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Counting Peaks of Solutions to Some Quasilinear Elliptic Equations with Large Exponents

XIAOFENG REN* AND JUNCHENG WEI†

School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church Street S.E., Minneapolis, Minnesota 55455

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We consider the asymptotic behavior of certain solutions to a quasilinear problem with large exponent in the nonlinearity. Starting with the investigation of a Sobolev embedding, we get a sharp estimate for the embedding constant. Then we obtain a crucial L^1-estimate for the N-Laplacian operators in R^N. Using these estimates we prove that the solutions obtained by the standard variational method will develop a spiky pattern of peaks as the nonlinear exponent gets large, and we also have an upper bound depending on N only of the number of peaks. Stronger results for some special convex domains and some special solutions are also achieved. © 1995 Academic Press, Inc.

1. INTRODUCTION

In this paper we shall study the asymptotic behavior of certain solutions, as $p \to \infty$, of the quasilinear elliptic equation

$$\begin{cases} \Delta_N u + u^p = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0,\ u > 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

where $p > 1$, $N \geqslant 2$, $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2} \nabla u)$ is the N-Laplacian operator and $\Omega \subset R^N$ is a smooth bounded domain. We shall only focus on the solutions of the problem obtained by the following variational method. Let

$$\mathscr{A}_p = \{ v \in W_0^{1,N}(\Omega) : \|v\|_{p+1} = 1 \}$$

be the admissible set and define

$$J_p : \mathscr{A}_p \to R \tag{1.2}$$

by

$$J_p(v) = \int_\Omega |\nabla v|^N.$$

* Current address: Department of Mathematics, Brigham Young University, Provo, UT 84602.
† Current address: SISSA, Via Beirut 2-4, 34013 Trieste, Italy.

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Clearly J_p is bounded from below. Standard arguments show that J_p has at least one nonnegative minimizer in $\mathscr{A}_p$. If we denote such a minimizer by u'_p, then a suitable multiple of u'_p, say u_p, solves (1.1) and

$$c_p(N) := \inf\left\{\left[\int_\Omega |\nabla u|^N\right]^{1/N} : u \in \mathscr{A}_p\right\} = \frac{\|\nabla u_p\|_{L^N(\Omega)}}{\|u_p\|_{L^{p+1}(\Omega)}}. \tag{1.3}$$

A Hopf-type boundary lemma, see Guedda and Veron [7], shows that u_p is positive in Ω. It is also known that the solutions of (1.1) are $C^{1,\alpha}$ functions. We refer the reader to [7, 17, and 16] for the regularity, comparison principle, and Hopf boundary lemma for N-Laplacian operators.

Our goal is to understand the asymptotic behavior of the variational solutions u_p obtained above when p, serving as a parameter, gets large. The case where $N = 2$ is studied in our earlier work [12]. In that article, we proved that $\|u_p\|_{L^\infty}$ are bounded both from below and above as p tends to infinity. We also proved that u_p approach zero except at one or two points. u_p hence develop a pattern of peaks in Ω. In this paper we shall show that our method developed there can be successfully extended to higher dimensional cases with Δ replaced by Δ_N. Our first result is

THEOREM 1.1. *Let u_p be a variational solution of (1.1) obtained above. Then there exist positive C_1, C_2, independent of p, such that*

$$0 < C_1 < \|u_p\|_{L^\infty} < C_2 < \infty$$

for p large.

To state the second theorem, let

$$v_p = \frac{u_p}{\left(\int_\Omega u_p^p\right)^{1/(N-1)}}. \tag{1.4}$$

For a sequence $\{v_{p_n}\}$ of v_p we define the blow-up set $\mathscr{B}$ of $\{v_{p_n}\}$ to be the subset of $\bar{\Omega}$ such that $x \in \mathscr{B}$ if there exist a subsequence, still denoted by v_{p_n}, and a sequence x_n in Ω with

$$v_{p_n}(x_n) \to \infty \qquad \text{and} \qquad x_n \to x. \tag{1.5}$$

We also define, with respect to $\{v_{p_n}\}$,

$$\begin{aligned} S &= \mathscr{B} \cap \Omega, \\ S' &= \mathscr{B} \cap \partial\Omega. \end{aligned} \tag{1.6}$$

We use $\#\mathscr{B}$ ($\#S$, $\#S'$), to denote the cardinality of $\mathscr{B}$ (S, S' respectively). It turns out later that $\mathscr{B}(S, S')$ will be the set of global (interior, boundary) peaks of the subsequence v_{p_n} respectively. We also call them global (interior, boundary) peak sets.

THEOREM 1.2. *Let $N \geq 2$. Then for any sequence $\{v_{p_n}\}$ of v_p with $p_n \to \infty$, the global peak set $\mathscr{B}$ of v_{p_n} is not empty and there exists a subsequence of v_{p_n} such that the interior peak set S of the subsequence has the property*

$$0 \leq \# S \leq \left[\frac{1}{d_N} \left(\frac{N}{N-1} \right)^{N-1} \right]$$

where

$$d_N = \inf_{X \neq Y \in R^N} \frac{(|X|^{N-2} X - |Y|^{N-2} Y)(X-Y)}{|X-Y|^N}$$

is a positive number depending on N only.

From the above results, we see that the variational solutions develop a spiky pattern as p approaches infinity and the number of peaks is controlled in Theorem 1.2. If we impose more conditions on the domain as well as solutions, we can prove that they develop a single peak in the interior of the domain. We note that single-peak spiky patterns also appear in the works of Ni and co-workers [8–10] and Pan [11] where some biological pattern formation problems are considered.

Our paper is organized as follows. In Section 2, we prove a crucial sharp estimate for $c_p(N)$ defined in (1.3). Theorem 1.1 will be proved in Section 3. In Section 4 we extend an estimate of Brezis and Merle [1] to the N-Laplacian cases using the level set method. Theorem 1.2 will then be proved in Section 5. Stronger conclusions for some special convex domains and some special variational solutions u_p are obtained in Section 6; namely, $\# S = 1$ and $S' = \varnothing$.

2. AN ESTIMATE FOR $c_p(N)$

Recall $c_p(N)$ defined in (1.3). We first prove

LEMMA 2.1. *For every $t \geq 2$ there is D_t such that*

$$\|u\|_{L^t} \leq D_t t^{(N-1)/N} \|\nabla u\|_{L^N}$$

for all $u \in W_0^{1,N}(\Omega)$ where Ω is a bounded domain in R^N, furthermore

$$\lim_{t \to \infty} D_t = (\alpha_N)^{-(N-1)/N} \left(\frac{N-1}{Ne} \right)^{(N-1)/N}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the area of unit $(N-1)$-sphere in R^N.

Proof. Let $u \in W_0^{1,N}(\Omega)$. We know

$$\frac{1}{\Gamma(s+1)} x^s \leqslant e^x$$

for all $x \geqslant 0$, $s \geqslant 0$ where Γ is the Γ function. From Moser's sharp form of Trudinger's inequality (see [5, p. 160; 6]), we have

$$\int_\Omega \exp\left[\alpha_N \left(\frac{u}{\|\nabla u\|_{L^N}}\right)^{N/(N-1)}\right] dx \leqslant C\,|\Omega|$$

where α_N is defined in Lemma 2.1, C depends on N only, and $|\Omega|$ is the Lebesgue measure of Ω. Therefore

$$\begin{aligned}
&\left(1/\Gamma\left(\frac{N-1}{N}t+1\right)\right)\int_\Omega u^t\,dx\\
&\quad=\left(1/\Gamma\left(\frac{N-1}{N}t+1\right)\right)\int_\Omega \left[\alpha_N\left(\frac{u}{\|\nabla u\|_{L^N}}\right)^{N/(N-1)}\right]^{((N-1)/N)t}\\
&\qquad\times dx(\alpha_N)^{-((N-1)/N)t}\,\|\nabla u\|_{L^N}^t\\
&\quad\leqslant \int_\Omega \exp\left[\alpha_N\left(\frac{u}{\|\nabla u\|_{L^N}}\right)^{N/(N-1)}\right] dx(\alpha_N)^{-((N-1)/N)t}\,\|\nabla u\|_{L^N}^t\\
&\quad\leqslant C\,|\Omega|\,(\alpha_N)^{-((N-1)/N)t}\,\|\nabla u\|_{L^N}^t.
\end{aligned}$$

Hence

$$\left(\int_\Omega u^t\,dx\right)^{1/t} \leqslant \left(\Gamma\left(\frac{N-1}{N}t+1\right)\right)^{1/t} (C\,|\Omega|)^{1/t}\,\alpha_N^{-(N-2)/N}\,\|\nabla u\|_{L^N(\Omega)}.$$

Note that according to Stirling's formula,

$$\left(\Gamma\left(\frac{N-1}{N}t+1\right)\right)^{1/t} \sim \left(\frac{N-1}{Ne}\right)^{(N-1)/N} t^{(N-1)/N}.$$

Choosing D_t to be

$$\left(\Gamma\left(\frac{N-1}{N}t+1\right)\right)^{1/t} (C\,|\Omega|)^{1/t}\,\alpha_N^{-(N-1)/N} t^{-(N-1)/N}$$

we get the desired result. ∎

We then prove a sharp estimate for $c_p(N)$.

LEMMA 2.2.

$$\lim_{p\to\infty} \frac{c_p(N)}{p^{-(N-1)/N}} = \left(\frac{N}{N-1}\alpha_N e\right)^{(N-1)/N}.$$

Proof. Without loss of generality, we assume $0\in\Omega$. Let $L>0$ be such that $B_L\subset\Omega$ where B_L is the ball of radius L centered at the origin. For $0<l<L$, consider the so-called Moser function

$$m_l(x) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} \left(\log \dfrac{L}{l}\right)^{(N-1)/N}, & 0\leqslant |x|\leqslant l \\ \dfrac{\log(L/|x|)}{[\log(L/l)]^{1/N}}, & l\leqslant |x|\leqslant L \\ 0, & |x|\geqslant L. \end{cases}$$

Then $m_l\in W_0^{1,N}(\Omega)$ and $\|\nabla u\|_{L^N}=1$. Now

$$\begin{aligned} &\left(\int_\Omega m_l^{p+1}(x)\,dx\right)^{1/(p+1)} \\ &\quad\geqslant \left(\int_{B_l} m_l^{p+1}(x)\,dx\right)^{1/(p+1)} \\ &\quad= \frac{1}{\omega_{N-1}^{1/N}} \left(\log\frac{L}{l}\right)^{(N-1)/N} \left(\frac{1}{N} l^N \omega_{N-1}\right)^{1/(p+1)}. \end{aligned}$$

Choosing $l=L\exp(-((N-1)/N^2)(p+1)))$, we have

$$\begin{aligned} \|m_l\|_{p+1} \geqslant \frac{1}{\omega_{N-1}^{1/N}} \exp\left(-\frac{N-1}{N}\right)\left(\frac{N-1}{N^2}\right)^{(N-1)/N} \\ \times (p+1)^{(N-1)/N} \left(\frac{1}{N}\omega_{N-1}L^N\right)^{1/(p+1)}. \end{aligned}$$

Therefore

$$\begin{aligned} c_p(N) \leqslant \omega_{N-1}^{1/N} \exp\left(\frac{N-1}{N}\right)\left(\frac{N^2}{N-1}\right)^{(N-1)/N} \\ \times (p+1)^{-(N-1)/N} \left(\frac{1}{N}\omega_{N-1}L^N\right)^{-1/(p+1)}. \end{aligned}$$

Combining this with Lemma 2.1, we get the conclusion. ∎

By the construction of the variational solutions u_p in Section 1, we have

$$c_p(N) = \frac{\|\nabla u_p\|_{L^N(\Omega)}}{\|u_p\|_{L^{p+1}(\Omega)}}.$$

If we multiply Eq. (1.1) by u_p and integrate both sides on Ω, we have

$$\int_\Omega |\nabla u_p|^N = \int_\Omega u_p^{p+1}.$$

Hence we derive from Lemma 2.2

COROLLARY 2.3.

$$\lim_{p\to\infty} p^{N-1} \int_\Omega u_p^{p+1} = \left(\frac{N\alpha_N e}{N-1}\right)^{N-1},$$

$$\lim_{p\to\infty} p^{N-1} \int_\Omega |\nabla u_p|^N = \left(\frac{N\alpha_N e}{N-1}\right)^{N-1}.$$

Define

$$v_p = \left[\int_\Omega u_p^p\right]^{1/(N-1)},$$

$$L'_0 = \overline{\lim_{p\to\infty}} \frac{p v_p}{e}, \tag{2.1}$$

$$L_0 = L'_0 \, d_N^{-1/(N-1)},$$

where d_N is as defined in Theorem 1.2. We have the following rough estimates for L_0 and L'_0.

COROLLARY 2.4. *For any smooth bounded domain Ω in R^N,*

$$L'_0 \leqslant \frac{N}{N-1}\alpha_N, \qquad L_0 \leqslant \frac{N}{N-1}\alpha_N d_N^{-1/(N-1)}$$

Proof. From Corollary 2.3 we have by Holder's inequality

$$L'_0 = \overline{\lim_{p\to\infty}} \frac{p v_p}{e} \leqslant \overline{\lim_{p\to\infty}} \, p \left[\int_\Omega u_p^{p+1}\right]^{(p/(p+1))(1/(N-1))} |\Omega|^{(1/(p+1))(1/(N-1))} e^{-1}$$

$$\leqslant \frac{N\alpha_N}{N-1} \quad ∎$$

3. Proof of Theorem 1.1

To get a lower bound for $\|u_p\|_{L^\infty}$, we define

$$\lambda = \inf\left\{\frac{\|\nabla u\|_{L^N}}{\|u\|_{L^N}} : u \in W_0^{1,N}(\Omega),\ u \neq 0\right\}.$$

From Poincaré's inequality, we have $0 < \lambda < \infty$. For u_p we have

$$\int_\Omega u_p^{p+1} = \int_\Omega |\nabla u_p|^N \geqslant \lambda^N \int_\Omega u_p^N,$$

$$\int_\Omega (u_p^{p+1} - \lambda^N u_p^N) \geqslant 0.$$

Therefore

$$\|u_p\|_{L^\infty}^{p+1-N} \geqslant \lambda^N.$$

Letting $p \gg N-1$, we obtain

$$\|u_p\|_{L^\infty} \geqslant \lambda^{N/(p+1-N)} \geqslant C_1 > 0.$$

To get a upper bound for $\|u_p\|_{L^\infty}$, let

$$\begin{aligned} \gamma_p &= \max_{x \in \bar{\Omega}} u_p(x), \\ A &= \{x : u_p(x) > \gamma_p/2\}, \qquad (3.1) \\ \Omega_t &= \{x : u_p(x) > t\}. \end{aligned}$$

Both A and Ω_t depend on p. From Lemma 2.1 and Corollary 2.3, we have

$$\begin{aligned} \|u_p\|_{L^{Np/(N-1)}} &\leqslant D_{Np/(N-1)} \left(\frac{Np}{N-1}\right)^{(N-1)/N} \|\nabla u\|_{L^N} \\ &\leqslant C \left(\frac{Np}{N-1}\right)^{(N-1)/N} p^{-(N-1)/N} < M, \end{aligned}$$

where M is a constant independent of p. Then

$$\left(\frac{\gamma_p}{2}\right)^{Np/(N-1)} |A| \leqslant \int_\Omega u_p^{Np/(N-1)} \leqslant M^{Np/(N-1)}. \qquad (3.2)$$

On the other hand,

$$\int_{\Omega_t} u_p^p = -\int_{\Omega_t} \operatorname{div}(|\nabla u_p|^{N-2} \nabla u_p) = \int_{\partial\Omega_t} |\nabla u_p|^{N-1}\, ds$$

and

$$-\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t} \frac{ds}{|\nabla u_p|},$$

where the second is the co-area formula (see Federer [3]). By the Schwartz inequality and the isoperimetric inequality we have

$$\begin{aligned}
&\left(-\frac{d}{dt}|\Omega_t|\right)^{N-1} \int_{\Omega_t} u_p^p \\
&= \left(\int_{\partial\Omega_t} \frac{ds}{|\nabla u_p|}\right)^{N-1} \left(\int_{\partial\Omega_t} |\nabla u_p|^{N-1}\, ds\right) \\
&\geqslant \left(\int_{\partial\Omega_t} \frac{ds}{|\nabla u_p|}\right)^{N-1} \left(\int_{\partial\Omega_t} |\nabla u_p|\right)^{N-1} |\partial\Omega_t|^{-(N-2)} \\
&\geqslant |\partial\Omega_t|^{2(N-1)}\, |\partial\Omega_t|^{-(N-2)} = |\partial\Omega_t|^N \geqslant C_N\, |\Omega_t|^{N-1},
\end{aligned}$$

where $|\partial\Omega_t|$ denotes the $(N-1)$-dimensional Hausdorff measure of $\partial\Omega_t$ and C_N is the best constant in the isoperimetric inequality (we refer to [3] for more information about the Hausdorff measures and the isoperimetric inequality). Now we define $r(t)$ for $0 \leqslant t \leqslant \gamma_p$ such that

$$|\Omega_t| = \frac{1}{N}\omega_{N-1} r^N(t);$$

then

$$\frac{d}{dt}|\Omega_t| = \omega_{N-1} r^{N-1}(t)\frac{dr}{dt}.$$

Hence we have

$$\left(-\omega_{N-1} r^{N-1}(t)\frac{dr}{dt}\right)^{N-1} \int_{\Omega_t} u_p^p(x)\, dx \geqslant C_N \left(\frac{1}{N}\omega_{N-1} r^N(t)\right)^{N-1};$$

$$\left(-\frac{dr}{dt}\right)^{N-1} \int_{\Omega_t} u_p^p(x)\, dx \geqslant C_N r^{N-1};$$

$$\begin{aligned}
-\frac{dt}{dr} &\leqslant C'_N \frac{1}{r}\left(\int_{\Omega_t} u_p^p(x)\, dx\right)^{1/(N-1)} \\
&\leqslant C'_N \frac{1}{r} \gamma_p^{p/(N-1)}\, |\Omega_t|^{1/(N-1)} \\
&= C'_N \gamma_p^{p/(N-1)} r^{1/(N-1)}.
\end{aligned}$$

Integrating the inequality from 0 to r_0, we have

$$t(0) - t(r_0) \leqslant C''_N \gamma_p^{p/(N-1)} r_0^{N/(N-1)}.$$

Choosing r_0 so that $t(r_0) = \gamma_p/2$, we get

$$\gamma_p \leqslant C'_N \gamma_p^{p/(N-1)} r_0^{N/(N-1)};$$

$$\gamma_p \leqslant C'_N \gamma_p^{p/(N-1)} |A|^{1/(N-1)}.$$

Combining this with (3.2), we obtain

$$\gamma_p \leqslant C'_N \gamma_p^{p/(N-1)} \left(\frac{(2M)^{Np/(N-1)}}{\gamma_p^{Np/(N-1)}} \right)^{1/(N-1)};$$

$$\gamma_p \leqslant C^{1/(1+Np/(N-1)^2 - p/(N-1))} (2M)^{Np/(1+Np/(N-1)^2 - p/(N-1))}$$

$$\leqslant C'$$

for p large enough where the last C' is a constant independent of large p. This proves Theorem 1.1.

We derive a consequence of Theorem 1.1 which will be used later.

COROLLARY 3.1. *There exist C_1 and C_2 independent of p such that*

$$\frac{C_1}{p^{N-1}} \leqslant \int_\Omega u_p^p \leqslant \frac{C_2}{p^{N-1}}$$

for large p.

Proof. The first inequality follows from Theorem 1.1 and the first limit of Corollary 2.3; the second inequality follows from the first limit of Corollary 2.3 by an interpolation. ∎

4. A PRIORI ESTIMATES FOR N-LAPLACIAN OPERATORS

In this section we extend the L^1 estimate of Brezis and Merle [1] to N-Laplacian operators. Due to the nonlinearity of N-Laplacian operators for $N \geqslant 3$, we use the level set argument here.

LEMMA 4.1. *Let u be a $C^{1,\alpha}$ solution of*

$$\begin{cases} -\Delta_N u = f(x) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $f \in L^1(\Omega)$, $f \geqslant 0$. *Then for every* $\delta \in (0, N\omega_{N-1}^{1/(N-1)}) = (0, \alpha_N)$ *we have*

$$\int_\Omega \exp\left[\frac{(\alpha_N - \delta)\,|u(x)|}{\|f\|_{L^1}^{1/(N-1)}}\right] dx \leqslant \frac{\alpha_N}{\delta}\,|\Omega|$$

where $|\Omega|$ *denotes the volume of* Ω.

Proof. We prove this by the symmetrization method. Consider the symmetrized problem

$$\begin{cases} -\operatorname{div}(|\nabla U|^{N-2}\,\nabla U) = F(x) & \text{in } \Omega^* \\ U|_{\partial\Omega^*} = 0 \end{cases}$$

where Ω^* is the ball centered at the origin with the same volume as Ω and F is the symmetric decreasing rearrangement of f. We refer the reader to Talenti [14, 15] for properties of the rearrangement. According to [15], we have

$$u^* \leqslant U$$

where u^* is the symmetric decreasing rearrangement of u. U clearly satisfies the following ODE.

$$\begin{cases} (|U'|^{N-2}\,U')' + \dfrac{N-1}{r}\,|U'|^{N-2}\,U' + F(r) = 0 \\ U'(0) = 0, \qquad U(R) = 0. \end{cases}$$

Therefore

$$-U'(r) = \frac{(\int_0^r s^{N-1} F(s)\,ds)^{1/(N-1)}}{r} \leqslant \frac{1}{\omega_{N-1}^{1/(N-1)}}\,\frac{1}{r}\,\|F\|_{L^1(\Omega^*)}^{1/(N-1)}.$$

Hence

$$|U(r)| \leqslant \frac{1}{\omega_{N-1}^{1/(N-1)}}\,\|F\|_{L^1(\Omega^*)}^{1/(N-1)} \log\frac{R}{r};$$

$$\begin{aligned} \int_{\Omega^*} \exp\left[(N-\varepsilon)\,\omega_{N-1}^{1/(N-1)}\,\frac{U}{\|F\|_{L^1}^{1/(N-1)}}\right] dx &\leqslant \int_{B(R)} \exp\log\left(\frac{R}{|x|}\right)^{N-\varepsilon} dx \\ &= \omega_{N-1}\int_0^R \left(\frac{R}{r}\right)^{N-\varepsilon} r^{N-1}\,dr \\ &= \varepsilon^{-1}\omega_{N-1} R^N. \end{aligned}$$

Letting $\varepsilon\omega_{N-1}^{1/(N-1)}=\delta$, we have

$$\int_{\Omega^*}\exp\left[(\alpha_N-\delta)\frac{U(r)}{\|F\|_{L^1}^{1/(N-1)}}\right]\leqslant\frac{\omega_{N-1}^{N/(N-1)}}{\delta}R^N.$$

According to the properties of the symmetric decreasing function, we have

$$\|F\|_{L^1(\Omega^*)}=\|f\|_{L^1(\Omega)},$$

$$\begin{aligned}\int_\Omega\exp\left[(\alpha_N-\delta)\frac{u(x)}{\|f\|_{L^1(\Omega)}}\right]dx&=\int_{\Omega^*}\exp\left[(\alpha_N-\delta)\frac{u^*(x)}{\|f\|_{L^1}^{1/(N-1)}}\right]\\&\leqslant\int_{\Omega^*}\exp\left[(\alpha_N-\delta)\frac{U(r)}{\|F\|_{L^1}^{1/(N-1)}}\right]\\&\leqslant\frac{\omega_{N-1}^{N/(N-1)}}{\delta}R^N=\frac{\alpha_N}{\delta}|\Omega|.\quad\blacksquare\end{aligned}$$

An interesting consequence is

COROLLARY 4.2. *Let u_n be a sequence of $C^{1,\alpha}$ solutions of*

$$\begin{cases}\Delta_N u_n+V_n e^{u_n}=0 & \text{in } \Omega\\ u_n|_{\partial\Omega}=0\end{cases}$$

such that

$$\|V_n\|_{L^q}\leqslant C_1;$$

$$\int_\Omega|V_n|\,e^{u_n}\leqslant\varepsilon_0<\frac{\alpha_N}{q'}$$

for some $1<q<\infty$ and $q'=q/(q-1)$. Then

$$\|u_n\|_{L^\infty(\Omega)}\leqslant C$$

where C depends on N, C_1, $|\Omega|$, and ε_0 only.

Proof. Fix $\delta>0$ so that $\alpha_N-\delta>\varepsilon_0(q'+\delta)$. By Lemma 4.1 we have

$$\int_\Omega\exp[(q'+\delta)\,|u_n|]\leqslant C$$

for some C independent of n. Therefore e^{u_n} is bounded in $L^{q'+\delta}(\Omega)$; hence $V_n e^{u_n}$ is bounded in $L^{1+\varepsilon_0}(\Omega)$. Then the standard Moser iteration method implies that u_n is bounded in $L^\infty(\Omega)$. ∎

Next we give a version of Lemma 4.1 without homogeneous boundary condition.

LEMMA 4.3. *Let u and φ be $C^{1,\alpha}(\bar{\Omega})$ solutions of*

$$\Delta_N u + f(x) = 0 \qquad \text{in } \Omega, \qquad f > 0,$$

and

$$\begin{cases} \Delta_N \varphi = 0 & \text{in } \Omega \\ \varphi\,|_{\partial\Omega} = u, \end{cases}$$

respectively. Then there exists a constant C depending on Ω only such that

$$\int_\Omega \exp\left[\frac{(\alpha_N - \delta)\, d_N^{1/(N-1)}}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}(u - \varphi)\right] \leqslant \frac{C}{\delta}$$

where d_N is defined in Theorem 1.2.

Proof. Let u_ε and φ_ε be solutions of the nondegenerate equations

$$\begin{cases} -\operatorname{div}((\varepsilon + |\nabla u_\varepsilon|^2)^{(N-2)/2}\, \nabla u_\varepsilon) = f & \text{in } \Omega, \qquad f > 0 \\ u_\varepsilon\,|_{\partial\Omega} = u \end{cases}$$

and

$$\begin{cases} -\operatorname{div}((\varepsilon + |\nabla \varphi_\varepsilon|^2)^{(N-2)/2}\, \nabla \varphi_\varepsilon) = 0 & \text{in } \Omega \\ \varphi_\varepsilon\,|_{\partial\Omega} = u, \end{cases}$$

respectively. (These solutions are smooth and obtained easily by the variational method. Furthermore,

$$\lim_{\varepsilon \to 0} u_\varepsilon = u,$$

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon = \varphi$$

in $C^{1,\beta}$ for some β. See [16].) Let $\Omega_t = \{x \in \Omega : u_\varepsilon - \varphi_\varepsilon > t\}$.

Claim.

$$\frac{\partial u_\varepsilon(x)}{\partial \nu} < \frac{\partial \varphi_\varepsilon(x)}{\partial \nu}$$

on $\partial\Omega_t$ for almost all $t \geqslant 0$.

Let $x_0 \in \partial\Omega_t$. For almost all $t>0$ we can find a ball $B_\delta(x_1) \subset \Omega_t$ with $\overline{B_\delta(x_1)} \cap \overline{\Omega_t} = x_0$ by Sard's theorem. Let $w = u_\varepsilon - \varphi_\varepsilon - t$. Then w verifies

$$-\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) = f > 0,$$

where

$$a_{ij} = (\varepsilon + |t_i \nabla u_\varepsilon + (1-t_i)\nabla\varphi_\varepsilon|^2)^{(N-4)/2}\bigg\{\delta_{ij}(\varepsilon + |t_i \nabla u_\varepsilon + (1-t_i)\nabla\varphi_\varepsilon|^2)$$
$$+(N-2)\left(t_i\frac{\partial u_\varepsilon}{\partial x_i} + (1-t_i)\frac{\partial\varphi_\varepsilon}{\partial x_i}\right)\left(t_i\frac{\partial u_\varepsilon}{\partial x_j} + (1-t_i)\frac{\partial\varphi_\varepsilon}{\partial x_j}\right)\bigg\}$$

and $t_i \in (0, 1)$. Because this equation is nondegenerate, we can apply Hopf's lemma. Therefore

$$\frac{\partial w}{\partial v} < 0;$$

hence we prove the claim.

Following the standard level set argument, we have

$$\begin{aligned}\int_{\Omega_t} f(x) &= -\int_{\Omega_t} \operatorname{div}((\varepsilon + |\nabla u_\varepsilon|^2)^{(N-2)/2}\nabla u_\varepsilon)\\
&\quad+\int_{\Omega_t} \operatorname{div}((\varepsilon + |\nabla\varphi_\varepsilon|^2)^{(N-2)/2}\nabla\varphi_\varepsilon)\\
&= \int_{\partial\Omega_t} ((\varepsilon + |\nabla u_\varepsilon|^2)^{(N-2)/2}\nabla u_\varepsilon\\
&\quad-(\varepsilon\,|\nabla\varphi_\varepsilon|^2)^{(N-2)/2}\nabla\varphi_\varepsilon)\frac{(\nabla u_\varepsilon - \nabla\varphi_\varepsilon)}{|\nabla u_\varepsilon - \nabla\varphi_\varepsilon|}\\
&\geqslant d_N^\varepsilon \int_{\partial\Omega_t} |\nabla u_\varepsilon - \nabla\varphi_\varepsilon|^{N-1},\end{aligned}$$

where

$$d_N^\varepsilon = \inf_{X \neq Y \in R^N} \frac{((\varepsilon + |X|^2)^{(N-2)/2} X - (\varepsilon + |Y|^2)^{(N-2)/2} Y)(X-Y)}{|X-Y|^N}$$

is a positive number,

$$\lim_{\varepsilon \to 0} d_N^\varepsilon = d_N$$

and d_N is as defined in Theorem 1.2. Also by the co-area formula we have

$$-\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t} \frac{ds}{|\nabla u_\varepsilon - \nabla\varphi_\varepsilon|}.$$

Hence by the Schwartz inequality and the isoperimetric inequality,

$$\begin{aligned}
&\left(-\frac{d}{dt}|\Omega_t|\right)^{N-1} \int_{\Omega_t} f(x) \\
&\geqslant \left(\int_{\partial\Omega_t} \frac{ds}{|\nabla u_\varepsilon - \nabla\varphi_\varepsilon|}\right)^{N-1} d_N^\varepsilon \left(\int_{\partial\Omega_t} |\nabla u_\varepsilon - \nabla\varphi_\varepsilon|^{N-1}\right) \\
&\geqslant \left(\int_{\partial\Omega_t} \frac{ds}{|\nabla u_\varepsilon - \nabla\varphi_\varepsilon|}\right)^{N-1} d_N^\varepsilon \left(\int_{\partial\Omega_t} |\nabla u_\varepsilon - \nabla\varphi_\varepsilon|\right)^{N-1} |\partial\Omega_t|^{-(N-2)} \\
&\geqslant d_N^\varepsilon\, |\partial\Omega_t|^{2(N-1)}\, |\partial\Omega_t|^{-(N-2)} = d_N^\varepsilon\, |\partial\Omega_t|^N \\
&\geqslant d_N^\varepsilon \omega_{N-1} N^{N-1}\, |\Omega_t|^{N-1} = d_N^\varepsilon \alpha_N^{N-1}\, |\Omega_t|^{N-1}.
\end{aligned}$$

Define $r(t)$ so that

$$|\Omega_t| = \frac{1}{N}\omega_{N-1} r^{N-1}(t);$$

then

$$\frac{d\,|\Omega_t|}{dt} = \frac{1}{N}\omega_{N-1} N r^{N-1}(t)\frac{dr}{dt} = \omega_{N-1} r^{N-1}(t)\frac{dr}{dt}.$$

Hence we have from the above that

$$\begin{aligned}
\left(-\omega_{N-1} r^{N-1}(t)\frac{dr}{dt}\right)^{N-1} \int_{\Omega_t} f(x)\,dx &\geqslant d_N^\varepsilon N^{N-1}\omega_{N-1}\left(\frac{1}{N}\omega_{N-1} r^N(t)\right)^{N-1}; \\
\left(-\frac{dr}{dt}\right)^{N-1} \int_{\Omega_t} f(x)\,dx &\geqslant d_N^\varepsilon \omega_{N-1} r^{N-1}; \\
\left(-\frac{dt}{dr}\right)^{N-1} &\leqslant \frac{1}{d_N^\varepsilon \omega_{N-1}} \frac{1}{r^{N-1}} \int_{\Omega_t} f(x)\,dx \\
&\leqslant \frac{1}{d_N^\varepsilon \omega_{N-1}} \frac{1}{r^{N-1}} \|f\|_{L^1(\Omega)}; \\
-\frac{dt}{dr} &\leqslant \frac{1}{(d_N^\varepsilon)^{1/(N-1)}\,\omega_{N-1}^{1/(N-1)}} \|f\|_{L^1(\Omega)}^{1/(N-1)} \frac{1}{r}.
\end{aligned}$$

Integrating the last inequality over (r, R) (note that $|\Omega| = (1/N)\,\omega_{N-1} R^N$), we have

$$t(r) \leqslant \frac{1}{(d_N^\varepsilon)^{1/(N-1)}\,\omega_{N-1}^{1/(N-1)}} \|f\|_{L^1(\Omega)}^{1/(N-1)} \log\frac{R}{r};$$

$$\exp\left(\frac{(d_N^\varepsilon)^{1/(N-1)}\,\omega_{N-1}^{1/(N-1)}(N-\varepsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}\,t(r)\right) \leqslant \left(\frac{R}{r}\right)^{N-\varepsilon_0};$$

$$\int_0^R \exp\left(\frac{(d_N^\varepsilon)^{1/(N-1)}\,\omega_{N-1}^{1/(N-1)}(N-\varepsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}\,t(r)\right) r^{N-1}\,dr$$

$$\leqslant \int_0^R \left(\frac{R}{r}\right)^{N-\varepsilon_0} r^{N-1}\,dr = \frac{C}{\varepsilon_0}.$$

However, the left-hand side of the last inequality,

$$\int_0^R \exp\left(\frac{(d_N^\varepsilon)^{1/(N-1)}\,\omega_{N-1}^{1/(N-1)}(N-\varepsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}\,t(r)\right) r^{N-1}\,dr$$

$$= \int_\infty^0 \exp\left(\frac{(d_N^\varepsilon)^{1/(N-1)}\,\omega_{N-1}^{1/(N-1)}(N-\varepsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}\,t\right)\frac{1}{\omega_{N-1}}\,d\,|\Omega_t|$$

$$= \frac{1}{\omega_{N-1}} \int_\Omega \exp\left(\frac{(d_N^\varepsilon)^{1/(N-1)}\,\omega_{N-1}^{1/(N-1)}(N-\varepsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}\,(u_\varepsilon - \varphi_\varepsilon)\right) dx.$$

Letting $\delta = \omega_{N-1}^{1/(N-1)}\varepsilon_0$, we have the desired estimate for u_ε and φ_ε. Finally, letting $\varepsilon \to 0$, we get the estimate for u and φ themselves. ∎

In order to have a local analogy of Corollary 4.2, we state a result from Serrin [13] which can be proved following Moser's iteration scheme.

PROPOSITION 4.4. *Let u be a weak solution of*

$$\Delta_N u + f(x) = 0$$

in $B_{2R} \subset \Omega$ and $f \in L^{N/(N-\varepsilon)}(B_{2R})$. Then we have

$$\|u\|_{L^\infty(B_R)} \leqslant CR^{-1}(\|u\|_{L^N(B_{2R})} + KR)$$

where

$$K = (R^\varepsilon \|f\|_{L^{N/(N-\varepsilon)}(B_{2R})})^{1/(N-1)}$$

and C depends on N only.

COROLLARY 4.5. *Let*

$$\Delta_N u_n + V_n e^{u_n} = 0 \qquad in\ \Omega$$

and

$$\|u_n\|_{L^N(\Omega)} \leqslant C_1, \qquad \|V_n\|_{L^q(B_R)} \leqslant C_2.$$

where $1 < q < \infty$ *and* B_R *is a ball compactly contained in* Ω. *Assuming*

$$\int_{B_R} V_n e^{u_n} \leqslant \varepsilon_0 < \frac{\alpha_N d_N^{1/(N-1)}}{q'}$$

where $q' = q/(q-1)$, *we have*

$$\|u_n\|_{L^\infty(B_{R/4})} \leqslant C$$

for some C *depending on* N, C_1, C_2, R, *and* ε_0 *only*.

Proof. Consider on B_R

$$\begin{cases} \Delta_N \varphi_n = 0 & \text{in } B_R \\ \varphi_n|_{\partial B_R} = u_n|_{\partial B_R}. \end{cases}$$

By the comparison principle in [7], we have

$$\varphi_n \leqslant u_n; \qquad \|\varphi_n\|_{L^N(B_R)} \leqslant C_1.$$

Using Proposition 4.4, we conclude

$$\|\varphi_n\|_{L^\infty(B_{R/2})} \leqslant C \tag{4.1}$$

for some constant C depending on N, C_1, C_2, and R only. From Lemma 4.3 we also know

$$\int_{B_R} \exp\left[\frac{(\alpha_N - \delta)\, d_N^{1/(N-1)}}{\varepsilon_0}(u_n - \varphi_n)\right] \leqslant \int_{B_R} \exp\left[\frac{(\alpha_N - \delta)\, d_N^{1/(N-1)}}{\|V_n e^{u_n}\|_{L^1(B_R)}}(u_n - \varphi_n)\right] \leqslant \frac{C}{\delta}.$$

Combining this with (4.1), we obtain

$$\int_{B_{R/2}} \exp\left[\frac{(\alpha_N - \delta)\, d_N^{1/(N-1)}}{\varepsilon_0} u_n\right] \leqslant \frac{C}{\delta}. \tag{4.2}$$

Choosing δ small enough so that

$$(\alpha_N - \delta)\, d_N^{1/(N-1)} > \varepsilon_0(q' + \delta),$$

we get from (4.2)

$$\|\exp u_n\|_{L^{q'+\delta}(B_{R/2})} \leqslant C.$$

Therefore

$$\| V_n \exp u_n\|_{L^{1+\varepsilon_1}(B_{R/2})} \leqslant C$$

for some $\varepsilon_1 > 0$. Using Proposition 4.4 again, we finally conclude

$$\|u_n\|_{L^\infty(B_{R/4})} \leqslant C. \quad \blacksquare$$

We close this section with a positive lower bound for d_N.

PROPOSITION 4.6. *Let*

$$d_N = \inf_{X \neq Y \in R^N} \frac{(|X|^{N-2} X - |Y|^{N-2} Y)(X-Y)}{|X-Y|^N}.$$

Then

$$d_N \geqslant \frac{2}{N}\left(\frac{1}{2}\right)^{N-2},$$

in particular $d_2 = 1$.

Proof. Without loss of generality, let $0 \leqslant |Y| \leqslant |X|$, $X \neq Y$, and $X \neq 0$. Let

$$t = \frac{|Y|}{|X|}, \qquad \cos\theta = \frac{\langle X, Y\rangle}{|X|\,|Y|}.$$

Then

$$\frac{(|X|^{N-2} X - |Y|^{N-2} Y)(X-Y)}{|X-Y|^N} = \frac{1-(t^{N-1}+t)\cos\theta + t^N}{(1-2t\cos\theta + t^2)^{N/2}}.$$

Let

$$f(t, x) = \frac{1-(t^{N-1}+t)\,x + t^N}{(1-2tx+t^2)^{N/2}}$$

for $0 \leqslant t \leqslant 1$ and $-1 \leqslant x \leqslant 1$. Fix t and set

$$\frac{\partial f}{\partial x} = 0.$$

Then

$$1-(t^{N-1}+t)\,x + t^N = \frac{t^{N-2}+1}{N}(1-2tx+t^2).$$

Therefore at the critical points x of $f(t, \ )$,

$$f(t, x) = \frac{t^{N-2}+1}{N} \frac{1}{(1-2tx+t^2)^{(N-2)/2}}$$
$$= \frac{1}{N} \frac{t^{N-2}+1}{(1-2tx+t^2)^{(N-2)/2}} \geqslant \frac{1}{N} \frac{t^{N-2}+1}{(t+1)^{N-2}}$$
$$\geqslant \frac{1}{N} \min_{0 \leqslant t \leqslant 1} \frac{t^{N-2}+1}{(t+1)^{N-2}}.$$

Let

$$g(t) = \frac{t^{N-2}+1}{(t+1)^{N-2}}.$$

Then

$$g'(t) = \frac{(t+1)^{N-3}}{(t+1)^{2(N-2)}} (N-2)(t^{N-3}-1) \leqslant 0$$

and

$$\min_{0 \leqslant t \leqslant 1} g(t) = g(1) = \frac{2}{2^{N-2}}.$$

Hence

$$d_N \geqslant \frac{2}{N} \left(\frac{1}{2}\right)^{N-2}. \quad \blacksquare$$

Remark 4.7. An upper bound for $\# S$ in Theorem 1.2 can therefore be

$$\frac{N}{4} \left(\frac{2N}{N-1}\right)^{N-1},$$

which equals 2 when $N = 2$.

5. Proof of Theorem 1.2

Recall (1.4) and (2.1)

$$v_p = \frac{u_p}{(\int_\Omega u_p^p)^{1/(N-1)}}, \qquad \nu_p = \left[\int_\Omega u_p^p\right]^{1/(N-1)}.$$

Define

$$f_p = \frac{u_p^p}{\int_\Omega u_p^p} = \nu_p^{p-(N-1)} v_p^p. \tag{5.1}$$

Then we have

$$\Delta_N v_p + f_p = 0. \tag{5.2}$$

We first prove $\mathscr{B} \neq \emptyset$ for any sequence $\{v_n\} = \{v_{p_n}\}$ of v_p with $p_n \to \infty$. Let x_n be such that

$$v_n(x_n) = \max_{x \in \bar{\Omega}} v_n(x) = \frac{\max u_n(x)}{(\int_\Omega u_n^{p_n})^{1/(N-1)}}$$

$$\geqslant \frac{C_1}{(\int_\Omega u_n^{p_n})^{1/(N-1)}} \to \infty$$

by Theorem 1.1 and Corollary 3.1. Therefore cluster points of $\{x_n\}$ belong to $\mathscr{B}$; hence $\mathscr{B} \neq \emptyset$.

Since $\int_\Omega f_p = 1$ and $f_p > 0$, for any sequence of $\{f_p\}$ we can subtract a subsequence

$$\{f_n\} = \{f_{p_n}\}$$

which converges to a measure μ weakly in $M(\Omega)$ where $M(\Omega)$ is the space of real bounded measures on Ω and μ is a positive measure with $\mu(\Omega) \leqslant 1$. From now on in the rest of this section we shall work on this subsequence $\{f_n\}$ and the corresponding $\{v_n\} = \{v_{p_n}\}$. For any $\delta > 0$, we call $x_0 \in \Omega$ a δ-regular point if there is a function $\varphi \in C_0(\Omega)$, $0 \leqslant \varphi \leqslant 1$ with $\varphi = 1$ in a neighborhood of x_0, such that

$$\int_\Omega \varphi \, d\mu < \left(\frac{\alpha_N}{L_0 + 3\delta}\right)^{N-1} \tag{5.3}$$

where L_0 is as defined in (2.1). We also define the δ-irregular set

$$\Sigma(\delta) = \{y_0 \colon y_0 \text{ is not a } \delta\text{-regular point}\}.$$

Clearly

$$\mu(y_0) \geqslant \left(\frac{\alpha_N}{L_0 + 3\delta}\right)^{N-1} \tag{5.4}$$

for all $y_0 \in \Sigma(\delta)$. We shall frequently say "regular" and "irregular," not mentioning δ if there is no confusion.

LEMMA 5.1. *If x_0 is a regular point, then v_n is uniformly bounded in $L^\infty(B_{R_0(x_0)})$ for some R_0.*

Proof. Let x_0 be a regular point. From (5.3), we can find $R_1 > 0$ such that

$$\int_{B_{R_1}(x_0)} f_n < \left(\frac{\alpha_N}{L_0 + 2\delta}\right)^{N-1}. \tag{5.5}$$

Applying Lemma 4.1 to f_n on Ω (note that $\|f_n\|_{L^1(\Omega)} = 1$), we have

$$\int_\Omega \exp[(\alpha_N - \varepsilon)\, v_n]\, dx \leqslant \frac{C}{\varepsilon},$$

especially $\|v_n\|_{L^N(B_{R_1}(x_0))} \leqslant C$ for some C independent of n.

Let φ_n be a solution of

$$\begin{cases} -\Delta_N \varphi_n = 0 & \text{in } B_{R_1}(x_0), \\ \varphi_n|_{\partial B_{R_1}(x_0)} = v_n|_{\partial B_{R_1}(x_0)}. \end{cases}$$

Then by Proposition 4.4, we have (note that $\varphi_n \leqslant v_n$ by the comparison principle)

$$\|\varphi_n\|_{L^\infty(B_{R_1/2}(x_0))} \leqslant C.$$

By Lemma 4.3 and (5.5), if we choose "δ" in Lemma 4.3 small enough,

$$\int_{B_{R_1}(x_0)} \exp[(L_0 + \delta)\, d_N^{1/(N-1)} (v_n - \varphi_n)] \leqslant C;$$

hence

$$\int_{B_{R_1/2}(x_0)} \exp[(L_0 + \delta)\, d_N^{1/(N-1)}\, v_n]\, dx \leqslant C. \tag{5.6}$$

Let $t = L_0' + d_N^{1/(N-1)}\, \delta/2$. Observe

$$\log x \leqslant \frac{x}{e}$$

for $x > 0$. We get

$$\begin{aligned} p_n \log \frac{u_n}{\nu_n^{(N-1)/p_n}} &\leqslant \frac{p_n}{e} \frac{u_n}{\nu_n^{(N-1)/p_n}} \\ &\leqslant \frac{L_0' + d_N^{1/(N-1)}\, \delta/3}{\nu_n} \frac{u_n}{\nu_n^{(N-1)/p_n}} = \frac{t - d_N^{1/(N-1)}\, \delta/6}{\nu_n^{(N-1)/p_n}} \frac{u_n}{\nu_n} \\ &\leqslant t \frac{u_n}{\nu_n} = t v_n \end{aligned}$$

for n large enough where $v_n = v_{p_n}$ is defind in (2.1) and the last inequality is based on

$$\lim_{n \to \infty} v_n^{(N-1)/p_n} = 1$$

which follows from Corollary 3.1. Hence

$$f_n \leqslant e^{tv_n}.$$

Note that

$$t = L_0' + d_N^{1/(N-1)} \delta/2 = (L_0 + \delta/2)\, d_N^{1/(N-1)} < (L_0 + \delta)\, d_N^{1/(N-1)};$$

hence with the aid of (5.6) we see that f_n is bounded in $L^q(B_{R_1/2}(x_0))$ where

$$q = \frac{L_0 + \delta}{L_0 + \delta/2} > 1.$$

Using Proposition 4.4 again, we conclude that for large n there exists $C > 0$ such that

$$\|v_n\|_{L^\infty(B_{R_1/4}(x_0))} \leqslant C.$$

This proves Lemma 5.1 if we choose $R_0 = R_1/4$. ∎

Back to the proof of Theorem 1.2, we claim $S = \Sigma(\delta)$ for any $\delta > 0$ where S is the interior peak set with respect to $\{v_n\}$ defined in (1.6).

Clearly, $S \subset \Sigma$. In fact, letting $x_0 \notin \Sigma$, then we know that x_0 is a regular point. Hence by Lemma 5.1, $\{v_n\}$ is uniformly bounded in a neighborhood of x_0. Therefore $x_0 \notin S$. Conversely, suppose $x_0 \in \Sigma$. Then we have for every $R > 0$,

$$\lim_{n \to \infty} \|v_n\|_{L^\infty(B_R(x_0))} = \infty.$$

Otherwise, there would be some $R_0 > 0$ and a subsequence of v_n, again denoted by v_n, such that

$$\|v_n\|_{L^\infty(B_{R_0}(x_0))} < C$$

for some C independent of n. Then

$$\begin{aligned} f_n &= v_n^{p_n-(N-1)} v_n^{p_n} < v_n^{p_n-(N-1)} C^{p_n} \\ &\to 0 \end{aligned}$$

uniformly on $B_{R_0}(x_0)$ as $n \to \infty$. Then

$$\int_{B_{R_0}(x_0)} f_n = \int_{B_{R_0}(x_0)} v_n^{p_n - (N-1)} v_n^{p_n} \leqslant \varepsilon_0 < \left(\frac{\alpha_N}{L_0 + 3\delta}\right)^{N-1}$$

for large n which implies that x_0 is a regular point. This proves the claim.

Back to the measure μ defined earlier in this section. We have from (5.4)

$$1 \geqslant \mu(\Omega) \geqslant \left(\frac{\alpha_N}{L_0 + 3\delta}\right)^{N-1} \# \Sigma(\delta) = \left(\frac{\alpha_N}{L_0 + 3\delta}\right)^{N-1} \# S.$$

Hence

$$0 \leqslant \# S \leqslant \left(\frac{L_0 + 3\delta}{\alpha_N}\right)^{N-1}.$$

Letting $\delta \to 0$, we get with the aid of Corollary 2.4

$$0 \leqslant \# S \leqslant \left(\frac{L_0}{\alpha_N}\right)^{N-1} \leqslant \left(\frac{N}{N-1}\right)^{N-1} d_N^{-1}.$$

This proves Theorem 1.2.

Remark 5.2. From the proof of the theorem we see that the measure μ is atomic. Actually,

$$\mu = \sum_{k=1}^{\# S} a_k \, \delta(x_k)$$

where $S = \{x_1, x_2, ..., x_{\# S}\}$ and

$$a_k \geqslant \left(\frac{\alpha_N}{L_0}\right)^{N-1}.$$

The subsequence v_n approaches a function G in $C^{1,\alpha}_{\text{loc}}(\Omega \backslash S)$ and G is N-harmonic in $\Omega \backslash S$ but singular on S.

Remark 5.3. It is also clear from the proof of Theorem 1.2 and Corollary 3.1 that the subsequence

$$u_n \to 0$$

in $L^{\infty}_{\text{loc}}(\Omega \backslash S)$.

6. Further Results

So far, we have not touched the boundary peak sets S' yet. Our next result shows that when Ω is strictly convex and u_p are generic in some sense, S' is empty; i.e., $\mathscr{B} = S$.

Recall that u_p are solutions of (1.1) obtained by minimizing

$$J_p(v) = \int_\Omega |\nabla v|^N$$

in the class

$$\mathscr{A}_p = \{v \in W_0^{1,N}(\Omega) \colon \|u\|_{p+1} = 1\}.$$

Let

$$J_p^\varepsilon \colon \mathscr{A}_p \to R \tag{6.1}$$

defined by

$$J_p^\varepsilon(v) = \int_\Omega (\varepsilon + |\nabla v|^2)^{N/2}.$$

We call u_p a generic solution if there exist a sequence ε_n of ε with

$$\varepsilon_n \to 0$$

and a sequence of positive minimizers $\{u'_{p\varepsilon_n}\}$ of $J_p^{\varepsilon_n}$ such that $\{u'_{p\varepsilon_n}\}$ converges to u'_p weakly in $W^{1,N}(\Omega)$ as $\varepsilon_n \to 0$ where $u_p = cu'_p$ for some scalar c. Clearly $\{u'_{p\varepsilon_n}\}$ is a minimizing sequence of J_p in $\mathscr{A}_p$. Actually, any sequence $\{u'_{p\varepsilon_n}\}$ of minimizers of $J_p^{\varepsilon_n}$ is a minimizing sequence of J_p, so generic solutions exist for all smooth bounded domains.

THEOREM 6.1. *Let Ω be a strict convex domain. Assume u_p are generic solutions for all p. Then S', the boundary peak set of $\{u_p\}$, is empty; i.e., $\mathscr{B} = S$.*

Proof. Let

$$\{u'_{p\varepsilon_n}\} \to u'_p$$

weakly in $W^{1,N}(\Omega)$ where $u'_p = cu_p$ for some c and $u'_{p\varepsilon_n}$ are positive minimizers of $J_p^{\varepsilon_n}$. Then $u'_{p\varepsilon_n}$ solve

$$\begin{cases} \operatorname{div}((\varepsilon + |\nabla u'_{p\varepsilon_n}|^2)^{(N-2)/2} \nabla u'_{p\varepsilon_n}) + \lambda_n u'_{p\varepsilon_n} = 0 & \text{in } \Omega \\ u'_{p\varepsilon_n}|_{\partial\Omega} = 0 \end{cases}$$

for some $\lambda_n > 0$.

Therefore, using the moving plane method for nondegenerate equations developed by Gidas, Ni, and Nirenberg in [4], we can find a neighborhood ω of $\partial\Omega$ and a cone Γ of fixed size, both depending on Ω only such that

$$u'_{p\varepsilon_n}(x) \leqslant \frac{1}{|\Gamma|} \int_\Omega u'_{p\varepsilon_n}(x)\, dx \tag{6.2}$$

for all $x \in \omega$. We refer the reader to DeFigueiredo *et al.* [2] for details of this trick.

Since $\{u'_{p\varepsilon_n}\} \to u'_p$ weakly in $W^{1,N}(\Omega)$, we have

$$\begin{aligned} u'_{p\varepsilon_n} &\to u'_p \qquad \text{strongly in } L^1(\Omega);\\ u'_{p\varepsilon_n} &\to u'_p \qquad \text{almost everywhere.} \end{aligned} \tag{6.3}$$

Hence passing limit in (6.2), we get

$$u'_p(x) \leqslant \frac{1}{|\Gamma|} \int_\Omega u'_p(x)\, dx$$

for almost all $x \in \omega$. Therefore

$$u_p(x) \leqslant \frac{1}{|\Gamma|} \int_\Omega u_p(x)\, dx$$

and

$$v_p(x) \leqslant \frac{1}{|\Gamma|} \int_\Omega v_p(x)\, dx$$

for almost all $x \in \omega$. But $\int_\Omega v_p(x)\, dx \leqslant C$ by Lemma 4.1. Therefore v_p are uniformly bounded in $L^\infty(\omega)$; hence $S' = \varnothing$. ▮

It is interesting to see when the peak set $\mathscr{B}$ contains one point only.

THEOREM 6.2. *Let Ω be a strict convex domain and u_{p_n} be a sequence of generic solutions. If we further assume*

$$\int_{\partial\Omega} \frac{ds}{\langle x - y, n(x)\rangle^{N-1}} < (2d_N)^N (e\alpha_N)^{N-1} \left(\frac{N-1}{N^2}\right)^{N-1} \left(\frac{N-1}{N}\right)^{(N-1)^2},$$

then there exists a subsequence of u_{p_n}, again denoted by u_{p_n}, such that the peak set $\mathscr{B}$ of the subsequence equals the interior peak set S and it contains one point only.

Proof. The assertion $\mathscr{B} = S$ follows from Theorem 6.1. We also know $\#\mathscr{B} \geqslant 1$ from Theorem 1.2.

Now we state a Pohozaev-type identity for (1.1). The proof of this integral identity can be found in ([7, Theorem 1.1]). Let $u \in L^{\infty}(\Omega) \cap W_{1,q}(\Omega)$ solve

$$\begin{cases} -\operatorname{div}(|\nabla u|^{q-2} \nabla u) = g(x, u) & \text{in } \Omega \\ u_{\partial\Omega} = 0 \end{cases}$$

where g is smooth with its growth bounded either by $|u|^{(Nq-N+q)/(N-q)}$ if $q < N$ or like a polynomial in u if $q = N$. Let $G(x, u) = \int_0^u g(x, r)\, dr$. Then

$$\int_\Omega NG(x, u)\, dx + \left(1 - \frac{N}{q}\right) \int_\Omega ug(x, u)\, dx + \int_\Omega \langle x - y, \nabla_x G(x, u) \rangle\, dx$$
$$= \left(1 - \frac{1}{q}\right) \int_{\partial\Omega} \langle x - y, n(x) \rangle \left|\frac{\partial u}{\partial n}\right|^q ds$$

for all $y \in R^N$.

Apply it to (1.1). Let "y" in the integral identity be "y" in the statement of Theorem 6.2. Without loss of generality, we can assume $y = 0$. Then

$$\frac{N}{p+1} \int_\Omega u_p^{p+1}\, dx = \left(1 - \frac{1}{N}\right) \int_\Omega \langle x, n(x) \rangle \left|\frac{\partial u_p}{\partial n}\right|^N ds. \tag{6.5}$$

On the other hand,

$$\int_\Omega u_p^p\, dx = \int_{\partial\Omega} \left|\frac{\partial u_p}{\partial n}\right|^{N-1} ds.$$

Hence by Holder's inequality,

$$\int_\Omega u_p^p\, dx \leqslant \left(\int_{\partial\Omega} \frac{1}{\langle x, n(x) \rangle^{N-1}}\, ds\right)^{1/N} \left(\int_{\partial\Omega} \langle x, n(x) \rangle \left|\frac{\partial u_p}{\partial n}\right|^N ds\right)^{(N-1)/N}$$
$$= \left(\int_{\partial\Omega} \frac{1}{\langle x, n(x) \rangle^{N-1}}\, ds\right)^{1/N} \left(\frac{N^2}{(N-1)(p+1)} \int_\Omega u^{p+1}\, dx\right)^{(N-1)/N}.$$

Therefore

$$\left(\frac{L_0}{\alpha_N}\right)^{N-1} = \frac{1}{d_N} \left(\frac{L_0'}{\alpha_N}\right)^{N-1}$$
$$= \frac{1}{d_N} \lim_{p \to \infty} \left(\frac{p}{e\alpha_N} \left(\int_\Omega u_p^p\, dx\right)^{1/(N-1)}\right)^{N-1}$$
$$= \frac{1}{d_N} \overline{\lim_{p \to \infty}} \frac{p^{N-1} \int_\Omega u_p^p\, dx}{(e\alpha_N)^{N-1}}$$

$$
\begin{aligned}
&\leqslant \varlimsup_{p \to \infty} \frac{1}{d_N} \frac{1}{(e\alpha_N)^{N-1}} \left(\int_{\partial\Omega} \frac{1}{\langle x, n(x) \rangle^{N-1}} \, ds \right)^{1/N} \\
&\quad \times p^{N-1} \left(\frac{N^2}{N-1} \right)^{(N-1)/N} \left(\frac{1}{p+1} \frac{1}{p^{N-1}} \left(\frac{N\alpha_N e}{N-1} \right)^{N-1} \right)^{(N-1)/N} \\
&= \frac{1}{d_N} \frac{1}{(e_N)^{N-1}} \left(\frac{N^2}{N-1} \right)^{(N-1)/N} \left(\frac{N\alpha_N e}{N-1} \right)^{(N-1)^2/N} \\
&\quad \times \left(\int_{\partial\Omega} \frac{1}{\langle x, n(x) \rangle^{N-1}} \, ds \right)^{1/N} \\
&< 2.
\end{aligned}
$$

Hence from the last inequality in the proof of Theorem 1.2 we have $\# S \leqslant 1$, and in turn $\# S = 1$. ∎

Remark 6.3. It turns out that when $N = 2$ the assumptions that Ω is strict convex and that u_p are generic solutions are both superfluous for Theorems 6.1 and 6.2. In our earlier article [12], we proved the corresponding results of Theorems 6.1 and 6.2 without these two conditions. In that work we used Kelvin transform to take care of non-convex domains and we applied the moving plane method to u_p directly since the equations (1.1) are nondegenerate when $N = 2$.

Finally we confine ourselves to the problem when $\Omega = B_R$, the ball of radius R centered at the origin. We also consider generic solutions. Applying the moving plane method to each approximate solution $u_{p\varepsilon_n}$ of u_p, we conclude that $u_{p\varepsilon_n}$ are all radially symmetric, and so are u_p. Therefore u_p solve the following ODE.

$$
\begin{cases}
(|u'|^{N-2} u')' + \dfrac{N-1}{r} |u'|^{N-2} u' + u^p = 0 & \text{in } (0, R) \\
u'(0) = 0, \qquad u(R) = 0.
\end{cases}
$$

Applying Theorem 1.2, we know $\mathscr{B} = S = \{0\}$; otherwise there would be infinitely many peaks by the symmetry. A straightforward argument shows

$$
f_p \to \delta
$$

in the sense of distribution where f_p is defined in (5.1) and δ is the Dirac mass at 0. We can actually prove the following. We leave the proof to the reader.

THEOREM 6.4. *Let u_p be generic variational solutions of* (1.1) *on B_R, the ball of radius R. Then as $p \to \infty$,*

$$v_p = \frac{u_p}{\left(\int_{B_R} u_p^p\right)^{1/(N-1)}} \to \frac{1}{\omega_{N-1}^{1/(N-1)}} \log\left(\frac{R}{r}\right)$$

in $C^{1,\alpha}_{\mathrm{loc}}(\bar{B}_R \backslash \{0\})$ for some $\alpha > 0$ and also in the sense of distribution on B_R.

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