NONDEGENERACY OF NONRADIAL NODAL SOLUTIONS TO YAMABE PROBLEM

MONICA MUSSO AND JUNCHENG WEI

Abstract: We prove the existence of a sequence of *nondegenerate*, in the sense of Duyckaerts-Kenig-Merle [9], nodal nonradial solutions to the critical Yamabe problem

$$-\Delta Q = |Q|^{\frac{2}{n-2}} Q, \quad Q \in \mathcal{D}^{1,2}(\mathbb{R}^n).$$

This is the first example in the literature of nondegeneracy for nodal nonradial solutions of nonlinear elliptic equations and it is also the only nontrivial example for which the result of Duyckaerts-Kenig-Merle [9] applies.

1. Introduction

In this paper we consider the critical Yamabe problem

$$-\Delta u = \frac{n(n-2)}{4} |u|^{\frac{4}{n-2}} u, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n) \tag{1.1}$$

where $n \geq 3$ and $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm $\sqrt{\int_{\mathbb{R}^n} |\nabla u|^2}$.

If $u > 0$ Problem (1.1) is the conformally invariant Yamabe problem. For sign-changing u Problem (1.1) corresponds to the steady state of the energy-critical focusing nonlinear wave equation

$$\partial_t^2 u - \Delta u - |u|^{\frac{4}{n-2}} u = 0, \ (t, x) \in \mathbb{R} \times \mathbb{R}^n. \tag{1.2}$$

These are classical problems that have attracted the attention of several researchers in order to understand the structure and properties of the solutions to Problems (1.1) and (1.2).

Denote the set of non-zero finite energy solutions to Problem (1.1) by

$$\Sigma := \left\{ Q \in \mathcal{D}^{1,2}(\mathbb{R}^n) \backslash \{0\} : \ -\Delta Q = \frac{n(n-2)}{4} |Q|^{\frac{4}{n-2}} Q \right\}. \tag{1.3}$$

This set has been completely characterized in the class of positive solutions to Problem (1.1) by the classical work of Caffarelli-Gidas-Spruck [5] (see also [2, 24, 31]): all positive solutions to (1.1) are radially symmetric around some point $a \in \mathbb{R}^n$ and are of the form

$$W_{\lambda,a}(x) = \Big(\frac{\lambda}{\lambda^2 + |x-a|^2} \Big)^{\frac{n-2}{2}}, \ \lambda > 0. \tag{1.4}$$

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Much less is known in the sign-changing case. A direct application of Pohozaev's identity gives that all sign-changing solutions to Problem (1.1) are non-radial. The existence of elements of Σ that are nonradial sign-changing, and with arbitrary large energy was first proved by Ding [6] using Ljusternik-Schnirelman category theory. Indeed, via stereographic projection to S^n Problem (1.1) becomes

$$\Delta_{S^n} v + \frac{n(n-2)}{4}(|v|^{\frac{4}{n-2}} v - v) = 0 \quad \text{in } S^n,$$

(see for instance [30], [14]) and Ding showed the existence of infinitely many critical points to the associated energy functional within functions of the form

$$v(x) = v(|x_1|, |x_2|), \quad x = (x_1, x_2) \in S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^k \times \mathbb{R}^{n+1-k}, \quad k \geq 2,$$

where compactness of critical Sobolev's embedding holds, for any $n \geq 3$. No other qualitative properties are known for the corresponding solutions. Recently more explicit constructions of sign changing solutions to Problem (1.1) have been obtained by del Pino-Musso-Pacard-Pistoia [7, 8]. However so far only existence is available, and there are no rigidity results on these solutions.

The main purpose of this paper is to prove that these solutions are rigid, up to the transformations of the equation. In other words, these solutions are *nondegenerate*, in the sense of the definition introduced by Duyckaerts-Kenig-Merle in [9]. Following [9], we first find out all possible invariances of the equation (1.1). Equation (1.1) is invariant under the following four transformations:

(1) (translation): If $Q \in \Sigma$ then $Q(x + a) \in \Sigma, \forall a \in \mathbb{R}^n$;

(2) (dilation): If $Q \in \Sigma$ then $\lambda^{\frac{n-2}{2}} Q(\lambda x) \in \Sigma, \forall \lambda > 0$;

(3) (orthogonal transformation): If $Q \in \Sigma$ then $Q(Px) \in \Sigma$ where $P \in O_n$ and O_n is the classical orthogonal group;

(4) (Kelvin transformation): If $Q \in \Sigma$ then $|x|^{2-N} Q(\frac{x}{|x|^2}) \in \Sigma$.

If we denote by $\mathcal{M}$ the group of isometries of $\mathcal{D}^{1,2}(\mathbb{R}^n)$ generated by the previous four transformations, a result of Duyckaerts-Kenig-Merle [Lemma 3.8,[9]] states that $\mathcal{M}$ generates an $N-$parameter family of transformations in a neighborhood of the identity, where the dimension N is given by

$$N = 2n + 1 + \frac{n(n-1)}{2}. \tag{1.5}$$

In other words, if $Q \in \Sigma$ we denote

$$L_Q := -\Delta - \frac{n(n+2)}{4} |Q|^{\frac{4}{n-2}}$$

the linearized operator around Q. Define the null space of L_Q

$$\mathcal{Z}_Q = \left\{ f \in \mathcal{D}^{1,2}(\mathbb{R}^n) : \ L_Q f = 0 \right\} \tag{1.6}$$

The elements in $\mathcal{Z}_Q$ generated by the family of transformations $\mathcal{M}$ define the following vector space

$$\tilde{\mathcal{Z}}_Q = \text{span}\begin{Bmatrix} (2-n)x_jQ + |x|^2\partial_{x_j}Q - 2x_jx\cdot\nabla Q, & \partial_{x_j}Q, & 1\le j\le n, \\ \\ (x_j\partial_{x_k} - x_k\partial_{x_j})Q, \quad 1\le j<k\le n, & \frac{n-2}{2}Q + x\cdot Q & \end{Bmatrix}. \tag{1.7}$$

Observe that the dimension of $\tilde{\mathcal{Z}}_Q$ is at most N, but in principle it could be strictly less than N. For example in the case of the positive solutions $Q = W$, it turns out that the dimension of $\tilde{\mathcal{Z}}_Q$ is $n+1$ as a consequence of being Q radially symmetric. Indeed, it is known that

$$\tilde{\mathcal{Z}}_W = \left\{\frac{n-2}{2}W + x\cdot\nabla W, \quad \partial_{x_j}W, \quad 1\le j\le n\right\}. \tag{1.8}$$

Duyckaerts-Kenig-Merle [9] introduced the following definition of nondegeneracy for a solution of Problem (1.1): $Q\in\Sigma$ is said to be *nondegenerate* if

$$\mathcal{Z}_Q = \tilde{\mathcal{Z}}_Q. \tag{1.9}$$

So far the only nondegeneracy example of $Q\in\Sigma$ is the positive solution W. The proof of this fact relies heavily on the radial symmetry of W and it is straightforward: In fact since $Q = W$ is radially symmetric (around some point) one can decompose the linearized operator into Fourier modes, getting (1.9) as consequence of a simple ode analysis. See also [27]. In the case of nodal (nonradial) solutions this strategy no longer works out. In fact, as far as the authors know, there are no results in the literature on nondegeneracy of nodal nonradial solutions for nonlinear elliptic equations in the whole space. For positive radial solutions there have been many results. We refer to Frank-Lenzmann [12], Frank-Lenzmann-Silvestre [13], Kwong [21] and the references therein.

The knowledge of nondegeneracy is a crucial ingredient to show the soliton resolution for a solution to the energy-critical wave equation (1.2) with the *compactness property* obtained by Kenig and Merle in [16, 17]. If the dimension n is $3, 4$ or 5, and under the above nondegeneracy assumption, they prove that any non zero such solution is a sum of stationary solutions and solitary waves that are Lorentz transforms of the former. See also Duyckaerts, Kenig and Merle [10, 11]. Nondegeneracy also plays a vital role in the study of Type II blow-up solutions of (1.2). We refer to Krieger, Schlag and Tataru [20], Rodnianski and Sterbenz [26] and the references therein.

The main result of this paper can be stated as follows:

Main Result: *There exists a sequence of nodal solutions to (1.1), with arbitrary large energy, such that they are nondegenerate in the sense of (1.9).*

Now let us be more precise.

Let

$$f(t) = \gamma\,|t|^{p-1}\,t, \quad \text{for} \quad t\in\mathbb{R}, \quad \text{and} \quad p = \frac{n+2}{n-2}. \tag{1.10}$$

The constant $\gamma > 0$ is chosen for normalization purposes to be

$$\gamma = \frac{n(n-2)}{4}.$$

In [7], del Pino, Musso, Pacard and Pistoia showed that Problem

$$\Delta u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{1.11}$$

admits a sequence of entire non radial sign changing solutions with finite energy. To give a first description of these solutions, let us introduce some notations. Fix an integer k. For any integer $l = 1, \ldots, k$, we define angles θ_l and vectors n_l, t_l by

$$\theta_l = \frac{2\pi}{k}(l-1), \quad \mathsf{n}_l = (\cos\theta_l, \sin\theta_l, \mathsf{0}), \quad \mathsf{t}_l = (-\sin\theta_l, \cos\theta_l, \mathsf{0}). \tag{1.12}$$

Here $\mathsf{0}$ stands for the zero vector in $\mathbb{R}^{n-2}$. Notice that $\theta_1 = 0$, $\mathsf{n}_1 = (1, 0, \mathsf{0})$, and $\mathsf{t}_1 = (0, 1, \mathsf{0})$.

In [7] it was proved that there exists k_0 such that for all integer $k > k_0$ there exists a solution u_k to (1.11) that can be described as follows

$$u_k(x) = U_*(x) + \tilde\phi(x) \tag{1.13}$$

where

$$U_*(x) = U(x) - \sum_{j=1}^k U_j(x), \tag{1.14}$$

while $\tilde\phi$ is smaller than U_*. The functions U and U_j are positive solutions to (1.11), respectively defined as

$$U(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}, \quad U_j(x) = \mu_k^{-\frac{n-2}{2}} U(\mu_k^{-1}(x - \xi_j)). \tag{1.15}$$

For any integer k large, the parameters $\mu_k > 0$ and the k points ξ_l, $l = 1, \ldots, k$ are given by

$$\left[\sum_{l>1}^k \frac{1}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\right] \mu_k^{\frac{n-2}{2}} = \left(1 + O(\frac{1}{k})\right), \quad \text{for} \quad k \to \infty \tag{1.16}$$

in particular, as $k \to \infty$, we have

$$\mu_k \sim k^{-2} \quad \text{if} \quad n \geq 4, \quad \mu_k \sim k^{-2} |\log k|^{-2} \quad \text{if} \quad n = 3.$$

Furthermore

$$\xi_l = \sqrt{1-\mu^2}\,(\mathsf{n}_l, 0). \tag{1.17}$$

The functions U, U_j and U_* are invariant under rotation of angle $\frac{2\pi}{k}$ in the x_1, x_2 plane, namely

$$U(e^{\frac{2\pi}{k}} \bar x, x') = U(\bar x, x'), \quad \bar x = (x_1, x_3), \quad x' = (x_3, \ldots, x_n). \tag{1.18}$$

They are even in the x_j-coordinates, for any $j = 2, \ldots, n$

$$U(x_1, \ldots, x_j, \ldots, x_n) = U(x_1, \ldots, -x_j, \ldots, x_n), \quad j = 2, \ldots, n \tag{1.19}$$

and they respect invariance under Kelvin's transform:

$$U(x) \ = \ |x|^{2-n} U(|x|^{-2} x) \ . \tag{1.20}$$

In (1.13) the function $\tilde{\phi}$ is a small function when compared with U_*. We will further describe the function u, and in particular the function $\tilde{\phi}$ in Section 2. Let us just mention that $\tilde{\phi}$ satisfies all the symmetry properties (1.18), (1.19) and (1.20).

Recall that Problem (1.11) is invariant under the four transformations mentioned before: translation, dilation, rotation and Kelvin transformation. These invariances will be reflected in the element of the kernel of the linear operator

$$L(\varphi) := \Delta\varphi + f'(u_k)\varphi = \Delta\varphi + p\gamma|u_k|^{p-2}u\,\varphi \tag{1.21}$$

which is the linearized equation associated to (1.11) around u_k.

From now on, for simplicity we will drop the label k in u_k, so that u will denote the solution to Problem (1.11) described in (1.13).

Let us introduce the following set of $3n$ functions

$$z_0(x) = \frac{n-2}{2} u(x) + \nabla u(x) \cdot x, \tag{1.22}$$

$$z_\alpha(x) = \frac{\partial}{\partial x_\alpha} u(x), \quad \text{for} \quad \alpha = 1, \dots, n, \tag{1.23}$$

and

$$z_{n+1}(x) = -x_2 \frac{\partial}{\partial x_1} u(x) + x_1 \frac{\partial}{\partial x_2} u(x) \tag{1.24}$$

where u is the solution to (1.11) described in (1.13). Observe that z_{n+1} is given by

$$z_{n+1}(x) = \frac{\partial}{\partial \theta}[u(R_\theta x)]_{|\theta=0}$$

where R_θ is the rotation in the x_1, x_2 plane of angle θ. Furthermore,

$$z_{n+2}(x) = -2x_1 z_0(x) + |x|^2 z_1(x), \quad z_{n+3}(x) = -2x_2 z_0(x) + |x|^2 z_2(x) \tag{1.25}$$

for $l = 3, \dots, n$

$$z_{n+l+1}(x) = -x_l z_1(x) + x_1 z_l(x), \quad u_{2n+l-1}(x) = -x_l z_2(x) + x_2 z_l(x). \tag{1.26}$$

The functions defined in (1.25) are related to the invariance of Problem (1.11) under Kelvin transformation, while the functions defined in (1.26) are related to the invariance under rotation in the (x_1, x_l) plane and in the (x_2, x_l) plane respectively.

The invariance of Problem (1.11) under scaling, translation, rotation and Kelvin transformation gives that the set $\tilde{\mathcal{Z}}_Q$ (introduced in (1.7)) associated to the linear operator L introduced in (1.21) has dimension at least $3n$, since

$$L(z_\alpha) = 0, \quad \alpha = 0, \dots, 3n-1. \tag{1.27}$$

We shall show that these functions are the *only* bounded elements of the kernel of the operator L. In other words, the sign changing solutions (1.13) to Problem (1.11) constructed in [7] are non degenerate in the sense of Duyckaerts-Kenig-Merle [9].

To state our result, we introduce the following function: For any positive integer i, we define

$$P_i(x) = \sum_{l=1}^{\infty} \frac{\cos(l\,x)}{l^i} \quad \text{and} \quad Q_i(x) = \sum_{l=1}^{\infty} \frac{\sin(l\,x)}{l^i}.$$

Up to a normalization constant, when n is even, P_n and Q_n are related to the Fourier series of the Bernoulli polynomial $B_n(x)$, and when n is odd P_n and Q_n are related to the Fourier series of the Euler polynomial $E_n(x)$. We refer to [1] for further details.

We now define

$$g(x) = \sum_{j=1}^{\infty} \frac{1 - \cos(jx)}{j^n}, \quad 0 \le x \le \pi \tag{1.28}$$

which can be rewritten as

$$g(x) = P_n(0) - P_n(x).$$

Observe that

$$g'(x) = Q_{n-1}(x), \quad g''(x) = P_{n-2}(x).$$

Theorem 1.1. *Assume that*

$$g''(x) < \frac{n-2}{n-1} \frac{(g'(x))^2}{g(x)} \quad \forall x \in (0, \pi). \tag{1.29}$$

Then all bounded solutions to the equation

$$L(\varphi) = 0$$

are a linear combination of the functions $z_\alpha(x)$, for $\alpha = 0, \ldots, 3n-1$.

When $n = 3$, condition (1.29) is satisfied. Indeed, in this case we observe that $g''(x) = -\ln(2 \sin \frac{x}{2})$. Thus, if we call $\rho(x) = g''(x)g(x) - \frac{1}{2}(g'(x))^2$, we get $\rho'(x) = g'''(x)g(x) = -\frac{1}{2}\cot(\frac{x}{2})g(x) < 0$. Since $\rho(0) = 0$, condition (1.29) is satisfied.

When $n = 4$, let us check the condition (1.29): let $x = 2\pi t, t \in (0, \frac{1}{2})$. Using the explicit formula for the Bernoulli polynomial B_4 we find that

$$g(t) = t^2(1-t)^2 \tag{1.30}$$

and hence (1.29) is reduced to showing

$$12t^2 - 12t + 2 < \frac{8}{3}(1+t)^2, \quad t \in (0, \frac{1}{2}) \tag{1.31}$$

which is trivial to verify.

In general we believe that condition (1.29) should be true for any dimension $n \ge 4$. In fact, we have checked (1.29) numerically, up to dimension $n \le 48$. Nevertheless, let us mention that even if (1.29) fails, our result is still valid for a *subsequence* u_{k_j}, $k_j \to +\infty$, of solutions (1.13) to Problem (1.11). Indeed, also in this case, our proof can still go through by choosing *a subsequence* $k_j \to +\infty$ in order to avoid the resonance.

We end this section with some remarks.

First: very few results are known on sign-changing solutions to the Yamabe problem. In the critical exponent case and $n = 3$ the topology of lower energy level sets was analyzed in Bahri-Chanillo [3] and Bahri-Xu [4]. For the construction of sign-changing bubbling solutions we refer to Hebey-Vaugon [15], Robert-Vetois [28, 29], Vaira [32] and the references therein. We believe that the non-degeneracy property established in Theorem 1.1 may be used to obtain new type of constructions for sign changing bubbling solutions.

Second: as far as we know the kernels due to the Kelvin transform (i.e. $-2x_jz_0 + |x|^2 z_j$) were first used by Korevaar-Mazzeo-Pacard-Schoen [18] and Mazzeo-Pacard ([23]) in the construction of isolated singularities for Yamabe problem by using a gluing procedure. An interesting question is to determine if and how the non-degenerate sign-changing solutions can used in gluing methods.

Third: for the sign-changing solutions considered in this paper, the dimension of the kernel equals $3n$ which is strictly less than $N = 2n + 1 + \frac{n(n-1)}{2}$. An open question is whether or not there are sign-changing solutions whose dimension of kernel equals N.

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2. Description of the solutions

In this section we describe the solutions u_k in (1.13), recalling some properties that have already been established in [7], and adding some further properties that will be useful for later purpose.

In terms of the function $\tilde{\phi}$ in the decomposition (1.13), equation (1.11) gets re-written as

$$\Delta\tilde{\phi} + p\gamma|U_*|^{p-1}\tilde{\phi} + E + \gamma N(\tilde{\phi}) = 0 \tag{2.1}$$

where E is defined by

$$E = \Delta U_* + f(U_*) \tag{2.2}$$

and

$$N(\phi) = |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\phi.$$

One has a precise control of the size of the function E when measured for instance in the following norm. Let us fix a number q, with $\frac{n}{2} < q < n$, and consider the weighted L^q norm

$$\|h\|_{**} = \|\,(1 + |y|)^{n+2-\frac{2n}{q}}h\|_{L^q(\mathbb{R}^n)}. \tag{2.3}$$

In [7] it is proved that there exist an integer k_0 and a positive constant C such that for all $k \geq k_0$ the following estimates hold true

$$\|E\|_{**} \leq Ck^{1-\frac{n}{q}} \quad \text{if} \quad n \geq 4, \quad \|E\|_{**} \leq \frac{C}{\log k} \quad \text{if} \quad n = 3. \tag{2.4}$$

To be more precise, we have estimates for the $\|\cdot\|_{**}$-norm of the error term E first in the *exterior region* $\bigcap_{j=1}^k \{|y-\xi_j| > \frac{\eta}{k}\}$, and also in the *interior regions* $\{|y-\xi_j| < \frac{\eta}{k}\}$, for any $j = 1,\dots,k$. Here $\eta > 0$ is a positive and small constant, independent of k.

In the exterior region. We have

$$\|\,(1+|y|)^{n+2-\frac{2n}{q}}\,E(y)\|_{L^q(\bigcap_{j=1}^k\{|y-\xi_j|>\frac{\eta}{k}\})} \le Ck^{1-\frac{n}{q}}$$

if $n \ge 4$, while

$$\|\,(1+|y|)^{n+2-\frac{2n}{q}}\,E(y)\|_{L^q(\bigcap_{j=1}^k\{|y-\xi_j|>\frac{\eta}{k}\})} \le \frac{C}{\log k}$$

if $n = 3$.

In the interior regions. Now, let $|y-\xi_j| < \frac{\eta}{k}$ for some $j \in \{1,\dots,k\}$ fixed. It is convenient to measure the error after a change of scale. Define

$$\tilde E_j(y) := \mu^{\frac{n+2}{2}} E(\xi_j + \mu y), \quad |y| < \frac{\eta}{\mu k}.$$

We have

$$\|\,(1+|y|)^{n+2-\frac{2n}{q}}\,\tilde E_j(y)\|_{L^q(|y-\xi_j|<\frac{\eta}{\mu k})} \le Ck^{-\frac{n}{q}} \quad \text{if} \quad n \ge 4$$

and

$$\|\,(1+|y|)^{n+2-\frac{2n}{q}}\,\tilde E_j(y)\|_{L^q(|y-\xi_j|<\frac{\eta}{\mu k})} \le \frac{C}{k\log k} \quad \text{if} \quad n = 3.$$

We refer the readers to [7].

The function $\tilde\phi$ in (1.13) can be further decomposed. Let us introduce some cut-off functions ζ_j to be defined as follows. Let $\zeta(s)$ be a smooth function such that $\zeta(s) = 1$ for $s < 1$ and $\zeta(s) = 0$ for $s > 2$. We also let $\zeta^-(s) = \zeta(2s)$. Then we set

$$\zeta_j(y) = \begin{cases} \zeta(\,k\eta^{-1}|y|^{-2}|(y-\xi|y|)\,|\,) & \text{if } |y| > 1\,, \\ \zeta(\,k\eta^{-1}\,|y-\xi|\,) & \text{if } |y| \le 1\,, \end{cases}$$

in such a way that

$$\zeta_j(y) = \zeta_j(y/|y|^2).$$

The function $\tilde\phi$ has the form

$$\tilde\phi \;=\; \sum_{j=1}^k \tilde\phi_j + \psi. \tag{2.5}$$

In the decomposition (2.5) the functions $\tilde\phi_j$, for $j > 1$, are defined in terms of $\tilde\phi_1$

$$\tilde\phi_j(\bar y, y') = \tilde\phi_1(e^{\frac{2\pi j}{k}i}\bar y, y'), \quad j = 1,\dots,k-1. \tag{2.6}$$

Each function $\tilde\phi_j$, $j = 1,\dots,k$, is constructed to be a solution in the whole $\mathbb{R}^n$ to the problem

$$\Delta\tilde\phi_j + p\gamma|U_*|^{p-1}\zeta_j\tilde\phi_j + \zeta_j[\,p\gamma|U_*|^{p-1}\psi + E + \gamma N(\tilde\phi_j + \Sigma_{i\ne j}\tilde\phi_i + \psi)] = 0, \tag{2.7}$$

while ψ solves in $\mathbb{R}^n$

$$\Delta\psi + p\gamma U^{p-1}\psi + \,[\, p\gamma\,(|U_*|^{p-1} - U^{p-1})(1-\Sigma_{j=1}^k\zeta_j) + p\gamma U^{p-1}\Sigma_{j=1}^k\zeta_j\,]\,\psi$$

$$+\, p\gamma|U_*|^{p-1}\sum_j(1-\zeta_j)\tilde\phi_j + (1-\Sigma_{j=1}^k\zeta_j)\,(\,E+\gamma N(\Sigma_{j=1}^k\tilde\phi_j+\psi)\,) \,=\, 0. \tag{2.8}$$

Define now $\phi_1(y) = \mu^{\frac{n-2}{2}}\tilde\phi_1(\mu y+\xi_1)$. Then ϕ_1 solves the equation

$$\Delta\phi_1 + f'(U)\phi_1 + \chi_1(\xi_1+\mu y)\mu^{\frac{n+2}{2}}E(\xi_1+\mu y)$$

$$+\gamma\mu^{\frac{n+2}{2}}\mathcal{N}(\phi_1)(\xi_1+\mu y) = 0 \quad \text{in} \quad \mathbb{R}^n \tag{2.9}$$

where

$$\mathcal{N}(\phi_1) = p(|U_*|^{p-1}\zeta_1 - U_1^{p-1})\tilde\phi_1 + \zeta_1[p|U_*|^{p-1}\Psi(\phi_1)$$

$$+N(\tilde\phi_1 + \sum_{j\neq 1}\tilde\phi_j + \Psi(\phi_1))] \tag{2.10}$$

In [7] it is shown that the following estimate on the function ψ holds true:

$$\|\psi\|_{n-2} \le Ck^{1-\frac{n}{q}} \quad \text{if} \quad n\ge 4, \quad \|\psi\|_{n-2} \le \frac{C}{\log k} \quad \text{if} \quad n=3, \tag{2.11}$$

where

$$\|\phi\|_{n-2} := \|\,(1+|y|^{n-2})\phi\,\|_{L^\infty(\mathbb{R}^n)}\,. \tag{2.12}$$

On the other hand, if we rescale and translate the function $\tilde\phi_1$

$$\phi_1(y) = \mu^{\frac{n-2}{2}}\tilde\phi_1(\xi_1+\mu y) \tag{2.13}$$

we have the validity of the following estimate for ϕ_1

$$\|\phi_1\|_{n-2} \le Ck^{-\frac{n}{q}} \quad \text{if} \quad n\ge 4, \quad \|\phi_1\|_{n-2} \le \frac{C}{k\log k} \quad \text{if} \quad n=3. \tag{2.14}$$

Furthermore, we have

$$\|\mathcal{N}(\phi_1)\|_{**} \le Ck^{-\frac{2n}{q}} \quad \text{if} \quad n\ge 4, \quad \|\mathcal{N}(\phi_1)\|_{**} \le C(k\log k)^{-2} \quad \text{if} \quad n=3, \tag{2.15}$$

see (2.10). Let us now define the following functions

$$\pi_\alpha(y) = \frac{\partial}{\partial y_\alpha}\tilde\phi(y), \quad \text{for} \quad \alpha = 1,\dots,n;$$

$$\pi_0(y) = \frac{n-2}{2}\tilde\phi(y) + \nabla\tilde\phi(y)\cdot y. \tag{2.16}$$

In the above formula $\tilde\phi$ is the function defined in (1.13) and described in (2.5). Observe that the function π_0 is even in each of its variables, namely

$$\pi_0(y_1,\dots,y_j,\dots,y_n) = \pi_0(y_1,\dots,-y_j,\dots,y_n) \quad \forall j=1,\dots,n,$$

while π_α, for $\alpha=1,\dots,n$ is odd in the y_α variable, while it is even in all the other variables. Furthermore, all functions π_α are invariant under rotation of $\frac{2\pi}{k}$ in the first two coordinates, namely they satisfy (1.18). The functions π_α can be further described, as follows.

Proposition 2.1. *The functions π_α can be decomposed into*

$$\pi_\alpha(y) = \sum_{j=1}^{k} \tilde\pi_{\alpha,j}(y) + \hat\pi_\alpha(y) \tag{2.17}$$

where

$$\tilde\pi_{\alpha,j}(y) = \tilde\pi_{\alpha,1}(e^{\frac{2\pi}{k}ji}\bar y, y').$$

Furthermore, there exists a positive constant C so that

$$\|\hat\pi_0\|_{n-2} \le Ck^{1-\frac{n}{q}}, \quad \|\hat\pi_j\|_{n-1} \le Ck^{1-\frac{n}{q}}, \quad j = 1, \dots, k,$$

if $n \ge 4$, and

$$\|\hat\pi_0\|_{n-2} \le \frac{C}{\log k}, \quad \|\hat\pi_j\|_{n-1} \le \frac{C}{\log k}, \quad j = 1, \dots, k,$$

if $n = 3$. Furthermore, if we denote $\pi_{\alpha,1}(y) = \mu^{\frac{n-2}{2}}\tilde\pi_{\alpha,1}(\xi_1 + \mu y)$, then

$$\|\pi_{0,1}\|_{n-2} \le Ck^{-\frac{n}{q}}, \quad \|\pi_{\alpha,1}\|_{n-1} \le Ck^{-\frac{n}{q}}, \quad \alpha = 1, \dots, n$$

if $n \ge 4$, and

$$\|\pi_{0,1}\|_{n-2} \le \frac{C}{k\log k}, \quad \|\pi_{\alpha,1}\|_{n-1} \le C\frac{C}{k\log k}, \quad \alpha = 1, \dots, 3$$

if $n = 3$.

The proof of this result can be obtained using similar arguments as the ones used in [7]. We leave the details to the reader.

3. Scheme of the proof

Let φ be a bounded function satisfying $L(\varphi) = 0$, where L is the linear operator defined in (1.21). We write our function φ as

$$\varphi(x) = \sum_{\alpha=0}^{3n-1} a_\alpha z_\alpha(x) + \tilde\varphi(x) \tag{3.1}$$

where the functions $z_\alpha(x)$ are defined in (1.22), (1.23), (1.24) (1.25), (1.26) respectively, while the constants a_α are chosen so that

$$\int u^{p-1} z_\alpha \tilde\varphi = 0, \quad \alpha = 0, \dots, 3n-1. \tag{3.2}$$

Observe that $L(\tilde\varphi) = 0$. Our aim is to show that, if $\tilde\varphi$ is bounded, then $\tilde\varphi \equiv 0$.

For this purpose, recall that

$$u(x) = U(x) - \sum_{j=1}^{k} U_j(x) + \tilde\phi(x), \quad \text{with} \quad U(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}$$

and

$$U_j(x) = \mu_k^{-\frac{n-2}{2}} U(\mu_k^{-1}(x - \xi_j)).$$

We introduce the following functions

$$Z_0(x) = \frac{n-2}{2}U(x) + \nabla U(x) \cdot x, \tag{3.3}$$

and

$$Z_\alpha(x) = \frac{\partial}{\partial x_\alpha}U(x), \quad \text{for} \quad \alpha = 1, \dots, n. \tag{3.4}$$

Moreover, for any $l = 1, \dots, k$, we define

$$Z_{0l}(x) = \frac{n-2}{2}U_l(x) + \nabla U_l(x) \cdot (x - \xi_l). \tag{3.5}$$

Observe that, as a consequence of (1.22) and (1.23), we have that

$$\begin{aligned} z_0(x) = Z_0(x) - \sum_{l=1}^{k} \Bigg[& Z_{0,l}(x) + \sqrt{1-\mu^2}\cos\theta_l \frac{\partial}{\partial x_1}U_l(x) \\ & + \sqrt{1-\mu^2}\sin\theta_l \frac{\partial}{\partial x_2}U_l(x)\Bigg] + \pi_0(x), \end{aligned}$$

where π_0 is defined in (2.16). Define, for $l = 1, \dots, k$,

$$Z_{1l}(x) = \sqrt{1-\mu^2}\left[\cos\theta_l \frac{\partial}{\partial x_1}U_l(x) + \sin\theta_l \frac{\partial}{\partial x_2}U_l(x)\right] \tag{3.6}$$

$$Z_{2l}(x) = \sqrt{1-\mu^2}\left[-\sin\theta_l \frac{\partial}{\partial x_1}U_l(x) + \cos\theta_l \frac{\partial}{\partial x_2}U_l(x)\right] \tag{3.7}$$

where $\theta_l = \frac{2\pi}{k}(l-1)$. Furthermore, for any $l = 1, \dots, k$,

$$Z_{\alpha,l}(x) = \frac{\partial}{\partial x_\alpha}U(x), \quad \text{for} \quad \alpha = 3, \dots, n. \tag{3.8}$$

Thus, we can write

$$z_0(x) = Z_0(x) - \sum_{l=1}^{k}\left[Z_{0,l}(x) + Z_{1,l}(x)\right] + \pi_0(x), \tag{3.9}$$

$$\begin{aligned} z_1(x) &= Z_1(x) - \sum_{l=1}^{k} \frac{\partial}{\partial x_1}U_l(x) + \pi_1(x) \\ &= Z_1(x) - \sum_{l=1}^{k} \frac{[\cos\theta_l Z_{1l}(x) - \sin\theta_l Z_{2,l}(x)]}{\sqrt{1-\mu^2}} + \pi_1(x) \end{aligned} \tag{3.10}$$

$$\begin{aligned} z_2(x) &= Z_2(x) - \sum_{l=1}^{k} \frac{\partial}{\partial x_2}U_2(x) + \pi_2(x) \\ &= Z_2(x) - \sum_{l=1}^{k} \frac{[\sin\theta_l Z_{1l}(x) + \cos\theta_l Z_{2,l}(x)]}{\sqrt{1-\mu^2}} + \pi_2(x) \end{aligned} \tag{3.11}$$

and, for $\alpha = 3, \ldots, n$,

$$z_\alpha(x) = Z_\alpha(x) - \sum_{l=1}^{k} Z_{\alpha,l} + \pi_\alpha(x) \tag{3.12}$$

Furthermore

$$z_{n+1}(x) = \sum_{l=1}^{k} Z_{2l}(x) + x_2\pi_1(x) - x_1\pi_2(x) \tag{3.13}$$

$$\begin{aligned} z_{n+2}(x) = \sum_{l=1}^{k} \sqrt{1-\mu^2} \ \cos\theta_l Z_{0l}(x) - \sum_{l=1}^{k} \sqrt{1-\mu^2} \cos\theta_l Z_{1l}(x) \\ -2x_1\pi_0(x) + |x|^2\pi_1(x) \end{aligned} \tag{3.14}$$

$$\begin{aligned} z_{n+3}(x) = \sum_{l=1}^{k} \sqrt{1-\mu^2} \sin\theta_l Z_{0l}(x) - \sum_{l=1}^{k} \sqrt{1-\mu^2} \sin\theta_l Z_{1l}(x) \\ - 2x_2\pi_0(x) + |x|^2\pi_2(x) \end{aligned} \tag{3.15}$$

and, for $\alpha = 3, \ldots, n$,

$$z_{n+\alpha+1}(x) = \sqrt{1-\mu^2} \sum_{l=1}^{k} \cos\theta_l Z_{\alpha l}(x) + x_1\pi_\alpha(x) \tag{3.16}$$

$$z_{2n+\alpha-1}(x) = \sqrt{1-\mu^2} \sum_{l=1}^{k} \sin\theta_l Z_{\alpha l}(x) + x_2\pi_\alpha(x). \tag{3.17}$$

Let

$$Z_{\alpha 0}(x) = Z_\alpha(x) + \pi_\alpha(x), \quad \alpha = 0, \ldots, n, \tag{3.18}$$

and introduce the $(k+1)$-dimensional vector functions

$$\Pi_\alpha(x) = \begin{bmatrix} Z_{\alpha,0}(x) \\ Z_{\alpha 1}(x) \\ Z_{\alpha 2}(x) \\ \cdot\cdot \\ Z_{\alpha k}(x) \end{bmatrix} \quad \text{for} \quad \alpha = 0, 1, \ldots, n.$$

For a given real vector $\bar{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot\cdot \\ c_k \end{bmatrix} \in \mathbb{R}^{k+1}$, we write

$$\bar{c} \cdot \Pi_\alpha(x) = \sum_{l=0}^{k} c_l Z_{\alpha l}(x).$$

With this in mind, we write our function $\tilde\varphi$ as

$$\tilde\varphi(x) = \sum_{\alpha=0}^{n} \mathsf{c}_\alpha \cdot \Pi_\alpha(x) + \varphi^\perp(x) \tag{3.19}$$

where $\mathsf{c}_\alpha = \begin{bmatrix} c_{\alpha 0} \\ c_{\alpha 1} \\ \dots \\ c_{\alpha k} \end{bmatrix}$, $\alpha = 0, 1, \dots, n$, are $(n+1)$ vectors in $\mathbb{R}^{k+1}$ defined so that

$$\int U_l^{p-1}(x) Z_{\alpha l}(x) \varphi^\perp(x)\, dx = 0, \quad \text{for all} \quad l = 0, 1, \dots, k, \quad \alpha = 0, \dots, n.$$

Observe that

$$\mathsf{c}_\alpha = 0 \quad \text{for all} \quad \alpha \quad \text{and} \quad \varphi^\perp \equiv 0 \Longrightarrow \tilde\varphi \equiv 0. \tag{3.20}$$

Hence, our purpose is to show that all vector c_α are zero vectors and that $\varphi^\perp \equiv 0$. This will be consequence of the following three facts.

Fact 1. The orthogonality conditions (3.2) take the form

$$\sum_{\alpha=0}^{n} \mathsf{c}_\alpha \cdot \int \Pi_\alpha u^{p-1} z_\beta = \sum_{\alpha=0}^{n} \sum_{l=0}^{k} c_{\alpha l} \int Z_{\alpha l} u^{p-1} z_\beta = -\int \varphi^\perp u^{p-1} z_\beta \tag{3.21}$$

for $\beta = 0, \dots, 3n-1$. Equation (3.21) is a system of $(n+2)$ linear equations ($\beta = 0, \dots, 3n-1$) in the $(n+1) \times (k+1)$ variables $c_{\alpha l}$.

Let us introduce the following three vectors in $\mathbb{R}^k$

$$\mathbf{1_k} = \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}, \quad \mathbf{cos} = \begin{bmatrix} 1 \\ \cos\theta_2 \\ \dots \\ \cos\theta_{k-1} \end{bmatrix}, \quad \mathbf{sin} = \begin{bmatrix} 0 \\ \sin\theta_2 \\ \dots \\ \sin\theta_{k-1} \end{bmatrix}. \tag{3.22}$$

Let us write

$$\mathsf{c}_\alpha = \begin{bmatrix} c_{\alpha,0} \\ \bar{\mathsf{c}}_\alpha \end{bmatrix}, \quad \text{with} \quad c_{\alpha,0} \in \mathbb{R},\ \bar{\mathsf{c}}_\alpha \in \mathbb{R}^k, \quad \alpha = 0, 1, \dots, n,$$

and

$$\bar{\mathsf{c}} = \begin{bmatrix} \bar{\mathsf{c}}_0 \\ .. \\ \bar{\mathsf{c}}_n \end{bmatrix} \in \mathbb{R}^{n(k+1)}, \quad \hat{\mathsf{c}} = \begin{bmatrix} c_{0,0} \\ .. \\ c_{n,0} \end{bmatrix} \in \mathbb{R}^{n+1}$$

We have the validity of the following

Proposition 3.1. *The system (3.21) reduces to the following* $3n$ *linear conditions of the vectors* c_α*:*

$$c_0 \cdot \begin{bmatrix} 1 \\ -\mathbf{1_k} \end{bmatrix} + c_1 \cdot \begin{bmatrix} 0 \\ -\mathbf{1_k} \end{bmatrix} = t_0 + \Theta_k^1 \mathcal{L}_0(\bar{\mathsf{c}}) + \Theta_k^2 \hat{\mathcal{L}}_0(\hat{\mathsf{c}}), \tag{3.23}$$

$$c_1 \cdot \begin{bmatrix} 1 \\ -\mathbf{cos} \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ \mathbf{sin} \end{bmatrix} = t_1 + \Theta_k^1 \mathcal{L}_1(\bar{\mathsf{c}}) + \Theta_k^2 \hat{\mathcal{L}}_1(\hat{\mathsf{c}}), \tag{3.24}$$

$$c_1 \cdot \begin{bmatrix} 0 \\ -\mathbf{sin} \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ -\mathbf{cos} \end{bmatrix} = t_2 + \Theta_k^1 \, \mathcal{L}_2(\bar{c}) + \Theta_k^2 \, \hat{\mathcal{L}}_2(\hat{c}), \tag{3.25}$$

for $\alpha = 3, \ldots, n$

$$c_\alpha \cdot \begin{bmatrix} 1 \\ -\mathbf{1_k} \end{bmatrix} = t_\alpha + \Theta_k^1 \, \mathcal{L}_\alpha(\bar{c}) + \Theta_k^2 \, \hat{\mathcal{L}}_\alpha(\hat{c}), \tag{3.26}$$

$$c_2 \cdot \begin{bmatrix} 0 \\ \mathbf{1_k} \end{bmatrix} = t_{n+1} + \Theta_k^1 \, \mathcal{L}_{n+1}(\bar{c}) + \Theta_k^2 \, \hat{\mathcal{L}}_{n+1}(\hat{c}), \tag{3.27}$$

$$c_0 \cdot \begin{bmatrix} 0 \\ \mathbf{cos} \end{bmatrix} - c_1 \cdot \begin{bmatrix} 0 \\ \mathbf{cos} \end{bmatrix} = t_{n+2} + \Theta_k^1 \, \mathcal{L}_{n+2}(\bar{c}) + \Theta_k^2 \, \hat{\mathcal{L}}_{n+2}(\hat{c}), \tag{3.28}$$

$$c_0 \cdot \begin{bmatrix} 0 \\ \mathbf{sin} \end{bmatrix} - c_1 \cdot \begin{bmatrix} 0 \\ \mathbf{sin} \end{bmatrix} = t_{n+3} + \Theta_k^1 \, \mathcal{L}_{n+3}(\bar{c}) + \Theta_k^2 \, \hat{\mathcal{L}}_{n+3}(\hat{c}), \tag{3.29}$$

for $\alpha = 3, \ldots, n$,

$$c_\alpha \cdot \begin{bmatrix} 0 \\ \mathbf{cos} \end{bmatrix} = t_{n+\alpha+1} + \Theta_k^1 \, \mathcal{L}_{n+\alpha+1}(\bar{c}) + \Theta_k^2 \, \hat{\mathcal{L}}_{n+\alpha+1}(\hat{c}), \tag{3.30}$$

$$c_\alpha \cdot \begin{bmatrix} 0 \\ \mathbf{sin} \end{bmatrix} = t_{2n+\alpha-1} + \Theta_k^1 \, \mathcal{L}_{2n+\alpha-1}(\bar{c}) + \Theta_k^2 \, \hat{\mathcal{L}}_{2n+\alpha-1}(\hat{c}), \tag{3.31}$$

In the above expansions, $\begin{bmatrix} t_0 \\ t_1 \\ \cdots \\ t_n \end{bmatrix}$ *is a fixed vector with*

$$\left\| \begin{bmatrix} t_0 \\ t_1 \\ \cdots \\ t_n \end{bmatrix} \right\| \leq C \|\varphi^\perp\|_*$$

and $\mathcal{L}_j : \mathbb{R}^{k(n+1)} \to \mathbb{R}^{3n}$, $\hat{\mathcal{L}}_j : \mathbb{R}^n \to \mathbb{R}^{3n}$ *are linear functions, whose coefficients are constants uniformly bounded as* $k \to \infty$. *The number* q, *with* $\frac{n}{2} < q < n$, *is the one already fixed in (2.3). Furthermore,* Θ_k^1 *and* Θ_k^2 *denote quantities which can be described respectively as*

$$\Theta_k^1 = k^{-\frac{n}{q}} O(1), \quad \textit{if} \quad n \geq 4, \quad \Theta_k^1 = (k \log k)^{-1} O(1), \quad \textit{if} \quad n = 3,$$

and

$$\Theta_k^1 = k^{1-\frac{n}{q}} O(1), \quad \textit{if} \quad n \geq 4, \quad \Theta_k^1 = (\log k)^{-1} O(1), \quad \textit{if} \quad n = 3,$$

where $O(1)$ *stands for a quantity which is uniformly bounded as* $k \to \infty$.

We shall prove (3.23)–(3.31) in Section 8.

Fact 2. Since $L(\tilde{\varphi}) = 0$, we have that

$$\sum_{\alpha=0}^{n} \mathsf{c}_\alpha \cdot L(\Pi_\alpha(x)) = \sum_{\alpha=0}^{n}\sum_{l=0}^{k} c_{\alpha l} L(Z_{\alpha,l}) = -L(\varphi^\perp) \tag{3.32}$$

Let $\varphi^\perp = \varphi_0^\perp + \sum_{l=1}^k \varphi_l^\perp$ where

$$-L(\varphi_0^\perp) = \sum_{\alpha=0}^{n} c_{\alpha 0} L(Z_{\alpha,0})$$

and for any $l = 1, \dots, k$

$$-L(\varphi_l^\perp) = \sum_{\alpha=0}^{n} c_{\alpha l} L(Z_{\alpha,l}).$$

Furthermore, let

$$\tilde{\varphi}_l^\perp(y) = \mu^{\frac{n-2}{2}} \varphi_l^\perp(\mu y + \xi_l),$$

and define

$$\|\varphi^\perp\|_* = \|\varphi^\perp\|_{n-2} + \sum_{l=1}^{k} \|\tilde{\varphi}_l^\perp\|_{n-2} \tag{3.33}$$

where the $\|\cdot\|_{n-2}$ is defined in (2.12). A first consequence of (3.32) is that there exists a positive constant C such that

$$\|\varphi^\perp\|_* \le C\mu^{\frac{1}{2}} \sum_{\alpha=0}^{n} \|\mathsf{c}_\alpha\| \tag{3.34}$$

for all k large. We postpone the proof of (3.34) to Section 9.

Fact 3. Let us now multiply (3.32) against $Z_{\beta l}$, for $\beta = 0, \dots, n$ and $l = 0, 1, \dots, k$. After integrating in $\mathbb{R}^n$ we get a linear system of $(n+1)\times(k+1)$ equations in the $(n+1)\times(k+1)$ constants $c_{\alpha j}$ of the form

$$M \begin{bmatrix} \mathsf{c}_0 \\ \mathsf{c}_1 \\ .. \\ \mathsf{c}_n \end{bmatrix} = - \begin{bmatrix} \mathsf{r}_0 \\ \mathsf{r}_1 \\ .. \\ \mathsf{r}_n \end{bmatrix}, \quad \text{with} \quad \mathsf{r}_\alpha = \begin{bmatrix} \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{\alpha,0} \\ \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{\alpha,1} \\ .. \\ \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{\alpha,k} \end{bmatrix} \tag{3.35}$$

Observe first that relation (3.9) together with the fact that $L(z_\alpha) = 0$ for all $\alpha = 0, \dots, n$, allow us to say that the vectors r_α have the form

$$\mathrm{row}_1(\mathsf{r}_0) = \sum_{l=2}^{k+1} [\mathrm{row}_l(\mathsf{r}_0) + \mathrm{row}_l(\mathsf{r}_1)] \tag{3.36}$$

$$\mathrm{row}_1(\mathsf{r}_1) = \frac{1}{\sqrt{1-\mu^2}} \sum_{l=2}^{k+1} [\cos\theta_l \mathrm{row}_l(\mathsf{r}_1) - \sin\theta_l \mathrm{row}_l(\mathsf{r}_2)], \tag{3.37}$$

$$\text{row}_1\,(\mathsf{r}_2) = \frac{1}{\sqrt{1-\mu^2}}\sum_{l=2}^{k+1}[\sin\theta_l \text{row}_l(\mathsf{r}_1) + \cos\theta_l \text{row}_l(\mathsf{r}_2)] \tag{3.38}$$

$$\text{row}_1\,(\mathsf{r}_\alpha) = \sum_{l=2}^{k+1}\text{row}_l\,(\mathsf{r}_\alpha) \quad \text{for all} \quad \alpha = 3,\ldots,n. \tag{3.39}$$

Here with row_l we denote the l-th row.

The matrix M in (3.35) is a square matrix of dimension $[(n+1)\times(k+1)]^2$. The entries of M are numbers of the form

$$\int_{\mathbb{R}^n} L(Z_{\alpha l})Z_{\beta j}\,dy$$

for $\alpha,\beta = 0,\ldots,n$ and $l,j = 0,1,\ldots,k$.

A first observation is that, if α is any of the indeces $\{0,1,2\}$, and β is any of the index in $\{3,\ldots,n\}$, then by symmetry the above integrals are zero, namely

$$\int_{\mathbb{R}^n} L(Z_{\alpha l})Z_{\beta j}\,dy = 0 \quad \text{for any} \quad l,j = 0,\ldots,k$$

This fact implies that the matrix M has the form

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \tag{3.40}$$

where M_1 is a square matrix of dimension $(3\times(k+1))^2$ and M_2 is a square matrix of dimension $[(n-2)\times(k+1)]^2$.

Since

$$\int_{\mathbb{R}^n} L(Z_{\alpha l})Z_{\beta j}\,dy = \int_{\mathbb{R}^n} L(Z_{\beta j})Z_{\alpha l}\,dy$$

for $\alpha,\beta = 0,\ldots,n$ and $l,j = 0,1,\ldots,k$, we can write

$$M_1 = \begin{bmatrix} \bar{A} & \bar{B} & \bar{C} \\ \bar{B}^T & \bar{F} & \bar{D} \\ \bar{C}^T & \bar{D}^T & \bar{G} \end{bmatrix} \tag{3.41}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{F}$ and $\bar{G}$ are square matrices of dimension $(k+1)^2$, with $\bar{A}$, $\bar{F}$ and $\bar{G}$ symmetric. More precisely,

$$\bar{A} = \left(\int L(Z_{0i})Z_{0j}\right)_{i,j=0,1,\ldots,k}, \bar{F} = \left(\int L(Z_{1i})Z_{1j}\right)_{i,j=0,1,\ldots,k}, \tag{3.42}$$

$$\bar{G} = \left(\int L(Z_{2i})Z_{2j}\right)_{i,j=0,1,\ldots,k}, \bar{B} = \left(\int L(Z_{0i})Z_{1j}\right)_{i,j=0,1,\ldots,k}, \tag{3.43}$$

and

$$\bar{C} = \left(\int L(Z_{0i})Z_{2j}\right)_{i,j=0,1,\ldots,k}, \bar{D} = \left(\int L(Z_{1i})Z_{2j}\right)_{i,j=0,1,\ldots,k} \tag{3.44}$$

Furthermore, again by symmetry, since

$$\int L(Z_{\alpha i})Z_{\beta j}\,dx = 0, \quad \text{if} \quad \alpha\neq\beta, \quad \alpha,\beta = 3,\ldots,n$$

the matrix M_2 has the form

$$M_2 = \begin{bmatrix} \bar H_3 & 0 & 0 & 0 & 0 \\ 0 & \bar H_4 & 0 & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & \bar H_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \bar H_n \end{bmatrix} \tag{3.45}$$

where $\bar H_j$ are square matrices of dimension $(k+1)^2$, and each of them is symmetric. The matrices $\bar H_\alpha$ are defined by

$$\bar H_\alpha = \left(\int L(Z_{\alpha i}) Z_{\alpha j} \right)_{i,j=0,1,\ldots,k}, \quad \alpha = 3, \ldots, n. \tag{3.46}$$

Thus, given the form of the matrix M as described in (3.40), (3.41) and (3.45), system (3.35) is equivalent to

$$M_1 \begin{bmatrix} \mathsf{c}_0 \\ \mathsf{c}_1 \\ \mathsf{c}_2 \end{bmatrix} = \begin{bmatrix} \mathsf{r}_0 \\ \mathsf{r}_1 \\ \mathsf{r}_2 \end{bmatrix}, \quad \bar H_\alpha \mathsf{c}_\alpha = \mathsf{r}_\alpha \quad \text{for} \quad \alpha = 3, \ldots, n, \tag{3.47}$$

where the vectors r_α are defined in (3.47).

Observe that system (3.47) impose $(n+1) \times (k+1)$ linear conditions on the $(n+1) \times (k+1)$ constants $c_{\alpha j}$. We shall show that $3n$ equations in (3.47) are linearly dependent. Thus in reality system (3.47) reduce to only $(n+1)\times(k+1) - 3n$ linearly independent conditions on the $(n+1) \times (k+1)$ constants $c_{\alpha j}$. We shall also show that system (3.47) is solvable. Indeed we have the validity of the following

Proposition 3.2. *There exist k_0 and C such that, for all $k > k_0$ System (3.47) is solvable. Furthermore, the solution has the form*

$$\begin{bmatrix} \mathsf{c}_0 \\ \mathsf{c}_1 \\ \mathsf{c}_2 \end{bmatrix} = \begin{bmatrix} \mathsf{v}_0 \\ \mathsf{v}_1 \\ \mathsf{v}_2 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ -\mathbf{1_k} \\ 0 \\ -\mathbf{1_k} \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{cos} \\ 0 \\ \frac{1}{\sqrt{1-\mu^2}}\mathbf{sin} \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{sin} \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{cos} \end{bmatrix}$$

$$+ s_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \\ \mathbf{1_k} \end{bmatrix} + s_5 \begin{bmatrix} 0 \\ \mathbf{cos} \\ 0 \\ -\mathbf{cos} \\ 0 \\ 0 \end{bmatrix} + s_6 \begin{bmatrix} 0 \\ \mathbf{sin} \\ 0 \\ -\mathbf{sin} \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathsf{c}_\alpha = \mathsf{v}_\alpha + s_{\alpha 1} \begin{bmatrix} 1 \\ -\mathbf{1_k} \end{bmatrix} + s_{\alpha 2} \begin{bmatrix} 0 \\ \mathbf{cos} \end{bmatrix} + s_{\alpha 3} \begin{bmatrix} 0 \\ \mathbf{sin} \end{bmatrix}, \quad \alpha = 3, \ldots, n$$

for any $s_1, \ldots, s_6, s_{\alpha 1}, s_{\alpha 2}, s_{\alpha 3} \in \mathbb{R}$, where the vectors $\mathbf{v}_\alpha$ are fixed vectors with

$$\|\mathsf{v}_\alpha\| \le C \|\varphi^\perp\|, \quad \alpha = 0, 1, \ldots, n.$$

Conditions (3.23)–(3.31) guarantees that the solution c_α to (3.47) is indeed unique. Furthermore, we shall show that there exists a positive constant C such that

$$\sum_{\alpha=0}^{n} \|\mathsf{c}_\alpha\| \le C\|\varphi^\perp\|_*. \tag{3.48}$$

Here $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^k$.

Estimates (3.48) combined with (3.34) gives that

$$\mathsf{c}_\alpha = 0 \quad \forall\alpha = 0, \dots, n. \tag{3.49}$$

Replacing equation (3.49) into (3.34) we finally get (3.20), namely

$$\mathsf{c}_\alpha = 0 \quad \text{for all} \quad \alpha \quad \text{and} \quad \varphi^\perp \equiv 0.$$

Scheme of the paper: In Section 4 we discuss and simplify system (3.47). In Section 5 we establish an invertibility theory for solving (3.47). Section 6 is devoted to prove Proposition 3.2. In Section 7 we prove Theorem 1.1. Section 8 is devoted to the proof of Proposition 3.1, while Section 9 is devoted to the proof of (3.34). Section 10 is devoted to the detailed proofs of several computations.

4. A first simplification of the system (3.47)

Let us consider system (3.47) and let us fix $\alpha \in \{3, \dots, n\}$. Recall that the function z_α defined in (1.23) satisfies $L(z_\alpha) = 0$. Hence, by (3.9), (3.18) and (3.46) we have that

$$\mathrm{row}_1(\bar H_\alpha) = \sum_{l=2}^{k+1} \mathrm{row}_l(\bar H_\alpha).$$

This implies that $\begin{bmatrix} 1 \\ -\mathbf{1_k} \end{bmatrix} \in \mathrm{kernel}(\bar H_\alpha)$ and thus that the system $\bar H_\alpha(\mathsf{c}_\alpha) = \mathsf{r}_\alpha$ is solvable only if $\mathrm{r}_\alpha \cdot \begin{bmatrix} 1 \\ -\mathbf{1_k} \end{bmatrix} = 0$. On the other hand, this last solvability condition is satisfied as consequence of (3.39). Thus $\bar H_\alpha \mathsf{c}_\alpha = \mathsf{r}_\alpha$ is solvable.

Arguing similarly, we get that

$$\mathrm{row}_1(M_1) = \sum_{l=2}^{k+1} \mathrm{row}_l(M_1) + \sum_{l=k+3}^{2k+2} \mathrm{row}_l(M_1),$$

$$\mathrm{row}_{k+2}(M_1) = \frac{1}{\sqrt{1-\mu^2}} \left[\sum_{l=1}^{k} \cos\theta_l \mathrm{row}_{k+2+l}(M_1) - \sum_{l=1}^{k} \sin\theta_l \mathrm{row}_{2k+3+l}(M_1) \right],$$

and

$$\mathrm{row}_{2k+3}(M_1) = \frac{1}{\sqrt{1-\mu^2}} \left[\sum_{l=1}^{k} \sin\theta_l \mathrm{row}_{k+2+l}(M_1) + \sum_{l=1}^{k} \cos\theta_l \mathrm{row}_{2k+3+l}(M_1) \right].$$

This implies that the vectors

$$\mathbf{w_0} = \begin{bmatrix} 1 \\ -\mathbf{1_k} \\ 0 \\ -\mathbf{1_k} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{cos} \\ 0 \\ \frac{1}{\sqrt{1-\mu^2}}\mathbf{sin} \end{bmatrix}, \quad \mathbf{w_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{sin} \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{cos} \end{bmatrix} \in \text{kernel}(M_1)$$

and thus that the system $M_1 \begin{bmatrix} \mathsf{c}_0 \\ \mathsf{c}_1 \\ \mathsf{c}_2 \end{bmatrix} = \begin{bmatrix} \mathsf{r}_0 \\ \mathsf{r}_1 \\ \mathsf{r}_2 \end{bmatrix}$ is solvable only if $\begin{bmatrix} \mathsf{r}_0 \\ \mathsf{r}_1 \\ \mathsf{r}_2 \end{bmatrix} \cdot \mathbf{w_j} = \mathbf{0}$, for $j = 0, 1, 2$. On the other hand, this last solvability condition is satisfied as consequence of (3.36), (3.37) and (3.38).

We thus conclude that system (3.47) is solvable and the solution has the form

$$\begin{bmatrix} \mathsf{c}_0 \\ \mathsf{c}_1 \\ \mathsf{c}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{\mathsf{c}}_0 \\ 0 \\ \bar{\mathsf{c}}_1 \\ 0 \\ \bar{\mathsf{c}}_\alpha \end{bmatrix} + t\mathbf{w_0} + s\mathbf{w_1} + r\mathbf{w_2} \quad \text{for all} \quad t, s, r \in \mathbb{R} \tag{4.1}$$

and, if $\alpha = 3, \dots, n$

$$\mathsf{c}_\alpha = \begin{bmatrix} 0 \\ \bar{\mathsf{c}}_\alpha \end{bmatrix} + t \begin{bmatrix} 1 \\ -\mathbf{1_k} \end{bmatrix} \quad \text{for all} \quad t \in \mathbb{R} \tag{4.2}$$

In (4.1)-(4.2), $\bar{\mathsf{c}}_\alpha$ for $\alpha = 0, \dots, n$, are $(n+1)$ vectors in $\mathbb{R}^k$, respectively given by

$$\bar{\mathsf{c}}_\alpha = \begin{bmatrix} c_{\alpha 1} \\ c_{\alpha 2} \\ \dots \\ c_{\alpha k} \end{bmatrix}. \tag{4.3}$$

These vectors correspond to solutions of the systems

$$N \begin{bmatrix} \bar{\mathsf{c}}_0 \\ \bar{\mathsf{c}}_1 \\ \bar{\mathsf{c}}_2 \end{bmatrix} = \begin{bmatrix} \bar{\mathsf{r}}_0 \\ \bar{\mathsf{r}}_1 \\ \bar{\mathsf{r}}_2 \end{bmatrix}, \quad H_\alpha [\bar{\mathsf{c}}_\alpha] = \bar{\mathsf{r}}_\alpha \quad \text{for} \quad \alpha = 3, \dots, n. \tag{4.4}$$

In the above formula $\bar{\mathsf{r}}_\alpha$ for $\alpha = 0, \dots, n$, are $(n+1)$ vectors in $\mathbb{R}^k$, respectively given by

$$\bar{\mathsf{r}}_\alpha = \begin{bmatrix} \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{\alpha,1} \\ .. \\ \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{\alpha,k} \end{bmatrix}.$$

In (4.4) the matrix N is defined by

$$N := \begin{bmatrix} A & B & C \\ B^T & F & D \\ C^T & D^T & G \end{bmatrix} \tag{4.5}$$

where A, B, C, D, F, G are $k \times k$ matrices whose entrances are given respectively by

$$A = \left(\int L(Z_{0i}) Z_{0j} \right)_{i,j=1,\dots,k}, F = \left(\int L(Z_{1i}) Z_{1j} \right)_{i,j=1,\dots,k}, \tag{4.6}$$

$$G = \left(\int L(Z_{2i}) Z_{2j} \right)_{i,j=1,\dots,k}, B = \left(\int L(Z_{0i}) Z_{1j} \right)_{i,j=1,\dots,k}, \tag{4.7}$$

and

$$C = \left(\int L(Z_{0i}) Z_{2j} \right)_{i,j=1,\dots,k}, D = \left(\int L(Z_{1i}) Z_{2j} \right)_{i,j=1,\dots,k} \tag{4.8}$$

Furthermore, in (4.4) the matrix H_α is defined by

$$H_\alpha = \left(\int L(Z_{\alpha i}) Z_{\alpha j} \right)_{i,j=1,\dots,k}, \quad \alpha = 3, \dots, n. \tag{4.9}$$

The rest of this section is devoted to compute explicitely the entrances of the matrices A, B, C, D, F, G, H_α and their eigenvalues.

We start with the following observation: all matrices A, B, C, D, F, G and H_α in (4.4) are circulant matrices of dimension $k \times k$. For properties of circulant matrices, we refer to [19].

A circulant matrix X of dimension $k \times k$ has the form

$$X = \begin{bmatrix} x_0 & x_1 & \dots & \dots & x_{k-2} & x_{k-1} \\ x_{k-1} & x_0 & x_1 & \dots & \dots & x_{k-2} \\ \dots & x_{k-1} & x_0 & x_1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & x_1 \\ x_1 & \dots & \dots & \dots & x_{k-1} & x_0 \end{bmatrix},$$

or equivalently, if x_{ij}, $i, j = 1, \dots, k$ are the entrances of the matrix X, then

$$x_{i,j} = x_{1,|i-j|+1}.$$

In particular, in order to know a circulant matrix it is enough to know the entrances of its first row.

The eigenvalues of a circulant matrix X are given by the explicit formula

$$\eta_m = \sum_{l=0}^{k-1} x_l e^{\frac{2\pi m}{k} i l}, \quad m = 0, \dots, k-1 \tag{4.10}$$

and with corresponding normalized eigenvectors defined by

$$E_m = k^{-\frac{1}{2}} \begin{bmatrix} 1 \\ e^{\frac{2\pi m}{k} i} \\ e^{\frac{2\pi m}{k} i 2} \\ \dots \\ e^{\frac{2\pi m}{k} i (k-1)} \end{bmatrix} \quad m = 0, \dots, k-1. \tag{4.11}$$

Observe that any circulant matrix X can be diagonalized

$$X = P\, D_X\, P^T$$

where D_X is the diagonal matrix

$$D_X = \mathrm{diag}(\eta_0, \eta_1, \dots, \eta_{k-1}) \tag{4.12}$$

and P is the $k \times k$ invertible matrix defined by

$$P = \left[E_0 \,\middle|\, E_1 \,\middle|\, \dots\, E_{k-1} \right]. \tag{4.13}$$

The matrices A, B, C, D, F, G and H_α are circulant as a consequence of the invariance under rotation of an angle $\frac{2\pi}{k}$ in the (x_1, x_2)-plane of the functions $Z_{\alpha j}$. This is trivial in the case of Z_{0l} and $Z_{\alpha,l}$ for all $\alpha = 3, \dots, n$. On the other hand, if we denote by R_j the rotation in the (x_1, x_2) plane of angle $\frac{2\pi}{k}(j-1)$, then we get

$$\begin{aligned} Z_{1,j}(x) = \nabla U_j(x) \cdot \xi_j &= \mu^{-\frac{n-2}{2}} \nabla U\Big(\frac{R_j(y-\xi_1)}{\mu}\Big) \cdot R_j \xi_1 \\ &= \mu^{-\frac{n-2}{2}} R_j^{-1} U\Big(\frac{R_j(y-\xi_1)}{\mu}\Big) \cdot \xi_1, \quad x = R_j y. \end{aligned}$$

Thus, for instance

$$(F)_{jj} = \int L(Z_{1j}) Z_{1j} = \int L(Z_{11}) Z_{11} = (F)_{11}, \quad j = 1, \dots, k$$

and, after a rotation of an angle of $\frac{2\pi}{k}(|h-j|+1)$,

$$(F)_{hj} = \int L(Z_{1h}) Z_{1j} = \int L(Z_{11}) Z_{1(j-h+1)} = (F)_{1(|j-h|+1)}$$

In a similar way one can show that

$$Z_{2,j}(x) = \mu^{-\frac{n-2}{2}} R_j^{-1} U\Big(\frac{R_j(y-\xi_1)}{\mu}\Big) \cdot \xi_1^{\perp}, \quad x = R_j y.$$

With this in mind, it is straightforward to show that also the matrices B, C, D and G are circulant.

A second observation we want to make is that

$$A, B, F, G, H_\alpha \quad \text{are symmetric}$$

while

$$C, D \quad \text{are anti-symmetric}.$$

The fact that A, F, G and H_α are symmetric follows directly from their definition. On the other hand, we have

$$Z_{1j}(x) = \mu^{-\frac{n-2}{2}} R_{2j}^{-1} \nabla U(\frac{R_{2j}(y-\xi_{k-j+1})}{\mu}) \cdot \xi_{k-j+1}, \quad x = R_{2j} y$$

thus

$$B_{1,j} = \int L(Z_{0,1}) Z_{1,j} = \int L(Z_{0,1}) Z_{1,k-j+2} = B_{1,k-j+2}.$$

Furthermore,

$$Z_{2j}(x) = \mu^{-\frac{n-2}{2}} R_{2j}^{-1} \nabla U(\frac{R_{2j}(y-\xi_{k-j+1})}{\mu}) \cdot (-\xi_{k-j+1})^{\perp}, \quad x = R_{2j} y$$

and thus

$$C_{1,j} = \int L(Z_{0,1}) Z_{2,j} = -\int L(Z_{0,1}) Z_{2,k-j+2} = -C_{1,k-j+2},$$

and

$$D_{1,j} = \int L(Z_{1,1}) Z_{2,j} = -\int L(Z_{1,1}) Z_{2,k-j+2} = -D_{1,k-j+2},$$

for $j \geq 2$. Combining this property with the property of being circulant, we get that B is symmetric, while C and D are anti-symmetric.

Let us now introduce the following positive number

$$\Xi = p\,\gamma \frac{(n-2)}{2} \left(-\int y_1\, U^{p-1}\, Z_1(y)\, dy \right). \tag{4.14}$$

Next we describe the entrances of the matrices A, F, G, B, C , D and H_α, together with their eigenvalues. We refer the reader to Section 10 for the detailed proof of the following expansions. With $O(1)$ we denotes a quantity which is uniformly bounded, as $k \to \infty$.

The matrix A**.** The matrix $A = (A_{ij})_{i,j=1,\dots,k}$ defined by

$$A_{ij} = \int_{\mathbb{R}^n} L(Z_{0i}) Z_{0j}$$

is symmetric. We have

$$A_{11} = k^{n-2} \mu^{n-1} O(1) \tag{4.15}$$

and for any integer $l > 1$,

$$A_{1l} = \Xi \left[\frac{-\frac{(n-2)}{2}}{(1-\cos\theta_l)^{\frac{n-2}{2}}} \right] \mu^{n-2} + \mu^{n-1} k^{n-2} O(1), \tag{4.16}$$

where $O(1)$ is bounded as $k \to \infty$.

Eigenvalues for A: A direct application of (4.10) gives that the eigenvalues of the matrix A are given by

$$a_m = -\frac{n-2}{2}\,\Xi\,\mu^{n-2}\left[\sum_{l>1}^{k}\frac{\cos(m\theta_l)}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\right]\left(1+O(\frac{1}{k})\right)$$

$$= \Xi\,\bar{a}_m\,\mu^{n-2}\left(1+O(\frac{1}{k})\right) \tag{4.17}$$

for $m = 0, 1, \ldots, k-1$, where

$$\bar{a}_m = -\frac{n-2}{2}\frac{k^{n-2}}{(\sqrt{2}\pi)^{n-2}}g''(\frac{2\pi}{k}m) \tag{4.18}$$

where g is the function defined in (1.28).

The matrix F. The matrix $F = (F_{ij})_{i,j=1,\ldots,k}$ defined by

$$F_{ij} = \int_{\mathbb{R}^n} L(Z_{1i})Z_{1j}$$

is symmetric. We have

$$F_{11} = \Xi\left[\sum_{l>1}^{k}\frac{\cos\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}}\right]\mu^{\frac{n-2}{2}} + O(\mu^{\frac{n}{2}}) \tag{4.19}$$

and, for any $l > 1$

$$F_{1l} = \Xi\left[\frac{\frac{n-2}{2}\cos\theta_l - \frac{n}{2}}{(1-\cos\theta_l)^{\frac{n}{2}}}\right]\mu^{n-2} + O(\mu^{\frac{n}{2}}) \tag{4.20}$$

where $O(1)$ is bounded as $k \to 0$.

Eigenvalues for F. For any $m = 0, \ldots, k-1$, the eigenvalues of F are

$$f_m = \Xi\,\bar{f}_m\,\mu^{n-2}. \tag{4.21}$$

where

$$\bar{f}_m = \left[\sum_{l>1}^{k}\frac{\cos\theta_l}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\right.$$

$$\left. + \sum_{l>1}^{k}\frac{\frac{n-2}{2}\cos\theta_l - \frac{n}{2}}{(1-\cos\theta_l)^{\frac{n}{2}}}\cos m\theta_l\right]\left(1+O(\frac{1}{k})\right). \tag{4.22}$$

The matrix G. The matrix $G = (G_{ij})_{i,j=1,\ldots,k}$ defined by

$$G_{ij} = \int_{\mathbb{R}^n} L(Z_{2i})Z_{2j}$$

is symmetric. We have

$$G_{11} = \Xi\left[\sum_{l>1}^{k}\frac{\frac{n-2}{2}\cos\theta_l + \frac{n}{2}}{(1-\cos\theta_l)^{\frac{n}{2}}}\right]\mu^{\frac{n-2}{2}} + \mu^{\frac{n}{2}}O(1) \tag{4.23}$$

and, for $l > 1$,

$$G_{1l} = -\,\Xi\left[\frac{\frac{n-2}{2}\cos\theta_l + \frac{n}{2}}{(1-\cos\theta_l)^{\frac{n}{2}}}\right]\mu^{n-2} \;+\; O(\mu^{\frac{n}{2}}) \tag{4.24}$$

Again $O(1)$ is bounded as $k \to \infty$.

Eigenvalues for G. The eigenvalues of G are given by

$$\begin{aligned} g_m &= -\,\Xi\;\mu^{n-2}\left[\sum_{l>1}^{k}\frac{\left(\frac{n-2}{2}\cos\theta_l + \frac{n}{2}\right)(1-\cos m\theta_l)}{(1-\cos\theta_l)^{\frac{n}{2}}}\right]\left(1+O(\frac{1}{k})\right) \\ &= \Xi\bar{g}_m\,\mu^{n-2}\left(1+O(\frac{1}{k})\right), \end{aligned} \tag{4.25}$$

for $m = 0,\dots,k-1$ where

$$\bar{g}_m = \frac{k^n}{(\sqrt{2}\pi)^n}\,(n-1)\,g(\frac{2\pi}{k}m) \tag{4.26}$$

see (1.28) for the definition of g.

The matrix B. The matrix $B = (B_{ij})_{i,j=1,\dots,k}$ defined by

$$B_{ij} = \int_{\mathbb{R}^n} L(Z_{0i})Z_{1j}$$

is symmetric. We have

$$B_{11} = \mu^{n-1}k^{n-2}O(1) \tag{4.27}$$

and, for any $l > 1$,

$$B_{1l} = \;\Xi\left[\frac{\frac{n-2}{2}}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\right]\mu^{n-2} + \mu^{n-1}k^{n-2}O(1). \tag{4.28}$$

Eigenvalues for B. For any $m = 0,\dots,k-1$

$$\begin{aligned} b_m &= \Xi\,\mu^{n-2}\frac{n-2}{2}\sum_{l>1}^{k}\frac{\cos m\theta_l}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\;\left(1+O(\frac{1}{k})\right) \\ &= \Xi\,\bar{b}_m\,\mu^{n-2}\;\left(1+O(\frac{1}{k})\right) \end{aligned} \tag{4.29}$$

with

$$\bar{b}_m = \frac{n-2}{2}\,\frac{k^{n-2}}{(\sqrt{2}\pi)^{n-2}}\,g''(\frac{2\pi}{k}m) \tag{4.30}$$

see (1.28) for the definition of g.

The matrix C. The matrix $C = (C_{ij})_{i,j=1,\dots,k}$ defined by

$$C_{ij} = \int_{\mathbb{R}^n} L(Z_{0i})Z_{2j}$$

is anti symmetric. We have

$$C_{11} = k^{n-2}\mu^{n-1}O(1) \tag{4.31}$$

and, for $l > 1$,

$$C_{1l} = \Xi \left[\frac{\frac{n-2}{2} \sin\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}} \right] \mu^{n-2} + k^{n-2}\mu^{n-1}O(1). \tag{4.32}$$

Eigenvalues for C. For any $m = 0, \dots, k-1$

$$\begin{aligned} c_m &= \Xi\, i\, \mu^{n-2} \frac{n-2}{2} \left[\sum_{l>1}^{k} \frac{\sin\theta_l \, \sin m\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}} \right] \left(1 + O(\frac{1}{k})\right) \\ &= \Xi\, i\, \bar{c}_m \, \mu^{n-2} \left(1 + O(\frac{1}{k})\right) \end{aligned} \tag{4.33}$$

where

$$\bar{c}_m = \frac{n-2}{2} \frac{\sqrt{2}k^{n-1}}{(\sqrt{2}\pi)^{n-1}} g'(\frac{2\pi}{k} m) \tag{4.34}$$

see (1.28) for the definition of g.

The matrix D. The matrix $D = (D_{ij})_{i,j=1,\dots,k}$

$$D_{ij} = \int_{\mathbb{R}^n} L(Z_{1i}) Z_{2j}$$

is anti symmetric. We have

$$D_{11} = k^{n-1}\mu^{n-1} O(1) \tag{4.35}$$

and, for $l > 1$,

$$D_{1l} = -\,\Xi \left[\frac{\frac{n-2}{2} \sin\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}} \right] \mu^{n-3} + k^{n-1}\mu^{n} O(1). \tag{4.36}$$

Eigenvalues for D. For any $m = 0, \dots, k-1$

$$d_m = i\; \Xi \mu^{n-2} \, \frac{n-2}{2} \left[\sum_{l>1}^{k} \frac{\sin\theta_l \sin m\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}} \right] \left(1 + O(\frac{1}{k})\right) \tag{4.37}$$

$$= -i\; \Xi\, \bar{d}_m \, \mu^{n-2} \left(1 + O(\frac{1}{k})\right) \tag{4.38}$$

with

$$\bar{d}_m = -\frac{n-2}{2} \frac{\sqrt{2}k^{n-1}}{(\sqrt{2}\pi)^{n-1}} g'(\frac{2\pi}{k} m)$$

see (1.28) for the definition of g.

The matrix H_α, for $\alpha = 3, \dots, n$. Fix $\alpha = 3$. The other dimensions can be treated in the same way. The matrix $H_3 = (H_{3,ij})_{i,j=1,\dots,k}$ defined by

$$H_{3,ij} = \int_{\mathbb{R}^n} L(Z_{3i}) Z_{3j}$$

is symmetric. We have

$$H_{3,11} = \Xi \mu^{\frac{n-2}{2}} \left[\sum_{l>1}^{k} \frac{-\cos\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}} \right] + O(\mu^{\frac{n}{2}}) \tag{4.39}$$

and, for $l > 1$,

$$H_{3,1l} = \Xi \left[\frac{1}{(1-\cos\theta_l)^{\frac{n}{2}}} \right] \mu^{n-2} + O(\mu^{\frac{n}{2}}). \tag{4.40}$$

Eigenvalues for H_3. For any $m = 0, \dots, k-1$

$$h_{3,m} = \Xi \, \bar h_{3,m} \, \mu^{n-2} \tag{4.41}$$

where

$$\bar h_{3,m} = \left[\sum_{l>1}^{k} \frac{-\cos\theta_l + \cos m\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}} \right] \left(1 + O(\frac{1}{k}) \right).$$

5. Solving a linear system.

This section is devoted to solve system (4.4), namely

$$N \begin{bmatrix} \bar{\mathtt{c}}_0 \\ \bar{\mathtt{c}}_1 \\ \bar{\mathtt{c}}_2 \end{bmatrix} = \begin{bmatrix} \bar{\mathtt{s}}_0 \\ \bar{\mathtt{s}}_1 \\ \bar{\mathtt{s}}_2 \end{bmatrix}, \quad H_\alpha [\bar{\mathtt{c}}_\alpha] = \bar{\mathtt{s}}_\alpha \quad \text{for} \quad \alpha = 3, \dots, n.$$

for a given right hand side $\begin{bmatrix} \bar{\mathtt{s}}_0 \\ \bar{\mathtt{s}}_1 \\ \bar{\mathtt{s}}_2 \end{bmatrix} \in \mathbb{R}^{3k}$, and $\bar{\mathtt{s}}_\alpha \in \mathbb{R}^k$, where N is the matrix defined in (4.5) and H_α are the matrices defined in (4.9).

Let

$$\Upsilon = \frac{(\sqrt{2}\pi)^{n-2}}{p\gamma \frac{n-2}{2} \Xi}, \tag{5.1}$$

where Ξ is defined in (4.14). We have the validity of the following

Proposition 5.1. *Part a.*
There exist k_0 and $C > 0$ such that, for all $k > k_0$, System

$$N \begin{bmatrix} \bar{\mathtt{c}}_0 \\ \bar{\mathtt{c}}_1 \\ \bar{\mathtt{c}}_2 \end{bmatrix} = \begin{bmatrix} \bar{\mathtt{s}}_0 \\ \bar{\mathtt{s}}_1 \\ \bar{\mathtt{s}}_2 \end{bmatrix} \tag{5.2}$$

is solvable if

$$\bar{\mathtt{s}}_2 \cdot \mathbf{1}_k = (\bar{\mathtt{s}}_0 + \bar{\mathtt{s}}_1) \cdot \mathbf{cos} = (\bar{\mathtt{s}}_0 + \bar{\mathtt{s}}_1) \cdot \mathbf{sin} = 0. \tag{5.3}$$

Furthermore, the solutions of System (5.2) has the form

$$\begin{bmatrix} \bar{\mathtt{c}}_0 \\ \bar{\mathtt{c}}_1 \\ \bar{\mathtt{c}}_2 \end{bmatrix} = \begin{bmatrix} \bar{\mathtt{w}}_0 \\ \bar{\mathtt{w}}_1 \\ \bar{\mathtt{w}}_2 \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ 0 \\ \mathbf{1}_k \end{bmatrix} + t_2 \begin{bmatrix} \mathbf{cos} \\ -\mathbf{cos} \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} \mathbf{sin} \\ -\mathbf{sin} \\ 0 \end{bmatrix} \tag{5.4}$$

for all $t_1, t_2, t_3 \in \mathbb{R}$, *and with* $\begin{bmatrix} \bar{\mathsf{w}}_0 \\ \bar{\mathsf{w}}_1 \\ \bar{\mathsf{w}}_2 \end{bmatrix}$ *a fixed vector such that*

$$\left\| \begin{bmatrix} \bar{\mathsf{w}}_0 \\ \bar{\mathsf{w}}_1 \\ \bar{\mathsf{w}}_2 \end{bmatrix} \right\| \leq \frac{C}{k^n \mu^{n-2}} \left\| \begin{bmatrix} \bar{\mathsf{s}}_0 \\ \bar{\mathsf{s}}_1 \\ \bar{\mathsf{s}}_2 \end{bmatrix} \right\|. \tag{5.5}$$

Part b. *Let* $\alpha = 3, \dots, n$. *There exist* k_0 *and* C *such that, for any* $k > k_0$, *system*

$$H_\alpha [\bar{\mathsf{c}}_\alpha] = \bar{\mathsf{s}}_\alpha \tag{5.6}$$

is solvable only if

$$\bar{\mathsf{s}}_\alpha \cdot \mathbf{cos} = \bar{\mathsf{s}}_\alpha \cdot \mathbf{sin} = 0. \tag{5.7}$$

Furthermore, the solutions of System (5.6) has the form

$$\bar{\mathsf{c}}_\alpha = \bar{\mathsf{w}}_\alpha + t_1 \mathbf{cos} + t_2 \mathbf{sin} \tag{5.8}$$

for all $t_1, t_2 \in \mathbb{R}$, *and with* $\left[\bar{\mathsf{w}}_\alpha \right]$ *a fixed vector such that*

$$\left\| \left[\bar{\mathsf{w}}_\alpha \right] \right\| \leq \frac{C}{k^n \mu^{n-2}} \left\| \left[\bar{\mathsf{s}}_\alpha \right] \right\|. \tag{5.9}$$

Proof. Part a.

Define

$$\mathcal{P} = \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix}$$

where P is defined in (4.13), a simple algebra gives that

$$N = \mathcal{P} \mathcal{D} \mathcal{P}^T$$

where

$$\mathcal{D} = \begin{bmatrix} D_A & D_B & D_C \\ D_B & D_F & D_D \\ D_{-C} & D_{-D} & D_G \end{bmatrix}.$$

Here D_X denotes the diagonal matrix of dimension $k \times k$ whose entrances are given by the eigenvalues of X. For instance

$$D_A = \operatorname{diag}(a_0, a_1, \dots, a_{k-1})$$

where a_j are the eigenvalues of the matrix A, defined in (4.17). Using the change of variables

$$\begin{bmatrix} \bar{\mathsf{y}}_0 \\ \bar{\mathsf{y}}_1 \\ \bar{\mathsf{y}}_2 \end{bmatrix} = \mathcal{P}^T \begin{bmatrix} \bar{\mathsf{c}}_0 \\ \bar{\mathsf{c}}_1 \\ \bar{\mathsf{c}}_2 \end{bmatrix}; \quad \begin{bmatrix} \bar{\mathsf{s}}_0 \\ \bar{\mathsf{s}}_1 \\ \bar{\mathsf{s}}_2 \end{bmatrix} = \mathcal{P} \begin{bmatrix} \bar{\mathsf{h}}_0 \\ \bar{\mathsf{h}}_1 \\ \bar{\mathsf{h}}_2 \end{bmatrix}, \tag{5.10}$$

with $\bar{\mathsf{y}}_\alpha = \begin{bmatrix} y_{\alpha,1} \\ y_{\alpha,2} \\ \dots \\ y_{\alpha,k} \end{bmatrix}$, $\bar{\mathsf{h}}_\alpha = \begin{bmatrix} h_{\alpha,1} \\ h_{\alpha,2} \\ \dots \\ h_{\alpha,k} \end{bmatrix} \in \mathbb{R}^k$, $\alpha = 0, 1, 2$, one sees that solving

$$N \begin{bmatrix} \bar{\mathsf{c}}_0 \\ \bar{\mathsf{c}}_1 \\ \bar{\mathsf{c}}_2 \end{bmatrix} = \begin{bmatrix} \bar{\mathsf{s}}_0 \\ \bar{\mathsf{s}}_1 \\ \bar{\mathsf{s}}_2 \end{bmatrix}$$

is equivalent to solving

$$\mathcal{D} \begin{bmatrix} \bar{\mathsf{y}}_0 \\ \bar{\mathsf{y}}_1 \\ \bar{\mathsf{y}}_2 \end{bmatrix} = \begin{bmatrix} \bar{\mathsf{h}}_0 \\ \bar{\mathsf{h}}_1 \\ \bar{\mathsf{h}}_2 \end{bmatrix}. \tag{5.11}$$

Furthermore, observe that

$$\|\bar{\mathsf{y}}_\alpha\| = \|\bar{\mathsf{c}}_\alpha\|, \quad \text{and} \quad \|\bar{\mathsf{h}}_\alpha\| = \|\bar{\mathsf{s}}_\alpha\|, \quad \alpha = 0, 1, 2. \tag{5.12}$$

Let us now introduce the matrix

$$D = \begin{bmatrix} D_0 & 0 \dots & & 0 \\ 0 & D_1 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & D_{k-1} \end{bmatrix}$$

where for any $m = 0, \dots, k-1$, D_m is the 3×3 matrix given by

$$D_m = \begin{bmatrix} a_m & b_m & c_m \\ b_m & f_m & d_m \\ -c_m & -d_m & g_m \end{bmatrix} = \Xi \mu^{n-2} \begin{bmatrix} \bar{a}_m & \bar{b}_m & i\bar{c}_m \\ \bar{b}_m & \bar{f}_m & i\bar{d}_m \\ -i\bar{c}_m & -i\bar{d}_m & \bar{g}_m \end{bmatrix} \tag{5.13}$$

where a_m, b_m, c_m, f_m, g_m, d_m are the eigenvalues of the matrices A, B, C, F, G and D respectively. In the above formula we have used the computation for the eigenvalues a_m, b_m, c_m, d_m, f_m and g_m that we obtained in (4.17), (4.29), (4.33), (4.37), (4.21) and (4.25).

An easy argument implies that system (5.11) can be re written in the form

$$D_m \begin{bmatrix} y_{0,m+1} \\ y_{1,m+1} \\ y_{2,m+1} \end{bmatrix} = \begin{bmatrix} h_{0,m+1} \\ h_{1,m+1} \\ h_{2,m+1} \end{bmatrix} \quad m = 0, 1, \dots, k-1. \tag{5.14}$$

Taking into account that $\bar{a}_m = -\bar{b}_m$ and $\bar{c}_m = -\bar{d}_m$, a direct algebraic manipulation of the system gives that (5.14) reduces to the simplified system

$$\begin{bmatrix} -\bar{b}_m & 0 & i\bar{c}_m \\ 0 & \bar{f}_m + \bar{b}_m & 0 \\ -i\bar{c}_m & 0 & \bar{g}_m \end{bmatrix} \begin{bmatrix} y_{0,m+1} - y_{1,m+1} \\ y_{1,m+1} \\ y_{2,m+1} \end{bmatrix} = \frac{1}{\Xi \mu^{n-2}} \begin{bmatrix} h_{0,m+1} \\ h_{1,m+1} + h_{0,m+1} \\ h_{2,m+1} \end{bmatrix}. \tag{5.15}$$

Let, for any $m = 0, \dots, k-1$,

$$\ell_m := -\left(\bar{b}_m + \bar{f}_m\right) \left[\bar{g}_m \bar{b}_m + \bar{c}_m^2\right], \tag{5.16}$$

being ℓ_m the determinant of the above matrix.

We have the following cases

Case 1. If $m = 0$, we have that $\bar{g}_0 = \bar{c}_0 = 0$ and so $\ell_0 = 0$. Furthermore,

$$\bar{b}_0 = \frac{n-2}{2}\frac{k^{n-2}}{(\sqrt{2}\pi)^{n-2}}g''(0)\left(1 + O(\frac{1}{k})\right)$$

and

$$\bar{f}_0 + \bar{b}_0 = -\frac{k^{n-2}}{(\sqrt{2}\pi)^{n-2}}g''(0)\left(1 + O(\frac{1}{k})\right).$$

We conclude that System (5.15) for $m = 0$ is solvable if

$$h_{21} = 0$$

and there exists a positive constant C, independent of k, such that the solution has the form

$$\begin{bmatrix} y_{0,1} \\ y_{1,1} \\ y_{2,1} \end{bmatrix} = \begin{bmatrix} \hat{y}_{0,1} \\ \hat{y}_{1,1} \\ \hat{y}_{2,1} \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for any $t \in \mathbb{R}$ and for a fixed vector $\begin{bmatrix} \hat{y}_{0,1} \\ \hat{y}_{1,1} \\ \hat{y}_{2,1} \end{bmatrix}$ with

$$\left\| \begin{bmatrix} \hat{y}_{0,1} \\ \hat{y}_{1,1} \\ \hat{y}_{2,1} \end{bmatrix} \right\| \leq \frac{C}{\mu^{n-2}k^{n-2}} \left\| \begin{bmatrix} h_{0,1} \\ h_{1,1} \\ h_{2,1} \end{bmatrix} \right\|.$$

Case 2. If $m = 1$, we have that $\bar{f}_1 + \bar{b}_1 = 0$. By symmetry, for $m = k-1$ we also have $\bar{f}_{k-1} + \bar{b}_{k-1} = 0$. Furthermore

$$\bar{b}_1 = \bar{b}_{k-1} = \frac{n-2}{2}\frac{k^{n-2}}{(\sqrt{2}\pi)^{n-2}}g''(0)\left(1 + O(\frac{1}{k})\right),$$

$$\bar{g}_1 = \bar{g}_{k-1} = -(n-1)\frac{k^{n-2}}{(\sqrt{2}\pi)^{n-2}}g''(0)\left(1 + O(\frac{1}{k})\right),$$

and

$$\bar{c}_1 = -\bar{c}_{k-1} = (n-2)\frac{k^{n-2}}{(\sqrt{2}\pi)^{n-2}}g''(0)\left(1 + O(\frac{1}{k})\right).$$

We conclude that System (5.15) for $m = 1$ is solvable if

$$h_{02} + h_{12} = 0$$

and there exists a positive constant C, independent of k, such that the solution has the form

$$\begin{bmatrix} y_{0,2} \\ y_{1,2} \\ y_{2,2} \end{bmatrix} = \begin{bmatrix} \hat{y}_{0,2} \\ \hat{y}_{1,2} \\ \hat{y}_{2,2} \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

for any $t \in \mathbb{R}$ and for a fixed vector $\begin{bmatrix} \hat y_{0,2} \\ \hat y_{1,2} \\ \hat y_{2,2} \end{bmatrix}$ with

$$\| \begin{bmatrix} \hat y_{0,2} \\ \hat y_{1,2} \\ \hat y_{2,2} \end{bmatrix} \| \leq \frac{C}{\mu^{n-2} k^{n-2}} \| \begin{bmatrix} h_{0,2} \\ h_{1,2} \\ h_{2,2} \end{bmatrix} \|.$$

On the other hand, when $m = k-1$ System (5.15) is solvable if

$$h_{0,k} + h_{1,k} = 0$$

and there exists a positive constant C, independent of k, such that the solution has the form

$$\begin{bmatrix} y_{0,k} \\ y_{1,k} \\ y_{2,k} \end{bmatrix} = \begin{bmatrix} \hat y_{0,k} \\ \hat y_{1,k} \\ \hat y_{2,k} \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

for any $t \in \mathbb{R}$ and for a fixed vector $\begin{bmatrix} \hat y_{0,k} \\ \hat y_{1,k} \\ \hat y_{2,k} \end{bmatrix}$ with

$$\| \begin{bmatrix} \hat y_{0,k} \\ \hat y_{1,k} \\ \hat y_{2,k} \end{bmatrix} \| \leq \frac{C}{\mu^{n-2} k^{n-2}} \| \begin{bmatrix} h_{0,k} \\ h_{1,k} \\ h_{2,k} \end{bmatrix} \|.$$

Case 3. Let now m be $\neq 0, 1, k-1$. In this case we have

$$\bar b_m = \frac{n-2}{2} \frac{k^{n-2}}{(\sqrt{2}\pi)^{n-2}} g''(\frac{2\pi}{k} m) \left(1 + O(\frac{1}{k})\right),$$

$$\bar f_m + \bar b_m = \frac{k^n}{(\sqrt{2}\pi)^n} g''(\frac{2\pi}{k} m) \left(1 + O(\frac{1}{k})\right),$$

$$\bar g_m = -(n-1) \frac{k^n}{(\sqrt{2}\pi)^n} g(\frac{2\pi}{k} m) \left(1 + O(\frac{1}{k})\right),$$

and

$$\bar c_m = \frac{n-2}{2} \frac{\sqrt{2} k^{n-1}}{(\sqrt{2}\pi)^{n-1}} g'(\frac{2\pi}{k} m) \left(1 + O(\frac{1}{k})\right).$$

In particular

$$\ell_m = -\frac{n-2}{2} \frac{k^{3n-2}}{(\sqrt{2}\pi)^{3n-2}} g''(\frac{2\pi}{m}) \times$$
$$\left[-(n-1) g(\frac{2\pi}{k} m) g''(\frac{2\pi}{k} m) + (n-2) (g'(\frac{2\pi}{k} m))^2 \right] \left(1 + O(\frac{1}{k})\right)$$

Thus under condition (1.29), we have that

$$\ell_m < 0 \quad \forall m = 2, \ldots, k-2.$$

Hence, for all $m \neq 0, 1, k-1$, System (5.15) is uniquely solvable and there exists a positive constant C, independent of k, such that the solution $\begin{bmatrix} \hat{y}_{0,1} \\ \hat{y}_{1,1} \\ \hat{y}_{2,1} \end{bmatrix}$ satisfies

$$\| \begin{bmatrix} \hat{y}_{0,1} \\ \hat{y}_{1,1} \\ \hat{y}_{2,1} \end{bmatrix} \| \leq \frac{C}{\mu^{n-2} k^n} \| \begin{bmatrix} h_{0,1} \\ h_{1,1} \\ h_{2,1} \end{bmatrix} \|.$$

Going back to the original variables, and applying a fixed point argument for contraction mappings we get the validity of Part a of Proposition 5.1.

Part b. Fix $\alpha = 3, \ldots, n$. We have

$$H_\alpha = P D_\alpha P^T$$

where P is defined in (4.13), and

$$D_\alpha = \mathrm{diag}\left(h_{\alpha,0}, h_{\alpha,1}, \ldots, h_{\alpha,k-1}\right)$$

where $h_{\alpha,j}$ are the eigenvalues of the matrix H_α, defined in (4.41). Using the change of variables $\bar{\mathsf{y}}_\alpha = P^T \mathsf{c}_\alpha$ and $\bar{\mathsf{s}}_\alpha = P^T \mathsf{h}_\alpha$, we have to solve $D_\alpha \mathsf{y}_\alpha = \mathsf{h}_\alpha$.

Recall that, for any $m = 0, \ldots, k-1$

$$h_{\alpha,m} = \Xi \, \bar{h}_{\alpha,m} \, \mu^{n-2}$$

where

$$\bar{h}_{\alpha,m} = \left[\sum_{l>1}^{k} \frac{-\cos\theta_l + \cos m\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}} \right] \left(1 + O(\frac{1}{k})\right).$$

If $m = 1$ or $m = k-1$, we have that $\sum_{l>1}^{k} \frac{-\cos\theta_l + \cos m\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}} = 0$, so the system is solvable only if $h_{\alpha,2} = h_{\alpha,k-1} =$. On the other hand we have

$$h_{\alpha,0} = \Xi \mu^{n-2} \frac{k^{n-2}}{(\sqrt{2}\pi)^{n-2}} \left(1 + O(\frac{1}{k})\right)$$

and for $m = 2, \ldots, k-2$

$$h_{\alpha,m} = \Xi \mu^{n-2} \frac{k^n}{(\sqrt{2}\pi)^n} g(\frac{2\pi}{k} m) \left(1 + O(\frac{1}{k})\right)$$

Going back to the original variables, we get the validity of Part b, and this concludes the proof of Proposition 5.1. □

6. Proof of Proposition 3.2

A key ingredient to prove Proposition 3.2 is the estimates on the right hand sides of sistems (4.4). We have

Proposition 6.1. *There exists a positive constant C such that, for any $\alpha = 0, 1, \dots, n$,*

$$\|\bar{\mathrm{r}}_\alpha\| \le C\,\mu^{\frac{n-2}{2}}\,\|\varphi^\perp\|_* \tag{6.1}$$

for any k sufficiently large.

Proof. We prove (6.1), only for $\alpha = 0$.

Recall that

$$\bar{\mathrm{r}}_0 = \begin{bmatrix} \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{01} \\ \dots \\ \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{0k} \end{bmatrix}.$$

Then estimate (6.1) will follows from

$$\left| \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{0j} \right| \le C\,\mu^{\frac{n-2}{2}}\,\|\varphi\|_*, \tag{6.2}$$

for any $j = 1, \dots, k$. To prove (6.2), we fix $j = 1$ and we write

$$\int_{\mathbb{R}^n} L(\varphi^\perp) Z_{01}\,dx = \int_{\mathbb{R}^n} L(Z_{01})\varphi^\perp$$
$$= \int_{\mathbb{R}^n \setminus \cup B(\xi_j, \frac{\eta}{k^{1+\sigma}})} L(Z_{01})\varphi^\perp + \sum_{j=1}^k \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} L(Z_{01})\varphi^\perp$$

where η and σ are small positive numbers, independent of k.

We start to estimate $\int_{B(\xi_1, \frac{\eta}{k^{1+\sigma}})} L(Z_{01})\varphi^\perp$. We have $L(Z_{01}) = [f'(u) - f'(U_1)]\,Z_{01}$. As we have already observed very close to ξ_1, $U_1(x) = O(\mu^{-\frac{n-2}{2}})$ and so in $B(\xi_1, \frac{\eta}{k^{1+\sigma}})$ the function U_1 dominates globally the other terms, provided η is chosen small enough. Thus, after the change of variable $x = \xi_1 + \mu y$,

$$\left| \int_{B(\xi_1, \frac{\eta}{k^{1+\sigma}})} L(Z_{01})\varphi^\perp \right| \le C \int_{B(0, \frac{\eta}{k^{1+\sigma}\mu})} f''(U)|\Upsilon(y)| Z_0(y) [\mu^{\frac{n-2}{2}} |\varphi^\perp(\xi_1 + \mu y)|]\,dy$$
$$\le C\|\varphi^\perp\|_* \int_{B(0, \frac{\eta}{k^{1+\sigma}\mu})} f''(U)|\Upsilon(y)| Z_0(y)\,dy$$

where

$$\Upsilon(y) = \mu^{\frac{n-2}{2}} U(\xi_1 + \mu y) + \sum_{l \ne 1} U(y + \mu^{-1}(\xi_1 - \xi_l))$$

A direct consequence of (10.2) is then that

$$\left| \int_{B(\xi_1, \frac{\eta}{k^{1+\sigma}})} L(Z_{01})\varphi^\perp \right| \le C\mu^{\frac{n-2}{2}} \|\varphi^\perp\|_*.$$

Let now $j \neq 1$ and consider $\int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} L(Z_{01})\varphi^\perp$. In this case, after the change of variables $x = \xi_j + \mu y$, we get

$$
\begin{aligned}
&\left| \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} L(Z_{01})\varphi^\perp \right| \\
&\leq C \int_{B(0, \frac{\eta}{\mu k^{1+\sigma}})} U^{p-1} Z_1(y + \mu^{-1}(\xi_1 - \xi_j))[\mu^{-\frac{n-2}{2}} \varphi^\perp(\xi_j + \mu y)] \\
&\leq C\|\varphi^\perp\|_* \left(\int_{\mathbb{R}^n} U^{p-1} \frac{1}{(1+|y|)^{n-2}} \right) \frac{\mu^{n-2}}{(1-\cos\theta_j)^{\frac{n-2}{2}}}
\end{aligned}
$$

where we used (10.5). Thus we estimate

$$
\left| \sum_{j>1} \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} L(Z_{01})\varphi^\perp \right| \leq C\mu^{\frac{n-2}{2}} \|\varphi^\perp\|_*.
$$

Finally, in the exterior region $\mathbb{R}^n \setminus \cup B(\xi_j, \frac{\eta}{k^{1+\sigma}})$ we can estimate

$$
\begin{aligned}
\left| \int_{\mathbb{R}^n \setminus \cup B(\xi_j, \frac{\eta}{k^{1+\sigma}})} L(Z_{01})\varphi^\perp \right| &\leq C\|\varphi^\perp\|_* \int_{\mathbb{R}^n \setminus \cup B(\xi_j, \frac{\eta}{k^{1+\sigma}})} \frac{U^{p-1}}{(1+|y|)^{n-2}} Z_{01}(y)\, dy \\
&\leq C\mu^{\frac{n}{2}} \|\varphi^\perp\|_*.
\end{aligned}
$$

Thus we have proven (6.1) for $\alpha = 0$. The other cases can be treated similarly. □

We have now the tools for the

Proof of Proposition 3.2. System (3.47) is solvable only if the following orthogonality conditions are satisfied:

$$
\begin{bmatrix} \bar{\mathrm{r}}_0 \\ \bar{\mathrm{r}}_1 \\ \bar{\mathrm{r}}_2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -\mathbf{1_k} \\ 0 \\ -\mathbf{1_k} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{\mathrm{r}}_0 \\ \bar{\mathrm{r}}_1 \\ \bar{\mathrm{r}}_2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{cos} \\ 0 \\ \frac{1}{\sqrt{1-\mu^2}}\mathbf{sin} \end{bmatrix} = \begin{bmatrix} \bar{\mathrm{r}}_0 \\ \bar{\mathrm{r}}_1 \\ \bar{\mathrm{r}}_2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{sin} \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{cos} \end{bmatrix} = 0, \tag{6.3}
$$

$$
\begin{bmatrix} \bar{\mathrm{r}}_0 \\ \bar{\mathrm{r}}_1 \\ \bar{\mathrm{r}}_2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \\ \mathbf{1_k} \end{bmatrix} = \begin{bmatrix} \bar{\mathrm{r}}_0 \\ \bar{\mathrm{r}}_1 \\ \bar{\mathrm{r}}_2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \mathbf{cos} \\ 0 \\ -\mathbf{cos} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{\mathrm{r}}_0 \\ \bar{\mathrm{r}}_1 \\ \bar{\mathrm{r}}_2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \mathbf{sin} \\ 0 \\ -\mathbf{sin} \\ 0 \\ 0 \end{bmatrix} = 0 \tag{6.4}
$$

and

$$
\bar{\mathrm{r}}_\alpha \cdot \begin{bmatrix} 1 \\ -\mathbf{1_k} \end{bmatrix} = \bar{\mathrm{r}}_\alpha \cdot \begin{bmatrix} 0 \\ \mathbf{cos} \end{bmatrix} = \bar{\mathrm{r}}_\alpha \cdot \begin{bmatrix} 0 \\ \mathbf{sin} \end{bmatrix} = 0 \quad \alpha = 3, \ldots, n \tag{6.5}
$$

We recall that $\bar{\mathfrak{r}}_\alpha = \begin{bmatrix} \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{\alpha,1} \\ .. \\ \int_{\mathbb{R}^n} L(\varphi^\perp) Z_{\alpha,k} \end{bmatrix}$. As we already mentioned at the beginning of Section 4, the orthogonality conditions (6.3) are satisfied as consequence of (3.36), (3.37) and (3.38). Similarly, the first orthogonality condition in (6.5) is satisfied as consequence of (3.39).

Let us recall from (3.32) that

$$L(\varphi^\perp) = -\sum_{\alpha=0}^{n} \sum_{l=0}^{k} c_{\alpha l} L(Z_{\alpha,l}).$$

Thus the function $x \to L(\varphi^\perp)(x)$ is invariant under rotation of angle $\frac{2\pi}{k}$ in the (x_1, x_2)-plane. Thus

$$0 = \sum_{l=1}^{k} \int L(\varphi^\perp) Z_{2l}(x)\, dx = \bar{\mathfrak{r}}_2 \cdot \mathbf{1_k}$$

and, for all $\alpha = 3, \ldots, n$,

$$\sum_{l=1}^{k} \cos\theta_l \int L(\varphi^\perp) Z_{\alpha l}(x)\, dx = \left(\int L(\varphi^\perp) Z_{\alpha 1}(x)\, dx \right) \left(\sum_{l=1}^{k} \cos\theta_l \right) = 0,$$

thus $\bar{\mathfrak{r}}_\alpha \cdot \mathbf{cos} = \mathbf{0}$, and similarly

$$0 = \sum_{l=1}^{k} \sin\theta_l \int L(\varphi^\perp) Z_{\alpha l}(x)\, dx = \bar{\mathfrak{r}}_\alpha \cdot \mathbf{sin}$$

namely the first orthogonality condition in (6.4) and the remaining orthogonality conditions in (6.5) are satisfied. Let us check that also the last two orthogonality conditions in (6.4) are verified.

Observe that $L(\varphi^\perp)(x) = |x|^{-2-n} L(\varphi^\perp)(\frac{x}{|x|^2})$. The remaining orthogonality conditions in (6.4) are consequence of the following

Lemma 6.1. *Let h be a function in $\mathbb{R}^n$ such that $h(y) = |y|^{-n-2} h(\frac{y}{|y|^2})$. Then*

$$\mu \int_{\mathbb{R}^n} \frac{\partial}{\partial \mu} \Big(U_\mu(x - \xi_l) \Big) h(y)\, dy = \xi_l \cdot \int_{\mathbb{R}^n} \nabla U_\mu(x - \xi_l) h(y)\, dx \tag{6.6}$$

We postpone the proof of the above Lemma to the end of this Section.

Combining the result of Proposition 5.1 and the a-priori estimates in Proposition 6.1, a direct application of a fixed point theorem for contraction mapping readily gives the proof of Proposition 3.2.

We conclude this section with

Proof of Lemma 6.1.

Proof of (6.6). Assume $l = 1$. Define

$$I(t) = \int_{\mathbb{R}^n} \omega_\mu(y - t\xi_1)h(y)\,dy \quad \text{where} \quad \omega_\mu(y - t\,\xi_1) = \mu^{-\frac{n-2}{2}} U(\frac{y - t\,\xi_1}{\mu}).$$

We have

$$\frac{d}{dt}I(t) = -\int_{\mathbb{R}^n} \nabla\omega_\mu(y - t\,\xi_1)\cdot\xi_1\,h(y)\,dy, \tag{6.7}$$

and

$$\left(\frac{d}{dt}I(t)\right)_{t=1} = -\int_{\mathbb{R}^n} \nabla\omega_\mu(y - \xi_1)\cdot\xi_1\,h(y)\,dy.$$

On the other hand, using the change of variables $y = \frac{x}{|x|^2}$, we have

$$I(t) = \int_{\mathbb{R}^n} \omega_\mu(\frac{x}{|x|^2} - t\,\xi_1)h(\frac{x}{|x|^2})|x|^{-2n}\,dx = \int_{\mathbb{R}^n} \omega_\mu(\frac{x}{|x|^2} - t\,\xi_1)h(x)|x|^{2-n}\,dx =$$

$$= \int_{\mathbb{R}^n} \omega_{\bar\mu}(x - \bar p)h(x)\,dx$$

where

$$\bar\mu(t) = \frac{\mu}{\mu^2 + t^2|\xi_1|^2}, \quad \bar p(t) = \frac{t}{\mu^2 + t^2|\xi_1|^2}\,\xi_1.$$

Observe that $\bar\mu(1) = \mu$, $\bar p(1) = \xi_1$,

$$\frac{d}{dt}\bar\mu(t) = \frac{-2t\mu}{\mu^2 + t^2|\xi_1|^2}, \quad \frac{d}{dt}\bar p(t) = \left[\frac{1}{\mu^2 + t^2|\xi_1|^2} - \frac{2t^2|\xi_1|^2}{\mu^2 + t^2|\xi_1|^2}\right]\xi_1.$$

Hence

$$\frac{d}{dt}I(t) = \frac{d}{dt}\bar\mu(t)\int_{\mathbb{R}^n} \frac{\partial}{\partial\bar\mu}\omega_{\bar\mu}(x - \bar p)h(x)\,dx - \frac{d}{dt}\bar p(t)\int_{\mathbb{R}^n} \nabla\omega_{\bar\mu}(x - \bar p)h(x)\,dx.$$

This gives

$$\left(\frac{d}{dt}I(t)\right)_{t=1} = -2\mu|\xi_1|^2\int_{\mathbb{R}^n} \frac{\partial}{\partial\mu}\omega_\mu(x - \xi_1)h(x)\,dx$$

$$-(1 - 2|\xi_1|^2)\int_{\mathbb{R}^n} \nabla\omega_\mu(x - \xi_1)\cdot\xi_1\,h(x)\,dx. \tag{6.8}$$

¿From (6.7) and (6.8) we conclude with the validity of (6.6).

If $l > 1$ in (6.6), the same arguments hold true. The thus conclude with the proof of the Lemma. □

7. Final argument.

Let $\begin{bmatrix} \bar{\mathsf{c}}_0 \\ \bar{\mathsf{c}}_1 \\ \bar{\mathsf{c}}_2 \\ \bar{\mathsf{c}}_3 \\ \dots \\ \bar{\mathsf{c}}_n \end{bmatrix}$ be the solution to (3.47) predicted by Proposition 3.2, given by

$$\begin{bmatrix} \mathsf{c}_0 \\ \mathsf{c}_1 \\ \mathsf{c}_2 \end{bmatrix} = \begin{bmatrix} \mathsf{v}_0 \\ \mathsf{v}_1 \\ \mathsf{v}_2 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ -\mathbf{1_k} \\ 0 \\ -\mathbf{1_k} \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{cos} \\ 0 \\ \frac{1}{\sqrt{1-\mu^2}}\mathbf{sin} \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{sin} \\ 1 \\ -\frac{1}{\sqrt{1-\mu^2}}\mathbf{cos} \end{bmatrix}$$

$$+ s_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \\ \mathbf{1_k} \end{bmatrix} + s_5 \begin{bmatrix} 0 \\ \mathbf{cos} \\ 0 \\ -\mathbf{cos} \\ 0 \\ 0 \end{bmatrix} + s_6 \begin{bmatrix} 0 \\ \mathbf{sin} \\ 0 \\ -\mathbf{sin} \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathsf{c}_\alpha = \mathsf{v}_\alpha + s_{\alpha 1} \begin{bmatrix} 1 \\ -\mathbf{1_k} \end{bmatrix} + s_{\alpha 2} \begin{bmatrix} 0 \\ \mathbf{cos} \end{bmatrix} + s_{\alpha 3} \begin{bmatrix} 0 \\ \mathbf{sin} \end{bmatrix}, \quad \alpha = 3, \dots, n$$

A direct computation shows that there exists a unique

$$(s_1^*, \dots, s_6^*, s_{3,1}^*, s_{3,2}^*, s_{3,3}^*, \dots, s_{n,1}^*, s_{n,2}^*, s_{n,3}^*) \in \mathbb{R}^{2n}$$

for which the above solution satisfies all the $2n$ conditions of Proposition 3.1. Furthermore, one can see that

$$\|(s_1^*, \dots, s_6^*, s_{3,1}^*, s_{3,2}^*, s_{3,3}^*, \dots, s_{n,1}^*, s_{n,2}^*, s_{n,3}^*)\| \leq C\sqrt{\mu}\|\varphi^\perp\|_*.$$

Hence, there exists a unique solution $\begin{bmatrix} \bar{\mathsf{c}}_0 \\ \bar{\mathsf{c}}_1 \\ \bar{\mathsf{c}}_2 \\ \bar{\mathsf{c}}_3 \\ \dots \\ \bar{\mathsf{c}}_n \end{bmatrix}$ to systems (4.4), satisfying estimates in Proposition 3.1. Furthermore, one has

$$\left\| \begin{bmatrix} \bar{\mathsf{c}}_0 \\ \bar{\mathsf{c}}_1 \\ \bar{\mathsf{c}}_2 \\ \bar{\mathsf{c}}_3 \\ \dots \\ \bar{\mathsf{c}}_n \end{bmatrix} \right\| \leq C\|\varphi^\perp\|_*$$

for some positive constant C independent of k. On the other hand, from (3.34) we conclude that

$$\|\varphi^\perp\|_* \le C\mu^{\frac{1}{2}} \| \begin{bmatrix} \bar{\mathsf{c}}_0 \\ \bar{\mathsf{c}}_1 \\ \bar{\mathsf{c}}_2 \\ \bar{\mathsf{c}}_3 \\ \cdots \\ \bar{\mathsf{c}}_n \end{bmatrix} \| \tag{7.1}$$

where again C denotes a positive constant, independent of k. Thus we conclude that

$$c_{\alpha,j} = 0, \quad \text{for all} \quad \alpha = 0, 1, \ldots, n, \quad j = 0, \ldots, k.$$

Plugging this information into (7.1), we conclude that $\varphi^\perp \equiv 0$ and this proves Theorem 1.1.

8. Proof of Proposition 3.1

We will give the proof of Proposition 3.1 when dimension $n \geq 4$. The estimates for dimension $n = 3$ can be obtained with similar arguments.

The key ingredient to prove Proposition 3.1 are the folllowing estimates

$$\begin{aligned} \int |u|^{p-1} Z_{\alpha,l} Z_0 &= \int U^{p-1} Z_0^2 \, dy + O(\mu^{\frac{n-2}{2}}) \quad \text{if} \quad \alpha = 0, l = 0 \\ &= O(\mu^{\frac{n-2}{2}}) \quad \text{otherwise} \end{aligned} \tag{8.1}$$

$$\begin{aligned} \int |u|^{p-1} Z_{\alpha,l} Z_\beta &= \int U^{p-1} Z_1^2 \, dy + O(\mu^{\frac{n-2}{2}}) \quad \text{if} \quad \alpha = \beta, l = 0 \\ &= O(\mu^{\frac{n-2}{2}}) \quad \text{otherwise} \end{aligned} \tag{8.2}$$

$$\begin{aligned} \int |u|^{p-1} Z_{\alpha,l} Z_{0,j} &= \int U^{p-1} Z_0^2 \, dy + O(\mu^{\frac{n-2}{2}}) \quad \text{if} \quad \alpha = 0, l = j \\ &= O(\mu^{\frac{n-2}{2}}) \quad \text{otherwise} \end{aligned} \tag{8.3}$$

$$\begin{aligned} \int |u|^{p-1} Z_{\alpha,l} Z_{\beta,j} &= \int U^{p-1} Z_1^2 \, dy + O(\mu^{\frac{n-2}{2}}) \quad \text{if} \quad \alpha = \beta, l = j \\ &= O(\mu^{\frac{n-2}{2}}) \quad \text{otherwise} \end{aligned} \tag{8.4}$$

We prove (8.3).

Let $\eta > 0$ be a small number, fixed independently from k. We write

$$\begin{aligned} \int |u|^{p-1} Z_{\alpha l} Z_{0j} &= \int_{B(\xi_l, \frac{\eta}{k})} |u|^{p-1} Z_{\alpha l} Z_{0l} + \int_{\mathbb{R}^n \setminus B(\xi_l, \frac{\eta}{k})} |u|^{p-1} Z_{\alpha l} Z_{0,j} \\ &= i_1 + i_2. \end{aligned}$$

We claim that the main term is i_1. Performing the change of variable $x = \xi_l + \mu y$, we get

$$i_1 = \int_{B(0,\frac{\eta}{\mu k})} |u|^{p-1}(\xi_l + \mu y) Z_\alpha(y) Z_0(y) dy$$
$$= \left(\int U^{p-1} Z_0^2 + O((\mu k)^n) \right) \quad \text{if} \quad \alpha = 0$$
$$= 0 \quad \text{if} \quad \alpha \neq 0.$$

On the other hand, to estimate i_2, we write

$$i_2 = \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j, \frac{\eta}{k})} |u|^{p-1} Z_{\alpha l} Z_{0,j} + \sum_{j \neq l} \int_{B(\xi_j, \frac{\eta}{k})} u^{p-1} Z_{\alpha l} Z_{0,j} = i_{21} + i_{22}$$

The first integral can be estimated as follows

$$|i_{21}| \leq C \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j, \frac{\eta}{k})} \frac{\mu^{\frac{n+2}{2}}}{|x - \xi_l|^{n-2}} \frac{1}{(1 + |x|)^{n+2}} \, dx \leq C \mu^{\frac{n-2}{2}}$$

while the second integral can be estimated by

$$|i_{22}| \leq C \sum_{j \neq l} \int_{B(\xi_j, \frac{\eta}{k})} \frac{\mu^{\frac{n-2}{2}}}{|x - \xi_l|^n} |u|^{p-1} Z_{0j} \, dx \leq C \mu^{\frac{n-2}{2}}$$

where again C denotes an arbitrary positive constant, independent of k. This concludes the proof of (8.3). The proofs of (8.1), (8.2) and (8.4) are similar, and left to the reader.

Now we claim that

$$\int U^{p-1} Z_0^2 = \int U^{p-1} Z_1^2 = 2^{\frac{n-4}{2}} \, n \, (n-2)^2 \frac{\Gamma(\frac{n}{2})^2}{\Gamma(n+2)}. \tag{8.5}$$

The proof of identity (8.5) is postponed to the end of this section.

Let us now consider (3.21) with $\beta = 0$, that is

$$\sum_{\alpha=0}^n \sum_{l=0}^k c_{\alpha l} \int Z_{\alpha l} u^{p-1} z_0 = - \int \varphi^\perp u^{p-1} z_0.$$

First we write $t_0 = -\frac{1}{\int U^{p-1} Z_0^2} \int \varphi^\perp u^{p-1} z_0$. A straightforward computation gives that $|t_0| \leq C \|\varphi^\perp\|_*$, for a certain constant C independent from k. Second, we observe that, direct consequence of (8.1) – (8.4), of (3.9) and Proposition 2.1 is that

$$\sum_{\alpha=0}^{n}\sum_{l=0}^{k} c_{\alpha l}\int Z_{\alpha l}u^{p-1}z_0 = c_{00}\int U^{p-1}Z_0^2 - \sum_{l=1}^{k}\left[c_{0l}\int U^{p-1}Z_0^2 - c_{1l}\int U^{p-1}Z_1^2\right] + O(k^{-\frac{n}{q}})\mathcal{L}(\begin{bmatrix}\bar{c}_0\\ \bar{c}_1\\ \cdots\\ \bar{c}_n\end{bmatrix}) + O(k^{1-\frac{n}{q}})\hat{\mathcal{L}}(\begin{bmatrix}c_{00}\\ c_{10}\\ \cdots\\ c_{n0}\end{bmatrix})$$

where $\mathcal{L}$ and $\hat{\mathcal{L}}$ are linear function, whose coefficients are uniformly bounded in k, as $k \to \infty$. Here we have used the fact that there exists a positive constant C independent of k such that

$$\left|\int |u|^{p-1}Z_{\alpha l}\pi_0(x)\,dx\right| \leq C\|\hat{\pi}_0\|_{n-2}$$

and

$$\left|\int |u|^{p-1}Z_{\alpha l}\pi_0(x)\,dx\right| \leq C\|\hat{\pi}_{01}\|_{n-2},$$

together with the result in Proposition 2.1. The condition (3.23) follows readily. The proof of (3.24) – (3.31) is similar to that performed above, and we leave it to the reader.

We conclude this section with the proof of (8.5). Using the definition of Z_0 and Z_1, we have that

$$\int U^{p-1}Z_1^2 = a_n\frac{(n-2)^2}{n}\int\frac{|x|^2}{(1+|x|^2)^{n+2}}\,dx$$

and

$$\int U^{p-1}Z_0^2 = a_n\frac{(n-2)^2}{4}\int\frac{(1-|x|)^2}{(1+|x|^2)^{n+2}}\,dx,$$

for a certain positive number a_n that depends only on n. Using the formula

$$\int_0^\infty\left(\frac{r}{1+r^2}\right)^q\frac{1}{r^{1+\alpha}}\,dr = \frac{\Gamma(\frac{q+\alpha}{2})\Gamma(\frac{q-\alpha}{2})}{2\Gamma(q)}$$

we get

$$\int\frac{1}{(1+|x|^2)^{n+2}}\,dx = \frac{\frac{n}{2}(\frac{n}{2}+1)\Gamma(\frac{n}{2})^2}{2\Gamma(n+2)}, \tag{8.6}$$

and

$$\int\frac{|x|^2}{(1+|x|^2)^{n+2}}\,dx = \frac{(\frac{n}{2})^2\Gamma(\frac{n}{2})^2}{2\Gamma(n+2)}. \tag{8.7}$$

Replacing (8.6), (8.7) in $\int U^{p-1}Z_1^2$ and $\int U^{p-1}Z_0^2$ we obtain

$$\int U^{p-1}Z_1^2 - \int U^{p-1}Z_0^2 = (n-2)^2\, a_n\, \frac{\frac{n}{2}\Gamma(\frac{n}{2})^2}{2\Gamma(n+2)} \times \left[\frac{1}{2} - \frac{1}{4}(\frac{n}{2}+1) + \frac{n}{4} - \frac{1}{4}(\frac{n}{2}+1)\right] = 0,$$

thus (8.5) is proven.

9. Proof of (3.34).

We start with the following

Proposition 9.1. *Let*

$$L_0(\phi) = \Delta\phi + p\gamma U^{p-1}\phi + a(y)\phi \quad \text{in} \quad \mathbb{R}^n.$$

Assume that $a \in L^{\frac{n}{2}}(\mathbb{R}^n)$. Assume furthermore that h is a function in $\mathbb{R}^n$ with $\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$ bounded and such that $|y|^{-n-2}h(|y|^{-2}y) = \pm h(y)$. Then there exists a positive constant C such that any solution ϕ to

$$L_0(\phi) = h \tag{9.1}$$

satisfies

$$\|\phi\|_{n-2} \le C\|h\|_{**}.$$

Proof. Since $a \in L^{\frac{n}{2}}(\mathbb{R}^n)$ and $U^{p-1} = O(1+|y|^4)$, the operator L_0 is a compact perturbation of the Laplace operator in the space $D^{1,2}(\mathbb{R}^n)$. Thus standard argument gives that

$$\|\nabla\phi\|_{L^2(\mathbb{R}^n)} + \|\phi\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \le C\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$$

where the last inequality is a direct consequence of Holder inequality. Being ϕ a weak solution to (9.1), local elliptic estimates yields

$$\|D^2\phi\|_{L^q(B_1)} + \|D\phi\|_{L^q(B_1)} + \|\phi\|_{L^\infty(B_1)} \le C\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}.$$

Consider now the Kelvin's transform of ϕ, $\hat\phi(y) = |y|^{2-n}\phi(|y|^{-2}y)$. This function satisfies

$$\Delta\hat\phi + pU^{p-1}\hat\phi + |y|^{-4}a(|y|^{-2}y)\hat\phi = \hat h \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \tag{9.2}$$

where $\hat h(y) = |y|^{-n-2}h(|y|^{-2}y)$. We observe that

$$\|\hat h\|_{L^q(|y|<2)} = \|\,|y|^{n+2-\frac{2n}{q}}\, h\|_{L^q(|y|>\frac{1}{2}} \le C\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)},$$

$$\|\,|y|^{-4}\, a(|y|^{-2}y)\|_{L^{\frac{n}{2}}(|y|<2)} = \|a\|_{L^{\frac{n}{2}}(|y|>\frac{1}{2})}$$

and

$$\|\nabla\hat\phi\|_{L^2(\mathbb{R}^n)} + \|\hat\phi\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \le C\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}.$$

Applying then elliptic estimates to (9.2), we get

$$\|D^2\hat\phi\|_{L^q(B_1)} + \|D\hat\phi\|_{L^q(B_1)} + \|\hat\phi\|_{L^\infty(B_1)} \le CC\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}.$$

This concludes the proof of the proposition since $\|\hat\phi\|_{L^\infty(B_1)} = \|\phi\|_{L^\infty(\mathbb{R}^n\setminus B_1)}$. $\square$

We have now the tools to give the
Proof of (3.34). We start with the estimate on $\varphi_0^\perp$. We write

$$\varphi_0^\perp = \sum_{\alpha=0}^{n} c_{\alpha 0}\varphi_{\alpha 0}^\perp$$

where

$$L(\varphi_{\alpha 0}^\perp) = -L(Z_{\alpha 0}).$$

We write the above equation in the following way

$$\Delta(\varphi_{\alpha 0}^\perp) + p\gamma U^{p-1}(\varphi_{\alpha 0}^\perp) + \underbrace{p(|u|^{p-1} - U^{p-1})}_{:=a_0(y)}\varphi_{\alpha 0}^\perp = -L(Z_{\alpha 0}).$$

Observe that

$$|y|^{-n-2}\, L(Z_{0,0})(\,|y|^{-2}\, y) = -L(Z_{0,0})(y),$$

while

$$|y|^{-n-2}\, L(Z_{\alpha,0})(\,|y|^{-2}\, y) = L(Z_{\alpha,0})(y) \quad \alpha = 1, \dots, n.$$

We claim that $a_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$,

$$\|a_0\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \le C k^{\frac{2}{n}}, \quad \text{and} \quad \|L(Z_{\alpha 0})\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \le C\mu^{\frac{n-1}{n}}, \tag{9.3}$$

where we take into account that $\|h\|_{**} \le C\|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$. Let $\eta > 0$ be a fixed positive number, independent of k. We split the integral all over $\mathbb{R}^n$ into a first integral over $\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j, \frac{\eta}{k})$ and a second integral over $\bigcup_{j=1}^k B(\xi_j, \frac{\eta}{k})$. We write then

$$\begin{aligned}\|a_0\|_{L^{\frac{n}{2}}(\mathbb{R}^n)}^{\frac{n}{2}} &= \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j, \frac{\eta}{k})} |a_0(y)|^{\frac{n}{2}}\, dy + \sum_{j=1}^{k} \int_{B(\xi_j, \frac{\eta}{k})} |a_0(y)|^{\frac{n}{2}}\, dy \\ &= i_1 + i_2. \end{aligned} \tag{9.4}$$

In the region $\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j, \frac{\eta}{k})$, we have that

$$|a_0(y)| = p\,\big||u|^{p-1} - U^{p-1}\big| \le C U^{p-2} \sum_{j=1}^{k} \frac{\mu^{\frac{n-2}{2}}}{|y - \xi_j|^{n-2}},$$

for some positive convenient constant C. Thus

$$\begin{aligned}\int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j, \frac{\eta}{k})} |a_0(y)|^{\frac{n}{2}}\, dy &\le C\mu^{\frac{n-2}{2}\frac{n}{2}} \sum_{j=1}^{k} \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j, \frac{\eta}{k})} U^{(p-2)\frac{n}{2}} \frac{1}{|y - \xi_j|^{(n-2)\frac{n}{2}}}\, dy \\ &\le C\, k\, \mu^{\frac{n-2}{2}\frac{n}{2}} \int_{\frac{1}{k}}^{1} \frac{t^{n-1}}{t^{(n-2)\frac{n}{2}}}\, dt \le C\, k\, \mu^{\frac{n-2}{2}\frac{n}{2}} k^{\frac{n}{2}(n-2)-n}.\end{aligned}$$

We conclude that

$$\int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j, \frac{\eta}{k})} |a_0(y)|^{\frac{n}{2}}\, dy \le C\mu^{\frac{n-1}{2}} \tag{9.5}$$

Let us now fix $j \in \{1, \dots, k\}$ and consider $y \in B(\xi_j, \frac{\eta}{k})$. In this region we have

$$|a_0(y)| \le C|U_j|^{p-1},$$

for some proper positive constant C. Recalling that $U_j(y) = \mu^{-\frac{n-2}{2}} U(\frac{y-\xi_j}{\mu})$, we easily get

$$\int_{B(\xi_j,\frac{\eta}{k})} |a_0(y)|^{\frac{n}{2}}\, dy \le C$$

and thus

$$\sum_{j=1}^{k} \int_{B(\xi_j,\frac{\eta}{k})} |a_0(y)|^{\frac{n}{2}}\, dy \le Ck. \tag{9.6}$$

We conclude then that $a_0 \in L^{\frac{n}{2}}(\mathbb{R}^n)$, and from (9.4), (9.5) and (9.6) we conclude the first estimate in (9.3).

We prove the second estimate in (9.3) for $\alpha = 0$. Analogous computations give the estimate for $\alpha \ne 0$. We write

$$\int_{\mathbb{R}^n} |L(Z_{00})|^{\frac{2n}{n+2}}\, dy = \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j,\frac{\eta}{k})} + \sum_{j=1}^{k} \int_{B(\xi_j,\frac{\eta}{k})} = i_1 + i_2 \tag{9.7}$$

Since $L(Z_{00}) = p(|u|^{p-1} - U^{p-1})\, Z_{00} = a_0(y) Z_{00}$, a direct application of Holder inequality gives

$$|i_1| \le C \left(\int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j,\frac{\eta}{k})} |a_0(y)|^{\frac{n}{2}} \right)^{\frac{4}{n+2}} \left(\int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j,\frac{\eta}{k})} |Z_{00}(y)|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n+2}}$$

Taking into account that $\left(\int_{\mathbb{R}^n \setminus \bigcup_{j=1}^k B(\xi_j,\frac{\eta}{k})} |Z_{00}(y)|^{\frac{2n}{n-2}} \right) \le \left(\int_{\mathbb{R}^n} |Z_{00}(y)|^{\frac{2n}{n-2}} \right)$ and the validity of (9.5), we get

$$|i_1| \le C\mu^{2\frac{n-1}{n+2}}. \tag{9.8}$$

Let us fix now $j \in \{1, \dots, k\}$. Using now that

$$\left| \int_{B(\xi_j,\frac{\eta}{k})} |L(Z_{00})|^{\frac{2n}{n+2}} \right| \le C \left(\int_{B(\xi_j,\frac{\eta}{k})} |a_0(y)|^{\frac{n}{2}} \right)^{\frac{4}{n+2}} \left(\int_{B(\xi_j,\frac{\eta}{k})} |Z_{00}(y)|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n+2}}$$

together with the fact that

$$\int_{B(\xi_j,\frac{\eta}{k})} |Z_{00}(y)|^{\frac{2n}{n-2}} \le Ck^{-n},$$

we conclude that

$$|i_2| \le C\mu^{\frac{n}{2}\frac{n-2}{n+2} - \frac{1}{2}} \tag{9.9}$$

¿From (9.7), (9.8) and (9.9) we conclude that

$$\|L(Z_{00})\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \le C\mu^{\frac{n-1}{n}}$$

thus completing the proof of (9.3).

Let us now fix $l \in \{1, \ldots, k\}$. Say $l = 1$. We write

$$\varphi_1^{\perp} = \sum_{\alpha=0}^{n} c_{\alpha 1} \varphi_{\alpha 1}^{\perp}$$

where

$$L(\varphi_{\alpha 1}^{\perp}) = -L(Z_{\alpha 1}).$$

After the change of variable $\tilde{\varphi}_{\alpha 1}^{\perp}(y) = \mu^{\frac{n-2}{2}} \varphi_{\alpha 1}^{\perp}(\mu y + \xi_1)$, the above equation gets rewritten as

$$\Delta(\tilde{\varphi}_{\alpha 1}^{\perp}) + pU^{p-1}(\tilde{\varphi}_{\alpha 1}^{\perp}) + \underbrace{p[(\mu^{-\frac{n-2}{2}}|u|(\mu y + \xi_1))^{p-1} - U^{p-1}]}_{:=a_1(y)} \tilde{\varphi}_{\alpha 1}^{\perp} = h(y)$$

where

$$h(y) = -\mu^{\frac{n+2}{2}} L(Z_{\alpha 1})(\mu y + \xi_1).$$

We claim that $a_1 \in L^{\frac{n}{2}}(\mathbb{R}^n)$.

$$\|a_1\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \leq C\mu, \quad \text{and} \quad \|h\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C\mu. \tag{9.10}$$

We leave the details to the reader. The proof of (3.34) follows by (9.3), (9.10) and a direct application of Proposition 9.1.

10. Appendix

In this section we perform the computations of the entrances of the matrices A, F, G, B, C, D and H_α, $\alpha = 3, \ldots, n$. The results of this section are valid for any dimension $n \geq 3$. We start with proving some usefull expansions and a formula.

Some usefull expansions.
Let $\eta > 0$ and $\sigma > 0$ be small and fixed numbers, independent of k. Assume that $y \in B(0, \frac{\eta}{\mu k^{1+\sigma}})$. We will provide usefull expansions of some functions in this region.

We start with the function, for $y \in B(0, \frac{\eta}{\mu k^{1+\sigma}})$,

$$\Upsilon(y) := \mu^{\frac{n-2}{2}} U(\xi_1 + \mu y) - \sum_{l>1} U(y + \mu^{-1}(\xi_1 - \xi_l)). \tag{10.1}$$

We have the validity of the following expansion

$$\Upsilon(y) = -\frac{n-2}{2}\mu^{\frac{n}{2}}\left[y_1 - \mu^{\frac{n-2}{2}}\sum_{l>1}^{k}\frac{1}{(1-\cos\theta_l)^{\frac{n-2}{2}}}(y_1 - \frac{\sin\theta_l}{1-\cos\theta_l}y_2)\right]\times$$
$$(1+\mu^2 O(|y|))$$

$$+\frac{n-2}{4}\mu^{\frac{n+2}{2}}\left[\frac{ny_1^2}{2} - |y|^2 - \mu^{\frac{n-2}{2}}\sum_{l>1}\frac{1}{(1-\cos\theta_l)^{\frac{n}{2}}}\times\right.$$
$$\left.\left(-1-|y|^2+\frac{n}{2}(1-\cos\theta_l)y_1^2+\frac{n}{2}(1+\cos\theta_l)y_2^2+n\sin\theta_l y_1y_2\right)\right]\times$$
$$(1+\mu^2 O(|y|^2))$$

$$+\mu^{\frac{n+4}{2}}O(1+|y|^3)+O(\mu^{\frac{n+2}{2}}) \tag{10.2}$$

for a fixed constant A. Formula (10.2) is a direct application of the fact that

$$\mu^{\frac{n-2}{2}}\left(\frac{2}{1+|\xi_1|^2}\right)^{\frac{n-2}{2}} - \mu^{n-2}\sum_{l>1}^{k}\frac{1}{(1-\cos\theta_l)^{\frac{n-2}{2}}} = O(\mu^{\frac{n+2}{2}})$$

and of Taylor expansion applied separatedly to $\mu^{\frac{n-2}{2}}U(\xi_1+\mu y)$ and $\sum_{l>1}^k U(y+\mu^{-1}(\xi_1-\xi_l))$ in the considered region $y\in B(0,\frac{\eta}{\mu k^{1+\sigma}})$. Indeed, we have

$$U(\xi_1+\mu y) = \mu^{\frac{n-2}{2}}\left(\frac{2}{(1+|\xi_1|^2)}\right)^{\frac{n-2}{2}}\left[1-\frac{(n-2)}{2}y_1\mu+\frac{n-2}{4}\left(\frac{n(y\cdot\xi_1)^2}{2}-|y|^2\right)\mu^2\right.$$
$$\left.+\mu^3 O(|y|^3)\right](1+O(\mu^2)) \tag{10.3}$$

and

$$U(y+\mu^{-1}(\xi_1-\xi_l)) = \frac{\mu^{n-2}}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\left[1-\frac{(n-2)}{2}\frac{(\xi_1-\xi_l)\cdot y}{(1-\cos\theta_l)}\mu\right.$$
$$+\frac{n-2}{4}\frac{\mu^2}{(1-\cos\theta_l)}\left(-1-|y|^2+n\left(\frac{(\xi_1-\xi_l)\cdot y}{|\xi_1-\xi_l|}\right)^2\right) \tag{10.4}$$
$$\left.+\mu^3\frac{O(1+|y|^3)}{|\xi_1-\xi_l|^3}\right].$$

Recall now the definition of the functions Z_α, $\alpha=0,\dots,n$ in (3.3). In the region $y\in B(0,\frac{\eta}{\mu k^{1+\sigma}})$, we need to describe the functions

$$Z_\alpha(y+\mu^{-1}(\xi_l-\xi_1)),\quad \alpha=0,1,\dots,n.$$

A direct application of Taylor expansion gives

$$Z_0(y+\mu^{-1}(\xi_l-\xi_1)) = -\frac{n-2}{2}\frac{\mu^{n-2}}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\Big[1-(n-2)\frac{(\xi_l-\xi_1)\cdot y}{|\xi_l-\xi_1|^2}\mu + \frac{\mu^2}{|\xi_l-\xi_1|^2}O(1+|y|^2)\Big], \tag{10.5}$$

$$Z_1(y+\mu^{-1}(\xi_l-\xi_1)) = -\frac{n-2}{2}\frac{\mu^{n}}{(1-\cos\theta_l)^{\frac{n}{2}}}\Big[\mu^{-1}(\cos\theta_l-1)+[1-\frac{n}{2}(1-\cos\theta_l)]\,y_1 - \frac{n}{2}\sin\theta_l\, y_2+\mu O(1+|y|)\Big] \tag{10.6}$$

$$Z_2(y+\mu^{-1}(\xi_l-\xi_1)) = -\frac{n-2}{2}\frac{\mu^{n}}{(1-\cos\theta_l)^{\frac{n}{2}}}\Big[\mu^{-1}\sin\theta_l+[1-\frac{n}{2}(1+\cos\theta_l)]\,y_2 + \frac{n}{2}\sin\theta_l\, y_1+\mu O(1+|y|)\Big] \tag{10.7}$$

and for $\alpha = 3,\dots,n$

$$Z_\alpha(y+\mu^{-1}(\xi_l-\xi_1)) = -\frac{n-2}{2}\frac{\mu^{n}}{(1-\cos\theta_l)^{\frac{n}{2}}}\,y_\alpha\,(1+\mu^2 O(1+|y|)). \tag{10.8}$$

We have now the tools to give the proofs of (4.15), (4.16), (4.19), (10.23), (4.23), (4.24), (4.27), (4.28), (4.31), (4.32), (4.35) and (4.36).

Computation of A_{11}. Let $\eta>0$ and $\sigma>0$ be small and fixed numbers. We write

$$\begin{aligned} A_{11} &= \int_{\mathbb{R}^n}(f'(u)-f'(U_1))Z_{01}^2 \\ &= [\int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})} + \int_{\mathbb{R}^n\setminus B(\xi_1,\frac{\eta}{k^{1+\sigma}})}](f'(u)-f'(U_1))Z_{01}^2 \\ &= I_1+I_2 \end{aligned}$$

We claim that the main part of the above expansion is I_1. Note that very close to ξ_1, $U_1(x) = O(\mu^{-\frac{n-2}{2}})$. More in general, taking η small if necessary, we have that U_1 dominates globally the other terms. We thus have

$$
\begin{aligned}
I_1 &= \int_{B(\xi_1, \frac{\eta}{k^{1+\sigma}})} f''(U_1)[U(x) - \sum_{l>1} U_l(x) + \tilde{\phi}(x)] Z_{01}^2(x)\, dx + O(k^{n-2}\mu^{n-1}) \\
&(\ x = \xi_1 + \mu y) \\
&= \int_{B(0, \frac{\eta}{\mu k^{1+\sigma}})} f''(U)\, [\Upsilon(y)]\, Z_0^2\, dx \\
&+ \int_{B(0, \frac{\eta}{\mu k^{1+\sigma}})} f''(U) \left[\mu^{\frac{n-2}{2}} \tilde{\phi}(\mu y + \xi_1)]\right] Z_0^2\, dy + O(k^{n-2}\mu^{n-1}) \\
&= \int_{B(0, \frac{\eta}{\mu k^{1+\sigma}})} f''(U)\, [\Upsilon(y)]\, Z_0^2 \\
&+ p(p-1)\gamma \int_{B(0, \frac{\eta}{\mu k^{1+\sigma}})} U^{p-2} \phi_1(y) Z_0^2\, dx + O(k^{n-2}\mu^{n-1})
\end{aligned}
$$

where $\Upsilon(y)$ is defined in (10.1) and $\phi_1(y) = \mu^{\frac{n-2}{2}} \tilde{\phi}(\mu y + \xi_1)$. Using (1.16), expansion (10.3) and (10.4), we get

$$I_1 = O(k^{n-2}\mu^{n-1}).$$

On the other hand, we have that

$$I_2 = O(k^{n-2}\mu^{n-1}) \tag{10.9}$$

Indeed, we first write

$$I_2 = [\sum_{j>1} \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} + \int_{\mathbb{R}^n \setminus \bigcup_{j\geq 1} B(\xi_j, \frac{\eta}{k^{1+\sigma}})}](f'(u) - f'(U_1)) Z_{01}^2$$

Fix now $j > 1$. In the ball $B(\xi_j, \frac{\eta}{k^{1+\sigma}})$, $u \sim U_j = O(\mu^{-\frac{n-2}{2}})$ and U_j dominates all the other terms. Taking this into consideration, we have that

$$
\begin{aligned}
\left| \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} [f'(u) - f'(U_1)] Z_{01}^2 \right| &\leq \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} f'(U_j) Z_{01}^2 \\
&\leq C \int_{B(0, \frac{\eta}{\mu k^{1+\sigma}})} \frac{1}{(1+|y|^2)^2} Z_0^2(y + \mu^{-1}(\xi_j - \xi_1))\, dy \\
&\quad (\text{using}(10.5)) \\
&\leq C \frac{\mu^{2(n-2)}}{(1-\cos\theta_j)^{n-2}} \int_{B(0, \frac{\eta}{\mu k^{1+\sigma}})} \frac{1}{(1+|y|^2)^2}\, dy \\
&\leq C \frac{\mu^{2(n-2)}}{(1-\cos\theta_j)^{n-2}} \frac{1}{(\mu k^{1+\sigma})^{n-4}}
\end{aligned}
$$

where C is an appropiate positive constant independent of k. Thus we conclude that

$$\left| \sum_{j>1} \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} [f'(u) - f'(U_1)] Z_{01}^2 \right| \leq C \mu^{n-1} k^{n-2}, \tag{10.10}$$

where again C is an appropiate positive constant independent of k.

On the other hand

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1}B(\xi_j,\frac{\eta}{k^{1+\sigma}})}(f'(u)-f'(U_1))Z_{01}^2\right| &\leq C\mu^{-n+2}\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1}B(\xi_j,\frac{\eta}{k^{1+\sigma}})}\frac{1}{(1+|x|^2)^2}Z_0^2(\frac{x-\xi_1}{\mu})\\
&\leq C\mu^{n-2}\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1}B(\xi_j,\frac{\eta}{k^{1+\sigma}})}\frac{1}{(1+|x|^2)^2}\frac{1}{|x-\xi_1|^{2(n-2)}}\,dx\\
&\leq C\mu^{n-2}k^{(n-4)(1+\sigma)}
\end{aligned}
$$

Thus we conclude that

$$
\left|\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1}B(\xi_j,\frac{\eta}{k^{1+\sigma}})}(f'(u)-f'(U_1))Z_{01}^2\right|\leq C\mu^{n-1}k^{n-2} \tag{10.11}
$$

Formulas (10.10) and (10.11) imply (10.9). Thus we get (4.15).

Computation of A_{1l}. Let $l>1$ be fixed. Let again $\eta>0$ and $\sigma>0$ be small and fixed numbers. In this case we write

$$
\begin{aligned}
A_{1l} &= \int_{\mathbb{R}^n}(f'(u)-f'(U_1))Z_{01}Z_{0l}\\
&= [\int_{B(\xi_l,\frac{\eta}{k^{1+\sigma}})}+\int_{\mathbb{R}^n\setminus B(\xi_l,\frac{\eta}{k^{1+\sigma}})}](f'(u)-f'(U_1))Z_{01}Z_{0l}\\
&= I_1+I_2
\end{aligned}
$$

We start with the expansion of I_1. Using again the fact that in $B(\xi_l,\frac{\eta}{k^{1+\sigma}})$ the leading term in u is U_l, which is of order $\mu^{-\frac{n-2}{2}}$, and dominates all the other terms in the definition of u, we get that

$$
\begin{aligned}
I_1 &= \int_{B(\xi_l,\frac{\eta}{k})}[f'(u)-f'(U_1)]Z_{01}Z_{0l}\,dx\\
&= -p\gamma\int_{B(\xi_l,\frac{\eta}{k})}[\mu^{-\frac{n-2}{2}}U(\frac{x-\xi_l}{\mu})]^{p-1}\mu^{-n+2}Z_0(\frac{x-\xi_1}{\mu})Z_0(\frac{x-\xi_l}{\mu})+R_1\\
&\quad (x=\mu y+\xi_l)\\
&= -p\gamma\int_{B(0,\frac{\eta}{\mu k})}U^{p-1}(y)\,Z_0(y)\,Z_0(y+\mu^{-1}(\xi_l-\xi_1))\,dy+R_1
\end{aligned}
$$

where $R_1=I_1-p\gamma\int_{B(\xi_l,\frac{\eta}{k})}[\mu^{-\frac{n-2}{2}}U(\frac{x-\xi_l}{\mu})]^{p-1}\mu^{-n+2}Z_0(\frac{x-\xi_1}{\mu})Z_0(\frac{x-\xi_l}{\mu})$. Now using the expansion (10.5), together with formula (10.13), we get, for any integer $l>1$

$$
I_1=-p\gamma\frac{n-2}{2}(-\int_{\mathbb{R}^n}U^{p-1}Z_0\,dy)\left[\frac{1}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\right]\mu^{n-2}+O(\mu^{n-1}k^{n-2}). \tag{10.12}
$$

Observe that

$$
\int_{\mathbb{R}^n}U^{p-1}Z_0\,dy=-\frac{n-2}{2}(-\int_{\mathbb{R}^n}y_1U^{p-1}Z_1(y)\,dy) \tag{10.13}
$$

Indeed,

$$\int_{\mathbb{R}^n} U^{p-1} Z_0\, dy = \frac{n-2}{2}\int U^p + U^{p-1}\nabla U\cdot y$$

$$= \frac{n-2}{2}\int U^p + n\int U^{p-1} y_1 Z_1(y)\, dy \tag{10.14}$$

On the other hand, we have

$$p\int U^{p-1} y_1 Z_1(y) = -\int U^p$$

We thus conclude (10.13) from (10.14). Replacing (10.13) in (10.12) we get

$$I_1 = p\gamma(\frac{n-2}{2})^2(-\int_{\mathbb{R}^n} U^{p-1} y_1 Z_1\, dy)\left[\frac{1}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\right]\mu^{n-2} + O(\mu^{n-1}k^{n-2}). \tag{10.15}$$

On the other hand, a direct computation gives that

$$R_1 = O(\mu^{n-1}k^{n-2}). \tag{10.16}$$

We now estimate the term I_2. We write

$$I_2 = [\sum_{j\neq l}\int_{B(\xi_j,\frac{\eta}{k^{1+\sigma}})} + \int_{\mathbb{R}^n\setminus\bigcup_j B(\xi_j,\frac{\eta}{k^{1+\sigma}})}](f'(u) - f'(U_1))Z_{01}Z_{0l}$$

Fix now $j \neq l$. In the ball $B(\xi_j, \frac{\eta}{k^{1+\sigma}})$, $u \sim U_j = O(\mu^{-\frac{n-2}{2}})$ and U_j dominates all the other terms. Taking this into consideration, we have that

$$\left|\int_{B(\xi_j,\frac{\eta}{k^{1+\sigma}})}[f'(u) - f'(U_1)]Z_{01}Z_{0l}\right| \leq \int_{B(\xi_j,\frac{\eta}{k^{1+\sigma}})} f'(U_j)Z_{01}Z_{0l}$$

$$\leq C\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}\frac{1}{(1+|y|^2)^2}Z_0(y+\mu^{-1}(\xi_j-\xi_1))\,Z_0(y+\mu^{-1}(\xi_j-\xi_l))\,dy$$

(using(10.5))

$$\leq C\frac{\mu^{2(n-2)}}{(1-\cos\theta_j)^{n-2}}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}\frac{1}{(1+|y|^2)^2}\,dy \quad \text{if} \quad j\neq 1$$

while

$$\leq C\frac{\mu^{2(n-2)}}{(1-\cos\theta_j)^{\frac{n-2}{2}}} \quad \text{if} \quad j=1$$

Thus we conclude that

$$\left|\sum_{j\neq l}\int_{B(\xi_j,\frac{\eta}{k^{1+\sigma}})}[f'(u) - f'(U_1)]Z_{01}Z_{0l}\right| \leq C\mu^{n-1}k^{n-2}, \tag{10.17}$$

where again C is an appropiate positive constant independent of k.

On the other hand

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1} B(\xi_j,\frac{\eta}{k^{1+\sigma}})}(f'(u)-f'(U_1))Z_{01}Z_{0l}\right| \\
&\leq C\mu^{-n+2}\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1} B(\xi_j,\frac{\eta}{k^{1+\sigma}})}\frac{1}{(1+|x|^2)^2}Z_0(\frac{x-\xi_1}{\mu})Z_0(\frac{x-\xi_l}{\mu}) \\
&\leq C\mu^{n-2}\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1} B(\xi_j,\frac{\eta}{k^{1+\sigma}})}\frac{1}{(1+|x|^2)^2}\frac{1}{|x-\xi_1|^{(n-2)}}\frac{1}{|x-\xi_l|^{(n-2)}}\,dx
\end{aligned}
$$

Thus we conclude that

$$
\left|\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1} B(\xi_j,\frac{\eta}{k^{1+\sigma}})}(f'(u)-f'(U_1))Z_{01}^2\right|\leq C\mu^{n-1}k^{n-2} \tag{10.18}
$$

Summing up the information in (10.15), (10.16), (10.24) and (10.25), we conclude that the validity of (4.16).

Computation of F_{11}. Let $\eta>0$ and $\sigma>0$ be small and fixed numbers. We write

$$
F_{11}=\int_{\mathbb{R}^n}[f'(u)-f'(U_1)]Z_{11}^2\,dx
$$

$$
=\left[\int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})}+\int_{\mathbb{R}^n\setminus B(\xi_1,\frac{\eta}{k^{1+\sigma}})}\right][f'(u)-f'(U_1)]Z_{11}^2\,dx=I_1+I_2
$$

We claim that the main part of the above expansion is I_1. In $B(\xi_1,\frac{\eta}{k^{1+\sigma}})$, the main part in u is given by U_1, which is of size $\mu^{-\frac{n-2}{2}}$ in this region, and which dominates all the other terms of u. Thus we can perform a Taylor expansion of the function

$$
f'(u)-f'(U_1)=f''(U_1+s(u-U_1))[u-U_1]\quad\text{for some}\quad 0<s<1,
$$

so we write

$$
I_1=\int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})}f''(U_1)\left[U(x)-\sum_{l>1}^{k}U_l(x)+\tilde{\phi}(x)\right]Z_{11}^2\,dx+R_1,
$$

Performing the change of variables $x=\xi_1+\mu y$, and recalling that $Z_{11}(x)=\mu^{-\frac{n}{2}}Z_1(\frac{x-\xi_1}{\mu})(1+O(\mu^2))$, we get

$$
I_1-R_1=\mu^{-2}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}f''(U_1)\Upsilon(y)\,Z_1^2(y)\,dy
$$

$$
+\mu^{-2}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}f''(U_1)\mu^{\frac{n-2}{2}}\tilde{\phi}(\xi_1+\mu y)\,Z_1^2(y)\,dy+O(\mu^{\frac{n}{2}})
$$

where we recall that

$$
\Upsilon(y)=\left[\mu^{\frac{n-2}{2}}U(\xi_1+\mu y)-\sum_{l>1}^{k}U(y+\mu^{-1}(\xi_1-\xi_l))\right].
$$

Recall now that $\tilde\phi_1(y) = \mu^{\frac{n-2}{2}}\tilde\phi_1(\mu y+\xi_1)$ solves the equation

$$\Delta\phi_1 + f'(U)\phi_1 + \chi_1(\xi_1+\mu y)\mu^{\frac{n+2}{2}}E(\xi_1+\mu y) + \gamma\mu^{\frac{n+2}{2}}\mathcal{N}(\phi_1)(\xi_1+\mu y) = 0 \quad \text{in} \quad \mathbb{R}^n$$

Hence we observe that

$$p(p-1)\gamma\int_{\mathbb{R}^n} U^{p-2}\phi_1 Z_1^2 = p\gamma\int_{\mathbb{R}^n}\frac{\partial}{\partial y_1}(U^{p-1})\phi_1 Z_1$$

$$= -p\gamma\int_{\mathbb{R}^n} U^{p-1}\phi_1(\partial_1 Z_1) - p\gamma\int_{\mathbb{R}^n} U^{p-1}(\partial_1\phi_1)Z_1$$

$$= \int_{\mathbb{R}^n}\chi_1(\xi_1+\mu y)\mu^{\frac{n+2}{2}}E(\xi_1+\mu y)\partial_1 Z_1\,dy + \gamma\mu^{\frac{n+2}{2}}\int_{\mathbb{R}^n}\mathcal{N}(\phi_1)(\xi_1+\mu y)\partial_1 Z_1$$

$$+ \underbrace{\int_{\mathbb{R}^n}[\Delta\phi_1(\partial_1 Z_1) + \Delta Z_1(\partial_1\phi_1)]}_{=0}$$

$$= \mu^{\frac{n+2}{2}}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} E(\xi_1+\mu y)(\partial_1 Z_1)\,dy + \gamma\mu^{\frac{n+2}{2}}\int_{\mathbb{R}^n}\mathcal{N}(\phi_1)(\xi_1+\mu y)(\partial_1 Z_1) + O(\mu^{\frac{n}{2}})$$

Taking this into account, we first observe that

$$I_1 - R_1 = p\gamma\mu^{-2}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}\Upsilon(y)\,\partial_1(U^{p-1}Z_1)\,dy + O(\mu^{\frac{n}{2}})$$

On the other hand recall that

$$R_1 = \int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})}[f''(U_1+s(u-U_1)) - f''(U_1)]\left[U(x) - \sum_{l>1}^{k} U_l(x) + \tilde\phi(x)\right]Z_{11}^2\,dx$$

Thus we have

$$|R_1| \le C\int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})} U_1^{p-2}|(1+sU_1^{-1}(u-U_1))^{p-2}-1|\left|U(x) - \sum_{l>1}^{k} U_l(x) + \tilde\phi(x)\right|Z_{11}^2\,dx$$

$$\le C\mu^{\frac{n-2}{2}}\int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})} U_1^{p-2}\left|U(x) - \sum_{l>1}^{k} U_l(x) + \tilde\phi(x)\right|^2 Z_{11}^2\,dx$$

Arguing as before, we get that

$$R_1 = \mu^{\frac{n}{2}}O(1)$$

where $O(1)$ is bounded as $k \to 0$. Using the definition of μ and the expansions (10.3), (10.4) we conclude that

$$\begin{aligned}
I_1 &= p\gamma\mu^{\frac{n-2}{2}}\frac{n-2}{4}\int_{\mathbb{R}^n}(\frac{n}{2}y_1^2 - |y|^2)\partial_1(U^{p-1}Z_1)\\
&- \mu^{n-2}\frac{n-2}{4}\sum_{l>1}\frac{1}{(1-\cos\theta_l)^{\frac{n}{2}}}\int_{\mathbb{R}^n}[-1-|y|^2+\frac{n}{2}(1-\cos\theta_l)y_1^2+\frac{n}{2}(1+\cos\theta_l)y_2^2]\partial_1(U^{p-1}Z_1)\\
&+ O(\mu^{\frac{n}{2}})\\
&= p\,\gamma\,\frac{n-2}{4}\,\mu^{\frac{n-2}{2}}\left[(n-2)+\mu^{\frac{n-2}{2}}\sum_{l>1}^{k}\frac{n\cos\theta_l-(n-2)}{(1-\cos\theta_l)^{\frac{n}{2}}}\right](-\int_{\mathbb{R}^n}y_1U^{p-1}Z_1)\\
&+ O(\mu^{\frac{n}{2}})\,(10.19)
\end{aligned}$$

On the other hand, we have that

$$I_2 = \mu^{\frac{n}{2}}O(1) \tag{10.20}$$

where $O(1)$ is bounded as $k \to 0$. Indeed, we first write

$$I_2 = [\sum_{j>1}\int_{B(\xi_j,\frac{\eta}{k^{1+\sigma}})} + \int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1}B(\xi_j,\frac{\eta}{k^{1+\sigma}})}](f'(u)-f'(U_1))Z_{11}^2$$

Fix now $j>1$. In the ball $B(\xi_j,\frac{\eta}{k^{1+\sigma}})$, $u\sim U_j = O(\mu^{-\frac{n-2}{2}})$ and U_j dominates all the other terms. Taking this into consideration, we have that

$$\begin{aligned}
\left|\int_{B(\xi_j,\frac{\eta}{k^{1+\sigma}})}[f'(u)-f'(U_1)]Z_{11}^2\right| &\leq \int_{B(\xi_j,\frac{\eta}{k^{1+\sigma}})} f'(U_j)Z_{11}^2\\
&\leq C\mu^{-2}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}\frac{1}{(1+|y|^2)^2}Z_1^2(y+\mu^{-1}(\xi_j-\xi_1))\,dy\\
&\quad \text{(using(10.5))}\\
&\leq C\frac{\mu^{2n-2}}{(1-\cos\theta_j)^n}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}\frac{1}{(1+|y|^2)^2}\,dy\\
&\leq C\frac{\mu^{2n-2}}{(1-\cos\theta_j)^n}\frac{1}{(\mu k^{1+\sigma})^{n-4}}
\end{aligned}$$

where C is an appropiate positive constant independent of k. Thus we conclude that

$$\left|\sum_{j>1}\int_{B(\xi_j,\frac{\eta}{k^{1+\sigma}})}[f'(u)-f'(U_1)]Z_{11}^2\right| \leq C\mu^{\frac{n}{2}}, \tag{10.21}$$

where again C is an appropiate positive constant independent of k.

On the other hand

$$\left|\int_{\mathbb{R}^n\setminus\bigcup_{j\ge1}B(\xi_j,\frac{\eta}{k^{1+\sigma}})}(f'(u)-f'(U_1))Z_{11}^2\right|\le C\mu^{-n}\int_{\mathbb{R}^n\setminus\bigcup_{j\ge1}B(\xi_j,\frac{\eta}{k^{1+\sigma}})}\frac{1}{(1+|x|^2)^2}Z_1^2(\frac{x-\xi_1}{\mu})$$
$$\le C\mu^{n-2}\int_{\mathbb{R}^n\setminus\bigcup_{j\ge1}B(\xi_j,\frac{\eta}{k^{1+\sigma}})}\frac{1}{(1+|x|^2)^2}\frac{1}{|x-\xi_1|^{2(n-1)}}\,dx$$
$$\le C\mu^{n-2}k^{(n-2)(1+\sigma)}$$

Thus we conclude that

$$\left|\int_{\mathbb{R}^n\setminus\bigcup_{j\ge1}B(\xi_j,\frac{\eta}{k^{1+\sigma}})}(f'(u)-f'(U_1))Z_{11}^2\right|\le C\mu^{\frac{n}{2}} \tag{10.22}$$

¿From (10.21) and (10.22) we get (10.20). From (10.19) and (10.20) we conclude (4.19).

Computation of F_{1l}. Let $l>1$ be fixed. Let again $\eta>0$ and $\sigma>0$ be small and fixed numbers. In this case we write

$$\begin{aligned}F_{1l}&=\int_{\mathbb{R}^n}(f'(u)-f'(U_1))Z_{11}Z_{1l}\\&=[\int_{B(\xi_l,\frac{\eta}{k^{1+\sigma}})}+\int_{\mathbb{R}^n\setminus B(\xi_l,\frac{\eta}{k^{1+\sigma}})}](f'(u)-f'(U_1))Z_{11}Z_{1l}\\&=I_1+I_2\end{aligned}$$

We start with the expansion of I_1. Recall that

$$Z_{1l}(x)=\left[\cos\theta_l\mu^{-\frac{n}{2}}Z_1(\frac{x-\xi_l}{\mu})+\sin\theta_l\mu^{-\frac{n}{2}}Z_2(\frac{x-\xi_l}{\mu})\right]\left(1+O(\mu^2)\right).$$

Using again the fact that in $B(\xi_l,\frac{\eta}{k^{1+\sigma}})$ the leading term in u is U_l, which is of order $\mu^{-\frac{n-2}{2}}$, and dominates all the other terms in the definition of u, we get that

$$\begin{aligned}I_1&=-p\cos\theta_l\int_{B(\xi_l,\frac{\eta}{k})}[\mu^{-\frac{n-2}{2}}U(\frac{x-\xi_l}{\mu})]^{p-1}\mu^{-n}Z_1(\frac{x-\xi_1}{\mu})Z_1(\frac{x-\xi_l}{\mu})\\&\quad-p\sin\theta_l\int_{B(\xi_l,\frac{\eta}{k})}[\mu^{-\frac{n-2}{2}}U(\frac{x-\xi_l}{\mu})]^{p-1}\mu^{-n}Z_1(\frac{x-\xi_1}{\mu})Z_2(\frac{x-\xi_l}{\mu})+R_1\\&\quad(x=\mu y+\xi_l)\\&=-p\gamma\mu^{-2}\cos\theta_l\int_{B(0,\frac{\eta}{\mu k})}U^{p-1}Z_1Z_1(y+\mu^{-1}(\xi_l-\xi_1))\,dy\\&\quad-p\gamma\mu^{-2}\sin\theta_l\int_{B(0,\frac{\eta}{\mu k})}U^{p-1}Z_1Z_2(y+\mu^{-1}(\xi_l-\xi_1))\,dy+R_1\end{aligned}$$

Now using the expansion (10.7) we get, for any $l > 1$

$$
\begin{aligned}
I_1 - R_1 &= p\gamma \frac{n-2}{4}\, \Xi \, \cos\theta_l \left[\frac{n-2-n\cos\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}}\right]\mu^{n-2} \\
&\quad - p\gamma \frac{n-2}{4}\, \Xi \, \sin\theta_l \left[\frac{n\sin\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}}\right]\mu^{n-2} + O(\mu^{\frac{n}{2}}) \\
&= p\gamma \frac{n-2}{2}\, \Xi \left[\frac{\frac{n-2}{2}\cos\theta_l - \frac{n}{2}}{(1-\cos\theta_l)^{\frac{n}{2}}}\right]\mu^{n-2} + O(\mu^{\frac{n}{2}}) \qquad (10.23)
\end{aligned}
$$

On the other hand we directly compute

$$R_1 = \mu^{\frac{n}{2}} O(1)$$

where $O(1)$ is bounded as $k \to 0$. We now estimate the term I_2. We write

$$I_2 = [\sum_{j\neq l} \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} + \int_{\mathbb{R}^n \setminus \bigcup_j B(\xi_j, \frac{\eta}{k^{1+\sigma}})}](f'(u) - f'(U_1))Z_{11}Z_{1l}$$

Fix now $j \neq l$. In the ball $B(\xi_j, \frac{\eta}{k^{1+\sigma}})$, $u \sim U_j = O(\mu^{-\frac{n-2}{2}})$ and U_j dominates all the other terms. Taking this into consideration, we have that

$$
\begin{aligned}
&\left|\int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} [f'(u) - f'(U_1)]Z_{01}Z_{0l}\right| \leq \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} f'(U_j)Z_{11}Z_{1l} \\
&\leq C\mu^{-2} \int_{B(0, \frac{\eta}{\mu k^{1+\sigma}})} \frac{1}{(1+|y|^2)^2} Z_1(y + \mu^{-1}(\xi_j - \xi_1))\, Z_1(y + \mu^{-1}(\xi_j - \xi_l))\, dy \\
&\text{(using(10.7))} \\
&\leq C \frac{\mu^{2n-2}}{(1-\cos\theta_j)^n} \int_{B(0, \frac{\eta}{\mu k^{1+\sigma}})} \frac{1}{(1+|y|^2)^2}\, dy \quad \text{if} \quad j \neq 1 \\
&\text{while} \\
&\leq C \frac{\mu^{2n-2}}{(1-\cos\theta_j)^{\frac{n}{2}}} \quad \text{if} \quad j = 1
\end{aligned}
$$

Thus we conclude that

$$\left|\sum_{j\neq l} \int_{B(\xi_j, \frac{\eta}{k^{1+\sigma}})} [f'(u) - f'(U_1)]Z_{11}Z_{1l}\right| \leq C\mu^{\frac{n}{2}}, \qquad (10.24)$$

where again C is an appropiate positive constant independent of k.

On the other hand

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1} B(\xi_j,\frac{\eta}{k^{1+\sigma}})}(f'(u)-f'(U_1))Z_{11}Z_{1l}\right|\\
&\leq C\mu^{-n}\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1} B(\xi_j,\frac{\eta}{k^{1+\sigma}})}\frac{1}{(1+|x|^2)^2}Z_1(\frac{x-\xi_1}{\mu})Z_1(\frac{x-\xi_l}{\mu})\\
&\leq C\mu^{n-4}\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1} B(\xi_j,\frac{\eta}{k^{1+\sigma}})}\frac{1}{(1+|x|^2)^2}\frac{1}{|x-\xi_1|^{(n-1)}}\frac{1}{|x-\xi_l|^{(n-1)}}\,dx
\end{aligned}
$$

Thus we conclude that

$$
\left|\int_{\mathbb{R}^n\setminus\bigcup_{j\geq 1} B(\xi_j,\frac{\eta}{k^{1+\sigma}})}(f'(u)-f'(U_1))Z_{01}^2\right|\leq C\mu^{\frac{n}{2}} \tag{10.25}
$$

Computation of G_{11}. Let $\eta>0$ and $\sigma>0$ be small and fixed numbers. We write

$$
\begin{aligned}
G_{11} &= \int_{\mathbb{R}^n}(f'(u)-f'(U_1))Z_{21}^2\\
&= [\int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})}+\int_{\mathbb{R}^n\setminus B(\xi_1,\frac{\eta}{k^{1+\sigma}})}](f'(u)-f'(U_1))Z_{21}^2\\
&= I_1+I_2
\end{aligned}
$$

Recall that $Z_{21}(x)=\mu^{-\frac{n}{2}}Z_2(\frac{x-\xi_1}{\mu})$. We claim that the main part of the above expansion is I_1. Arguing as in the expansion of F_{11}, in the set $B(\xi_1,\frac{\eta}{k^{1+\sigma}})$ we perform a Taylor expansion of the function $(f'(u)-f'(U_1))$ so that

$$
\begin{aligned}
I_1 &= \int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})}f''(U_1)[U(x)-\sum_{l>1}^{k}U_l(x)+\tilde\phi(x)]Z_{21}^2(x)\,dx+R_1\\
&\quad(\text{ changing variables } x=\xi_1+\mu y)\\
&= \mu^{-2}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}f''(U)\Upsilon(y)\,Z_2^2\\
&\quad+\mu^{-2}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}f''(U)\mu^{\frac{n-2}{2}}\tilde\phi(\mu y+\xi_1)Z_2^2\,dx+R_1\\
&= \mu^{-2}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}f''(U)\Upsilon(y)Z_2^2\\
&\quad+p(p-1)\gamma\mu^{-2}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})}U^{p-2}\phi_1(y)Z_2^2\,dx+R_1
\end{aligned}
$$

where $\phi_1(y)=\mu^{\frac{n-2}{2}}\tilde\phi(\mu y+\xi_1)$ and $\Upsilon(y)=\left[\mu^{\frac{n-2}{2}}U(\xi_1+\mu y)-\sum_{l>1}^{k}U(y+\mu^{-1}(\xi_1-\xi_l))\right]$.

Using the equation satisfied by ϕ_1 and by Z_2 in $\mathbb{R}^n$, we get

$$
\begin{aligned}
& p(p-1)\gamma \int U^{p-2}\phi_1 Z_2^2 = p\gamma \int \frac{\partial}{\partial y_2} U^{p-1}\, \phi_1 Z_2 \\
&= -p\gamma \int U^{p-1}\partial_{y_2}\phi_1 Z_2 - p\gamma \int U^{p-1}\phi_1 \partial_{y_2} Z_2 \\
&= \int \zeta_1(\xi_1+\mu y)\mu^{\frac{n+2}{2}} E(\xi_1+\mu y)\partial_{y_2} Z_2 + \gamma\mu^{\frac{n+2}{2}} \int \mathcal{N}(\phi_1)(\xi_1+\mu y)\partial_{y_2} Z_2 \\
&= p\gamma \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} U^{p-1}\left[\mu^{\frac{n-2}{2}} U(\xi_1+\mu y) - \sum_{l>1} U(y+\mu^{-1}(\xi_1-\xi_l))\right]\partial_{y_2} Z_2 \\
&+ O(\mu^{\frac{n}{2}})
\end{aligned}
$$

Thus we conclude that

$$
I_1 = p\gamma\mu^{-2} \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} \Upsilon(y)\, \partial_{y_2}\left(U^{p-1}Z_2\right) dy + O(\mu^{\frac{n}{2}})
$$

Using the definition of μ in (1.16), we see that the first order term in expansions (10.3) and (10.4) gives a lower order contribution to I_1. Furthermore, by symmetry, also the second order term in the expansions (10.3) and (10.4) gives a small contribution. Thus, the third order term in the above mentioned expansions is the one that counts. We get indeed

$$
\begin{aligned}
I_1 &= p\,\gamma\, \frac{n-2}{4}\,\mu^{\frac{n-2}{2}} \int [\frac{n}{2} y_1^2 - |y|^2]\partial_{y_2}(U^{p-1}Z_2) \\
&- p\,\gamma\,\frac{n-2}{4}\,\mu^{n-2} \sum_{l>1}^{k} \frac{1}{(1-\cos\theta_l)^{\frac{n}{2}}} \int \Big[-1-|y|^2 \\
&+ \frac{n}{2}(1-\cos\theta_l)y_1^2 + \frac{n}{2}(1+\cos\theta_l)y_2^2\Big]\partial_{y_2}(U^{p-1}Z_2) + O(\mu^{\frac{n}{2}}) \\
&= -p\gamma\frac{n-2}{4}\mu^{\frac{n-2}{2}}\left[2+\mu^{\frac{n-2}{2}}\sum_{l>1}^{k}\frac{-2+n(1+\cos\theta_l)}{(1-\cos\theta_l)^{\frac{n}{2}}}\right](-\int y_2 U^{p-1}Z_2) + O(\mu^{\frac{n}{2}})
\end{aligned}
$$

On the other hand, arguing as in the proof of estimate (10.20), we have that

$$
I_2 = \mu^{\frac{n}{2}} O(1)
$$

where $O(1)$ is bounded as $k \to \infty$. Thus we conclude (4.23).

Computation of G_{1l}. Let $l > 1$ be fixed. Arguing as in the computation of F_{1l}, we first observe that

$$
G_{1l} = \int_{B(\xi_l,\frac{\eta}{k})} [f'(u) - f'(U_1)]Z_{21}Z_{2l}\, dy + O(\mu^{\frac{n}{2}})
$$

Recall that

$$
Z_{2l}(x) = \left[-\sin\theta_l\,\mu^{-\frac{n}{2}} Z_1(\frac{x-\xi_l}{\mu}) + \cos\theta_l\,\mu^{-\frac{n}{2}}\, Z_2(\frac{x-\xi_l}{\mu})\right]\left(1+O(\mu^2)\right).
$$

In the ball $B(\xi_l, \frac{\eta}{k})$, we expand as before in Taylor, and we get

$$
\begin{aligned}
G_{1l} = & -p\,\gamma \cos\theta_l \int_{B(\xi_l,\frac{\eta}{k})} [\mu^{-\frac{n-2}{2}} U(\frac{x-\xi_l}{\mu})]^{p-1} \mu^{-n} Z_2(\frac{x-\xi_1}{\mu}) Z_2(\frac{x-\xi_l}{\mu}) \\
& + p\,\gamma \sin\theta_l \int_{B(\xi_l,\frac{\eta}{k})} [\mu^{-\frac{n-2}{2}} U(\frac{x-\xi_l}{\mu})]^{p-1} \mu^{-n} Z_2(\frac{x-\xi_1}{\mu}) Z_1(\frac{x-\xi_l}{\mu}) \\
& + O(\mu^{\frac{n}{2}}) \\
& \quad (x = \mu y + \xi_l) \\
& = -p\,\gamma\,\mu^{-2} \, \sin\theta_l \int_{B(0,\frac{\eta}{\mu k})} U^{p-1} Z_2 Z_2(y + \mu^{-1}(\xi_l - \xi_1))\, dy \\
& + p\,\gamma\,\mu^{-2} \, \cos\theta_l \int_{B(0,\frac{\eta}{\mu k})} U^{p-1} Z_2 Z_1(y + \mu^{-1}(\xi_l - \xi_1))\, dy + O(\mu^{\frac{n}{2}}).
\end{aligned}
$$

Now using the expansion (10.7) we get, for any $l > 1$, the validity of (4.24).

Computation of B_{11}. Let $\eta > 0$ and $\sigma > 0$ be small and fixed numbers. We write

$$
\begin{aligned}
B_{11} &= \int_{\mathbb{R}^n} (f'(u) - f'(U_1)) Z_{01} Z_{11} \\
&= [\int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})} + \int_{\mathbb{R}^n \setminus B(\xi_1,\frac{\eta}{k^{1+\sigma}})}](f'(u) - f'(U_1)) Z_{01} Z_{11} \\
&= I_1 + I_2
\end{aligned}
$$

We claim that the main part of the above expansion is I_1. We have

$$
\begin{aligned}
I_1 = & \int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})} f''(U_1)[U(x) - \sum_{l>1}^{k} U_l(x) + \tilde{\phi}(x)] Z_{01} Z_{11}\, dx \; + O(\mu^{\frac{n}{2}}) \\
& (\; x = \xi_1 + \mu y) \\
& = \mu^{-2} \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} f''(U) \Upsilon(y)\, Z_0 Z_1\, dy \\
& + \mu^{-2} \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} f''(U) \left[\mu^{\frac{n-2}{2}} \tilde{\phi}(\mu y + \xi_1)] Z_0 Z_1\, dx \right] + O(\mu^{\frac{n}{2}}) \\
& = \mu^{-2} \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} f''(U) [\mu^{\frac{n-2}{2}} U(\xi_1 + \mu y) - \sum_{l>1}^{k} U(y + \mu^{-1}(\xi_1 - \xi_l))] Z_0 Z_1\, dy \\
& + p(p-1)\gamma\mu^{-2} \left[\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} U^{p-2} \phi_1(y) Z_0 Z_1\, dy \right] + O(\mu^{\frac{n}{2}})
\end{aligned}
$$

where $\phi_1(y) = \mu^{\frac{n-2}{2}} \tilde{\phi}(\mu y + \xi_1)$.

Using the equation satisfied by ϕ_1 and by Z_0 , Z_1 in $\mathbb{R}^n$, we have that

$$
\begin{aligned}
p(p-1)\gamma \quad \int U^{p-2}\phi_1 Z_0 Z_1 &= p\gamma \int \frac{\partial}{\partial y_1} U^{p-1}\,\phi_1 Z_0 \\
&= -p\gamma \int U^{p-1}\partial_{y_1}\phi_1 Z_0 - p\gamma \int U^{p-1}\phi_1 \partial_{y_1} Z_0 \\
&= p\gamma \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} U^{p-1}\Upsilon(y)\partial_{y_1}(U^{p-1}Z_0)
\end{aligned}
$$

where $\Upsilon(y) = \Big[\mu^{\frac{n-2}{2}} U(\xi_1+\mu y) - \sum_{l>1}^k U(y+\mu^{-1}(\xi_1-\xi_l))\Big]$. Using expansions (10.3) and (10.4), and taking into account that $\partial_{y_1}\left(U^{p-1}Z_0\right) = (p-1)U^{p-1}Z_0Z_1 + U^{p-1}\partial_{y_1}Z_0$,

$$
\begin{aligned}
I_1 &= p\gamma\mu^{-2}\int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} \Upsilon(y)\partial_{y_1}\left(U^{p-1}Z_0\right) \\
&= p\gamma\frac{n-2}{2}\Bigg[-\mu^{\frac{n-4}{2}}\int y_1\partial_{y_1}(U^{p-1}Z_0)\,dy \\
&+ \mu^{n-3}\sum_{l>1}^{k}\frac{1}{(1-\cos\theta_l)^{\frac{n-2}{2}}}\int y_1\partial_{y_1}(U^{p-1}Z_0)\Bigg] + O(\mu^{\frac{n}{2}}) \\
&= O(\mu^{\frac{n}{2}}).
\end{aligned}
$$

On the other hand, arguing as in the expansion of A_{11}, one can easily prove that

$$
I_2 = O(\mu^{\frac{n}{2}}).
$$

Taking into account (10.13), we conclude (4.27).
Computation of B_{1l}. Let $l > 1$ be fixed. We have

$$
\begin{aligned}
B_{1l} &= \int_{B(\xi_l,\frac{\eta}{k})} [f'(u) - f'(U_1)]Z_{01}Z_{1l}\,dx + O(\mu^{\frac{n}{2}}) \\
&= -p\gamma\,\cos\theta_l \int_{B(\xi_l,\frac{\eta}{k})} [\mu^{-\frac{n-2}{2}} U(\frac{x-\xi_l}{\mu})]^{p-1}\,\mu^{-n} Z_0(\frac{x-\xi_1}{\mu}) Z_1(\frac{x-\xi_l}{\mu}) \\
&- p\gamma\,\sin\theta_l \int_{B(\xi_l,\frac{\eta}{k})} [\mu^{-\frac{n-2}{2}} U(\frac{x-\xi_l}{\mu})]^{p-1}\,\mu^{-n} Z_0(\frac{x-\xi_1}{\mu}) Z_2(\frac{x-\xi_l}{\mu}) \\
&+ O(\mu^{\frac{n}{2}}) \\
&\quad (x = \mu y + \xi_l) \\
&= -p\gamma\mu^{-1}\cos\theta_l \int_{B(0,\frac{\eta}{\mu k})} U^{p-1}Z_1 Z_0(y+\mu^{-1}(\xi_l-\xi_1))\,dy \\
&- p\gamma\mu^{-1}\sin\theta_l \int_{B(0,\frac{\eta}{\mu k})} U^{p-1}Z_2 Z_0(y+\mu^{-1}(\xi_l-\xi_1))\,dy + O(\mu^{\frac{n}{2}}).
\end{aligned}
$$

Now using the expansion (10.5) we get, for any $l > 1$, (4.28).

Computation of C_{11}. Arguing as in the computation of G_{11}, we are led to

$$
\begin{aligned}
C_{11} &= p\gamma\mu^{-1}\left[\int_{B(0,\frac{\eta}{\mu k})} \Upsilon(y)\,\partial_{y_2}(U^{p-1}Z_0)\,dy\right](1+O(\mu)) \\
&+ k^{n-2}\mu^{n-1}O(1) \\
&= -p\gamma\mu^n\left(\sum_{l>1}^{k}\frac{\sin\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}}\right)\int U^{p-1}Z_0 \ + k^{n-2}\mu^{n-1}O(1),
\end{aligned}
$$

where

$$
\Upsilon(y) = \left[\mu^{\frac{n-2}{2}}U(\xi_1+\mu y) - \sum_{l>1}^{k} U(y+\mu^{-1}(\xi_1-\xi_l))\right]
$$

so that we conclude, by cancellation, the validity of (4.31).

Computation of C_{1l}. Let $l>1$ be fixed. We have

$$
\begin{aligned}
C_{1l} = & \int_{B(\xi_l,\frac{\eta}{k})}[f'(u)-f'(U_1)]Z_{01}Z_{2l}\,dx + k^{n-2}\mu^{n-1}O(1) \\
&= -p\gamma\ \cos\theta_l\int_{B(\xi_l,\frac{\eta}{k})}[\mu^{-\frac{n-2}{2}}U(\frac{x-\xi_l}{\mu})]^{p-1}\,\mu^{-n+}Z_0(\frac{x-\xi_1}{\mu})Z_2(\frac{x-\xi_l}{\mu})\,dx \\
&+ p\gamma\ \sin\theta_l\int_{B(\xi_l,\frac{\eta}{k})}[\mu^{-\frac{n-2}{2}}U(\frac{x-\xi_l}{\mu})]^{p-1}\,\mu^{-n+1}Z_0(\frac{x-\xi_1}{\mu})Z_1(\frac{x-\xi_l}{\mu})\,dx \\
&+ k^{n-2}\mu^{n-1}O(1) \\
&\quad (x=\mu y+\xi_l) \\
&= -p\gamma\mu^{-1}\ \cos\theta_l\int_{B(0,\frac{\eta}{\mu k})}U^{p-1}Z_2Z_0(y+\mu^{-1}(\xi_l-\xi_1))\,dy \\
&+ p\gamma\mu^{-1}\ \sin\theta_l\int_{B(0,\frac{\eta}{\mu k})}U^{p-1}Z_2Z_0(y+\mu^{-1}(\xi_l-\xi_1))\,dy + k^{n-2}\mu^{n-1}O(1).
\end{aligned}
$$

Now using the expansion (10.5) we get, for any $l>1$, (4.32).

Computation of D_{11}. Arguing as in the computation of G_{11}, we are led to

$$
\begin{aligned}
D_{11} &= p\gamma\mu^{-2}\int\Upsilon(y)\,\partial_{y_1}(U^{p-1}Z_2)\,dy \\
&+ k^{n-1}\mu^nO(1) \\
&= p\gamma\frac{n-2}{4}n\mu^{n-2}\left(\sum_{l>1}^{k}\frac{\sin\theta_l}{(1-\cos\theta_l)^{\frac{n}{2}}}\right)\int y_2U^{p-1}Z_2 + k^{n-1}\mu^nO(1)
\end{aligned}
$$

so that we conclude (4.35).

Computation of D_{1l}. Let $l > 1$ be fixed. We have

$$
\begin{aligned}
D_{1l} &= \int_{B(\xi_l,\frac{\eta}{k})} [f'(u) - f'(U_1)] Z_{11} Z_{2l}\, dx \ + k^{n-1}\mu^n O(1) \\
&= -p\gamma \ \cos\theta_l \int_{B(\xi_l,\frac{\eta}{k})} [\mu^{-\frac{n-2}{2}} U(\frac{x-\xi_l}{\mu})]^{p-1} \mu^{-n} Z_1(\frac{x-\xi_1}{\mu}) Z_2(\frac{x-\xi_l}{\mu})\, dx \\
&+ p\gamma \ \sin\theta_l \int_{B(\xi_l,\frac{\eta}{k})} [\mu^{-\frac{n-2}{2}} U(\frac{x-\xi_l}{\mu})]^{p-1} \mu^{-n} Z_1(\frac{x-\xi_1}{\mu}) Z_1(\frac{x-\xi_l}{\mu})\, dx \\
&+ k^{n-1}\mu^n O(1) \\
&\quad (x = \mu y + \xi_l) \\
&= -p\gamma\mu^{-2} \ \cos\theta_l \int_{B(0,\frac{\eta}{\mu k})} U^{p-1} Z_2 Z_1(y + \mu^{-1}(\xi_l - \xi_1))\, dy \\
&+ p\gamma\mu^{-2} \ \sin\theta_l \int_{B(0,\frac{\eta}{\mu k})} U^{p-1} Z_2 Z_2(y + \mu^{-1}(\xi_l - \xi_1))\, dy + k^{n-1}\mu^n O(1).
\end{aligned}
$$

Now using the expansion (10.7) we get (4.36).

Computation of $H_{3,11}$. Let $\eta > 0$ and $\sigma > 0$ be small and fixed numbers. We write

$$
\begin{aligned}
H_{3,11} &= \int_{\mathbb{R}^n} (f'(u) - f'(U_1)) Z_{31}^2 \\
&= [\int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})} + \int_{\mathbb{R}^n \setminus B(\xi_1,\frac{\eta}{k^{1+\sigma}})}](f'(u) - f'(U_1)) Z_{31}^2 \\
&= I_1 + I_2
\end{aligned}
$$

Arguing as before one can show that

$$
I_2 = O(\mu^{\frac{n}{2}}).
$$

In $B(\xi_1, \frac{\eta}{k^{1+\sigma}})$ we can perform a Taylor expansion of the function $(f'(u) - f'(U_1))$ so that

$$
\begin{aligned}
I_1 &= \int_{B(\xi_1,\frac{\eta}{k^{1+\sigma}})} f''(U_1)[U(x) - \sum_{l>1}^{k} U_l(x) + \tilde{\phi}(x)] Z_{31}^2(x)\, dx + O(\mu^{\frac{n}{2}}) \\
&\quad (\ x = \xi_1 + \mu y) \\
&= \mu^{-2} \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} f''(U) \Upsilon(y)\, Z_3^2 \\
&+ \mu^{-2} \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} f''(U) \mu^{\frac{n-2}{2}} \tilde{\phi}(\mu y + \xi_1) Z_3^2\, dx + O(\mu^{\frac{n}{2}}) \\
&= \mu^{-2} \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} f''(U) \Upsilon(y) Z_3^2 \\
&+ p(p-1)\gamma\mu^{-2} \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} U^{p-2} \phi_1(y) Z_3^2\, dx + O(\mu^{\frac{n}{2}})
\end{aligned}
$$

where $\phi_1(y) = \mu^{\frac{n-2}{2}} \tilde\phi(\mu y + \xi_1)$. Using the equation satisfied by ϕ_1 and by z_2 in $\mathbb{R}^n$, and arguing as in the previous steps, we get

$$p(p-1)\gamma \int U^{p-2}\phi_1 Z_3^2 = p\gamma \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} U^{p-1}\Upsilon(y)\partial_{y_3} Z_3$$

where we recall that

$$\Upsilon(y) = \left[\mu^{\frac{n-2}{2}} U(\xi_1 + \mu y) - \sum_{l>1}^{k} U(y + \mu^{-1}(\xi_1 - \xi_l))\right].$$

Thus we conclude that

$$I_1 = p\gamma\mu^{-2} \int_{B(0,\frac{\eta}{\mu k^{1+\sigma}})} \Upsilon(y)\, \partial_{y_3}\left(U^{p-1}Z_3\right) dy + O(\mu^{\frac{n}{2}}).$$

Using the definition of μ in (1.16), we see that the first order term in expansions (10.3) and (10.4) gives a lower order contribution to I_1. Furthermore, by symmetry, also the second order term in the expansions (10.3) and (10.4) gives a small contribution. Thus, the third order term in the above mentioned expansions is the one that counts. We get indeed

$$\begin{aligned}
I_1 &= p\,\gamma\,\frac{n-2}{4}\,\mu^{\frac{n-2}{2}} \int [\frac{n}{2}y_1^2 - |y|^2]\partial_{y_3}(U^{p-1}z_3) \\
&- p\,\gamma\,\frac{n-2}{4}\,\mu^{n-2} \sum_{l>1}^{k} \frac{1}{(1-\cos\theta_l)^{\frac{n}{2}}} \int \Big[-1 - |y|^2 + \frac{n}{2}(1-\cos\theta_l)y_1^2 \\
&+ \frac{n}{2}(1+\cos\theta_l)y_2^2\Big]\partial_{y_3}(U^{p-1}Z_3) + O(\mu^{\frac{n}{2}}) \\
&= p\gamma\frac{n-2}{2}\mu^{\frac{n-2}{2}}\left[1 - \mu^{\frac{n-2}{2}} \sum_{l>1}^{k} \frac{1}{(1-\cos\theta_l)^{\frac{n}{2}}}\right](-\int y_3 U^{p-1}Z_3) + O(\mu^{\frac{n}{2}})
\end{aligned}$$

Thus we conclude (4.39).

Computation of $H_{3,1l}$. Let $l > 1$ be fixed. Arguing as before, we get

$$\begin{aligned}
H_{3,1l} &= \int_{B(\xi_l,\frac{\eta}{k})} [f'(u) - f'(U_1)]Z_{31}Z_{3l} + O(\mu^{\frac{n}{2}}) \\
&= -p\,\gamma \int_{B(\xi_l,\frac{\eta}{k})} [\mu^{-\frac{n-2}{2}} U(\frac{x-\xi_l}{\mu})]^{p-1}\,\mu^{-n}Z_3(\frac{x-\xi_1}{\mu})Z_3(\frac{x-\xi_l}{\mu}) \\
&+ O(\mu^{\frac{n}{2}}) \\
&\quad (x = \mu y + \xi_l) \\
&= -p\,\gamma\,\mu^{-2} \int_{B(0,\frac{\eta}{\mu k})} U^{p-1}Z_3(y)Z_3(y + \mu^{-1}(\xi_l - \xi_1))\,dy + O(\mu^{\frac{n}{2}})
\end{aligned}$$

Now using the expansion (10.8) we get (4.40).

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M. Musso - Departamento de Matemática, Pontificia Universidad Catolica de Chile, Avda. Vicuña Mackenna 4860, Macul, Chile

E-mail address: mmusso@mat.puc.cl

J. Wei -Department of Mathematics, University of British Columbia, Vancouver, BC V6T1Z2, Canada

E-mail address: jcwei@math.ubc.ca