NONDEGENERACY OF NONRADIAL SIGN-CHANGING SOLUTIONS TO THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We prove that the non-radial sign-changing solutions to the nonlinear Schrödinger equation

$$\Delta u - u + |u|^{p-1}u = 0 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N)$$

constructed by Musso, Pacard, and Wei [19] are non-degenerate. This provides the first example of a non-degenerate sign-changing solution to the above nonlinear Schrödinger equation with finite energy.

1. Introduction and statement of main results

In this paper, we consider the nonlinear semilinear elliptic equation

$$\Delta u - u + |u|^{p-1}u = 0 \text{ in } \mathbb{R}^N, \ u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where p satisfies $1 < p < \infty$ if $N = 2$, and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$. Equation (1.1) arises in various models in mathematical physics and biology. In particular, the study of standing waves for the nonlinear Klein-Gordon or Schrödinger equations reduces to (1.1). We refer the reader to the papers of Berestycki and Lions [2], [3], and Bartsch and Willem [4] for further references and motivation.

Denote the set of non-zero finite energy solutions to (1.1) by

$$\Sigma := \{u \in H^1(\mathbb{R}^N) : \Delta u - u + |u|^{p-1}u = 0\}.$$

If $u \in \Sigma$ and $u > 0$, then the classical result of Gidas, Ni, and Nirenberg [11] asserts that u is radially symmetric. Indeed, it is known ([12, 16]) that there exists a unique radially symmetric (in fact radially decreasing) positive solution for

$$\Delta w - w + w^p = 0 \quad \text{in} \quad \mathbb{R}^N,$$

which tends to 0 as $|x|$ tends to ∞. All of the other positive solutions to (1.1) belonging to Σ are translations of w.

Let L_0 be the linearized operator around w, defined by

$$L_0 := \Delta - 1 + p\, w^{p-1}\,. \tag{1.2}$$

Then, the natural invariance of problem (1.1) under the group of isometries in $\mathbb{R}^N$ reduces to the fact that the functions

$$\partial_{x_1} w, \ldots, \partial_{x_N} w \tag{1.3}$$

naturally belong to the kernel of the operator L_0. The solution w is *non-degenerate* in the sense that the L^∞-kernel of the operator L_0 is spanned by the functions given in (1.3). For further details, see [21].

No other example is known of a *non-degenerate* solution to (1.1) in Σ. The purpose of this paper is to provide the first example other than w of a *non-degenerate* solution to (1.1) in Σ.

Concerning the existence of other solutions to (1.1) in Σ, several results are available in the literature. Berestycki and Lions [2], [3] and Struwe [23] have demonstrated the existence of infinitely many radially symmetric sign-changing solutions. The proofs of these results are based on the use of variational

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methods. The existence of non-radial sign-changing solutions was first proved by Bartsch and Willem [4] in dimensions $N = 4$ and $N \geq 6$. In this case, the result is also proved by means of variational methods, and the key idea is to look for solutions that are invariant under the action of $O(2)\times O(N-2)$, in order to recover a certain compactness property. Subsequently, the result was generalized by Lorca and Ubilla [17] to the $N = 5$ dimensional case. Besides the symmetry property of the solutions, the mentioned results do not provide any other qualitative properties of the solutions. A different approach, and a different construction, have been developed in [19] and [1], where new types of non-radial sign-changing finite-energy solutions to (1.1) are constructed, and a detailed description of these solutions is provided.

The main purpose of this paper is to prove that the solutions constructed by Musso, Pacard, and Wei in [19] are rigid, up to transformations of the equation. In other words, these solutions are *non-degenerate*, in the sense of the definition introduced by Duyckaerts, Kenig, and Merle [9].

To explain the meaning of a *non-degenerate* solution for a given $u \in \Sigma$, we recall all of the possible invariance of equation (1.1). We have that equation (1.1) is invariant under the following two transformations:

(1) (translation): If $u \in \Sigma$, then $u(x + a) \in \Sigma\ \forall a \in \mathbb{R}^N$;

(2) (rotation): If $u \in \Sigma$, then $u(Px) \in \Sigma$, where $P \in O_N$, and O_N is the classical orthogonal group.

If $u \in \Sigma$, then by

$$L_u = \Delta - 1 + p|u|^{p-1} \tag{1.4}$$

we denote the linearized operator around u. We define the null space of L_u as

$$\mathcal{Z}_u = \{f \in H^1(\mathbb{R}^N) : L_u f = 0\}. \tag{1.5}$$

If we denote the group of isometries of $H^1(\mathbb{R}^N)$ generated by the previous two transformations by $\mathcal{M}$, then the elements in $\mathcal{Z}_u$ generated by the family of transformations $\mathcal{M}$ define the following vector space:

$$\tilde{\mathcal{Z}}_u = \text{span}\left\{ \begin{array}{l} \partial_{x_j} u,\ 1 \leq j \leq N \\ (x_j\partial_{x_k} - x_k\partial_{x_j})u,\ 1 \leq j < k \leq N \end{array} \right\}. \tag{1.6}$$

A solution u of (1.1) is *non-degenerate* in the sense of Duyckaerts, Kenig, and Merle [9] if

$$\mathcal{Z}_u = \tilde{\mathcal{Z}}_u. \tag{1.7}$$

As we already mentioned, the only non-degenerate example of $u \in \Sigma$ known so far is the positive solution w. In fact, in this case

$$(x_j\partial_{x_k} - x_k\partial_{x_j})w = 0, \quad \forall \quad 1 \leq j < k \leq N,$$

and hence

$$\mathcal{Z}_w = \text{span}\left\{\partial_{x_j} w,\ 1 \leq j \leq N\right\}.$$

The proof of the non-degeneracy of w relies heavily on the radial symmetry of w. For non-radial solutions, the strategy used to prove non-degeneracy in the radial case is no longer applicable. Thus, a new strategy is required for non-radially symmetric solutions.

A similar problem has arisen in the study of non-radial sign-changing finite-energy solutions for the Yamabe type problem

$$\Delta u + |u|^{\frac{4}{N-2}} u = 0 \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),$$

for $N \geq 3$. In [20], the second and third authors of the present paper introduced some new ideas for dealing with non-degeneracy in non-radial sign-changing solutions to the above problem. Indeed, they successfully analyzed the non-degeneracy of some non-radial solutions to the Yamabe problem that were previously constructed in [6]. For other constructions, we refer the reader to [7]. In this paper, we will adopt the idea developed in [20] to analyze the non-degeneracy of the solutions of (1.1) constructed by Musso, Pacard, and Wei [19].

The main result of this paper can be stated as follows.

Theorem 1.1. *There exists a sequence of non-radial sign-changing solutions to (1.1) with arbitrarily large energy, and each solution is* non-degenerate *in the sense of (1.7).*

We believe that the non-degeneracy property of the solutions in Theorem 1.1 can be used to obtain new types of constructions for sign-changing solutions to (1.1), or related problems in bounded domains with Dirichlet or Neumann boundary conditions. We will address this problem in future work.

This paper is organized as follows. In Section 2, we introduce the solutions constructed by Musso, Pacard, and Wei in [19]. In Section 3, we sketch the main steps, and present the proof of Theorem 1.1. Sections 4 to 8 are devoted to the proof of properties required for the proof of Theorem 1.1.

2. Description of the solutions

In this section, we describe the solutions u_ℓ constructed in [19], and recall some properties that will be useful later. To provide the description of the solutions, we introduce some notations. The canonical basis of $\mathbb{R}^N$ will be denoted by

$$\mathbf{e}_1 = (1, 0, \cdots, 0),\ \mathbf{e}_2 = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e}_N = (0, \cdots, 0, 1). \tag{2.1}$$

Let k be an integer, and assume we are given two positive integers m, n and two positive real numbers $\ell, \bar{\ell}$, which are related by

$$2 \sin \frac{\pi}{k} m\ell = (2n-1)\bar{\ell}. \tag{2.2}$$

We shall comment on the possible choices of these parameters later on. Consider the regular polygon in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$ with k edges whose vertices are given by the orbit of the point

$$y_1 = \frac{\bar{\ell}}{2 \sin \frac{\pi}{k}} \mathbf{e}_1 \in \mathbb{R}^N$$

under the action of the group generated by R_k. Here, $R_k \in O(2) \times O(N-2)$ is the rotation through the angle $\frac{2\pi}{k}$ in the (x_1, x_2) plane. By construction, the edges of this polygon have length $\bar{\ell}$. We refer to this polygon as the *inner polygon*. We define the *outer polygon* to be the regular polygon with k edges whose vertices are the orbit of the point

$$y_{m+1} = y_1 + m\ell \mathbf{e}_1$$

under the group generated by R_k. Observe that the distance from y_{m+1} to the origin is given by $m\ell + \frac{\bar{\ell}}{2\sin\frac{\pi}{k}}$, and thanks to (2.2), the edges of the *outer* polygon have length $2n\bar{\ell}$.

By construction, the distance between the points y_1 and y_{m+1} is equal to $m\ell$, and by y_j, for $j = 2, \cdots, m$, we denote the evenly distributed points on the segment between these two points. Namely,

$$y_j = y_1 + (j-1)\ell \mathbf{e}_1 \quad \text{for} \quad j = 2, \cdots, m.$$

As mentioned above, the edges of the *outer* polygon have length $2n\bar{\ell}$, and we evenly distribute points y_j, $j = m+2, \cdots, m+2n$, along this segment. More precisely, if we define

$$\mathbf{t} = -\sin\frac{\pi}{k}\mathbf{e}_1 + \cos\frac{\pi}{k}\mathbf{e}_2 \in \mathbb{R}^N,$$

then the points y_j are given by

$$y_j = y_{m+1} + (j-m-1)\bar{\ell}\mathbf{t} \quad \text{for} \quad j = m+2, \cdots, m+2n.$$

We also denote

$$z_h = y_j \text{ for } h = 1, \cdots, 2n-1, \quad \text{where } h = j-m-1.$$

Let

$$\Pi = \bigcup_{i=0}^{k-1} \big(\{R_k^i y_j \,:\, j = 1, \ldots, m+1\}\big) \cup \big(\{R_k^i z_h \,:\, h = 1, \ldots, 2n-1\}\big) \tag{2.3}$$

Let us introduce the function w to be the unique solution of the following equation:

$$\begin{cases} \Delta u - u + u^p = 0,\ u > 0 \text{ in } \mathbb{R}^N \\ \max_{x \in \mathbb{R}^N} u(x) = u(0), \end{cases}$$

whose existence and properties are obtained in the classical works ([11, 12, 16]).

In [19], the authors constructed solutions of (1.1) that can be viewed as the sum of positive copies of w centered at the points $y_j,\ j=1,\cdots,m+1$, together with their images by rotations $R_k^i = R_k\circ\cdots R_k$ (composition of R_k, i times) for $i=1,\cdots,k-1$, as well as copies of $(-1)^h w$ (hence with the opposite sign) centered at the points $z_h,\ h=1,\cdots,2n-1$ and their images by the rotations R_k^i for $i=1,\cdots,k-1$. More precisely, the solution can be described as follows:

$$u(x)\sim U(x):=\sum_{i=0}^{k-1}\Big(\sum_{j=1}^{m+1}w(x-R_k^iy_j)+\sum_{j=m+2}^{m+2n}(-1)^{j-m-1}w(x-R_k^iy_j)\Big). \tag{2.4}$$

These solutions admit the following invariance:

$$u(x)=u(Rx),\ \text{ for } R\in\{I_2\}\times O(N-2), \tag{2.5}$$

and

$$u(R_kx)=u(x)\quad \text{ and }\quad u(\Gamma x)=u(x), \tag{2.6}$$

where $\Gamma\in O(2)\times I(N-2)$ is the symmetry with respect to the hyperplane $x_2=0$.

The numbers $m,n,\ell,\bar\ell$ are related by equation (2.2) and by the following equation:

$$\Psi(\ell)=(2\sin\frac{\pi}{k})\Psi(\bar\ell), \tag{2.7}$$

where $\Psi(s)$ is the so-called interaction function which is defined by

$$\Psi(s)=-\int w(x-s\mathbf{e})div(w^p(x)\mathbf{e})dx, \tag{2.8}$$

and $\mathbf{e}\in\mathbb{R}^N$ is any unit vector. The definition of Ψ is independent of $\mathbf{e}$.

The constraint (2.2) on the parameters $m,n,\ \ell,\bar\ell$ is easy to understand: it is to make sure that an outer polygon is formed. The second constraint (2.7) is not so easy to see. As mentioned in [19], the relation between ℓ and $\bar\ell$ can be understood as a balancing condition, which is a consequence of a conservation law for solutions of (1.1). Alternatively, it can be understood as a condition that ensures that the approximate solution U is close enough to a genuine solution u of (1.1).

The main theorem in [19] is as follows:

Theorem A. Let k be an integer number with $k\ge 7$ and let $\tau>0$ be a fixed real number. Then, there exists a positive number $\ell_0>0$ such that for all $\ell>\ell_0$, if $\bar\ell$ is the solution of (2.7), and m,n are positive integers satisfying (2.2), and

$$m\le \ell^\tau, \tag{2.9}$$

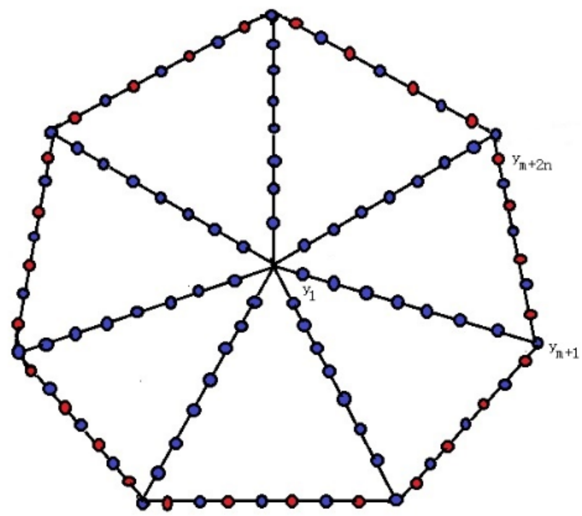
then (1.1) admits a sign-changing solution u_ℓ that satisfies the symmetry conditions given in (2.5) and (2.6). Moreover,

$$u_\ell(x)=U(x)+\phi, \tag{2.10}$$

where U is defined in (2.4) and $\phi=o(1)\to 0$ as $\ell\to\infty$. The energy of u is finite and can be expanded as

$$\mathcal{E}(u_\ell)=(2n+m)\,k\,\mathcal{E}(w)+o(1),$$

where $o(1)\to 0$ as $\ell\to\infty$.



An example of a configuration with $k = 7$ edges, $m = 7$ interior points on any radius, $n = 4$ for $2n-1$ interior points on any edge.

As mentioned in Remark 1.2, [19], (2.9) is a technical condition. Observe that, once ℓ is fixed large enough, from (2.7) we see that $\bar{\ell}$ is a function of ℓ which can be expanded as

$$\bar{\ell} = \ell + \ln(2 \sin \frac{\pi}{k}) + O(\frac{1}{\ell}),$$

since $-(\log \Psi)'(s) = 1 + \frac{N-1}{2s} + O(\frac{1}{s^2})$. Inserting this information in (2.2), one gets that

$$\frac{2n-1}{m} = 2 \sin \frac{\pi}{k} \, \left(1 - \ln 2(\sin \frac{\pi}{k})\ell^{-1} + O(\ell^{-2})\right), \quad \text{as} \quad \ell \to \infty.$$

The authors in [19] provide examples of possible choices for sequences of m and n satisfying the above expansion. For instance: for any integer m one can choose an integer n so that

$$1 \leq 2n - 1 - 2 \sin \frac{\pi}{k} m < 3.$$

Then, if m is sufficiently large, there exists a unique $\ell > \ell_0$ so that (2.7) and (2.2) are satisfied, and $c_1 \ell \leq m \leq c_2 \ell$, for some constants c_1, c_2. Thus Theorem A. guarantees the existence of a solution of the form (2.4) for any such integer m.

Equation (1.1) can be rewritten in terms of the function ϕ in (2.10) as

$$\Delta \phi - \phi + p|U|^{p-1}\phi + E + N(\phi) = 0, \tag{2.11}$$

where

$$E = \Delta U - U + |U|^{p-1}U \tag{2.12}$$

and

$$N(\phi) = |U + \phi|^{p-1}(U + \phi) - |U|^{p-1}U - p|U|^{p-1}\phi. \tag{2.13}$$

One has precise control over the size of the error function E when measured in the following weighted norm. Let us fix a number $-1 < \eta < 0$, and define the weighted norm

$$\|h\|_* = \sup_{x \in \mathbb{R}^N} |(\sum_{y \in \Pi} e^{\eta|x-y|})^{-1} h(x)|, \tag{2.14}$$

where

$$\Pi = \cup_{i=0}^{k-1}\{R_k^i y_j : j = 1, \cdots, m+2n\} \tag{2.15}$$

is the set of the concentration points. In [19], it is proved that there exist $\ell_0 > 0$ and $\xi > 0$ such that for $\ell > \ell_0$, ϕ satisfies the following estimate (Proposition 4.1 in [19]):

$$\|\phi\|_* \le Ce^{-\frac{1+\xi}{2}\ell}, \tag{2.16}$$

which gives a considerably smaller bound than U.

We now define the following functions:

$$\pi_\alpha(x) = \frac{\partial}{\partial x_\alpha}\phi(x) \text{ for } \alpha = 1, \cdots, N. \tag{2.17}$$

Then, the function π_α can be described as follows.

Proposition 2.1. *The function π_α satisfies the following estimates:*

$$\|\pi_\alpha\|_* \le Ce^{-\frac{1+\xi}{2}\ell}, \tag{2.18}$$

for some positive constants C and ξ that are independent of ℓ.

Recall that problem (1.1) is invariant under the two transformations mentioned in Section 1, translation and rotation. This invariance will be reflected in the element of the kernel of the linearized operator

$$L(\psi) := \Delta\psi - \psi + p|u_\ell|^{p-1}\psi, \tag{2.19}$$

which is the linearized equation associated to (1.1) around u_ℓ.

From this point on, we will drop the ℓ in u_ℓ, for simplicity. Let us now introduce the following $3N-3$ functions:

$$z_\alpha(x) = \frac{\partial}{\partial x_\alpha}u(x), \text{ for } \alpha = 1, \cdots, N, \tag{2.20}$$

and

$$z_{N+1}(x) = x_1\frac{\partial}{\partial x_2}u(x) - x_2\frac{\partial}{\partial x_1}u(x). \tag{2.21}$$

Observe that $z_{N+1} = \frac{\partial}{\partial\theta}[u(R_\theta x)]|_{\theta=0}$, where R_θ is the rotation in the x_1x_2 plane by the angle θ.

Furthermore, for $\alpha = 3, \cdots, N$,

$$z_{N+\alpha-1}(x) = x_1z_\alpha - x_\alpha z_1,\ z_{2N+\alpha-3}(x) = x_2z_\alpha - x_\alpha z_2. \tag{2.22}$$

Observe that the functions defined in (2.20) are related to the invariance of (1.1) under translation, while the functions defined in (2.21) and (2.22) are related to the invariance of (1.1) under rotations in the (x_1, x_2), (x_1, x_α), and (x_2, x_α) planes, respectively.

The invariance of problem (1.1) under translation and rotation implies that the set $\tilde{Z}_u$ (as introduced in (1.6)) associated to the linear operator L introduced in (2.19) has dimension at least $3N-3$, because

$$L(z_\alpha) = 0,\ \alpha = 1, \cdots, 3N-3. \tag{2.23}$$

We will show that these functions are the only bounded elements of the kernel of the operator L. In other words, the sign-changing non-radial solutions (2.10) to problem (1.1) constructed in [19] are non-degenerate in the sense of [9].

3. Scheme of the proof

In this section we describe the main ingredients which constitute the proof of our result.

Assume that φ is a bounded function satisfying

$$L(\varphi) = 0, \tag{3.1}$$

where L is the linear operator defined by (1.4). We write our function φ as

$$\varphi(x) = \sum_{\alpha=1}^{3N-3} a_\alpha z_\alpha(x) + \tilde{\varphi}(x), \tag{3.2}$$

where the functions $z_\alpha(x)$ are defined by (2.20), (2.21), and (2.22) respectively, while the constants a_α are chosen such that

$$\int z_\alpha \tilde{\varphi} = 0, \quad \alpha = 1, \cdots, 3N-3. \tag{3.3}$$

Observe that $L(\tilde{\varphi}) = 0$. Our aim is to show that **under the conditions (3.1)-(3.3), if $\tilde{\varphi}$ is bounded, then $\tilde{\varphi} = 0$.**

3.1. **Introduction of the approximate kernels and some notation.** In order to explain our idea, we first introduce some functions. These functions $\tilde{Z}^i_{j,\alpha}$ correspond to the approximate kernels around each spike.

For $j = 1, \cdots, m+1$, $i = 0, \cdots, k-1$,

$$\tilde{Z}^i_{j,1} = \mathbf{R}_i \cdot \nabla w(x - R^i_k y_j), \qquad \tilde{Z}^i_{j,2} = \mathbf{R}^\perp_i \cdot \nabla w(x - R^i_k y_j),$$

and, for $\alpha = 3, \cdots, N$,

$$\tilde{Z}^i_{j,\alpha} = \frac{\partial}{\partial x_\alpha} w(x - R^i_k y_j).$$

Moreover, for $j = m+2, \cdots, m+2n$, $i = 0, \cdots, k-1$, we define

$$\tilde{Z}^i_{j,1} = (-1)^{j-m-1} \mathbf{t}_i \cdot \nabla w(x - R^i_k y_j), \qquad \tilde{Z}^i_{j,2} = (-1)^{j-m-1} \mathbf{n}_i \cdot \nabla w(x - R^i_k y_j),$$

and, for $\alpha = 3, \cdots, N$,

$$\tilde{Z}^i_{j,\alpha} = (-1)^{j-m-1} \frac{\partial}{\partial x_\alpha} w(x - R^i_k y_j).$$

In the above formulas, we denoted $\theta_i = \frac{2\pi i}{k}$ and

$$\mathbf{R}_i = (\cos\theta_i, \sin\theta_i, 0), \qquad \mathbf{R}^\perp_i = (\sin\theta_i, -\cos\theta_i, 0),$$

$$\mathbf{t}_i = (-\sin(\theta_i + \frac{\pi}{k}), \cos(\theta_i + \frac{\pi}{k}), 0), \qquad \mathbf{n}_i = (\cos(\theta_i + \frac{\pi}{k}), \sin(\theta_i + \frac{\pi}{k}), 0).$$

Recall that the solutions constructed in [19] take the form $u = U + \phi$ given in (2.4)-(2.10), and recall further the definition of π_α in (2.17). From this, one can obtain a more precise expression for the real kernels z_α mentioned at the end of Section 2. Indeed

$$\begin{aligned}
z_1(x) \quad &= \frac{\partial u}{\partial x_1} = \pi_1 + \frac{\partial U}{\partial x_1} \\
&= \pi_1 + \sum_{i=0}^{k-1} \Big(\sum_{j=1}^{m+1} (\cos\theta_i \tilde{Z}^i_{j,1} + \sin\theta_i \tilde{Z}^i_{j,2}) \\
&\quad - \sum_{j=m+2}^{2n+m} \big(\sin(\theta_i + \frac{\pi}{k}) \tilde{Z}^i_{j,1} - \cos(\theta_i + \frac{\pi}{k}) \tilde{Z}^i_{j,2} \big) \Big),
\end{aligned}$$

$$\begin{aligned}
z_2(x) \quad &= \frac{\partial u}{\partial x_2} = \pi_2 + \frac{\partial U}{\partial x_2} \\
&= \pi_2 + \sum_{i=0}^{k-1} \Big(\sum_{j=1}^{m+1} (\sin\theta_i \tilde{Z}^i_{j,1} - \cos\theta_i \tilde{Z}^i_{j,2}) \\
&\quad + \sum_{j=m+2}^{2n+m} \big(\cos(\theta_i + \frac{\pi}{k}) \tilde{Z}^i_{j,1} + \sin(\theta_i + \frac{\pi}{k}) \tilde{Z}^i_{j,2} \big) \Big),
\end{aligned}$$

and, for $\alpha = 3, \cdots, N$,

$$z_\alpha = \frac{\partial u}{\partial x_\alpha} = \pi_\alpha + \sum_{i=0}^{k-1} (\sum_{j=1}^{m+2n} \tilde{Z}^i_{j,\alpha}).$$

Furthermore,

$$
\begin{aligned}
z_{N+1} &= x_1z_2 - x_2z_1 = x_1\pi_2 - x_2\pi_1 + x_1\frac{\partial U}{\partial x_2} - x_2\frac{\partial U}{\partial x_1}\\
&= x_1\pi_2 - x_2\pi_1 + \sum_{i=0}^{k-1}\Big(\sum_{j=1}^{m+1}|y_j|\big(\cos\theta_i\frac{\partial}{\partial x_2} - \sin\theta_i\frac{\partial}{\partial x_1}\big)w(x-R_k^iy_j)\\
&+ \sum_{j=m+2}^{2n+m}\big(R_k^iy_j\cdot\mathbf{n}_i\tilde{Z}_{j,1}^i - R_k^iy_j\cdot\mathbf{t}_i\tilde{Z}_{j,2}^i\big)\Big)
\end{aligned}
$$

and for $\alpha = 3,\cdots,N$,

$$
\begin{aligned}
z_{N+\alpha-1} &= x_1z_\alpha - x_\alpha z_1 = x_1\pi_\alpha - x_\alpha\pi_1 + x_1\frac{\partial U}{\partial x_\alpha} - x_\alpha\frac{\partial U}{\partial x_1}\\
&= x_1\pi_\alpha - x_\alpha\pi_1 + \sum_{i=0}^{k-1}\Big(\sum_{j=1}^{m+1}|y_j|\cos\theta_i\tilde{Z}_{j,\alpha}^i\\
&+ \sum_{j=m+2}^{2n+m}\big(R_k^iy_j\cdot\mathbf{n}_i\cos(\theta_i+\frac{\pi}{k}) - R_k^iy_j\cdot\mathbf{t}_i\sin(\theta_i+\frac{\pi}{k})\big)\tilde{Z}_{j,\alpha}^i\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
z_{2N+\alpha-3} &= x_2z_\alpha - x_\alpha z_2\\
&= x_2\pi_\alpha - x_\alpha\pi_2 + x_2\frac{\partial U}{\partial x_\alpha} - x_\alpha\frac{\partial U}{\partial x_2}\\
&= x_2\pi_\alpha - x_\alpha\pi_2 + \sum_{i=0}^{k-1}\Big(\sum_{j=1}^{m+1}|y_j|\sin\theta_i\tilde{Z}_{j,\alpha}^i\\
&+ \sum_{j=m+2}^{2n+m}\big(R_k^iy_j\cdot\mathbf{n}_i\sin(\theta_i+\frac{\pi}{k}) + R_k^iy_j\cdot\mathbf{t}_i\cos(\theta_i+\frac{\pi}{k})\big)\tilde{Z}_{j,\alpha}^i\Big).
\end{aligned}
$$

Let us further define the following functions. For $i = 0,\cdots,k-1$,

$$
Z_{1,1}^i = \cos\theta_i\Big(\frac{\partial w(x-R_k^iy_1)}{\partial x_1} + \frac{\pi_1}{k}\Big) + \sin\theta_i\Big(\frac{\partial w(x-R_k^iy_1)}{x_2} + \frac{\pi_2}{k}\Big), \tag{3.4}
$$

$$
Z_{1,2}^i = \sin\theta_i\Big(\frac{\partial w(x-R_k^iy_1)}{\partial x_1} + \frac{\pi_1}{k}\Big) - \cos\theta_i\Big(\frac{\partial w(x-R_k^iy_1)}{x_2} + \frac{\pi_2}{k}\Big), \tag{3.5}
$$

and, for $\alpha = 3,\cdots,N$,

$$
Z_{1,\alpha}^i = \frac{\partial w(x-R_k^iy_1)}{x_\alpha} + \frac{\pi_\alpha}{k}. \tag{3.6}
$$

Moreover, we define the following functions:

$$
Z_{j,\alpha}^i = \tilde{Z}_{j,\alpha}^i \text{ for } i = 0,\cdots,k-1, \quad j = 2,\cdots,2n+m, \quad \alpha = 1,\cdots,N. \tag{3.7}
$$

In the following, we will always deal with the kernels in vector form. These column vectors simply represent rearrangements of the approximate kernels $Z_{j,\alpha}^i$:

$$
Z_{\mathbf{v},\alpha} = (Z_{1,\alpha}^0,\cdots,Z_{1,\alpha}^{k-1},Z_{m+1,\alpha}^0,\cdots,Z_{m+1,\alpha}^{k-1})^t \in \mathbb{R}^{2k}.
$$

This contains the kernels around the vertices of the inner and outer polygons. Furthermore, define

$$
Z_{Y_1,\alpha}^i = (Z_{2,\alpha}^i,\cdots,Z_{m,\alpha}^i)^t \in \mathbb{R}^{m-1},\ Z_{Y_2,\alpha}^i = (Z_{m+2,\alpha}^i,\cdots,Z_{2n+m,\alpha}^i)^t \in \mathbb{R}^{2n-1},
$$

which correspond to the spikes on the line joining the inner polygon vertex $R_k^iy_1$ and the outer polygon vertex $R_k^iy_{m+1}$, and the spikes on the edge joining $R_k^iy_{m+1}$ and $R_k^{i+1}y_{m+1}$ of the outer polygon, respectively. Then, we define

$$\mathbf{Z}_\alpha = \begin{pmatrix} Z_{\mathbf{v},\alpha} \\ Z^0_{Y_1,\alpha} \\ \vdots \\ Z^{k-1}_{Y_1,\alpha} \\ Z^0_{Y_2,\alpha} \\ \vdots \\ Z^{k-1}_{Y_2,\alpha} \end{pmatrix} \in R^{k\times(2n+m)},\ \mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 \\ \vdots \\ \mathbf{Z}_N \end{pmatrix} \in R^{k\times N\times(2n+m)}.$$

3.2. **Reduction of the main problem.** As explained in the beginning of this section, our aim is to show that the function $\tilde{\varphi}$ defined by (3.1)-(3.3) is identically zero, $\tilde{\varphi} \equiv 0$.

With the notations introduced in Section 3.1 in mind, we write our function $\tilde{\varphi}$ as

$$\tilde{\varphi} = \sum_{\alpha=1}^{N} \mathbf{c}_\alpha \cdot \mathbf{Z}_\alpha + \varphi^\perp(x),$$

where $\mathbf{c}_\alpha = \begin{pmatrix} c_{\mathbf{v},\alpha} \\ c^0_{Y_1,\alpha} \\ \vdots \\ c^{k-1}_{Y_1,\alpha} \\ c^0_{Y_2,\alpha} \\ \vdots \\ c^{k-1}_{Y_2,\alpha} \end{pmatrix} = \begin{pmatrix} c_{1,\alpha} \\ \vdots \\ c_{(m+2n)\times k,\alpha} \end{pmatrix}$, $\alpha = 1, \cdots, N$, are N vectors in $\mathbb{R}^{(m+2n)\times k}$ defined such that

$$\int Z^i_{j,\alpha}\varphi^\perp = 0 \text{ for all } \alpha = 1, \cdots, N,\ i = 0, \cdots, k-1,\ j = 1, \cdots, m+2n. \tag{3.8}$$

Observe that, **if we prove that**

$$\mathbf{c}_\alpha = 0 \textbf{ for all } \alpha,\ \textbf{and } \varphi^\perp = 0$$

then we have that $\tilde{\varphi} = 0$. Hence, our aim is to show that **all vectors** $\mathbf{c}_\alpha$ **and** $\varphi^\perp$ **are zero in the above decomposition.** This will be a consequence of the following three facts.

Fact 1: Since $L(\tilde{\varphi}) = 0$, we have that

$$L(\varphi^\perp) = -\sum_{\alpha=1}^{N} \mathbf{c}_\alpha \cdot L(\mathbf{Z}_\alpha). \tag{3.9}$$

Our first result shows that $\varphi^\perp$ can be controlled by $\mathbf{c}_\alpha$, and we have the following a priori estimate of $\varphi^\perp$:

$$\|\varphi^\perp\|_* \leq Ce^{-\frac{1+\xi}{2}\ell} \sum_{\alpha=1}^{N} \|\mathbf{c}_\alpha\|. \tag{3.10}$$

The proof is deferred to Section 5.

Fact 2. The orthogonality condition (3.3) takes the form

$$\sum_{\alpha=1}^{N} \mathbf{c}_\alpha \cdot \int \mathbf{Z}_\alpha z_\beta = -\int \varphi^\perp z_\beta, \tag{3.11}$$

for $\beta = 1, \cdots, 3N-3$.

Our second result implies that the above orthogonality condition can be reduced to $3N-3$ linear conditions on the vectors $\mathbf{c}_\alpha$. Let us introduce the following notations:

$$\mathbf{cos}_k = \begin{pmatrix} \cos\theta_0 \\ \vdots \\ \cos\theta_{k-1} \end{pmatrix} \in \mathbb{R}^k, \ \mathbf{sin}_k = \begin{pmatrix} \sin\theta_0 \\ \vdots \\ \sin\theta_{k-1} \end{pmatrix} \in \mathbb{R}^k,$$

are two k-dimensional vectors,

$$\mathbf{cos}(\theta_{\mathbf{i}}) = \begin{pmatrix} \cos\theta_i \\ \vdots \\ \cos\theta_i \end{pmatrix} \in \mathbb{R}^{m-1}, \quad \mathbf{sin}(\theta_{\mathbf{i}}) = \begin{pmatrix} \sin\theta_i \\ \vdots \\ \sin\theta_i \end{pmatrix} \in \mathbb{R}^{m-1},$$

are two $(m-1)$-dimensional vectors, and

$$\mathbf{cos}(\theta_{\mathbf{i}} + \frac{\pi}{\mathbf{k}}) = \begin{pmatrix} \cos(\theta_i + \frac{\pi}{k}) \\ \vdots \\ \cos(\theta_i + \frac{\pi}{k}) \end{pmatrix} \in \mathbb{R}^{2n-1}, \quad \mathbf{sin}(\theta_{\mathbf{i}} + \frac{\pi}{\mathbf{k}}) = \begin{pmatrix} \sin(\theta_i + \frac{\pi}{k}) \\ \vdots \\ \sin(\theta_i + \frac{\pi}{k}) \end{pmatrix} \in \mathbb{R}^{2n-1}.$$

are two $(2n-1)$-dimensional vectors. Furthermore,

$$\mathbf{d}_l = \begin{pmatrix} d \\ \vdots \\ d \end{pmatrix} \in \mathbb{R}^l, \ |\mathbf{y_i}|_k = \begin{pmatrix} |y_i| \\ \vdots \\ |y_i| \end{pmatrix} \in \mathbb{R}^k$$

are constant vectors, where $|y_i|$ denotes the distance from the point y_i to the origin.

For any unit vector $\mathbf{e} \in \mathbb{R}^N$, we denote

$$\mathbf{R^i_k y_j} \cdot \mathbf{e} = \begin{pmatrix} R^i_k y_2 \cdot \mathbf{e} \\ \vdots \\ R^i_k y_m \cdot \mathbf{e} \end{pmatrix} \in \mathbb{R}^{m-1}, \ \mathbf{R^i_k z_h} \cdot \mathbf{e} = \begin{pmatrix} R^i_k z_1 \cdot \mathbf{e} \\ \vdots \\ R^i_k z_{2n-1} \cdot \mathbf{e} \end{pmatrix} \in \mathbb{R}^{2n-1}$$

We have the validity of the following

Proposition 3.1. *The system (3.11) reduces to the following* $3N-3$ *linear conditions on the vectors* $\mathbf{c}_\alpha$*:*

$$\mathbf{c}_1 \cdot \begin{pmatrix} \mathbf{cos}_k \\ \mathbf{cos}_k \\ \mathbf{cos}(\theta_\mathbf{0}) \\ \vdots \\ \mathbf{cos}(\theta_{\mathbf{k-1}}) \\ -\mathbf{sin}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \\ \vdots \\ -\mathbf{sin}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \end{pmatrix} + \mathbf{c}_2 \cdot \begin{pmatrix} \mathbf{sin}_k \\ \mathbf{sin}_k \\ \mathbf{sin}(\theta_\mathbf{0}) \\ \vdots \\ \mathbf{sin}(\theta_{\mathbf{k-1}}) \\ \mathbf{cos}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \\ \vdots \\ \mathbf{cos}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \end{pmatrix} = f_1 + O(e^{-\frac{(1+\xi)\ell}{2}})\mathcal{L}_1 \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{pmatrix} \tag{3.12}$$

$$\mathbf{c}_1 \cdot \begin{pmatrix} \mathbf{sin}_k \\ \mathbf{sin}_k \\ \mathbf{sin}(\theta_\mathbf{0}) \\ \vdots \\ \mathbf{sin}(\theta_{\mathbf{k-1}}) \\ \mathbf{cos}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \\ \vdots \\ \mathbf{cos}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \end{pmatrix} + \mathbf{c}_2 \cdot \begin{pmatrix} -\mathbf{cos}_k \\ -\mathbf{cos}_k \\ -\mathbf{cos}(\theta_\mathbf{0}) \\ \vdots \\ -\mathbf{cos}(\theta_{\mathbf{k-1}}) \\ \mathbf{sin}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \\ \vdots \\ \mathbf{sin}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \end{pmatrix} = f_2 + O(e^{-\frac{(1+\xi)\ell}{2}})\mathcal{L}_2 \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{pmatrix} \tag{3.13}$$

$$\mathbf{c}_\alpha \cdot \mathbf{1}_{(2n+m)\times k} = f_\alpha + O(e^{-\frac{(1+\xi)\ell}{2}})\mathcal{L}_\alpha \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{pmatrix} \tag{3.14}$$

$$\mathbf{c}_1 \cdot \begin{pmatrix} \mathbf{0}_k \\ \mathbf{0}_k \\ \mathbf{R_k^0 y_j \cdot R_0^\perp} \\ \vdots \\ \mathbf{R_k^{k-1} y_j \cdot R_{k-1}^\perp} \\ \mathbf{R_k^0 z_h \cdot n_0} \\ \vdots \\ \mathbf{R_k^{k-1} z_h \cdot n_{k-1}} \end{pmatrix} - \mathbf{c}_2 \cdot \begin{pmatrix} |\mathbf{y_1}|_k \\ |\mathbf{y_{m+1}}|_k \\ \mathbf{R_k^0 y_j \cdot R_0} \\ \vdots \\ \mathbf{R_k^{k-1} y_j \cdot R_{k-1}} \\ \mathbf{R_k^0 z_h \cdot t_0} \\ \vdots \\ \mathbf{R_k^{k-1} z_h \cdot t_{k-1}} \end{pmatrix} = f_{N+1} + O(e^{-\frac{(1+\xi)\ell}{2}})\mathcal{L}_{N+1} \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{pmatrix} \tag{3.15}$$

and for $\alpha = 3, \cdots, N$,

$$\mathbf{c}_\alpha \cdot \begin{pmatrix} |y_1|\mathbf{cos}_k \\ |y_{m+1}|\mathbf{cos}_k \\ \mathbf{R_k^0 y_j \cdot e_1} \\ \vdots \\ \mathbf{R_k^{k-1} y_j \cdot e_1} \\ \mathbf{R_k^0 z_h \cdot e_1} \\ \vdots \\ \mathbf{R_k^{k-1} z_h \cdot e_1} \end{pmatrix} = f_{N+\alpha-1} + O(e^{-\frac{(1+\xi)\ell}{2}})\mathcal{L}_{N+\alpha-1} \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{pmatrix} \tag{3.16}$$

$$\mathbf{c}_\alpha \cdot \begin{pmatrix} |y_1|\mathbf{sin}_k \\ |y_{m+1}|\mathbf{sin}_k \\ \mathbf{R_k^0 y_j \cdot e_2} \\ \vdots \\ \mathbf{R_k^{k-1} y_j \cdot e_2} \\ \mathbf{R_k^0 z_h \cdot e_2} \\ \vdots \\ \mathbf{R_k^{k-1} z_h \cdot e_2} \end{pmatrix} = f_{2N+\alpha-3} + O(e^{-\frac{(1+\xi)\ell}{2}})\mathcal{L}_{2N+\alpha-3} \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{pmatrix}, \tag{3.17}$$

for $\alpha = 3, \cdots, N$. *In the above expansions,* $\begin{pmatrix} f_1 \\ \vdots \\ f_{3N-3} \end{pmatrix}$ *is a fixed vector with*

$$\| \begin{pmatrix} f_1 \\ \vdots \\ f_{3N-3} \end{pmatrix} \| \le \ell^\tau \|\varphi^\perp\|_*$$

for some positive constant τ *that is independent of* ℓ. *Here,* $\mathcal{L}_i : \mathbb{R}^{(2n+m)\times k} \to \mathbb{R}$ *are linear functions whose coefficients are constants uniformly bounded as* $\ell \to \infty$.

The proof is deferred to Section 6.

Fact 3: Let us now multiply (3.9) by $Z^i_{j,\alpha}$, for $i = 0, \cdots, k-1$, $j = 1, \cdots, m+2n$, and $\alpha = 1, \cdots, N$. After integrating in $\mathbb{R}^N$, we obtain a linear system of $(2n+m)\times k\times N$ equations in the $(2n+m)\times k\times N$

coefficients $\mathbf{c}$ of the form

$$M\begin{pmatrix}\mathbf{c}_1\\ \vdots\\ \mathbf{c}_N\end{pmatrix}=-\begin{pmatrix}\mathbf{r}_1\\ \vdots\\ \mathbf{r}_N\end{pmatrix}\text{ with }\mathbf{r}_\alpha=\begin{pmatrix}\int L(\varphi^\perp)Z_{\mathbf{v},\alpha}\\ \int L(\varphi^\perp)Z^0_{Y_1,\alpha}\\ \vdots\\ \int L(\varphi^\perp)Z^{k-1}_{Y_1,\alpha}\\ \int L(\varphi^\perp)Z^0_{Y_2,\alpha}\\ \vdots\\ \int L(\varphi^\perp)Z^{k-1}_{Y_2,\alpha}\end{pmatrix}, \tag{3.18}$$

where $M=(\int L(Z^t_{i,\alpha})Z^s_{j,\beta}dx)$ is a square matrix of dimension $[(2n+m)\times k\times N]^2$. More detailed information and an analysis of the matrix M is provided in Section 4.

Our third result concerns the solvability of the above matrix equation. We can show the validity of the following statement:

Proposition 3.2. *There exists $\ell_0>0$ and C such that for $\ell>\ell_0$, system (3.18) is solvable. Furthermore, the general solution is*

$$\begin{pmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{pmatrix}=\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}$$

$$+s_1\begin{pmatrix}\mathbf{cos}_k\\ \mathbf{cos}_k\\ \mathbf{cos}(\theta_\mathbf{0})\\ \vdots\\ \mathbf{cos}(\theta_\mathbf{k-1})\\ -\mathbf{sin}(\theta_\mathbf{0}+\frac{\pi}{\mathbf{k}})\\ \vdots\\ -\mathbf{sin}(\theta_\mathbf{0}+\frac{\pi}{\mathbf{k}})\\ \mathbf{sin}_k\\ \mathbf{sin}_k\\ \mathbf{sin}(\theta_\mathbf{0})\\ \vdots\\ \mathbf{sin}(\theta_\mathbf{k-1})\\ \mathbf{cos}(\theta_\mathbf{0}+\frac{\pi}{\mathbf{k}})\\ \vdots\\ \mathbf{cos}(\theta_\mathbf{0}+\frac{\pi}{\mathbf{k}})\end{pmatrix}+s_2\begin{pmatrix}\mathbf{sin}_k\\ \mathbf{sin}_k\\ \mathbf{sin}(\theta_\mathbf{0})\\ \vdots\\ \mathbf{sin}(\theta_\mathbf{k-1})\\ \mathbf{cos}(\theta_\mathbf{0}+\frac{\pi}{\mathbf{k}})\\ \vdots\\ \mathbf{cos}(\theta_\mathbf{0}+\frac{\pi}{\mathbf{k}})\\ -\mathbf{cos}_k\\ -\mathbf{cos}_k\\ -\mathbf{cos}(\theta_\mathbf{0})\\ \vdots\\ -\mathbf{cos}(\theta_\mathbf{k-1})\\ \mathbf{sin}(\theta_\mathbf{0}+\frac{\pi}{\mathbf{k}})\\ \vdots\\ \mathbf{sin}(\theta_\mathbf{0}+\frac{\pi}{\mathbf{k}})\end{pmatrix}+s_3\begin{pmatrix}\mathbf{0}_k\\ \mathbf{0}_k\\ \mathbf{R^0_k y_j\cdot R_0^\perp}\\ \vdots\\ \mathbf{R^{k-1}_k y_j\cdot R_{k-1}^\perp}\\ \mathbf{R^0_k z_h\cdot n_0}\\ \vdots\\ \mathbf{R^{k-1}_k z_h\cdot n_{k-1}}\\ -|\mathbf{y_1}|_k\\ -|\mathbf{y_{m+1}}|_k\\ -\mathbf{R^0_k y_j\cdot R_0}\\ \vdots\\ -\mathbf{R^{k-1}_k y_j\cdot R_{k-1}}\\ -\mathbf{R^0_k z_h\cdot t_0}\\ \vdots\\ -\mathbf{R^{k-1}_k z_h\cdot t_{k-1}}\end{pmatrix}$$

$$:=\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{pmatrix}+s_1\mathbf{w}_1+s_2\mathbf{w}_1+s_3\mathbf{w}_3$$

and

$$\begin{aligned}\mathbf{c}_\alpha\quad &=\mathbf{v}_\alpha+s_{\alpha1}\mathbf{1}_{(2n+m)\times k}+s_{\alpha2}\begin{pmatrix}|y_1|\mathbf{cos}_k\\ |y_{m+1}|\mathbf{cos}_k\\ \mathbf{R^0_k y_j\cdot e_1}\\ \vdots\\ \mathbf{R^{k-1}_k y_j\cdot e_1}\\ \mathbf{R^0_k z_h\cdot e_1}\\ \vdots\\ \mathbf{R^{k-1}_k z_h\cdot e_1}\end{pmatrix}+s_{\alpha3}\begin{pmatrix}|y_1|\mathbf{sin}_k\\ |y_{m+1|}\mathbf{sin}_k\\ \mathbf{R^0_k y_j\cdot e_2}\\ \vdots\\ \mathbf{R^{k-1}_k y_j\cdot e_2}\\ \mathbf{R^0_k z_h\cdot e_2}\\ \vdots\\ \mathbf{R^{k-1}_k z_h\cdot e_2}\end{pmatrix}\\ &:=\mathbf{v}_\alpha+s_{\alpha1}\mathbf{w}_4+s_{\alpha2}\mathbf{w}_5+s_{\alpha3}\mathbf{w}_6\end{aligned}$$

for any $s_1, s_2, s_3, s_{\alpha 1}, s_{\alpha 2}, s_{\alpha 3} \in \mathbb{R}$, where the vectors $\mathbf{v}_\alpha$ are fixed, and satisfy

$$\|\mathbf{v}_\alpha\| \leq C\ell^\tau e^{\frac{1-\xi}{2}\ell}\|\varphi^\perp\|_* \tag{3.19}$$

for some $\tau, \xi > 0$.

The proof is deferred to Section 7.

3.3. **Final argument and proof of Theorem 1.1.** Let $\begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{pmatrix}$ denote the solutions to (3.18) predicted by Proposition 3.2, given explicitly by

$$\begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} + s_1\mathbf{w}_1 + s_2\mathbf{w}_2 + s_3\mathbf{w}_3$$

and

$$\mathbf{c}_\alpha = \mathbf{v}_\alpha + s_{\alpha 1}\mathbf{w}_4 + s_{\alpha 2}\mathbf{w}_5 + s_{\alpha 3}\mathbf{w}_6, \ \alpha = 3, \cdots, N.$$

Replace the above expressions for $\mathbf{c}_\alpha$, $\alpha = 1, 2, 3, \ldots, N$ into (3.12)–(3.17). This leads to a system of $(3N-3)$ non linear conditions on the $(3N-3)$ coefficients s_j, $s_{\alpha j}$ for $j = 1, 2, 3$, $\alpha = 3, \ldots, N$. Taking advantage of the explicit form of the vectors $\mathbf{w}_i$, $i = 1, \ldots, 6$, one can show that there exists a unique

$$(s_1^*, \cdots, s_3^*, s_{31}^*, \cdots, s_{N3}^*) \in \mathbb{R}^{3N-3}$$

for which the above solutions satisfy all $3N-3$ conditions of Proposition 3.1, which furthermore satisfy

$$\|(s_1^*, \cdots, s_3^*, s_{31}^*, \cdots, s_{N3}^*)\| \leq C\ell^\tau\|\varphi^\perp\|_*.$$

Hence, there exists a unique solution $\begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{pmatrix}$ to system (3.18), satisfying conditions (3.12)–(3.17) and the estimate

$$\sum_{\alpha=1}^N \|\mathbf{c}_\alpha\| \leq C\ell^\tau e^{\frac{1-\xi}{2}\ell}\|\varphi^\perp\|_*.$$

On the other hand, from (3.10) in **Fact 1**, we conclude that

$$\|\varphi^\perp\|_* \leq Ce^{-\frac{1+\xi}{2}\ell}\sum_{\alpha=1}^N \|\mathbf{c}_\alpha\|.$$

Thus, by combining the above two estimates we conclude that

$$c_{j,\alpha}^i = 0, \ \varphi^\perp = 0,$$

which implies that for $\tilde{\varphi}$ defined by (3.1)-(3.3), it holds that $\tilde{\varphi} = 0$. This proves Theorem 1.1.

4. Analysis of the matrix M

This section is devoted to the analysis of the matrix M defined in Section 3. We first derive a simplified form of M. Our first observation is that if α is either of the indices $\{1, 2\}$ and β is any of the indices in $\{3, \cdots, N\}$, then

$$\int L(Z_{i,\beta}^t)Z_{j,\alpha}^s = 0 \text{ for any } i, j = 1, \cdots, 2n+m, \ s, t = 0, \cdots, k-1.$$

This fact implies that the matrix M has the form

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \tag{4.1}$$

where M_1 is a matrix of dimension $(2 \times (2n+m) \times k)^2$ and M_2 is a matrix of dimension $((N-2) \times (2n+m) \times k)^2$.

Because

$$\int L(Z^s_{i,\alpha})Z^t_{j,\beta} = \int L(Z^t_{j,\beta})Z^s_{i,\alpha},$$

we can write that

$$M_1 = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \tag{4.2}$$

where A, B, C are square matrices of dimension $((2n+m)\times k)^2$, with A, C symmetric. More precisely,

$$A = (\int L(Z^s_{i,1})Z^t_{j,1})_{i,j=1,\cdots,2n+m,\ s,t=0,\cdots,k-1},$$

$$B = (\int L(Z^s_{i,1})Z^t_{j,2})_{i,j=1,\cdots,2n+m,\ s,t=0,\cdots,k-1},$$

$$C = (\int L(Z^s_{i,2})Z^t_{j,2})_{i,j=1,\cdots,2n+m,\ s,t=0,\cdots,k-1}.$$

Furthermore, again by symmetry, because

$$\int L(Z^s_{i,\alpha})Z^t_{j,\beta} = 0, \text{ if } \alpha \neq \beta, \alpha, \beta = 3, \cdots, N,$$

the matrix M_2 has the form

$$M_2 = \begin{pmatrix} H_3 & 0 & 0 & 0 & 0 \\ 0 & H_4 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & H_N \end{pmatrix} \tag{4.3}$$

where H_α are square matrices of dimension $[(2n+m)\times k]^2$, and each of them is symmetric. The matrices H_α are defined by

$$H_\alpha = (\int L(Z^s_{i,\alpha})Z^t_{j,\alpha})^{s,t=0,\cdots,k-1}_{i,j=1,\cdots,2n+m} \text{ for } \alpha = 3, \cdots, N. \tag{4.4}$$

Thus, given the form of the matrix M as described in (4.1), (4.2), and (4.3), system (3.18) is equivalent to

$$M_1 \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = - \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix},\ H_\alpha \mathbf{c}_\alpha = -\mathbf{r}_\alpha,\ \text{for } \alpha = 3, \cdots, N.$$

where the vectors $\mathbf{r}_\alpha$ are defined in (3.18).

This section is devoted to the analysis of the kernels and eigenvalues of the matrices A, B, C, H_α. The main result of this section is the following solvability condition for the matrix M.

Proposition 4.1. *Part a.*

There exists $\ell_0 > 0$ such that for $\ell > \ell_0$, the system

$$M_1 \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix}$$

is solvable if

$$\begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} \cdot \mathbf{w}_1 = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} \cdot \mathbf{w}_2 = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} \cdot \mathbf{w}_3 = 0.$$

Furthermore, the general solution is

$$\begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} + s_1\mathbf{w}_1 + s_2\mathbf{w}_2 + s_3\mathbf{w}_3 \tag{4.5}$$

for all $s_i \in \mathbb{R}$, with $\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$ being a fixed vector such that

$$\| \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \| \leq C\ell^\tau e^\ell \| \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} \|. \tag{4.6}$$

Part b.

Let $\alpha = 3, \cdots, N$. *Then. there exists* $\ell_0 > 0$ *such that for any* $\ell > \ell_0$,

$$H_\alpha(\mathbf{c}_\alpha) = \mathbf{r}_\alpha \tag{4.7}$$

is solvable if

$$\mathbf{r}_\alpha \cdot \mathbf{w}_4 = \mathbf{r}_\alpha \cdot \mathbf{w}_5 = \mathbf{r}_\alpha \cdot \mathbf{w}_6 = 0.$$

Furthermore, the solution has the form

$$\mathbf{c}_\alpha = \mathbf{v}_\alpha + s_{\alpha 1} w_4 + s_{\alpha 2} w_5 + s_{\alpha 3} w_6, \tag{4.8}$$

for all $s_{\alpha i} \in \mathbb{R}$, *where* $\mathbf{v}_\alpha$ *a fixed vector such that*

$$\|\mathbf{v}_\alpha\| \le C \ell^\tau e^\ell \|\mathbf{r}_\alpha\|. \tag{4.9}$$

Remark 4.1. *From the statement of the above proposition, because* M_1, M_2 *are symmetric matrices, one need only show that* M_1 *has* $3-$*dimensional kernels spanned by* $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, *while* H_α *has* $3-$*dimensional kernels spanned by* $\mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_6$.

Before we prove the above proposition, we first need to introduce some notation.

For all $\bar{n} \ge 2$, we define the $\bar{n} \times \bar{n}$ matrix

$$T_{\bar{n}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{4.10}$$

In practice, the integer $\bar{n}$ will be equal to $m-1$ or $2n-1$.

It is easy to check that the inverse of $T_{\bar{n}}$ is the matrix whose entries are given by

$$(T_{\bar{n}}^{-1})_{ij} = \min(i,j) - \frac{ij}{\bar{n}+1}. \tag{4.11}$$

We define the vectors $S^\downarrow$ and $S^\uparrow$ by

$$T_{\bar{n}}\, S^\downarrow := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{\bar{n}} \qquad T_{\bar{n}}\, S^\uparrow := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{\bar{n}}\,. \tag{4.12}$$

It is simple to check that

$$S^\uparrow := \begin{pmatrix} \frac{\bar{n}}{\bar{n}+1} \\ \frac{\bar{n}-1}{\bar{n}+1} \\ \vdots \\ \frac{2}{\bar{n}+1} \\ \frac{1}{\bar{n}+1} \end{pmatrix} \in \mathbb{R}^{\bar{n}} \qquad S^\downarrow := \begin{pmatrix} \frac{1}{\bar{n}+1} \\ \frac{2}{\bar{n}+1} \\ \vdots \\ \frac{\bar{n}-1}{\bar{n}+1} \\ \frac{\bar{n}}{\bar{n}+1} \end{pmatrix} \in \mathbb{R}^{\bar{n}}\,. \tag{4.13}$$

We also introduce the following vectors:

$$\mathbf{d}_{L,\bar{n}} = (c, 0 \cdots, 0) \in \mathbb{R}^{\bar{n}}, \quad \mathbf{d}_{R,\bar{n}} = (0, \cdots, 0, c) \in \mathbb{R}^{\bar{n}}$$

and

$$\mathbf{d}_{\bar{n}} = (d, d, \cdots, d) \in \mathbb{R}^{\bar{n}}. \tag{4.14}$$

In practice, $\bar{n} = m-1$ or $2n-1$.

We will see below that a circulant matrix will play an important role in our proof. We recall the definition of a circulant matrix.

A circulant matrix X of dimension $k \times k$ has the form

$$X = \begin{pmatrix} x_0 & x_1 & \cdots & x_{k-2} & x_{k-1} \\ x_{k-1} & x_0 & x_1 & \cdots & x_{k-2} \\ \cdots & x_{k-1} & x_0 & x_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x_1 \\ x_1 & \cdots & \cdots & x_{k-1} & x_0 \end{pmatrix}, \tag{4.15}$$

or equivalently, if $x_{ij}, i, j = 1, \cdots, k$ are the entries of the matrix X, then

$$x_{ij} = x_{1,|i-j|+1}.$$

In particular, in order to determine a circulant matrix it is sufficient to know the entries of the first row. By

$$X = \mathrm{Cir}\{(x_0, x_1, \cdots, x_{k-1})\}, \tag{4.16}$$

we denote the above mentioned circulant matrix. For the properties of circulant matrices, we refer the reader to [15]. The eigenvalues of a circulant matrix X are given by the explicit formula

$$\eta_s = \sum_{l=1}^{k-1} x_l e^{\frac{2\pi s}{k} il}, \ s = 0, \cdots, k-1 \tag{4.17}$$

with corresponding normalized eigenvectors defined by

$$E_s = k^{-\frac{1}{2}} \begin{pmatrix} 1 \\ e^{\frac{2\pi s}{k} i} \\ e^{\frac{2\pi s}{k} i2} \\ \vdots \\ e^{\frac{2\pi s}{k} i(k-1)} \end{pmatrix}. \tag{4.18}$$

Observe that any circulant matrix X can be diagonalized as

$$X = P D_X P^t,$$

where D_X is the diagonal matrix

$$D_X = \mathrm{diag}(\eta_0, \eta_1, \cdots, \eta_{k-1})$$

and P is the $k \times k$ matrix defined by

$$P = (E_0 | E_1 | \cdots | E_{k-1}). \tag{4.19}$$

From this point on, we begin to analyze the components of the matrix M, A, B, C, H_α, for which explicit expressions are given in Section 8. First, observe that M is a symmetric matrix.

4.1. **Analysis of H_α and proof of part (b) of Proposition 4.1.** We first analyze the kernels of the matrix H_α.

First, we denote

$$\frac{\Psi_2(\bar{\ell})}{\Psi_2(\ell)} = \frac{\delta_2}{2 \sin \frac{\pi}{k}}$$

where Ψ_2 is defined in (8.2).

By dividing both sides of the equation $H_\alpha(\mathbf{c}_\alpha) = 0$ by $\Psi_2(\ell)$, we obtain that

$$\bar{H}_\alpha(\mathbf{c}_\alpha) = 0,$$

where $\bar{H}_\alpha = \frac{H_\alpha}{\Psi_2(\ell)}$.

From the computations regarding H_α in Section 7, we know that $\bar{H}_\alpha$ has the form

$$\bar{H}_\alpha = \begin{pmatrix} H_{\alpha,1} & 0 & H_{\alpha,2} & 0 \\ 0 & H_{\alpha,3} & H_{\alpha,4} & H_{\alpha,5} \\ H^t_{\alpha,2} & H^t_{\alpha,4} & H_{\alpha,6} & 0 \\ 0 & H^t_{\alpha,5} & 0 & H_{\alpha,7} \end{pmatrix} + O(e^{-\xi\ell}), \tag{4.20}$$

where

$$H_{\alpha,1} = \begin{pmatrix} -1-\frac{\delta_2}{\sin\frac{\pi}{k}} & \frac{\delta_2}{2\sin\frac{\pi}{k}} & 0 & \cdots & 0 & \frac{\delta_2}{2\sin\frac{\pi}{k}} \\ \frac{\delta_2}{2\sin\frac{\pi}{k}} & -1-\frac{\delta_2}{\sin\frac{\pi}{k}} & \frac{\delta_2}{2\sin\frac{\pi}{k}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\delta_2}{2\sin\frac{\pi}{k}} & -1-\frac{\delta_2}{\sin\frac{\pi}{k}} & \frac{\delta_2}{2\sin\frac{\pi}{k}} \\ \frac{\delta_2}{2\sin\frac{\pi}{k}} & 0 & \cdots & 0 & \frac{\delta_2}{2\sin\frac{\pi}{k}} & -1-\frac{\delta_2}{\sin\frac{\pi}{k}} \end{pmatrix}_{k\times k},$$

$$H_{\alpha,2} = \begin{pmatrix} \mathbf{1}_{L,m-1} & \mathbf{0}_{m-1} & \cdots & \mathbf{0}_{m-1} \\ \mathbf{0}_{m-1} & \mathbf{1}_{L,m-1} & \cdots & \mathbf{0}_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{m-1} & \cdots & \cdots & \mathbf{1}_{L,m-1} \end{pmatrix}_{[(m-1)\times k]\times k},$$

$$H_{\alpha,3} = \begin{pmatrix} \frac{\delta_2}{\sin\frac{\pi}{k}}-1 & 0 & \cdots & 0 \\ 0 & \frac{\delta_2}{\sin\frac{\pi}{k}}-1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \frac{\delta_2}{\sin\frac{\pi}{k}}-1 \end{pmatrix}_{k\times k},$$

$$H_{\alpha,4} = \begin{pmatrix} \mathbf{1}_{R,m-1} & \mathbf{0}_{m-1} & \cdots & \mathbf{0}_{m-1} \\ \mathbf{0}_{m-1} & \mathbf{1}_{R,m-1} & \cdots & \mathbf{0}_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{m-1} & \cdots & \cdots & \mathbf{1}_{R,m-1} \end{pmatrix}_{[(m-1)\times k]\times k},$$

$$H_{\alpha,5} = \begin{pmatrix} -(\frac{\boldsymbol{\delta_2}}{\mathbf{2}\sin\frac{\pi}{\mathbf{k}}})_{L,2n-1} & \mathbf{0}_{2n-1} & \cdots & -(\frac{\boldsymbol{\delta_2}}{\mathbf{2}\sin\frac{\pi}{\mathbf{k}}})_{R,2n-1} \\ -(\frac{\boldsymbol{\delta_2}}{\mathbf{2}\sin\frac{\pi}{\mathbf{k}}})_{R,2n-1} & -(\frac{\boldsymbol{\delta_2}}{\mathbf{2}\sin\frac{\pi}{\mathbf{k}}})_{L,2n-1} & \cdots & \mathbf{0}_{2n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{2n-1} & \cdots & -(\frac{\boldsymbol{\delta_2}}{\mathbf{2}\sin\frac{\pi}{\mathbf{k}}})_{R,2n-1} & -(\frac{\boldsymbol{\delta_2}}{\mathbf{2}\sin\frac{\pi}{\mathbf{k}}})_{L,2n-1} \end{pmatrix}_{[(2n-1)\times k]\times k},$$

$$H_{\alpha,6} = \begin{pmatrix} -T_{m-1} & 0 & \cdots & 0 \\ 0 & -T_{m-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & -T_{m-1} \end{pmatrix}_{[(m-1)\times k]^2},$$

and

$$H_{\alpha,7} = \begin{pmatrix} \frac{\delta_2}{2\sin\frac{\pi}{k}}T_{2n-1} & 0 & \cdots & 0 \\ 0 & \frac{\delta_2}{2\sin\frac{\pi}{k}}T_{2n-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \frac{\delta_2}{2\sin\frac{\pi}{k}}T_{2n-1} \end{pmatrix}_{[(2n-1)\times k]^2}.$$

We want to analyze the eigenvalues of the matrix $\bar{H}_\alpha$. Thus, we assume that

$$\bar{H}_\alpha(\mathbf{a}) = 0. \tag{4.21}$$

First, by considering the third row of the matrix $\bar{H}_\alpha$ written in the form (4.20), one can obtain that

$$(-T_{m-1}+O(e^{-\xi\ell}))(\mathbf{a}_{Y_1,i})+\begin{pmatrix} a_1^i \\ 0 \\ \vdots \\ 0 \\ a_{m+1}^i \end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v$$

and

$$(T_{2n-1}+O(e^{-\xi\ell}))(\mathbf{a}_{Y_2}^i)-\begin{pmatrix} a_{m+1}^i \\ 0 \\ \vdots \\ 0 \\ a_{m+1}^{i+1} \end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v.$$

Here, $\mathbf{a}_v$ corresponds to the unknown variables around the vertices of the inner and outer polygons. In other words, these equations imply that once $\mathbf{a}_v$ is fixed, the other unknown variables will be determined. From the above two equations, and using (4.13), one has that for $i=0,\cdots,k-1$,

$$\begin{cases} a_2^i=\frac{1}{m}\big((m-1)a_1^i+a_{m+1}^i\big)+O(e^{-\xi\ell})\mathbf{a}_v \\ \\ a_m^i=\frac{1}{m}\big(a_1^i+(m-1)a_{m+1}^i\big)+O(e^{-\xi\ell})\mathbf{a}_v, \end{cases} \tag{4.22}$$

and

$$\begin{cases} a_{m+2}^i=\frac{1}{2n}\big((2n-1)a_{m+1}^i+a_{m+1}^{i+1}\big)+O(e^{-\xi\ell})\mathbf{a}_v \\ \\ a_{m+2n}^i=\frac{1}{2n}\big(a_{m+1}^i+(2n-1)a_{m+1}^{i+1}\big)+O(e^{-\xi\ell})\mathbf{a}_v. \end{cases} \tag{4.23}$$

Next, we consider the first and second rows of the matrix $\bar{H}_\alpha$ in (4.20). We can obtain that

$$H_{\alpha,1}\begin{pmatrix} a_1^0 \\ a_1^1 \\ \vdots \\ a_1^{k-1} \end{pmatrix}+\begin{pmatrix} a_2^0 \\ a_2^1 \\ \vdots \\ a_2^{k-1} \end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v$$

and

$$H_{\alpha,3}\begin{pmatrix} a_{m+1}^0 \\ a_{m+1}^1 \\ \vdots \\ a_{m+1}^{k-1} \end{pmatrix}+\begin{pmatrix} a_m^0-\frac{\delta_2}{2\sin\frac{\pi}{k}}(a_{m+2}^0+a_{m+2n}^{k-1}) \\ a_m^1-\frac{\delta_2}{2\sin\frac{\pi}{k}}(a_{m+2}^1-a_{m+2n}^0) \\ \vdots \\ a_m^{k-1}-\frac{\delta_2}{2\sin\frac{\pi}{k}}(a_{m+2}^{k-1}-a_{m+2n}^{k-2}) \end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v.$$

By using the above two equations (4.22) and (4.23) for $a_2^i, a_m^i, a_{m+2}^i, a_{m+2n}^i$, the above two equations are reduced to a $2k$ system of $2k$ unknowns a_1^i, a_{m+1}^i for $i=0,\cdots,k-1$:

$$\tilde{H}_\alpha\begin{pmatrix} a_1^0 \\ \vdots \\ a_{1^{k-1}} \\ a_{m+1}^0 \\ \vdots \\ a_{m+1}^{k-1} \end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v, \quad \text{where} \quad \tilde{H}_\alpha=\begin{pmatrix} \tilde{H}_{\alpha,1} & \tilde{H}_{\alpha,2} \\ \tilde{H}_{\alpha,2}^t & \tilde{H}_{\alpha,3} \end{pmatrix} \tag{4.24}$$

and $\tilde{H}_{\alpha,i}$ are all circulant matrices with

$$\tilde{H}_{\alpha,1}=\mathrm{Cir}\{(-\frac{1}{m}-\frac{\delta_2}{\sin\frac{\pi}{k}},\frac{\delta_2}{2\sin\frac{\pi}{k}},0,\cdots,0,\frac{\delta_2}{2\sin\frac{\pi}{k}})\}, \tag{4.25}$$

$$\tilde{H}_{\alpha,2} = \mathrm{Cir}\{(\frac{1}{m}, 0\cdots, 0)\}, \tag{4.26}$$

$$\tilde{H}_{\alpha,3} = \mathrm{Cir}\{(-\frac{1}{m} + \frac{\delta_2}{2n\sin\frac{\pi}{k}}, -\frac{\delta_2}{4n\sin\frac{\pi}{k}}, 0, \cdots, 0, -\frac{\delta_2}{4n\sin\frac{\pi}{k}})\}. \tag{4.27}$$

From the above analysis, equation (4.21) is equivalent to equation (4.24) above for the $2k$ variables $\mathbf{a}_v$.

Next, we begin to analyze the matrix $\tilde{H}_\alpha$.

Eigenvalues of $\tilde{H}_{\alpha,1}$: A direct application of (4.17) gives that the eigenvalues of the matrix $\tilde{H}_{\alpha,1}$ are given by

$$h_{1,i} = \frac{\delta_2}{\sin\frac{\pi}{k}}(\cos\frac{2\pi i}{k} - 1) - \frac{1}{m} \tag{4.28}$$

for $i = 0, \cdots, k-1$.

Eigenvalues of $\tilde{H}_{\alpha,3}$: The eigenvalues of the matrix $\tilde{H}_{\alpha,3}$ are given by

$$h_{3,i} = \frac{\delta_2}{4n\sin\frac{\pi}{k}}(2 - 2\cos\frac{2\pi i}{k}) - \frac{1}{m}. \tag{4.29}$$

for $i = 0, \cdots, k-1$.

Define

$$\mathcal{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}. \tag{4.30}$$

Then, simple algebra gives that

$$\tilde{H}_\alpha = \mathcal{P}\begin{pmatrix} D_1 & D_2 \\ D_2 & D_3 \end{pmatrix}\mathcal{P}^t,$$

where

$$D_1 = \mathrm{diag}(h_{1,0}, \cdots, h_{1,k-1}), \quad D_2 = \mathrm{diag}(\frac{1}{m}, \cdots, \frac{1}{m}),$$
$$D_3 = \mathrm{diag}(h_{3,0}, \cdots, h_{3,k-1}).$$

We consider the matrix

$$\mathcal{D}_i = \begin{pmatrix} h_{1,i} & \frac{1}{m} \\ \frac{1}{m} & h_{3,i} \end{pmatrix}. \tag{4.31}$$

The determinant of $\mathcal{D}_i$ is given by

$$\mathrm{Det}(\mathcal{D}_i) = \frac{2\delta_2}{n}\sin^2\frac{\pi i}{k}(1 - \frac{\sin^2\frac{\pi i}{k}}{\sin^2\frac{\pi}{k}})(1 + O(\frac{1}{\ell})). \tag{4.32}$$

One can check that $\mathrm{Det}(\mathcal{D}_i) = 0$ for $i = 0, 1, k-1$, and $|\mathrm{Det}(\mathcal{D}_i)| \geq \frac{c}{n}$ for $2 \leq i \leq k-2$.

From the above analysis, one can see that the matrix $\bar{H}_\alpha$ has at most three kernels, and other than zero eigenvalues the eigenvalues will have a lower bound of $\frac{C}{\ell^\tau}$ for some $\tau > 0$. Moreover, one can check directly that $\mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_6$ are in the kernels of $\bar{H}_\alpha$, and so one can obtain part (b) of Proposition 4.1.

4.2. **Analysis of the matrix M_1 and proof of part (a) of Proposition 4.1.** First, we denote

$$\bar{M}_1 = \frac{1}{\Psi_1(\ell)}M_1,$$

where Ψ_1 is defined in (8.1), and we introduce the following notations:

$$\Psi_2(\ell) = \frac{\sigma_1}{\ell}\Psi_1(\ell), \qquad \Psi_2(\bar{\ell}) = \frac{\sigma_2}{\ell}\Psi_1(\bar{\ell}) \tag{4.33}$$

and

$$\frac{\Psi_1(\bar{\ell})}{\Psi_1(\ell)} = \frac{\sigma_3}{2\sin\frac{\pi}{k}} \tag{4.34}$$

for $\sigma_1, \sigma_2, \sigma_3 > 0$ positive constants. In fact, from (8.5) one can obtain that $\sigma_3 = (\frac{d\bar{\ell}}{d\ell})^{-1}$. By the computation in Section 8, we know that $\bar{M}_1$ can be written in the following form:

$$\bar{M}_1 = \begin{pmatrix} A_{11}^1 & 0 & A_{12}^1 & 0 & B_{11}^1 & 0 & 0 & 0 \\ 0 & A_{11}^3 & A_{12}^2 & A_{12}^3 & 0 & 0 & 0 & B_{12}^1 \\ A_{12}^{1,t} & A_{12}^{2,t} & A_{13}^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{12}^{3,t} & 0 & A_{13}^2 & 0 & B_{21}^1 & 0 & 0 \\ B_{11}^{1,t} & 0 & 0 & 0 & C_{11}^1 & 0 & C_{12}^1 & 0 \\ 0 & 0 & 0 & B_{21}^{1,t} & 0 & C_{11}^3 & C_{12}^2 & C_{12}^3 \\ 0 & 0 & 0 & 0 & C_{12}^{1,t} & C_{12}^{2,t} & C_{13}^1 & 0 \\ 0 & B_{12}^{1,t} & 0 & 0 & 0 & C_{12}^{3,t} & 0 & C_{13}^2 \end{pmatrix}, \tag{4.35}$$

where A_{11}^1, A_{11}^3 are $k \times k$ circulant matrices:

$$A_{11}^1 = \mathrm{Cir}\{(A_{11,0}^1, A_{11,1}^1, 0, \cdots, 0, A_{11,k-1}^1)\},$$

$$A_{11,0}^1 = -1 - \frac{\sigma_3}{\sin\frac{\pi}{k}}(\sin^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k}),$$

$$A_{11,1}^1 = A_{11,k-1}^1 = \frac{\sigma_3}{2\sin\frac{\pi}{k}}(-\sin^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k}),$$

$$A_{11}^3 = \mathrm{Cir}\{(-1 + \frac{\sigma_3}{\sin\frac{\pi}{k}}(\sin^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k}), 0, \cdots, 0)\},$$

$$A_{12}^1 = \begin{pmatrix} \mathbf{1}_{L,m-1} & \mathbf{0}_{m-1} & \cdots & \mathbf{0}_{m-1} \\ \mathbf{0}_{m-1} & \mathbf{1}_{L,m-1} & \cdots & \mathbf{0}_{m-1} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}_{m-1} & \cdots & \cdots & \mathbf{1}_{L,m-1} \end{pmatrix}_{[(m-1)\times k]\times k},$$

$$A_{12}^2 = \begin{pmatrix} \mathbf{1}_{R,m-1} & \mathbf{0}_{m-1} & \cdots & \mathbf{0}_{m-1} \\ \mathbf{0}_{m-1} & \mathbf{1}_{R,m-1} & \cdots & \mathbf{0}_{m-1} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}_{m-1} & \cdots & \cdots & \mathbf{1}_{R,m-1} \end{pmatrix}_{[(m-1)\times k]\times k},$$

$$A_{12}^3 = \begin{pmatrix} \boldsymbol{\frac{\sigma_3}{2}}_{L,2n-1} & \mathbf{0}_{2n-1} & \cdots & -\boldsymbol{\frac{\sigma_3}{2}}_{R,2n-1} \\ -\boldsymbol{\frac{\sigma_3}{2}}_{R,2n-1} & \boldsymbol{\frac{\sigma_3}{2}}_{L,2n-1} & \cdots & \mathbf{0}_{2n-1} \\ \mathbf{0}_{2n-1} & \ddots & \ddots & \mathbf{0}_{2n-1} \\ \mathbf{0}_{2n-1} & \cdots & -\boldsymbol{\frac{\sigma_3}{2}}_{R,2n-1} & \boldsymbol{\frac{\sigma_3}{2}}_{L,2n-1} \end{pmatrix}_{[(2n-1)\times k]\times k},$$

$$A_{13}^1 = \begin{pmatrix} -T_{m-1} & 0 & \cdots & 0 \\ 0 & -T_{m-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & -T_{m-1} \end{pmatrix}_{[(m-1)\times k]^2},$$

$$A_{13}^2 = \begin{pmatrix} \frac{\sigma_3}{2\sin\frac{\pi}{k}}T_{2n-1} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_3}{2\sin\frac{\pi}{k}}T_{2n-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \frac{\sigma_3}{2\sin\frac{\pi}{k}}T_{2n-1} \end{pmatrix}_{[(2n-1)\times k]^2}.$$

For the matrix B, we have that B_{11}^1 is a $k \times k$ circulant matrix:

$$B_{11}^1 = \mathrm{Cir}\{(0, \frac{\sigma_3\cos\frac{\pi}{k}}{2}(1 + \frac{\sigma_2}{\ell}), 0, \cdots, \frac{\sigma_3\cos\frac{\pi}{k}}{2}(1 + \frac{\sigma_2}{\ell}))\},$$

$$B_{12}^1=\frac{\sigma_2\sigma_3\cos\frac{\pi}{k}}{2\ell\sin\frac{\pi}{k}}\begin{pmatrix}-\mathbf{1}_{L,2n-1} & \mathbf{0}_{2n-1} & \cdots & -\mathbf{1}_{R,2n-1}\\ -\mathbf{1}_{R,2n-1} & -\mathbf{1}_{L,2n-1} & \cdots & \mathbf{0}_{2n-1}\\ \mathbf{0}_{2n-1} & \ddots & \ddots & \mathbf{0}_{2n-1}\\ \mathbf{0}_{2n-1} & \cdots & -\mathbf{1}_{R,2n-1} & -\mathbf{1}_{L,2n-1}\end{pmatrix}_{[(2n-1)\times k]\times k},$$

and

$$B_{21}^{1,t}=\frac{\sigma_3\cos\frac{\pi}{k}}{2\sin\frac{\pi}{k}}\begin{pmatrix}\mathbf{1}_{L,2n-1} & \mathbf{0}_{2n-1} & \cdots & \mathbf{1}_{R,2n-1}\\ \mathbf{1}_{R,2n-1} & \mathbf{1}_{L,2n-1} & \cdots & \mathbf{0}_{2n-1}\\ \mathbf{0}_{2n-1} & \ddots & \ddots & \mathbf{0}_{2n-1}\\ \mathbf{0}_{2n-1} & \cdots & \mathbf{1}_{R,2n-1} & \mathbf{1}_{L,2n-1}\end{pmatrix}_{[(2n-1)\times k]\times k}.$$

The matrices C_{11}^1, C_{11}^3 are $k\times k$ circulant matrices:

$$C_{11}^1=\mathrm{Cir}\{(C_{11,0}^1, C_{11,1}^1, 0\cdots,0, C_{11,k-1}^1)\},$$
$$C_{11,0}^1=-\frac{\sigma_1}{\ell}-\frac{\sigma_3}{\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k}+\frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k}),$$
$$C_{11,1}^1=C_{11,k-1}^1=\frac{\sigma_3}{2\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k}-\frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k}),$$
$$C_{11}^3=\mathrm{Cir}\{(-\frac{\sigma_1}{\ell}+\frac{\sigma_3}{\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k}+\frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k}),0,\cdots,0)\}.$$

$$C_{12}^1=\begin{pmatrix}\frac{\boldsymbol{\sigma_1}}{\boldsymbol{\ell}}{}_{L,m-1} & \mathbf{0}_{m-1} & \cdots & \mathbf{0}_{m-1}\\ \mathbf{0}_{m-1} & \frac{\boldsymbol{\sigma_1}}{\boldsymbol{\ell}}{}_{L,m-1} & \cdots & \mathbf{0}_{m-1}\\ \vdots & \ddots & \ddots & \vdots\\ \mathbf{0}_{m-1} & \cdots & \cdots & \frac{\boldsymbol{\sigma_1}}{\boldsymbol{\ell}}{}_{L,m-1}\end{pmatrix}_{[(m-1)\times k]\times k}$$

$$C_{12}^2=\begin{pmatrix}\frac{\boldsymbol{\sigma_1}}{\boldsymbol{\ell}}{}_{R,m-1} & \mathbf{0}_{m-1} & \cdots & \mathbf{0}_{m-1}\\ \mathbf{0}_{m-1} & \frac{\boldsymbol{\sigma_1}}{\boldsymbol{\ell}}{}_{R,m-1} & \cdots & \mathbf{0}_{m-1}\\ \vdots & \ddots & \ddots & \vdots\\ \mathbf{0}_{m-1} & \cdots & \cdots & \frac{\boldsymbol{\sigma_1}}{\boldsymbol{\ell}}{}_{R,m-1}\end{pmatrix}_{[(m-1)\times k]\times k},$$

$$C_{12}^3=\begin{pmatrix}(\frac{\boldsymbol{\sigma_2\sigma_3}}{\boldsymbol{2\ell}})_{L,2n-1} & \mathbf{0}_{2n-1} & \cdots & -(\frac{\boldsymbol{\sigma_2\sigma_3}}{\boldsymbol{2\ell}})_{R,2n-1}\\ -(\frac{\boldsymbol{\sigma_2\sigma_3}}{\boldsymbol{2\ell}})_{R,2n-1} & (\frac{\boldsymbol{\sigma_2\sigma_3}}{\boldsymbol{2\ell}})_{L,2n-1} & \cdots & \mathbf{0}_{2n-1}\\ \mathbf{0}_{2n-1} & \ddots & \ddots & \mathbf{0}_{2n-1}\\ \mathbf{0}_{2n-1} & \cdots & -(\frac{\boldsymbol{\sigma_2\sigma_3}}{\boldsymbol{2\ell}})_{R,2n-1} & (\frac{\boldsymbol{\sigma_2\sigma_3}}{\boldsymbol{2\ell}})_{L,2n-1}\end{pmatrix}_{[(2n-1)\times k]\times k},$$

$$C_{13}^1=\begin{pmatrix}-\frac{\sigma_1}{\ell}T_{m-1} & 0 & \cdots & 0\\ 0 & -\frac{\sigma_1}{\ell}T_{m-1} & \cdots & 0\\ \vdots & \vdots & \vdots & -\frac{\sigma_1}{\ell}T_{m-1}\end{pmatrix}_{[(m-1)\times k]^2},$$

and

$$C_{13}^2=\begin{pmatrix}\frac{\sigma_2\sigma_3}{2\ell\sin\frac{\pi}{k}}T_{2n-1} & 0 & \cdots & 0\\ 0 & \frac{\sigma_2\sigma_3}{2\ell\sin\frac{\pi}{k}}T_{2n-1} & \cdots & 0\\ \vdots & \vdots & \vdots & \frac{\sigma_2\sigma_3}{2\ell\sin\frac{\pi}{k}}T_{2n-1}\end{pmatrix}_{[(2n-1)\times k]^2}.$$

The strategy for dealing with the matrix M_1 is similar to that for H_α. The main idea is that once the variables for the inner and outer vertices are fixed, the other variables will be determined. Thus, we will reduce the problem $\bar{M}_1(\mathbf{a})=0$ to a $4k\times 4k$ matrix equation for the variables around the $2k$ vertices.

First, by considering the third and fourth rows of $\bar{M}_1(\mathbf{a})=0$ in the form (4.35), one can obtain that

$$(-T_{m-1}+O(e^{-\xi\ell})\begin{pmatrix} a^i_{2,1}\\ a^i_{3,1}\\ \vdots \\ a^i_{m-1,1}\\ a^i_{m,1}\end{pmatrix}+\begin{pmatrix} a^i_{1,1}\\ 0\\ \vdots\\ 0\\ a^i_{m+1,1}\end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v, \tag{4.36}$$

$$T_{2n-1}\begin{pmatrix} a^i_{m+2,1}\\ a^i_{m+3,1}\\ \vdots\\ a^i_{m+2n-1,1}\\ a^i_{m+2n,1}\end{pmatrix}+\sin\frac{\pi}{k}\begin{pmatrix} a^i_{m+1,1}\\ 0\\ \vdots\\ 0\\ -a^{i+1}_{m+1,1}\end{pmatrix}+\cos\frac{\pi}{k}\begin{pmatrix} a^i_{m+1,2}\\ 0\\ \vdots\\ 0\\ a^i_{m+1,2}\end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v. \tag{4.37}$$

From the seventh and eighth rows of the matrix $\bar{M}_1$ in (4.35), we can obtain that

$$(-T_{m-1}+O(e^{-\xi\ell})\begin{pmatrix} a^i_{2,2}\\ a^i_{3,2}\\ \vdots \\ a^i_{m-1,2}\\ a^i_{m,2}\end{pmatrix}+\begin{pmatrix} a^i_{1,2}\\ 0\\ \vdots\\ 0\\ a^i_{m+1,2}\end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v, \tag{4.38}$$

$$T_{2n-1}\begin{pmatrix} a^i_{m+2,}\\ a^i_{m+3,2}\\ \vdots\\ a^i_{m+2n-1,2}\\ a^i_{m+2n,2}\end{pmatrix}+\sin\frac{\pi}{k}\begin{pmatrix} a^i_{m+1,2}\\ 0\\ \vdots\\ 0\\ -a^{i+1}_{m+1,2}\end{pmatrix}-\cos\frac{\pi}{k}\begin{pmatrix} a^i_{m+1,1}\\ 0\\ \vdots\\ 0\\ a^i_{m+1,1}\end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v. \tag{4.39}$$

From the above four systems, using (4.13) one can solve $\mathbf{a}^i_{Y_1,j},\mathbf{a}^i_{Y_2,j}$ in terms of $\mathbf{a}_v$ for $i=0,\cdots,k-1$, $j=1,2$. In particular, we can obtain that

$$\begin{cases} a^i_{2,1}=\frac{1}{m}((m-1)a^i_{1,1}+a^i_{m+1,1})+O(e^{-\xi\ell})\mathbf{a}_v,\\ \\ a^i_{m,1}=\frac{1}{m}(a^i_{1,1}+(m-1)a^i_{m+1,1})+O(e^{-\xi\ell})\mathbf{a}_v,\end{cases} \tag{4.40}$$

$$\begin{cases} a^i_{m+2,1}=-\frac{\sin\frac{\pi}{k}}{2n}((2n-1)a^i_{m+1,1}-a^{i+1}_{m+1,1})\\ \qquad\qquad -\frac{\cos\frac{\pi}{k}}{2n}((2n-1)a^i_{m+1,2}+a^{i+1}_{m+1,2})+O(e^{-\xi\ell})\mathbf{a}_v\\ \\ a^i_{m+2n,1}=-\frac{\sin\frac{\pi}{k}}{2n}(a^i_{m+1,1}-(2n-1)a^{i+1}_{m+1,1})\\ \qquad\qquad -\frac{\cos\frac{\pi}{k}}{2n}(a^i_{m+1,2}+(2n-1)a^{i+1}_{m+1,2})+O(e^{-\xi\ell})\mathbf{a}_v,\end{cases} \tag{4.41}$$

$$\begin{cases} a^i_{2,2}=\frac{1}{m}((m-1)a^i_{1,2}+a^i_{m+1,2})+O(e^{-\xi\ell})\mathbf{a}_v,\\ \\ a^i_{m,2}=\frac{1}{m}(a^i_{1,2}+(m-1)a^i_{m+1,2})+O(e^{-\xi\ell})\mathbf{a}_v,\end{cases} \tag{4.42}$$

and

$$\begin{cases} a^i_{m+2,2}=-\frac{\sin\frac{\pi}{k}}{2n}((2n-1)a^i_{m+1,2}-a^{i+1}_{m+1,2})\\ \qquad\qquad +\frac{\cos\frac{\pi}{k}}{2n}((2n-1)a^i_{m+1,1}+a^{i+1}_{m+1,1})+O(e^{-\xi\ell})\mathbf{a}_v\\ \\ a^i_{m+2n,2}=-\frac{\sin\frac{\pi}{k}}{2n}(a^i_{m+1,2}-(2n-1)a^{i+1}_{m+1,2})\\ \qquad\qquad +\frac{\cos\frac{\pi}{k}}{2n}(a^i_{m+1,1}+(2n-1)a^{i+1}_{m+1,1})+O(e^{-\xi\ell})\mathbf{a}_v.\end{cases} \tag{4.43}$$

From the first and second rows of the equation in (4.35), one can obtain that

$$A_{11}^1\begin{pmatrix} a_{1,1}^0\\ \vdots\\ a_{1,1}^{k-1}\end{pmatrix}+\begin{pmatrix} a_{2,1}^0\\ \vdots\\ a_{2,1}^{k-1}\end{pmatrix}+\frac{\sigma_3\cos\frac{\pi}{k}}{2}(1+\frac{\sigma_2}{\ell})\begin{pmatrix} a_{1,2}^1-a_{1,2}^{k-1}\\ a_{1,2}^2-a_{1,2}^0\\ \vdots\\ a_{1,2}^0-a_{1,2}^{k-2}\end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v, \tag{4.44}$$

$$\begin{aligned} A_{11}^3\begin{pmatrix} a_{m+1,1}^0\\ \vdots\\ a_{m+1,1}^{k-1}\end{pmatrix}+\begin{pmatrix} a_{m,1}^0\\ \vdots\\ a_{m,1}^{k-1}\end{pmatrix}\quad &+\frac{\sigma_3}{2}\begin{pmatrix} a_{m+2,1}^0-a_{m+2n,1}^{k-1}\\ a_{m+2,1}^1-a_{m+2n,1}^0\\ \vdots\\ a_{m+2,1}^{k-1}-a_{m+2n,1}^{k-2}\end{pmatrix}\\ &-\frac{\sigma_2\sigma_3\cos\frac{\pi}{k}}{2\ell\sin\frac{\pi}{k}}\begin{pmatrix} a_{m+2,2}^0+a_{m+2n,2}^{k-1}\\ a_{m+2,2}^1+a_{m+2n,2}^0\\ \vdots\\ a_{m+2,2}^{k-1}+a_{m+2n,2}^{k-2}\end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v. \end{aligned} \tag{4.45}$$

From the fifth and sixth rows of (4.35), one can obtain that

$$C_{11}^1\begin{pmatrix} a_{1,2}^0\\ \vdots\\ a_{1,2}^{k-1}\end{pmatrix}+\frac{c_1}{\ell}\begin{pmatrix} a_{2,2}^0\\ \vdots\\ a_{2,2}^{k-1}\end{pmatrix}-\frac{\sigma_3\cos\frac{\pi}{k}}{2}(1+\frac{c_2}{\ell})\begin{pmatrix} a_{1,1}^1-a_{1,1}^{k-1}\\ a_{1,1}^2-a_{1,1}^0\\ \vdots\\ a_{1,1}^0-a_{1,1}^{k-2}\end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v, \tag{4.46}$$

$$\begin{aligned} C_{11}^3\begin{pmatrix} a_{m+1,2}^0\\ \vdots\\ a_{m+1,2}^{k-1}\end{pmatrix}+\frac{\sigma_1}{\ell}\begin{pmatrix} a_{m,2}^0\\ \vdots\\ a_{m,2}^{k-1}\end{pmatrix}\quad &+\frac{\sigma_2\sigma_3}{2\ell}\begin{pmatrix} a_{m+2,2}^0-a_{m+2n,2}^{k-1}\\ a_{m+2,2}^1-a_{m+2n,2}^0\\ \vdots\\ a_{m+2,2}^{k-1}-a_{m+2n,2}^{k-2}\end{pmatrix}\\ &+\frac{\sigma_3\cos\frac{\pi}{k}}{2\sin\frac{\pi}{k}}\begin{pmatrix} a_{m+2,1}^0+a_{m+2n,1}^{k-1}\\ a_{m+2,1}^1+a_{m+2n,1}^0\\ \vdots\\ a_{m+2,1}^{k-1}+a_{m+2n,1}^{k-2}\end{pmatrix}=O(e^{-\xi\ell})\mathbf{a}_v \end{aligned} \tag{4.47}$$

Using the equations for $a_{2,j}^i, a_{m,j}^i$ and $a_{m+2,j}^i, a_{m+2n,j}^i$ (4.40)-(4.43), the above system (4.44)-(4.47) can be reduced to $4k$ equations in terms of $4k$ unknowns $a_{1,1}^0,\cdots,a_{1,1}^{k-1}$, $a_{m+1,1}^0,\cdots,a_{m+1,1}^{k-1}$, $a_{1,2}^0,\cdots,a_{1,2}^{k-1}$, and $a^0_{m+1,2},\cdots,a^{k-1}_{m+1,2}$:

$$\begin{pmatrix} F_{11} & F_{12} & F_{13} & 0 \\ F_{12}^t & F_{22} & 0 & F_{24} \\ F_{13}^t & 0 & F_{33} & F_{34} \\ 0 & F_{24}^t & F_{34}^t & F_{44} \end{pmatrix} \begin{pmatrix} a^0_{1,1} \\ \vdots \\ a^{k-1}_{1,1} \\ a^0_{m+1,1} \\ \vdots \\ a^{k-1}_{m+1,1} \\ a^0_{1,2} \\ \vdots \\ a^{k-1}_{1,2} \\ a^0_{m+1,2} \\ \vdots \\ a^{k-1}_{m+1,2} \end{pmatrix} = O(e^{-\xi\ell})\mathbf{a}_v \tag{4.48}$$

where F_{ij} are $k\times k$ circulant matrices given below:

The matrix F_{11}. F_{11} is defined by

$$F_{11} = \mathrm{Cir}\{(F_{11,0}, F_{11,1}, 0, \cdots, 0, F_{11,k-1})\}, \tag{4.49}$$

where

$$F_{11,0} = -\frac{1}{m} - \frac{\sigma_3}{\sin\frac{\pi}{k}}(\sin^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k})$$

and

$$F_{11,1} = F_{11,k-1} = \frac{\sigma_3}{2\sin\frac{\pi}{k}}(-\sin^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k}).$$

Eigenvalues of F_{11}. For any $l = 0,\cdots,k-1$, the eigenvalues of F_{11} are

$$f_{11,l} = -\frac{1}{m} - \frac{\sigma_3}{\sin\frac{\pi}{k}}(\sin^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k}) + \frac{\sigma_3}{\sin\frac{\pi}{k}}(-\sin^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k})\cos\frac{2l\pi}{k}.$$

The matrix F_{12}. The matrix F_{12} is defined by

$$F_{12} = \mathrm{Cir}\{(\frac{1}{m}, 0, \cdots, 0)\}$$

Eigenvalues of F_{12}. For any $l = 0,\cdots,k-1$, the eigenvalues of F_{12} are

$$f_{12,l} = \frac{1}{m}.$$

The matrix F_{13}. The matrix F_{13} is defined by

$$F_{13} = \mathrm{Cir}\{(0, F_{13,1}, 0, \cdots, 0, F_{13,k-1})\},$$

where

$$F_{13,1} = \frac{\sigma_3\cos\frac{\pi}{k}}{2}(1+\frac{\sigma_2}{\ell}),\ F_{13,k-1} = -\frac{\sigma_3\cos\frac{\pi}{k}}{2}(1+\frac{\sigma_2}{\ell}).$$

Eigenvalues of F_{13}. For any $l = 0,\cdots,k-1$, the eigenvalues of F_{12} are

$$f_{13,l} = i\sigma_3(1+\frac{\sigma_2}{\ell})\cos\frac{\pi}{k}\sin\frac{2l\pi}{k}. \tag{4.50}$$

The matrix F_{22}. The matrix F_{22} is defined by

$$F_{22} = \mathrm{Cir}\{(F_{22,0}, F_{22,1}, 0, \cdots, 0, F_{22,k-1})\},$$

where

$$F_{22,0} = -\frac{1}{m} + \frac{\sigma_3}{2n\sin\frac{\pi}{k}}(\sin^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k})$$

and

$$F_{22,1} = F_{22,k-1} = \frac{\sigma_3}{4n\sin\frac{\pi}{k}}(\sin^2\frac{\pi}{k} - \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k}).$$

Eigenvalues of F_{22}. For any $l = 0, \cdots, k-1$, the eigenvalues of F_{22} are

$$f_{22,l} = -\frac{1}{m} + \frac{\sigma_3}{2n\sin\frac{\pi}{k}}(\sin^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k}) + \frac{\sigma_3}{2n\sin\frac{\pi}{k}}(\sin^2\frac{\pi}{k} - \frac{\sigma_2}{\ell}\cos^2\frac{\pi}{k})\cos\frac{2l\pi}{k}. \tag{4.51}$$

The matrix F_{24}. The matrix F_{24} is defined by

$$F_{24} = \mathrm{Cir}\{(0, F_{24,1}, 0, \cdots, 0, F_{24,k-1})\},$$

where

$$F_{24,1} = -\frac{\sigma_3\sin\frac{\pi}{k}}{4n}(1+\frac{\sigma_2}{\ell}),\ F_{24,k-1} = \frac{\sigma_3\sin\frac{\pi}{k}}{4n}(1+\frac{\sigma_2}{\ell}).$$

Eigenvalues of F_{24}. For any $l = 0, \cdots, k-1$, the eigenvalues of F_{24} are

$$f_{24,l} = -\frac{i\sigma_3}{2n}(1+\frac{\sigma_2}{\ell})\cos\frac{\pi}{k}\sin\frac{2l\pi}{k}.$$

The matrix F_{33}. The matrix F_{33} is defined by

$$F_{33} = \mathrm{Cir}\{(F_{33,0}, F_{33,1}, 0, \cdots, 0, F_{33,k-1})\},$$

where

$$F_{33,0} = -\frac{\sigma_1}{m\ell} - \frac{\sigma_3}{\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k})$$

and

$$F_{33,1} = F_{33,k-1} = \frac{\sigma_3}{2\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k} - \frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k}).$$

Eigenvalues of F_{33}. For any $l = 0, \cdots, k-1$, the eigenvalues of F_{33} are

$$f_{33,l} = -\frac{\sigma_1}{m\ell} - \frac{\sigma_3}{\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k}) + \frac{\sigma_3}{\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k} - \frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k})\cos\frac{2l\pi}{k}.$$

The matrix F_{34}. The matrix F_{34} is defined by

$$F_{34} = \mathrm{Cir}\{(\frac{c_1}{m\ell}, 0, \cdots, 0)\}.$$

Eigenvalues of F_{34}. For any $l = 0, \cdots, k-1$, the eigenvalues of F_{34} are

$$f_{34,l} = \frac{\sigma_1}{m\ell}.$$

The matrix F_{44}. The matrix F_{44} is defined by

$$F_{44} = \mathrm{Cir}\{(F_{44,0}, F_{44,1}, 0, \cdots, 0, F_{44,k-1})\},$$

where

$$F_{44,0} = -\frac{\sigma_1}{m\ell} + \frac{\sigma_3}{2n\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k})$$

and

$$F_{44,1} = F_{44,k-1} = -\frac{\sigma_3}{4n\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k} - \frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k}).$$

Eigenvalues of F_{44}. For any $l = 0, \cdots, k-1$, the eigenvalues of F_{44} are

$$f_{44,l} = -\frac{\sigma_1}{m\ell} + \frac{\sigma_3}{2n\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k} + \frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k}) - \frac{\sigma_3}{2n\sin\frac{\pi}{k}}(\cos^2\frac{\pi}{k} - \frac{\sigma_2}{\ell}\sin^2\frac{\pi}{k})\cos\frac{2l\pi}{k}.$$

The final part of this section is devoted to the analysis of the matrix

$$F = \begin{pmatrix} F_{11} & F_{12} & F_{13} & 0 \\ F_{12}^t & F_{22} & 0 & F_{24} \\ F_{13}^t & 0 & F_{33} & F_{34} \\ 0 & F_{24}^t & F_{34}^t & F_{44} \end{pmatrix}. \tag{4.52}$$

Define

$$\mathcal{P}_1 = \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 00 & 0 & P \end{pmatrix},$$

where P is defined in (4.19). Simple algebra gives that

$$F = \mathcal{P}_1 \begin{pmatrix} D_{F_{11}} & D_{F_{12}} & D_{F_{13}} & 0 \\ D_{F_{12}^t} & D_{F_{22}} & 0 & D_{F_{24}} \\ D_{F_{13}^t} & 0 & D_{F_{33}} & D_{F_{34}} \\ 0 & D_{F_{24}^t} & D_{F_{34}^t} & D_{F_{44}} \end{pmatrix} \mathcal{P}_1^t.$$

Here, D_X denotes the diagonal matrix of dimension $k \times k$ whose entries are given by the eigenvalues of X.

Let us now introduce the following matrix:

$$\mathcal{D}_F = \begin{pmatrix} \mathcal{D}_{f_0} & 0 & \cdots & 0 \\ 0 & \mathcal{D}_{f_1} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & \mathcal{D}_{f_{k-1}} \end{pmatrix},$$

where

$$\mathcal{D}_{f_i} = \begin{pmatrix} f_{11,i} & f_{12,i} & f_{13,i} & 0 \\ f_{12,i} & f_{22,i} & 0 & f_{24,i} \\ -f_{13,i} & 0 & f_{33,i} & f_{34,i} \\ 0 & -f_{24,i} & f_{34,i} & f_{44,i} \end{pmatrix}.$$

By direct calculation, one can check that for $j = 0$,

$$\mathrm{Det}(\mathcal{D}_{f_0}) = 0,$$

and $\mathcal{D}_{f_0}$ has only one kernel. The other eigenvalues of $\mathcal{D}_{f_0}$ will satisfy $|\lambda_{0,i}| \geq \frac{C}{\ell^\tau}$ for some constant $C, \tau > 0$.

For $j \geq 1$, we have that

$$\begin{aligned} \mathrm{Det}(\mathcal{D}_{f_j}) &= \frac{n}{2\sigma_3^2(1+\bar{d}_2)^2(1-\bar{a})\bar{b}_j(\bar{a}(1-\bar{b}_j)+\bar{d}_2(1-\bar{a})\bar{d}_j)(\bar{b}_j - \frac{(1-\bar{a})\bar{b}_j\bar{d}_2}{\bar{a}})} \\ &\times \frac{(\bar{a}-\bar{b}_j)^2\bar{d}_2(\bar{b}_j\bar{d}_2 + \bar{a}^2(1+\bar{d}_1)(1+\bar{d}_2) - \bar{a}(1+(2+\bar{d}_1)\bar{d}_2))}{\bar{a}\bar{b}_j(1-\bar{a})(1+\bar{d}_2)^2(\bar{a}+(\bar{a}-1)\bar{d}_2)(\bar{a}(\bar{b}_j-1)+(\bar{a}-1)\bar{b}_j\bar{d}_2)} \\ &= \frac{(\bar{a}-\bar{b}_j)^2 n\bar{d}_2}{2\sigma_3^2\bar{a}^3\bar{b}_j^3(1-\bar{a})[(\bar{b}_j-1)^2+c_0 d_2]}(1+O(\frac{1}{\ell})), \end{aligned}$$

where

$$\bar{a} = \sin^2\frac{\pi}{k},\ \bar{b}_j = \sin^2\frac{j\pi}{k},\ \bar{d}_1 = \frac{\sigma_1}{\ell},\ \bar{d}_2 = \frac{\sigma_2}{\ell}.$$

From the above computation, we know that for $j = 1, k-1$, $\mathrm{Det}(\mathcal{D}_{f_j}) = 0$, the matrix $\mathcal{D}_{f_j}$ has one kernel, and all the other eigenvalues have a lower bound $\frac{C}{\ell^\tau}$. For $j \neq 0, 1, k-1$, the matrix $\mathcal{D}_{f_j}$ is non-degenerate, and the eigenvalues have a lower bound $\frac{C}{\ell^\tau}$.

From the above analysis, we know that $\bar{M}_1$ has three kernels. Furthermore, the other eigenvalues of $\bar{M}_1$ have a lower bound $\frac{C}{\ell^\tau}$. Moreover, we can check that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are in the kernels of $\bar{M}_1$. Thus, we have proved part (a) of Proposition 4.1.

5. Proof of (3.10)

First, we have the following a priori estimate on the linear equation.

Proposition 5.1. *Assume that h is a function such that $\|h\|_* < \infty$, where the norm $\|\cdot\|_*$ is defined in (2.14). Let ϕ be a solution of the following equation:*

$$L(\phi) = h, \int \phi Z^i_{j,\alpha} = 0 \text{ for } i = 0, \cdots, k-1, j = 1, \cdots, m+2n, \alpha = 1, \cdots, N. \tag{5.1}$$

Then, for large ℓ, there exists a constant C independent of ℓ such that ϕ satisfies the following estimate:

$$\|\phi\|_* \le C\|h\|_*. \tag{5.2}$$

Proof. This can be proved by contradiction. Assume that there exists $\ell_n \to \infty$ and ϕ_n, h_n corresponding to (5.1) such that

$$\|\phi_n\|_* = 1, \ \|h_n\|_* \to 0 \text{ as } n \to \infty. \tag{5.3}$$

In the following, we omit the index n in the absence of ambiguity. Following the argument in Proposition 3.1 in [19], one can obtain that there exists $R^i_k y_j$ such that

$$\|\phi\|_{L^\infty(B(R^i_k y_j,\rho))} \ge C > 0, \tag{5.4}$$

for some fixed C and large ρ. By employing elliptic estimates together with the Ascoli-Arzela theorem, we can find a sequence $R^i_k y_j$ such that $\phi(x + R^i_k y_j)$ converges to ϕ_∞, which is a solution of

$$\Delta\phi_\infty - \phi_\infty + pw^{p-1}\phi_\infty = 0,$$

and satisfies the following orthogonality conditions:

$$\int \phi_\infty \frac{\partial w}{\partial x_\alpha} = 0, \ \alpha = 1, \cdots, N.$$

Thus, $\phi_\infty = 0$. This contradicts (5.4), and so this completes the proof. □

Because

$$L(Z^i_{j,\alpha}) = p(|u|^{p-1} - w^{p-1}(x - R^i_k y_j))Z^i_{j,\alpha} + O(e^{-\frac{1+\xi}{2}\ell}),$$

one can easily verify that

$$\|L(Z^i_{j,\alpha})\|_* \le Ce^{-\frac{1+\xi}{2}\ell} \tag{5.5}$$

for some ξ that is independent of ℓ, which is assumed to be large, where we have applied the estimate (2.18).

Thus, from Proposition 5.1 and the estimate (5.5), we obtain that

$$\|\varphi^\perp\|_* \le Ce^{-\frac{1+\xi}{2}\ell}\sum_{\alpha=1}^N \|\mathbf{c}_\alpha\|. \tag{5.6}$$

6. Proof of Proposition 3.1

Let us consider (3.11), with $\beta = 1$. That is,

$$\sum_{\alpha=1}^N \mathbf{c}_\alpha \cdot \int \mathbf{Z}_\alpha z_1 = -\int \varphi^\perp z_1. \tag{6.1}$$

First, we write $f_1 = -\frac{\int \varphi^\perp z_1}{\int (\frac{\partial w(x)}{\partial x_1})^2}$. A straightforward computation gives that $|f_1| \le C(2n+m) \times k\|\varphi^\perp\|_* \le \ell^\tau\|\varphi^\perp\|_*$, for a certain constant τ that is independent of ℓ, which is assumed to be large, where we have assumed that m, n satisfy (2.9). Second, by direct computation we find for $j = 1, \cdots, m+1$ that

$$\begin{aligned}
\int Z^i_{j,1} z_1 \quad &= \int \frac{\partial u}{\partial x_1} \mathbf{R}_i \cdot \nabla w(x - R^i_k y_j) dx + O(e^{-\frac{1+\xi}{2}\ell}) \\
&= \int_{B_{\frac{\ell}{2}}(R^i_k y_j)} \frac{\partial u}{\partial x_1} \mathbf{R}_i \cdot \nabla w(x - R^i_k y_j) dx \\
&+ \int_{\mathbb{R}^N - B_{\frac{\ell}{2}}(R^i_k y_j)} \frac{\partial u}{\partial x_1} \mathbf{R}_i \cdot \nabla w(x - R^i_k y_j) dx + O(e^{-\frac{1+\xi}{2}\ell}) \\
&= \cos\theta_i \int (\frac{\partial w}{\partial x_1})^2 dx + O(e^{-\frac{(1+\xi)\ell}{2}}).
\end{aligned}$$

Similarly, one can obtain that

$$\begin{aligned}
\int Z^i_{j,2} z_1 \quad &= \int \frac{\partial u}{\partial x_1} \mathbf{R}_i^{\perp} \cdot \nabla w(x - R^i_k y_j) dx + O(e^{-\frac{1+\xi}{2}\ell}) \\
&= \sin\theta_i \int (\frac{\partial w}{\partial x_1})^2 dx + O(e^{-\frac{(1+\xi)\ell}{2}}),
\end{aligned}$$

and for $\alpha = 3, \cdots, N$,

$$\int Z^i_{j,\alpha} z_1 = \int \frac{\partial u}{\partial x_1} \frac{\partial}{\partial x_\alpha} w(x - R^i_k y_j) = 0,$$

by the evenness of u in x_α.

Moreover, for $j = m+2, \cdots, m+2n$ we have that

$$\int Z^i_{j,1} z_1 = -\sin(\theta_i + \frac{\pi}{k}) \int (\frac{\partial w}{\partial x_1})^2 + O(e^{-\frac{(1+\xi)\ell}{2}}),$$

$$\int Z^i_{j,2} z_1 = \cos(\theta_i + \frac{\pi}{k}) \int (\frac{\partial w}{\partial x_1})^2 + O(e^{-\frac{(1+\xi)\ell}{2}}),$$

and

$$\int Z^i_{j,\alpha} z_1 = 0$$

for $\alpha = 3, \cdots, N$.

A direct consequence of the above calculation is that

$$\begin{aligned}
\sum_{\alpha=1}^{N} \mathbf{c}_\alpha \cdot \int \mathbf{Z}_\alpha z_1 \quad &= \mathbf{c_1} \cdot \begin{pmatrix} \mathbf{cos}_k \\ \mathbf{cos}_k \\ \mathbf{cos}(\theta_\mathbf{0}) \\ \vdots \\ \mathbf{cos}(\theta_\mathbf{k-1}) \\ -\mathbf{sin}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \\ \vdots \\ -\mathbf{sin}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \end{pmatrix} + \mathbf{c}_2 \cdot \begin{pmatrix} \mathbf{sin}_k \\ \mathbf{sin}_k \\ \mathbf{sin}(\theta_\mathbf{0}) \\ \vdots \\ \mathbf{sin}(\theta_\mathbf{k-1}) \\ \mathbf{cos}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \\ \vdots \\ \mathbf{cos}(\theta_\mathbf{0} + \frac{\pi}{\mathbf{k}}) \end{pmatrix} \\
&+ O(e^{-\frac{(1+\xi)\ell}{2}}) \mathcal{L} \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_N \end{pmatrix},
\end{aligned}$$

where $\mathcal{L}$ is a linear function whose coefficients are uniformly bounded in ℓ as ℓ to ∞. Thus, (3.12) follows straightforwardly. The proofs of (3.13) to (3.17) are similar, and left to the reader.

7. Proof of Proposition 3.2

In this section, we prove Proposition 3.2. A key ingredient in the proof is the estimates on the right hand side of (3.18). We have the following result.

Proposition 7.1. *There exist positive constants C and ξ such that for $\alpha = 1, \cdots, N$, it holds that*

$$\|\mathbf{r}_\alpha\| \le Ce^{-\frac{1+\xi}{2}\ell}\|\varphi^\perp\|_* \tag{7.1}$$

for any sufficiently large ℓ.

Proof. Recall that

$$\mathbf{r}_\alpha = \begin{pmatrix} \int L(\varphi^\perp)Z_{v,\alpha} \\ \int L(\varphi^\perp)Z^0_{Y_1,\alpha} \\ \vdots \\ \int L(\varphi^\perp)Z^{k-1}_{Y_1,\alpha} \\ \int L(\varphi^\perp)Z^0_{Y_2,\alpha} \\ \vdots \\ \int L(\psi^\perp)Z^{k-1}_{Y_2,\alpha} \end{pmatrix}.$$

Then, the estimate follows from

$$|\int L(\varphi^\perp)Z^i_{j,\alpha}| \le Ce^{-\frac{1+\xi}{2}\ell}\|\varphi^\perp\|_*.$$

To prove the above estimate, we fix, for example $j = 1, i = 0, \alpha = 3$, and we write

$$\begin{aligned}
\int L(\varphi^\perp)Z^0_{1,3} &= \int L(Z^0_{1,3})\varphi^\perp \\
&= \int L(\frac{\partial w(x-y_1)}{\partial x_3})\varphi^\perp + L(\frac{\pi_3}{k})\varphi^\perp \\
&= \int p(|u|^{p-1} - w^{p-1}(x-y_1))\frac{\partial w(x-y_1)}{\partial x_3}\varphi^\perp + O(e^{-\frac{1+\xi}{2}}\ell)\|\varphi^\perp\|_* \\
&\le C\int w^{p-2}(x-y_1)|\frac{\partial w(x-y_1)}{\partial x_3}|\sum_{z\in\Pi_{y_1}} w(x-z) + O(e^{-\frac{1+\xi}{2}}\ell)\|\varphi^\perp\|_* \\
&\le Ce^{-\frac{1+\xi}{2}}\ell\|\varphi^\perp\|_*
\end{aligned}$$

for some $\xi > 0$ that is independent of ℓ, which is assumed to be large, where we have used the estimate for ϕ in (2.18), i.e.,

$$\|\phi\|_* \le Ce^{-\frac{1+\xi}{2}\ell}.$$

Thus, we have proved the estimate for $\alpha = 3$. The other cases can be treated similarly.

□

We have now the tools for the following proof.

Proof of Proposition 3.2. By Proposition 4.1, we need only show the following orthogonality conditions:

$$\begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} \cdot \mathbf{w}_1 = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} \cdot \mathbf{w}_2 = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} \cdot \mathbf{w}_3 = 0 \tag{7.2}$$

and

$$\mathbf{r}_\alpha \cdot \mathbf{w}_4 = \mathbf{r}_\alpha \cdot \mathbf{w}_5 = \mathbf{r}_\alpha \cdot \mathbf{w}_6 = 0. \tag{7.3}$$

First, recall that $L(z_1)=0$. Then, we have that

$$\int L(z_1)\varphi^{\perp}=\int L(\varphi^{\perp})z_1=0.$$

This gives us precisely the first orthogonality condition in (7.2). Similarly, from $L(z_2)=0$ one can obtain that

$$\int L(z_2)\varphi^{\perp}=\int L(\varphi^{\perp})z_2=0.$$

This gives us precisely the second orthogonality condition in (7.2).

For $\alpha=3,\cdots,N$, it follows from $L(z_\alpha)=0$ that

$$\int L(z_\alpha)\varphi^{\perp}=\int L(\varphi^{\perp})z_\alpha=0.$$

This gives us precisely the first orthogonality condition in (7.3).

Second, let us recall that

$$L(\varphi^{\perp})=-\sum_{\alpha=1}^{N}\mathbf{c}_\alpha\cdot L(\mathbf{Z}_\alpha). \tag{7.4}$$

Thus, the function $x\to L(\varphi^{\perp})(x)$ is invariant under a rotation by the angle $\frac{2\pi}{k}$ in the (x_1,x_2) plane. Therefore, we can obtain that

$$\sum_{i=0}^{k-1}(\sum_{j=1}^{m+1}\int L(\varphi^{\perp})|y_j|Z_{j,2}^i+\sum_{j=m+2}^{m+2n}\int L(\varphi^{\perp})(R_k^iy_j\cdot\mathbf{n}_iZ_{j,1}^i-R_k^iy_j\cdot\mathbf{t}_iZ_{j,2}^i)=0.$$

This gives us the third orthogonality condition in (7.2).

For $\alpha=3,\cdots,N$, we can obtain that

$$\sum_{i=0}^{k-1}\sum_{j=1}^{2m+n}\int L(\varphi^{\perp})Z_{j,\alpha}^iR_k^iy_j\cdot\mathbf{e}_1=0,$$

and

$$\sum_{i=0}^{k-1}\sum_{j=1}^{2m+n}\int L(\varphi^{\perp})Z_{j,\alpha}^iR_k^iy_j\cdot\mathbf{e}_2=0.$$

These give the final two orthogonality conditions in (7.3).

By combining the results of Proposition 4.1 and the a priori estimates in (7.1), we obtain the proof of Proposition 3.2.

8. Some useful computations

In this section, we compute the entries of the matrices A, B, C, and H_α for $\alpha=3,\cdots,N$.

We first introduce the following useful functions:

$$\Psi_1(\ell)=\int div(w^p(x)\mathbf{e})div(w(x-\ell\mathbf{e})\mathbf{e})dx, \tag{8.1}$$

$$\Psi_2(\ell)=\int div(w^p(x)\mathbf{e}^{\perp})div(w(x-\ell\mathbf{e})\mathbf{e}^{\perp})dx, \tag{8.2}$$

where $\mathbf{e}$ is any unit vector. It is simple to check that this definition is independent of the choice of the unit vector $\mathbf{e}$. It is known that

$$\Psi_1(\ell)=C_{N,p,1}e^{-\ell}\ell^{-\frac{N-1}{2}}(1+O(\frac{1}{\ell})) \tag{8.3}$$

and

$$\Psi_2(\ell)=C_{N,p,2}e^{-\ell}\ell^{-\frac{N+1}{2}}(1+O(\frac{1}{\ell})), \tag{8.4}$$

where $C_{N,p,i}>0$ are constants that depend only on p and N. See, for example, [19, 18] for details.

In fact, one can see from the definitions of Ψ_1 and Ψ that

$$\Psi_1'(\ell) = 2\sin\frac{\pi}{k}\Psi_1'(\bar\ell)\frac{d\bar\ell}{d\ell}. \tag{8.5}$$

By these two definitions, one can easily obtain that

$$\int pw^{p-1}\mathbf{a}\cdot\nabla w(x)\mathbf{b}\cdot\nabla w(x-\ell\mathbf{e}) = (\mathbf{a}\cdot\mathbf{e})(\mathbf{b}\cdot\mathbf{e})\Psi_1(\ell) + (\mathbf{a}\cdot\mathbf{e}^\perp)(\mathbf{b}\cdot\mathbf{e}^\perp)\Psi_2(\ell). \tag{8.6}$$

Computation of A.

$$\begin{aligned}\int L(Z_{1,1}^0)Z_{1,1}^0dx &= \int p(|u|^{p-1} - w^{p-1}(x-y_1))(\frac{\partial w(x-y_1)}{\partial x_1})^2\\ &= p(p-1)\int w^{p-2}(x-y_1)(\phi + \sum_{z\in\Pi_{y_1}} w(x-z))(\frac{\partial w(x-y_1)}{\partial x_1})^2\\ &+ O(e^{-(1+\xi)\ell}),\end{aligned}$$

where Π_{y_1} denotes the set of closest neighbors of y_1 of Π defined in (2.3). Recall that ϕ solves the following equation:

$$\Delta\phi - \phi + p|U|^{p-1}\phi + E + N(\phi) = 0, \tag{8.7}$$

where

$$E = \Delta U - U + |U|^{p-1}U$$

and

$$N(\phi) = |U+\phi|^{p-1}(U+\phi) - |U|^{p-1}U - p|U|^{p-1}\phi.$$

Hence, one has that

$$\begin{aligned}&p(p-1)\int w^{p-2}\phi(\frac{\partial w(x-y_1)}{\partial x_1})^2\\ &= \int \frac{\partial}{\partial x_1}(pw^{p-1}(x-y_1))\phi\frac{\partial}{\partial x_1}w(x-y_1)\\ &= -\int pw^{p-1}(x-y_1)\frac{\partial}{\partial x_1}(\phi\frac{\partial w(x-y_1)}{\partial x_1})\\ &= -\int pw^{p-1}(x-y_1)\frac{\partial w(x-y_1)}{\partial x_1}\frac{\partial\phi}{\partial x_1} - \int pw^{p-1}(x-y_1)\phi\frac{\partial^2 w(x-y_1)}{\partial x_1^2}\\ &= \int [p(|U|^{p-1} - w^{p-1}(x-y_1))\phi + E + N(\phi)]\frac{\partial^2 w(x-y_1)}{\partial x_1^2}\\ &= \int E\frac{\partial^2 w(x-y_1)}{\partial x_1^2} + O(e^{-(1+\xi)\ell})\\ &= \int (|U|^{p-1}U - w^p(x-y_1))\frac{\partial^2 w(x-y_1)}{\partial x_1^2} + O(e^{-(1+\xi)\ell})\\ &= \int pw^{p-1}(x-y_1)(\sum_{z\in\Pi_{y_1}} w(x-z))\frac{\partial^2 w(x-y_1)}{\partial x_1^2} + O(e^{-(1+\xi)\ell}).\end{aligned}$$

Taking this into account, we have that

$$\begin{aligned}\int L(Z^0_{1,1})Z^0_{1,1}dx &= p(p-1)\int w^{p-2}(\sum_{z\in\Pi_{y_1}} w(x-z))(\frac{\partial w(x-y_1)}{\partial x_1})^2\\ &+ \int pw^{p-1}(x-y_1)(\sum_{z\in\Pi_{y_1}} w(x-z))\frac{\partial^2 w(x-y_1)}{\partial x_1^2}+O(e^{-(1+\xi)\ell})\\ &= \int \frac{\partial}{\partial x_1}(pw^{p-1}(x-y_1)\frac{\partial w(x-y_1)}{\partial x_1})\sum_{z\in\Pi_{y_1}} w(x-z))+O(e^{-(1+\xi)\ell})\\ &= -p\int w^{p-1}(x-y_1)\frac{\partial w(x-y_1)}{\partial x_1}\sum_{z\in\Pi_{y_1}}\frac{\partial w(x-z)}{\partial x_1})+O(e^{-(1+\xi)\ell}).\end{aligned}$$

By (8.6), we can obtain that

$$\int L(Z^0_{1,\alpha})Z^0_{1,\alpha}dx = -[\Psi_1(\ell)+2(\Psi_1(\bar\ell)\sin^2\frac{\pi}{k}+\Psi_2(\bar\ell)\cos^2\theta_0)].$$

Next, we consider

$$\begin{aligned}\int L(Z^0_{1,1})Z^1_{1,1} &= \int(p|u|^{p-1}-pw^{p-1}(x-y_1))\frac{\partial w(x-y_1)}{\partial x_1}R^1_k\cdot\nabla w(x-R^1_k y_1)\\ &= \int pw^{p-1}(x-R^1_k y_1)\frac{\partial w(x-y_1)}{\partial x_1}R^1_k\cdot\nabla w(x-R^1_k y_1)+O(e^{-(1+\xi)\ell})\\ &= -\sin^2\frac{\pi}{k}\Psi_1(\bar\ell)+\cos^2\frac{\pi}{k}\Psi_2(\bar\ell)+O(e^{-(1+\xi)\ell}).\end{aligned}$$

Similarly, one can obtain that

$$\begin{aligned}&\int L(Z^0_{1,1})Z^{k-1}_{1,1} = -\sin^2\frac{\pi}{k}\Psi_1(\bar\ell)+\cos^2\frac{\pi}{k}\Psi_2(\bar\ell)+O(e^{-(1+\xi)\ell}),\\ &\int L(Z^0_{1,1})Z^j_{i,1} = O(e^{-(1+\xi)\ell}) \text{ for } (i,j)\neq(1,0),(1,1),(1,k-1),(2,0).\end{aligned}$$

Another observation is that

$$\int L(Z^s_{i,1})Z^t_{j,1} = \int L(Z^0_{i,1})Z^{t-s}_{j,1},$$

where we use the notation $Z^{t-s}_{j,1}=Z^{k+t-s}_{j,1}$ if $t-s<0$.

Moreover, for $i\geq 2$, it holds that

$$\int L(Z^s_{i,1})Z^t_{j,1} = \begin{cases} -2\Psi_1(\ell)+O(e^{-(1+\xi)\ell}), \text{ if } i=j, s=t, i\leq m,\\ \Psi_1(\ell)+O(e^{-(1+\xi)\ell}) \text{ if } j=i-1 \text{ or } i+1, s=t\ i\leq m,\\ 2\Psi_1(\bar\ell)+O(e^{-(1+\xi)\ell}) \text{ if } i=j, s=t,\ m+2\leq i\leq 2n+m,\\ \Psi_1(\bar\ell)+O(e^{-(1+\xi)\ell}) \text{ if } j=i-1 \text{ or } i+1, s=t,\ m+2\leq i\leq 2n+m,\\ -[\Psi_1(\ell)-2(\Psi_1(\bar\ell)\sin^2\frac{\pi}{k}+\Psi_2(\bar\ell)\cos^2\frac{\pi}{k})]+O(e^{-(1+\xi)\ell})\\ \qquad\qquad \text{if } i,j=m+1, s=t,\\ \Psi_1(\bar\ell)\sin\frac{\pi}{k}+O(e^{-(1+\xi)\ell}) \text{ if } (i,j)=(m+1,m+2), s=t,\\ -\Psi_1(\bar\ell)\sin\frac{\pi}{k}+O(e^{-(1+\xi)\ell}) \text{ if } (i,j)=(m+1,m+2n), t=s-1,\\ O(e^{-(1+\xi)\ell}) \quad \text{otherwise}.\end{cases}$$

Computation of C: Similarly, we have that

$$\begin{aligned}\int L(Z^0_{1,2})Z^0_{1,2} &= -p\int w^{p-1}(x-y_1)\frac{\partial w(x-y_1)}{\partial x_2}\sum_{z\in\Pi_{y_1}}\frac{\partial w(x-z)}{\partial x_2}) + O(e^{-(1+\xi)\ell})\\ &= -\Big[\Psi_2(\ell) + 2\Big(\Psi_1(\bar\ell)\cos^2\frac{\pi}{k} + \Psi_2(\bar\ell)\sin^2\frac{\pi}{k}\Big)\Big] + O(e^{-(1+\xi)\ell})\end{aligned}$$

and

$$\int L(Z^0_{1,2})Z^j_{i,2} = \begin{cases} \Psi_1(\bar\ell)\cos^2\frac{\pi}{k} - \Psi_2(\bar\ell)\sin^2\frac{\pi}{k} + O(e^{-(1+\xi)\ell}) \\ \qquad\qquad\qquad\qquad\qquad \text{if } (i,j) = (1,1) \text{ or } (1,k-1), \\ O(e^{-(1+\xi)\ell}) \quad \text{otherwise.}\end{cases} \tag{8.8}$$

Furthermore, for $i \geq 2$ one can obtain that

$$\int L(Z^s_{i,2})Z^t_{j,2} = \begin{cases} -2\Psi_2(\ell) + O(e^{-(1+\xi)\ell}), \text{ if } i=j, s=t, i\leq m, \\ \Psi_2(\ell) + O(e^{-(1+\xi)\ell}) \text{ if } j = i-1 \text{ or } i+1, s=t\ i\leq m, \\ 2\Psi_2(\bar\ell) + O(e^{-(1+\xi)\ell}) \text{ if } i=j, s=t,\ m+2\leq i\leq 2n+m, \\ \Psi_2(\bar\ell) + O(e^{-(1+\xi)\ell}) \text{ if } j=i-1 \text{ or } i+1, s=t,\ m+2\leq i\leq 2n+m, \\ -[\Psi_2(\ell) - 2(\Psi_1(\bar\ell)\cos^2\frac{\pi}{k} + \Psi_2(\bar\ell)\sin^2\frac{\pi}{k})] + O(e^{-(1+\xi)\ell}) \\ \qquad\qquad\qquad\qquad \text{if } i,j = m+1, s=t, \\ \Psi_2(\bar\ell)\sin\frac{\pi}{k} + O(e^{-(1+\xi)\ell}) \text{ if } (i,j) = (m+1,m+2), s=t, \\ -\Psi_2(\bar\ell)\sin\frac{\pi}{k} + O(e^{-(1+\xi)\ell}) \text{ if } (i,j) = (m+1, m+2n), t = s-1, \\ O(e^{-(1+\xi)\ell}) \quad \text{otherwise .}\end{cases}$$

Computation of B: Next, we consider $\int L(Z^s_{i,1})Z^t_{j,2}$ and $\int L(Z^s_{i,2})Z^t_{j,1}$. First, by the symmetry we have that

$$\int L(Z^0_{1,1})Z^0_{1,2} = 0$$

and

$$\begin{aligned}\int L(Z^0_{1,1})Z^1_{1,2} &= \int pw^{p-1}(x-R^1_k y_1)\frac{\partial w(x-y_1)}{\partial x_1}R^{1,\perp}_k\cdot\nabla w(x-R^1_k y_1)\\ &= \sin\frac{\pi}{k}\cos\frac{\pi}{k}\Big(\Psi_1(\bar\ell)+\Psi_2(\bar\ell)\Big) + O(e^{-(1+\xi)\ell}).\end{aligned}$$

Similarly, we can obtain that

$$\int L(Z^0_{1,1})Z^{k-1}_{1,2} = -\sin\frac{\pi}{k}\cos\frac{\pi}{k}\Big(\Psi_1(\bar\ell)+\Psi_2(\bar\ell)\Big) + O(e^{-(1+\xi)\ell}),$$

$$\int L(Z^0_{m+1,1})Z^0_{m+2,2} = \int L(Z^0_{m+1,1})Z^{k-1}_{2n+m,2} = -\Psi_2(\bar\ell)\cos\frac{\pi}{k} + O(e^{-(1+\xi)\ell}),$$

and

$$\int L(Z^s_{i,1})Z^t_{j,2} = O(e^{-(1+\xi)\ell}) \quad \text{otherwise .}$$

Similarly, we have the following expansion for $\int L(Z_{i,2}^s)Z_{j,1}^t$:

$$\int L(Z_{i,2}^s)Z_{j,1}^t = \begin{cases} -\sin\frac{\pi}{k}\cos\frac{\pi}{k}\Big(\Psi_1(\bar\ell)+\Psi_2(\bar\ell)\Big)+O(e^{-(1+\xi)\ell}) \text{ if } i,j=1, t=s-1, \\ \sin\frac{\pi}{k}\cos\frac{\pi}{k}\Big(\Psi_1(\bar\ell)+\Psi_2(\bar\ell)\Big)+O(e^{-(1+\xi)\ell}) \text{ if } i,j=1, t=s+1, \\ \Psi_1(\bar\ell)\cos\frac{\pi}{k}+O(e^{-(1+\xi)\ell}) \text{ if } (i,j)=(m+2,m+1), s=t \text{ or} \\ \qquad (i,j)=(2n+m,m+1), t=s+1, \\ O(e^{-(1+\xi)\ell}) \quad \text{otherwise}. \end{cases}$$

Computation of H_α: For the matrix H_α for $\alpha=3,\cdots,N$, the computation is simpler, we directly employ (8.6) to obtain the following expansion:

$$\int L(Z_{i,\alpha}^s)Z_{j,\alpha}^t = \begin{cases} -\Big(\Psi_2(\ell)+2\Psi_2(\bar\ell)\Big)+O(e^{-(1+\xi)\ell}) \text{ if } (i,j)=(1,1), s=t, \\ \Psi_2(\bar\ell)+O(e^{-(1+\xi)\ell}) \text{ if } (i,j)=(1,1), t=s-1 \text{ or } s+1, \\ \Big(2\Psi_2(\bar\ell)-\Psi_2(\ell)\Big)+O(e^{-(1+\xi)\ell}) \text{ if } (i,j)=(m+1,m+1), s=t, \\ -2\Psi_2(\ell)+O(e^{-(1+\xi)\ell}) \text{ if } i=j, s=t, 2\le i\le m, \\ \Psi_2(\ell)+O(e^{-(1+\xi)\ell}) \text{ if } j=i+1 \text{ or } i-1, s=t, 2\le i\le m, \\ 2\Psi_2(\bar\ell)+O(e^{-(1+\xi)\ell}) \text{ if } i=j, s=t, m+2\le i\le m+2n, \\ \Psi_2(\bar\ell)+O(e^{-(1+\xi)\ell}) \text{ if } j=i+1 \text{ or } i-1, s=t, m+2\le i\le m+2n, \\ O(e^{-(1+\xi)\ell}) \quad \text{otherwise}. \end{cases}$$

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