q|e^{\mathrm{i}\theta}-e^{\mathrm{i}\omega}|}+\frac{-4+3d_0^m+4d_1^m}{8}-\frac{d_1^m\cos(\theta-\omega)}{2}+\Big(\frac{-4-8A+3d_0^m}{16}\Big)(\cos 2\theta+\cos 2\omega) \\
&\quad +\Big(\frac{-2+d_0^m}{4}\Big)\cos\theta\cos\omega\Big]\varphi(\omega)\,d\omega \\
&\quad +q^2\Big[\Big(\frac{1}{4}+\frac{5\cos 2\theta}{16}\Big)\log\frac{8}{q}-\frac{11}{16}+\Big(-\frac{155}{192}+\frac{A}{2}\Big)\cos 2\theta\Big]\varphi(\theta).
\end{aligned}
\tag{4.17}
$$

In the $m=0$ case, one finds similar operators $\hat{\mathcal{L}}_0^0$, $\hat{\mathcal{L}}_1^0$, $\hat{\mathcal{L}}_2^0$,..., $\check{\mathcal{L}}_0^0$, $\check{\mathcal{L}}_1^0$, $\check{\mathcal{L}}_2^0$, ..., except that one must include an additional functional in each operator so that the average of the outcome vanishes. We have seen this already in (2.57) where $-\mathrm{Av}(...)$ is the functional.

By the expansion (3.22) for γ where $\Gamma'=0$, one obtains

$$\mathcal{L}_0^m = \hat{\mathcal{L}}_0^m+\frac{\Gamma}{q^3}\check{\mathcal{L}}_0^m \tag{4.18}$$

$$\mathcal{L}_1^m = \hat{\mathcal{L}}_1^m+\frac{\Gamma}{q^3}\check{\mathcal{L}}_1^m \tag{4.19}$$

$$\mathcal{L}_2^m = \hat{\mathcal{L}}_2^m+\frac{\Gamma}{q^3}\check{\mathcal{L}}_2^m+\frac{\Gamma''}{q}\check{\mathcal{L}}_0^m. \tag{4.20}$$

This expansion is used to approximate $\mathcal{L}^m$, but it does not approximate uniformly with respect to all m. In the next section, we restrict to the small m's where the approximation is uniform. The large and medium m's will be treated differently in later sections.

5 Small modes

Consider $\mathcal{L}_0^m$, the leading order approximation of $\mathcal{L}^m$,

$$\mathcal{L}_0^m\varphi=\frac{1}{2q^3}\big(-\varphi_{\theta\theta}-\varphi\big)+\frac{\Gamma}{q^3}\int_0^{2\pi}\Big(\frac{1}{2\pi}\log\frac{8}{q|e^{\mathrm{i}\theta}-e^{\mathrm{i}\omega}|}+\frac{d_0^m}{2\pi}\Big)\varphi(\omega)\,d\omega-\frac{\Gamma\varphi(\theta)}{2q^3}\ . \tag{5.1}$$

In the case $m=0$, we have seen $\mathcal{L}_0^0$ before in (3.25) as the leading order approximation for $\mathcal{S}'(v_0+v_1)$ in the axially symmetric class. The same $\mathcal{L}_0^0$ is now the leading order approximation for $\mathcal{S}'(v)$ in the axially symmetric class.

If $m=0$, the eigenvalues and the corresponding eigenfunctions of $\mathcal{L}_0^0$ are

$$\mu_k^0=\frac{1}{q^3}\Big(\frac{k^2-1}{2}+\Gamma\big(\frac{1}{2k}-\frac{1}{2}\big)\Big),\quad \varphi_k^0=\cos k\theta \text{ or } \sin k\theta,\quad k=1,2,3,... \tag{5.2}$$

If $m\geq 1$, the eigenvalues and the corresponding eigenfunctions of $\mathcal{L}_0^m$ are

$$\mu_k^m=\begin{cases}\frac{1}{q^3}\Big(-\frac{1}{2}+\Gamma\big(\log\frac{8}{q}+d_0^m-\frac{1}{2}\big)\Big), & \varphi_0^m=1, \qquad\qquad \text{if } k=0\\ \frac{1}{q^3}\Big(\frac{k^2-1}{2}+\Gamma\big(\frac{1}{2k}-\frac{1}{2}\big)\Big), & \varphi_k^m=\cos k\theta \text{ or } \sin k\theta, \quad \text{if } k=1,2,3,...\end{cases} \tag{5.3}$$

As the eigenvalues of the $\mathcal{L}_0^m$'s, these μ_k^m's serve as the leading order approximations of the eigenvalues of the $\mathcal{L}^m$'s. Clearly

$$\mu_1^m=0,\ m=0,1,2,... \tag{5.4}$$

Positive μ_k^m's lead to positive eigenvalues of the $\mathcal{L}^m$'s. Note that if $\Gamma < 6$, then

$$\mu_k^m > 0, \text{ for all } k \geq 2 \text{ and } m = 0, 1, 2, ... \tag{5.5}$$

Since Γ is small if q is small by (3.22), (5.5) holds for sufficiently small q.

The remaining eigenvalues μ_0^m, $m \geq 1$, are rather interesting. Since only small modes are considered in this section, i.e. $m \in \{1, 2, ..., M_i\}$, the μ_0^m's are positive, if q is sufficiently small. The situation will be a lot more complicated if m is a medium mode or a large mode. For now in the small mode case, we only need to investigate eigenvalues of $\mathcal{L}^m$ whose leading order approximations are μ_1^m, $m = 0, 1, 2, ..., M_i$.

One remark is in order. There are five degrees of freedom associated with the profile Ω: translations of Ω in x, y, or z directions respectively, and rotations of Ω about x axis or y axis. So $\mathcal{L}$ must have a kernel whose dimension is at least 5. It turns out that $(\mu_1^0 = 0, \varphi_1^0 = \sin\theta)$ is the eigenpair that is related to the translation in z direction; $(\mu_1^1 = 0, \varphi_1^1 = \cos\theta)$ pair is related to the translation in x direction and the translation in y direction; $(\mu_1^1 = 0, \varphi_1^1 = \sin\theta)$ is related to the rotation about the x axis and the rotation about the y axis.

Now we restate Part 1 of Theorem 1.1 and prove it in the succeeding subsections.

Theorem 5.1
1. *There exists $\tilde{q} > 0$ such that when $q \in (0, \tilde{q})$, one of $\mathcal{L}^0$'s eigenvalues is zero with multiplicity one, and all other eigenvalues are positive.*
2. *There exists $\widetilde{q} > 0$ such that when $q \in (0, \widetilde{q})$, one of $\mathcal{L}^1$'s eigenvalues is zero with multiplicity two, and all other eigenvalues are positive.*
3. *For every $M_i > 0$, there exists $\tilde{q}_i > 0$ depending on M_i such that when $q \in (0, \tilde{q}_i)$ and $m \in \{2, 3, ..., M_i\}$, all of $\mathcal{L}^m$'s eigenvalues are positive.*

5.1 $m \geq 1$ and $f_0 = \cos\theta$

The eigenvalue $\mu_1^m = 0$ of $\mathcal{L}_0^m$ has multiplicity 2 which makes perturbation analysis for $\mathcal{L}^m$ more complicated. Fortunately this problem can be circumvented. Since v is even (Ω is symmetric about the xy-plane), $\mathcal{L}^m$ maps even functions to even functions and odd functions to odd functions. Define

$$L^2_{even}(S^1) = \Big\{\varphi \in L^2(S^1):\ \varphi(-\theta) = \varphi(\theta)\Big\} \text{ and } L^2_{odd}(S^1) = \Big\{\varphi \in L^2(S^1):\ \varphi(-\theta) = -\varphi(\theta)\Big\}, \tag{5.6}$$

so that

$$L^2(S^1) = L^2_{even}(S^1) \oplus L^2_{odd}(S^1). \tag{5.7}$$

Then for $m \geq 1$, $\mathcal{L}^m$ (as well as $\mathcal{L}_0^m$, $\mathcal{L}_1^m$, $\mathcal{L}_2^m$, etc) maps $H^2(S^1) \cap L^2_{even}(S^1)$ into $L^2_{even}(S^1)$ and $H^2(S^1) \cap L^2_{odd}(S^1)$ into $L^2_{odd}(S^1)$. When $\mathcal{L}_0^m$ is restricted to each of the two subspaces, 0 becomes a simple eigenvalue.

In this subsection we consider $\mathcal{L}^m$ on the even subspace, $(\mu_1^m = 0, \varphi_1^m = \cos\theta)$, with $m \geq 1$. Denote by (λ, f) the eigen pair of $\mathcal{L}^m$ whose leading order approximation is $(\mu_1^m, \cos\theta)$. Expand f as

$$f = f_0 + f_1 + f_2 + ... \tag{5.8}$$

in terms of q and the corresponding λ as

$$\lambda = \frac{\lambda_0}{q^3} + \frac{\lambda_1}{q^2} + \frac{\lambda_2}{q} + ... \tag{5.9}$$

Furthermore one requires that

$$\|f\|_{L^2} = \|f_0\|_{L^2}. \tag{5.10}$$

which implies that

$$\langle f_0, f_1\rangle = 0. \tag{5.11}$$

From

$$(\mathcal{L}_0^m + \mathcal{L}_1^m + \mathcal{L}_2^m + ...)(f_0 + f_1 + f_2 + ...) = \Big(\frac{\lambda_0}{q^3} + \frac{\lambda_1}{q^2} + \frac{\lambda_2}{q} + ...\Big)(f_0 + f_1 + f_2 + ...), \tag{5.12}$$

the leading order gives

$$\mathcal{L}_0^m f_0 = \frac{\lambda_0}{q^3} f_0 \tag{5.13}$$

whose solution is chosen to be

$$\lambda_0 = 0 \text{ and } f_0 = \cos\theta \tag{5.14}$$

in this subsection.

The second order of (5.12) is

$$\mathcal{L}_0^m f_1 + \mathcal{L}_1^m f_0 = \frac{\lambda_0}{q^3} f_1 + \frac{\lambda_1}{q^2} f_0 \tag{5.15}$$

which by (5.14) is reduced to

$$\mathcal{L}_0^m f_1 + \mathcal{L}_1^m f_0 = \frac{\lambda_1}{q^2} f_0. \tag{5.16}$$

Compute

$$\begin{aligned}\mathcal{L}_1^m f_0 &= \hat{\mathcal{L}}_1^m(\cos\theta) + \frac{\Gamma}{q^3}\check{\mathcal{L}}_1^m(\cos\theta) \\ &= -\frac{\cos 2\theta}{2q^2} + \frac{\Gamma}{q^2}\Big(-\frac{3}{8}\log\frac{8}{q} - \frac{\cos 2\theta}{8}\log\frac{8}{q} + \frac{19}{32} - \frac{d_0^m}{4} + \frac{9\cos 2\theta}{32}\Big)\end{aligned} \tag{5.17}$$

by (4.6) and (4.16). Then

$$\langle \mathcal{L}_1^m f_0, f_0\rangle = 0, \tag{5.18}$$

and taking the inner product of (5.16) with f_0, we deduce

$$\lambda_1 = 0. \tag{5.19}$$

This further reduces (5.16) to

$$\mathcal{L}_0^m f_1 + \mathcal{L}_1^m f_0 = 0. \tag{5.20}$$

Then by (5.11) and (5.17), f_1 must be of the form

$$f_1(\theta) = q(B + D\cos 2\theta). \tag{5.21}$$

Since

$$\mathcal{L}_0^m(B + D\cos 2\theta) = \frac{B}{q^3}\Big[-\frac{1}{2} + \Gamma\Big(\log\frac{8}{q} + d_0^m - \frac{1}{2}\Big)\Big] + \frac{D}{q^3}\Big[\frac{3\cos 2\theta}{2} - \Gamma\frac{\cos 2\theta}{4}\Big], \tag{5.22}$$

(5.17), (5.20) and (5.22) imply

$$B = \frac{-\Gamma(-\frac{3}{8}\log\frac{8}{q} + \frac{19}{32} - \frac{d_0^m}{4})}{-\frac{1}{2} + \Gamma(\log\frac{8}{q} + d_0^m - \frac{1}{2})} \tag{5.23}$$

$$D = \frac{\frac{1}{2} - \Gamma(-\frac{1}{8}\log\frac{8}{q} + \frac{9}{32})}{\frac{3}{2} - \Gamma\frac{1}{4}} \tag{5.24}$$

The third order of (5.12) is

$$\mathcal{L}_0^m f_2 + \mathcal{L}_1^m f_1 + \mathcal{L}_2^m f_0 = \frac{\lambda_0}{q^3} f_2 + \frac{\lambda_1}{q^2} f_1 + \frac{\lambda_2}{q} f_0 \tag{5.25}$$

which is reduced to

$$\mathcal{L}_0^m f_2 + \mathcal{L}_1^m f_1 + \mathcal{L}_2^m f_0 = \frac{\lambda_2}{q} f_0 \tag{5.26}$$

by (5.14) and (5.19). Taking the inner product of (5.26) with f_0 yields

$$\langle \mathcal{L}_1^m f_1, f_0\rangle + \langle \mathcal{L}_2^m f_0, f_0\rangle = \frac{\lambda_2}{q}\|f_0\|_{L^2}^2 \tag{5.27}$$

It remains to calculate

$$\begin{aligned}
\langle \mathcal{L}_1^m f_1, f_0\rangle &= \langle \mathcal{L}_1^m f_0, f_1\rangle \\
&= \frac{2\pi B}{q}\Gamma\Big(-\frac{3}{8}\log\frac{8}{q}+\frac{19}{32}-\frac{d_0^m}{4}\Big)+\frac{\pi D}{q}\Big(-\frac{1}{2}+\Gamma\Big(-\frac{1}{8}\log\frac{8}{q}+\frac{9}{32}\Big)\Big) \\
&= -\Big(\frac{2\pi}{q}\Big)\frac{\big(\Gamma(-\frac{3}{8}\log\frac{8}{q}+\frac{19}{32}-\frac{d_0^m}{4})\big)^2}{-\frac{1}{2}+\Gamma(\log\frac{8}{q}+d_0^m-\frac{1}{2})}-\Big(\frac{\pi}{q}\Big)\frac{\big(-\frac{1}{2}+\Gamma(-\frac{1}{8}\log\frac{8}{q}+\frac{9}{32})\big)^2}{\frac{3}{2}-\Gamma\frac{1}{4}} \qquad (5.28)\\
\langle \mathcal{L}_2^m f_0, f_0\rangle &= \langle \hat{\mathcal{L}}_2^m f_0, f_0\rangle + \frac{\Gamma}{q^3}\langle \check{\mathcal{L}}_2^m f_0, f_0\rangle \\
&= \frac{\pi}{q}\Big(\frac{m^2}{2}-\frac{3A}{2}\Big)+\frac{\pi\Gamma}{q}\Big(\Big(-\frac{m^2}{4}+\frac{19}{32}\Big)\log\frac{8}{q}+\frac{3m^2}{16}+\frac{d_0^m-2d_1^m}{8}-\frac{419}{384}+\frac{A}{4}\Big) \qquad (5.29)
\end{aligned}$$

by (4.14) and (4.17). For $\langle \mathcal{L}_2^m f_0, f_0\rangle$, we have used (4.20) and $\check{\mathcal{L}}_0^m f_0 = 0$. Following (5.28) and (5.29), one derives

$$\begin{aligned}
&\langle \mathcal{L}_1^m f_1, f_0\rangle + \langle \mathcal{L}_2^m f_0, f_0\rangle \\
&= \Big(\frac{\pi}{q}\Big)\frac{\log\frac{8}{q}\Big(24d_0^m-48d_1^m-24m^2+60\Big)-64d_0^m d_1^m-(54+32m^2)d_0^m+12d_1^m+6m^2-143}{\Big(8\log\frac{8}{q}-10\Big)\Big(12\log\frac{8}{q}+16d_0^m-3\Big)} \\
&= \Big(\frac{\pi}{q}\Big)\frac{1}{8\log\frac{8}{q}-10}\Big(-2m^2+2d_0^m-4d_1^m+5-\frac{32(d_0^m+2)^2}{12\log\frac{8}{q}+16d_0^m-3}\Big) \qquad (5.30)
\end{aligned}$$

where we have used (3.22) and (3.29) for Γ and A. One further expands this with respect to $\frac{1}{\log\frac{8}{q}}$ to obtain

$$\begin{aligned}
&\langle \mathcal{L}_1^m f_1, f_0\rangle + \langle \mathcal{L}_2^m f_0, f_0\rangle \\
&= \Big(\frac{1}{2}-\frac{1}{2}\int_0^{\pi/2}\frac{\sin^2 m\tau}{\sin\tau}\,d\tau+\frac{1}{2}\int_0^{\pi/2}\frac{m^2\sin^2\tau-\sin^2 m\tau}{\sin^3\tau}\,d\tau\Big)\frac{\pi}{q\log\frac{8}{q}}+O\Big(\frac{1}{q(\log\frac{8}{q})^2}\Big) \qquad (5.31)
\end{aligned}$$

The first term in the last line is positive when $m \geq 2$ by Part 2 of Lemma 3.5. Hence for each $m \geq 2$

$$\lambda_2 > 0 \tag{5.32}$$

when q is small. Consequently for each $m \geq 2$

$$\lambda > 0 \tag{5.33}$$

if q is small.

So far in this subsection we have shown that for each $m \geq 2$ there exists $\tilde{q}_m$ such that the eigenvalue of $\mathcal{L}^m$, whose leading order eigen pair approximation is $(\mu_1^m, \cos\theta)$, is positive if $q \in (0, \tilde{q}_m)$. Note that $\tilde{q}_m$ depends on m. Unfortunately it is not possible to make $\tilde{q}_m$ independent of m. We can at most assert that there exists $\tilde{q}_i$ such that if $m \in \{2, 3, ..., M_i\}$ and $q \in (0, \tilde{q}_i)$, then the eigenvalue of $\mathcal{L}^m$ associated with $(\mu_1^m, \cos\theta)$ is positive.

When $m = 1$, (5.30) shows that $\lambda_2 = 0$. Actually in this case $\lambda = 0$. This is due to the translation invariance in the x and y directions. One can translate Ω in the x or y direction by a displacement ε and obtain a shifted profile Ω_ε represented by v_ε. Since v_ε is still a profile, $\mathcal{S}(v_\varepsilon) = 0$. Differentiating the equation with respect to ε and setting $\varepsilon = 0$ yield a solution of $\mathcal{L}\phi = 0$. This ϕ has the form $\phi = \varphi(\theta)\cos\sigma$ or $\varphi(\theta)\sin\sigma$. Then $\mathcal{L}^1\varphi = 0$. The leading order approximation of φ is $\cos\theta$ so φ is just f in the case of $m = 1$. The corresponding eigenvalue of $\mathcal{L}^1$ is identically 0, so

$$\lambda = 0 \text{ when } m = 1. \tag{5.34}$$

5.2 $m \geq 1$ and $f_0 = \sin\theta$

In this subsection we consider $\mathcal{L}^m$, $m \geq 1$, from the odd subspace $H^2(S^1) \cap L^2_{odd}(S^1)$ to $L^2_{odd}(S^1)$ and denote by (λ, f) the eigen pair of $\mathcal{L}^m$ whose leading order approximation is $(\mu_1^m, \sin\theta)$. Then in the expansion $f = f_0 + f_1 + f_2 + \ldots$,

$$f_0 = \sin\theta, \tag{5.35}$$

and in the expansion $\lambda = \frac{\lambda_0}{q^3} + \frac{\lambda_1}{q^2} + \frac{\lambda_2}{q} + \ldots$,

$$\lambda_0 = 0. \tag{5.36}$$

Next compute

$$\begin{aligned}
\mathcal{L}_1^m f_0 &= \hat{\mathcal{L}}_1^m(\sin\theta) + \frac{\Gamma}{q^3}\check{\mathcal{L}}_1^m(\sin\theta) \\
&= -\frac{\sin 2\theta}{2q^2} + \frac{\Gamma}{q^2}\Big(-\frac{1}{8}\log\frac{8}{q} + \frac{9}{32}\Big)\sin 2\theta \\
&= \frac{1}{q^2}\Big[-\frac{1}{2} + \Gamma\Big(-\frac{1}{8}\log\frac{8}{q} + \frac{9}{32}\Big)\Big]\sin 2\theta
\end{aligned} \tag{5.37}$$

by (4.6) and (4.16). Hence

$$\langle \mathcal{L}_1^m f_0, f_0\rangle = 0. \tag{5.38}$$

This implies

$$\lambda_1 = 0. \tag{5.39}$$

The next order f_1 of the eigenfunction satisfies

$$\mathcal{L}_0^m f_1 + \mathcal{L}_1^m f_0 = 0, \tag{5.40}$$

so f_1 has the form

$$f_1(\theta) = qE\sin 2\theta. \tag{5.41}$$

Since

$$\mathcal{L}_0^m(E\sin 2\theta) = \frac{E}{q^3}\Big(\frac{3}{2} - \frac{\Gamma}{4}\Big)\sin 2\theta, \tag{5.42}$$

it follows that

$$E = \frac{\frac{1}{2} - \Gamma(-\frac{1}{8}\log\frac{8}{q} + \frac{9}{32})}{\frac{3}{2} - \Gamma\frac{1}{4}}\ . \tag{5.43}$$

It remains to calculate

$$\langle \mathcal{L}_1^m f_1, f_0\rangle + \langle \mathcal{L}_2^m f_0, f_0\rangle \tag{5.44}$$

to find λ_2. Calculations show

$$\begin{aligned}
\langle \mathcal{L}_1^m f_1, f_0\rangle &= \langle \mathcal{L}_1^m f_0, f_1\rangle \\
&= \frac{\pi E}{q}\Big(-\frac{1}{2} + \Gamma\Big(-\frac{1}{8}\log\frac{8}{q} + \frac{9}{32}\Big)\Big) \\
&= \Big(\frac{\pi}{q}\Big)\frac{-\big(-\frac{1}{2} + \Gamma(-\frac{1}{8}\log\frac{8}{q} + \frac{9}{32})\big)^2}{\frac{3}{2} - \Gamma\frac{1}{4}}
\end{aligned} \tag{5.45}$$

$$\begin{aligned}
\langle \mathcal{L}_2^m f_0, f_0\rangle &= \langle \hat{\mathcal{L}}_2^m f_0, f_0\rangle + \frac{\Gamma}{q^3}\langle \check{\mathcal{L}}_2^m f_0, f_0\rangle \\
&= \frac{\pi}{q}\Big(\frac{m^2}{2} + \frac{3A}{2}\Big) + \frac{\pi\Gamma}{q}\Big(\big(-\frac{m^2}{4} + \frac{5}{32}\big)\log\frac{8}{q} + \frac{3m^2}{16} - \frac{d_1^m}{4} - \frac{85}{384} - \frac{A}{4}\Big)
\end{aligned} \tag{5.46}$$

by (4.14) and (4.17), and hence

$$\begin{aligned}\langle \mathcal{L}_1^m f_1, f_0\rangle + \langle \mathcal{L}_2^m f_0, f_0\rangle &= \Big(\frac{\pi}{q}\Big)\frac{-4d_1^m - 2m^2 + 1}{8\log\frac{8}{q} - 10} \\ &= \Big(\frac{1}{2}\int_0^{\pi/2} \frac{m^2\sin^2\tau - \sin^2 m\tau}{\sin^3\tau}\, d\tau\Big)\frac{\pi}{q\log\frac{8}{q}} + O\Big(\frac{1}{q(\log\frac{8}{q})^2}\Big) \qquad (5.47)\end{aligned}$$

by (3.22) and (3.29). The first term in the last line is positive when $m \geq 2$ by Part 1 of Lemma 3.5. Hence for each $m \geq 2$,

$$\lambda_2 > 0 \qquad (5.48)$$

if q is small, and consequently for each $m \geq 2$,

$$\lambda > 0 \qquad (5.49)$$

if q is sufficiently small.

Because of the rotation invariance of Ω about x and y axes,

$$\lambda = 0 \text{ when } m = 1. \qquad (5.50)$$

5.3 $m = 0$ and $f_0 = \cos\theta$

As in the $m \geq 1$ case, $\mathcal{L}^0$ maps even functions to even functions and odd functions to odd functions. In this subsection we consider $\mathcal{L}^0$ on even functions and let (λ, f) be the eigen pair of $\mathcal{L}^0$ whose leading order approximation is $(\mu_1^0, \cos\theta)$. Expand f to

$$f = f_0 + f_1 + f_2 + ... \qquad (5.51)$$

with

$$f_0 = \cos\theta, \qquad (5.52)$$

and expand λ to

$$\lambda = \frac{\lambda_0}{q^3} + \frac{\lambda_1}{q^2} + \frac{\lambda_2}{q} + ... \qquad (5.53)$$

where

$$\lambda_0 = 0. \qquad (5.54)$$

The calculations in this case remain largely similar to the ones in the $m \geq 1$ and $f_0 = \cos\theta$ case, provided one lets

$$m = 0, \;\; d_0^0 = 0, \;\; d_1^0 = \frac{1}{4}\;. \qquad (5.55)$$

Compute

$$\mathcal{L}_1^0 f_0 = -\frac{\cos 2\theta}{2q^2} + \frac{\Gamma}{q^2}\Big(-\frac{\cos 2\theta}{8}\log\frac{8}{q} + \frac{9\cos 2\theta}{32}\Big). \qquad (5.56)$$

Note that $\mathcal{L}_1^0 f_0$ in (5.56) differs from (5.17). Here the expression does not have constant terms because the average of $\mathcal{L}_1^0 f_0$ must vanish. Since $\langle \mathcal{L}_1^0 f_0, f_0\rangle = 0$,

$$\lambda_1 = 0. \qquad (5.57)$$

One then finds

$$f_1 = qD\cos 2\theta \qquad (5.58)$$

where

$$D = \frac{\frac{1}{2} - \Gamma(-\frac{1}{8}\log\frac{8}{q} + \frac{9}{32})}{\frac{3}{2} - \Gamma\frac{1}{4}}. \qquad (5.59)$$

It follows that

$$\begin{aligned}\langle \mathcal{L}_1^0 f_1, f_0\rangle &= \langle \mathcal{L}_1^0 f_0, f_1\rangle \\ &= \Big(\frac{\pi}{q}\Big)\frac{-\big(-\frac{1}{2}+\Gamma(-\frac{1}{8}\log\frac{8}{q}+\frac{9}{32})\big)^2}{\frac{3}{2}-\Gamma\frac{1}{4}} \qquad (5.60)\\ \langle \mathcal{L}_2^0 f_0, f_0\rangle &= \langle \hat{\mathcal{L}}_2^0 f_0, f_0\rangle + \frac{\Gamma}{q^3}\langle \check{\mathcal{L}}_2^m f_0, f_0\rangle \\ &= \frac{\pi}{q}\Big(-\frac{3A}{2}\Big)+\frac{\pi\Gamma}{q}\Big(\frac{19}{32}\log\frac{8}{q}-\frac{443}{384}+\frac{A}{4}\Big). \qquad (5.61)\end{aligned}$$

Then

$$\begin{aligned}\langle \mathcal{L}_1^0 f_1, f_0\rangle + \langle \mathcal{L}_2^0 f_0, f_0\rangle &= \Big(\frac{\pi}{q}\Big)\frac{12\log\frac{8}{q}-27}{16\log\frac{8}{q}-20} \\ &= \Big(\frac{\pi}{q}\Big)\Big(\frac{3}{4}-\frac{3}{4}\frac{1}{\log\frac{8}{q}}\Big)+O\Big(\frac{1}{q(\log\frac{8}{q})^2}\Big). \qquad (5.62)\end{aligned}$$

The first term on the left side is positive if q is small. Hence

$$\lambda_2 > 0 \qquad (5.63)$$

if q is small, and consequently

$$\lambda > 0 \qquad (5.64)$$

if q is small. Note that this λ_2 is positive in the leading $\frac{1}{q}$ order while the earlier positive λ_2's are positive in the $\frac{1}{q\log\frac{8}{q}}$ order.

5.4 $m = 0$ and $f_0 = \sin\theta$

Finally consider the eigenfunction $f = f_0 + f_1 + ...$ of $\mathcal{L}^0$ with

$$f_0 = \sin\theta. \qquad (5.65)$$

The calculations are identical to the $m \geq 1$ and $f_0 = \sin\theta$ case, and one finds

$$\lambda_1 = 0, \qquad (5.66)$$

and

$$f_1 = qE\sin 2\theta \qquad (5.67)$$

with

$$E = \frac{\frac{1}{2}-\Gamma(-\frac{1}{8}\log\frac{8}{q}+\frac{9}{32})}{\frac{3}{2}-\Gamma\frac{1}{4}} \qquad (5.68)$$

as before. Lastly

$$\begin{aligned}\langle \mathcal{L}_1^0 f_1, f_0\rangle + \langle \mathcal{L}_2^0 f_0, f_0\rangle &= \Big(\frac{\pi}{q}\Big)\frac{-\big(-\frac{1}{2}+\Gamma(-\frac{1}{8}\log\frac{8}{q}+\frac{9}{32})\big)^2}{\frac{3}{2}-\Gamma\frac{1}{4}}+\frac{\pi}{q}\Big(\frac{3A}{2}\Big)+\frac{\pi\Gamma}{q}\Big(\frac{5}{32}\log\frac{8}{q}-\frac{109}{384}-\frac{A}{4}\Big) \\ &= 0 \qquad (5.69)\end{aligned}$$

where (5.69) is obtained after one inserts (3.22) for Γ and (3.29) for A. At this point we have deduced that

$$\lambda_0 = \lambda_1 = \lambda_2 = 0. \qquad (5.70)$$

If one exploits the translation invariance in the z direction, then

$$\lambda = 0. \qquad (5.71)$$

In summary, Part 1 of Theorem 5.1 follows from (5.64) and (5.71); Part 2 follows from (5.34) and (5.50); Part 3 follows from (5.33) and (5.49).

6 Large modes

Below is a restatement of Part 2 of Theorem 1.1 and the rest of this section is devoted to its proof.

Theorem 6.1 *There exist $M_{ii} > 0$ and $\tilde{q}_{ii} > 0$ such that when $q \in (0, \tilde{q}_{ii})$ and $m \geq \frac{M_{ii}}{q}$, $\mathcal{L}^m$ is positive definite.*

Proof. Since

$$\Big| \int_0^{\pi/2} \frac{\cos 2m\tau \, d\tau}{\sqrt{\beta + \sin^2 \tau}} \Big| \leq \int_0^{\pi/2} \frac{d\tau}{\sqrt{\beta + (2\tau/\pi)^2}} = \frac{\pi}{2} \log \frac{1 + \sqrt{1+\beta}}{\sqrt{\beta}} \ ,$$

one finds a simple upper bound for G^m of (3.4), uniform with respect to m,

$$|G^m(r, z, \rho, \zeta)| \leq \frac{1}{4}\sqrt{\frac{\rho}{r}} \log \frac{1 + \sqrt{1+\beta}}{\sqrt{\beta}}, \text{ where } \beta = \frac{(r-\rho)^2 + (z-\zeta)^2}{4r\rho}. \tag{6.1}$$

Regarding the operator $\hat{\mathcal{L}}^m$, note that, since u does not depend on m, among the quadratic terms (2.50)-(2.55) only (2.53) depends on m. This dependence comes simply through the multiple m^2. Subsequent findings of $\hat{\mathcal{L}}^m_0$, $\hat{\mathcal{L}}^m_1$, and $\hat{\mathcal{L}}^m_2$ show that

$$\begin{aligned} \langle \hat{\mathcal{L}}^m \varphi, \varphi \rangle \ &\geq \ \Big(\frac{1 - Cq}{2q^3}\Big) \int_0^{2\pi} \varphi_\theta^2 \, d\theta - \Big(\frac{1 + Cq^2}{2q^3}\Big) \int_0^{2\pi} \varphi^2 \, d\theta + \Big(\frac{m^2 - Cq}{2q}\Big) \int_0^{2\pi} \varphi^2 \, d\theta \\ &\geq \ -\frac{1}{q^3} \int_0^{2\pi} \varphi^2 \, d\theta + \Big(\frac{m^2 - Cq}{2q}\Big) \int_0^{2\pi} \varphi^2 \, d\theta. \end{aligned} \tag{6.2}$$

Here the constant C is independent of m and independent of small q.

For the operator $\check{\mathcal{L}}^m$, denote the two parts by $\mathcal{E}$ and $\mathcal{F}$ respectively:

$$\check{\mathcal{L}}^m \varphi \ = \ \mathcal{E}\varphi + \mathcal{F}\varphi \tag{6.3}$$

$$\mathcal{E}\varphi \ = \ \int_0^{2\pi} \frac{G^m(1 + u(\theta)e^{\mathrm{i}\theta}, 1 + u(\omega)e^{\mathrm{i}\omega})\varphi(\omega)}{1 + u(\omega)\cos\omega} \, d\omega \tag{6.4}$$

$$\mathcal{F}\varphi \ = \ \frac{\varphi(\theta)}{u + u^2\cos\theta} \int_{\Omega'_v} \nabla G^0(1 + ue^{\mathrm{i}\theta}, \rho, \zeta) \cdot e^{i\theta} \, d\rho d\zeta \ . \tag{6.5}$$

The multiplication operator $\mathcal{F}$ does not depend on m and the findings of $\check{\mathcal{L}}^m_0$, $\check{\mathcal{L}}^m_1$, and $\check{\mathcal{L}}^m_2$ show that

$$\|\mathcal{F}\varphi\| \leq \Big(\frac{1 + Cq}{2}\Big)\|\varphi\|_{L^2} \tag{6.6}$$

for all m when q is small. The integral operator $\mathcal{E}$ depends on m in a more subtle way and we have to treat it differently. The estimate (6.1) implies

$$\begin{aligned} |G^m(1 + u(\theta)e^{\mathrm{i}\theta}, 1 + u(\omega)e^{\mathrm{i}\omega})| \ &\leq \ C + C\Big| \log |u(\theta)e^{\mathrm{i}\theta} - u(\omega)e^{\mathrm{i}\omega}| \ \Big| \\ &\leq \ C + C\log\frac{8}{q} + C\Big| \log\Big|\frac{u(\theta)}{q}e^{\mathrm{i}\theta} - \frac{u(\omega)}{q}e^{\mathrm{i}\omega}\Big| \ \Big|. \end{aligned} \tag{6.7}$$

For the last term in (6.7),

$$\Big\| \log\Big|\frac{u(\theta)}{q}e^{\mathrm{i}\theta} - \frac{u(\cdot)}{q}e^{\mathrm{i}(\cdot)}\Big| \Big\|_{L^1} \leq C, \ \forall \theta \in S^1.$$

Then the norm of the integral operator on $L^2(S^1)$ with $\Big| \log\Big|\frac{u(\theta)}{q}e^{\mathrm{i}\theta} - \frac{u(\omega)}{q}e^{\mathrm{i}\omega}\Big| \ \Big|$ as the kernel is bounded by C, independent of q. This leads to a bound on $\mathcal{E}$ of the form

$$\|\mathcal{E}\varphi\|_{L^2} \leq C\Big(\log\frac{8}{q}\Big)\|\varphi\|_{L^2} \tag{6.8}$$

uniformly with respect to m. By (6.6) and (6.8),

$$\|\check{\mathcal{L}}^m \varphi\|_{L^2} \leq C\Big(\log\frac{8}{q}\Big)\|\varphi\|_{L^2}. \tag{6.9}$$

Therefore by (3.22), (6.2), and (6.9) there exists $\tilde{q}_{ii} > 0$ such that for $q \in (0, \tilde{q}_{ii})$

$$\langle \mathcal{L}^m \varphi, \varphi\rangle \geq \frac{m^2}{2q}\int_0^{2\pi} \varphi^2\, d\theta - \frac{C}{q^3}\int_0^{2\pi}\varphi^2\, d\theta \tag{6.10}$$

holds for all m. This estimate is useful if m is large compared to $1/q$. Let

$$M_{ii} > 0,\ \ m \geq \frac{M_{ii}}{q},\ \text{ and } q \in (0, \tilde{q}_{ii}). \tag{6.11}$$

Then the operator $\mathcal{L}^m$ is positive definite since

$$\langle \mathcal{L}^m \varphi, \varphi\rangle \geq \frac{M_{ii}^2 - 2C}{2q^3}\|\varphi\|_{L^2}^2 \geq \frac{M_{ii}^2}{4q^3}\|\varphi\|_{L^2}^2 \tag{6.12}$$

if M_{ii} is sufficiently large. □

7 Medium modes

Consider $\mathcal{L}_0^m$ given in (5.1) with $m \geq 1$. Note that with $\Gamma = \frac{8}{4\log\frac{8}{q} - 5}$ given by (3.22),

$$\mathcal{L}_0^m(1) = -\frac{1}{2q^3} + \frac{\Gamma}{q^3}\Big(\log\frac{8}{q} + d_0^m - \frac{1}{2}\Big) = \frac{1}{q^3}\Big(\frac{12\log\frac{8}{q} + 16 d_0^m - 3}{8\log\frac{8}{q} - 10}\Big).$$

Hence $\varphi_0^m = 1$ is an eigenfunction of $\mathcal{L}_0^m$ and the corresponding eigenvalue is

$$\mu_0^m = \frac{1}{q^3}\Big(\frac{12\log\frac{8}{q} + 16 d_0^m - 3}{8\log\frac{8}{q} - 10}\Big). \tag{7.1}$$

Although $\mathcal{L}_0^m$ is an approximation of $\mathcal{L}^m$ only when m is fixed and q is small, we still consider $\mathcal{L}_0^m$ in the scenario that m varies with q. Since

$$d_0^m = -\log m + O(1) \ \text{ for large } m \tag{7.2}$$

by Part 1 of Lemma 3.3. We can make μ_0^m negative if we choose m sufficiently large. The borderline between negative μ_0^m and positive μ_0^m occurs at

$$12\log\frac{8}{q} + 16 d_0^m - 3 = 0, \tag{7.3}$$

which implies, for small q,

$$m \approx \Big(\frac{8}{q}\Big)^{3/4}. \tag{7.4}$$

This order puts m in the medium range, and $\mathcal{L}_0^m$ cannot be used to approximate $\mathcal{L}^m$ under (7.4). Nevertheless there is indeed a negative eigenvalue for $\mathcal{L}^m$ when m is in the range suggested by (7.4). This is the content of Part 3 of Theorem 1.1 which is restated below as a reminder.

Theorem 7.1 *Let M_i and M_{ii} be positive numbers. There exists $\tilde{q}_{i,ii} > 0$ depending on M_i and M_{ii} such that for every $q \in (0, \tilde{q}_{i,ii})$, there is $m \in (M_i, \frac{M_{ii}}{q})$ for which $\mathcal{L}^m$ has a negative eigenvalue. Moreover, as q tends to 0, m grows to infinity like $(\frac{8}{q})^{3/4}$.*

Proof. We proceed to show that at 1, the quadratic form

$$\langle \mathcal{L}^m(1), 1\rangle = \langle \hat{\mathcal{L}}^m(1), 1\rangle + \gamma\langle \check{\mathcal{L}}^m(1), 1\rangle \tag{7.5}$$

is negative for a properly chosen m, from which Theorem 7.1 follows.

For $\langle \hat{\mathcal{L}}^m(1), 1\rangle$, $m \geq 2$, note that only the term (2.53) among (2.50) to (2.55) depends on m. Let

$$\langle \hat{\mathcal{L}}^m(1), 1\rangle = m^2 Q_1(q) + Q_2(q) \tag{7.6}$$

where

$$\begin{aligned}
Q_1(q) &= \int_0^{2\pi} \frac{1}{2(u_\theta^2+u^2)^{1/2}(1+u\cos\theta)^3}\,d\theta && (7.7)\\
&= \frac{\pi}{q}(1+O(q)) && (7.8)\\
Q_2(q) &= \int_0^{2\pi} \Big[\frac{(1+u\cos\theta)u_\theta}{(u_\theta^2+u^2)^{1/2}}\Big(\frac{-(1+2u\cos\theta)}{2(u+u^2\cos\theta)^3}\Big)_\theta - \frac{(u+2u^2\cos\theta+u_\theta^2\cos\theta)(1+2u\cos\theta)}{2(u_\theta^2+u^2)^{1/2}(u+u^2\cos\theta)^3}\\
&\quad + \frac{(1+u\cos\theta)u^2}{2(u_\theta^2+u^2)^{3/2}}\Big(\frac{1}{u+u^2\cos\theta}\Big)_\theta^2 + \frac{-uu_\theta+u_\theta^3\cos\theta}{(u_\theta^2+u^2)^{3/2}(u+u^2\cos\theta)}\Big(\frac{1}{u+u^2\cos\theta}\Big)_\theta\\
&\quad + \frac{(2u^3\cos\theta+u_\theta^2+3uu_\theta^2\cos\theta)}{2(u_\theta^2+u^2)^{3/2}(u+u^2\cos\theta)^2}\Big]\,d\theta && (7.9)\\
&= -\frac{\pi}{q^3}(1+O(q^2)). && (7.10)
\end{aligned}$$

These calculations make use of the quadratic forms derived in Section 4. Neither Q_1 nor Q_2 depends on m.

To study $\langle \check{\mathcal{L}}^m(1), 1\rangle$, reorganize G^m into

$$G^m(r,z,\rho,\zeta) = F_1(r,z,\rho,\zeta) + m^2F_2(r,z,\rho,\zeta) + d_0^m F_3(r,z,\rho,\zeta) + d_1^m F_4(r,z,\rho,\zeta) + \tilde{F}^m(r,z,\rho,\zeta) \tag{7.11}$$

accodring to its depencece on m shown in Lemma 3.4, where

$$\begin{aligned}
F_1(r,z,\rho,\zeta) &= \frac{1}{2\pi}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} - \frac{r-\rho}{4\pi\rho}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|}\\
&\quad + \frac{5(r-\rho)^2-(z-\zeta)^2}{32\pi\rho^2}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} && (7.12)\\
F_2(r,z,\rho,\zeta) &= \frac{(r-\rho)^2+(z-\zeta)^2}{8\pi\rho^2}\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} && (7.13)\\
F_3(r,z,\rho,\zeta) &= \frac{1}{2\pi}\sqrt{\frac{\rho}{r}} && (7.14)\\
F_4(r,z,\rho,\zeta) &= \Big(\frac{1}{2\pi}\Big)\frac{(r-\rho)^2+(z-\zeta)^2}{4\rho^2} && (7.15)\\
\tilde{F}^m(r,z,\rho,\zeta) &= O(m^2q^3\log\frac{8}{q}) + O((m^2\log m)q^3) + O\big(m^{2+2\alpha}q^{2+2\alpha}\big). && (7.16)
\end{aligned}$$

Note that F_1 through F_4 do not depend on m. $\tilde{F}^m$ depends on m but is small, and one lets

$$\alpha \in (0, 1/2) \tag{7.17}$$

to further simplify it to

$$\tilde{F}^m(r,z,\rho,\zeta) = O\big(m^{2+2\alpha}q^{2+2\alpha}\big). \tag{7.18}$$

Then write

$$\langle \check{\mathcal{L}}^m(1), 1\rangle = m^2Q_3(q) + d_0^mQ_4(q) + d_1^mQ_5(Q) + Q_6(q) + \tilde{Q}(q,m) \tag{7.19}$$

where

$$Q_3(q) = \int_0^{2\pi}\int_0^{2\pi} \frac{F_2(1+u(\theta)e^{i\theta}, 1+u(\omega)e^{i\omega})}{1+u(\omega)\cos\omega}\, d\omega d\theta = \pi q^2(1+O(q)) \tag{7.20}$$

$$Q_4(q) = \int_0^{2\pi}\int_0^{2\pi} \frac{F_3(1+u(\theta)e^{i\theta}, 1+u(\omega)e^{i\omega})}{1+u(\omega)\cos\omega}\, d\omega d\theta = 2\pi(1+O(q^2)) \tag{7.21}$$

$$Q_5(q) = \int_0^{2\pi}\int_0^{2\pi} \frac{F_4(1+u(\theta)e^{i\theta}, 1+u(\omega)e^{i\omega})}{1+u(\omega)\cos\omega}\, d\omega d\theta = \pi q^2(1+O(q)) \tag{7.22}$$

$$\begin{aligned} Q_6(q) &= \int_0^{2\pi}\int_0^{2\pi} \frac{F_1(1+u(\theta)e^{i\theta}, 1+u(\omega)e^{i\omega})}{1+u(\omega)\cos\omega}\, d\omega d\theta \\ &\quad + \int_0^{2\pi} \frac{1}{u+u^2\cos\theta} \int_{\Omega'_v} \nabla G^0(1+ue^{i\theta}, \rho, \zeta)\cdot e^{i\theta}\, d\rho d\zeta = \pi\Big(2\log\frac{8}{q}-1\Big)(1+O(q)) \end{aligned} \tag{7.23}$$

$$\tilde{Q}^m(q) = \int_0^{2\pi}\int_0^{2\pi} \frac{\tilde{F}^m(1+u(\theta)e^{i\theta}, 1+u(\omega)e^{i\omega})}{1+u(\omega)\cos\omega}\, d\omega d\theta = O\big(m^{2+2\alpha} q^{2+2\alpha}\big). \tag{7.24}$$

Again note that Q_3 through Q_6 are independent of m and $\tilde{Q}^m$ is small.

Now take m to be of the order $(\frac{8}{q})^{3/4}$; namely, consider

$$m \in \Big(\frac{1}{2}\big(\frac{8}{q}\big)^{3/4}, 2\big(\frac{8}{q}\big)^{3/4}\Big) \tag{7.25}$$

for small q. One deduces that

$$m^2 Q_1(q) = O(q^{-2.5}) \tag{7.26}$$

$$Q_2(q) = -\frac{\pi}{q^3}(1+O(q^2)) \tag{7.27}$$

$$\gamma m^2 Q_3(q) = O\Big(\frac{1}{q^{2.5}\log\frac{8}{q}}\Big) \tag{7.28}$$

$$\gamma d_0^m Q_4(q) = \frac{2\pi d_0^m \Gamma}{q^3}(1+O(q)) \tag{7.29}$$

$$\gamma d_1^m Q_5(q) = O\Big(q^{-2.5}\Big) \tag{7.30}$$

$$\gamma Q_6(q) = \frac{\Gamma}{q^3}\Big(2\pi\log\frac{8}{q} - \pi\Big)(1+O(q)) \tag{7.31}$$

$$\gamma \tilde{Q}^m(q) = O\Big(\frac{1}{q^{3-\frac{1+\alpha}{2}}\log\frac{8}{q}}\Big). \tag{7.32}$$

Here to reach (7.30) we have used the estimate $d_1^m = O(m^2\log m)$ from Part 2 of Lemma 3.3. Hence

$$\begin{aligned} \langle \mathcal{L}^m(1), 1\rangle &= \frac{\pi}{q^3}\Big(-1 + 2d_0^m\Gamma + \big(2\log\frac{8}{q}-1\big)\Gamma + o(1)\Big) \\ &= \frac{\pi}{q^3}\Big(\frac{12\log\frac{8}{q} + 16 d_0^m - 3}{4\log\frac{8}{q} - 5} + o(1)\Big) \end{aligned} \tag{7.33}$$

Let m be of the order $(\frac{8}{q})^{3/4}$ and above the threshold (7.3). Then (7.33) is negative. $\square$

Appendix

A. Proof of Lemma 3.2

Let

$$\int_0^{\pi/2} \frac{\cos 2m\tau\, d\tau}{\sqrt{\beta+\sin^2\tau}} = \int_0^{\pi/2} \frac{1-2m^2\sin^2\tau}{\sqrt{\beta+\sin^2\tau}}\, d\tau + 2\int_0^{\pi/2} \frac{m^2\sin^2\tau - \sin^2 m\tau}{\sqrt{\beta+\sin^2\tau}}\, d\tau. \tag{A.1}$$

First we estimate

$$\begin{aligned}\int_0^{\pi/2} \frac{1-2m^2\sin^2\tau}{\sqrt{\beta+\sin^2\tau}}\, d\tau &= \int_0^{\pi/2} \frac{1-2m^2\cos^2\tau}{\sqrt{\beta+\cos^2\tau}}\, d\tau \\ &= \int_0^{\pi/2} \frac{1-2m^2+2m^2\sin^2\tau}{\sqrt{\beta+1-\sin^2\tau}}\, d\tau \\ &= \frac{1}{\sqrt{1+\beta}}\Big(\big(1-2m^2+\frac{2m^2}{k^2}\big)K(k) - \frac{2m^2}{k^2}E(k)\Big) \end{aligned}\tag{A.2}$$

where

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\tau}}\, d\tau = \log\frac{4}{k'} + \frac{1}{4}\big(\log\frac{4}{k'} - 1\big)k'^2 + O\big(k'^4\log\frac{1}{k'}\big) \tag{A.3}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\tau}\, d\tau = 1 + \frac{1}{2}\big(\log\frac{4}{k'} - \frac{1}{2}\big)k'^2 + O\big(k'^4\log\frac{1}{k'}\big) \tag{A.4}$$

$$k^2 = \frac{1}{1+\beta},\quad k'^2 = \frac{\beta}{1+\beta}\ . \tag{A.5}$$

Here $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and the second kind respectively, whose asymptotic expansions (A.3) and (A.4) can be found in [4, 8.113 and 8.114]. Then (A.2) becomes

$$\begin{aligned}&\int_0^{\pi/2} \frac{1-2m^2\sin^2\tau}{\sqrt{\beta+\sin^2\tau}}\, d\tau \\ &= \Big(1-\frac{\beta}{2}+O(\beta^2)\Big)\Big[\Big(1-2m^2+\frac{2m^2}{k^2}\Big)\Big(\log\frac{4}{k'}+\frac{1}{4}\big(\log\frac{4}{k'}-1\big)k'^2\Big) \\ &\quad -\frac{2m^2}{k^2}\Big(1+\frac{1}{2}\big(\log\frac{4}{k'}-\frac{1}{2}\big)k'^2\Big) + O\big((m^2+1)k'^4\log\frac{1}{k'}\big)\Big] \\ &= \Big(\frac{1}{2}+\big(\frac{m^2}{2}-\frac{1}{8}\big)\beta\Big)\log\frac{16}{\beta} - 2m^2 + \Big(\frac{1}{4}-\frac{m^2}{2}\Big)\beta + O\big((m^2+1)\beta^2\log\frac{1}{\beta}\big). \end{aligned}\tag{A.6}$$

Next we estimate

$$\int_0^{\pi/2} \frac{m^2\sin^2\tau-\sin^2 m\tau}{\sqrt{\beta+\sin^2\tau}}\, d\tau.$$

Obviously this term vanishes if $m=0$ or $m=1$, and Part 1 of the lemma follows from (A.6). When $m\geq 2$, this integral depends on m in a more subtle way. Let $\alpha\in(0,1)$. Expand $\frac{1}{\sqrt{\beta+\sin^2\tau}}$ with respect to $\beta\in[0,1]$, using a two term Taylor's expansion with the remainder controlled by a Holder semi-norm

$$\Big|\frac{1}{\sqrt{\beta+\sin^2\tau}} - \frac{1}{\sin\tau} + \frac{\beta}{2\sin^3\tau}\Big| \leq \Big[\Big| -\frac{1}{2(\cdot+\sin^2\tau)^{3/2}}\Big|\Big]_\alpha \frac{\beta^{1+\alpha}}{1+\alpha} \tag{A.7}$$

where the semi-norm is

$$\Big[\Big| -\frac{1}{2(\cdot+\sin^2\tau)^{3/2}}\Big|\Big]_\alpha = \sup_{0\leq b_1<b_2\leq 1} \frac{\Big|\frac{1}{2(b_2+\sin^2\tau)^{3/2}} - \frac{1}{2(b_1+\sin^2\tau)^{3/2}}\Big|}{|b_2-b_1|^\alpha}. \tag{A.8}$$

Then we estimate this semi-norm. If $b_2 - b_1 \le \sin^2\tau$, one uses the mean value theorem to deduce

$$\begin{aligned}
\frac{\left|\frac{1}{2(b_2+\sin^2\tau)^{3/2}} - \frac{1}{2(b_1+\sin^2\tau)^{3/2}}\right|}{|b_2-b_1|^\alpha} &= |b_2-b_1|^{1-\alpha}\Big|\frac{3}{4}(\eta b_1 + (1-\eta)b_2 + \sin^2\tau)^{-5/2}\Big|, \quad \text{where } \eta\in(0,1),\\
&\le \sin^{2-2\alpha}\tau\Big|\frac{3}{4}(\sin^2\tau)^{-5/2}\Big|\\
&= \frac{3}{4\sin^{3+2\alpha}\tau}\ ;
\end{aligned}$$

if $b_2 - b_1 > \sin^2\tau$, one has

$$\begin{aligned}
\frac{\left|\frac{1}{2(b_2+\sin^2\tau)^{3/2}} - \frac{1}{2(b_1+\sin^2\tau)^{3/2}}\right|}{|b_2-b_1|^\alpha} &\le |b_2-b_1|^{-\alpha}\Big|\frac{1}{2(b_1+\sin^2\tau)^{3/2}}\Big|\\
&\le \sin^{-2\alpha}\tau\Big|\frac{1}{2(\sin^2\tau)^{3/2}}\Big|\\
&= \frac{1}{2\sin^{3+2\alpha}\tau}\ .
\end{aligned}$$

Hence one finds a bound on the Holder semi-norm

$$\Big[\Big| - \frac{1}{2(\cdot+\sin^2\tau)^{3/2}}\Big|\Big]_\alpha \le \frac{3}{4\sin^{3+2\alpha}\tau} \tag{A.9}$$

and it turns (A.7) to

$$\Big|\frac{1}{\sqrt{\beta+\sin^2\tau}} - \frac{1}{\sin\tau} + \frac{\beta}{2\sin^3\tau}\Big| \le \frac{3\beta^{1+\alpha}}{4(1+\alpha)\sin^{3+2\alpha}\tau}\ . \tag{A.10}$$

It follows that

$$\begin{aligned}
&\Big|\int_0^{\pi/2}\frac{m^2\sin^2\tau - \sin^2 m\tau}{\sqrt{\beta+\sin^2\tau}}\,d\tau - \int_0^{\pi/2}\frac{m^2\sin^2\tau-\sin^2 m\tau}{\sin\tau}\,d\tau + \beta\int_0^{\pi/2}\frac{m^2\sin^2\tau-\sin^2 m\tau}{2\sin^3\tau}\,d\tau\Big|\\
&\qquad\le \frac{3\beta^{1+\alpha}}{4(1+\alpha)}\int_0^{\pi/2}\frac{|m^2\sin^2\tau-\sin^2 m\tau|}{\sin^{3+2\alpha}\tau}\,d\tau.
\end{aligned} \tag{A.11}$$

To estimate (A.11), write

$$\int_0^{\pi/2}\frac{|m^2\sin^2\tau-\sin^2 m\tau|}{\sin^{3+2\alpha}\tau}\,d\tau = \int_0^{\pi/2}\frac{|m^2\sin^2(\frac{t}{m})-\sin^2 t|}{m\sin^{3+2\alpha}(\frac{t}{m})}\,dt + \int_{\pi/2}^{m\pi/2}\frac{|m^2\sin^2(\frac{t}{m})-\sin^2 t|}{m\sin^{3+2\alpha}(\frac{t}{m})}\,dt. \tag{A.12}$$

For the first term on the right side of (A.12), use $|m^2\sin^2(\frac{t}{m}) - \sin^2 t| \le Ct^4$ where $C>0$ is independent of m to derive

$$\int_0^{\pi/2}\frac{|m^2\sin^2(\frac{t}{m})-\sin^2 t|}{m\sin^{3+2\alpha}(\frac{t}{m})}\,dt \le C\int_0^{\pi/2}\frac{t^4}{m(\frac{2t}{\pi m})^{3+2\alpha}}\,dt \le Cm^{2+2\alpha}. \tag{A.13}$$

Note that the last integral is convergent since $\alpha<1$. For the second term on the right side of (A.12)

$$\int_{\pi/2}^{m\pi/2}\frac{|m^2\sin^2(\frac{t}{m})-\sin^2 t|}{m\sin^{3+2\alpha}(\frac{t}{m})}\,dt \le \int_{\pi/2}^{m\pi/2}\frac{m^2(\frac{t}{m})^2+t^2}{m(\frac{2t}{\pi m})^{3+2\alpha}}\,dt \le \int_{\pi/2}^{\infty}\frac{2t^2}{m^{-2-2\alpha}(2t)^{3+2\alpha}}\,dt \le Cm^{2+2\alpha} \tag{A.14}$$

This time the last integral is convergent since $\alpha>0$. Therefore (A.11) becomes

$$\int_0^{\pi/2}\frac{m^2\sin^2\tau-\sin^2 m\tau}{\sqrt{\beta+\sin^2\tau}}\,d\tau = m^2 - \int_0^{\pi/2}\frac{\sin^2 m\tau}{\sin\tau}\,d\tau - \frac{\beta}{2}\int_0^{\pi/2}\frac{m^2\sin^2\tau-\sin^2 m\tau}{\sin^3\tau}\,d\tau + O(m^{2+2\alpha}\beta^{1+\alpha}). \tag{A.15}$$

Part 2 of the lemma follows from (A.1), (A.6) and (A.15).

B. Proof of Lemma 3.3

Let $\tau = \frac{t}{m}$ so that

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin^2 m\tau}{\sin\tau}\, d\tau &= \frac{1}{m}\int_0^{m\pi/2} \frac{\sin^2 t}{\sin(\frac{t}{m})}\, dt \\ &= \frac{1}{m}\int_0^{\pi/2} \frac{\sin^2 t}{\sin\frac{t}{m}}\, dt + \frac{1}{m}\int_{\pi/2}^{m\pi/2} \frac{\sin^2 t}{\sin(\frac{t}{m})}\, dt. \end{aligned} \tag{B.1}$$

Since $\sin x = x + O(x^3)$ when $x \in (0, \pi/2)$,

$$\frac{\sin^2 t}{\sin(\frac{t}{m})} = \frac{\sin^2 t}{\frac{t}{m}}\Big[1 + O\Big((\frac{t}{m})^2\Big)\Big] = \frac{\sin^2 t}{\frac{t}{m}} + \big(\sin^2 t\big)\ O\Big(\frac{t}{m}\Big),$$

which shows that the first term in the last line of (B.1) is

$$\frac{1}{m}\int_0^{\pi/2} \frac{\sin^2 t}{\sin\frac{t}{m}}\, dt = \int_0^{\pi/2} \frac{\sin^2 t}{t}\, dt + O\Big(\frac{1}{m^2}\Big) = O(1). \tag{B.2}$$

For the second term in the last line of (B.1),

$$\begin{aligned}\frac{1}{m}\int_{\pi/2}^{m\pi/2} \frac{\sin^2 t}{\sin(\frac{t}{m})}\, dt &= \frac{1}{m}\int_{\pi/2}^{m\pi/2} \frac{\sin^2 t}{\frac{t}{m}}\Big[1 + O\Big((\frac{t}{m})^2\Big)\Big]\, dt \\ &= \frac{1}{m}\int_{\pi/2}^{m\pi/2} \frac{\sin^2 t}{\frac{t}{m}}\, dt + \Big(\frac{1}{m}\int_{\pi/2}^{m\pi/2} \frac{t}{m}\, dt\Big)\, O(1) \\ &= \int_{\pi/2}^{m\pi/2} \frac{\sin^2 t}{t}\, dt + O(1). \end{aligned} \tag{B.3}$$

With (B.2) and (B.3) one turns (B.1) to

$$\int_0^{\pi/2} \frac{\sin^2 m\tau}{\sin\tau}\, d\tau = \int_{\pi/2}^{m\pi/2} \frac{\sin^2 t}{t}\, dt + O(1) = \sum_{j=2}^{m} \int_{(j-1)\pi/2}^{j\pi/2} \frac{\sin^2 t}{t}\, dt + O(1). \tag{B.4}$$

Note that

$$\frac{1}{2j} = \int_{(j-1)\pi/2}^{j\pi/2} \frac{\sin^2 t}{j\pi/2}\, dt \le \int_{(j-1)\pi/2}^{j\pi/2} \frac{\sin^2 t}{t}\, dt \le \int_{(j-1)\pi/2}^{j\pi/2} \frac{\sin^2 t}{(j-1)\pi/2}\, dt = \frac{1}{2(j-1)}.$$

Since $\sum_{j=2}^{m} \frac{1}{j} = \log m + O(1)$ and $\sum_{j=2}^{m} \frac{1}{j-1} = \log m + O(1)$ by the Euler's constant formula,

$$\sum_{j=2}^{m} \int_{(j-1)\pi/2}^{j\pi/2} \frac{\sin^2 t}{t}\, dt = \frac{1}{2}\log m + O(1).$$

Part 1 of the lemma then follows from (B.4).

Next consider

$$\int_0^{\pi/2} \frac{m^2\sin^2\tau - \sin^2 m\tau}{\sin^3\tau}\, d\tau = \frac{1}{m}\int_0^{\pi/2} \frac{m^2\sin^2(\frac{t}{m}) - \sin^2 t}{\sin^3(\frac{t}{m})}\, dt + \frac{1}{m}\int_{\pi/2}^{m\pi/2} \frac{m^2\sin^2(\frac{t}{m}) - \sin^2 t}{\sin^3(\frac{t}{m})}\, dt. \tag{B.5}$$

For $t \in (0, \pi/2)$, one deduces that

$$\begin{aligned}
\frac{m^2 \sin^2(\frac{t}{m}) - \sin^2 t}{\sin^3(\frac{t}{m})} &= \frac{m^2\Big(\frac{t}{m} + O\big((\frac{t}{m})^3\big)\Big)^2 - (t + O(t^3))^2}{\Big(\frac{t}{m} + O\big((\frac{t}{m})^3\big)\Big)^3} \\
&= \frac{m^2}{\frac{t}{m}}\Big(1 + \ O\big((\frac{t}{m})^2\big)\Big) - \frac{(t+O(t^3))^2}{(\frac{t}{m})^3}\Big(1 + O\big((\frac{t}{m})^2\big)\Big) \\
&= O(t\, m^3) + O(t\, m),
\end{aligned}$$

which leads to

$$\frac{1}{m}\int_0^{\pi/2} \frac{m^2 \sin^2(\frac{t}{m}) - \sin^2 t}{\sin^3(\frac{t}{m})}\, dt = O(m^2). \tag{B.6}$$

For $t \in (\pi/2, m\pi/2)$, one argues differently that

$$\begin{aligned}
\frac{m^2 \sin^2(\frac{t}{m}) - \sin^2 t}{\sin^3(\frac{t}{m})} &= \frac{m^2\Big(\frac{t}{m} + O\big((\frac{t}{m})^3\big)\Big)^2 - \sin^2 t}{\Big(\frac{t}{m} + O\big((\frac{t}{m})^3\big)\Big)^3} \\
&= \frac{m^2}{\frac{t}{m}}\Big(1 + \ O\big((\frac{t}{m})^2\big)\Big) - \frac{\sin^2 t}{(\frac{t}{m})^3}\Big(1 + O\big((\frac{t}{m})^2\big)\Big) \\
&= \frac{m^3}{t} + O(t\, m) + \Big(\frac{m^3}{t^3}\Big)\, O(1).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{m}\int_{\pi/2}^{m\pi/2} \frac{m^2 \sin^2(\frac{t}{m}) - \sin^2 t}{\sin^3(\frac{t}{m})}\, dt &= m^2 \int_{\pi/2}^{m\pi/2} \frac{1}{t}\, dt + \Big(\int_{\pi/2}^{m\pi/2} t\, dt\Big)\, O(1) + \Big(\int_{\pi/2}^{m\pi/2} \frac{1}{t^3}\, dt\Big)\, O(m^2) \\
&= m^2 \log m + O(m^2).
\end{aligned} \tag{B.7}$$

The second part of the lemma follows from (B.5), (B.6), and (B.7).

C. Proof of Lemma 3.4

Expand

$$\sqrt{\frac{\rho}{r}} = 1 - \frac{r-\rho}{2\rho} + \frac{3(r-\rho)^2}{8\rho^2} + O(|r-\rho|^3). \tag{C.1}$$

Note that $|r - \rho| = O(q)$ and $\beta = O(q^2)$. For $m \geq 2$, by Lemmas 3.2 and 3.3, one finds that

$$\begin{aligned}
G^m(r, z, \rho, \zeta) &= \frac{1}{2\pi}\sqrt{\frac{\rho}{r}} \int_0^{\pi/2} \frac{\cos 2m\tau\, d\tau}{\sqrt{\beta + \sin^2 \tau}} \\
&= \frac{1}{2\pi}\sqrt{\frac{\rho}{r}}\Big(\frac{1}{2} + (\frac{m^2}{2} - \frac{1}{8})\beta\Big) \log \frac{16}{\beta} + \frac{1}{2\pi}\sqrt{\frac{\rho}{r}}\big(d_0^m + d_1^m \beta\big) + O\big(m^{2+2\alpha} q^{2+2\alpha}\big) \\
&= \frac{1}{2\pi}\Big(1 - \frac{r-\rho}{2\rho} + \frac{3(r-\rho)^2}{8\rho^2}\Big)\Big(1 + (m^2 - \frac{1}{4})\beta\Big) \log \frac{8\sqrt{r\rho}}{|(r,z) - (\rho,\zeta)|} \\
&\quad + O\Big(q^3 \log \frac{8}{q}\Big) + O\Big(m^2 q^5 \log \frac{8}{q}\Big) + \frac{1}{2\pi}\sqrt{\frac{\rho}{r}}\big(d_0^m + d_1^m \beta\big) + O(m^{2+2\alpha} q^{2+2\alpha}) \\
&= \frac{1}{2\pi}\Big(1 - \frac{r-\rho}{2\rho} + \frac{3(r-\rho)^2}{8\rho^2} + (m^2 - \frac{1}{4})\frac{(r-\rho)^2 + (z-\zeta)^2}{4r\rho}\Big) \log \frac{8\sqrt{r\rho}}{|(r,z) - (\rho,\zeta)|}
\end{aligned}$$

$$
\begin{aligned}
& +O(m^2q^3\log\frac{8}{q}) + O(q^3\log\frac{8}{q}) + \frac{1}{2\pi}\sqrt{\frac{\rho}{r}}\big(d_0^m + d_1^m\beta\big) + O\big(m^{2+2\alpha}q^{2+2\alpha}\big) \\
= \; & \frac{1}{2\pi}\Big(1 - \frac{r-\rho}{2\rho} + \frac{(4m^2+5)(r-\rho)^2 + (4m^2-1)(z-\zeta)^2}{16\rho^2}\Big)\log\frac{8\sqrt{r\rho}}{|(r,z)-(\rho,\zeta)|} \\
& + \frac{d_0^m}{2\pi}\sqrt{\frac{\rho}{r}} + \Big(\frac{d_1^m}{2\pi}\Big)\frac{(r-\rho)^2+(z-\zeta)^2}{4\rho^2} \\
& + O(m^2q^3\log\frac{8}{q}) + O((m^2\log m)q^3) + O\big(m^{2+2\alpha}q^{2+2\alpha}\big).
\end{aligned}
$$

The proof for the $m = 0$ and $m = 1$ cases are similar.

D. Proof of Lemma 3.5

Let

$$\delta = \frac{\pi}{2m} \tag{D.1}$$

and write

$$\int_0^{\pi/2}\frac{m^2\sin^2\tau - \sin^2 m\tau}{\sin^3\tau}\,d\tau = \int_0^{\delta}\frac{m^2\sin^2\tau - \sin^2 m\tau}{\sin^3\tau}\,d\tau + \int_\delta^{\pi/2}\frac{m^2\sin^2\tau - \sin^2 m\tau}{\sin^3\tau}\,d\tau := I_1 + I_2. \tag{D.2}$$

Using

$$\tau - \frac{\tau^3}{6} < \sin\tau < \tau, \text{ and } \frac{2\tau}{\pi} < \sin\tau \tag{D.3}$$

we estimate the first term as

$$
\begin{aligned}
I_1 \; & > \; \int_0^{\delta}\frac{m^2(\tau-\tau^3/6)^2 - \sin^2 m\tau}{\sin^3\tau}\,d\tau \\
& > \; \int_0^{\delta}\frac{m^2\tau^2 - \sin^2 m\tau}{\sin^3\tau} - \int_0^{\delta}\frac{m^2\tau^4}{3\sin^3\tau}\,d\tau \\
& > \; 0 - \int_0^{\delta}\frac{m^2\tau^4}{3(2\tau/\pi)^3}\,d\tau \\
& = \; -\frac{\pi^5}{192}
\end{aligned}
\tag{D.4}
$$

and the second term as

$$
\begin{aligned}
I_2 \; & > \; \int_\delta^{\pi/2}\frac{m^2(2\tau/\pi)^2 - 1}{\tau^3}\,d\tau \\
& = \; \frac{4m^2}{\pi^2}\log m + \frac{2}{\pi^2}(1-m^2).
\end{aligned}
\tag{D.5}
$$

By (D.4) and (D.5), one obtains

$$I_1 + I_2 > \frac{4m^2}{\pi^2}\log m + \frac{2}{\pi^2}(1-m^2) - \frac{\pi^5}{192}. \tag{D.6}$$

The right side of (D.6) is increasing with respect to m, and when $m = 3$ it is $0.7923... > 0$. So Part 1 of this lemma holds for $m \geq 3$. When $m = 2$,

$$\int_0^{\pi/2}\frac{2^2\sin^2\tau - \sin^2 2\tau}{\sin^3\tau}\,d\tau = \int_0^{\pi/2} 4\sin\tau\,d\tau = 4. \tag{D.7}$$

Hence Part 1 holds for all $m \geq 2$.

Regarding Part 2, let

$$\delta = \frac{1}{m} \tag{D.8}$$

and estimate

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin^2 m\tau}{\sin\tau}\, d\tau &= \int_0^{\delta} \frac{\sin^2 m\tau}{\sin\tau}\, d\tau + \int_{\delta}^{\pi/2} \frac{\sin^2 m\tau}{\sin\tau}\, d\tau \\ &< \int_0^{\delta} \frac{(m\tau)^2}{2\tau/\pi}\, d\tau + \int_{\delta}^{\pi/2} \frac{1}{2\tau/\pi}\, d\tau \\ &= \frac{\pi}{4} + \frac{\pi}{2}\log\frac{m\pi}{2}. \end{aligned} \tag{D.9}$$

Combining this with (D.6) one sees that the quantity in Part 2 is bounded below by

$$1 - \Big(\frac{\pi}{4} + \frac{\pi}{2}\log\frac{m\pi}{2}\Big) + \frac{4m^2}{\pi^2}\log m + \frac{2}{\pi^2}(1-m^2) - \frac{\pi^5}{192}. \tag{D.10}$$

The quantity (D.10) is increasing with respect to m when $m \geq 2$. And if $m = 4$, it is equal to $1.6837... > 0$, so Part 2 holds if $m \geq 4$. If $m = 2$,

$$1 - \int_0^{\pi/2} \frac{\sin^2 2\tau}{\sin\tau}\, d\tau + \int_0^{\pi/2} \frac{2^2\sin^2\tau - \sin^2 2\tau}{\sin^3\tau}\, d\tau = 1 - \frac{4}{3} + 4 = \frac{11}{3} > 0. \tag{D.11}$$

In the $m = 3$ case, since

$$-\int_0^{\pi/2} \frac{\sin^2 3\tau}{\sin\tau}\, d\tau = -\frac{23}{15} \tag{D.12}$$

and the quantity in Part 1 is bounded below by $0.7923...$, seen after (D.6), one has

$$1 - \int_0^{\pi/2} \frac{\sin^2 3\tau}{\sin\tau}\, d\tau + \int_0^{\pi/2} \frac{3^2\sin^2\tau - \sin^2 3\tau}{\sin^3\tau}\, d\tau > 1 - \frac{23}{15} + 0.7923... = 0.2590... > 0. \tag{D.13}$$

Hence Part 2 holds for all $m \geq 2$.

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