BOUNDARY CLUSTERED INTERFACES FOR THE ALLEN-CAHN EQUATION

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Abstract. We consider the Allen-Cahn equation

$$\varepsilon^2 \Delta u + u - u^3 = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

where $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^n$ and $\varepsilon > 0$ is a small parameter. We prove the existence of a radial solution u_ε which has N interfaces $\{u_\varepsilon(r) = 0\} = \cup_{j=1}^N \{r = r_j^\varepsilon\}$, where $1 > r_1^\varepsilon > r_2^\varepsilon > ... > r_N^\varepsilon$ are such that $1 - r_1^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, r_{j-1}^\varepsilon - r_j^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, j = 2, ..., N$. Moreover, the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is exactly N.

1. Introduction

The aim of this paper is to construct a family of *clustered* transitional layered solutions to the Allen-Cahn equation

$$\begin{cases} \varepsilon^2 \Delta u + u - u^3 = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^n$, $\varepsilon > 0$ is a small parameter, and $\nu(x)$ denotes the unit outer normal at $x \in \partial\Omega$.

Problem (1.1) and its parabolic counterpart have been a subject of extensive research for many years. In order to describe some known results, we define the Allen-Cahn functional (see [2])

$$J_\varepsilon[u] = \int_\Omega [\frac{\varepsilon^2}{2} |\nabla u|^2 - F(u)], \text{ where } F(u) = -\frac{1}{4}(1 - u^2)^2.$$

The set $\{x \in \Omega \mid u(x) = 0\}$ is called the interface of u. Let $\mathrm{Per}_\Omega(A)$ be the relative perimeter of the set $A \subset \Omega$. Using $\Gamma-$convergence techniques (see [15]), Kohn and Sternberg in [13] showed a general result stating that in a neighborhood of an isolated local minimizer of Per_Ω there exists a local minimizer to the functional J_ε. They further used this idea to show the existence of a stable solution for (1.1) in two dimensional, non-convex domains, such as a dumb-bell. Since then, the existence of solutions with a single interface intersecting the boundary has been established and studied by many authors. See [1], [5], [8], [12], [19], [22] and the references

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therein. However, the existence of multiple interfaces is only proved (see [17] and [18]) in the one-dimensional case for the following Allen-Cahn equation with inhomogeneous terms

$$\varepsilon^2 u'' + a(x)(u-u^3) = 0, -1 < x < 1, u'(\pm 1) = 0 \tag{1.2}$$

and in the higher-dimensional case ([7]) for the following nonlinear equation with bistable nonlinearity and inhomogeneous term

$$\varepsilon^2 \Delta u + u(u - a(|x|))(1-u) = 0 \text{ in } B_1(0), \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_1(0). \tag{1.3}$$

The results of [7] states that if $a(r)$ has a critical point $r_0 \in (0,1)$ such that $a(r_0) = \frac{1}{2}, a'(r_0) = 0, a''(r_0) < 0$, then there exists a clustered interior layer solutions to (1.3). All the papers ([7], [17], [18]) use the properties of the inhomogeneous terms to construct multiple (interior) interfaces. (For Allen-Cahn equation with inhomogeneity $\Delta u + a(x)(u-u^3) = 0$ in $\mathbb{R}^2$, we refer to [20] and [21].)

Here, we continue our study, initiated in [14], in the study of clustered layered solutions for semilinear elliptic equations and show that the *homogeneous* Allen-Cahn equation *itself* can generate multiple clustered interfaces near the boundary. In [14], we showed that the following singularly perturbed Neumann problem

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

has a clustered layered solution near the boundary. (The existence of one layer solution to (1.4) near the boundary was first established in [3], [4].) The purpose of this paper is to show that a similar phenomenon happens to the Allen-Cahn equation. In particular, we establish the existence of clustered interfaces – the so-called "*phantom interfaces*" – in higher dimensions. Moreover we show that for each fixed positive integer N, there exists a solution to (1.1) with Morse index N (in the space of radial functions).

Our main result is the following.

Theorem 1.1. *Let N be a fixed positive integer. Then there exists $\varepsilon_N > 0$ such that for all $\varepsilon < \varepsilon_N$, problem (1.1) admits a radially symmetric solution u_ε with the following properties*

(1) the set of interfaces $\{u_\varepsilon(r) = 0\}$ contains N spheres $\{r = r_j^\varepsilon\}, j = 1, ..., N$ with

$$1 - r_1^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \qquad r_{j-1}^\varepsilon - r_j^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \quad j = 2, ..., N. \tag{1.5}$$

More precisely, we have $u_\varepsilon(r_j^\varepsilon + \varepsilon y) \to (-1)^j H(y)$, where $H(y)$ is the unique heteroclinic solution of

$$H'' + H - H^3 = 0, \ H(0) = 0, \ H(\pm\infty) = \pm 1. \tag{1.6}$$

(2) u_ε has the following energy bound

$$J_\varepsilon[u_\varepsilon] = \omega_{n-1} N \varepsilon I[H] + o(\varepsilon), \tag{1.7}$$

where

$$I[H] = \int_{\mathbb{R}} \left(\frac{1}{2}(H')^2 - F(H) \right),$$

and where ω_{n-1} denotes the volume of S^{n-1}.

(3) The Morse index of u_ε in $H^1_r(\Omega)$ is exactly N, where $H^1_r(\Omega)$ denotes the space of radial functions in $H^1(\Omega)$.

Remark: By a simple transformation, Theorem 1.1 readily extends to (1.3) with $a(r) \equiv \frac{1}{2}$.

Our approach is similar to that of [14], where a finite dimensional reduction procedure combined with a variational approach is used. Such a method has been used successfully in many other papers, see e.g. [3], [4], [6], [9], [10] and [11].

In the rest of section, we introduce some notation which are used later.

By the scaling $x = \varepsilon y$, problem (1.1) is reduced to the ODE

$$\begin{cases} u_{rr} + \frac{n-1}{r} u_r + f(u) = 0, r \in (0, \frac{1}{\varepsilon}), \\ u'(0) = u'(\frac{1}{\varepsilon}) = 0, \end{cases} \tag{1.8}$$

where $f(u) = u - u^3$. From now on, we will work with (1.8).

Let $H(y)$ be the unique solution to (1.6). Set

$$\Omega_\varepsilon = \frac{1}{\varepsilon} B_1(0) = B_{\frac{1}{\varepsilon}}(0), \quad I_\varepsilon = \left(0, \ \frac{1}{\varepsilon}\right). \tag{1.9}$$

For $u \in C^2(\Omega_\varepsilon)$ and $u = u(r)$, we have

$$\Delta u = u'' + \frac{n-1}{r} u'. \tag{1.10}$$

For $k \in \mathbb{N}$, we denote by $H^k_r(\Omega_\varepsilon)$ the space of radial functions in $H^k(\Omega_\varepsilon)$. On $H^1_r(\Omega_\varepsilon)$, we define an inner product as follows:

$$(u, v)_\varepsilon = \int_0^{\frac{1}{\varepsilon}} (u'v' + 2uv) r^{n-1} dr. \tag{1.11}$$

Similarly, the inner product on $L^2_r(\Omega_\varepsilon)$ can be defined by

$$< u, v >_\varepsilon = \int_0^{\frac{1}{\varepsilon}} (uv) r^{n-1} dr. \tag{1.12}$$

We also introduce a new energy functional which, up to a positive multiplicative constant, is equivalent to J_ε

$$\mathcal{E}_\varepsilon[u] = \frac{1}{2}\int_0^{\frac{1}{\varepsilon}} |u'|^2 r^{n-1} - \int_0^{\frac{1}{\varepsilon}} F(u) r^{n-1} dr, \quad u \in H^1_r(\Omega_\varepsilon). \tag{1.13}$$

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. The notation $A_\varepsilon >> B_\varepsilon$ means that $\lim_{\varepsilon\to 0} \frac{|B_\varepsilon|}{|A_\varepsilon|} = 0$, while $A_\varepsilon << B_\varepsilon$ means $\frac{1}{A_\varepsilon} >> \frac{1}{B_\varepsilon}$.

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2. Some Preliminary Analysis

In this section we introduce a family of approximate solutions to (1.8) and derive some useful estimates.

Let H be the unique solution of (1.6). It is easy to see that

$$\begin{cases} H(y) - 1 = -A_0 e^{-\sqrt{2}|y|} + O(e^{-(2\sqrt{2})|y|}) \text{ for } y > 1; \\ H(y) + 1 = A_0 e^{-\sqrt{2}|y|} + O(e^{-(2\sqrt{2})|y|}) \text{ for } y < -1; \\ H'(y) = \sqrt{2} A_0 e^{-\sqrt{2}|y|} + O(e^{-(2\sqrt{2})|y|}) \text{ for } |y| > 1, \end{cases} \tag{2.1}$$

where $A_0 > 0$ is a fixed constant.

We state the following well-known lemma on H. For a proof, see Lemma 4.1 of [16].

Lemma 2.1. *For the following eigenvalue problem*

$$\phi'' + f'(H)\phi = \lambda\phi, |\phi| \leq 1, \text{ in } \mathbb{R}, \tag{2.2}$$

there holds

$$\lambda_1 = 0, \ \phi_1 = cH'; \lambda_2 < 0. \tag{2.3}$$

For $u \in H^2_r(\Omega_\varepsilon)$, we define the operator

$$\mathcal{S}_\varepsilon[u] := u_{rr} + \frac{n-1}{r} u_r + f(u). \tag{2.4}$$

We introduce the following set

$$\Lambda = \left\{ \mathbf{t} = (t_1, ..., t_N) \middle| \begin{array}{l} t_N > 1 - \varepsilon(\log\frac{1}{\varepsilon})^2, \quad 1 - t_1 \geq \eta\,\varepsilon \log\frac{1}{\varepsilon}, \\ t_{j-1} - t_j > \eta\,\varepsilon\log\frac{1}{\varepsilon}, j = 2, ..., N \end{array} \right\}, \tag{2.5}$$

where $\eta \in (0, \frac{1}{8\sqrt{2}})$ is a fixed number.

Let $\chi(s)$ be a cut-off function such that $\chi(s) = 1$ for $s \le \frac{1}{4}$ and $\chi(s) = 0$ for $s \ge \frac{1}{2}$. For $t \in \left(\frac{3}{4}, 1\right)$, we define

$$\rho_\varepsilon(t) = H'(\frac{1-t}{\varepsilon}); \qquad \beta_\varepsilon(r) = \frac{1}{\sqrt{2}} e^{\sqrt{2}(r - \frac{1}{\varepsilon})}, \quad r \in \left[0, \frac{1}{\varepsilon}\right], \tag{2.6}$$

and

$$H_t(r) = H(r - \frac{t}{\varepsilon}), \quad H_{\varepsilon,t}(r) = \left(H(r - \frac{t}{\varepsilon}) - \rho_\varepsilon(t)\beta_\varepsilon(r) \right)(1 - \chi(\varepsilon r)) - \chi(\varepsilon r). \tag{2.7}$$

Then it is easy to see that for $\frac{1-t}{\varepsilon} >> 1$

$$\rho_\varepsilon(t) = \sqrt{2} A_0 e^{-\frac{\sqrt{2}(1-t)}{\varepsilon}} + O(e^{-2\sqrt{2}\frac{1-t}{\varepsilon}}). \tag{2.8}$$

We first assume that N is *odd*. For $\mathbf{t} \in \Lambda$, we now define our approximate function:

$$H_{\varepsilon,\mathbf{t}}(r) = \sum_{j=1}^{N} (-1)^j H_{\varepsilon,t_j}(r). \tag{2.9}$$

If N is even, we set

$$H_{\varepsilon,\mathbf{t}}(r) = \sum_{j=1}^{N} (-1)^j H_{\varepsilon,t}(r) - 1 = \sum_{j=1}^{N+1} (-1)^j H_{\varepsilon,t_j}(r) \tag{2.10}$$

where we use the convention $H_{\varepsilon,t_{N+1}} = 1$. So without loss of generality we can assume that N is *odd*.

Note that for $r \le \frac{1}{2\varepsilon}$, there holds

$$|H_{\varepsilon,t}(r) - (-1)^N| + |H'_{\varepsilon,t}(r)| + |H''_{\varepsilon,t}(r)| \le e^{-\frac{1}{C\varepsilon}}. \tag{2.11}$$

Observe also that, by construction, $H_{\varepsilon,\mathbf{t}}$ satisfies the Neumann boundary condition, namely $H'_{\varepsilon,\mathbf{t}}(0) = H'_{\varepsilon,\mathbf{t}}(\frac{1}{\varepsilon}) = 0$. Furthermore, $H_{\varepsilon,\mathbf{t}}$ depends smoothly on $\mathbf{t}$ as a map with values in $C^2\left(\left[0, \frac{1}{\varepsilon}\right]\right)$.

The following lemma shows that $H_{\varepsilon,\mathbf{t}}$ is a good approximate function to (1.8).

Lemma 2.2. *For ε sufficiently small and $\mathbf{t} \in \mathbf{\Lambda}$, one has*

$$\|\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\|_{L^\infty} + \varepsilon^{n-1} \int_0^{\frac{1}{\varepsilon}} |\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]| r^{n-1} dr \le C \left[\varepsilon + \sum_{j=1}^{N} (\rho_\varepsilon(t_j))^2 + \sum_{i \ne j} e^{-\frac{\sqrt{2}|t_i - t_j|}{\varepsilon}} \right]. \tag{2.12}$$

Proof . Using (1.6) it is easy to see that

$$\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}] = \frac{n-1}{r} H'_{\varepsilon,\mathbf{t}} + f(H_{\varepsilon,\mathbf{t}}) - \sum_{l=1}^{N} (-1)^l f(H_{t_l}) - 2 \sum_{l=1}^{N} (-1)^l \rho_\varepsilon(t_l)\beta_\varepsilon(r) + O\left(e^{-\frac{1}{C\varepsilon}}\right). \tag{2.13}$$

The first term in right hand side of (2.13) can be estimated as follows

$$\frac{1}{r}H'_{\varepsilon,\mathbf{t}} = \frac{1}{r}\sum_{j=1}^{N}(-1)^j(H'_{t_j} - \rho_\varepsilon(t_j)\beta'_\varepsilon(r)) + O\left(e^{-\frac{1}{C\varepsilon}}\right).$$

From the decay of H' and β_ε we deduce that

$$\left\|\frac{1}{r}H'_{\varepsilon,\mathbf{t}}\right\|_\infty + \varepsilon^{n-1}\int_0^{\frac{1}{\varepsilon}}\frac{1}{r}|H'_{\varepsilon,\mathbf{t}}|r^{n-1}dr \le C\varepsilon. \tag{2.14}$$

Next, we note that

$$\left|f(H_{\varepsilon,\mathbf{t}}) - \sum_{l=1}^{N}f((-1)^lH_{t_l}) - 2\sum_{l=1}^{N}(-1)^l\rho_\varepsilon(t_l)\beta_\varepsilon(r)\right| \le S_1 + S_2,$$

where

$$S_1 = \left|f(\sum_{j=1}^{N}(-1)^jH_{t_j}) - \sum_{j=1}^{N}f((-1)^jH_{t_j})\right|;$$

$$S_2 = \left|f(\sum_{j=1}^{N}(-1)^jH_{\varepsilon,t_j}) - f(\sum_{j=1}^{N}(-1)^jH_{t_j}) - 2\sum_{j=1}^{N}(-1)^j\rho_\varepsilon(t_j)\beta_\varepsilon(r)\right|.$$

To estimate S_1 and S_2, we divide the domain $I_\varepsilon = (0,\frac{1}{\varepsilon})$ into the N intervals $I_{\varepsilon,1},\dots,I_{\varepsilon,N}$ defined by

$$I_{\varepsilon,1} = \left[\frac{t_1+t_2}{2\varepsilon},\frac{1}{\varepsilon}\right), I_{\varepsilon,j} = \left[\frac{t_j+t_{j+1}}{2\varepsilon},\frac{t_j+t_{j-1}}{2\varepsilon}\right), j=2,...,N-1, I_{\varepsilon,N} = \left(0,\frac{t_N+t_{N-1}}{2\varepsilon}\right). \tag{2.15}$$

Let us choose $t_0 = 2-t_1, t_{N+1} = -t_N$ so that

$$I_{\varepsilon,j} = \left[\frac{t_j+t_{j+1}}{2\varepsilon},\frac{t_j+t_{j-1}}{2\varepsilon}\right), \qquad j=1,...,N,\ I_\varepsilon = \cup_{j=1}^N I_{\varepsilon,j}. \tag{2.16}$$

For $r \in I_{\varepsilon,j}$, we note that for $j<l$,

$$H_{t_l}(r) = 1 + O(e^{-\sqrt{2}|r-\frac{t_j}{\varepsilon}|})$$

while for $j>l$,

$$H_{t_l}(r) = -1 + O(e^{-\sqrt{2}|r-\frac{t_j}{\varepsilon}|}).$$

Since N is odd, we see that

$$\sum_{l\ne j}(-1)^lH_{t_l} = \sum_{l<j}(-1)^l(H_{t_l}+1) + \sum_{l>j}(-1)^l(H_{t_l}-1). \tag{2.17}$$

Thus we can rewrite S_1 as:

$$S_1 = f\left(\sum_{l<j}(-1)^l(H_{t_l}+1) + (-1)^jH_{t_j} + \sum_{l>j}(-1)^l(H_{t_l}-1)\right) - (-1)^jf(H_{t_j}) - \sum_{l\ne j}(-1)^lf(H_{t_l})$$

$$= f'((-1)^j H_{t_j}) \left(\sum_{l<j} (-1)^l (H_{t_l}+1) + \sum_{l>j} (-1)^l (H_{t_l}-1) \right) - \sum_{l \neq j} (-1)^l f(H_{t_l})$$
$$+O\left(\sum_{l<j} (H_{t_l}+1)^2 + \sum_{l>j} (H_{t_l}-1)^2 \right)$$

This quantity can also be written in the following way

$$S_1 = [f'((-1)^j H_{t_j}) - f'(1)] \left(\sum_{l<j} (-1)^l (H_{t_l}+1) + \sum_{l>j} (-1)^l (H_{t_l}-1) \right)$$
$$+O\left(\sum_{l<j} (H_{t_l}+1)^2 + \sum_{l>j} (H_{t_l}-1)^2 \right)$$
$$= O(\min\{H_{t_j}+1, H_{t_j}-1\}) \left(\sum_{l<j} (H_{t_l}+1) + \sum_{l>j} (H_{t_l}-1) \right) + O\left(\sum_{l<j} (H_{t_l}+1)^2 + \sum_{l>j} (H_{t_l}-1)^2 \right).$$

Then with some elementary computations one finds

$$\|S_1\|_{L^\infty(I_{\varepsilon,j})} + \varepsilon^{n-1} \int_{I_{\varepsilon,j}} |S_1(r)| r^{n-1} dr \leq C \sum_{i \neq j} e^{-\frac{\sqrt{2}|t_i - t_j|}{\varepsilon}}. \tag{2.18}$$

It remains to estimate S_2. To this purpose, we note that for $r \in I_{\varepsilon,j}$, $j \geq 2$, we have

$$\rho_\varepsilon(t_j)\beta_\varepsilon(r) = O(e^{-\sqrt{2}\frac{1-t_1}{\varepsilon}} e^{\sqrt{2}(r-\frac{1}{\varepsilon})}),$$

from which it follows that

$$\|S_2\|_{L^\infty(I_{\varepsilon,j})} + \varepsilon^{n-1} \int_{I_{\varepsilon,j}} |S_2(r)| r^{n-1} dr = O(e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}}) = O(\sum_{j=1}^N (\rho_\varepsilon(t_j))^2), \quad j \geq 2.$$

Therefore, we just need to consider the case $r \in I_{\varepsilon,1}$. But since $f'(\pm 1) = -2$, we have

$$S_2 = f\left(\sum_{l=1}^N (-1)^l H_{t_l} - \sum_{l=1}^N (-1)^l \rho_\varepsilon(t_l)\beta_\varepsilon(r) \right) - f(\sum_{l=1}^N (-1)^l H_{t_l}) - f'(-1) \sum_{l=1}^N (-1)^l \rho_\varepsilon(t_l)\beta_\varepsilon(r)$$
$$= \left[f'(\sum_{l=1}^N (-1)^l H_{t_l}) - f'(-1) \right] \sum_{l=1}^N (-1)^l \rho_\varepsilon(t_l)\beta_\varepsilon(r) + O\left(\sum_{l=1}^N \rho_\varepsilon(t_l)^2 \beta_\varepsilon(r)^2 \right)$$
$$= O\left(\sum_{l=1}^N e^{-\sqrt{2}|r - \frac{t_l}{\varepsilon}|} \right) \left(\sum_{l=1}^N \rho_\varepsilon(t_l)\beta_\varepsilon(r) \right) + O\left(\sum_{l=1}^N \rho_\varepsilon(t_l)^2 \beta_\varepsilon(r)^2 \right).$$

Hence we also get

$$\|S_2\|_{L^\infty(I_1)} + \varepsilon^{n-1} \int_{I_1} |S_2(r)| r^{n-1} dr \leq C \rho_\varepsilon^2(t_1). \tag{2.19}$$

□

The proof of the next lemma is postponed to the appendix.

Lemma 2.3. *Let* $\mathbf{t} \in \Lambda$. *Then for* ε *sufficiently small we have*

$$\begin{aligned} \mathcal{E}_\varepsilon[\sum_{j=1}^{N}(-1)^j H_{\varepsilon,t_j}] &= I[H]\sum_{i=1}^{N}\left(\frac{t_i}{\varepsilon}\right)^{n-1} - \left(\frac{1}{\varepsilon}\right)^{n-1}(\sqrt{2}A_0^2+o(1))e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}} \\ &- \sum_{j=2}^{N}\left(\frac{t_j}{\varepsilon}\right)^{n-1}(\sqrt{2}A_0^2+o(1))e^{-\sqrt{2}\frac{|t_j-t_{j-1}|}{\varepsilon}} + O(\varepsilon^{2-n}), \end{aligned} \tag{2.20}$$

where $A_0 > 0$ *is defined in (2.1).*

3. Lyapunov-Schmidt Process: Finite-Dimensional Reduction

In this section we outline the so-called Lyapunov-Schmidt reduction process. Since this can be proved along the same ideas of Sections 3 of [14], we skip some of the details.

Fix $\mathbf{t} \in \Lambda$. Integrating by parts, one can show that orthogonality to $\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}$ in $H_r^1(\Omega_\varepsilon)$, $j = 1, \dots, N$, is equivalent to orthogonality in $L^2(\Omega_\varepsilon)$ to the following functions

$$Z_{\varepsilon,t_j} = \Delta(\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}) - 2\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}, \qquad j = 1, ..., N. \tag{3.1}$$

By elementary computations, differentiating (1.6) we obtain

$$\frac{\partial H_{\varepsilon,t_j}}{\partial t_j} = -\frac{1}{\varepsilon}H'(r - \frac{t_j}{\varepsilon}) + \frac{1}{\varepsilon}H''(\frac{1-t_j}{\varepsilon})\beta_\varepsilon(r) + O(e^{-\frac{1}{C\varepsilon}}), \tag{3.2}$$

$$Z_{\varepsilon,t_j} = (f'(H_{t_j}) - f'(\pm 1))\frac{\partial H_{\varepsilon,t_j}}{\partial t_j} + \frac{n-1}{r}\left(\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}\right)' = -\frac{1}{\varepsilon}H'_{t_j}(f'(H_{t_j}) - f'(\pm 1)) + o\left(\frac{1}{\varepsilon}\right), \tag{3.3}$$

where $O(e^{-\frac{1}{C\varepsilon}})$ and $o(\frac{1}{\varepsilon})$ are intended both in the C^1 and H_r^1 sense.

We consider first the following linear problem. Given $h \in L^\infty(\Omega_\varepsilon)$, find a function ϕ satisfying

$$\begin{cases} L_\varepsilon[\phi] := \phi'' + \frac{n-1}{r}\phi' + f'(H_{\varepsilon,\mathbf{t}})\phi = h + \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}; \\ \phi'(0) = \phi'(\frac{1}{\varepsilon}) = 0; \quad <\phi, Z_{\varepsilon,t_j}>_\varepsilon = 0, \quad j = 1, ..., N, \end{cases} \tag{3.4}$$

for some constants $c_j, j = 1, ..., N$. To this purpose, define the norm

$$\|\phi\|_* = \sup_{r\in(0,\frac{1}{\varepsilon})} |\phi(r)|. \tag{3.5}$$

Assuming a solution to (3.4) exists, we have the following estimate on ϕ:

Proposition 3.1. *Let* ϕ *satisfy (3.4). Then for* ε *sufficiently small, we have*

$$\|\phi\|_* \leq C\|h\|_*, \tag{3.6}$$

where C *is a positive constant independent of* ε *and* $\mathbf{t} \in \Lambda$.

Proof : The proof is similar in spirit of that of Proposition 3.1 of [14]. For sake of completeness, we include a proof here.

Arguing by contradiction, assume that

$$\|\phi\|_* = 1; \qquad \|h\|_* = o(1). \tag{3.7}$$

We multiply (3.4) by $\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}$ and integrate over Ω_ε to obtain

$$\begin{aligned} \sum_{i=1}^{N} c_i < Z_{\varepsilon,t_i}, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} >_\varepsilon &= - < h, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} >_\varepsilon \\ &+ < \Delta\phi + f'(H_{\varepsilon,\mathbf{t}})\phi, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} >_\varepsilon . \end{aligned} \tag{3.8}$$

From the exponential decay of H' one finds

$$< h, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} >_\varepsilon = \int_0^{\frac{1}{\varepsilon}} h \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} r^{n-1} dr = O(\|h\|_* \varepsilon^{-n}).$$

Moreover, integrating by parts, using (3.2) and (3.3) we deduce

$$\begin{aligned} < \Delta\phi + f'(H_{\varepsilon,\mathbf{t}})\phi, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} >_\varepsilon &= < Z_{\varepsilon,t_j} + f'(H_{\varepsilon,\mathbf{t}}) \frac{\partial H_{\varepsilon,t_j}}{\partial t_j}, \phi >_\varepsilon \\ &= o(\varepsilon^{-n}\|\phi\|_*). \end{aligned}$$

From (3.2) and (3.3), we also see that

$$< Z_{\varepsilon,t_i}, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} >_\varepsilon = -\varepsilon^{-n-1} \left(t_i^{n-1} \delta_{ij} \int_{\mathbb{R}} f'(H)(H')^2 + o(1) \right), \tag{3.9}$$

where δ_{ij} denotes the Kronecker symbol. Note that, using the equation $H''' + f'(H)H' = 0$ we find

$$\int_{\mathbb{R}} f'(H)(H')^2 = \int_{\mathbb{R}} ((H'')^2) > 0.$$

This shows that the left hand side of the equation (3.8) is diagonally dominant in the indexes i, j, and hence by (3.7) we have

$$c_i = O(\varepsilon\|h\|_*) + o(\varepsilon\|\phi\|_*) = o(\varepsilon), \qquad i = 1, ..., N. \tag{3.10}$$

Also, since we are assuming that $\|h\|_* = o(1)$ and since $\|Z_{\varepsilon,t_j}\|_* = O\left(\frac{1}{\varepsilon}\right)$, there holds

$$\|h + \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}\|_* = o(1). \tag{3.11}$$

Thus (3.4) yields

$$\begin{cases} \phi'' + \frac{n-1}{r}\phi' + f'(\pm 1) + (f'(H_{\varepsilon,\mathbf{t}}) - f'(\pm 1))\phi = o(1); \\ \phi'(0) = \phi'(\frac{1}{\varepsilon}) = 0; \quad < \phi, Z_{\varepsilon,t_j} >_\varepsilon = 0, \quad j = 1, ..., N, \end{cases} \tag{3.12}$$

where $o(1)$ is in the sense of $L^\infty(0, \frac{1}{\varepsilon})$.

We show that (3.12) is incompatible with our assumption $\|\phi\|_* = 1$. First we claim that

$$|\phi| \to 0 \quad \text{on} \quad y \in \bigcup_{j=1}^{N} \left(\frac{t_j}{\varepsilon} - R, \frac{t_j}{\varepsilon} + R \right) \qquad \text{as } \varepsilon \to 0, \tag{3.13}$$

where R is any fixed positive constant.

Indeed, assuming the contrary, there exist $\delta_0 > 0$, $j \in \{1, \dots, N\}$ and sequences $\varepsilon_k, \phi_k, y_k \in \left(\frac{t_j}{\varepsilon} - R, \frac{t_j}{\varepsilon} + R \right)$ such that ϕ_k satisfies (3.4) and

$$|\phi_k(y_k)| \geq \delta_0. \tag{3.14}$$

Let $\tilde{\phi}_k = \phi_k(y - \frac{t_j}{\varepsilon_k})$. Then using (3.12) and $\|\phi\|_* = 1$, as $\varepsilon_k \to 0$ $\tilde{\phi}_k$ converges weakly in $H^2_{loc}(\mathbb{R})$ and strongly in $C^1_{loc}(\mathbb{R})$ to a bounded function ϕ_0 which satisfies

$$\phi_0'' + f'(H)\phi_0 = 0 \qquad \text{in } \mathbb{R}, \ |\phi_0| \leq C.$$

By Lemma 2.1, there holds $\phi_0 = cH'$ for some c. Since $\tilde{\phi}_k \perp Z_{\varepsilon, t_j}$, we conclude that $\int_{\mathbb{R}} \phi_0 f'(H)(H')^2(y) = 0$, which yields $c = 0$. Hence $\phi_0 = 0$ and $\tilde{\phi}_k \to 0$ in $B_{2R}(0)$. This contradicts (3.14), so (3.13) holds true.

Given $\delta > 0$, the decay of $f'(H) - f'(\pm 1)$ and (3.13) (with R sufficiently large) imply

$$\|(f'(H_{\varepsilon,\mathbf{t}}) - f'(\pm 1))\phi\|_* \leq \delta + \frac{1}{2}\|\phi\|_*. \tag{3.15}$$

Using (3.12) and the Maximum Principle one finds

$$\begin{aligned} \|\phi\|_* &\leq \|(f'(H_{\varepsilon,\mathbf{t}}) - f'(\pm 1))\phi\|_* + \sum_{j=1}^{N} |c_j| \|Z_{\varepsilon,t_j}\|_* + \|h\|_* \\ &\leq 2\delta + \frac{1}{2}\|\phi\|_*, \end{aligned}$$

and hence

$$\|\phi\|_* \leq 4\delta < 1$$

if we choose $\delta < \frac{1}{4}$. This contradicts (3.7). □

Next, we consider the following nonlinear problem: find a function ϕ such that for some constants $c_j, j = 1, ..., N$, the following equation holds true

$$\begin{cases} \Delta(H_{\varepsilon,\mathbf{t}} + \phi) + f(H_{\varepsilon,\mathbf{t}} + \phi) = \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j} \text{ in } \Omega_\varepsilon, \\ \phi'(0) = \phi'(\frac{1}{\varepsilon}) = 0, < \phi, Z_{\varepsilon,t_j} >_\varepsilon = 0, j = 1, ..., N. \end{cases} \tag{3.16}$$

We have the following result, whose proof follows the same lines of Proposition 4.2 of [14].

Proposition 3.2. *For* $\mathbf{t} \in \Lambda$ *and* ε *sufficiently small, there exists a unique* $\phi = \phi_{\varepsilon,\mathbf{t}}$ *such that (3.16) holds. Moreover,* $t \mapsto \phi_{\varepsilon,\mathbf{t}}$ *is of class* C^1 *as a map into* $H^1_r(\Omega_\varepsilon)$*, and we have*

$$\|\phi_{\varepsilon,\mathbf{t}}\|_* \leq C\left(\varepsilon + \sum_{j=1}^{n} e^{-\frac{3}{2}\sqrt{2}\frac{1-t_j}{\varepsilon}} + \sum_{i\neq j} e^{-\frac{\frac{3}{4}\sqrt{2}|t_i-t_j|}{\varepsilon}}\right). \tag{3.17}$$

4. Energy computation for reduced energy functional

In this section we expand the quantity

$$\mathcal{M}_\varepsilon(\mathbf{t}) := \varepsilon^{n-1}\mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] : \Lambda \to \mathbb{R} \tag{4.1}$$

in ε and $\mathbf{t}$, where $\phi_{\varepsilon,\mathbf{t}}$ is given by Proposition 3.2. Up to negligible error terms, the same expansion of Lemma 2.3 holds true.

Lemma 4.1. *For* $\mathbf{t} \in \Lambda$ *and* ε *sufficiently small, we have*

$$\begin{aligned}
\mathcal{M}_\varepsilon(\mathbf{t}) &= \varepsilon^{n-1}\mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] \\
&= I[H]\sum_{j=1}^{N} t_j^{n-1} - (\sqrt{2}A_0^2 + o(1))e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}} \\
&- (\sqrt{2}A_0^2 + o(1))\sum_{j=2}^{N} t_j^{n-1} e^{-\sqrt{2}\frac{|t_j-t_{j-1}|}{\varepsilon}} + O(\varepsilon).
\end{aligned} \tag{4.2}$$

Proof. It is sufficient to show that

$$\mathcal{M}_\varepsilon(\mathbf{t}) = \varepsilon^{n-1}\mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}}] + o\left(\sum_{j=1}^{N} e^{-2\sqrt{2}\frac{1-t_j}{\varepsilon}} + \sum_{i\neq j} e^{-\sqrt{2}\frac{|t_i-t_j|}{\varepsilon}}\right) + O(\varepsilon),$$

and to apply Lemma 2.3. In order to do this, we write

$$\varepsilon^{1-n}\mathcal{M}_\varepsilon = \mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}}] + K_1 + K_2 - K_3,$$

where

$$K_1 = \int_0^{\frac{1}{\varepsilon}} \left[H'_{\varepsilon,\mathbf{t}}\phi'_{\varepsilon,\mathbf{t}} - f(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\right] r^{n-1}dr;$$

$$K_2 = \frac{1}{2}\int_0^{\frac{1}{\varepsilon}} \left[|\phi'_{\varepsilon,\mathbf{t}}|^2 - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}^2\right] r^{n-1}dr;$$

$$K_3 = \int_0^{\frac{1}{\varepsilon}} \left[F(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - F(H_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} - \frac{1}{2}f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}^2\right] r^{n-1}dr.$$

Integrating by parts, using Lemmas 2.2 and Proposition 3.1, we find

$$
\begin{aligned}
|K_1| &= \left| \int_0^{\frac{1}{\varepsilon}} \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}} r^{n-1} dr \right| \le C\|\phi_{\varepsilon,\mathbf{t}}\|_* \int_0^{\frac{1}{\varepsilon}} |\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]| r^{n-1} dr \\
&\le C\varepsilon^{1-n}\left(\varepsilon^2 + \sum_{j=1}^N (\rho_\varepsilon(t_j))^{2+\frac{3}{2}} + \sum_{i\neq j} e^{-\frac{7}{4}\sqrt{2}|t_i-t_j|/\varepsilon} \right). \qquad (4.3)
\end{aligned}
$$

To estimate K_2, we note that $\phi_{\varepsilon,\mathbf{t}}$ satisfies

$$
\Delta\phi_{\varepsilon,\mathbf{t}} + f(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) + \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] = \sum_{j=1}^N c_j Z_{\varepsilon,t_j}. \qquad (4.4)
$$

Multiplying (4.4) by $\phi_{\varepsilon,\mathbf{t}} r^{n-1}$ and integrating over I_ε, we obtain

$$
\begin{aligned}
\int_{I_\varepsilon} \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}} r^{n-1} dr &= \int_{I_\varepsilon} \left(|\phi'_{\varepsilon,\mathbf{t}}|^2 - f'(H_{\varepsilon,\mathbf{t}})\phi^2_{\varepsilon,\mathbf{t}} \right) r^{n-1} dr \\
&+ \int_{I_\varepsilon} \left[f(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} \right] \phi_{\varepsilon,\mathbf{t}} r^{n-1} dr. \qquad (4.5)
\end{aligned}
$$

Hence we find

$$
2K_2 = -\int_{I_\varepsilon} \left[f(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} \right] \phi_{\varepsilon,\mathbf{t}} r^{n-1} dr + \int_{I_\varepsilon} \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}} r^{n-1} dr.
$$

From the Taylor's formula, we get

$$
|f(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}| \le C|\phi_{\varepsilon,\mathbf{t}}|^2,
$$

so we deduce

$$
|K_2| \le C\int_{I_\varepsilon} |\phi_{\varepsilon,\mathbf{t}}|^3 r^{n-1} dr + C\|\phi_{\varepsilon,\mathbf{t}}\|_* \int_{I_\varepsilon} \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}] r^{n-1} dr.
$$

From the exponential decay of $H(\pm y) - \pm 1$ one finds that $\phi_{\varepsilon,\mathbf{t}}(r)$ satisfies

$$
\phi''_{\varepsilon,\mathbf{t}} + \frac{n-1}{r}\phi'_{\varepsilon,\mathbf{t}} + f(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}})
$$

$$
= O\left(\sum_{j=1}^N e^{-\sqrt{2}\left|r - \frac{t_j}{\varepsilon}\right|} \right), \phi'_{\varepsilon,\mathbf{t}}(0) = \phi'_{\varepsilon,\mathbf{t}}(\frac{1}{\varepsilon}) = 0.
$$

From (4.4) and a comparison principle, we obtain

$$
|\phi_{\varepsilon,\mathbf{t}}(r)| \le C\sum_{j=1}^N e^{-\frac{\sqrt{2}}{\tilde{C}}\left|r - \frac{t_j}{\varepsilon}\right|} \qquad (4.6)
$$

for some $\tilde{C} < 1$.

Using Proposition 3.2 and (4.6), we get

$$|K_2| \le C\varepsilon^{1-n}\left(\varepsilon^2 + \sum_{j=1}^{N}(\rho_\varepsilon(t_j))^3 + \sum_{i\neq j} e^{-2\sqrt{2}|t_i-t_j|/\varepsilon}\right). \tag{4.7}$$

From the Hölder continuity of f' we deduce

$$\left|F(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - F(H_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} - \frac{1}{2}f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}^2\right| \le C|\phi_{\varepsilon,\mathbf{t}}|^3,$$

so, again, it follows that

$$|K_3| \le C\varepsilon^{1-n}\left(\varepsilon^2 + \sum_{j=1}^{N}(\rho_\varepsilon(t_j))^3 + \sum_{i\neq j} e^{-2\sqrt{2}|t_i-t_j|/\varepsilon}\right). \tag{4.8}$$

Combining with (2.20) of Lemma 2.2, we obtain the conclusion. □

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Fix $\mathbf{t} \in \overline{\Lambda}$ and let $\phi_{\varepsilon,\mathbf{t}}$ be given by Proposition 3.2. Let also $\mathcal{M}_\varepsilon(\mathbf{t})$ denote the reduced energy functional defined by (4.1).

Proposition 5.1. *For ε small, the following maximization problem*

$$\sup\{\mathcal{M}_\varepsilon(\mathbf{t}) : \mathbf{t} \in \Lambda\} \tag{5.1}$$

has a solution $\mathbf{t}^\varepsilon$ in the interior of Λ.

Proof: Since $\mathcal{M}_\varepsilon(\mathbf{t})$ is continuous in $\mathbf{t}$, it achieves a maximum in $\bar{\Lambda}$. Let $\mathbf{t}^\varepsilon$ be a maximum point. We claim that $\mathbf{t}^\varepsilon \in \Lambda$.

Let us argue by contradiction and assume that $\mathbf{t}^\varepsilon \in \partial\Lambda$. Then from the definition of Λ, there are three possibilities: either $1-t_1 = \eta\varepsilon\log\frac{1}{\varepsilon}$, or there exists $j \ge 2$ such that $t_{j-1}-t_j = \eta\varepsilon\log\frac{1}{\varepsilon}$, or $t_N = 1-\varepsilon(\log\frac{1}{\varepsilon})^2$.

In the first case, we have

$$\begin{aligned} I[H]t_1^{n-1} - (\sqrt{2}A_0^2+o(1))e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}} &= I[H]\left(1-\eta\varepsilon\log\frac{1}{\varepsilon}\right)^{n-1} - \sqrt{2}A_0^2 e^{-2\eta\sqrt{2}\log\frac{1}{\varepsilon}} + o(\varepsilon^{2\sqrt{2}\eta}) \\ &\le I[H] - A_0^2\varepsilon^{2\sqrt{2}\eta}. \end{aligned}$$

Since $\eta < \frac{1}{8\sqrt{2}}$, we obtain

$$\mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \le NI[H] - A_0^2\varepsilon^{2\sqrt{2}\eta}. \tag{5.2}$$

In the second case, there holds

$$\begin{aligned}\mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) &\leq I[H]\sum_{j=1}^{N} t_j^{n-1} - (\sqrt{2}A_0^2 + o(1))\varepsilon^{\sqrt{2}\eta} t_j^{n-1} \\ &\leq NI[H] - A_0^2 \varepsilon^{\sqrt{2}\eta}. \end{aligned} \tag{5.3}$$

In the latter case, we have $t_N = 1 - \varepsilon(\log \frac{1}{\varepsilon})^2$, and therefore

$$\mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \leq I[H](N - 1 + t_N^{n-1}) + O(\varepsilon) \leq I[H](N - (n-1)\varepsilon(\log \frac{1}{\varepsilon})^2) + O(\varepsilon). \tag{5.4}$$

On the other hand, choosing $t_j = 1 - \frac{j}{\sqrt{2}}\varepsilon \log \frac{1}{\varepsilon}$, $j = 1, ..., N$, we obtain

$$\sum_{j=1}^{N} t_j^{n-1} = 1 - \frac{N(N+1)(n-1)}{2\sqrt{2}} \varepsilon \log \frac{1}{\varepsilon} + O(\varepsilon^2 (\log \frac{1}{\varepsilon})^2);$$

$$e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}} = \varepsilon^2; \qquad e^{-\sqrt{2}\frac{|t_{j-1}-t_j|}{\varepsilon}} = \varepsilon, \tag{5.5}$$

and we find

$$\mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \geq NI[H] - \frac{N(N+1)(n-1)^2}{2\sqrt{2}} \varepsilon \log \frac{1}{\varepsilon} + O(\varepsilon)$$

which contradicts either (5.2), or (5.3), or (5.4). This completes the proof of Proposition 5.1. □

Remark: The above argument also shows that

$$1 - t_1^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \ t_{j-1}^\varepsilon - t_j^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}. \tag{5.6}$$

Finally, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 3.2, there exists ε_N such that for $\varepsilon < \varepsilon_N$ we have a C^1 map $\mathbf{t} \mapsto \phi_{\varepsilon,\mathbf{t}}$ from $\overline{\Lambda}$ into $C^2(I_\varepsilon)$ such that

$$\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] = \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j} \tag{5.7}$$

for some constants $\{c_j\} \subseteq \mathbb{R}$, which also are of class C^1 in $\mathbf{t}$.

By Proposition 5.1, there exists $\mathbf{t}^\varepsilon \in \Lambda$ achieving the maximum of $\mathcal{K}_\varepsilon : t \to \mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}]$. Let $u_\varepsilon = \sum_{i=1}^{N}(-1)^i H_{\varepsilon,t_i^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon} = H_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}$. Then we have

$$\partial_{t_i}|_{\mathbf{t}=\mathbf{t}^\varepsilon} \mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) = 0, i = 1, ..., N,$$

and hence

$$\int_{I_\varepsilon}\left[\nabla u_\varepsilon\nabla\partial_{t_i}(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}})+u_\varepsilon\partial_{t_i}(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}})-f(u_\varepsilon)\partial_{t_i}(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}})\right]\Bigg|_{\mathbf{t}=\mathbf{t}^\varepsilon} r^{n-1}dr=0.$$

Therefore, by (5.7) we find

$$\sum_{j=1}^{N} c_j\int_{I_\varepsilon}\left(Z_{\varepsilon,t_j}\partial_{t_i}(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}})\right)r^{n-1}dr=0. \tag{5.8}$$

Differentiating the equation $<\phi,Z_{\varepsilon,t_j}>_\varepsilon=0$ with respect to t_j, we get

$$<\partial_{t_i}\phi,Z_{\varepsilon,t_j}>_\varepsilon=-<\phi,\partial_{t_i}Z_{\varepsilon,t_j}>_\varepsilon=O(\|\phi\|_*)\varepsilon^{-n-1}.$$

Using (3.3), we see that (5.8) is diagonally dominant in the coefficients $\{c_i\}$, which implies $c_j=0$ for $j=1,...,N$. Hence $u_\varepsilon=H_{\varepsilon,\mathbf{t}^\varepsilon}+\phi_{\varepsilon,\mathbf{t}^\varepsilon}$ is a solution of (1.1).

By our construction, one can easily check that $\varepsilon^{n-1}\mathcal{E}_\varepsilon(u_\varepsilon)\to NI[H]$ as $\varepsilon\to 0$, and u_ε has only N zeroes $\frac{s_1^\varepsilon}{\varepsilon},...,\frac{s_N^\varepsilon}{\varepsilon}$. By the structure of u_ε we see that (up to a permutation) $s_i^\varepsilon-t_i^\varepsilon=o(1)$. This proves (1) and (2) of Theorem 1.1.

It remains to prove (3). First we note that $u_\varepsilon^{'}$ satisfies

$$\Delta u_\varepsilon^{'}+f^{'}(u_\varepsilon)u_\varepsilon^{'}=\frac{n-1}{r^2}u_\varepsilon^{'}. \tag{5.9}$$

By our construction, at each interval $(\frac{s_j^\varepsilon}{\varepsilon},\frac{s_{j-1}^\varepsilon}{\varepsilon})$ for $j=2,...,N$, there exists a point $\frac{\tilde{s}_{j-1}^\varepsilon}{\varepsilon}\in(\frac{s_j}{\varepsilon},\frac{s_{j-1}}{\varepsilon})$ such that $u_\varepsilon^{'}(\frac{\tilde{s}_{j-1}^\varepsilon}{\varepsilon})=0$. Now we set

$$\varphi_1(r)=\begin{cases}u_\varepsilon^{'}(r)\text{ for }r\in(\frac{\tilde{s}_1^\varepsilon}{\varepsilon},1),\\ 0,\text{ otherwise;}\end{cases}$$

$$\varphi_j(r)=\begin{cases}u_\varepsilon^{'}(r), & \text{for } r\in(\frac{\tilde{s}_j^\varepsilon}{\varepsilon},\frac{\tilde{s}_{j-1}^\varepsilon}{\varepsilon}),\\ 0, & \text{otherwise,}\end{cases}\qquad j=2,...,N-1;$$

$$\varphi_N(r)=\begin{cases}u_\varepsilon^{'}(r), & \text{for } r\in(\frac{1}{2\varepsilon},\frac{\tilde{s}_{N-1}^\varepsilon}{\varepsilon}),\\ 2\varepsilon(r-\frac{1}{4\varepsilon})u_\varepsilon^{'}(r), & \frac{1}{4\varepsilon}\le r\le\frac{1}{2\varepsilon},\\ 0, & \text{for } r<\frac{1}{4\varepsilon}\text{ or } r\ge\frac{\tilde{s}_{N-1}^\varepsilon}{\varepsilon}.\end{cases}$$

Next we define a quadratic functional

$$\mathbf{Q}[\phi]=\int_{I_\varepsilon}(|\nabla\phi|^2-f^{'}(u_\varepsilon)\phi^2)r^{n-1}dr. \tag{5.10}$$

It is easy to check that

$$\int_{I_\varepsilon}\varphi_i\varphi_j r^{n-1}dr=0\text{ for } i\neq j. \tag{5.11}$$

Using equation (5.9), we obtain

$$\mathbf{Q}[\varphi_i]=-\int_{I_\varepsilon}\varphi_i^2r^{n-3}dr<0, i=1,...,N-1. \tag{5.12}$$

When $i = N$, we have

$$\mathbf{Q}[\varphi_N] = -\int_{I_\varepsilon} \varphi_N^2 r^{n-3} dr + O(e^{-\frac{1}{C\varepsilon}}) < 0. \tag{5.13}$$

(5.12) and (5.13) imply that the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is at least N.

Finally we also show that the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is at most N. In fact, let us define

$$z_j^\varepsilon(r) = H'_{\varepsilon, t_j^\varepsilon} \chi\left(\frac{\varepsilon r - t_j^\varepsilon}{\varepsilon(\sqrt{|\log \frac{1}{\varepsilon}|})} \right), j = 1, ..., N \tag{5.14}$$

and consider the following minimization problem

$$\mu_j^\varepsilon = \inf_{\phi \in H^1(I_{\varepsilon,j}), \int_{I_{\varepsilon,j}} \phi z_j^\varepsilon r^{n-1} dr = 0} \frac{\int_{I_{\varepsilon,j}} (|\nabla \phi|^2 - f'(u_\varepsilon)\phi^2) r^{n-1} dr}{\int_{I_{\varepsilon,j}} \phi^2 r^{n-1} dr}. \tag{5.15}$$

Assume that $\mu_j^\varepsilon \leq 0$. By standard regularity theory, μ_j^ε is attained by a function ϕ_j^ε which satisfies

$$\Delta \phi_j^\varepsilon + f'(u_\varepsilon)\phi_j^\varepsilon = -\mu_j^\varepsilon \phi_j^\varepsilon + c_j^\varepsilon z_j^\varepsilon, \quad (\phi_j^\varepsilon)'|_{\partial I_{\varepsilon,j}} = 0, \quad \int_{I_{\varepsilon,j}} \phi_j^\varepsilon z_j r^{n-1} dr = 0 \tag{5.16}$$

where c_j^ε is a constant.

First, we notice that $c_j^\varepsilon = o(\|\phi_j^\varepsilon\|_*)$, which follows by reasoning as for (3.10) of Proposition 3.1. Then from Lemma 2.1 we deduce that $\mu_j^\varepsilon \to 0$ and moreover the same argument leading to Proposition 3.1 shows that $\phi_j^\varepsilon = 0$.

Thus $\mu_j^\varepsilon > 0$. Let $\phi = \phi(r)$ be such that $\int_{I_\varepsilon} \phi z_j^\varepsilon r^{n-1} = 0, j = 1, ..., N$, which is equivalent to $\int_{I_{\varepsilon,j}} \phi z_j^\varepsilon r^{n-1} = 0$. This then implies

$$\int_{I_{\varepsilon,j}} (|\nabla \phi|^2 - f'(u_\varepsilon)\phi^2) r^{n-1} dr \geq \mu_j^\varepsilon \int_{I_{\varepsilon,j}} |\phi|^2 r^{n-1} dr, j = 1, ..., N, \tag{5.17}$$

and hence

(5.18)

$$\int_{I_\varepsilon} (|\nabla \phi|^2 - f'(u_\varepsilon)\phi^2) r^{n-1} dr = \sum_{j=1}^N \int_{I_{\varepsilon,j}} (|\nabla \phi|^2 - f'(u_\varepsilon)\phi^2) r^{n-1} dr \geq \min_{j=1,...,N} \mu_j^\varepsilon \int_{I_\varepsilon} |\phi|^2 r^{n-1} dr.$$

This yields

$$\lambda_{N+1} = \sup_{v_1, ..., v_N} \inf_{\int_{I_\varepsilon} \phi v_j r^{n-1} = 0, j = 1, ..., N} \frac{\int_{I_\varepsilon} (|\nabla u|^2 - f'(u_\varepsilon)\phi^2) r^{n-1}}{\int_{I_\varepsilon} \phi^2 r^{n-1}} \geq \min_{j=1,...,N} \mu_j^\varepsilon > 0 \tag{5.19}$$

and hence the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is at most N.

Combining the upper and lower bound for the Morse index, we see that the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is exactly N. This proves (3) of Theorem 1.1.

□

Appendix

In this appendix we expand the quantity $\mathcal{E}_\varepsilon[\sum_{j=1}^N (-1)^j H_{\varepsilon,t_j}]$ as a function of ε and $\mathbf{t}$. Several facts will be used repeatedly:

$$H(y) = 1 - A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}), \text{ for } y > 1;$$

$$H(y) = -1 + A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}), \text{ for } y < -1;$$

$$H'(y) = \sqrt{2} A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}), \text{ for } |y| > 1;$$

$$\rho_\varepsilon(t_1) = \sqrt{2}(A_0 + o(1)) e^{-\sqrt{2}\frac{1-t_1}{\varepsilon}};$$

$$\rho_\varepsilon(t_j) = o(\rho_\varepsilon(t_1)) \text{ for } j \geq 2.$$

From a Taylor's expansion we find

$$\mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}}] = I_1 + I_2 + I_3 + O(\varepsilon^{1-n}\rho_\varepsilon^3(t_1)),$$

where

$$I_1 = \mathcal{E}_\varepsilon[\sum_{j=1}^N (-1)^j H_{t_j}],$$

$$I_2 = -\sum_{l=1}^K (-1)^l \rho_\varepsilon(t_l) \int_{I_\varepsilon} \left[(\sum_{j=1}^N (-1)^j H_{t_j})' \beta_\varepsilon' - f(\sum_{j=1}^N (-1)^j H_{t_j}) \beta_\varepsilon \right] r^{n-1} dr,$$

$$I_3 = \frac{1}{2} (\sum_{l=1}^N (-1)^l \rho_\varepsilon(t_l))^2 \int_{I_\varepsilon} \left[|\beta_\varepsilon'|^2 - f'(\sum_{j=1}^N (-1)^j H_{t_j}) \beta_\varepsilon^2 \right] r^{n-1}.$$

Recalling that $f'(\pm 1) = -2$, the term I_3 can be estimated by

$$I_3 = \frac{1}{2} (\sum_{j=1}^N (-1)^j \rho_\varepsilon(t_j))^2 \int_{I_\varepsilon} \left[2 - f'(\sum_{j=1}^N (-1)^j H_{t_j}) \right] \beta_\varepsilon^2 r^{n-1} dr + o(\varepsilon^{1-n}\rho_\varepsilon^2(t_1))$$

$$= (\rho_\varepsilon(t_1))^2 \int_{I_\varepsilon} \beta_\varepsilon^2 r^{n-1} dr + o(\varepsilon^{1-n}\rho_\varepsilon^2(t_1)) = \frac{1}{2\sqrt{2}} \varepsilon^{1-n} (\rho_\varepsilon(t_1))^2 + o(\varepsilon^{1-n}\rho_\varepsilon^2(t_1))$$

$$= \frac{(A_0^2 + o(1))}{\sqrt{2}} \varepsilon^{1-n} e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}}.$$

Next we estimate the integral in I_2. There holds

$$\int_{I_\varepsilon} \left(\sum_{j=1}^N (-1)^j H_{t_j}' \beta_\varepsilon' - f(\sum_{j=1}^N (-1)^j H_{t_j}) \beta_\varepsilon \right) r^{n-1} dr$$

$$= \int_{I_\varepsilon} \left(\sqrt{2} \sum_{j=1}^N (-1)^j H_{t_j}' - f(\sum_{j=1}^N (-1)^j H_{t_j}) \right) \beta_\varepsilon r^{n-1} dr$$

$$= \int_{I_{\varepsilon,1}} (-\sqrt{2} H_{t_1}' - f(-H_{t_1})) \beta_\varepsilon r^{n-1} dr + o(\varepsilon^{1-n}\rho_\varepsilon(t_1))$$

$$= -\frac{1}{\sqrt{2}} e^{-\sqrt{2}\frac{1-t_1}{\varepsilon}} \int_{\mathbb{R}} (\sqrt{2}H' - f(H)) e^{\sqrt{2}y} dy (\frac{t_1}{\varepsilon})^{n-1} + o(\varepsilon^{1-n}\rho_\varepsilon(t_1))$$
$$= -A_0 e^{-\sqrt{2}\frac{1-t_1}{\varepsilon}} (\frac{t_1}{\varepsilon})^{n-1} + o(\varepsilon^{1-n}\rho_\varepsilon(t_1)),$$

since

$$\int_{\mathbb{R}} (\sqrt{2}H' - f(H)) e^{\sqrt{2}y} dy = (H' e^{\sqrt{2}y})|_{-\infty}^{+\infty} = \sqrt{2}A_0.$$

Thus

$$I_2 = -(\sqrt{2}A_0^2 + o(1)) e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}} (\frac{t_1}{\varepsilon})^{n-1} + o(\varepsilon^{1-n}\rho_\varepsilon(t_1)) + O(\varepsilon^{2-n}),$$

which implies

$$I_2 + I_3 = -\frac{(A_0^2 + o(1))}{\sqrt{2}} e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}} (\frac{t_1}{\varepsilon})^{n-1} + o(\varepsilon^{1-n}\rho_\varepsilon(t_1)) + O(\varepsilon^{2-n}) \tag{5.20}$$

since $t_1 = 1 + O(\varepsilon(\log\frac{1}{\varepsilon})^2)$.

It remains to consider I_1. For this purpose, we decompose it in the following way

$$I_1 = \sum_{j=1}^{N} E_{\varepsilon,j},$$

where

$$E_{\varepsilon,j} = \int_{I_{\varepsilon,j}} \left[\frac{1}{2}|\sum_{l=1}^{N}(-1)^l H_{t_l}'|^2 - F(\sum_{l=1}^{N}(-1)^l H_{t_l})\right] r^{n-1} dr$$
$$= \int_{I_{\varepsilon,j}} \left[\frac{1}{2}|H_{t_j}' + \sum_{l\neq j}(-1)^{j+l} H_{t_l}'|^2 - F(H_j + \sum_{l\neq j}(-1)^{j+l} H_{t_l})\right] r^{n-1} dr$$
$$= I_4 + I_5 + I_6 + o(\varepsilon^{1-n} \sum_{i\neq j} e^{-\sqrt{2}\frac{|t_i - t_j|}{\varepsilon}}),$$

with

$$I_4 = \int_{I_{\varepsilon,j}} \left[\frac{1}{2}|H_{t_j}'|^2 - F(H_{t_j})\right] r^{n-1} dr,$$
$$I_5 = \int_{I_{\varepsilon,j}} \left[H_{t_j}' \sum_{l\neq j}(-1)^{l+j} H_{t_l}' - f(H_{t_j}) \sum_{l\neq j}(-1)^{l+j} H_{t_l}\right] r^{n-1} dr,$$
$$I_6 = \frac{1}{2}\int_{I_{\varepsilon,j}} |\sum_{l\neq j}(-1)^{j+l} H_{t_l}|^2 (2 - f'((-1)^j H_{t_j})) r^{n-1} dr.$$

Using the fact that $|H'|^2 = 2F(H)$, for I_4 we find

$$I_4 = \int_{I_{\varepsilon,j}} |H_{t_j}'|^2 r^{n-1} dr$$
$$= \int_R |H'|^2 dy (\frac{t_j}{\varepsilon})^{n-1} - \frac{A_0^2 + o(1)}{\sqrt{2}} \left(e^{-\sqrt{2}\frac{|t_j - t_{j-1}|}{\varepsilon}} + e^{-\sqrt{2}\frac{|t_j - t_{j+1}|}{\varepsilon}}\right) (\frac{t_j}{\varepsilon})^{n-1} + O(\varepsilon^{2-n}).$$

For $j \geq 2$, I_5 can be estimated as (recalling the exponential decaying property of $H(y) \pm 1$)

$$I_5 = (\frac{t_j}{\varepsilon})^{n-1} H'_{t_j} \sum_{l \neq j} (-1)^{l+j} H_{t_l} \Big|_{\partial I_{\varepsilon,j}} + O(\varepsilon^{2-n})$$

$$= -(A_0^2 + o(1))\sqrt{2}\left(e^{-\sqrt{2}\frac{|t_j - t_{j-1}|}{\varepsilon}} + e^{-\sqrt{2}\frac{|t_j - t_{j+1}|}{\varepsilon}}\right)(\frac{t_j}{\varepsilon})^{n-1} + O(\varepsilon^{2-n}).$$

For $j = 1$, we have

$$I_5 = (\frac{t_1}{\varepsilon})^{n-1} H'_{t_j} \sum_{l > 1} (-1)^{l+1} H_{t_l} \Big|_{\partial I_{\varepsilon,1}} + O(\varepsilon^{2-n})$$

$$= -(A_0^2 + o(1))\sqrt{2}\left(e^{-\sqrt{2}\frac{|t_1 - t_2|}{\varepsilon}}\right)(\frac{t_j}{\varepsilon})^{n-1} + O(\varepsilon^{2-n}).$$

I_6 can be estimated similarly: for $j \geq 2$ we have

$$I_6 = 2\int_{I_{\varepsilon,j}} |\sum_{l \neq j} (-1)^{j+l} H_{t_l}|^2 r^{n-1} dr$$

$$= \frac{A_0^2 + o(1)}{\sqrt{2}}\left(e^{-\sqrt{2}\frac{|t_j - t_{j-1}|}{\varepsilon}} + e^{-\sqrt{2}\frac{|t_j - t_{j+1}|}{\varepsilon}}\right)(\frac{t_j}{\varepsilon})^{n-1} + O(\varepsilon^{2-n}),$$

while for $j = 1$

$$I_6 = 2\int_{I_{\varepsilon,1}} |\sum_{l > 1} (-1)^{l+1} H_{t_l}|^2 r^{n-1} dr$$

$$= \frac{A_0^2 + o(1)}{\sqrt{2}}\left(e^{-\sqrt{2}\frac{|t_1 - t_2|}{\varepsilon}}\right)(\frac{t_1}{\varepsilon})^{n-1} + O(\varepsilon^{2-n}).$$

Combining the estimates of I_4, I_5, and I_6, we obtain

$$I_1 = I[H]\sum_{j=1}^{N}(\frac{t_j}{\varepsilon})^{n-1} - \sqrt{2}(A_0^2 + o(1))\sum_{j=2}^{N}\left(e^{-\sqrt{2}\frac{|t_j - t_{j-1}|}{\varepsilon}}\right)(\frac{t_j}{\varepsilon})^{n-1} - \frac{A_0^2 + o(1)}{\sqrt{2}} e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}} + O(\varepsilon^{2-n})$$

$$= I[H]\sum_{j=1}^{N}(\frac{t_j}{\varepsilon})^{n-1} - \sqrt{2}(A_0^2 + o(1))\sum_{j=2}^{N} e^{-\sqrt{2}\frac{|t_j - t_{j-1}|}{\varepsilon}}(\frac{t_j}{\varepsilon})^{n-1}$$

$$(5.21) \qquad - \frac{(A_0^2 + o(1))}{\sqrt{2}} e^{-2\sqrt{2}\frac{1-t_1}{\varepsilon}}(\frac{t_1}{\varepsilon})^{n-1} + O(\varepsilon^{2-n}).$$

Adding the estimates in (5.21) and (5.20), we obtain the asymptotic expansion (2.20) of $\mathcal{E}_\varepsilon[\sum_{j=1}^{N}(-1)^j H_{\varepsilon,t_j}]$.

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