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Classification and nondegeneracy of $SU(n+1)$ Toda system with singular sources

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Abstract We consider the following Toda system

$$\Delta u_i + \sum_{j=1}^{n} a_{ij} e^{u_j} = 4\pi\gamma_i \delta_0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{u_i} dx < \infty, \quad \forall\, 1 \le i \le n,$$

where $\gamma_i > -1$, δ_0 is Dirac measure at 0, and the coefficients a_{ij} form the standard tri-diagonal Cartan matrix. In this paper, (i) we completely classify the solutions and obtain the quantization result:

$$\sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} e^{u_j} dx = 4\pi(2 + \gamma_i + \gamma_{n+1-i}), \quad \forall\, 1 \le i \le n.$$

This generalizes the classification result by Jost and Wang for $\gamma_i = 0$, $\forall\, 1 \le i \le n$. (ii) We prove that if $\gamma_i + \gamma_{i+1} + \cdots + \gamma_j \notin \mathbb{Z}$ for all $1 \le i \le j \le n$, then any solution u_i is *radially symmetric* w.r.t. 0. (iii) We prove that the linearized

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equation at any solution is *non-degenerate*. These are fundamental results in order to understand the bubbling behavior of the Toda system.

1 Introduction

In this article, we consider the 2-dimensional (open) Toda system for $SU(n+1)$:

$$\begin{cases} \triangle u_i + \sum_{j=1}^{n} a_{ij} e^{u_j} = 4\pi \sum_{j=1}^{m} \gamma_{ij} \delta_{P_j} \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_i} dx < +\infty \end{cases} \tag{1.1}$$

for $i = 1, 2, \ldots, n$, where $\gamma_{ij} > -1$, P_j are distinct points and $A = (a_{ij})$ is the Cartan matrix for $SU(n+1)$, given by

$$A := (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & \ldots & -1 & 2 & -1 \\ 0 & \ldots & & -1 & 2 \end{pmatrix}. \tag{1.2}$$

Here δ_P denotes the Dirac measure at P. For $n = 1$, system (1.1) is reduced to the Liouville equation

$$\triangle u + 2e^u = 4\pi \sum_{j=1}^{m} \gamma_j \delta_{P_j} \tag{1.3}$$

which has been extensively studied for the past three decades. The Toda system (1.1) and the Liouville equation (1.3) arise in many physical and geometric problems. For example, in the Chern-Simons theory, the Liouville equation is related to abelian gauge field theory, while the Toda system is related to nonabelian gauge, see [11, 12, 14, 20, 21, 30–32, 36, 37] and references therein. On the geometric side, the Liouville equation with or without singular sources is related to the problem of prescribing Gaussian curvature proposed by Nirenberg, or related to the existence of the metrics with conic singularities. As for the Toda system, there have been a vast literature to discuss the relationship to holomorphic curves in $\mathbb{CP}^n$, flat $SU(n+1)$ connection, complete integrability and harmonic sequences. For example, see [2, 3, 5, 9, 10, 16, 21]. In this paper, we want to study the Toda system from the analytic viewpoint. For the past thirty years, the Liouville equation has been extensively studied by the method of nonlinear partial differential equations, see [4, 6–8, 22, 25, 31, 32, 34] and references therein. Recently, the analytic studies of the Toda system can be found in [17–19, 23, 29, 32, 33, 35, 36]. For the generalized Liouville system, see [26] and [27].

From the viewpoint of PDE, we are interested not only in the Toda system itself, but also in the situation with non-constant coefficients. One of such examples is the Toda system of mean field type:

$$\Delta u_i(x) + \sum_{j=1}^{n} a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j}} - \frac{1}{|\Sigma|} \right) = 4\pi \sum_{j=1}^{m} \gamma_{ij} \left(\delta_{P_j} - \frac{1}{|\Sigma|} \right), \tag{1.4}$$

where P_j are distinct points, $\gamma_{ij} > -1$ and h_j are positive smooth functions in a compact Riemann surface Σ. When $n = 1$, the equation becomes the following mean field equation:

$$\Delta u(x) + \rho \left(\frac{h e^{u}}{\int_\Sigma h e^{u}} - \frac{1}{\Sigma} \right) = 4\pi \sum_{j=1}^{m} \gamma_j \left(\delta_{P_j} - \frac{1}{|\Sigma|} \right) \quad \text{in } \Sigma. \tag{1.5}$$

This type of equations has many applications in different areas of research, and has been extensively investigated. One of main issues for (1.5) is to determine the set of parameter ρ (non-critical parameters) such that the a priori estimate exists for solutions of (1.5). After establishing a priori estimate, we then go to compute the topological degree of (1.5) for those non-critical parameters. In this way, we are able to solve (1.5) and understand the structure of the solution set. For the past ten years, those projects have been successfully carried out for (1.5). See [6–8, 22]. While carrying out those projects for (1.4), there often appears a sequence of bubbling solutions and the difficult issue is how to understand the behavior of bubbling solutions near blowup points. For that purpose, the fundamental question is to completely classify all entire solutions of the Toda system with a single singular source:

$$\Delta u_i + \sum_{j=1}^{n} a_{ij} e^{u_j} = 4\pi \gamma_i \delta_0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{u_i} dx < \infty, \quad 1 \le i \le n, \tag{1.6}$$

where δ_0 is the Dirac measure at 0, and $\gamma_i > -1$. When all γ_i are zero, the classification has been done by Jost-Wang [18]. However, when $\gamma_i \neq 0$ for some i, the classification has not been proved and has remained a long-standing open problem for many years. It is the purpose of this article to settle this open problem.

To state our result, we should introduce some notations. For any solution $u = (u_1, \dots, u_n)$ of (1.6), we define $U = (U_1, U_2, \dots, U_n)$ by

$$U_i = \sum_{j=1}^{n} a^{ij} u_j, \tag{1.7}$$

where (a^{ij}) is the inverse matrix of A. By (1.7), U satisfies

$$\triangle U_i + e^{u_i} = 4\pi\alpha_i\delta_0 \quad \text{in } \mathbb{R}^2, \text{ where } \alpha_i = \sum_{j=1}^{n} a^{ij}\gamma_j. \tag{1.8}$$

By direct computations, we have

$$a^{ij} = \frac{j(n+1-i)}{n+1}, \quad \forall\, n \geq i \geq j \geq 1 \quad \text{and} \quad u_i = \sum_{j=1}^{n} a_{ij}U_j.$$

Our first result is the following classification theorem.

Theorem 1.1 *Let $\gamma_i > -1$ for $1 \leq i \leq n$ and $U = (U_1, \ldots, U_n)$ be defined by (1.7) via a solution u of (1.6). Then U_1 can be expressed by*

$$e^{-U_1} = |z|^{-2\alpha_1}\left(\lambda_0 + \sum_{i=1}^{n} \lambda_i |P_i(z)|^2\right), \tag{1.9}$$

where

$$P_i(z) = z^{\mu_1+\cdots+\mu_i} + \sum_{j=0}^{i-1} c_{ij} z^{\mu_1+\cdots\mu_j}, \tag{1.10}$$

$\mu_i = 1 + \gamma_i > 0$, c_{ij} are complex numbers and $\lambda_i > 0$, $0 \leq i \leq n$, satisfy

$$\lambda_0 \cdots \lambda_n = 2^{-n(n+1)} \prod_{1 \leq i \leq j \leq n} \left(\sum_{k=i}^{j} \mu_k\right)^{-2}. \tag{1.11}$$

Furthermore, if $\mu_{j+1} + \cdots + \mu_i \notin \mathbb{N}$ for some $j < i$, then $c_{ij} = 0$.

In particular, we have the following theorem, generalizing a result by Prajapat-Tarantello [34] for the singular Liouville equation, $n = 1$.

Corollary 1.2 *Suppose $\mu_j + \cdots + \mu_i \notin \mathbb{N}$ for all $1 \leq j \leq i \leq n$. Then any solution of (1.6) is radially symmetric with respect to the origin.*

We note that once U_1 is known, U_2 can be determined uniquely by (1.8), i.e., $e^{-U_2} = e^{-2U_1}\triangle U_1$. In general, U_{i+1} can be solved via (1.8) by the induction on i. See the formula (5.16). In the Appendix, we shall apply Theorem 1.1 to give all the explicit solutions in the case of $n = 2$. Conversely, in Sect. 5, we will prove any expression of (1.9) satisfying (1.11) can generate a solution of (1.6). See Theorem 5.3. Thus, the number of free parameters depends on all the Dirac masses γ_j. For example if all $\mu_j \in \mathbb{N}$, then the number of free parameters is $n(n+2)$. And if all $\mu_i + \cdots + \mu_j \notin \mathbb{N}$ for $1 \le i \le j \le n$, thus the number of free parameters is n only. We let $N(\gamma)$ denote the real dimension of the solution set of the system (1.6).

Next, we will show the quantization of the integral of e^{u_i} over $\mathbb{R}^2$ and the non-degeneracy of the linearized system. For the Liouville equation with single singular source:

$$\triangle u + e^u = 4\pi\gamma\delta_0, \qquad \int_{\mathbb{R}^2} e^u dx < +\infty, \quad \gamma > -1,$$

it was proved in [34] that any solution u satisfies the following quantization:

$$\int_{\mathbb{R}^2} e^u dx = 8\pi(1+\gamma),$$

and in [13] that for any $\gamma \in \mathbb{N}$, the linearized operator around any solution u is nondegenerate. Both the quantization and the non-degeneracy are important when we come to study the Toda system of mean field type. In particular, this nondegeneracy plays a fundamental role as far as sharp estimates of bubbling solutions are concerned. See [1] and [8].

Theorem 1.3 *Suppose $u = (u_1, \ldots, u_n)$ is a solution of* (1.6). *Then the followings hold*:

(i) *Quantization*: *we have*, *for any* $1 \le i \le n$,

$$\sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} e^{u_j} dx = 4\pi(2 + \gamma_i + \gamma_{n+1-i})$$

and $u_i(z) = -(4 + 2\gamma_{n+1-i}) \log|z| + O(1)$ *as* $|z| \to \infty$.

(ii) *Nondegeneracy*: *The dimension of the null space of the linearized operator at u is equal to* $N(\gamma)$.

In the absence of singular sources, i.e., $\gamma_i = 0$ for all i, Theorem 1.1 was obtained by Jost and Wang [18]. By applying the holonomy theory, and identifying $S^2 = \mathbb{C} \cup \{\infty\}$, they could prove that any solution u can be extended to be a totally unramified holomorphic curve from S^2 to $\mathbb{CP}^n$, and then Theorem 1.1 can be obtained via a classic result in algebraic geometry, which says that any totally unramified holomorphic curve of S^2 into $\mathbb{CP}^n$ is a rational normal curve. Our proof does not use the classical result from algebraic geometry. As a consequence, we give a proof of this classic theorem in algebraic geometry by using nonlinear partial differential equations. In fact, our analytic method can be used to prove a generalization of this classic theorem.

For a holomorphic curve f of S^2 into $\mathbb{CP}^n$, we recall the k-th associated curve $f_k : S^2 \to GL(k, n+1)$ for $k = 1, 2, \ldots, n$ with $f_1 = f$ and $f_k = [f \wedge \cdots \wedge f^{(k-1)}]$. A point $p \in S^2$ is called a ramificated point if the pull-back metric $f_k^*(\omega_k) = |z - p|^{2\gamma_k} h(z) dz \wedge d\bar{z}$ with $h > 0$ at p for some $\gamma_k > 0$ where

$$\omega_k \text{ is the Fubini-Study metric on } GL(k, n+1) \subseteq \mathbb{CP}^{N_k - 1}, \quad N_k = \binom{n+1}{k}. \tag{1.12}$$

The positive integer $\gamma_k(p)$ is called the ramification index of f_k at p. See [15].

Corollary 1.4 *Let f be a holomorphic curve of S^2 into $\mathbb{CP}^n$. Suppose f has exactly two ramificated points P_1 and P_2 and $\gamma_j(P_i)$ are the ramification index of f_j at P_i, where f_j is the j-th associated curve for $1 \le j \le n$. Then $\gamma_j(P_1) = \gamma_{n+1-j}(P_2)$. Furthermore, if f and g are two such curves with the same ramificated points and ramification index, then g can be obtained via f by a linear map of $\mathbb{CP}^n$.*

It is well-known that the Liouville equation as well as the Toda system are completely integrable system, a fact known since Liouville [28]. Roughly speaking, any solution of (1.1) without singular sources in a simply connected domain Ω arises from a holomorphic map from Ω into $\mathbb{CP}^n$. See [2, 3, 5, 9, 10, 16, 21, 38]. For $n = 1$, The classic Liouville theorem says that if a smooth solution u satisfies $\Delta u + e^u = 0$ in a simply connected domain $\Omega \subset \mathbb{R}^2$, then $u(z)$ can be expressed in terms of a holomorphic function f in Ω:

$$u(z) = \log \frac{8|f'(z)|^2}{(1 + |f(z)|^2)^2} \quad \text{in } \Omega. \tag{1.13}$$

Similarly, system (1.1) has a very close relationship with holomorphic curves in $\mathbb{C}P^n$. Let f be a holomorphic curve from Ω into $\mathbb{CP}^n$. Lift locally f to $\mathbb{C}^{n+1}$ and denote the lift by $\nu = (\nu_0, \nu_1, \ldots, \nu_n)$. The k-th associated curve of f is defined by

$$f_k : \Omega \to G(k, n+1) \subset \mathbb{CP}^{N_k - 1}, \quad f_k(z) = \left[\nu(z) \wedge \nu'(z) \wedge \cdots \nu^{(k-1)}(z)\right], \tag{1.14}$$

where N_k is given by (1.12), $\nu^{(j)}$ stands for the j-th derivative of ν w.r.t. z. Let $\Lambda_k = \nu(z) \wedge \cdots \nu^{(k-1)}(z)$, then the well-known infinitesimal Plücker formulas (see [15]) is

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\Lambda_k\|^2 = \frac{\|\Lambda_{k-1}\|^2 \|\Lambda_{k+1}\|^2}{\|\Lambda_k\|^4} \quad \text{for } k = 1, 2, \ldots, n, \tag{1.15}$$

where conventionally we put $\|\Lambda_0\|^2 = 1$. Of course, this formula holds only for $\|\Lambda_k\| > 0$, i.e. for all unramificated points. By normalizing $\|\Lambda_{n+1}\| = 1$, and letting

$$U_k(z) = -\log \|\Lambda_k(z)\|^2 + k(n-k+1)\log 2, \quad 1 \le k \le n \tag{1.16}$$

at an unramificated point z. Since $\sum_{1\le k\le n} a_{ik}k(n-k+1) = 2$, (1.15) gives

$$-\Delta U_i = \exp\left(\sum_{j=1}^n a_{ij}U_j\right) \quad \text{in } \Omega \setminus \{P_1, \dots, P_m\},$$

where $\{P_1, \dots, P_m\}$ are the set of ramificated points of f in Ω. Since f is smooth at P_j, we have $U_i = -2\alpha_{ij}\log|z-P_j| + O(1)$ near P_j. Thus, U_i satisfies

$$\Delta U_i + \exp\left(\sum_{j=1}^n a_{ij}U_j\right) = 4\pi\sum_{j=1}^n \alpha_{ij}\delta_{P_j} \quad \text{in } \Omega.$$

The constants α_{ij} can be expressed by the total ramification index at P_j by the following arguments.

By the Plücker formulas (1.15), we have

$$f_i^*(\omega_i) = \frac{\sqrt{-1}}{2}\exp\left(\sum_{j=1}^n a_{ij}U_j\right) dz \wedge d\bar{z}.$$

Thus, the ramification index γ_{ij} at f_i at P_j is

$$\gamma_{ij} = \sum_{k=1}^n a_{ik}\alpha_{kj}. \tag{1.17}$$

Set

$$u_i = \sum_{j=1}^n a_{ij}U_j. \tag{1.18}$$

Then it is easy to see that u_i satisfies (1.1) with γ_{ij} the total ramification index of f_i at P_j.

Conversely, suppose $u = (u_1, \dots, u_n)$ is a smooth solution of (1.1) in a simply connected domain Ω. We introduce w_j $(0 \le j \le n)$ by

$$u_i = 2(w_i - w_{i-1}), \quad \sum_{i=0}^n w_i = 0. \tag{1.19}$$

Obviously, w_i can be uniquely determined by u and satisfies

$$\begin{pmatrix} w_0 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{pmatrix}_{z\bar{z}} = \frac{1}{8}\begin{pmatrix} e^{2(w_1-w_0)} \\ \vdots \\ e^{2(w_{i+1}-w_i)} - e^{2(w_i-w_{i-1})} \\ \vdots \\ -e^{2(w_n-w_{n-1})} \end{pmatrix}. \tag{1.20}$$

For a solution (w_i), we set

$$U = \begin{pmatrix} w_{0,z} & 0 & \dots & 0 \\ 0 & w_{1,z} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & w_{n,z} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & \dots & 0 \\ e^{w_1-w_0} & 0 & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & e^{w_n-w_{n-1}} & 0 \end{pmatrix}$$

and

$$V = -\begin{pmatrix} w_{0,\bar{z}} & 0 & \dots & 0 \\ 0 & w_{1,\bar{z}} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & w_{n,\bar{z}} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & e^{w_1-w_0} & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & \ddots & \ddots & e^{w_n-w_{n-1}} \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

where

$$w_z = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \quad \text{and} \quad w_{\bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) \quad \text{with } z = x + iy.$$

A straightforward computation shows that (w_i) is a solution of (1.20) if and only if U, V satisfy the Lax pair condition: $U_{\bar{z}} - V_z - [U, V] = 0$. Furthermore, this integrability condition implies the existence of a smooth map $\Phi : \Omega \to SU(n+1, \mathbb{C})$ satisfying

$$\Phi_z = \Phi U, \qquad \Phi_{\bar{z}} = \Phi V \tag{1.21}$$

or equivalently, Φ satisfies $\Phi^{-1}d\Phi = Udz + Vd\bar{z}$. Let $\Phi = (\Phi_0, \Phi_1, \dots, \Phi_n)$. By (1.21),

$$d\Phi_0 = \left(w_{0,z}\Phi_0 + \frac{1}{2}e^{w_1-w_0}\Phi_1\right)dz - w_{0,\bar{z}}\Phi_0 d\bar{z},$$

which implies

$$d(e^{w_0}\Phi_0) = e^{w_0}d\Phi_0 + e^{w_0}\Phi_0 dw_0 = \left(2w_{0,z}e^{w_0}\Phi_0 + \frac{1}{2}e^{w_1}\Phi_1\right)dz. \tag{1.22}$$

Therefore, $e^{w_0}\Phi_0$ is a holomorphic map from Ω to $\mathbb{C}^{n+1}$. We let $\nu(z) = 2^{\frac{n}{2}} e^{w_0}\Phi_0$. By using (1.21), we have $\nu^{(k)}(z) = 2^{\frac{n}{2}-k} e^{w_k}\Phi_k$ for $k = 1, 2, \dots, n$. Since $w_0 + \cdots + w_n = 0$, we have $\|\nu \wedge \nu' \wedge \cdots \nu^{(n)}(z)\| = 1$. Note that

$$w_0 = -\frac{1}{2}\sum_{j=1}^{n} \frac{(n-j+1)}{n+1} u_j = -\frac{U_1}{2},$$

hence we have $e^{-U_1} = e^{2w_0} = 2^{-n}\|\nu\|^2$. Thus, (1.16) implies U_1 is identical to the solution deriving from the holomorphic curve $\nu(z)$. Therefore, the space of smooth solutions of the system (1.1) (without singular sources) in a simply connected domain Ω is identical to the space of unramificated holomorphic curves from Ω into $\mathbb{CP}^n$.

However, if the system (1.1) has singular sources, then $\mathbb{R}^2 \setminus \{P_1, \dots, P_m\}$ is not simply connected. So, it is natural to ask whether in the case $\gamma_{ij} \in \mathbb{N}$, the space of solutions u of (1.1) can be identical to the space of holomorphic curves of $\mathbb{R}^2$ into $\mathbb{CP}^n$ which ramificates at $P_1, \dots, P_m$, with the given ramification index γ_{ij} at P_j. The following theorem answers this question affirmatively.

Theorem 1.5 *Let $\gamma_{ij} \in \mathbb{N}$ and $P_j \in \mathbb{R}^2$. Then for any solution u of* (1.1), *there exists a holomorphic curve f of $\mathbb{C}$ into $\mathbb{CP}^n$ with ramificated points P_j and the total ramification index γ_{ij} at P_j such that for $1 \le k \le n$,*

$$e^{-U_k} = 2^{-k(n+1-k)} \left\| \nu(z) \wedge \cdots \wedge \nu^{(k-1)}(z) \right\|^2 \quad \text{in } \mathbb{C} \setminus \{P_1, \dots, P_m\},$$

where $\nu(z)$ is a lift of f in $\mathbb{C}^{n+1}$ satisfying

$$\left\| \nu(z) \wedge \cdots \wedge \nu^{(n)}(z) \right\| = 1.$$

Furthermore, f can be extended smoothly to a holomorphic curve of S^2 into $\mathbb{CP}^n$.

We note that if (1.1) is defined in a Riemann surface rather than $\mathbb{C}$ or S^2, then the identity of the solution space of (1.1) with holomorphic curves in $\mathbb{CP}^n$ generally does not hold. For example, if (1.1) is defined on a torus, then even for $n = 1$, a solution of (1.1) would be not necessarily associated with a holomorphic curve from the torus into $\mathbb{CP}^1$. See [24].

The paper is organized as follows. In Sect. 2, we will show some invariants associated with a solution of the Toda system. Those invariants allow us to classify all the solutions of (1.6) without singular sources, thus it gives another proof of the classification due to Jost and Wang. In Sect. 5, those invariants can be extended to be meromorphic invariants for the case with singular sources. By using those invariants, we can prove e^{-U_1} satisfies an ODE in $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, the proof will be given in Sect. 5. In Sect. 4 and Sect. 6, we will prove the quantization and the non-degeneracy of the linearized equation of (1.6) for the case without or with singular sources. In the final section, we give a proof of Theorem 1.5. Explicits solutions in the case of $SU(3)$ are given in the Appendix.

2 Invariants for solutions of Toda system

In this section, we derive some invariants for the Toda system. Denote $A^{-1} = (a^{jk})$, the inverse matrix of A. Let

$$U_j = \sum_{k=1}^{n} a^{jk} u_k, \quad \forall\, 1 \le j \le n. \tag{2.1}$$

Since $\Delta = 4\partial_{z\bar{z}}$, it is easy to see that the system (1.6) is equivalent to for all $1 \le i \le n$,

$$\begin{aligned} &-4U_{i,z\bar{z}} = \exp\left(\sum_{j=1}^{n} a_{ij} U_j\right) - 4\pi\alpha_i\delta_0 \quad \text{in } \mathbb{R}^2, \\ &\int_{\mathbb{R}^2} \exp\left(\sum_{j=1}^{n} a_{ij} U_j\right) dx < \infty, \end{aligned}$$

where $\alpha_i = \sum_{1\le j\le n} a^{ij}\gamma_j$ for $1 \le i \le n$. Define

$$\begin{aligned} &W_1^j = -e^{U_1}\left(e^{-U_1}\right)^{(j+1)} \quad \text{for } 1 \le j \le n \quad \text{and} \\ &W_{k+1}^j = -\frac{W_{k,\bar{z}}^j}{U_{k,z\bar{z}}} \quad \text{for } 1 \le k \le j-1. \end{aligned} \tag{2.2}$$

We will prove that all these quantities W_k^j, $1 \le k \le j \le n$, are invariants for solutions of $SU(n+1)$, more precisely, W_k^j are a part of some specific holomorphic or meromorphic functions, which are determined explicitly by the Toda system.

Lemma 2.1 *For any classical solution of* (1.1), *there holds*:

$$W_k^k = \sum_{i=1}^{k} (U_{i,zz} - U_{i,z}^2) + \sum_{i=1}^{k-1} U_{i,z}U_{i+1,z} \quad \textit{for } 1 \le k \le n, \tag{2.3}$$

$$W^k_{k,\bar z} = -U_{k,z\bar z}U_{k+1,z} \quad for\ 1 \le k \le n-1, \tag{2.4}$$

$$W^j_k = (U_{k-1,z} - U_{k,z})W^{j-1}_k + W^{j-1}_{k,z} + W^{j-1}_{k-1} \quad for\ 1 \le k < j \le n, \tag{2.5}$$

where for convenience $U_0 = 0$ *and* $W^j_0 = 0$ *for all* j.

Proof First, we show that (2.3) implies (2.4). By the equation for U_j,

$$U_{j,z\bar z z} = U_{j,z\bar z}(2U_{j,z} - U_{j+1,z} - U_{j-1,z}), \quad \forall\, 1 \le j \le n, \tag{2.6}$$

where for the convenience, $U_{n+1} = 0$ is also used. Thus,

$$\begin{aligned} &-U_{j,z\bar z}U_{j+1,z} + U_{j-1,z\bar z}U_{j,z} \\ &\quad = U_{j,z\bar z z} - U_{j,z\bar z}\left(2U_{j,z} - U_{j-1,z}\right) + U_{j-1,z\bar z}U_{j,z} \\ &\quad = \left(U_{j,zz} - U^2_{j,z} + U_{j,z}U_{j-1,z}\right)_{\bar z}. \end{aligned} \tag{2.7}$$

Taking the sum of (2.7) for j from 1 to k, we get

$$-U_{k,z\bar z}U_{k+1,z} = \sum_{j=1}^{k}\left(U_{j,zz} - U^2_{j,z} + U_{j,z}U_{j-1,z}\right)_{\bar z} = W^k_{k,\bar z},$$

where (2.3) is used.

Next, we will prove (2.3)–(2.5) by the induction on k. Obviously, (2.3) holds for $k = 1$. By the definition of W^j_1, for $j \ge 2$, we have

$$W^j_1 = -e^{U_1}(e^{-U_1})^{(j+1)} = e^{U_1}\left(e^{-U_1}W^{j-1}_1\right)_z = W^{j-1}_{1,z} - W^{j-1}_1 U_{1,z},$$

which is (2.5) for $k = 1$. To compute W^{k+1}_{k+1}, (2.5) with index k implies

$$\begin{aligned} -U_{k,z\bar z}W^{k+1}_{k+1} = W^{k+1}_{k,\bar z} &= (U_{k-1,z\bar z} - U_{k,z\bar z})W^k_k + (U_{k-1,z} - U_{k,z})W^k_{k,\bar z} \\ &\quad + W^k_{k,z\bar z} + W^k_{k-1,\bar z}. \end{aligned}$$

Since $U_{k-1,z\bar z}W^k_k + W^k_{k-1,\bar z} = 0$, the above identity leads by (2.4) with index k,

$$\begin{aligned} W^{k+1}_{k,\bar z} &= -U_{k,z\bar z}W^k_k - (U_{k-1,z} - U_{k,z})U_{k,z\bar z}U_{k+1,z} - (U_{k,z\bar z}U_{k+1,z})_z \\ &= -U_{k,z\bar z}W^k_k - (U_{k-1,z} - U_{k,z})U_{k,z\bar z}U_{k+1,z} - U_{k,z\bar z}(2U_{k,z} - U_{k+1,z} \\ &\quad - U_{k-1,z})U_{k+1,z} - U_{k,z\bar z}U_{k+1,zz} \\ &= -U_{k,z\bar z}\left(W^k_k + U_{k+1,zz} - U^2_{k+1,z} + U_{k+1,z}U_{k,z}\right), \end{aligned}$$

where (2.6) is used. Hence

$$W_{k+1}^{k+1} = W_k^k + U_{k+1,zz} - U_{k+1,z}^2 + U_{k+1,z}U_{k,z},$$

and then (2.3) is proved for $k+1$.

To compute W_{k+1}^j for $j \geq k+2$, we have $j-1 \geq k+1$ and by similar calculations:

$$\begin{aligned}
W_{k,\bar{z}}^j &= (U_{k-1,z\bar{z}} - U_{k,z\bar{z}})W_k^{j-1} + (U_{k-1,z} - U_{k,z})W_{k,\bar{z}}^{j-1} + W_{k,z\bar{z}}^{j-1} + W_{k-1,\bar{z}}^{j-1} \\
&= -U_{k,z\bar{z}}W_k^{j-1} - (U_{k-1,z} - U_{k,z})U_{k,z\bar{z}}W_{k+1}^{j-1} - \left(U_{k,z\bar{z}}W_{k+1}^{j-1}\right)_z \\
&= -U_{k,z\bar{z}}W_k^{j-1} - (U_{k-1,z} - U_{k,z})U_{k,z\bar{z}}W_{k+1}^{j-1} - U_{k,z\bar{z}}(2U_{k,\bar{z}} \\
&\quad - U_{k+1,z} - U_{k-1,z})W_k^{j-1} - U_{k,z\bar{z}}W_{k+1,z}^{j-1} \\
&= -U_{k,z\bar{z}}\left[(U_{k,z} - U_{k+1,z})W_{k+1}^{j-1} + W_{k+1,z}^{j-1}W_k^{j-1}\right],
\end{aligned}$$

which leads to

$$W_{k+1}^j = (U_{k,z} - U_{k+1,z})W_{k+1}^{j-1} + W_{k+1,z}^{j-1} + W_k^{j-1}.$$

Therefore, Lemma 2.1 is proved. □

3 Classification of solutions of $SU(n+1)$ with $m = 0$

Here we show a new proof of the classification result of Jost-Wang [18]. That is, all classical solutions of (1.1) with $m = 0$ is given by a $n(n+2)$ manifold $\mathcal{M}$. Our idea is to use the invariants W_j^n for solutions of $SU(n+1)$. Consider

$$-\Delta u_i = \sum_{j=1}^n a_{ij}e^{u_j} \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{u_i}dx < \infty, \quad \forall\, 1 \leq i \leq n. \tag{3.1}$$

Theorem 3.1 *For any classical solution of* (3.1), *let* U_j, W_j^n *be defined by* (2.1) *and* (2.2), *then*

$$W_j^n \equiv 0 \quad \mathit{in}\ \mathbb{R}^2, \forall\, 1 \leq j \leq n.$$

Remark 3.2 The fact $W_n^n = 0$ has been proved by Jost and Wang in an equivalent form, which is just the function f in the proof of Proposition 2.2 in [18].

Proof The proof is based on the following observation:

$$W_{n,\bar{z}}^n = 0 \quad \text{in } \mathbb{R}^2 \text{ for any solution of (3.1)}. \tag{3.2}$$

In fact, using formula (2.3) and the equations of U_i,

$$
\begin{aligned}
W^n_{n,\bar z} &= \sum_{i=1}^{n}(U_{i,z\bar z})_z - 2\sum_{i=1}^{n} U_{i,z}U_{i,z\bar z} + \sum_{i=1}^{n-1}\left(U_{i,z\bar z}U_{i+1,z} + U_{i,z}U_{i+1,z\bar z}\right)\\
&= \sum_{i=1}^{n} U_{i,z\bar z}\left[\sum_{j=1}^{n}\left(a_{ij}U_{j,z}\right) - 2U_{i,z} + U_{i+1,z} + U_{i-1,z}\right]\\
&= 0.
\end{aligned} \tag{3.3}
$$

Here we used again the convention $U_0 = U_{n+1} = 0$ for $SU(n+1)$.

Furthermore, $e^{u_i} \in L^1(\mathbb{R}^2)$ implies that for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\int_{\mathbb{R}^2\setminus B_{R_\epsilon}} e^{u_i} dz \le \epsilon, \quad 1 \le i \le n.$$

For sufficient small $\epsilon > 0$, applying Brezis-Merle's argument [4] to the system u_i, we can prove $u_i(z) \le C$ for $|z| \ge R_\epsilon$, i.e. u_i is bounded from the above over $\mathbb{C}$. Thus, u_i can be represented by the following integral formulas:

$$u_i(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{|z'|}{|z-z'|}\sum_{j=1}^{n} a_{ij}e^{u_j(z')}dz' + c_i, \quad \forall\, 1 \le i \le n, \tag{3.4}$$

for some real constants c_i.

This gives us the asymptotic behavior of u_i and their derivatives at infinity. In particular, for any $k \ge 1$, $\nabla^k u_i = O(|z|^{-k})$ as $|z|$ goes to ∞. So $\nabla^k U_i = O(|z|^{-k})$ as $|z| \to \infty$, for $k \ge 1$. Therefore, W^n_n is a entire holomorphic function, which tends to zero at infinity, so $W^n_n \equiv 0$ in $\mathbb{R}^2$ by classical Liouville theorem. As $W^n_{n-1,\bar z} = -U_{n-1,z\bar z}W^n_n$, we obtain $W^n_{n-1,\bar z} = 0$ in $\mathbb{R}^2$. By (2.3) and (2.5), it is not difficult to see that for $1 \le i \le n-1$, W^n_i are also polynomials of $\nabla^k U_i$ with $k \ge 1$, so they tend to 0 at infinity, hence $W^n_{n-1} = 0$ in $\mathbb{R}^2$. We can complete the proof of Theorem 3.1 by induction. □

Furthermore, we know that e^{-U_1} can be computed as a square of some holomorphic curves in $\mathbb{CP}^n$, see the Introduction. Thus, there is a holomorphic map $\nu(z) = (\nu_0(z), \ldots, \nu_n(z))$ from $\mathbb{C}$ into $\mathbb{C}^{n+1}$ satisfying

$$\left\|\nu \wedge \nu' \cdots \wedge \nu^{(n)}(z)\right\| = 1 \quad \text{and} \quad e^{-U_1(z)} = \sum_{i=0}^{n}|\nu_i(z)|^2 \quad \text{in } \mathbb{C}.$$

Since $W^n_1 \equiv 0$ in $\mathbb{R}^2$ yields $(e^{-U_1})^{(n+1)} = 0$, we have $\nu_i^{(n+1)}(z) = 0$. By the asymptotic behavior of u_i, we know that e^{-U_1} is of polynomial growth as

$|z| \to \infty$. Hence $\nu_i(z)$ is a polynomial and $\nu_0, \ldots, \nu_n$ is a set of fundamental holomorphic solutions of $f^{(n+1)} = 0$. Thus

$$\nu_i(z) = \sum_{j=0}^{n} c_{ij} z^j \quad \text{with } \det(c_{ij}) \neq 0. \tag{3.5}$$

By a linear transformation, we have

$$\nu(z) = \lambda(1, z, z^2, \ldots, z^n), \quad \lambda \in \mathbb{C}$$

and $[\nu]$ is the rational normal curve of S^2 into $\mathbb{CP}^n$. Hence we have proved the classification theorem of Jost and Wang.

Remark 3.3 Here we use the integrability of the Toda system. In Sect. 5, we actually prove the classification theorem without use of the integrability.

Remark 3.4 The invariants W_j^n are called W-symmetries or conservation laws, see [21]. It is claimed that for the Cartan matrix there are n linearly independent W-symmetries, see [38]. However, as far as we are aware, we cannot find the explicit formulas in the literature (except for $n = 2$ [35]). Here we give explicit formula for the n invariants.

4 Nondegeneracy of solutions of $SU(n+1)$ without sources

Let $\mathcal{M}$ be the collection of entire solutions of (3.1). In the previous section, we know that $\mathcal{M}$ is a smooth manifold of $n(n+2)$ dimension. Fixing a solution $u = (u_1, \ldots, u_n)$ of (3.1), we consider $LSU(n+1)$, the linearized system of (3.1) at u:

$$\triangle \phi_i + \sum_{j=1}^{n} a_{ij} e^{u_j} \phi_j = 0 \quad \text{in } \mathbb{R}^2. \tag{4.1}$$

Let $s \in \mathbb{R}$ be any parameter appearing in (3.5) and $u(z; s)$ be a solution of (3.1) continuously depending on s such that $u(z; 0) = u(z)$. Thus $\phi(z) = \frac{\partial}{\partial s} u(z; s)|_{s=0}$ is a solution of (4.1) satisfying $\phi \in L^\infty(\mathbb{R}^2)$. Let $T_u\mathcal{M}$ denote the tangent space of $\mathcal{M}$ at u. The nondegeneracy of the linearized system is equivalent to showing that any bounded solution $\phi = (\phi_1, \ldots, \phi_n)$ of (4.1) belongs to this space.

Theorem 4.1 *Suppose u is a solution of* (3.1) *and ϕ is a bounded solution of* (4.1). *Then $\phi \in T_u\mathcal{M}$.*

Proof For any solution $\phi = (\phi_1, \ldots, \phi_n)$ of (4.1), we define

$$\Phi_j = \sum_{k=1}^{n} a^{jk}\phi_k, \quad \forall\, 1 \le j \le n. \tag{4.2}$$

We have readily that bounded (ϕ_i) solves (4.1) if and only if (Φ_i) is a solution of

$$-4\Phi_{i,z\bar{z}} = \exp\left(\sum_{j=1}^{n} a_{ij}U_j\right) \times \sum_{j=1}^{n} a_{ij}\Phi_j \quad \text{in } \mathbb{R}^2,\ \Phi_i \in L^\infty(\mathbb{R}^2)\, \forall\, 1 \le i \le n. \tag{4.3}$$

Our idea is also to find some invariants which characterize all solutions of (4.3). Indeed, we find them by linearizing the above quantities W_k^n for U_i. Let

$$Y_1^n = e^{U_1}\left[\left(e^{-U_1}\Phi_1\right)^{(n+1)} - \left(e^{-U_1}\right)^{(n+1)}\Phi_1\right]$$

and

$$Y_{k+1}^n = -\frac{Y_{k,\bar{z}}^n + W_{k+1}^n\Phi_{k,z\bar{z}}}{U_{k,z\bar{z}}} \quad \text{for } 1 \le k \le n-1.$$

The quantities Y_k^n are well defined and we can prove by induction the following formula: *With any solution of* $LSU(n+1)$, there hold

$$\begin{aligned}
&Y_1^n = Y_{1,z}^{n-1} - Y_1^{n-1}U_{1,z} - W_1^{n-1}\Phi_{1,z},\\
&Y_k^n = \left(U_{k-1,z} - U_{k,z}\right)Y_k^{n-1} + Y_{k,z}^{n-1} + Y_{k-1}^{n-1} + \left(\Phi_{k-1,z} - \Phi_{k,z}\right)W_k^{n-1}\\
&\quad \text{for } 2 \le k \le n.
\end{aligned}$$

Moreover, for any solution of (4.3), we have

$$Y_n^n = \sum_{i=1}^{n}\Phi_{i,zz} - 2\sum_{i=1}^{n}U_{i,z}\Phi_{i,z} + \sum_{i=1}^{n-1}\left(\Phi_{i,z}U_{i+1,z} + U_{i,z}\Phi_{i+1,z}\right). \tag{4.4}$$

The proof is very similar as above for W_j^n, since each quantity Y_j^n is just the *linearized* version of W_j^n with respect to (U_i), as well as the involved equations, so we leave the details for interested readers.

Applying (4.3), it can be checked easily that

$$Y_{n,\bar{z}}^n = 0 \quad \text{in } \mathbb{R}^2,\ \text{for any solution of } LSU(n+1)\ (4.1).$$

Using the classification of u_i in Sect. 3 (see also [18]), we know that $e^{u_i} = O(z^{-4})$ at ∞. Since $\phi_i \in L^\infty(\mathbb{R}^2)$, the function $\sum_{1\le j\le n} a_{ij}e^{u_j}\phi_j \in L^1(\mathbb{R}^2)$. As before, we can express ϕ_i by integral representation and prove that $\lim_{|z|\to\infty} \nabla^k\phi_i = 0$ for any $k \ge 1$. Hence $\lim_{|z|\to\infty} \nabla^k\Phi_i = 0$ for any $k \ge 1$.

By similar argument as above, this implies that $Y_n^n = 0$ in $\mathbb{R}^2$ for any solution of (4.3), and we get successively $Y_k^n = 0$ in $\mathbb{R}^2$ for $1 \le k \le n-1$, recalling just $Y_{k,\bar z}^n = -U_{k,z\bar z}Y_{k+1}^n - \Phi_{k,z\bar z}W_{k+1}^n$ and $W_j^n = 0$ in $\mathbb{R}^2$ for any classical solution of (3.1). Since

$$0 = Y_1^n = e^{U_1}\left(e^{-U_1}\Phi_1\right)^{(n+1)} + W_1^n\Phi_1 = e^{U_1}\left(e^{-U_1}\Phi_1\right)^{(n+1)},$$

we conclude then $(e^{-U_1}\Phi_1)^{(n+1)} = 0$ in $\mathbb{R}^2$. As $e^{-U_1}\Phi_1$ is a real smooth function, we get

$$e^{-U_1}\Phi_1 = \sum_{i,j=0}^{n} b_{ij}z^i\bar z^j$$

with $b_{ij} = \overline{b_{ji}}$ for all $0 \le i, j \le n$. This yields

$$\Phi_1 \in \mathcal{L} = \left\{ e^{U_1}\left[\sum_{i,j=0}^{n} b_{ij}z^i\bar z^j\right],\ b_{ij}\in\mathbb{C},\ b_{ij} = \overline{b_{ji}},\ \forall\, 0\le i,j\le n \right\},$$

a linear space of dimension $(n+1)^2$. Once Φ_1 is fixed, as $-\Delta\Phi_1 = e^{u_1}(2\Phi_1 - \Phi_2)$ in $\mathbb{R}^2$, Φ_2 is uniquely determined, successively all Φ_i are uniquely determined, so is ϕ_i.

Moreover, the expression of e^{-U_1} given by the last section yields that the constant functions belong to $\mathcal{L}$. If $\Phi_1 \equiv \ell_1 \in \mathbb{R}$, by (4.3), successively we obtain $\Phi_i \equiv \ell_i \in \mathbb{R}$ for all $2 \le i \le n$. Using again the system (4.3), we must have

$$\sum_{j=1}^{n} a_{ij}\ell_j = 0, \quad \forall\, 1 \le i \le n,$$

which implies $\ell_j = 0$ for any $1 \le j \le n$, hence (Φ_i) can only be the trivial solution. Therefore, we need only to consider Φ_1 belonging to the algebraic complementary of $\mathbb{R}$ in $\mathcal{L}$, a linear subspace of dimension $n(n+2)$.

Finally, it is known that $T_u\mathcal{M}$, the tangent space of $u = (u_i)$ to the solution manifold $\mathcal{M}$ provides us a $n(n+2)$ dimensional family of bounded solutions to $LSU(n+1)$, so we can conclude that all the solutions of (4.1) form exactly a linear space of dimension $n(n+2)$. Theorem 4.1 is then proved. □

Remark 4.2 We note by the proof that Theorem 4.1 remains valid if we relax the condition $\phi_i \in L^\infty(\mathbb{R}^2)$ to the growth condition $\phi_i(z) = O(|z|^{1+\alpha})$ at infinity with $\alpha \in (0,1)$.

5 Classification of singular Toda system with one source

For the Toda system $SU(n+1)$ with one singular source (1.6), denote $A^{-1} = (a^{jk})$, the inverse matrix of A and define as before

$$U_j = \sum_{k=1}^{n} a^{jk} u_k, \quad \alpha_j = \sum_{k=1}^{n} a^{jk}\gamma_k \ \forall\, 1 \le j \le n, \tag{5.1}$$

where $u = (u_1, \ldots, u_n)$ is a solution of (1.6). So

$$-\Delta U_i = \exp\left(\sum_{j=1}^{n} a_{ij} U_j\right) - 4\pi\alpha_i\delta_0 \quad \text{in } \mathbb{R}^2 \tag{5.2}$$

with

$$\int_{\mathbb{R}^2} \exp\left(\sum_{j=1}^{n} a_{ij} U_j\right) dx = \int_{\mathbb{R}^2} e^{u_i} dx < \infty, \quad \forall\, i.$$

In this section, we will completely classify all the solutions of (1.6), and prove in the next section the nondegenerency of the corresponding linearized system. Here is the classification result.

Theorem 5.1 *Suppose that $\gamma_i > -1$ for $1 \le i \le n$, and $U = (U_1, \ldots, U_n)$ is a solution of* (5.2), *then we have*

$$|z|^{2\alpha_1} e^{-U_1} = \lambda_0 + \sum_{1 \le i \le n} \lambda_i |P_i(z)|^2 \quad \textit{in } \mathbb{C}^*, \tag{5.3}$$

where

$$P_i(z) = c_{i0} + \sum_{j=1}^{i-1} c_{ij} z^{\mu_1+\mu_2+\cdots+\mu_j} + z^{\mu_1+\mu_2+\cdots+\mu_i}, \quad c_{ij} \in \mathbb{C}, \tag{5.4}$$

$\mu_i = \gamma_i + 1$, *and* $\lambda_i \in \mathbb{R}$ *satisfies*

$$\lambda_i > 0, \quad \lambda_0\lambda_1\cdots\lambda_n = 2^{-n(n+1)} \times \prod_{1 \le i \le j \le n} \left(\sum_{k=i}^{j} \mu_k\right)^{-2}. \tag{5.5}$$

Conversely, U_1 defined by (5.3)–(5.5) *generates a solution (U_i) of* (5.2).

The proof of Theorem 5.1 is divided in several steps. Suppose $U = (U_1, \dots, U_n)$ is a solution of (5.2).

5.1 Step 1

We will prove that $e^{-U_1} = f$ verifies the differential equation as follows:

$$f^{(n+1)} + \sum_{k=0}^{n-1} \frac{w_k}{z^{n+1-k}} f^{(k)} = 0 \quad \text{in } \mathbb{C}^*, \tag{5.6}$$

where w_k are real constants only depending on all γ_i and $f^{(i)}$ denotes the i-th order derivative of f w.r.t. z.

Lemma 5.2 *Let (U_j) be given by (5.1), with (u_i) a solution of (1.6). Define $Z_n = W_n^n$ and by iteration*

$$Z_k = W_k^n + U_{k,z} Z_{k+1} + \sum_{j=k}^{n-2} W_k^j Z_{j+2}, \quad \forall\, k = n-1, n-2, \dots, 1. \tag{5.7}$$

Then Z_k are holomorphic in $\mathbb{C}^*$. More precisely, there exist $w_k \in \mathbb{C}$ such that

$$Z_k = \frac{w_k}{z^{n+2-k}} \quad \text{in } \mathbb{C}^*, \text{for any } 1 \le k \le n,$$

where w_k only depends on γ_j.

Here W_k^j $(1 \le k \le j \le n)$, considered as functional of $(U_1, U_2, \dots, U_n)$ and their derivatives, are the invariants constructed in Sect. 2 for Toda system $SU(n+1)$.

Proof First, we recall that

$$\begin{aligned} W_1^m &= -e^{U_1}\left(e^{-U_1}\right)^{(m+1)} \quad \text{for } 1 \le m \le n, \\ W_{k+1}^m &= -\frac{W_{k,\bar z}^m}{U_{k,z\bar z}} \quad \text{for } 1 \le k \le m-1. \end{aligned} \tag{5.8}$$

Using (3.3), Z_n is holomorphic in $\mathbb{C}^*$ and by Lemma 2.1

$$W_{k,\bar z}^k = -U_{k,z\bar z} U_{k+1,z}, \quad \text{for any } 1 \le k \le n-1.$$

Consequently, in $\mathbb{C}^*$ there holds by (5.8),

$$\begin{aligned} 0 &= W_{n-1,\bar z}^n + U_{n-1,z\bar z} W_n = W_{n-1,\bar z}^n + U_{n-1,z\bar z} Z_n = \left(W_{n-1}^n + U_{n-1,z} Z_n\right)_{\bar z} \\ &= Z_{n-1,\bar z}. \end{aligned}$$

So Z_{n-1} is also holomorphic in $\mathbb{C}^*$. Suppose that $Z_{\ell+1}$ are holomorphic in $\mathbb{C}^*$ for $k \le \ell \le n-2$, then we have in $\mathbb{C}^*$,

$$\begin{aligned}
Z_{k,\bar{z}} &= \left(W_k^n + U_{k,z} Z_{k+1} + \sum_{j=k}^{n-2} W_k^j Z_{j+2} \right)_{\bar{z}} \\
&= W_{k,\bar{z}}^n + U_{k,z\bar{z}} Z_{k+1} + W_{k,\bar{z}}^k Z_{k+2} + \sum_{j=k+1}^{n-2} W_{k,\bar{z}}^j Z_{j+2} \\
&= -U_{k,z\bar{z}} W_{k+1}^n + U_{k,z\bar{z}} Z_{k+1} - U_{k,z\bar{z}} U_{k+1,z} Z_{k+2} \\
&\quad - \sum_{j=k+1}^{n-2} U_{k,z\bar{z}} W_{k+1}^j Z_{j+2} \\
&= U_{k,z\bar{z}} \left(Z_{k+1} - W_{k+1}^n - U_{k+1,z} Z_{k+2} - \sum_{j=k+1}^{n-2} W_{k+1}^j Z_{j+2} \right) = 0.
\end{aligned}$$

The last line comes from the definition of Z_{k+1}. Thus, Z_k is holomorphic in $\mathbb{C}^*$ for all $1 \le k \le n$.

Next, we want to show that

$$Z_k = \frac{w_k}{z^{n+2-k}} \tag{5.9}$$

for some real constant w_k depending on γ_j. Define

$$V_j = U_j - 2\alpha_j \log |z|, \quad \forall\, 1 \le j \le n. \tag{5.10}$$

So

$$\begin{aligned}
-\Delta V_i &= -4U_{i,z\bar{z}} + 4\pi \alpha_i \delta_0 \\
&= \exp\left(\sum_{j=1}^n a_{ij} U_j \right) + 4\pi \alpha_i \delta_0 - 4\pi \sum_{j=1}^n \left(a^{ij} \gamma_j \delta_0 \right) \\
&= |z|^{2\gamma_j} \exp\left(\sum_{j=1}^n a_{ij} V_j \right)
\end{aligned}$$

with

$$\begin{aligned}
\int_{\mathbb{R}^2} |z|^{2\gamma_j} \exp\left(\sum_{j=1}^n a_{ij} V_j \right) dx &= \int_{\mathbb{R}^2} \exp\left(\sum_{j=1}^n a_{ij} U_j \right) dx \\
&= \int_{\mathbb{R}^2} e^{u_i} dx < \infty, \quad \forall\, 1 \le i \le n.
\end{aligned}$$

As $\gamma_i > -1$, applying Brezis-Merle's argument in [4] to the system of V_i, we have $V_i \in C^{0,\alpha}$ in $\mathbb{C}$ for some $\alpha \in (0, 1)$ and they are upper bounded over $\mathbb{C}$. This implies that we can express V_i by the integral representation formula. Moreover, by scaling argument and elliptic estimates, we have for all $1 \leq i \leq n$,

$$\begin{aligned} &\nabla^k V_i(z) = O\left(1 + |z|^{2+2\gamma_i - k}\right) \quad \text{near } 0 \quad \text{and} \\ &\nabla^k V_i(z) = O\left(z^{-k}\right) \quad \text{near } \infty, \forall\, k \geq 1. \end{aligned} \tag{5.11}$$

By (2.3) and (5.11), it is obvious that

$$W_k^k(z) = \frac{C_k + o(1)}{z^2} \quad \text{near } 0 \quad \text{and} \quad W_k^k(z) = O\left(z^{-2}\right) \quad \text{near } \infty,$$

where C_k are real constants depending on γ_j only. Thus considering $z^2 W_k^k$, we get

$$W_k^k(z) = \frac{C_k}{z^2} \quad \text{in } \mathbb{C}. \tag{5.12}$$

In particular, Z_n is determined uniquely. To determine Z_k for $k < n$, we can do the induction step on k. By using (5.7), the definition of W_k^j, (2.5) and (5.11), we obtain

$$Z_k = \frac{w_k + o(1)}{z^{n+2-k}} \quad \text{near } 0 \quad \text{and} \quad Z_k = O\left(\frac{1}{z^{n+2-k}}\right) \quad \text{at } \infty,$$

where w_k is a real constant and depends only on γ_j. By the Liouville theorem, (5.9) is proved. □

Proof of (5.6) To prove that f satisfies the ODE, we use (5.9) with $k = 1$. By the above lemma, for $k = 1$,

$$\begin{aligned} \frac{w_1}{z^{n+1}} = Z_1 &= W_1^n + U_{1,z} Z_2 + \sum_{j=1}^{n-2} W_1^j Z_{j+2} \\ &= W_1^n + \frac{w_2}{z^n} U_{1,z} + \sum_{j=1}^{n-2} \frac{w_{j+2}}{z^{n-j}} W_1^j. \end{aligned}$$

As $f = e^{-U_1}$, we have $-U_{1,z} f = f'$ and $W_1^j f = -f^{(j+1)}$ by definition for all $1 \le j \le n$. Multiplying the above equation with f, we get

$$\frac{w_1}{z^{n+1}} f = -f^{(n+1)} - \frac{w_2}{z^n} f' - \sum_{j=1}^{n-2} \frac{w_{j+2}}{z^{n-j}} f^{(j+1)},$$

or equivalently

$$f^{(n+1)} + \sum_{k=0}^{n-1} Z_{k+1} f^{(k)} = f^{(n+1)} + \sum_{k=0}^{n-1} \frac{w_{k+1}}{z^{n+1-k}} f^{(k)} = 0.$$

Up to change the definition of w_k, we are done. □

5.2 Step 2

We will prove that the fundamental solutions for (5.6) are just given by $f_i(z) = z^{\beta_i}$ with

$$\beta_0 = -\alpha_1, \qquad \beta_i = \alpha_i - \alpha_{i+1} + i \quad \text{for } 1 \le i \le (n-1), \qquad \beta_n = \alpha_n + n, \tag{5.13}$$

or equivalently we have $P(\beta_i) = 0$ where

$$P(\beta) = \beta(\beta-1)\dots(\beta-n) + \sum_{i=0}^{n-1} w_k \beta(\beta-1)\dots(\beta-k+1).$$

By (5.13), β_i satisfies

$$\beta_i - \beta_{i-1} = \gamma_i + 1 > 0 \quad \text{for all } 1 \le i \le n. \tag{5.14}$$

Let

$$f = \lambda_0 |z|^{-2\alpha_1} + \sum_{i=1}^{n} \lambda_i |P_i(z)|^2, \tag{5.15}$$

with

$$P_i(z) = z^{(\mu_1+\mu_2+\dots+\mu_i-\alpha_1)} + \sum_{j=0}^{i-1} c_{ij} z^{\mu_1+\dots+\mu_j-\alpha_1},$$

where $\mu_i = 1 + \gamma_i > 0$. Note that

$$\frac{|P_i(z)|}{|z|^{\mu_1+\dots+\mu_i-\alpha_1}} = \left| 1 + \sum_{j=0}^{i-1} c_{ij} z^{-\mu_{j+1}-\dots-\mu_i} \right| \quad \text{in } \mathbb{C}^*.$$

Since $|P_i(z)|$ is a single-valued function, we have $c_{ij} = 0$ if $\mu_{j+1} + \cdots + \mu_i \notin \mathbb{N}$. In the following, we let $f^{(p,q)}$ denote $\partial_{\bar{z}}^q \partial_z^p f$. For any f of (5.15), we define, if possible, $U = (U_1, \ldots, U_n)$ by

$$e^{-U_1} = f \quad \text{and} \quad e^{-U_k} = 2^{k(k-1)} \det_k(f) \quad \text{for } 2 \le k \le n, \tag{5.16}$$

where

$$\det_k(f) = \det\left(f^{(p,q)}\right)_{0 \le p,q \le k-1} \qquad \text{for } 1 \le k \le n+1. \tag{5.17}$$

Theorem 5.3 *Let* $\det_k(f)$ *be defined by* (5.17) *with* f *given by* (5.15) *and* $\lambda_i > 0$ *for all* $0 \le i \le n$. *Then we have* $\det_k(f) > 0$ *in* $\mathbb{C}^*$, $\forall\, 1 \le k \le n$. *Furthermore,* $U = (U_1, \ldots, U_n)$ *defined by* (5.16) *satisfies* (5.2) *if and only if* (5.5) *holds.*

Before going into the details of proof of Theorem 5.3, we first explain how to construct solutions of Toda system from f via the formula (5.16). Here we follow the procedure from [37]. For any function f, we define $\det_k(f)$ by (5.17). Then we have

$$\det_{k+1}(f) = \frac{\det_k(f)\partial_{z\bar{z}} \det_k(f) - \partial_z \det_k(f)\partial_{\bar{z}} \det_k(f)}{\det_{k-1}(f)} \quad \text{for } k \ge 1. \tag{5.18}$$

The above formula comes from a general formula for the determinant of a $(k+1) \times (k+1)$ matrix. We explain it in the following. Let $\mathcal{N} = (c_{i,j})$ be a $(k+1) \times (k+1)$ matrix:

$$\mathcal{N} = \begin{pmatrix} M_1 & \vec{\mathbf{u}} & \vec{\mathbf{v}} \\ \vec{\mathbf{s}} & c_{k,k} & c_{k,k+1} \\ \vec{\mathbf{t}} & c_{k+1,k} & c_{k+1,k+1} \end{pmatrix}$$

where $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ stands for the column vectors consisting of first $(k-1)$ entries of the k-th column and $(k+1)$-th column respectively, and $\vec{\mathbf{s}}$ and $\vec{\mathbf{t}}$ stand for row vectors consisting of the first $(k-1)$ entries of the k-th row and $(k+1)$-th row respectively. We let

$$\mathcal{N}_1 = \begin{pmatrix} M_1 & \vec{\mathbf{u}} \\ \vec{\mathbf{s}} & c_{k,k} \end{pmatrix}, \qquad \mathcal{N}_2 = \begin{pmatrix} M_1 & \vec{\mathbf{v}} \\ \vec{\mathbf{t}} & c_{k+1,k+1} \end{pmatrix},$$

$$\mathcal{N}_1^* = \begin{pmatrix} M_1 & \vec{\mathbf{u}} \\ \vec{\mathbf{t}} & c_{k+1,k} \end{pmatrix}, \qquad \mathcal{N}_2^* = \begin{pmatrix} M_1 & \vec{\mathbf{v}} \\ \vec{\mathbf{s}} & c_{k,k+1} \end{pmatrix}.$$

Then we have

$$\det(\mathcal{N}) \det(M_1) = \det(\mathcal{N}_1) \det(\mathcal{N}_2) - \det(\mathcal{N}_1^*) \det(\mathcal{N}_2^*).$$

Since the proof is elementary, we omit it. Clearly, (5.18) follows from the above formula immediately.

Suppose that $\det_k(f) > 0$ for $1 \le k \le n$ and $\det_{n+1}(f) = 2^{-n(n+1)}$. Define U_1 by $f = e^{-U_1}$. As $-e^{-2U_1}U_{1,z\bar{z}} = ff_{z\bar{z}} - f_z f_{\bar{z}}$, then

$$-4U_{1,z\bar{z}} = e^{2U_1-U_2} \quad \text{if and only if} \quad e^{-U_2} = 4(ff_{z\bar{z}} - f_z f_{\bar{z}}) = 4\det_2(f).$$

By the induction on k, $2 \le k \le n$, we have

$$\begin{aligned}
-4e^{-2U_k}U_{k,z\bar{z}} &= 4e^{-2U_k}\big[\log\det_k(f)\big]_{z\bar{z}} \\
&= 4\cdot 2^{2k(k-1)}\big[\det_k(f)\partial_{z\bar{z}}\det_k(f) - \partial_z\det_k(f)\partial_{\bar{z}}\det_k(f)\big] \\
&= 2^{2k(k-1)+2}\det_{k+1}(f)\,\det_{k-1}(f) \\
&= 2^{(k+1)k}e^{-U_{k-1}}\det_{k+1}(f).
\end{aligned}$$

Thus, U_k satisfies $\triangle U_{k,z\bar{z}} + e^{2U_k-U_{k+1}-U_{k-1}} = 0$ in $\mathbb{C}^*$ if and only if $e^{-U_{k+1}} = 2^{(k+1)k}\det_{k+1}(f)$. For the last equation $k = n$, we have

$$-4e^{-2U_n}U_{n,z\bar{z}} = 2^{(n+1)n}e^{-U_{n-1}}\det_{n+1}(f).$$

Thus, U_n satisfies $\triangle U_n + e^{2U_n-U_{n-1}} = 0$ in $\mathbb{C}^*$ if and only if $\det_{n+1}(f) = 2^{-n(n+1)}$.

Therefore, assume that $U = (U_k)$ given by (5.16), (5.17) and (5.15) is a solution of the Toda system (5.2), to get the equality in (5.5), it is equivalent to show

$$\det_{n+1}(f) = \lambda_0\lambda_1\cdots\lambda_n \times \prod_{1\le i\le j\le n}\left(\sum_{k=i}^{j}\mu_k\right)^2 \tag{5.19}$$

for f given by (5.15). We have first

Lemma 5.4 *Let $g = |z|^{2\beta}f$ with $\beta \in \mathbb{R}$, and f be a complex analytic function in $\mathbb{C}^*$, there holds*

$$\det_k(g) = |z|^{2k\beta}\det_k(f) \quad \textit{in } \mathbb{C}^*, \forall\, k \in \mathbb{N}^*. \tag{5.20}$$

Proof This is obviously true for $k = 1$, we can check also easily for $k = 2$. Suppose that the above formula holds for $1 \le \ell \le k$, then by formula (5.18),

$$\begin{aligned}
\det_{k+1}(g) &= \frac{\det_k(g)\partial_{z\bar{z}}\det_k(g) - \partial_z\det_k(g)\partial_{\bar{z}}\det_k(g)}{\det_{k-1}(g)} \\
&= \frac{\det_2(\det_k(g))}{\det_{k-1}(g)}
\end{aligned}$$

$$= \frac{\det_2(|z|^{2k\beta}\det_k(f))}{|z|^{2(k-1)\beta}\det_{k-1}(f)}$$
$$= |z|^{2(k+1)\beta}\frac{\det_2(\det_k(f))}{\det_{k-1}(f)}$$
$$= |z|^{2(k+1)\beta}\det_{k+1}(f).$$

The equality (5.20) holds when $\det_{k-1}(f) \neq 0$. □

Thanks to (5.20), to prove (5.19), it is enough to prove the following: Let

$$\widetilde{f} = \lambda_0 + \sum_{i=1}^{n} \lambda_i |P_i(z)|^2 \quad \text{in } \mathbb{C} \tag{5.21}$$

with P_i given by (5.4), then

$$\det_{n+1}(\widetilde{f}) = \lambda_0\lambda_1\cdots\lambda_n \times \prod_{1\le i\le j\le n}\left(\sum_{k=i}^{j}\mu_k\right)^2 \times |z|^{2n\gamma_1+2(n-1)\gamma_2+\cdots+2\gamma_n}. \tag{5.22}$$

Here we used $(n+1)\alpha_1 = n\gamma_1 + (n-1)\gamma_2 + \cdots + \gamma_n$ for $SU(n+1)$.

Proof of (5.22) We proceed by induction. Let $n = 1$, we have $P_1 = c_0 + z^{\mu_1}$, so

$$\det_2(\widetilde{f}) = \det_2\left(\lambda_0 + \lambda_1|P_1|^2\right) = |z|^{-4\alpha_1}\lambda_0\lambda_1|P_1'|^2 = \lambda_0\lambda_1\mu_1^2|z|^{2(\mu_1-1)}$$
$$= \lambda_0\lambda_1\mu_1^2|z|^{2\gamma_1}$$

since $\mu_1 - 1 = \gamma_1$. Then (5.22) holds true for $n = 1$.

Suppose that (5.22) is true for some $(n-1) \in \mathbb{N}^*$, we will prove (5.22) for the range n. Define $L_k(P)$ to be the vertical vector $(P, \partial_z P, \ldots, \partial_z^k P) \in \mathbb{C}^{k+1}$ for any smooth function P and $k \in \mathbb{N}^*$. Denote $P_0 \equiv 1$, there holds

$$\det_{n+1}(\widetilde{f}) = \sum_{0\le i_k\le n, i_p\neq i_q} \lambda_{i_0}\lambda_{i_1}\cdots\lambda_{i_n}\det\Big(\overline{P_{i_0}}L_n(P_{i_0}), \partial_{\bar z}\overline{P_{i_1}}L_n(P_{i_1}), \ldots,$$
$$\partial_{\bar z}^n\overline{P_{i_n}}L_n(P_{i_n})\Big)$$
$$= \lambda_0\lambda_1\cdots\lambda_n \sum_{1\le i_k\le n, i_p\neq i_q} \det\Big(\overline{P_0}L_n(P_0), \partial_{\bar z}\overline{P_{i_1}}L_n(P_{i_1}), \ldots,$$
$$\partial_{\bar z}^n\overline{P_{i_n}}L_n(P_{i_n})\Big).$$

The last line is due to $P_0 \equiv 1$. Let e_1 be the vertical vector $(1, 0, \ldots, 0)$, we have

$$\begin{aligned}
&\det\left(\overline{P_0} L_n(P_0), \partial_{\bar z}\overline{P_{i_1}} L_n(P_{i_1}), \ldots, \partial_{\bar z}^n \overline{P_{i_n}} L_n(P_{i_n})\right)\\
&\quad = \det\left(e_1, \partial_{\bar z}\overline{P_{i_1}} L_n(P_{i_1}), \ldots, \partial_{\bar z}^n \overline{P_{i_n}} L_n(P_{i_n})\right)\\
&\quad = \det\left(\overline{P'_{i_1}} L_{n-1}(P'_{i_1}), \ldots, \overline{P'_{i_n}} L_{n-1}(P'_{i_n})\right).
\end{aligned}$$

Therefore $\det_{n+1}(\widetilde{f}) = \lambda_0\lambda_1\cdots\lambda_n \det_n(h)$ with $h = \sum_{1\le i\le n} |P'_i|^2$. Moreover, for $i \ge 1$,

$$\begin{aligned}
P'_i &= \sum_{k=1}^{i-1} (\mu_1 + \mu_2 + \cdots + \mu_k) c_{ik} z^{\mu_1+\mu_2+\cdots+\mu_k - 1}\\
&\quad + (\mu_1 + \mu_2 + \cdots + \mu_i) z^{\mu_1+\mu_2+\cdots+\mu_i - 1}\\
&= (\mu_1 + \mu_2 + \cdots + \mu_i) z^{\mu_1 - 1} \widetilde{P}_i,
\end{aligned}$$

where

$$\widetilde{P}_i = z^{\mu_2+\cdots+\mu_i} + \sum_{k=1}^{i-1} \widetilde{c}_{ik} z^{\mu_2+\cdots+\mu_k} \quad \text{with } \widetilde{c}_{ij} \in \mathbb{C}.$$

This means that

$$\begin{aligned}
h &= |z|^{2\gamma_1} \left[\sum_{i=1}^{n} (\mu_1 + \mu_2 + \cdots + \mu_i)^2 |\widetilde{P}_i|^2\right]\\
&= |z|^{2\gamma_1} \left[\mu_1^2 + \sum_{i=1}^{n-1} (\mu_1 + \mu_2 + \cdots + \mu_{i+1})^2 |\widetilde{P}_{i+1}|^2\right] := |z|^{2\gamma_1} \widetilde{h},
\end{aligned}$$

hence $\widetilde{h}$ is in the form of (5.21) with $(n-1)$. Consequently, by the induction hypothesis, we get

$$\begin{aligned}
\det_{n+1}(\widetilde{f}) &= \lambda_0\lambda_1\cdots\lambda_n \det_n(h)\\
&= \lambda_0\lambda_1\cdots\lambda_n |z|^{2n\gamma_1} \det_n(\widetilde{h})\\
&= \lambda_0\lambda_1\cdots\lambda_n |z|^{2n\gamma_1}\\
&\quad \times \prod_{1\le k\le n} (\mu_1 + \mu_2 + \cdots + \mu_k)^2 \times \prod_{2\le i\le j\le n} \left(\sum_{k=i}^{j} \mu_k\right)^2\\
&\quad \times |z|^{2(n-1)\gamma_2 + \cdots + 2\gamma_n},
\end{aligned}$$

which yields clearly the equality (5.22). $\square$

On the other hand, assume that (5.5) holds true, using the above analysis and (5.19), we see that U defined by (5.16) and (5.15) is a solution of (5.2) in $\mathbb{C}^*$ provided that $\det_k(f) > 0$ in $\mathbb{C}^*$.

First we make a general calculus of $\det_k(g)$ with

$$g = \sum_{i,j=0}^{n} m_{ij} f_i \overline{f_j}, \quad \text{where } m_{ij} = \overline{m_{ji}} \text{ for all } 0 \le i, j \le n, \tag{5.23}$$

where $f_i(z) = z^{\beta_i}$. Let $M = (m_{ij})_{0\le i,j\le n}$ and $J = (z_{ij})_{0\le i,j\le n}$ with $z_{ij} = (z^{\beta_j})^{(i)}$. Let $\mathcal{N}_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}$ be the $k\times k$ sub matrix $(b_{ij})_{i=i_1,\ldots,i_k,j=j_1,\ldots,j_k}$, for any matrix $\mathcal{N} = (b_{ij})$, we denote also $\mathcal{N}_{i_1,\ldots,i_k}$ the $k\times(n+1)$ sub matrix by taking the rows $i_1,\ldots,i_k$ of $\mathcal{N}$, and $\mathcal{N}^t$ means the transposed matrix of $\mathcal{N}$.

As $g^{(p,q)} = \sum m_{ij} f_i^{(p)} \overline{f_j^{(q)}}$. For $1\le k\le n$, we can check easily that

$$\left(g^{(p,q)}\right)_{0\le p,q\le k} = J_{0,1,\ldots,k} M \overline{J_{0,1,\ldots,k}}^{t},$$

and

$$\begin{aligned}
&\det\left(J_{0,1,\ldots,k} M \overline{J_{0,1,\ldots,k}}^{t}\right)\\
&\quad = \sum_{0\le i_0<i_1<\cdots<i_k\le n, 0\le j_0<j_1<\cdots<j_k\le n} \det\left(J_{0,1,\ldots,k}^{i_0,i_1,\ldots,i_k} M_{i_0,i_1,\ldots i_k}^{j_0,j_1,\ldots,j_k} \overline{J_{0,1,\ldots,k}^{j_0,j_1,\ldots,j_k}}^{t}\right)\\
&\quad = \sum_{0\le i_0<i_1<\cdots<i_k\le n, 0\le j_0<j_1<\cdots<j_k\le n} \det\left(M_{i_0,i_1,\ldots,i_k}^{j_0,j_1,\ldots,j_k}\right) \det\left(J_{0,1,\ldots,k}^{i_0,i_1,\ldots,i_k}\right)\\
&\qquad \times \overline{\det\left(J_{0,1,\ldots,k}^{j_0,j_1,\ldots,j_k}\right)}.
\end{aligned} \tag{5.24}$$

Moreover, exactly as for (5.22), by induction, we can prove that

$$\begin{aligned}
\det\left(J_{0,1,\ldots,k}^{i_0,i_1,\ldots,i_k}\right) &= \prod_{0\le p<q\le k} \left(\beta_{i_q} - \beta_{i_p}\right)\\
&\quad \times z^{(k+1)\beta_{i_0} + k(\beta_{i_1}-\beta_{i_0}-1)+\cdots+(\beta_{i_k}-\beta_{i_{k-1}}-1)}\\
&= \prod_{0\le p<q\le k} \left(\beta_{i_q} - \beta_{i_p}\right) \times z^{\beta_{i_0}+\beta_{i_1}+\cdots+\beta_{i_k} - \frac{k(k+1)}{2}}.
\end{aligned} \tag{5.25}$$

Given f by (5.15) with λ_i satisfying (5.5), we will prove that $\det_k(f)>0$ in $\mathbb{C}^*$. Clearly, $f>0$ in $\mathbb{C}^*$ and $f=\sum_{0\le i,j\le n} m_{ij} f_i \overline{f_j}$ where

$$M=(m_{ij})=B\overline{B}^t,$$
$$B=(b_{ij}) \text{ with } b_{ii}=\sqrt{\lambda_i}, b_{ij}=\sqrt{\lambda_i}c_{ji} \text{ for } j>i, b_{ij}=0 \text{ for } j<i.$$

For $1\le k\le n$, denote $\mathcal{B}=J_{0,1,\ldots,k}B$, we can check that

$$\begin{aligned}\det_{k+1}(f)&=\det\left(J_{0,1,\ldots,k}M\overline{J_{0,1,\ldots,k}}^t\right)\\&=\det\left(\mathcal{B}\overline{\mathcal{B}}^t\right)\\&=\sum_{0\le i_0<i_1<\cdots<i_k\le n}\det\left(\mathcal{B}^{i_0,i_1,\ldots,i_k}_{0,1,\ldots,k}\right)\det\left(\overline{\mathcal{B}^{i_0,i_1,\ldots,i_k}_{0,1,\ldots,k}}^t\right)\\&=\sum_{0\le i_0<i_1<\cdots<i_k\le n}\left|\det\left(\mathcal{B}^{i_0,i_1,\ldots,i_k}_{0,1,\ldots,k}\right)\right|^2.\end{aligned}$$

As $\det_{n+1}(f)=2^{-n(n+1)}\neq 0$ by (5.5) and (5.19), the rank of the matrix $\mathcal{B}$ must be $(k+1)$ in $\mathbb{C}^*$, hence for any $z\in\mathbb{C}^*$, we have $0\le i_0<i_1<\cdots<i_k\le n$, such that $\det(\mathcal{B}^{i_0,i_1,\ldots,i_k}_{0,1,\ldots,k})(z)\neq 0$, thus $\det_{k+1}(f)>0$ in $\mathbb{C}^*$.

To complete the proof of Theorem 5.3, it remains to compute the strength of the singularity. Notice that $M=B\overline{B}^t$ is a positive hermitian matrix, since $\lambda_i>0$. By the formulas (5.24), (5.25), as $i_p\ge p$, $j_p\ge p$, β_i are increasing and

$$\sum_{p=0}^{k}\beta_p-\frac{k(k+1)}{2}=-(k+1)\alpha_0+k\gamma_1+(k-1)\gamma_2+\cdots+\gamma_k=-\alpha_{k+1},$$

we get

$$\det_{k+1}(f)=\prod_{0\le p<q\le k}(\beta_q-\beta_p)|z|^{-2\alpha_{k+1}}\left[\zeta_k+o(1)\right]\quad\text{as } z\to 0,\tag{5.26}$$

with $\zeta_k=\det(M^{0,1,\ldots,k}_{0,1,\ldots,k})>0$. This implies

$$U_{k+1}=-2\alpha_{k+1}\log|z|+O(1)\quad\text{near } 0.$$

Hence $U=(U_1,\ldots,U_n)$ satisfies (5.2) in $\mathbb{C}$. This completes the proof of Theorem 5.3. $\square$

By Theorem 5.3, we have proved that any f given by (5.15) verifying (5.5) is a solution of (5.6), because $U=(U_1,\dots,U_n)$ defined by (5.16) is a solution of the Toda system. In particular, it is the case for $f=\sum_{0\le i\le n}\lambda_i|z|^{2\beta_i}$ satisfying (5.5), with β_i are given by (5.13). Let L denote the linear operator of the differential equation (5.6). Then

$$0=\overline{L}L(f)=\sum_{i=0}^{n}\lambda_i|L(z^{\beta_i})|^2,$$

which implies $L(z^{\beta_i})=0$, $\forall\, 0\le i\le n$. Thus Step 2 is proved.

5.3 Step 3

Suppose $U=(U_1,\dots,U_n)$ is a solution of (5.2), we will prove that $f=e^{-U_1}$ can be written as the form of (5.15). For any solution (U_i), as $f=e^{-U_1}>0$ satisfies (5.6), we have

$$f=\sum_{i,j=0}^{n}m_{ij}f_i\overline{f_j},\quad \text{where } m_{ij}=\overline{m_{ji}} \text{ for all } 0\le i,j\le n,$$

where $f_i(z)=z^{\beta_i}$ is a set of fundamental solutions of (5.6).

We want to prove that f can be written as a sum of $|P_i(z)|^2$, which is not true in general, because even a positive polynomial in $\mathbb{C}$ cannot be written always as sum of squares of module of polynomials. For example, it is the case for $2|z|^6-|z|^4-|z|^2+2$. It means that, we need to use further informations from the Toda system. In fact, we will prove that $M=(m_{ij})$ is a positive hermitian matrix.

With V_i given by (5.10),

$$\begin{aligned} e^{V_1}&=|z|^{2\alpha_1}e^{-U_1}=|z|^{2\alpha_1}f\\ &=m_{00}+\sum_{i=1}^{n}m_{ii}|z|^{2(\beta_i-\beta_0)}+2\sum_{0\le i<j\le n}\mathrm{Re}\left(m_{ij}\bar{z}^{\beta_j-\beta_i}\right)|z|^{2(\beta_i-\beta_0)}.\end{aligned}$$

Take $z=0$, we get $m_{00}>0$. Let $J=(z_{ij})_{0\le i,j\le n}$ with $z_{ij}=(z^{\beta_j})^{(i)}$ as in Step 2. Using (5.24), (5.25) and the monotonicity of β_i, exactly as before, we get, for $1\le k\le n-1$

$$\det{}_{k+1}(f)=\prod_{0\le p<q\le k}(\beta_q-\beta_p)|z|^{-2\alpha_{k+1}}\left[\det\left(M^{0,1,\dots,k}_{0,1,\dots,k}\right)+o(1)\right],$$

as $z\to 0$.

Recall that $e^{-U_{k+1}} = 2^{k(k+1)}\det_{k+1}(f)$ and V_{k+1} is defined by (5.10),

$$\frac{e^{-V_{k+1}(0)}}{2^{2(k+1)k}} = \left[|z|^{2\alpha_{k+1}}\det_{k+1}(f)\right]_{z=0} = \det\left(M^{0,1,\dots,k}_{0,1,\dots k}\right) \times \prod_{0\le p<q\le k} \left(\beta_q - \beta_p\right)^2,$$

which yields

$$\det\left(M^{0,1,\dots,k}_{0,1,\dots,k}\right) > 0, \quad \forall\, 1 \le k \le n-1. \tag{5.27}$$

Similarly, when $k = n$, noticing that

$$\sum_{p=0}^{n} \beta_p - \frac{n(n+1)}{2} = 0,$$

we obtain

$$2^{-n(n+1)} = \det_{n+1}(f) = \det(M) \times \prod_{0\le p<q\le n} \left(\beta_q - \beta_p\right)^2, \tag{5.28}$$

hence $\det(M) > 0$. Combining with (5.27) and $m_{00} > 0$, it is well known that M is a positive hermitian matrix. Consequently, we can decompose $M = B\overline{B}^t$ with a upper triangle matrix $B = (b_{ij})$ where $b_{ii} > 0$. To conclude, we have

$$f = \sum_{i,j=0}^{n} m_{ij} f_i \overline{f_j} = \sum_{k=0}^{n} |Q_k|^2, \quad \text{where } Q_k = \sum_{i=0}^{k} b_{ik} f_i.$$

It is equivalent to saying that f is in the form of (5.15) with $\lambda_i = b_{ii}^2 > 0$. Combining with Theorem 5.3, the proof of Theorem 5.1 is finished. □

6 Quantization and nondegeneracy

Here we will prove Theorem 1.3. We first prove the quantization of the integral of e^{u_i}. By (5.24), (5.25) and again the monotonicity of β_i with f given by (5.15), we have for $1 \le k \le n$,

$$e^{-U_k} = 2^{k(k-1)}\det_k(f) = |z|^{2(\beta_{n-k+1}+\dots+\beta_n)-k(k-1)}\left[c_k + o(1)\right], \quad \text{as } |z| \to \infty,$$

where

$$c_k = 2^{k(k-1)}\lambda_{n-k+1}\lambda_{n-k+2}\cdots\lambda_n \times \prod_{n-k+1\le q<p\le n} (\beta_p - \beta_q)^2 > 0.$$

Thus, as $-\Delta U_k = e^{u_k} - 4\pi\alpha_k\delta_0$

$$\int_{\mathbb{R}^2} e^{u_k}dx = 4\pi\alpha_k + \lim_{R\to+\infty}\int_{\partial B_R}\frac{\partial U_k}{\partial\nu}ds$$
$$= 4\pi\left[\alpha_k + \beta_{n-k+1} + \cdots + \beta_n - \frac{k(k-1)}{2}\right]$$
$$= 4\pi\Big[\alpha_k + \alpha_{n-k+1} + k(n-k+1)\Big].$$

Therefore,

$$\sum_{k=1}^{n} a_{ik}\int_{\mathbb{R}^2} e^{u_k}dx = 4\pi(2+\gamma_k+\gamma_{n+1-k}),$$

which implies

$$u_k(z) = -4\pi(2+\gamma_{n+1-k})\log|z| + O(1), \quad \text{for large } |z|.$$

This proves the quantization.

To prove the nondegeneracy, we let (u_i) be a solution of the singular Toda system $SU(n+1)$ (1.6) and ϕ_i be solutions of the linearized system $LSU(n+1)$:

$$-\Delta\phi_i = \sum_{j=1}^{n} a_{ij}e^{u_j}\phi_j \quad \text{in } \mathbb{R}^2, \phi_i \in L^\infty(\mathbb{R}^2)\ \forall\, 1\le i\le n, \tag{6.1}$$

or equivalently

$$-4\Phi_{i,z\bar z} = \exp\left(\sum_{j=1}^{n} a_{ij}U_j\right)\times\sum_{j=1}^{n}\big(a_{ij}\Phi_j\big)$$
$$\text{in } \mathbb{R}^2,\ \Phi_i\in L^\infty(\mathbb{R}^2),\ \forall\, 1\le i\le n,$$

where U_j are defined by (5.1) and Φ_j defined by (4.2).

We will use the quantities $Y_1^j = e^{U_1}[(e^{-U_1}\Phi_1)^{(j+1)} - (e^{-U_1})^{(j+1)}\Phi_1]$ for $1\le j\le n$, and

$$Y_{k+1}^j = -\frac{Y_{k,\bar z}^j + W_{k+1}^j\Phi_{k,z\bar z}}{U_{k,z\bar z}} \quad \text{for } 1\le k<j\le n.$$

Recall that $Y_{n,\bar z}^n = 0$ in $\mathbb{C}^*$ for solutions of $LSU(n+1)$, we can prove also (as for (2.4))

$$Y_{j,\bar z}^j = -\Phi_{j,z\bar z}U_{j+1,z} - U_{j,z\bar z}\Phi_{j+1,z} \quad \text{for solutions of } LSU(n+1) \text{ and } j<n. \tag{6.2}$$

Now we define some new invariants $\widetilde{Z}_k$ for solutions of (6.1), which correspond to Z_k for system $SU(n+1)$. Let

$$\widetilde{Z}_n = Y_n^n, \quad \text{and} \quad \widetilde{Z}_k = Y_k^n + \Phi_{k,z} Z_{k+1} + \sum_{j=k}^{n-2} Y_k^j Z_{j+2},$$
$$\forall\, k = n-1, n-2, \dots, 1.$$

The central argument is

Lemma 6.1 *For any solution of* (6.1), *we have* $\widetilde{Z}_k \equiv 0$ *in* $\mathbb{C}^*$ *for all* $1 \le k \le n$.

Proof By the same argument as in Sect. 4, we have that $\widetilde{Z}_n$ is holomorphic in $\mathbb{C}^*$, since

$$\widetilde{Z}_n = Y_n^n = \sum_{i=1}^{n} \Phi_{i,zz} - 2\sum_{i=1}^{n} U_{i,z}\Phi_{i,z} + \sum_{i=1}^{n-1} \left(\Phi_{i,z} U_{i+1,z} + U_{i,z}\Phi_{i+1,z}\right).$$

Using the integral representation formula for Φ_i, we see that $\nabla^k \Phi_i = O(z^{-k})$ as $|z| \to \infty$ for all $k \ge 1$, so $\widetilde{Z}_n = O(z^{-2})$ at infinity. On the other hand, since $\gamma_j > -1$ for all $1 \le j \le n$, we have $\Phi_i \in C^{0,\alpha}(\mathbb{C})$ with some $\alpha \in (0,1)$, for any $1 \le i \le n$. Again, by elliptic estimates, we can claim that

$$\nabla^k \Phi_i(z) = o\left(z^{-k}\right) \quad \text{as } z \to 0, \text{ for } k \ge 1,\ 1 \le i \le n.$$

By the behavior of U_i via (5.11), $\widetilde{Z}_n = o(z^{-2})$ near the origin, so $\widetilde{Z}_n \equiv 0$ in $\mathbb{C}^*$.

Combining the iterative relations on Y_k^j, the behaviors of Φ_i and U_j, we can claim that for all $k \le j \le n$,

$$Y_k^j = O\left(z^{k-j-2}\right) \quad \text{as } |z| \to \infty \quad \text{and} \quad Y_k^j = o\left(z^{k-j-2}\right) \quad \text{as } |z| \to 0. \tag{6.3}$$

Therefore (recalling that $Z_k = w_k z^{k-2-n}$ for any k), as $Z_n = W_n^n$ and $Y_n^n = 0$,

$$\begin{aligned}\widetilde{Z}_{n-1,\bar{z}} &= Y_{n-1,\bar{z}}^n + \Phi_{n-1,z\bar{z}} Z_n \\ &= -U_{n-1,\bar{z}} Y_n^n - \Phi_{n-1,z\bar{z}} W_n^n + \Phi_{n-1,z\bar{z}} Z_n = 0.\end{aligned}$$

So $\widetilde{Z}_{n-1}$ is holomorphic in $\mathbb{C}^*$. Using expression of Z_k, the asymptotic behavior of Φ_i and (6.3), we see that $\widetilde{Z}_{n-1} = O(z^{-3})$ at infinity and $\widetilde{Z}_{n-1} = o(z^{-3})$

near 0, hence $\widetilde{Z}_{n-1}=0$ in $\mathbb{C}^*$. For $k\le n-2$, suppose that $\widetilde{Z}_j=0$ for $j>k$, we have

$$\begin{aligned}
\widetilde{Z}_{k,\bar z} &= Y^n_{k,\bar z}+\Phi_{k,z\bar z}Z_{k+1}+Y^k_{k,\bar z}Z_{k+2}+\sum_{j=k+1}^{n-2}Y^j_{k,\bar z}Z_{j+2}\\
&= -U_{k,z\bar z}\left[Y^n_{k+1}+\Phi_{k+1,z}Z_{k+2}+\sum_{j=k+1}^{n-2}Y^j_{k+1}Z_{j+2}\right]\\
&\quad +\Phi_{k,z\bar z}\left[Z_{k+1}-W^n_{k+1}-U_{k+1,z}Z_{k+2}-\sum_{j=k+1}^{n-2}W^j_{k+1}Z_{j+2}\right]\\
&= -U_{k,z\bar z}\widetilde{Z}_{k+1}\\
&= 0.
\end{aligned}$$

Here we used the definition of Z_{k+1}. Similarly, the asymptotic behaviors yield that $\widetilde{Z}_k=0$ in $\mathbb{C}^*$. The backward induction finishes the proof. □

Let $g=f\Phi_1$ with $f=e^{-U_1}$. By the definition of Y^j_1, we see that $g^{(j+1)}=f^{(j+1)}\Phi_1+fY^j_1$ for any $1\le j\le n$. Finally,

$$\begin{aligned}
g^{(n+1)} &= f^{(n+1)}\Phi_1+fY^n_1\\
&= -\Phi_1\sum_{j=0}^{n-1}Z_{j+1}f^{(j)}+fY^j_1\\
&= -Z_1f\Phi_1-Z_2f'\Phi_1-\sum_{j=2}^{n-1}Z_{j+1}\left[g^{(j)}-fY^{j-1}_1\right]+fY^n_1\\
&= -\sum_{j=0}^{n-1}Z_{j+1}g^{(j)}+f\left[Y^n_1+\Phi_{1,z}Z_2-\sum_{j=1}^{n-2}Y^j_1Z_{j+2}\right]\\
&= -\sum_{j=0}^{n-1}Z_{j+1}g^{(j)}.
\end{aligned}$$

For the last line, we used $\widetilde{Z}_1=0$. Therefore g satisfies exactly the same differential equation (5.6) for f.

As g is a real function in $\mathbb{C}^*$, we get $g=\sum\widetilde{m}_{kl}f_k\overline{f_l}$ with a hermitian matrix $(\widetilde{m}_{kl})$. As before, the coefficients $\widetilde{m}_{kl}$ need to be zero if $\mu_{k+1}+\cdots+$

$\mu_l \notin \mathbb{N}$, $k < l$, because for $z = |z|e^{i\theta}$,

$$g = \sum_{k=0}^{n} \widetilde{m}_{kk}|z|^{2\beta_k} + 2\sum_{k} |z|^{2\beta_k}\mathrm{Re}\left(\sum_{k<l} \widetilde{m}_{kl} e^{i(\mu_{k+1}+\cdots+\mu_l)\theta}\right)$$

is a single-valued function in $\mathbb{C}^*$. Besides, we can also eliminate the subspace of constant functions for Φ_1 as in Sect. 4. We can conclude then the solution space for (6.1) has the same dimension for the solution manifold for (1.6), which means just the nondegeneracy. □

7 Proof of Theorem 1.5

Let u be a solution of (1.1). By the proof of Lemma 5.2, $f = e^{-U_1}$ satisfies the differential equation:

$$L(f) = f^{(n+1)} + \sum_{k=0}^{n-1} Z_{k+1} f^{(k)} = 0 \quad \text{in } \mathbb{C} \setminus \{P_1, \ldots, P_m\}, \tag{7.1}$$

where Z_{k+1} is a meromorphic function with poles at $\{P_1, \ldots, P_m\}$ and $Z_{k+1}(z) = O(|z|^{-n+k-1})$ at ∞.

From Lemma 2.1, the principal part of Z_k at P_j is

$$Z_k = \frac{w_k}{(z-P_j)^{n+1-k}} + O\left(\frac{1}{|z-P_j|^{n-k}}\right), \tag{7.2}$$

where the coefficient depends only on $\{\gamma_{ij}, 1 \le i \le n\}$.

As we knew in the Introduction, locally f can be written as a sum of $|\nu_i(z)|^2$, where $\nu_i(z)$ is a holomorphic function. Hence

$$0 = \overline{L}L(f) = \sum_{i=0}^{n} |L(\nu_i)|^2.$$

Therefore, $\{\nu_i\}_{0\le i\le n}$ is a set of fundamental solutions of (7.1), and by (7.1), $\|\nu \wedge \cdots \wedge \nu^{(n)}(z)\|$ remains a constant through its analytical continuation. The local exponents $\{\beta_{ij}, 1 \le i \le n\}$ of (7.1) at each P_j is completely determined by the principal part of Z_k. Hence by (7.2) and (5.13), we have

$$\beta_{0j} = -\alpha_{1j}, \qquad \beta_{ij} = \beta_{i-1,j} + \gamma_{ij} + 1.$$

Therefore, near each P_ℓ, $\ell = 1, 2, \ldots, m$, $\nu_i(P_\ell + z) = \sum_{0\le j\le n} c_{ij} z^{\beta_{j\ell}} g_j(z)$, where g_j is a holomorphic function in a neighborhood of P_ℓ. Since $\beta_{j\ell} - \beta_{0\ell}$ are positive integers, we have

$$\nu(P_\ell + ze^{2\pi i}) = e^{2\pi i\beta_{0\ell}}\nu(P_\ell + z), \tag{7.3}$$

i.e. the monodromy of ν near P_ℓ is $e^{2\pi i\beta_{0\ell}}I$, I is the identity matrix. Therefore, the monodromy group of (5.6) consists of scalar multiples of I only, which implies $[\nu(z)]$, as a map into $\mathbb{CP}^n$, is smooth at P_ℓ and well-defined in $\mathbb{C}$.

Applying the estimate of Brezis and Merle [4], we have

$$u_i(z) = -(4+2\gamma_i^*)\log|z| + O(1) \quad \text{at } \infty,$$

for some γ_i^*. To compute γ_i^*, we might use the Kelvin transformation, $\widehat{u}_i(z) = u_i(z|z|^{-2}) - 4\log|z|$. Then $\widehat{u}_i(z)$ also satisfies (1.1) with a new singularity at 0,

$$\widehat{u}_i(z) = -2\gamma_i^*\log|z| + O(1) \quad \text{near } 0.$$

The local exponent of ODE (7.1) corresponding to $\widehat{u}_i$ near 0 is β_i^* where $\beta_i^* - \beta_{i-1}^* = \gamma_i^* + 1$ for $1 \le i \le n$. Let $\widehat{\nu} = (\widehat{\nu}_1, \ldots, \widehat{\nu}_n)$ be a holomorphic curve corresponding to $\widehat{u}$, then

$$\widehat{\nu}_i(ze^{2\pi i}) = e^{2\pi i\beta_i^*}\widehat{\nu}_i(z).$$

Since the monodromy near 0 is a scalar multiple of the identity matrix, we conclude that $\beta_i^* - \beta_0^*$ must be integers and therefore, all γ_i^* are integers. By identifying $S^2 = \mathbb{C} \cup \{\infty\}$, we see $\nu(z)$ can be smoothly extended to be a holomorphic curve from S^2 into $\mathbb{CP}^n$ and ∞ might be a ramificated point with the total ramification index γ_i^*. This ends the proof of Theorem 1.5. $\square$

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Appendix: Explicit formula for $SU(3)$

For general $SU(n+1)$ (1.6), depending the values of $\gamma_i > -1$, we can have many different situations by Theorem 1.1. The solution manifolds have dimensions ranging from n to $n(n+2)$. On the other hand, with the expression of U_1 given by (1.9) and $f = e^{-U_1}$, we can obtain $U_2, \ldots, U_n$ using the formulas in (5.16). However the formulas for U_k, $2 \le k \le n$ are quite complicated in general.

In this appendix, we focus on the case of $SU(3)$ and give the explicit formulas for $n = 2$. Consider

$$\begin{aligned} &-\Delta u_1 = 2e^{u_1} - e^{u_2} - 4\pi\gamma_1\delta_0, -\Delta u_2 = 2e^{u_2} - e^{u_1} - 4\pi\gamma_2\delta_0 \quad \text{in } \mathbb{R}^2,\\ &\int_{\mathbb{R}^2} e^{u_i} < \infty, \quad i = 1, 2, \end{aligned} \tag{8.1}$$

with $\gamma_1, \gamma_2 > -1$. Our result is

Theorem 8.1 *Assume that* (u_1, u_2) *is solution of* (8.1).

- *If* $\gamma_1, \gamma_2 \in \mathbb{N}$. *The solution space is an eight dimensional smooth manifold. More precisely, we have*

$$e^{u_1} = 4\Gamma|z|^{2\gamma_1}\frac{Q}{P^2}, \qquad e^{u_2} = 4\Gamma|z|^{2\gamma_2}\frac{P}{Q^2} \quad in\ \mathbb{C} \tag{8.2}$$

  with $\Gamma = (\gamma_1+1)(\gamma_2+1)(\gamma_1+\gamma_2+2)$ *and*

$$\begin{aligned} P(z) &= (\gamma_2+1)\xi_1 + (\gamma_1+\gamma_2+2)\xi_2\left|z^{\gamma_1+1} - c_1\right|^2 \\ &\quad + \frac{\gamma_1+1}{\xi_1\xi_2}\left|z^{\gamma_1+\gamma_2+2} - c_2 z^{\gamma_1+1} - c_3\right|^2, \\ Q(z) &= (\gamma_1+1)\xi_1\xi_2 + \frac{\gamma_1+\gamma_2+2}{\xi_2}\left|z^{\gamma_2+1} - \frac{(\gamma_1+1)c_2}{\gamma_1+\gamma_2+2}\right|^2 \\ &\quad + \frac{\gamma_2+1}{\xi_1}\left|z^{\gamma_1+\gamma_2+2} - \frac{(\gamma_1+\gamma_2+2)c_1}{\gamma_2+1}z^{\gamma_1+1} + \frac{(\gamma_1+1)c_3}{\gamma_2+1}\right|^2, \end{aligned}$$

  where $c_1, c_2, c_3 \in \mathbb{C}$, $\xi_1, \xi_2 > 0$.
- *If now* $\gamma_1 \notin \mathbb{N}$, $\gamma_2 \notin \mathbb{N}$ *and* $\gamma_1+\gamma_2 \notin \mathbb{Z}$, *then* $c_1 = c_2 = c_3 = 0$, *the solution manifold to* (8.1) *is of two dimensions.*
- *If* $\gamma_1 \in \mathbb{N}$, $\gamma_2 \notin \mathbb{N}$, *then* $c_2 = c_3 = 0$; *if* $\gamma_1 \notin \mathbb{N}$, $\gamma_2 \in \mathbb{N}$, *there holds* $c_1 = c_3 = 0$; *we get a four dimensional solution manifold in both cases.*
- *If* $\gamma_1 \notin \mathbb{N}$, $\gamma_2 \notin \mathbb{N}$ *but* $\gamma_1+\gamma_2 \in \mathbb{Z}$, *then* $c_1 = c_2 = 0$, *the solution manifold to* (8.1) *is of four dimensions.*

In all cases, we have

$$\int_{\mathbb{R}^2} e^{u_1}dx = \int_{\mathbb{R}^2} e^{u_2}dx = 4\pi(\gamma_1+\gamma_2+2). \tag{8.3}$$

The proof follows directly from the formulas (1.9) and (5.16). Here in the below we give direct calculations instead of the general consideration in Sect. 5.

Define (U_1, U_2) and α_1, α_2 by (5.1). Denoting

$$W_1 = -e^{U_1}\left(e^{-U_1}\right)''' = U_{1,zzz} - 3U_{1,zz}U_{1,z} + U_{1,z}^3,$$

then $W_{1,\bar z} = -U_{1,z\bar z}[U_{1,zz} + U_{2,zz} - U_{1,z}^2 - U_{2,z}^2 + U_{1,z}U_{2,z}] := -U_{1,z\bar z}W_2$. As before, we can claim that $W_{2,\bar z} = 0$ in $\mathbb{C}^*$. By studying the behavior of W_2

at ∞, we get

$$W_2 = \frac{w_2}{z^2} \quad \text{in } \mathbb{C}^* \text{ where } w_2 = -\alpha_1^2 - \alpha_2^2 + \alpha_1\alpha_2 - \alpha_1 - \alpha_2.$$

As $(W_1 + U_{1,z}W_2)_{\bar{z}} = U_{1,z}W_{2,\bar{z}} = 0$ in $\mathbb{C}^*$, by considering $z^3(W_1 + U_{1,z}W_2)$, there holds

$$W_1 + U_{1,z}W_2 = \frac{w_1}{z^3} \quad \text{in } \mathbb{C}^* \text{ where } w_1 = 2\alpha_1 + 3\alpha_1^2 + \alpha_1^3 + \alpha_1 w_2.$$

Combine all these informations, the function $f := e^{-U_1}$ satisfies

$$f_{zzz} = -fW_1 = -\frac{w_1}{z^3}f + fU_{1,z}\frac{w_2}{z^2} = -\frac{w_2}{z^2}f_z - \frac{w_1}{z^3}f \quad \text{in } \mathbb{C}^*. \tag{8.4}$$

Consider special solution of (8.4) like z^β, then β should satisfy $\beta(\beta-1)(\beta-2) + w_2\beta + w_1 = 0$. We check readily that the equation of β has three roots: $\beta_1 = -\alpha_1$, $\beta_2 = \alpha_1 + 1 - \alpha_2$ and $\beta_3 = \alpha_2 + 2$. Hence $\beta_3 - \beta_2 = \gamma_2 + 1 > 0$ and $\beta_2 - \beta_1 = \gamma_1 + 1 > 0$. We obtain finally $f(z) = \sum_{1\le i,j\le 3} b_{ij} z^{\beta_i}\bar{z}^{\beta_j}$ with an hermitian matrix (b_{ij}).

In the following, we show how to get explicit formulas of U_i for two cases, and all the others can be treated similarly. The formulas of u_i or the quantization (8.3) of the integrals are clearly direct consequences of expression of U_i.

Case 1: $\gamma_i \notin \mathbb{N}$ and $\gamma_1 + \gamma_2 \notin \mathbb{Z}$. To get a well defined real function f in $\mathbb{C}^*$, we have $b_{ij} = 0$ for $i \ne j$, so that

$$f = e^{-U_1} = \sum_{i=1}^{3} a_i |z|^{2\beta_i} \quad \text{in } \mathbb{C}^*, \text{ with } a_i \in \mathbb{R}.$$

Therefore direct calculation yields

$$\frac{e^{-U_2}}{4} = -e^{-2U_1}U_{1,z\bar{z}} = ff_{z\bar{z}} - f_z f_{\bar{z}} = \sum_{1\le i<j\le 3} a_i a_j (\beta_i - \beta_j)^2 |z|^{2(\beta_i+\beta_j-1)}.$$

Moreover, there holds also $e^{-U_1} = -4e^{-2U_2}U_{2,z\bar{z}}$. With the explicit values of β_i, we can check that (U_1, U_2) is a solution if and only if

$$a_1a_2a_3\Gamma^2 = \frac{1}{64} \quad \text{where } \Gamma = (\gamma_1+1)(\gamma_2+1)(\gamma_1+\gamma_2+2), \tag{8.5}$$

or equivalently

$$a_1 = \frac{(\gamma_2+1)\xi_1}{4\Gamma}, \qquad a_2 = \frac{(\gamma_1+\gamma_2+2)\xi_1}{4\Gamma},$$
$$a_3 = \frac{(\gamma_1+1)}{4\Gamma\xi_1\xi_2} \quad \text{with } \xi_1, \xi_2 > 0.$$

Indeed, the positivity of e^{-U_1} in $\mathbb{C}^*$ implies that $a_1, a_3 > 0$, so is a_2 by (8.5).

Case 2: $\gamma_1 \in \mathbb{N}$ but $\gamma_2 \notin \mathbb{N}$. We get

$$e^{-U_1} = \sum_{i=1}^{3} a_i |z|^{2\beta_i} + \frac{\mathrm{Re}(\lambda z^{\gamma_1+1})}{|z|^{2\alpha_1}} \quad \text{in } \mathbb{C}^*, \text{ with } a_i \in \mathbb{R},\ \lambda \in \mathbb{C}.$$

If $a_2 \neq 0$, changing eventually the value of a_1, there exists $c_1 \in \mathbb{C}$ such that

$$e^{-U_1} = \frac{a_1 + a_2|z^{\gamma_1+1} - c_1|^2 + a_3|z|^{2(\gamma_1+\gamma_2+2)}}{|z|^{2\alpha_1}} \quad \text{in } \mathbb{C}^*.$$

We obtain then the expression of e^{-U_2} directly and we can check that the necessary and sufficient condition required to get solutions of (8.1) is always (8.5). We leave the details for interested readers. This yields

$$e^{-U_1} = \frac{1}{4\Gamma|z|^{2\alpha_1}} \Bigg[(\gamma_2+1)\xi_1 + (\gamma_1+\gamma_2+2)\xi_2 \left| z^{\gamma_1+1} - c_1 \right|^2 + \frac{\gamma_1+1}{\xi_1\xi_2}|z|^{2(\gamma_1+\gamma_2+2)} \Bigg]$$

and

$$e^{-U_2} = \frac{1}{4\Gamma|z|^{2\alpha_2}} \Bigg[(\gamma_1+1)\xi_1\xi_2 + \frac{\gamma_1+\gamma_2+2}{\xi_2}|z|^{2(\gamma_2+1)} + \frac{\gamma_2+1}{\xi_1}|z|^{2(\gamma_2+1)} \left| z^{\gamma_1+1} - \frac{(\gamma_1+\gamma_2+2)c_1}{\gamma_2+1} \right|^2 \Bigg].$$

So it remains to eliminate the case $a_2 = 0$. If $a_2 = 0$, we can rewrite

$$f = \frac{a_1 + \mathrm{Re}(\lambda z^{\gamma_1+1}) + a_3|z|^{2(\gamma_1+\gamma_2+2)}}{|z|^{2\alpha_1}} \quad \text{in } \mathbb{C}^*,$$

where $\lambda \in \mathbb{C}$. Direct calculation yields

$$\frac{e^{-U_2}}{4} = ff_{z\bar{z}} - f_z f_{\bar{z}} = |z|^{2(-\alpha_2+\gamma_2+1)} \left[c_1'|z|^{-2(\gamma_2+1)} + c_2' + c_3'\mathrm{Re}\left(\lambda z^{\gamma_1+1}\right) \right],$$

where

$$c_1' = -\frac{|\lambda|^2(\gamma_1+1)^2}{4}, \qquad c_2' = a_1a_3(\gamma_1+\gamma_2+2)^2,$$
$$c_3' = a_3(\gamma_1+\gamma_2+2)(\gamma_2+1).$$

As $e^{-U_2} > 0$, we must have $c_1' \geq 0$. So we get $\lambda = 0$, and we find the expression of f as in Case 1 with $a_2 = 0$. Then we need to verify (8.5). However this is impossible since $a_2 = 0$. Thus a_2 must be nonzero.

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