ON ANISOTROPIC CURVATURE FLOW EQUATIONS

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Abstract. We study the anisotropic flow $V = bk^{\sigma}$ where $b > 0$. We prove that if $\frac{1}{3} < \sigma \le 1$, only type I blow up occurs and if $0 < \sigma < \frac{1}{3}$ Type II blow up occurs. We also establish *a prior* estimates and existence of stationary solutions of self-similar curves and the existence of " Abresch-Langer" type curves.

1. Introduction

In this paper, we consider the following generalized anisotropic curvature flow equations

$$V = b(\theta)k^{\sigma} \tag{1.1}$$

where V be the inward velocity of embedded closed curves Γ_t of R^2 in the direction of its unit inward normal vector

$$n(\theta) = (cos\ \theta, sin\ \theta),$$

k is the inward curvature of Γ_t, $0 < \sigma \le 1$ and $b(\theta)$ is a positive 2π-periodic functions.

Problem has two significant interpretations. When $b = 1, \sigma = 1$, equation (1.1) is the well-known curve shortening problem and has been studied extensively, see [11], [12], [14]. Equation (1.1) can be seen as the generalization of the curve shortening problem to Minkowski

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geometry. It can also be regarded as a simplified model of the motion of the interface of a metal crystal as it melts , see [3], [11].

Note that when $\sigma = \frac{1}{3}$, problem (1.1) arises in the image processing, see [17]. It also arises in affine geometry, see [2].

The case when $\sigma = 1, b \neq 1$ has been studied by Gage [11] and Gage and Y. Li [13]. The first author and Taniyama in [16] considered the isotropic case $b = 1, 0 < \sigma \leq 1$. In this paper, we consider the anisotropic case, namely, $0 < \sigma \leq 1, b \neq 1$. To introduce our results, we first give some definitions.

If $k > 0$, then (1.1) is equivalent to the following

$$v_t = \sigma b(\theta)^{-\frac{1}{\sigma}} v^{1+\frac{1}{\sigma}} (v_{\theta\theta} + v) \tag{1.2}$$

with $v = bk^\sigma$.

It is well-known that starting from a convex curve, solutions of (1.1) will shrink to a point in a finite time $T < \infty$. To study the asymptotic behavior of the curves as $t \to T$, we introduce the following rescaled function

$$u(\theta, \tau) = (T - t)^{\frac{\sigma}{1+\sigma}} v(\theta, t), \tau = \log(\frac{1}{T - t})$$

We have that u satisfies

$$u_\tau = \sigma b^{-\frac{1}{\sigma}} u^{1+\frac{1}{\sigma}} (u_{\theta\theta} + u - \frac{1}{1+\sigma} b^{\frac{1}{\sigma}} u^{-\frac{1}{\sigma}}) \tag{1.3}$$

Let $a = \frac{1}{1+\sigma} b^{\frac{1}{\sigma}}$. Then a stationary solution of (1.3) is the following

$$u_{\theta\theta} + u - \frac{a}{u^\alpha} = 0, \theta \in R/(2\pi Z) \tag{1.4}$$

with $\alpha = \frac{1}{\sigma}$.

We call the following equation as $(1.4)_k$

$$u_{\theta\theta} + u - \frac{a}{u^\alpha} = 0, \theta \in R/(2k\pi Z)$$

We give the following definition.

Definition: Equation (1.1) has a type I blow up if solution of (1.3) remains bounded as $\tau \to \infty$ and has a type II blow up if solutions of (1.3) becomes bounded as $\tau \to \infty$.

One of the main results of this paper is the following classification of singularities theorem.

Theorem 1.1. *Let $b > 0$ be a smooth 2π periodic function. If $\frac{1}{3} < \sigma \leq 1$, then only Type I blow up occurs. More precisely, every convex curve which evolves by equation (1.1) evolves to a point in finite time and the renormalized curve has a convergent sequence which converges to the shape of a self-similar solution.*

If $0 < \sigma < \frac{1}{3}$, then there exists Type II blow up.

Remark: For $\frac{1}{3} < \sigma \leq 1$, Theorem 1.1 can be reinterpreted as follows

Corollary 1.1. *Every positive, smooth function $b(\theta)$ defined on the circle can be written as*

$$b(\theta) = h^{\frac{1}{\sigma}}(h_{\theta\theta} + h) = \frac{h^{\frac{1}{\sigma}}}{k} \tag{1.5}$$

where h is the support function of a convex set, $\frac{1}{3} < \sigma \leq 1$ and $k = (h_{\theta\theta} + h)^{-1}$ is the curvature of its boundary.

The case $\sigma = \frac{1}{3}$ is more complicated. Note that when $b = 1$, only Type I blow up occurs by results of [16]. But for general b, it is unclear. The following interesting result shows the difference between isotropic and anisotropic flows.

Theorem 1.2. *Let $\sigma = \frac{1}{3}, a(\theta) = \frac{1}{1+\sigma} b^{\frac{1}{\sigma}}$. If $a^{'} sin(2\theta - \theta_0)$ doesnot change sign for some θ_0, then only Type II blow up occurs.*

Examples of a satisfying conditions in Theorem 1.2 are $a(\theta) = 1 + \epsilon\ cos^{2k-1}(2\theta - \theta_0)(0 < \epsilon < 1, k \in Z, k \geq 1)$.

Note that for $\sigma = 1$ Theorem 1.1 gives a new proof of results in [13].

The proof of Theorem 1.1 relies on some monotonicity arguments and a detailed study of equation (1.4) as well as equation $(1.4)_k$.

We first prove *a prior* estimates for solutions of $(1.4)_k$. Let $\alpha = \frac{1}{\sigma}$ and $a(\theta) = \frac{1}{1+\sigma} b^{\frac{1}{\sigma}}$.

Theorem 1.3. *Assume that $a > 0$ is a continuous function and $1 \leq \alpha \neq 3$. Then for any $k \in N$, there exist $0 < m_k < M_k < \infty$ depending on $a(x)$ only such that for any solution of $(1.4)_k$, we have*

$$m_k \leq u(x) \leq M_k$$

Theorem 1.3 covers the results of [6] for $\alpha < 2$ and $k = 1$. However, our result is more general and the proof is much simpler.

From Theorem 1.3, we have the following result which is an extension of Corollary 1.1.

Corollary 1.2. *Let $\frac{1}{8} < \sigma \leq 1$ and $\sigma \neq \frac{1}{3}$. Then every positive, smooth function $b(\theta)$ defined on the circle can be written as*

$$b(\theta) = h^{\frac{1}{\sigma}}(h_{\theta\theta} + h) = \frac{h^{\frac{1}{\sigma}}}{k} \tag{1.6}$$

where h is the support function of a convex set, and $k = (h_{\theta\theta} + h)^{-1}$ is the curvature of its boundary.

The next two theorems address the question of the existence of large solutions (or so-called subharmonic solutions in the literature) of $(1.4)_k$ which is used to prove Theorem 1.1. Abresch and Langer in [4] first studied this kind of solutions for the isotropic case: $\sigma = 1, b = 1$.

Theorem 1.4. *Assume that $\alpha < 3$ and $a > 0$ is smooth. Then there exists a sequence of solutions u_k of $(1.4)_{w_k}$ with minimal period $2w_k\pi$ such that u_k has exactly p_k local maximum points, $y_1, ..., y_{p_k}$. Moreover, $2w_k > p_k, \frac{2w_k}{p_k} \to 1, w_k \to \infty, u(y_i) \to \infty, i = 1, ..., p_k$ as $k \to \infty$.*

Theorem 1.5. *Assume that $\alpha > 3$ and $a > 0$ is smooth. Then there exists a sequence of solutions u_k of $(1.4)_{w_k}$ with minimal period $2w_k\pi$ such that u_k has exactly p_k local maximum points, $y_1, ..., y_{p_k}$. Moreover, $2w_k < p_k, \frac{2w_k}{p_k} \to \pi, w_k \to \infty, u(y_i) \to \infty, i = 1, ..., p_k$ as $k \to \infty$.*

Remarks: 1. For the existence of subharmonic solutions, there are many results, see [7], [10], etc. Problem $(1.4)_k$ is not included in all the previous results on subharmonic solutions since our nonlinearity has a linear growth at infinity.

2. When $\sigma = \frac{1}{3}$, partial results on the existence of self similar solutions to (1.1) has been obtained in [5].

This paper is organized as follows: In Section 2, we prove Theorems 1.1 and 1.2 by some monotonicity arguments and the use of Theorem 1.3, 1.4 and 1.5. In Section 3, we discuss the intial value problem associated with (1.4) and obtain some basic estimates. We prove Theorem 1.3 in Section 4 and Theorems 1.4 and 1.5 in Section 5.

Throughout this paper, the constant $C, c, c_1, c_2, c_0, etc.$ will denote various constants which depend on b only. $B = O(A)$ means $|B| \leq CA$ and $B = o(A)$ means $|B|/|A| \to 0$. $A \sim B$ means there exist two generic constants $C_1 > 0, C_2 > 0$ such that $C_1 A \leq B \leq C_2 A$.

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2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We shall follow the main lines in [16].

Let $\gamma(0)$ be a convex curve and $\gamma(t)$ be the solution of (1.1). Since $\gamma(t)$ shrinks to a point in time T. We assume that $\gamma(t) \to x_0$ as $t \to T$.

We rescale γ as follows:

$$\tilde{\gamma}(\tau) = (T-t)^{-\frac{1}{1+\sigma}}(\gamma(t) - x_0), \tau = \log(\frac{1}{T-t})$$

Let $\tilde{k}(\tau)$ be the curvature of $\tilde{\gamma}(\tau)$.

We begin with the following Proposition.

Proposition 2.1. *The following two statements are equivalent*

(a) $\tilde{\gamma}(\tau)$ *remains bounded as* $\tau \to \infty$.

(b) $\tilde{k}(\tau)$ *remains bounded as* $\tau \to \infty$.

To prove Proposition 2.1, we introduce some notations first. We first introduce the so-called generalized " Catenary" or " Grim Reaper".

Notation: A curve satisfying

$$k = M(b^{-1}(\theta)cos(\theta - \theta_0))^{\frac{1}{\sigma}}, \theta \in (\theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2})$$

with vertex at P will be denoted by $\gamma^*_{M,\theta_0,P}$. Let $D^*_{M,\theta_0,P}$ denote the region occupied by $\gamma^*_{M,\theta_0,P}$.

For $\sigma = 1, b = 1$, it is called Grim Reaper. For $\sigma = \frac{1}{2}, b = 1$, this is Catenoid. For any $\sigma \in (0, 1], D^*_{M,\theta_0,P}$ is contained in the sector with angle α_M and vertex Q where Q lies on the line $\theta = \theta_0$ and $|PQ| = 1$. Note that $\alpha_M \to 0$ as $M \to \infty$.

We now state some technical lemmas.

Lemma 2.2. *Any line passing through the origin intersects $\tilde{\gamma}(\tau)$ for $\tau \geq \tau_0$ for some $\tau_0 \geq 0$.*

Proof: The proof is easy since a line is solution to (1.1). See [16]. □

Lemma 2.3. *If $\tilde{\gamma}(\tau)$ remains bounded as $\tau \to \infty$, then there exists $\delta > 0$ such that*

$$B_\delta(0) \subset \tilde{D}(\tau)$$

where $\tilde{D}(\tau)$ is the region occupied by $\tilde{\gamma}(\tau)$.

Proof: Suppose that there exists $\tau_1 < \tau_2 < ... < \tau_k \to \infty$ such that $dist(0, \tilde{\gamma}(\tau_k)) \to 0$ as $k \to \infty$. Then since $\tilde{\gamma}(\tau) \subset B_R(0)$ for R large and since $\tilde{\gamma}(\tau)$ is convex, then there exists a modified catenary $D^*_{M,\theta_0,P}$ such that

$$\tilde{\gamma}(\tau_k) \subset D^*_{M,\theta_0,P}$$

for some $M \geq c > 0$ and $|P|$ is very small.

Consider the solution of the following problem

$$v_\tau = \sigma b^{-\frac{1}{\sigma}} v^{1+\frac{1}{\sigma}}(v_{\theta\theta} + v - \frac{1}{1+\sigma} b^{\frac{1}{\sigma}} v^{-\frac{1}{\sigma}})$$

$$v(\theta, 0) = Mcos(\theta - \theta_0)$$

Then $v = e^{-\frac{\sigma}{1+\sigma}\tau} Mcos(\theta - \theta_0)$.

The equation for the peak point is

$$P^{'}(\tau) = v(\theta_0, \tau) = Me^{-\frac{\sigma}{1+\sigma}\tau}$$

The distance that $P(\tau)$ travels from 0 to ∞ is

$$\int_0^\infty P^{'}(\tau)d\tau = \frac{1+\sigma}{\sigma}M$$

Since $M \geq c > 0$ and P is very small, then after finite time 0 will be outside $\tilde{\gamma}(\tau)$ by comparison principle. A contradiction to Lemma 2.2. $\square$

We are now in a position to prove Proposition 2.1.

Proof of Proposition 2.1:

Suppose that $\tilde{\gamma}(\tau)$ is not bounded. Let

$$\rho(\tau) = \max_{y\in\tilde{\gamma}(\tau)} |y|$$

Let $x(\tau)$ be a point on $\tilde{\gamma}$ such that

$$|x(\tau)| = \rho(\tau)$$

Then $\rho(\tau)$ satisfies the following equation

$$\rho^{'}(\tau) = \frac{1}{1+\sigma}\rho(\tau) - b\tilde{k}^{\sigma}(\tau)$$

Now suppose $\tilde{k}(\tau)$ remains bounded, then there exists $M > 0$ such that

$$\tilde{k}(\tau) \leq M, \text{ for all } \tau \geq 0$$

Therefore $\rho^{'}(\tau) \geq \frac{1}{1+\sigma}\rho(\tau) - cM^{\sigma}$ for some constant $c > 0$ and $e^{-\frac{\tau}{1+\sigma}}(\rho(\tau) - c(1+\sigma)M^{\sigma})$ is increasing. Since $\overline{\lim}_{\tau\to\infty}\rho(\tau) = \infty$, there exists $\tau_0 \geq 0$ such that $\rho(\tau_0) - c(1+\sigma)M^{\sigma} \geq 1$, then

$$\rho(\tau) \geq c(1+\sigma)M^{\sigma} + e^{\frac{\tau-\tau_0}{1+\sigma}}$$

This means that

$$\overline{\lim}_{t\to T}(T-t)^{\frac{1}{1+\sigma}} \max_{x\in\gamma(t)} |x - x_0| \geq e^{-\frac{\tau_0}{1+\sigma}}$$

A contradiction to Lemma 2.2! Hence (b) implies (a).

Next suppose that $\tilde{k}(\tau)$ is unbounded while $\tilde{\gamma}(\tau)$ is bounded. In this case, we need a few technical lemmas.

Lemma 2.4. *Let γ_1, γ_2 be two closed convex curves in R^2. Suppose γ_i is parametrized by $k = g_i(\theta)$ where k is the curvature . Suppose that $g_1(\theta) \leq g_2(\theta)$ and γ_1 is tangential to γ_2 at some point (i.e. with the same inner normal direction). Then*

$$D(\gamma_2) \subset D(\gamma_1)$$

where $D(\gamma_i)$ denotes the region occupied by γ_i.

Proof: Geometrically, this is clear. For the proof, see [16]. $\square$

Lemma 2.5. *Let $k(\theta, t)$ be the positive solution solution of*

$$k_t = k^2((bk^\sigma)_{\theta\theta} + bk^\sigma) \tag{2.1}$$

Let $\theta^*(t)$ be such that

$$k(\theta^*(t), t) = k_{max}(t) = \max_{\theta \in [0,2\pi]} k(\theta, t)$$

Then there exists $M > 0$ such that

$$k(\theta, t) \geq k_{\max}(t)(\frac{1}{b(\theta)} cos(\theta - \theta^*(t))^{\frac{1}{\sigma}}, \theta \in [\theta^*(t) - \frac{\pi}{2}, \theta^*(t) + \frac{\pi}{2}]$$

whenever $k_{max}(t) \geq M$.

Proof: There exists $M > 0$ such that the graph of the function $M_1(b^{-1}cos(\theta - \theta_0))^{\frac{1}{\sigma}}$ intersects the graph of $k(\theta, 0)$ at precisely 2 points in the interval $[\theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2}]$ for any $M_1 \geq M$ and any θ_0. Consider the equation on the interval $[\theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2}]$. Since the function $w(\theta, t) = k(\theta, t) - M_1(b^{-1}cos(\theta - \theta_0))^{\frac{1}{\sigma}}$ doesnot change sign at the end point $\theta_0 \pm \frac{\pi}{2}$, then number of zeroes of $w(\theta, t)$ doesnot increase as time passes (hence at most 2). Now choose any t_1 such that $k_{max}(t_1) \geq M$ and take $\theta_0 = \theta^*(t_1), M_1 = k_{max}(t_1)$. Then w has a degenerate 0 at $\theta = \theta^*(t_1)$. Therefore w does not have a zero in $[\theta^*(t_1) - \frac{\pi}{2}, \theta^*(t_1)) \cup (\theta^*(t_1), \theta^*(t_1) + \frac{\pi}{2}]$. Since $w > 0$ at the end points, $\theta^*(t_1) \pm \frac{\pi}{2}$, the lemma is proved. $\square$

Lemma 2.6. *Let $k(\theta, t)$ be the positive solution solution of*

$$k_t = k^2((bk^\sigma)_{\theta\theta} + bk^\sigma) \tag{2.2}$$

Let $\theta^*(t)$ be a local maximum of $k(\theta, t)$ with $k(\theta^*, t) \geq M$, then

$$k(\theta, t) \geq k(\theta^*, t)(\frac{1}{b(\theta)} cos(\theta - \theta^*(t))^{\frac{1}{\sigma}}, \theta \in [\theta^*(t) - \frac{\pi}{2}, \theta^*(t) + \frac{\pi}{2}]$$

Proof: The proof is similar to that of Lemma 2.5. □

We then have the following corollaries

Corollary 2.1. *$k_{max}(t) = \max_\theta k(\theta, t)$ is strictly monotone increasing for t sufficiently close to T.*

Proof: Fix t_1 as in Lemma 2.5. Applying the strong Maximum Principle to w on the interval $[\theta^*(t) - \frac{\pi}{2}, \theta^*(t) + \frac{\pi}{2}]$. Then since $w > 0$ at $\theta = \theta^*(t) \pm \frac{\pi}{2}$ and since $w \geq 0$ for $t = t_1$. Therefore $w > 0$ for $t > t_1$. Hence $k_{max}(t) > k_{max}(t_1), t > t_1$. □

Note: The same result was found in [1] for $\sigma = 1, b = 1$ with a different proof.

Corollary 2.2. *Let $P(t)$ be a point of $\gamma(t)$ be such that k attains $k_{max}(t)$ at $P(t)$. If $k_{max}(t) > M$, then*

$$\gamma(t) \subset D^*_{k_{max}(t), \theta^*(t), P(t)}$$

Proof: By Lemma 2.5, $k(\theta, t) \leq k_{\max}(t)(b^{-1}(\theta) cos(\theta - \theta^*(t)))^{\frac{1}{\sigma}}$ for $\theta \in [\theta^*(t) - \frac{\pi}{2}, \theta^*(t) - \frac{\pi}{2}]$. Therefore $\gamma(t)$ is contained in $D^*_{k_{max}(t), \theta^*(t), P(t)}$ for $\theta \in [\theta^*(t) - \frac{\pi}{2}, \theta^*(t) - \frac{\pi}{2}]$. The remaining part is also contained in D^* since γ is convex. □

Corollary 2.3. *Let $P(t)$ be a point of $\gamma(t)$ be such that k attains a local maximum $k_{localmax}(t)$ at $P(t)$. If $k_{localmax}(t) > M$, then*

$$\gamma(t) \subset D^*_{k_{localmax}(t), \theta^*(t), P(t)}$$

Proof: the proof is the same as that in Corollary 2.2 by using Lemma 2.6 and Lemma 2.4. □

Proof of Proposition 2.1 (finished): Suppose that $\tilde{k}(\tau)$ is not bounded while $\tilde{\gamma}(\tau)$ remains bounded, i.e. $\tilde{\gamma}(\tau) \subset B_R(0), \tau \geq 0$.

Let $\tau_1 < \tau_2 < ... < \tau_k < ...$ be such that $\tilde{k}_{max}(\tau_j) \to \infty (j \to \infty)$. Let $\theta^*(\tau_j)$ be as in Lemma 2.5 and let $P(\tau)$ be a point on $\tilde{\gamma}(\tau)$ corresponding to $\theta = \theta^*(\tau)$. By Corollary 2.2,

$$\tilde{\gamma}(\tau_j) \subset D^*_{\tilde{k}_{max}(\tau_j),\theta^*(\tau_j),P(\tau_j)}$$

As $j \to \infty, \tilde{k}_{max}(\tau_j) \to \infty$. This is impossible by Lemma 2.3 since there is a fixed radius ball inside $\tilde{\gamma}(\tau)$. $\square$

Next we consider the lower bound for $\tilde{k}(\tau)$. We have

Lemma 2.7. $\overline{\lim}_{\tau\to\infty}\tilde{k}_{max}(\theta,\tau) < \infty$ *implies* $\underline{\lim}_{\tau\to\infty}\tilde{k}_{min}(\theta,\tau) > 0$.

Remark: For $\sigma = 1, b = 1$, Angenent [1] uses traveling waves to obtain a similar statement.

Proof: Let $M_2 > 0$ be such that

$$u_{max}(\tau) \le M_2$$

For each $\theta_0 \in R/(2\pi Z)$ and small $m > 0$, let $u^*_{\theta_0,m}(\theta)$ be the solution of

$$u^*_{\theta\theta} + u^* - a(u^*)^{-\alpha} = 0$$

$$u^*(\theta_0) = m, (u^*)'(\theta_0) = 0$$

Then there exist $\alpha_1, \alpha_2 > 0$ such that $(u^*_{\theta_0,m})'(\theta) < 0$ for $\theta \in (\theta_0 - \alpha_1, \theta_0)$ and $(u^*_{\theta_0,m})'(\theta) > 0$ for $\theta \in (\theta_0, \theta_0 + \alpha_2)$ as $m \to 0$, we have $\alpha_1, \alpha_2 \to \frac{\pi}{2}$ by Lemmas 3.1 and 3.2 in Section 3 uniformly with respect to θ_0. Furthermore, there exists $\delta > 0$ such that $u^*_{\theta_0,m}(\theta_0 - \alpha_1), u^*_{\theta_0,m}(\theta_0 + \alpha_2) > M_1$ for any $\theta_0 \in R/(2\pi Z)$ and $m \le \delta$. Moreover, we can choose δ sufficiently small so that the graph of $u^*_{\theta_0,m}$ intersects $u(\theta, 0)$ at precisely two points for any θ_0 and $m \le \delta$.

Now suppose that $u_{min}(\tau) < \delta$ and let $\theta_*(\tau)$ be such that $u(\theta_*(\tau), \tau) = u_{min}(\tau)$. Then by an argument similar to the proof of Lemma 2.5, we see that

$$u(\theta, \tau) \le u_{\theta_*(\tau), u_{min}(\tau)}(\theta)$$

for $\theta \in [\theta_*(\tau) - \alpha_1, \theta_*(\tau) + \alpha_2]$.

Then

$$\int_{\theta_*(\tau)}^{\theta_*(\tau)+\alpha_2} \frac{cos(\theta-\theta_*)}{\tilde{k}(\theta,\tau)}d\theta \geq \int_{\theta_*(\tau)}^{\theta_*(\tau)+\alpha_2} \frac{a(\theta)cos(\theta-\theta_*)}{(u^*)^\alpha}d\theta \to \infty$$

by simple computations.

This is a contradiction since we assume that $\tilde{k}$ is bounded (hence $\tilde{\gamma}$ is bounded by Proposition 2.1 and hence the diameter of $\tilde{\gamma}$ is bounded). □

From Lemma 2.7 and the Liapunov functional method (see [15]), we have

Proposition 2.8. *Type I blow up implies that $u(\theta,\tau)$ converges to an equilibrium solution (namely, a self similar solution).*

We next analyze Type II blow up.

By Lemma 2.2, $0 \in \tilde{\gamma}(\tau)$ for $\tau \geq \tau_0$. Let $\rho(\tau) = \max_\theta dist(0, \tilde{\gamma}(\tau))$. Assume that there exists $\tau_1 < \tau_2 < ... < \tau_k \to \infty$ such that $\rho(\tau_k) \to \infty$. Let $y(\tau_k)$ be a point on $\tilde{\gamma}(\tau_k)$ where $|y(\tau_k)| = \rho(\tau_k)$. Let e_k, e_k^* be the unit vector such that

$$e_k = \frac{1}{|y(\tau_k)|}y(\tau_k), e_k^* \cdot e_k = 0$$

Then we have

Lemma 2.9. *Suppose* $\max e_k \cdot \tilde{\gamma}(\tau_k) \to \infty$. *Then* $\min e_k \cdot \tilde{\gamma}(\tau_k) \to -\infty$.

Proof: We prove Lemma 2.9 by a comparison argument similar to that of Lemma 2.3. We use a modified catenary. Recall that a modified catenary is a solution of the following problem

$$k = M(b^{-1}(\theta)cos\ (\theta-\theta_0))^{\frac{1}{\sigma}}$$

We prove by contradiction. Suppose that $\min e_k \cdot \tilde{\gamma}(\tau_k)$ is bounded from below. Then there exists a modified catenary $D^*_{M,\theta_0,P}$ such that

$$\tilde{\gamma}(\tau_k) \subset D^*_{M,\theta_0,P}$$

Consider the solution of the following problem

$$v_\tau = \sigma b^{-\frac{1}{\sigma}} v^{1+\frac{1}{\sigma}}(v_{\theta\theta} + v - \frac{1}{1+\sigma} b^{\frac{1}{\sigma}} v^{-\frac{1}{\sigma}})$$

$$v(\theta,0) = M cos(\theta - \theta_0)$$

Then $v = e^{-\frac{\sigma}{1+\sigma}\tau} M cos(\theta - \theta_0)$.

The equation for the peak point is

$$P^{'}(\tau) = v(\theta_0,\tau) = Me^{-\frac{\sigma}{1+\sigma}\tau}$$

The distance that $P(\tau)$ travels from 0 to ∞ is

$$\int_0^\infty P^{'}(\tau)d\tau = \frac{1+\sigma}{\sigma}M$$

Let M be very large, then after finite time 0 will be outside $\tilde{\gamma}(\tau)$ by comparison principle. A contradiction to Lemma 2.2. $\square$

Lemma 2.10. *Suppose that* $\rho(\tau_1) < \rho(\tau_2) < ... \to \infty$. *Then* $\max |e_k^* \tilde{\gamma}(\tau_k)| \to 0$ *as* $k \to \infty$.

Remark: If $\sigma = 1$ then the area of $\tilde{D}(\tau)$ is constant. Because of this and the convexity of $\tilde{\gamma}(\tau)$, Lemma 2.10 is obvious in this case.

Proof: We need to prove that $\max |e_k^* \tilde{\gamma}(\tau_k)| \to 0$ as $k \to \infty$.

In fact, let $Q(\tau)$ be the point such that $\rho(\tau) = \max_{y \in \tilde{\gamma}(\tau)} |y|$ is attained. Let $\theta^*(\tau)$ be the angle corresponding to $Q(\tau)$. We can take such a subsequence such that

$$b(\theta^*(\tau_j))\tilde{k}^\sigma(\theta^*(\tau_j),\tau_j) \geq \frac{1+\sigma}{2}\rho(\tau_j)$$

We first claim that there exists a point P_j not far from $Q(\tau_j)$, where the curvature $\tilde{k}(\theta,\tau_j)$ atains a local maximum. In fact let θ_j be the point closest to $\theta^*(\tau_j)$ such that $\tilde{k}(\theta,\tau_j)$ attains a local maximum at θ_j and that $\tilde{k}(\theta_j,\tau_j) \geq \tilde{k}(\theta^*(\tau_j),\tau_j)$ (if such local maxima accumulates at $Q(\tau_j)$, then the conlusion is obvious).

Then

$$|P_j - Q(\tau_j)| = (\int_{\theta^*(\tau_j)}^{\theta_j} \frac{cos(\theta - \theta^*(\tau_j))}{\tilde{k}(\theta,\tau_j)}d\theta)^2 + (\int_{\theta^*(\tau_j)}^{\theta_j} \frac{sin(\theta - \theta^*(\tau_j))}{\tilde{k}(\theta,\tau_j)}d\theta)^2$$

$$\leq o(1)$$

since $\tilde{k}(\theta,\tau)$ is unbounded over $[\theta^*(\tau_j),\theta_j]$.

Next let $M_j^* = \tilde{k}(\theta_j, \tau_j)$. Then $M_j^* \to \infty$ and

$$M_j^* = \tilde{k}(\theta_j, \tau_j) \geq C\rho^{\frac{1}{\sigma}}(\tau_j)$$

For a generalized catenery, let $y \in D^*_{M_j^*,\theta^*,P}$ be a point on the line $\theta = \theta_0$. We denote $\xi = |yP|$ and η be the width at y (i.e., the length of the segment which is the intersection of the line perpendicular to yP and $D^*_{M,\theta^*,P}$). Let us now compute ξ and η.

Set $k^* = M_j^*(b^{-1}(\theta)cos(\theta-\theta_0))^{\frac{1}{\sigma}}$. Then

$$\xi = \int_{\theta_0}^{\theta} \frac{sin(\theta-\theta_0)}{k^*(\theta)} d\theta, \eta = 2\int_{\theta_0}^{\theta} \frac{cos(\theta-\theta_0)}{k^*(\theta)} d\theta$$

Hence

$$\xi \sim \frac{1}{M_j^*} \int_{\theta_0}^{\theta} \frac{sin(\theta-\theta_0)}{(cos(\theta-\theta_0))^{\frac{1}{\sigma}}}$$

$$= \frac{1}{M^*}\frac{\sigma}{1-\sigma}((cos(\theta-\theta_0))^{1-1/\sigma} - 1)$$

and

$$\eta \sim \frac{1}{M_j^*} \int_{\theta_0}^{\theta} \frac{cos(\theta-\theta_0)}{(cos(\theta-\theta_0))^{\frac{1}{\sigma}}}$$

Let $\delta = \theta_0 + \frac{\pi}{2} - \theta$. Then as $\delta \to 0$, we have

$$\eta \sim \frac{C}{M_j^*}\begin{cases} \frac{1}{\delta^{\frac{1}{\sigma}-2}} \text{ if } \sigma < \frac{1}{2} \\ log\frac{1}{\delta} \text{ if } \sigma = \frac{1}{2} \\ C \text{ if } \sigma > \frac{1}{2} \end{cases}$$

$$\xi \sim \frac{C}{M_j^*}\frac{1}{\delta^{\frac{1}{\sigma}} - 1} \text{ if } \sigma < 1$$

If $\sigma > \frac{1}{2}$, then $\eta \to 0$ if $M^* \to \infty$. Lemma 2.10 follows.

If $\sigma < \frac{1}{2}$, then

$$\xi \sim \frac{1}{M_j^*}\frac{1}{\delta^{\frac{1}{\sigma}-1}}, \eta \sim \frac{1}{M_j^*}\frac{1}{\delta^{\frac{1}{\sigma}-2}}$$

$$\delta\eta \sim \xi$$

Since P_j and $Q(\tau_j)$ are close, we have

$$\xi \sim \rho_j$$

Note that

$$M_j^* \geq C(\rho_j)^{\frac{1}{\sigma}}$$

Hence

$$\frac{1}{\delta^{\frac{1}{\sigma}-1}} \sim M_j^* \rho_j \geq C_2 \rho_j^{1+\frac{1}{\sigma}}$$

$$C \geq \delta^{\frac{1}{\sigma}-1} \rho_j^{1+\frac{1}{\sigma}}$$

Hence

$$\eta \sim \delta \xi_j \sim \delta \rho_j \leq (C\delta^2)^{\frac{\sigma}{1+\sigma}} \to 0$$

since $\delta \to 0$.

Therefore $\max |\tilde{\gamma}(\tau_j) \cdot e_j^*| \leq 2\eta_j \to 0$ by the convexity of $\tilde{\gamma}(\tau_j)$.

The case $\sigma = \frac{1}{2}$ is similar.

Lemma 2.10 is thus proved. □

Finally we prove Theorem 1.1. We need the following standard proposition.

Proposition 2.11. *If $\gamma_1(t)$ and $\gamma_2(t)$ are two solutions of (1.1) then the number of intersection points is nonincreasing.*

Proof of Theorem 1.1. Let $\frac{1}{3} < \sigma \leq 1$. By Theorem 1.4, there exists a family of solutions of (1.4) such that the corresponding curves (we denoted by γ_k) have the following properties;

1. the winding numbers of $\gamma_k = w_k$,
2. the number of peaks (maximum points) $p_k \leq 2w_k - 1$.
3. the diameter of $\gamma_k \to \infty$
4. distance from 0 to $\gamma_k \to 0$.

Let $\gamma(0)$ be a convex curve. Then by Theorem 1.4, the number of intersection points between $\tilde{\gamma}(0)$ and γ_k is $2p_k \leq 2(2w_k - 1)$ for k sufficiently large.

Now suppose Type II occurs for this initial data. Let k be large and fixed. By Lemma 2.9 and Lemma 2.10, $\tilde{\gamma}(\tau)$ tends to a two-fold line (at least for a sequence $\tau_1 < \tau_2 < ... < \tau_j \to \infty$). For each line L that passes through the origin and j large, the number of intersection points between γ_k and L is $2w_k$. Therefore the number of intersection points between γ_k and $\tilde{\gamma}(\tau_j)$ is great than or equal to $2 \times 2w_k = 4w_k$ for j large. This contradicts to Proposition 2.11, since $2p_k < 4w_k$.

Hence only type I occurs. By Proposition 2.1, we have that $\tilde{k}(\tau)$ is bounded. By Proposition 2.8, a subsequence of $\tilde{\gamma}(\tau)$ converges to a self-similar shape solution of (1.2).

This proves Theorem 1.1 for $\frac{1}{3} < \sigma \leq 1$.

We next consider the case $0 < \sigma < \frac{1}{3}$. Then by Theorem 1.3, the set S of stationary solutions of (1.4) is compact. By Theorem 1.5, there exists a family of solutions of $(1.4)_{w_k}$ such that the corresponding curves (we denoted by γ_k) have the following properties;

1. the winding numbers of $\gamma_k = w_k$,
2. the number of peaks (maximum points) $p_k \geq 2w_k + 1$.
3. the diameter of $\gamma_k \to \infty$
4. distance from 0 to $\gamma_k \to 0$.

Therefore for large enough k, the number of intersection points between γ_k and any solution $\Gamma \in S$ is $2p_k \geq 2(2w_k + 1)$.

Now fix k sufficiently large and take an initial data $\gamma(0)$ such that the number of intersection points between $\tilde{\gamma}(0)$ and γ_k is $4w_k$ (this is the case if $\tilde{\gamma}(0)$ is close to a long thin ellipse). If Type I occurs for this initial data, then $\tilde{\gamma}(\tau) \to \Gamma^* \in S$. Hence the number of intersection points between Γ^* and γ_k is $2p_k \geq 2(2w_k + 1)$. Therefore the number of intersection points between $\tilde{\gamma}(\tau)$ and γ_k is $2p_k$ for τ large. A contradiction to Proposition 2.11. So Type I blow up doesnot occur. $\square$

Proof of Theorem 1.3. For $\sigma = \frac{1}{3}$, if Type I occurs, then the rescaled curve will approach the shape of a self similar solution. Namely, problem (1.4) has a solution u. Note that u satisfies also the following equation

$$(\frac{u^2}{2})''' + 4(\frac{u^2}{2})' = a^{'}u^{-2}$$

Mutiplying both sides by $sin(2\theta - \theta_0)$, we have

$$\int_0^{2\pi} a^{'}(\theta)u^{-2}sin(2\theta - \theta_0)d\theta = 0$$

By our hypothesis on a, this is impossible ! Hence Type I does not occur and only Type II occurs in this case. $\square$

3. Preliminary Estimates

In the rest of sections, we shall study equations (1.4) and $(1.4)_k$ and prove Theorems 1.3 to 1.5.

In the present section, we obtain some basic estimates that we will need later. We always assume that $a > 0$ is smooth and $1 \leq \alpha \neq 3$.

We now consider the following initial value problem

$$\begin{cases} u^{''} + u = \frac{a(x)}{u^\alpha} \\ u(x) > 0, u(x_0) = M, u^{'}(x_0) = \beta, M > 0 \end{cases} \tag{3.1}$$

It is easy to see that the global solution exists and is unique. We denote the global solution as $u(x, M, \beta; x_0)$. If $\beta = 0$, we denote $u(x, M, 0; x_0)$ as $u(x, M; x_0)$. The purpose of this section is to understand the asymptotic behaviour of $u(x, M; x_0)$ as $M \to \infty$.

Let x_M be the first positive zero of $u^{'}(x, M; x_0)$ (it is easy to see that x_M exists). A key estimate is the asymptotic behaviour of $x_M - x_0$ as $M \to \infty$.

We first have

Lemma 3.1. *$x_M - x_0 \to \frac{\pi}{2}$ and $\frac{u}{M} \to$ cos $(x - x_0)$ for $x \in [x_0, x_0 + \frac{\pi}{2}]$ as $M \to \infty$.*

Proof: We assume that $1 < \alpha \neq 3$. The case $\alpha = 1$ can be treated similarly.

Let $m = u(x_M)$. Then $m \to 0$ as $M \to \infty$. In fact, we have

$$\frac{C_1}{u^\alpha} \leq u^{''} + u \leq \frac{C_2}{u^\alpha}, u^{'} \leq 0$$

for $x \in [x_0, x_M]$.

Set

$$f_C(u) = u^2 + C\frac{2}{\alpha - 1}u^{1-\alpha}$$

Then $(u^{'})^2 + f_{C_2}(u)$ is increasing over $[x_0, x_M]$ while $(u^{'})^2 + f_{C_1}(u)$ is decreasing over $[x_0, x_M]$. Hence

$$f_{C_2}(m) \geq f_{C_2}(M)$$

$$f_{C_1}(m) \leq f_{C_1}(M)$$

We have $M^2 \sim m^{1-\alpha}$ as $M \to \infty$. Hence $m \to 0$.

Let $f_{C_2}(u(z_M))) = f_{C_2}(M)$ and $x_0 < z_M < x_M$. Then

$$M^2 u(z_M)^{\alpha-1} \sim 1$$

Hence $u(z_M) \sim m$.

Over $[z_M, x_M]$, by equation (3.1), $u^{''} \sim m^{-\alpha}$. Hence $u^{'}(x) \sim m^{-\alpha}(x - z_M)$ for $x \in [z_M, x_M]$. Therefore

$$u(z_M) - u(x_M) = \int_{x_M}^{z_M} u^{'}(x)dx \sim m^{-\alpha}(x_M - z_M)^2$$

Hence

$$x_M - z_M = O(m^{\alpha+1})$$

On the other hand, we have

$$(u^{'})^2 \geq f_{C_1}(M) - f_{C_1}(u)$$

Hence

$$z_M - x_0 \leq \int_{u(z_M)}^{M} \frac{du}{\sqrt{f_{C_1}(M) - f_{C_1}(u)}} \to \frac{\pi}{2}$$

So

$$x_M - x_0 = x_M - z_M + z_M - x_0 \to \frac{\pi}{2}$$

as $M \to \infty$.

Finally $\frac{u}{M} \to cos\ (x - x_0)$ is an easy consequence of the above arguments. □

Let $-y_M$ be the first negative zero of $u(x, M, x_0)$. Then similarly we have $y_M \to \frac{\pi}{2}$.

In the following, it will be more convenient to assume that $y_M = 0, x_0 = y_M$. We omit the index M and x_0 in u.

Lemma 3.2. *For any solution of (3.1), we have*

$$u'(\frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \frac{a(x)}{u^\alpha} sin\ (x)dx - u(0) \tag{3.2}$$

For $\alpha \geq 2$ and $a \in C^1$, we have

(3.3)

$$2u(\frac{\pi}{2})u'(\frac{\pi}{2}) = (3-\alpha)\int_0^{\frac{\pi}{2}} \frac{a(x)u'}{u^\alpha} sin\ (2x)dx + \int_0^{\frac{\pi}{2}} \frac{a'(x)}{u^{\alpha-1}} sin\ (2x)dx$$

Proof: (3.2) follows by multiplying equation (3.1) by $sin\ x$ and integrating over $(0, \frac{\pi}{2})$.

For (3.3), we note that u satisfies

$$(\frac{u^2}{2})''' + 4(\frac{u^2}{2})' = (3-\alpha)\frac{au'}{u^\alpha} + \frac{a'}{u^{\alpha-1}} \tag{3.4}$$

Multiplying (3.4) by $sin\ (2x)$ and integrating over $(0, \frac{\pi}{2})$, we have (3.3).

□

Next we set

$$m = u(0)$$
$$\epsilon_M = \frac{1}{M^{\alpha+1}},\ \text{for } 1 \leq \alpha < 2$$
$$\epsilon_M = M^{-3}\log\ (M) \text{ for } \alpha = 2$$
$$\epsilon_M = \frac{1}{M^{\frac{\alpha+1}{\alpha-1}}},\ \text{for } 2 < \alpha \neq 3$$

We first consider the asymptotic behaviour of y_M for $1 \leq \alpha < 2$.

Lemma 3.3. *For $1 \leq \alpha < 2$, we have*

$$y_M = \frac{\pi}{2} + (\int_0^{\frac{\pi}{2}} \frac{a(x)}{sin^{\alpha-1}(x)} dx + o(1))\epsilon_M \tag{3.5}$$

Proof: By equation (3.1), we have

$$0 - u'(\frac{\pi}{2}) = u'(y_M) - u'(\frac{\pi}{2})$$
$$= \int_{\frac{\pi}{2}}^{y_M} u''$$

Hence

$$y_M = \frac{\pi}{2} + \frac{u'(\frac{\pi}{2})}{M}$$

By (3.2),

$$u'(\frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \frac{a(x)}{u^\alpha} sin\,(x) dx - m = \frac{1}{M^\alpha}\int_0^{\frac{\pi}{2}} \frac{a(x)}{sin^{\alpha-1}(x)} dx(1+o(1)) - m$$

$$= (\int_0^{\frac{\pi}{2}} \frac{a(x)}{sin^{\alpha-1}(x)} dx + o(1)) M^{-\alpha}$$

since $M^{-\alpha} \sim m^{\alpha(\alpha-1)/2}$ and $\alpha(\alpha-1)/2 < 1$ for $\alpha < 2$. □

We next consider the case when $2 < \alpha \neq 3$.

Lemma 3.4. *For $2 < \alpha \neq 3$, we have*

$$\int_0^{\frac{\pi}{2}} \frac{a(x)}{u^{\alpha-1}} dx = \sqrt{\frac{\alpha-1}{2a(0)}} \int_0^\infty \frac{dy}{v_0^{\alpha-1}} m^{\frac{3-\alpha}{2}} (1+o(1)) \tag{3.6}$$

where $m = u(0)$ and v_0 is the unique solution of

$$v_0'' = v_0^{-\alpha}, v_0(0) = 1, v_0'(0) = 0.$$

Proof: Let $\delta > 0$ be a small fixed number. Let x_1 be a point where $u(x) \leq \delta a(0)/u^\alpha(x)$ for $x \leq x_1$ and $u(x_1)^{\alpha+1} = \delta a(0)$. Then

$$(1-\delta)a(0)/u^\alpha(x) \leq u'' \leq (1+\delta)a(0)/u^\alpha(x)$$

for $x \in [0, x_1]$.

We have

$$\frac{(u')^2}{2} + \frac{(1-\delta)a(0)}{(\alpha-1)u^{\alpha-1}} \geq \frac{(1-\delta)a(0)}{(\alpha-1)m^{\alpha-1}}, \frac{(u')^2}{2} + \frac{(1+\delta)a(0)}{(\alpha-1)u^{\alpha-1}} \leq \frac{(1+\delta)a(0)}{(\alpha-1)m^{\alpha-1}}$$

$$(u')^2 \geq \frac{2(1-\delta)a(0)}{\alpha-1}(\frac{1}{m^{\alpha-1}} - \frac{1}{u^{\alpha-1}}), (u')^2 \leq \frac{2(1+\delta)a(0)}{\alpha-1}(\frac{1}{m^{\alpha-1}} - \frac{1}{u^{\alpha-1}})$$

Set $y = \sqrt{\frac{2(1-\delta)}{\alpha-1} m^{-\alpha-1} a(0)} x, u(x) = mv(y)$. Then we have v satisfies

$$v_0(y) \leq v(y)$$

for $y \in [0, y_1]$ where $y_1 = \sqrt{2(1-\delta)/(\alpha-1) m^{-\alpha-1} a(0)} x_1$.

Note that $v_0(y) \sim y$ as $|y| \to \infty$.

Hence

$$\int_0^{x_1} \frac{a(x)}{u^{\alpha-1}} dx \leq \sqrt{(\alpha-1)/2(1-\delta)} \sqrt{m^{\alpha+1}/a(0)} m^{1-\alpha} \int_0^{y_1} \frac{1}{v_0^{\alpha-1}(y)} dy$$

$$= \sqrt{(\alpha-1)/2(1-\delta)a(0)}m^{\frac{3-\alpha}{2}} \int_0^{y_1} \frac{1}{v_0^{\alpha-1}(y)} dy$$

where $y_1 = \sqrt{2(1-\delta)/(\alpha-1)m^{-\alpha-1}a(0)}x_1$.

Note that $v(y_1) = \frac{u(x_1)}{m} \to \infty$. We have $y_1 \to \infty$. Hence

$$\int_0^{x_1} \frac{a(x)}{u^{\alpha-1}(x)} \leq \sqrt{(\alpha-1)/2(1-\delta)a(0)}m^{\frac{3-\alpha}{2}} \int_0^{\infty} \frac{dy}{v_0^{\alpha}(y)}$$

Similarly

$$\int_0^{x_1} \frac{a(x)}{u^{\alpha-1}(x)} \geq \sqrt{(\alpha-1)/2(1+\delta)a(0)}m^{\frac{3-\alpha}{2}} \int_0^{\infty} \frac{dy}{v_0^{\alpha}(y)}$$

Note that

$$x_1 \geq \int_m^{u(x_1)} \frac{du}{\sqrt{\frac{2(1+\delta)a(0)}{\alpha-1}(m^{1-\alpha}-u^{1-\alpha})}}$$

$$\geq Cm^{\frac{\alpha-1}{2}}$$

Similarly we have $x_1 \leq Cm^{\frac{\alpha-1}{2}}$. Hence $x_1 \sim m^{\frac{\alpha-1}{2}} \sim M^{-1}$.

On the other hand, for fixed δ, we have for $\alpha > 2$,

$$\int_{x_1}^{\frac{\pi}{2}} \frac{dx}{u^{\alpha-1}} \leq \frac{1}{M^{\alpha-1}} \int_{x_1}^{1} \frac{1}{sin^{\alpha-1}(x)} dx \leq C_\delta/M \sim C_\delta m^{\frac{\alpha-1}{2}}$$

Since $3-\alpha < \alpha-1$, we have $m^{(\alpha-1)/2} = o(m^{(3-\alpha)/2})$.

Let $\delta \to 0$, we have

$$\int_0^{\frac{\pi}{2}} \frac{a(x)}{u^{\alpha-1}} dx = \sqrt{\frac{\alpha-1}{2a(0)}} \int_0^{\infty} \frac{dy}{v_0^{\alpha-1}} m^{\frac{3-\alpha}{2}}(1+o(1))$$

□

For $\alpha = 2$, we have

Lemma 3.5. *For $\alpha = 2$, we have*

$$\int_0^{\frac{\pi}{2}} \frac{a(x)}{u^{\alpha-1}} dx \sim M^{-1}\log(M) \tag{3.7}$$

Proof: We use the same notation as in Lemma 3.4.

$$\int_0^{x_1} \frac{a(x)}{u^{\alpha-1}(x)} \geq \sqrt{(\alpha-1)/2(1+\delta)a(0)}m^{\frac{3-\alpha}{2}} \int_0^{y_2} \frac{dy}{v_0^\alpha(y)}$$

where $y_2 = \sqrt{2(1+\delta)/(\alpha-1)m^{-\alpha-1}a(0)}x_1$.

Note that

$$x_1 \sim m^{\frac{\alpha-1}{2}}.$$

Thus

$$\int_0^{y_2} \frac{dy}{v_0^\alpha(y)} \geq C\log(1/m)$$

and

$$\int_0^{x_1} \frac{a(x)}{u^{\alpha-1}(x)} \geq CM^{-1}\log(M)$$

Similarly we have

$$\int_0^{x_1} \frac{a(x)}{u^{\alpha-1}(x)} \sim CM^{-1}\log(M)$$

Similar to the proof of Lemma 3.4, we have

$$\int_0^{x_1} \frac{a(x)}{u^{\alpha-1}(x)} \leq CM^{-1}\log(M)$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{a(x)}{u^{\alpha-1}(x)} \sim CM^{-1}\log(M)$$

□

Lemma 3.6. *For $2 < \alpha \neq 3$ and $a(x) > 0 \in C^1$, we have*

(3.8)

$$y_M = \frac{\pi}{2} + (3-\alpha)/(\alpha-1)((\alpha-1)/(2a(0))^{\frac{3-\alpha}{1-\alpha}+1/2} \int_0^\infty \frac{dy}{v_0^{\alpha-1}}\epsilon_M + o(\epsilon_M)$$

For $\alpha = 2$ we have

$$y_M - \frac{\pi}{2} \sim \epsilon_M \tag{3.9}$$

Proof: Similar to Lemma 3.3, we have

$$y_M = \frac{\pi}{2} + \frac{u'(\frac{\pi}{2})}{M}(1+o(1))$$

We use $l.o.t.$ to denote the lower order terms. By Lemma 3.2 we have

$$u^{'}(\frac{\pi}{2}) = \frac{3-\alpha}{2u(\frac{\pi}{2})}\int_0^{\frac{\pi}{2}} \frac{au^{'}sin\ 2x}{u^\alpha} + l.o.t.$$

$$= \frac{3-\alpha}{(\alpha-1)M}\int_0^{\frac{\pi}{2}} \frac{acos\ 2x}{u^{\alpha-1}} + l.o.t.$$

$$= \frac{3-\alpha}{(\alpha-1)M}\int_0^{\frac{\pi}{2}} \frac{a(x)}{u^{\alpha-1}} + l.o.t.$$

Denote $m = u(0)$. Then by equation (3.1), we have

$$(\frac{(u^{'})^2}{2})^{'} + (\frac{u^2}{2})^{'} + (\int_0^x \frac{au^{'}}{u^\alpha})^{'} = 0$$

$$\frac{M^2}{2} + \int_0^{y_M} \frac{au^{'}}{u^\alpha} = \frac{m^2}{2}$$

$$\frac{M^2}{2} + \frac{1}{\alpha-1}\int_0^{y_M} \frac{a^{'}}{u^{\alpha-1}} = \frac{m^2}{2} + \frac{a(0)m^{1-\alpha}}{\alpha-1}$$

Hence

$$M^2 = \frac{2a(0)}{\alpha-1}m^{1-\alpha}(1+o(1))$$

So by Lemma 3.4, we have

$$y_M = \frac{\pi}{2} + (3-\alpha)/(\alpha-1)M^{-2}\sqrt{(\alpha-1)/2a(0)}\int_0^\infty \frac{1}{v_0^\alpha}dy m^{(3-\alpha)/2}(1+o(1))$$

$$= \frac{\pi}{2} + (3-\alpha)/(\alpha-1)\sqrt{(\alpha-1)/(2a(0))}^{\frac{3-\alpha}{1-\alpha}+\frac{1}{2}}\int_0^\infty \frac{dy}{v_0^\alpha}M^{-\frac{\alpha+1}{\alpha-1}}(1+o(1))$$

The case for $\alpha = 2$ can be proved in a similar way.

The Lemma is proved. □

Corollary 3.1. *If $a(x) > 0$ is smooth, then there exists $c_0 > 0$ such that*

$$y_M > \frac{\pi}{2} + c_0\epsilon_M$$

for $\alpha < 3$ and

$$y_M < \frac{\pi}{2} - c_0\epsilon_M$$

for $\alpha > 3$.

In particular for $a(x) \equiv constant = a$, we have

Corollary 3.2. *Let $u_m(x)$ be the unique solution of*

$$u^{''} + u = \frac{a}{u^\alpha}, u^{'}(0) = 0, u(0) = m$$

Let y_m be the psotive zero of $u_m^{'}(x)$. Then for m sufficiently small and $M^2 = \frac{2a}{\alpha-1}m^{1-\alpha}$, we have

$$y_m > \frac{\pi}{2} + c_0\epsilon_M$$

for $\alpha < 3$ and

$$y_m < \frac{\pi}{2} - c_0\epsilon_M$$

for $\alpha > 3$

Next we estimate the difference between the local maximums. Let y_M, z_M be the two consecutive local maximum points. Let $u(y_M) = M_1, y(z_M) = M_2$. Let t_M be the local maximum point in between y_M, z_M.

Then we have

$$\frac{M_1^2}{2} - \int_{y_M}^{t_M} \frac{au^{'}}{u^\alpha} = \frac{M_2^2}{2} + \int_{z_M}^{t_M} \frac{au^{'}}{u^\alpha}$$

Note that

$$\int_{y_M}^{t_M} \frac{au^{'}}{u^\alpha} = \frac{a(t_M)u^{1-\alpha}(t_M)}{1-\alpha} - \frac{a(y_M)M_1^{1-\alpha}}{1-\alpha} + \frac{1}{\alpha-1}\int_{y_M}^{t_M} \frac{a^{'}}{u^{\alpha-1}}$$

Similarly we have

$$\int_{z_M}^{t_M} \frac{au^{'}}{u^\alpha} = \frac{a(z_M)u^{1-\alpha}(z_M)}{1-\alpha} - \frac{a(t_M)u^{1-\alpha}(t_M)}{1-\alpha} + \frac{1}{\alpha-1}\int_{z_M}^{t_M} \frac{a^{'}}{u^{\alpha-1}}$$

By Lemmas 3.3, 3.4 and 3.5, we have

$$\int_{y_M}^{t_M} \frac{a^{'}}{u^{\alpha-1}} = M_1^2 O(\epsilon_{M_1})$$

$$\int_{z_M}^{t_M} \frac{a^{'}}{u^{\alpha-1}} = M_2^2 O(\epsilon_{M_2})$$

Hence we have proved

Lemma 3.7. *For $1 \leq \alpha \neq 3$, we have*

$$M_2 = M_1(1 + O(\epsilon_{M_1}))$$

Namely we have for $M \geq M_0$

$$M_1(1 - c_1\epsilon_{M_1}) \leq M_2 \leq M_1(1 + c_1\epsilon_{M_1})$$

4. Proofs of Theorem 1.3 and Corollary 1.2

Theorem 1.3 can be proved based on Lemma 3.2 and the following Lemma.

Lemma 4.1. *If $a(x) > 0$ is continuous, then for M sufficiently large we have*

$$y_M > \frac{\pi}{2}$$

for $\alpha < 3$ and

$$y_M < \frac{\pi}{2}$$

for $\alpha > 3$, where y_M is defined as in Section 3.

Proof: We just need to consider the case for $\alpha > 2$. For $1 \leq \alpha < 2$, we have Lemma 3.3. Similar to the proof of Lemma 3.3, we have by (3.2) for $\alpha = 2$,

$$u^{'}(\frac{\pi}{2}) > 0$$

for M sufficiently large. Hence $y_M > \frac{\pi}{2}$ for M sufficiently large.

For $2 < \alpha \neq 3$, we have by (3.2)

$$u^{'}(\frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \frac{a(x)}{u^\alpha} sin\ (x)dx - u(0)$$

Let $\delta > 0$ be a small fixed number. Let x_1 be a point where $a(0)(1 - \delta) \leq a(x) \leq (1 + \delta)a(0)$ for $x \leq x_1$. Then

$$(1 - \delta)a(0)/u^\alpha(x) \leq u^{''} + u \leq (1 + \delta)a(0)/u^\alpha(x)$$

for $x \in [0, x_1]$.

Note that

$$\int_{x_1}^{\frac{\pi}{2}} \frac{a(x)}{u^\alpha} sin\ (x)dx = O(M^{-\alpha})$$

and over $[0, x_1]$ we have

$$u_m^1 \leq u \leq u_m^2$$

where u_m^1 is the unique solution of

$$u^{''} + u = (1-\delta)a(0)/u^\alpha, u^{'}(0) = 0, u(0) = m$$

and u_m^2 is the unique solution of

$$u^{''} + u = (1+\delta)a(0)/u^\alpha, u^{'}(0) = 0, u(0) = m$$

Then we have

$$u^{'}(\frac{\pi}{2}) \leq \int_0^{\frac{\pi}{2}} a(x)/(u_m^1)^\alpha - m + O(M^{-\alpha})$$

$$\leq \int_0^{\frac{\pi}{2}} \frac{a(x)-(1-\delta)a(0)}{(u_m^1)^\alpha} sin\ (x) + \leq \int_0^{\frac{\pi}{2}} \frac{(1-\delta)a(0)}{(u_m^1)^\alpha} sin\ (x) - m$$

Note that

$$\int_0^{\frac{\pi}{2}} \frac{(1-\delta)a(0)}{(u_m^1)^\alpha} sin\ (x) - m = (u_m^1)^{'}(\frac{\pi}{2})$$

and for $2 < \alpha < 3$, we have

$$(u_m^1)^{'}(\frac{\pi}{2}) \sim M^{\frac{2}{1-\alpha}} \sim m$$

for $\alpha > 3$, we have

$$(u_m^1)^{'}(\frac{\pi}{2}) \sim -M^{\frac{2}{1-\alpha}} \sim -m$$

Hence

$$u^{'}(\frac{\pi}{2}) > 0$$

for $2 < \alpha < 3$ and

$$u^{'}(\frac{\pi}{2}) < 0$$

for $\alpha > 3$.

The Lemma is thus proved. □

Proof of Theorem 1.3:

Let $k \geq 1$ be a fixed integer. Let u be a $2k\pi$ period solution of $(1.4)_k$. It is easy to see that we just need to proved that $\min_{0\leq x\leq 2k\pi} u(x) \geq c > 0$. Suppose on the contrary that we have a sequence of solutions u_j such that $\min_{0\leq x\leq 2k\pi} u_j(x) \to 0$. Then by Lemma 3.1, we have that u_j

has exactly $L+1$ local minimum points, say $0 < x_1 < x_2 < ... < x_{L-1} < x_L = 2k\pi$ such that $u_j(x_i) \to 0, i = 1, ..., L$. Moreover $x_{i+1} - x_i \to \pi$ as $j \to \infty$.

Therefore $L = k$ for j large. But by Corollary 3.3, $x_{i+1} - x_i > \pi$ for $\alpha < 3$ and $x_{i+1} - x_i < \pi$ for $\alpha < 3$. Hence

$2k\pi = \sum_{i=1}^{L}(x_i - x_{i-1}) > 2L\pi$ for $\alpha < 3$ and $2k\pi = \sum_{i=1}^{L}(x_i - x_{i-1}) < 2L\pi$ for $\alpha > 3$. A contradiction ! □

Proof of Corollary 1.2: We first note that for $\alpha < 8$ and $\alpha \neq 3$, the solution to the following equation

$$u_{\theta\theta} + u - \frac{1}{u^\alpha} = 0, \theta \in R/(2\pi Z) \tag{4.1}$$

is unique and in fact is equal to 1. The proof of the above fact can be done by using a first integral method (a proof of it can be seen in [16]). Moreover the solution $u_0 = 1$ is nondegenerate and has a nonzero degree. In fact let v be the solution of the linearized equation

$$v_{\theta\theta} + v + \alpha \frac{1}{u_0^{\alpha+1}} v = \lambda v, \theta \in R/(2\pi Z) \tag{4.2}$$

Then

$$1 + \alpha - \lambda = n^2, \lambda = 1 + \alpha - n^2$$

for some $n \in Z, n \neq 0$. Hence $\lambda \neq 0$ and there are only one or two positive eigenvalues with multiplicity 2.

By using Theorem 1.3 and a degree method, we can proceed as in Section 5 of [6] and obtain the existence of a solution to (1.4). □

5. Proof of Theorems 1.4 and 1.5

In this section, we apply Ding's version of the Poincare-Birkhoff Theorem to prove Theorems 1.4 and 1.5. We first state Ding's result.

Lemma 5.1. *([8]) Let A denote an anular region whose inner boundary C_1 and outer boundary C_2 of A are simple curves. Denote by D_i the open region bounded by $C_i, i = 1, 2$. Let $W : A \to W(A) \subset R^2 \backslash \{0\}$ be an area-preserving homeomorphism. Suppose that*

(1) The inner boundary curve C_1 is star-shaped about the origin.

(2) W has a lifting $\tilde{W}$ to the polar coordinates plane, that is $\tilde{W}$ satisfies $P \circ \tilde{W} = W \circ P$, where $P(\rho,\theta) = (\rho cos\ \theta, \rho sin\ \theta)$ such that if $\tilde{W}(\rho,\theta) = (R(\rho,\theta), \Theta(\rho,\theta))$ Then $\Theta(\rho-\theta) - \theta > 0 (< 0)$ on $P^{-1}(C_1)$ and $\Theta(\rho,\theta) - \theta > 0 (> 0)$ on $P^{-1}(C_2)$. The functions R and Θ are continuous and 2π-periodic in θ.

(3) W can be extended as an area-preserving homeomorphism $W : \overline{D_2} \to R^2$ so that $0 \in W(D_1)$.

Then $\tilde{W}$ has at least two fixed points such that their images under P are two different fixed points of W.

To apply the above theorem, we first set up some notations. Let

$$r(\alpha,\beta) = \alpha^2 + \alpha^{1-\alpha} + \beta^2$$

If a function $y \in C^1[0,T]$, where T is to be defined later, has only simple zeros (namely if $y(x_0) = 0, y^{'}(x_0) \neq 0$), we define its rotation number $\psi(y)$ as

$$\psi(y) = k\pi + \lim_{t\to 0+} tan^{-1}\frac{y^{'}(t)}{y(t)} - \lim_{t\to T-} tan^{-1}\frac{y^{'}(t)}{y(t)}$$

where k is the number of zeros of $y(t)$ in $(0,T)$. Geometrically $\psi(y)$ represents the total angle the vector from the origin to the point $(y(t), y^{'}(t))$ in R^2 describes as t goes from 0 to T, positive angles measures clockwise.

Let us go back to equation (3.1). Let $y_1, y_2, ..., y_k, ...$ be the local maximum points of $u(x, M; 0)$. Set

$$M_k = u(y_k), \quad \epsilon_k = \epsilon_{M_k}$$

By Lemma 3.7 of Section 3, we have

$$M_k(1 - c_1\epsilon_k) \leq M_{k+1} \leq M_k(1 - c_1\epsilon_k) \tag{5.1}$$

Let c_0 be defined in Corollary 3.5 and c_1 be defined in Lemma 3.6. We now define the following constants.

$$k(M)=\begin{cases} I(\frac{2^{-\alpha-1}/c_1}{\epsilon_1}) \text{ for } 1\le\alpha<2\\ I(\frac{2}{3c_1\epsilon_1}) \text{ for } \alpha=2\\ I(\frac{2^{-\frac{\alpha+1}{\alpha+1}}c_1}{\epsilon_1}) \text{ for } 2<\alpha\neq 3\end{cases}$$

where $I(a)$ means the largest integer less than or equal to a.

$$c_2=\begin{cases} 3^{-\alpha-1}/c_1 \text{ for } 1\le\alpha<2\\ \frac{1}{27c_1} \text{ for } \alpha=2\\ \frac{3^{-\frac{\alpha+1}{\alpha-1}}}{c_1} \text{ for } 2<\alpha\neq 3\end{cases}$$

$$c_3=\begin{cases} c_1 2^{\frac{2}{\alpha-1}} \text{ for } 1\le\alpha<2\\ 3c_1 \text{ for } \alpha=2\\ 2^{\alpha}c_1 \text{ for } 2<\alpha\neq 3\end{cases}$$

We begin with the following lemma.

Lemma 5.2. *For $M=M_1\ge M_0$ for $k\le k(M)$, we have*

$$\frac{1}{2}M\le M(1-c_3k\epsilon_1)\le M_k\le M(1+c_3k\epsilon_1)\le\frac{3}{2}M \tag{5.2}$$

Moreover

$$\sum_{k=1}^{k(M)}\epsilon_k\ge c_2 \tag{5.3}$$

Proof: We prove (5.2) only when $1\le\alpha<2$. The other cases are simlar. We prove it by induction. In fact suppose

$$M(1-2^{\alpha}c_1(k-1)\epsilon_1)\le M_k\le M(1+2^{\alpha}c_1(k-1)\epsilon_1)$$

Then by (5.1)

$$M_{k+1}\le M_k(1+c_1\epsilon_k)\le M+2^{\alpha}c_1(k-1)M\epsilon_1+c_1M\epsilon_k$$

$$\le M+2^{\alpha}c_1kM\epsilon_1+c_1M\epsilon_k-2^{\alpha}c_1M\epsilon_1$$

$$\le M+2^{\alpha}c_1kM\epsilon_1$$

since for $\alpha < 2$

$$c_1 M\epsilon_k = c_1 M_k^{-\alpha} \le c_1(M - 2^\alpha c_1 M)^{-\alpha}$$
$$\le c_1 2^\alpha M^{-\alpha}$$

Simlarly we have the other inequality.

We now have

$$\sum_{k=1}^{k(M)} \epsilon_k \ge \sum_{k=1}^{k(M)} (M + 2^\alpha k M^{-\alpha})^{-\alpha-1}$$
$$\ge 2^{-\alpha-1}3^{\alpha+1}/(2^{\alpha+1}c_1) = c_2$$

□

The next lemma is the key estimate.

Lemma 5.3. *Let* $M \ge 2^{N+4}M_0$ *and*

$$M^i = M_{k(M^{i-1})+1}, k_i = k(M^{i-1}), i = 2, ..., N$$

Then

$$\sum_{k=1}^{k_N} \epsilon_k \ge Nc_2$$

Proof: In fact it easy to see that

$$M^i \ge \frac{1}{2}M^{i-1} \ge 2^{-N}M \ge M_0, i = 2, ..., N$$

By Lemma 5.2, we have

$$\sum_{k=k_i+1}^{k_{i+1}} \epsilon_k \ge c_0$$

$$\sum_{k=1}^{k_N} \epsilon_k = \sum_{i=1}^{N} (\sum_{k=k_i+1}^{k_{i+1}} \epsilon_k) \ge Nc_2$$

□

Let $M = M_1$ and y_k be the local maximum points of $u(x, M, 0; 0)$. Note that $y_1 = 0$.Then we have

Corollary 5.1. *For $M \geq 2^{N+4} M_0$, we have*

$$\sum_{k=1}^{k_N} (y_{k+1} - y_k) \geq k_N \pi + N c_0 c_2$$

for $1 \leq \alpha < 3$ and

$$\sum_{k=1}^{k_N} (y_{k+1} - y_k) \leq k_N \pi - N c_0 c_2$$

for $\alpha > 3$.

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4: Let N be such that $N c_0 c_2 > 64\pi$. Let k_N be defined as in Lemma 5.3. Set

$$T = 2 k_N \pi$$

Consider the operator

$$W : (-1, \infty) \times R \to (-1, \infty) \times R$$

defined by

$$W(\alpha, \beta) = (u(T, 1+\alpha, \beta) - 1, u^{'}(T, 1+\alpha, \beta)$$

Standard arguments show that W is an area-preserving homeomorphism. Observe also that $u(x, 1+\alpha, \beta)$ is a $T-$periodic solution of $(1.4)_{k_N}$ if and only if (α, β) is a fixed point of W.

Let $r_0 > 4^{N+4} M_0$. For $r((1+\alpha), \beta) = (1+\alpha)^2 + (1+\alpha)^{1-\alpha} + \beta^2 = r_0^2$, we have that $\max_{0 \leq x \leq \pi} u(x, 1+\alpha, \beta) \sim r_0$. Then we have for $\alpha < 3$

$$\eta(1+\alpha, \beta) \leq 4 k_N \pi \frac{4 k_N \pi}{4 k_N \pi + \sum_{k=1}^{k_N} (y_{k+1} - y_k)}$$

$$\leq 4 k_N \pi \frac{4 k_N \pi}{4 k_N \pi + c_0 \sum_{k=1}^{k_N} \epsilon_k}$$

$$\leq 4 k_N \pi - \frac{1}{4} c_0 \sum_{k=1}^{k_N} \epsilon_k$$

$$\leq 4 k_N \pi - 16\pi$$

Let $n_0 = k_N - 4$. Then we have proved that

$$\eta(1+\alpha,\beta) \le 4n_0\pi$$

for $r(1+\alpha,\beta) = r_0^2$.

Since as $r \to \infty$, $\eta(1+\alpha,\beta) \to 4k_N\pi$, there exists $r_1 > r_0$ such that

$$\begin{aligned} &sup\ (\eta(1+\alpha,\beta)|r(1+\alpha,\beta) = r_0^2) \le 2n_0\pi \\ &< 4(k_N-1)\pi < inf\ (\eta(1+\alpha,\beta)|r(1+\alpha,\beta) = r_1^2) \end{aligned} \tag{5.4}$$

Let us now define the curves C_1 and C_2 in R^2 by

$$C_1 = \{(\alpha,\beta) \in (-1,\infty) \times R | r(1+\alpha,\beta) = r_0^2\}$$

$$C_2 = \{(\alpha,\beta) \in (-1,\infty) \times R | r(1+\alpha,\beta) = r_1^2\}$$

Then C_1, C_2 are closed simple curves and, in particular, C_1 is star-shaped around the origin. Thus define A to be the annular region between these two curves. It is standard that the restriction of W to A can be lifted to the plar coordinate plane through the usual covering map $P(\rho,\theta) = (\rho cos\ \theta, \rho sin\ \theta)$ to a map $\tilde{W}$ satisfying the periodicity condition

$$\tilde{W}(\rho,\theta+2\pi) = \tilde{W}(\rho,\theta) + (0,2\pi)$$

Then if we write $\tilde{W}(\rho,\theta) = (R(\rho,\theta),\Theta(\rho,\theta))$, we have an integrer $\tilde{n}$ such that

$$\Theta(\rho,\theta) - \theta = 2\tilde{n}\pi - \eta(1+\alpha,\beta)$$

for all $(\rho,\theta) \in P^{-1}(A)$ where $(\alpha,\beta) = P(\rho,\theta)$.

Now for each integer n such that $n_0 \le n < k_N - 1$ define

$$\tilde{W}_n(\rho,\theta) = \tilde{W}(\rho,\theta) + (0, 2(2n-\tilde{n})\pi)$$

Then $\tilde{W}_n$ is still a lifting of W via the covering map P. By (5.4), it is clear that $\tilde{W}_n$ satisfies conditon (2) of Lemma 5.1. The only thing left is to verify that $0 \in W(D_1)$. But this is equivalent to saying that if (α,β) is such that $u(T,\alpha,\beta) = 1, u'(T,\alpha,\beta) = 0$ then $r(\alpha,\beta) < r_0^2$. This is obvious if we choose r_0 sufficiently large.

It follows from Lemma 5.1 that $\tilde{W}_n$ possessses two fixed points yielding two distinct fixed points (α_n^1, β_n^1) and (α_n^2, β_n^2) of W.

Thus $u(x, \alpha_n^1+1, \beta_n^1)$ is a $2k_N\pi-$periodic solutions of $(1.4)_{k_N}$. Observing that $\eta(1+\alpha_n^1, \beta_n^1) = 4n\pi$ we have $u(x, \alpha_n^1+1, \beta_n^1)$ has exactly $P_n = 2n$ maximum points (note for r large two zeros of $u(x, 1+\alpha_n^1, \beta_n^1)-1$ correspond to one local maximum point) and $2(k_N - 1) > 2n \geq 2(k_N - 4)$. Moreover $r_0^2 < r(1+\alpha, \beta) < r_1^2$.

Therfore for each M we have $w(M), p(M)$ such that $w(M) \to \infty, 2(w(M)-1) > p(M) \geq 2(w(M) - 4)$ and a solution u_M with p_M local maximum points, $y_1, ..., y_{p_M}$. Moreover $u_j(y_i) > cM$ for some $c > 0, j = 1, ..., p_M$.

Let $M \to \infty$, we obtain a sequence of solutions satisfying Theoren 1.4

Theorem 1.5 can proved by similar methods. We omit the details.

$\square$

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