FROM KP-I LUMP SOLUTION TO TRAVELLING WAVES OF GROSS-PITAEVSKII EQUATION

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ABSTRACT. Let $q(x,y)$ be an nondegenerate lump solution to KP-I (Kadomtsev-Petviashvili-I) equation

$$\partial_x^4 q - 2\sqrt{2}\partial_x^2 q - 3\sqrt{2}\partial_x((\partial_x q)^2) - 2\partial_y^2 q = 0.$$

We prove the existence of a traveling wave solution $u_\varepsilon(x-ct,y)$ to GP (Gross-Pitaevskii) equation

$$i\partial_t\Psi + \Delta\Psi + (1-|\Psi|^2)\Psi = 0, \quad \text{in } \mathbb{R}^2$$

in the transonic limit

$$c = \sqrt{2} - \epsilon^2$$

with

$$u_\epsilon = 1 + i\epsilon q(x,y) + \mathcal{O}(\epsilon^2).$$

This proves the existence of finite energy solutions in the so-called Jones-Roberts program in the transonic range $c \in (\sqrt{2}-\epsilon^2, \sqrt{2})$. The main ingredients in our proof are detailed point-wise estimates of the Green function associated to a family of fourth order hypoelliptic operators

$$\partial_x^4 - (2\sqrt{2}-\varepsilon^2)\partial_x^2 - 2\partial_y^2 + \varepsilon^2\partial_x^2\partial_y^2 + \varepsilon^4\partial_y^4.$$

Keywords: KP-I lump solution, Gross-Pitaevskii equation, 4-th order anisotropic elliptic operator

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The GP (Gross-Pitaevskii) equation investigated in this paper has the form

$$i\partial_t\Psi + \Delta\Psi + (1-|\Psi|^2)\Psi = 0, \quad \text{in } \mathbb{R}^2 \tag{1.1}$$

where Ψ is a complex-valued function in $\mathbb{R}^2$. It is formally Hamiltonian and can be written as

$$\partial_t\Psi = -i\nabla E,$$

where E is the Hamiltonian energy given by

$$E(\Psi) := \int_{\mathbb{R}^2}\left[\frac{1}{2}|\nabla\Psi|^2 + \frac{1}{4}\left(|\Psi|^2-1\right)^2\right],$$

and $\nabla E := -\Delta\Psi - (1-|\Psi|^2)\Psi$ represents the L^2-gradient of E. In this paper, we are interested in the travelling wave solutions of (1.1). More precisely, let us consider solutions of (1.1) of the form $\Phi(x-ct,y)$. Then $\Phi(x,y)$ satisfies the following elliptic equation

$$ic\partial_x\Phi = \Delta\Phi + \left(1-|\Phi|^2\right)\Phi \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

Since Φ is complex valued, actually (1.2) is a system consisting of two elliptic equations.

Travelling wave solutions are expected to play central role in the long time dynamics of the GP equation. One of the main issue in the study of (1.1) is the existence and classification of solutions to the equation (1.2). We would like to mention that some types of classification results and universal bound of certain classes of solutions, including travelling wave solutions, to the GP equation has been obtained in [24]. Reversing the time if necessary, we may assume that the travelling speed c is nonnegative. Let us focus on the equation (1.2) imposed in the entire space $\mathbb{R}^n$. Without any assumption on the asymptotic behavior at infinity, the structure of the solution set of (1.2) could be complicated. Observe that the energy E is conserved the GP flow. We therefore focus on those finite energy solutions of (1.2). It follows from results of [20, 22] that any finite energy solution will satisfy, up to a multiplicative constant,

$$\Phi(z) \to 1, \text{ as } |z| \to +\infty.$$

More precise asymptotic behavior of finite energy solutions is also available. Indeed, by results of [23], for $c \in \left(0, \sqrt{2}\right)$, there holds

$$|z|\left(\Phi(z) - 1\right) - if\left(\frac{z}{|z|}\right) \to 0, \text{ as } z \to +\infty,$$

where f is a smooth function defined on S^{n-1}, the $n-1$-dimensional unit sphere. Particularly, when dimension $n = 2$, there exists constants α, β, such that

$$f\left(\frac{z}{|z|}\right) = \frac{(\alpha x + \beta y)\sqrt{x^2+y^2}}{x^2 + \left(1 - \frac{c^2}{2}\right)y^2}. \tag{1.3}$$

From the formula (1.3) we notice that the existence or nonexistece of nontrivial travelling wave solutions heavily depends on the speed c. For $c > \sqrt{2}$, it is proved by Gravejat [19] that any finite energy solution to (1.2) has to be a constant.

In the case of $c = \sqrt{2}$, it is now known [21] that there does not exist nonconstant finite energy solutions when $n = 2$. The problem is still open for $n \geq 3$.

In the subsonic regime, that is, when $c \in \left(0, \sqrt{2}\right)$, Jones et.al [26, 27] studied the equation from the physical point of view and obtained solutions with formal and numerical calculation. Later on, many works have been done to get the solutions in a more rigorous mathematical way. The methods applied in these works can be divided into two categories. The first one is variational, and the second one is Lyapunov-Schmidt reduction.

Variationally, there are several different strategies to tackle the problem. Mountain pass arguments were used in [7, 13] to prove the existence of solutions with small travelling speed. On the other hand, minimizing the energy functional with fixed momentum yields existence in dimension 2 for any constrained value of the momentum and existence in dimension 3 when the momentum $p > p_0$ for some threshold value p_0, see [9, 10]. In spite of all the above results, the question remains as for whether for all $c \in \left(0, \sqrt{2}\right)$, there is a solution. Minimizing the energy under a Pohozaev constraint, Maris [34] successfully obtained solutions in the full speed interval $\left(0, \sqrt{2}\right)$ for dimension $n > 2$. Unfortunately, this argument breaks down in 2D, thus leaving the existence problem open in this dimension. Recently, Bellazzini-Ruiz [4] proved the existence of almost all subsonic speed in 2D, using variational argument and Struwe's monotonicity trick. They also recovered the results of Maris in 3D.

Up to now, the variational arguments mentioned above have not been able to tackle the higher energy solutions. For these type of solutions, when the speed c is close to zero, in [3, 29], the second and third authors applied Lyapunov-Schmidt reduction method to show the existence by gluing suitable copies of the vortex solutions of the Ginzburg-Landau equation. In dimension two, the position of the vortices is determined by the Adler-Moser polynomials; in higher dimensions, the position is determined by a family polynomials, which we called generalized Adler-Moser polynomials. We also refer to [15] for the discussion of two vortices case.

When the speed c tends to the transonic limit $\sqrt{2}$, it is now known that a suitable rescaled travelling waves will converge to solutions of the KP-I equation

$$\partial_x^4 q - 2\sqrt{2}\partial_x^2 q - 3\sqrt{2}\partial_x((\partial_x q)^2) - 2\partial_y^2 q = 0, \tag{1.4}$$

which is an important integrable system. (See Section 2 below for the derivation.) We refer to Bethuel, Gravejet and Saut [8] for convergence result in dimension two, and Chiron and Maris [14] for convergence results in dimension three or higher. (See also [16, 17, 18] for detailed discussion and related results.) Numerical evidence of these results can be found in [17, 26]. An important open question is whether the converse is true, i.e., given a solution to the KP-I equation (1.4), whether or not a travelling wave solution to (1.2) close to the KP-I solution exists. For the KP-I equation, lump solutions have been obtained by variational methods in [11, 12, 28]. We also point out that the existence of least energy solutions for c close to $\sqrt{2}$ can be obtained by variational arguments. However, since we lack a clear classification of solutions to the KP-I equation, the precise form of these "transonic" solutions are not known. On the other hand, although the Lyapunov-Schmidt reduction type perturbation arguments have the disadvantage that it can not be used for the general case $c \in \left(0, \sqrt{2}\right)$, it is possible to handle the case of c close to $\sqrt{2}$. This will be carried out in this paper and our main result is the following

Theorem 1.1. *Let $q(x,y)$ be an nondegenerate lump solution to the following KP-I equation (1.4). For any $\varepsilon > 0$ sufficiently small, there exists a solution Φ_ε to the equation* (1.2) *with travelling speed $c = \sqrt{2} - \varepsilon^2$ and has the following asymptotic behavior*

$$\Phi_\varepsilon = 1 + i\varepsilon q(x,y) + \mathcal{O}(\varepsilon^2).$$

Remark 1.1. *Lump solutions to KP-I are a kind of rational function solutions, localized and decaying in all directions. A lump solution to KP-I is called nondegenerate if the corresponding linearized operator admits only translational kernels. For the standard lump solution to (1.4)*

$$q_0(x,y) == -\frac{2\sqrt{2}x}{x^2 + \sqrt{2}y^2 + \frac{3}{2\sqrt{2}}} \tag{1.5}$$

it was proved in [30] *that q_0 is nondegenerate and has Morse index one. (We note that the Morse index of the lump solution (1.5) has been shown numerically to be one (*[17]*).) There are many higher Morse index lump solutions to KP-I and their nondegeneracy are expected (but rigorous proofs are needed). See* [2, 31, 32, 33] *and the references therein.*

In the following let us sketch the main idea and explain the new ingredients of our proof. First of all, we assume our solution has the form $f + ig$, where $f,\ g$ are real valued functions with $f - 1 = O\left(\varepsilon^2\right)$ and $g = O\left(\varepsilon\right)$. Plugging the function into (1.1), we shall see that the original problem is connected with the KP-I equation in the second leading order. Particularly, f should essentially be determined by g, which satisfies a perturbed KP-I equation. However, we can not just simply build up a genuine solution by the nondegeneracy of the lump solution and the standard Lyapunov-Schmidt reduction method. The crucial point is that the perturbation terms would create some difficulties in obtaining the suitable decay property and studying the nonlinear problem. Therefore, we have to consider the full linearized problem including the higher order derivatives on y direction, but with small coefficients

$$\partial_x^4 - (2\sqrt{2} - \varepsilon^2)\partial_x^2 - 2\partial_y^2 + \varepsilon^2\partial_x^2\partial_y^2 + \varepsilon^4\partial_y^4. \tag{1.6}$$

The L^p boundedness of this anisotropic 4-th order elliptic operator has been derived and used in the convergence results in [8, 14]. However, in order to capture the decay property of the perturbations, we need to obtain *point-wise* behavior of the Green's function associated with the anisotropic 4-th order operator (1.6). Precisely, since the maximum principle can not be used to show the decay property for the fourth order anisotropic problem, we have to study the Green kernel and its pointwise derivatives by describing their asymptotic behaviors around the singular point and decay property at infinity instead. (It is also interesting to mention that for the kernel of the linear operator $\partial_x^4 - (2\sqrt{2} - \varepsilon^2)\partial_x^2 - 2\partial_y^2$, several important properties including the decay property and the asymptotic behavior around the singularity on the kernel K_0 have been obtained by Gravejat in [23].)

In most situations, the Green function would decay faster if we take more derivatives on it. However, the situation becomes quite different here, this is due to the fact that the Fourier integration of the Fourier inverse transform of the Green function has singularity in our problem. As a consequence, the higher derivatives of the Green function can decay at most $r^{-\frac{3}{2}}$ at infinity. Simultaneously, as the original problem is an anisotropic elliptic operator, we would get some huge coefficients on the derivatives of the Green function with respect to the y direction, see Theorem 3.3 in section 3. All the above difficulties make the whole problem quite sensitive, it would break down the Lyapunov-Schmidt reduction process easily in the nonlinear problem if we do not pursue the accurate asymptotic behaviors on its various derivatives. To solve this issue, we have to study the linear problem in a suitable space, see the definition of space in (4.21). The definition of this space turns out to be quite complicated, but it seems to us that most of the terms are necessary. Its norm is consisted of three parts, the first part $\|\cdot\|_a$ is used for studying the L^2 theory of the linearized problem, the second part (contain various mixed derivatives) is included to deal with the terms appeared in the error and nonlinear higher order perturbations. Precisely, consider the term $\partial_x\phi$ for example, we shall see that it appears in the error terms, such as $\partial_y(\partial_x g_1 \partial_y g_1)$, see (2.28) in section 2. When we deal with this term, we need at least one of $\partial_x\phi$ and $\partial_y\phi$ has decay up to $r^{-\frac{3}{2}}$ at infinity, while concerning the linear problem in weighted Sobolev space, we require $\partial_x\phi$ to be small, but sacrificing a little bit in its decay rate. Therefore we have to include both situations in the definition of $\|\cdot\|_*$. The third part, the anti-derivative terms are used for making the perturbations of f have enough decay to run the fixed point argument. After establishing the a-priori estimate for the full linearized problem, we study the existence of the nonlinear problem by contraction principle. Then the original result is established. We remark that the arguments presented in this paper is quite robust for treating other models with the finite Morse index solutions to (1.4).

The paper is organized as follows. In Section 2 we derive the equations satisfied by f and g. In Section 3, we study the Green function associated to the linear operator. In section 5, we analyzed the perturbed linearized KP-I operator around the lump solution. In Section 5, we solve the nonlinear problem using contraction mapping principle and finish the proof of Theorem 1.1.

Notation

Throughout the paper, the letter C will stand for positive constants which are allowed to vary among different formulas and even within the same lines. The letter $\varepsilon > 0$ will represent a small parameter.

2. KP-I AS A CONSISTENT CONDITION FOR GP

We are seeking for travelling wave solutions Φ in the transonic regime, that is, with speed close to $\sqrt{2}$. In this section, we compute the equation satisfied by the real and imaginary parts of Φ. Formal computation has already been done in [26]. We point out that in [8, 17], the transonic limit has been investigated in great details. The approach taken there is writing the solution Φ in the form $\rho e^{i\phi}$ and analyzing the equations satisfied by ρ and ϕ. Here we study the real and imaginary parts directly, because as we shall see later on, in this form, the real part can be determined from the imaginary part through an ODE, which is technically relatively easier.

Let us write Φ as $f + ig$, where $f = \operatorname{Re}\Phi$ and $g = \operatorname{Im}\Phi$. Then the travelling wave equation (1.2) is equivalent to the following system

$$\begin{cases} c\partial_{\tilde{x}} g = -\Delta f + \left(f^2 + g^2 - 1\right) f, \\ -c\partial_{\tilde{x}} f = -\Delta g + \left(f^2 + g^2 - 1\right) g. \end{cases}$$

Introducing new variables by

$$x = \varepsilon \tilde{x}, y = \varepsilon^2 \tilde{y},$$

we obtain

$$\begin{cases} c\varepsilon\partial_x g = -\varepsilon^4 \partial_y^2 f - \varepsilon^2 \partial_x^2 f + \left(f^2 + g^2 - 1\right) f, \\ \\ -c\varepsilon\partial_x f = -\varepsilon^4 \partial_y^2 g - \varepsilon^2 \partial_x^2 g + \left(f^2 + g^2 - 1\right) g. \end{cases} \tag{2.1}$$

We assume that the travelling speed c has the form $c = c_0 - \varepsilon^2$, where c_0 is a constant independent of ε.

We look for f and g in the form

$$f = 1 + \varepsilon^2 f_1 + \varepsilon^4 f_2,\ g = \varepsilon g_1,$$

where f_1, f_2, g_1 are unknown functions to be determined. Substituting this into the system (2.1) we get the following two equations to be solved:

$$\begin{aligned}\left(c_0-\varepsilon^2\right)\partial_x g_1 = &\left[2\left(f_1+\varepsilon^2 f_2\right)+g_1^2+\left(\varepsilon f_1+\varepsilon^3 f_2\right)^2\right]\left(1+\varepsilon^2 f_1+\varepsilon^4 f_2\right)\\ &-\varepsilon^4\left(\partial_y^2 f_1+\varepsilon^2\partial_y^2 f_2\right)-\left(\varepsilon^2\partial_x^2 f_1+\varepsilon^4\partial_x^2 f_2\right),\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}-\left(c_0-\varepsilon^2\right)\left(\partial_x f_1+\varepsilon^2\partial_x f_2\right) = &\left[2\left(f_1+\varepsilon^2 f_2\right)+g_1^2+\left(\varepsilon f_1+\varepsilon^3 f_2\right)^2\right] g_1\\ &-\varepsilon^2\partial_y^2 g_1-\partial_x^2 g_1.\end{aligned} \tag{2.3}$$

To solve these two equations, we will expand them into powers of ε.

2.1. **The $O(1)$ terms.** Comparing the $O(1)$ terms in (2.2) , we get

$$c_0\partial_x g_1 = 2f_1+g_1^2. \tag{2.4}$$

On the other hand, comparing the $O(1)$ term in (2.3) leads to

$$-c_0\partial_x f_1 = -\partial_x^2 g_1+g_1\left(2f_1+g_1^2\right). \tag{2.5}$$

The consistency of the two equations (2.4) and (2.5) requires that

$$-\frac{c_0}{2}\partial_x\left(c_0\partial_x g_1-g_1^2\right) = -\partial_x^2 g_1+c_0 g_1\partial_x g_1.$$

Hence it is natural to set $c_0=\sqrt{2}$. Equation (2.4) then has the form

$$f_1 = \frac{\sqrt{2}}{2}\partial_x g_1-\frac{g_1^2}{2}. \tag{2.6}$$

This means that f_1 is essentially determined by g_1.

2.2. **The $o(1)$ terms and the consistent condition.** With the $O(1)$ terms being understood, we now divide the $o(1)$ terms in the equation (2.2) by ε^2. This yields

$$\begin{aligned}-\partial_x g_1 = &-\varepsilon^2\left(\partial_y^2 f_1+\varepsilon^2\partial_y^2 f_2\right)-\left(\partial_x^2 f_1+\varepsilon^2\partial_x^2 f_2\right)+2f_2+\left(f_1+\varepsilon^2 f_2\right)^2\\ &+\left(f_1+\varepsilon^2 f_2\right)\left\{2f_1+g_1^2+2\varepsilon^2 f_2+\left(\varepsilon f_1+\varepsilon^3 f_2\right)^2\right\}.\end{aligned}$$

The above equation can be written into the following more compact form:

$$-\partial_x g_1 = -\partial_x^2 f_1+f_1\left(2f_1+g_1^2\right)+2f_2+f_1^2+\Pi_1, \tag{2.7}$$

where the perturbation term Π_1 is defined by

$$\Pi_1 = -\varepsilon^2\partial_y^2 f_1-\varepsilon^4\partial_y^2 f_2-\varepsilon^2\partial_x^2 f_2+6\varepsilon^2 f_1 f_2+\varepsilon^2(f_1+\varepsilon^2 f_2)^3+3\varepsilon^4 f_2^2+\varepsilon^2 f_2 g_1^2. \tag{2.8}$$

For convenience, we write the last four terms as

$$\Pi_2 = 6\varepsilon^2 f_1 f_2+\varepsilon^2(f_1+\varepsilon^2 f_2)^3+3\varepsilon^4 f_2^2+\varepsilon^2 f_2 g_1^2. \tag{2.9}$$

To proceed, we define the function

$$\Xi := 2f_2+f_1^2+\sqrt{2}\left(\partial_x\left(f_1 g_1\right)+g_1^2\partial_x g_1\right). \tag{2.10}$$

This definition is inspired by equations (A13) and (A14) of [26]. Note that Ξ involves f_1, g_1 and f_2, we can represent Ξ in the following way

Lemma 2.1. *Suppose* (2.6) *and* (2.7) *hold. Then we have*

$$\Xi = -\partial_x g_1 + \frac{1}{\sqrt{2}}\partial_x^3 g_1 - (\partial_x g_1)^2 - \Pi_1.$$

Proof. Eliminating f_2 in (2.10) by using equation (2.7) , we get

$$\Xi = -\partial_x g_1 + \partial_x^2 f_1 - f_1\left(2f_1 + g_1^2\right) + \sqrt{2}\left(\partial_x\left(f_1 g_1\right) + g_1^2\partial_x g_1\right) - \Pi_1.$$

Inserting (2.4) into the right hand side, we compute

$$\begin{aligned}\Xi =& -\partial_x g_1 + \partial_x^2\left(\frac{\partial_x g_1}{\sqrt{2}} - \frac{g_1^2}{2}\right) - \sqrt{2}\left(\frac{\partial_x g_1}{\sqrt{2}} - \frac{g_1^2}{2}\right)\partial_x g_1 \\ &+ \sqrt{2}\left(\partial_x\left(\frac{g_1\partial_x g_1}{\sqrt{2}} - \frac{g_1^3}{2}\right) + g_1^2\partial_x g_1\right) - \Pi_1 \\ =& -\partial_x g_1 + \frac{1}{\sqrt{2}}\partial_x^3 g_1 - \frac{1}{2}\partial_x^2\left(g_1^2\right) - (\partial_x g_1)^2 + \frac{1}{\sqrt{2}}g_1^2\partial_x g_1 \\ &+ \partial_x\left(g_1\partial_x g_1\right) - \frac{1}{\sqrt{2}}\partial_x\left(g_1^3\right) + \sqrt{2}g_1^2\partial_x g_1 - \Pi_1.\end{aligned}$$

Hence

$$\Xi = -\partial_x g_1 + \frac{1}{\sqrt{2}}\partial_x^3 g_1 - (\partial_x g_1)^2 - \Pi_1. \tag{2.11}$$

This finishes the proof. □

Next let us divide the $o(1)$ terms in (2.3) by ε^2. This leads to

$$-(\sqrt{2} - \varepsilon^2)\partial_x f_2 = -\partial_x f_1 - \partial_y^2 g_1 + g_1(2f_2 + f_1^2) + \Pi_3, \tag{2.12}$$

where the perturbation term Π_3 is given by

$$\Pi_3 = 2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2. \tag{2.13}$$

We shall use this information to compute $\partial_x\Xi$. We have the following

Lemma 2.2. *Suppose* (2.6), (2.7) , (2.12) *hold. Then*

$$\begin{aligned}\partial_x\Xi =& \partial_x^2 g_1\left(\frac{\sqrt{2}}{\sqrt{2} - \varepsilon^2} + 4\partial_x g_1\right) + \frac{2}{\sqrt{2} - \varepsilon^2}\left(\partial_y^2 g_1 + g_1\Pi_1 - \Pi_3\right) \\ &+ \frac{\varepsilon^2}{2 - \sqrt{2}\varepsilon^2}\partial_x g_1\partial_x(g_1^2) - \frac{\varepsilon^2}{2\sqrt{2} - 2\varepsilon^2}g_1^2\partial_x(g_1^2) - \frac{\sqrt{2}\varepsilon^2}{\sqrt{2} - \varepsilon^2}g_1\partial_x^2 f_1.\end{aligned}$$

Proof. First of all, using (2.12) to eliminate $\partial_x f_2$, we have

$$\begin{aligned}\partial_x\Xi &= \partial_x\left(2f_2 + f_1^2 + \sqrt{2}(\partial_x(f_1 g_1) + g_1^2\partial_x g_1)\right) \\ &= \frac{2}{\sqrt{2} - \varepsilon^2}\left(\partial_x f_1 + \partial_y^2 g_1 - g_1(2f_2 + f_1^2) - \Pi_3\right) \\ &\quad + \partial_x\left(f_1^2 + \sqrt{2}(\partial_x(f_1 g_1) + g_1^2\partial_x g_1)\right).\end{aligned}$$

Using equation (2.7), we can eliminate f_2 appeared on the right hand side. It follows that

$$\begin{aligned}
\partial_x\Xi =& \frac{2}{\sqrt{2}-\varepsilon^2}\left(\partial_x f_1 + \partial_y^2 g_1\right) - \frac{2}{\sqrt{2}-\varepsilon^2} g_1\left(-\partial_x g_1 + \partial_x^2 f_1 - f_1\left(2f_1 + g_1^2\right)\right)\\
&+ \frac{2}{\sqrt{2}-\varepsilon^2}(g_1\Pi_1 - \Pi_3) + \partial_x\left(f_1^2 + \sqrt{2}\left(\partial_x\left(f_1 g_1\right) + g_1^2\partial_x g_1\right)\right)\\
=& \frac{2}{\sqrt{2}-\varepsilon^2}\left(\partial_x f_1 + \partial_y^2 g_1 + g_1\partial_x g_1 - g_1\partial_x^2 f_1 + g_1 f_1(2f_1 + g_1^2)\right)\\
&+ \frac{2}{\sqrt{2}-\varepsilon^2}(g_1\Pi_1 - \Pi_3) + \partial_x(f_1^2) + \sqrt{2}\partial_x(g_1^2\partial_x g_1)\\
&+ \sqrt{2}\left(g_1\partial_x^2 f_1 + 2\partial_x g_1\partial_x f_1 + \partial_x^2 g_1 f_1\right)\\
=& \frac{2}{\sqrt{2}-\varepsilon^2}\left(\partial_x f_1 + \partial_y^2 g_1 + g_1\partial_x g_1 + g_1 f_1(2f_1 + g_1^2)\right) + \frac{2}{\sqrt{2}-\varepsilon^2}(g_1\Pi_1 - \Pi_3)\\
&+ \partial_x(f_1^2) + \sqrt{2}(f_1\partial_x^2 g_1 + 2\partial_x f_1\partial_x g_1) + \sqrt{2}\partial_x(g_1^2\partial_x g_1) - \frac{\sqrt{2}\varepsilon^2}{\sqrt{2}-\varepsilon^2} g_1\partial_x^2 f_1.
\end{aligned}$$

To further simplify the expression, we use (2.6) to compute

$$\begin{aligned}
\partial_x\Xi =& \frac{2}{\sqrt{2}-\varepsilon^2}\left(\partial_x f_1 + \partial_y^2 g_1 + g_1\partial_x g_1\right) + \sqrt{2} f_1\partial_x^2 g_1 + 2\sqrt{2}\partial_x f_1\partial_x g_1 + \partial_x(f_1^2)\\
&+ \frac{2\sqrt{2}}{\sqrt{2}-\varepsilon^2} f_1 g_1\partial_x g_1 + \frac{2}{\sqrt{2}-\varepsilon^2}(g_1\Pi_1 - \Pi_3) + \sqrt{2}\partial_x(g_1^2\partial_x g_1) - \frac{\sqrt{2}\varepsilon^2}{\sqrt{2}-\varepsilon^2} g_1\partial_x^2 f_1\\
=& \sqrt{2}\partial_x f_1\left(\frac{\sqrt{2}}{\sqrt{2}-\varepsilon^2} + 2\partial_x g_1 + \sqrt{2} f_1\right) + \frac{2}{\sqrt{2}-\varepsilon^2}\partial_y^2 g_1 + \frac{1}{\sqrt{2}-\varepsilon^2}\partial_x(g_1^2) + \sqrt{2}\partial_x(g_1^2\partial_x g_1)\\
&+ \sqrt{2} f_1\left(\partial_x^2 g_1 + \frac{2}{\sqrt{2}-\varepsilon^2} g_1\partial_x g_1\right) + \frac{2}{\sqrt{2}-\varepsilon^2}(g_1\Pi_1 - \Pi_3) - \frac{\sqrt{2}\varepsilon^2}{\sqrt{2}-\varepsilon^2} g_1\partial_x^2 f_1.
\end{aligned}$$

Hence

$$\begin{aligned}
\partial_x\Xi =& \left(\partial_x^2 g_1 - \frac{\sqrt{2}}{2}\partial_x\left(g_1^2\right)\right)\left(\frac{\sqrt{2}}{\sqrt{2}-\varepsilon^2} + 2\partial_x g_1 + \left(\partial_x g_1 - \frac{g_1^2}{\sqrt{2}}\right)\right)\\
&+ \frac{2}{\sqrt{2}-\varepsilon^2}\left(\partial_y^2 g_1 + g_1\partial_x g_1\right) + \frac{2}{\sqrt{2}-\varepsilon^2}\left(g_1\Pi_1 - \Pi_3\right) + \sqrt{2}\partial_x\left(g_1^2\partial_x g_1\right)\\
&+ \left(\partial_x g_1 - \frac{g_1^2}{\sqrt{2}}\right)\left(\partial_x^2 g_1 + \frac{2}{\sqrt{2}-\varepsilon^2} g_1\partial_x g_1\right) - \frac{\sqrt{2}\varepsilon^2}{\sqrt{2}-\varepsilon^2} g_1\partial_x^2 f_1\\
=& \partial_x^2 g_1\left(\frac{\sqrt{2}}{\sqrt{2}-\varepsilon^2} + 3\partial_x g_1 - \frac{g_1^2}{\sqrt{2}} + \partial_x g_1 - \frac{g_1^2}{\sqrt{2}} + \sqrt{2} g_1^2\right) + \frac{2}{\sqrt{2}-\varepsilon^2}\partial_y^2 g_1\\
&- \frac{\sqrt{2}}{2}\partial_x(g_1^2)\left(\frac{\sqrt{2}}{\sqrt{2}-\varepsilon^2} + 3\partial_x g_1 - \frac{g_1^2}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}-\varepsilon^2} - \frac{\sqrt{2}}{\sqrt{2}-\varepsilon^2}(\partial_x g_1 - \frac{g_1^2}{\sqrt{2}}) - 2\partial_x g_1\right)\\
&+ \frac{2}{\sqrt{2}-\varepsilon^2}(g_1\Pi_1 - \Pi_3) - \frac{\sqrt{2}\varepsilon^2}{\sqrt{2}-\varepsilon^2} g_1\partial_x^2 f_1.
\end{aligned}$$

Direct computation shows that

$$\begin{aligned}\partial_x\Xi &= \partial_x^2 g_1\left(\frac{\sqrt{2}}{\sqrt{2}-\varepsilon^2}+4\partial_x g_1\right)+\frac{2}{\sqrt{2}-\varepsilon^2}\partial_y^2 g_1+\frac{2}{\sqrt{2}-\varepsilon^2}(g_1\Pi_1-\Pi_3)\\ &\quad+\frac{\varepsilon^2}{2-\sqrt{2}\varepsilon^2}\partial_x g_1\partial_x(g_1^2)-\frac{\varepsilon^2}{2\sqrt{2}-2\varepsilon^2}g_1^2\partial_x(g_1^2)-\frac{\sqrt{2}\varepsilon^2}{\sqrt{2}-\varepsilon^2}g_1\partial_x^2 f_1.\end{aligned} \tag{2.14}$$

The proof is thus completed. □

Observe that we can also use Lemma 2.1 to compute $\partial_x\Xi$. Consistency for (2.11) and (2.14) requires that g_1 satisfies the following equation:

$$\begin{aligned}&\partial_x\left(-\partial_x g_1+\frac{1}{\sqrt{2}}\partial_x^3 g_1-(\partial_x g_1)^2-\Pi_1\right)\\ &=\partial_x^2 g_1\left(\frac{\sqrt{2}}{\sqrt{2}-\varepsilon^2}+4\partial_x g_1\right)+\frac{2}{\sqrt{2}-\varepsilon^2}\partial_y^2 g_1+\frac{2}{\sqrt{2}-\varepsilon^2}(g_1\Pi_1-\Pi_3)\\ &\quad+\frac{\varepsilon^2}{2-\sqrt{2}\varepsilon^2}\partial_x g_1\partial_x(g_1^2)-\frac{\varepsilon^2}{2\sqrt{2}-2\varepsilon^2}g_1^2\partial_x(g_1^2)-\frac{\sqrt{2}\varepsilon^2}{\sqrt{2}-\varepsilon^2}g_1\partial_x^2 f_1.\end{aligned}$$

It can be regarded as a perturbed KP-I equation:

$$\begin{aligned}&\frac{1}{\sqrt{2}}\partial_x^4 g_1-\frac{2\sqrt{2}-\varepsilon^2}{\sqrt{2}-\varepsilon^2}\partial_x^2 g_1-3\partial_x\left((\partial_x g_1)^2\right)-\frac{2}{\sqrt{2}-\varepsilon^2}\partial_y^2 g_1\\ &=\partial_x\Pi_1+\frac{2}{\sqrt{2}-\varepsilon^2}(g_1\Pi_1-\Pi_3)+\frac{\varepsilon^2}{2-\sqrt{2}\varepsilon^2}\partial_x g_1\partial_x(g_1^2)\\ &\quad-\frac{\varepsilon^2}{2\sqrt{2}-2\varepsilon^2}g_1^2\partial_x(g_1^2)-\frac{\sqrt{2}\varepsilon^2}{\sqrt{2}-\varepsilon^2}g_1\partial_x^2 f_1.\end{aligned} \tag{2.15}$$

Substituting (2.8) and (2.9) into $\partial_x\Pi_1+\frac{2}{\sqrt{2}-\varepsilon^2}g_1\Pi_1$ we have

$$\begin{aligned}\partial_x\Pi_1+\frac{2}{\sqrt{2}-\varepsilon^2}g_1\Pi_1 =& -\varepsilon^2\partial_x\partial_y^2 f_1-\varepsilon^4\partial_x\partial_y^2 f_2-\varepsilon^2\partial_x^3 f_2+\partial_x\Pi_2\\ &-\frac{2}{\sqrt{2}-\varepsilon^2}g_1\left(\varepsilon^2\partial_y^2 f_1+\varepsilon^4\partial_y^2 f_2+\varepsilon^2\partial_x^2 f_2-\Pi_2\right).\end{aligned} \tag{2.16}$$

Using (2.12), we can further write (2.16) as

$$\begin{aligned}&\partial_x\Pi_1+\frac{2}{\sqrt{2}-\varepsilon^2}g_1\Pi_1\\ &=-\varepsilon^2\partial_x\partial_y^2\left(\frac{\sqrt{2}}{2}\partial_x g_1-\frac{g_1^2}{2}\right)-\frac{2\varepsilon^2}{\sqrt{2}-\varepsilon^2}g_1\partial_y^2\left(\frac{\sqrt{2}}{2}\partial_x g_1-\frac{g_1^2}{2}\right)+\partial_x\Pi_2+\frac{2}{\sqrt{2}-\varepsilon^2}g_1\Pi_2\\ &\quad-\frac{\varepsilon^4}{\sqrt{2}-\varepsilon^2}\partial_y^2\left(\partial_x f_1+\partial_y^2 g_1-g_1 f_1^2-\Pi_3\right)+\frac{2\varepsilon^4}{\sqrt{2}-\varepsilon^2}\left(\partial_y^2 g_1 f_2+2\partial_y g_1\partial_y f_2\right)\\ &\quad-\frac{\varepsilon^2}{\sqrt{2}-\varepsilon^2}\partial_x^2\left(\partial_x f_1+\partial_y^2 g_1-g_1 f_1^2-\Pi_3\right)+\frac{2\varepsilon^2}{\sqrt{2}-\varepsilon^2}\left(\partial_x^2 g_1 f_2+2\partial_x g_1\partial_x f_2\right)\\ &=-\frac{2}{\sqrt{2}-\varepsilon^2}\varepsilon^2\partial_x^2\partial_y^2 g_1-\frac{\varepsilon^2}{2-\sqrt{2}\varepsilon^2}\partial_x^4 g_1-\frac{\varepsilon^4}{\sqrt{2}-\varepsilon^2}\partial_y^4 g_1+\Pi_4,\end{aligned} \tag{2.17}$$

where

$$
\begin{aligned}
\Pi_4 = & \frac{\varepsilon^2}{2}\partial_x\partial_y^2(g_1^2) - \frac{\sqrt{2}\varepsilon^2}{\sqrt{2}-\varepsilon^2} g_1\partial_x\partial_y^2 g_1 + \frac{\varepsilon^2}{\sqrt{2}-\varepsilon^2} g_1\partial_y^2 g_1^2 + \frac{\varepsilon^4}{\sqrt{2}-\varepsilon^2}\partial_y^2(g_1\partial_x g_1) \\
& + \frac{\varepsilon^2}{\sqrt{2}-\varepsilon^2}\partial_x^2(g_1\partial_x g_1) + \frac{\varepsilon^4}{\sqrt{2}-\varepsilon^2}\partial_y^2(g_1 f_1^2 + \Pi_3) + \frac{\varepsilon^2}{\sqrt{2}-\varepsilon^2}\partial_x^2(g_1 f_1^2 + \Pi_3) \\
& + \frac{2\varepsilon^4}{\sqrt{2}-\varepsilon^2}\left(\partial_y^2 g_1 f_2 + 2\partial_y g_1 \partial_y f_2\right) + \frac{2\varepsilon^2}{\sqrt{2}-\varepsilon^2}\left(\partial_x^2 g_1 f_2 + 2\partial_x g_1 \partial_x f_2\right) \\
& + \partial_x \Pi_2 + \frac{2}{\sqrt{2}-\varepsilon^2} g_1 \Pi_2.
\end{aligned}
\tag{2.18}
$$

Based on (2.17) and (2.18) we can write (2.15) as

$$
\begin{aligned}
& \partial_x^4 g_1 - (2\sqrt{2}-\varepsilon^2)\partial_x^2 g_1 - 3(\sqrt{2}-\varepsilon^2)\partial_x((\partial_x g_1)^2) - 2\partial_y^2 g_1 + 2\varepsilon^2\partial_x^2\partial_y^2 g_1 + \varepsilon^4\partial_y^4 g_1 \\
& = (\sqrt{2}-\varepsilon^2)\Pi_4 - 2\Pi_3 + \frac{1}{\sqrt{2}}\varepsilon^2\partial_x g_1 \partial_x(g_1^2) - \frac{1}{2}\varepsilon^2 g_1^2 \partial_x(g_1^2) - \sqrt{2}\varepsilon^2 g_1 \partial_x^2 f_1.
\end{aligned}
\tag{2.19}
$$

We denote the right hand side of (2.19) by P and it can be written as

$$
\begin{aligned}
P = & \left(\frac{\sqrt{2}-\varepsilon^2}{2}\right)\varepsilon^2\partial_x\partial_y^2(g_1^2) - \sqrt{2}\varepsilon^2 g_1\partial_x\partial_y^2 g_1 + \varepsilon^2 g_1\partial_y^2(g_1^2) + \varepsilon^4\partial_y^2(g_1\partial_x g_1) \\
& + \varepsilon^2\partial_x^2(g_1\partial_x g_1) + \varepsilon^4\partial_y^2\left(g_1 f_1^2 + 2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2\right) \\
& + \varepsilon^2\partial_x^2\left(g_1 f_1^2 + 2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2\right) + 2\varepsilon^4(\partial_y^2 g_1 f_2 + 2\partial_y g_1\partial_y f_2) \\
& + 2\varepsilon^2(\partial_x^2 g_1 f_2 + 2\partial_x g_1 \partial_x f_2) - 2(2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2) \\
& + (\sqrt{2}-\varepsilon^2)\partial_x\left(6\varepsilon^2 f_1 f_2 + \varepsilon^2(f_1+\varepsilon^2 f_2)^3 + 3\varepsilon^4 f_2^2 + \varepsilon^2 f_2 g_1^2\right) \\
& + 2g_1\left(6\varepsilon^2 f_1 f_2 + \varepsilon^2(f_1+\varepsilon^2 f_2)^3 + 3\varepsilon^4 f_2^2 + \varepsilon^2 f_2 g_1^2\right) \\
& + \frac{1}{\sqrt{2}}\varepsilon^2\partial_x g_1 \partial_x g_1^2 - \frac{1}{2}\varepsilon^2 g_1^2\partial_x(g_1^2) - \sqrt{2}\varepsilon^2 g_1\partial_x^2 f_1.
\end{aligned}
\tag{2.20}
$$

Among the terms in (2.20), we shall analyze the following four terms $g_1\partial_x\partial_y^2 g_1$, $g_1\partial_x^2 f_1$, $\partial_y^2 g_1 f_2 + 2\partial_y g_1 \partial_y f_2$ and $\partial_x^2 g_1 f_2 + 2\partial_x g_1\partial_x f_2$. For the first one we could rewrite it as

$$
g_1\partial_x\partial_y^2 g_1 = \partial_x(g_1\partial_y^2 g_1) - \partial_y(\partial_x g_1 \partial_y g_1) + \frac{1}{2}\partial_x((\partial_y g_1)^2). \tag{2.21}
$$

Using (2.6), we could write the second one as

$$
g_1\partial_x^2 f_1 = \partial_x(g_1\partial_x f_1) - \frac{\sqrt{2}}{4}\partial_x((\partial_x g_1)^2) + \frac{1}{2}\partial_x g_1 \partial_x(g_1^2). \tag{2.22}
$$

For the third term $\partial_y^2 g_1 f_2 + 2\partial_y g_1 \partial_y f_2$, based on (2.6) and (2.12) we obtain that

$$\begin{aligned}(\partial_y^2 g_1 f_2 + 2\partial_y g_1 \partial_y f_2) &= 2\partial_y(\partial_y g_1 f_2) - \partial_y^2 g_1 f_2 \\ &= 2\partial_y(\partial_y g_1 f_2) + \left(\partial_x f_1 - (\sqrt{2}-\varepsilon^2)\partial_x f_2 - g_1(2f_2 + f_1^2)\right) f_2 \\ &\quad - (2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2) f_2 \\ &= 2\partial_y(\partial_y g_1 f_2) - \frac{\sqrt{2}-\varepsilon^2}{2}\partial_x(f_2^2) - g_1(2f_2 + f_1^2) f_2 \\ &\quad - (2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2) f_2 + \partial_x(f_1 f_2) \\ &\quad + \frac{1}{\sqrt{2}-\varepsilon^2}\left(-\partial_x f_1 - \partial_y^2 g_1 + g_1(2f_2 + f_1^2)\right) f_1 \\ &\quad + \frac{1}{\sqrt{2}-\varepsilon^2}(2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2) f_1.\end{aligned} \tag{2.23}$$

By (2.6) we also notice that

$$\begin{aligned}\partial_y^2 g_1 f_1 &= \partial_y(\partial_y g_1 f_1) - \partial_y g_1 \left(\frac{1}{\sqrt{2}}\partial_x \partial_y g_1 - \frac{1}{2}\partial_y(g_1^2)\right) \\ &= \partial_y(\partial_y g_1 f_1) - \frac{1}{2\sqrt{2}}\partial_x((\partial_y g_1)^2) + \frac{1}{2}\partial_y g_1 \partial_y(g_1^2).\end{aligned}$$

Then we can write (2.23) as

$$\begin{aligned}(\partial_y^2 g_1 f_2 + 2\partial_y g_1 \partial_y f_2) &= 2\partial_y(\partial_y g_1 f_2) - \frac{\sqrt{2}-\varepsilon^2}{2}\partial_x(f_2^2) + \partial_x(f_1 f_2) - \frac{1}{2\sqrt{2}-2\varepsilon^2}\partial_x f_1^2 \\ &\quad - \frac{1}{\sqrt{2}-\varepsilon^2}\partial_y(\partial_y g_1 f_1) + \frac{1}{4-2\sqrt{2}\varepsilon^2}\partial_x((\partial_y g_1)^2) \\ &\quad - \frac{1}{2\sqrt{2}-2\varepsilon^2}\partial_y g_1 \partial_y(g_1^2) - g_1(2f_2 + f_1^2) f_2 \\ &\quad - (2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2) f_2 + \frac{1}{\sqrt{2}-\varepsilon^2} g_1(2f_2 + f_1^2) f_1 \\ &\quad + \frac{1}{\sqrt{2}-\varepsilon^2}(2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2) f_1.\end{aligned} \tag{2.24}$$

Using (2.6) and (2.12) again we rewrite $\partial_x^2 g_1 f_2 + 2\partial_x g_1 \partial_x f_2$ as

$$\begin{aligned}(\partial_x^2 g_1 f_2 + 2\partial_x g_1 \partial_x f_2) &= \partial_x(\partial_x g_1 f_2) + \frac{1}{\sqrt{2}-\varepsilon^2}\left(\partial_x f_1 + \partial_y^2 g_1 - g_1(2f_2 + f_1^2)\right)\partial_x g_1 \\ &\quad - \frac{1}{\sqrt{2}-\varepsilon^2}\partial_x g_1(2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2) \\ &= \partial_x(\partial_x g_1 f_2) + \frac{1}{4-2\sqrt{2}\varepsilon^2}\partial_x((\partial_x g_1)^2) + \frac{1}{\sqrt{2}-\varepsilon^2}\partial_y(\partial_y g_1 \partial_x g_1) \\ &\quad - \frac{1}{2\sqrt{2}-2\varepsilon^2}\partial_x((\partial_y g_1)^2) - \frac{1}{2(\sqrt{2}-\varepsilon^2)}\partial_x(g_1^2)\partial_x g_1 \\ &\quad - \frac{1}{\sqrt{2}-\varepsilon^2}\left(g_1(2f_2 + f_1^2)\partial_x g_1 + \partial_x g_1(2\varepsilon^2 f_1 f_2 g_1 + \varepsilon^4 g_1 f_2^2)\right).\end{aligned} \tag{2.25}$$

Substituting (2.21), (2.22), (2.24) and (2.25) into (2.20) we have

$$P = P_1 + P_2 + P_3, \tag{2.26}$$

where

$$\begin{aligned} P_1 &= \varepsilon^2\partial_x^2(g_1\partial_x g_1) + \varepsilon^2\partial_x^2(g_1f_1^2 + 2\varepsilon^2 f_1f_2g_1 + \varepsilon^4 g_1f_2^2) - (\sqrt{2}-\varepsilon^2)\varepsilon^4\partial_x(f_2^2) \\ &\quad - \frac{\varepsilon^4}{\sqrt{2}-\varepsilon^2}\partial_x f_1^2 + 2\varepsilon^2\partial_x(\partial_x g_1 f_2) + \frac{2\varepsilon^2 - \frac{\sqrt{2}}{2}\varepsilon^4}{2-\sqrt{2}\varepsilon^2}\partial_x(\partial_x g_1)^2 \\ &\quad + (\sqrt{2}-\varepsilon^2)\partial_x\left(6\varepsilon^2 f_1f_2 + \varepsilon^2(f_1+\varepsilon^2 f_2)^3 + 3\varepsilon^4 f_2^2 + \varepsilon^2 f_2 g_1^2\right) \\ &\quad + 2\varepsilon^4\partial_x(f_1f_2) - \sqrt{2}\varepsilon^2\partial_x(g_1\partial_x f_1), \end{aligned} \tag{2.27}$$

$$\begin{aligned} P_2 &= \sqrt{2}\varepsilon^2\partial_y(\partial_x g_1\partial_y g_1) + \varepsilon^4\partial_y^2\left(g_1f_1^2 + 2\varepsilon^2 f_1f_2g_1 + \varepsilon^4 g_1f_2^2\right) \\ &\quad + 4\varepsilon^4\partial_y(\partial_y g_1 f_2) - \frac{2\varepsilon^4}{\sqrt{2}-\varepsilon^2}\partial_y(\partial_y g_1 f_1) + \frac{2\varepsilon^2}{\sqrt{2}-\varepsilon^2}\partial_y(\partial_y g_1\partial_x g_1), \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} P_3 &= \varepsilon^2 g_1\partial_y^2(g_1^2) - \frac{\varepsilon^4}{\sqrt{2}-\varepsilon^2}\partial_y g_1\partial_y(g_1^2) - 2\varepsilon^4 g_1(2f_2+f_1^2)f_2 \\ &\quad - 2\varepsilon^4(2\varepsilon^2 f_1f_2g_1 + \varepsilon^4 g_1f_2^2)f_2 + \frac{2\varepsilon^4}{\sqrt{2}-\varepsilon^2}g_1(2f_2+f_1^2)f_1 \\ &\quad + \frac{2\varepsilon^4}{\sqrt{2}-\varepsilon^2}(2\varepsilon^2 f_1f_2g_1 + \varepsilon^4 g_1f_2^2)f_1 - \frac{\varepsilon^2}{\sqrt{2}-\varepsilon^2}\partial_x(g_1^2)\partial_x g_1 \\ &\quad - \frac{2\varepsilon^2}{\sqrt{2}-\varepsilon^2}\left(g_1(2f_2+f_1^2)\partial_x g_1 + \partial_x g_1(2\varepsilon^2 f_1f_2g_1 + \varepsilon^4 g_1f_2^2)\right) \\ &\quad + 2g_1\left(6\varepsilon^2 f_1f_2 + \varepsilon^2(f_1+\varepsilon^2 f_2)^3 + 3\varepsilon^4 f_2^2 + \varepsilon^2 f_2g_1^2\right) \\ &\quad - 2(2\varepsilon^2 f_1f_2g_1 + \varepsilon^4 g_1f_2^2) - \frac{1}{2}\varepsilon^2 g_1^2\partial_x(g_1^2). \end{aligned} \tag{2.29}$$

With (2.26), and (2.27)-(2.29) we write (2.19) as

$$\begin{aligned} &\partial_x^4 g_1 - (2\sqrt{2}-\varepsilon^2)\partial_x^2 g_1 - 3(\sqrt{2}-\varepsilon^2)\partial_x((\partial_x g_1)^2) - 2\partial_y^2 g_1 + 2\varepsilon^2\partial_x^2\partial_y^2 g_1 + \varepsilon^4\partial_y^4 g_1 \\ &= P_1 + P_2 + P_3. \end{aligned} \tag{2.30}$$

Observe that P_1 and P_2 can be regarded as two functions $\partial_x H_1$ and $\partial_y H_2$ respectively for some proper functions H_1 and H_2, and we shall show that P_3 decays faster than r^{-3} at infinity. The advantage of writing (2.19) as (2.30) is that we could derive a better decay for the perturbation function.

Concerning the left hand side of (2.30), we shall look for solution $g_1 = q + \phi$, where q is a lump solution of a rescaled KP-I

$$\partial_x^4 q - (2\sqrt{2}-\varepsilon^2)\partial_x^2 q - 3\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{5/2}\partial_x((\partial_x q)^2) - 2\partial_y^2 q = 0. \tag{2.31}$$

(Note that the constant in the third term in (2.31) is slightly different from the constant in (2.30) because we also collect the first term in P_1, for the convenience of proof.)

Actually, (2.31) is closely related to one of KP-I equation after suitable rescaling. In the rest of the proof we take the standard lump solution

$$q_\varepsilon(x,y) = -\left(\frac{2\sqrt{2}}{2\sqrt{2}-\varepsilon^2}\right)^2 \frac{\sqrt{8-2\sqrt{2}\varepsilon^2}x}{\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}x^2 + \frac{(2\sqrt{2}-\varepsilon^2)^2}{4\sqrt{2}}y^2 + \frac{3}{2\sqrt{2}}}, \tag{2.32}$$

whose nondegeneracy is known ([**?**]). The proof works exactly the same for other nondegenerate lump solutions. The associated unique negative eigenvalue and eigenfunction with respect to q_ε are well approximated by the ones of the limit equation (sending ε to 0)

$$\partial_x^4 q - 2\sqrt{2}\partial_x^2 q - 3\sqrt{2}\partial_x((\partial_x q)^2) - 2\partial_y^2 q = 0, \tag{2.33}$$

where the corresponding solution is

$$q_0(x,y) = -\frac{2\sqrt{2}x}{x^2 + \sqrt{2}y^2 + \frac{3}{2\sqrt{2}}}.$$

As we can see that the linearized problem of (2.30) is well approximated by the one of KP-I equation. However, we can not just ignore these two terms. Indeed, we have seen that $\partial_y^2 f_2$ appears in P and it is related to $\partial_y^4 g_1$ by (2.12), so we need to get some control on $\partial_y^4 g_1$ in showing the existence of f_2. In next section we shall study the Green kernel of the linear operator associated to (2.30).

3. The Green Function

In this section, we shall study the Green kernel $G(x,y)$ of the differential operator (introduced in section 2),

$$\partial_x^4 - (2\sqrt{2}-\varepsilon^2)\partial_x^2 - 2\partial_y^2 + \varepsilon^2\partial_x^2\partial_y^2 + \varepsilon^4\partial_y^4. \tag{3.1}$$

To simplify our discussion, we set all the constant coefficients to be 1 (after suitable scaling), i.e.,

$$\partial_x^4 - \partial_x^2 - \partial_y^2 + \varepsilon^2\partial_x^2\partial_y^2 + \varepsilon^4\partial_y^4.$$

It is not difficult to see that the associated Green kernel (denoted by $K(x,y)$) has similar asymptotic behavior at singularity and decay properties at infinity as $G(x,y)$.

The Fourier transform of $K(x,y)$ can be written as

$$\hat{K}(\xi_1,\xi_2) = \frac{1}{\xi_1^4 + \xi_1^2 + \xi_2^2 + \varepsilon^2\xi_1^2\xi_2^2 + \varepsilon^4\xi_2^4}.$$

The main purpose of this section is devoted to studying the decay property and asymptotic behavior of the derivatives of $K(x,y)$. Based on the elementary computation from Fourier analysis, we know that

$$\widehat{\partial_x^m\partial_y^n K}(\xi_1,\xi_2) = (-i)^{m+n}\xi_1^m\xi_2^n\hat{K}(\xi_1,\xi_2).$$

In order to study the asymptotic behavior of $\partial_x^m\partial_y^n K(x,y)$, it is enough to compute the following integration when $r = \sqrt{x^2+y^2}$ tends to ∞ and 0,

$$K_{m,n}(x,y) := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1^m\xi_2^n e^{ix\xi_1+iy\xi_2}}{\xi_1^4 + \xi_1^2 + \xi_2^2 + \varepsilon^2\xi_1^2\xi_2^2 + \varepsilon^4\xi_2^4} d\xi_1 d\xi_2. \tag{3.2}$$

In our problem, the behavior of $K_{m,n}(x,y)$ for the following cases are needed

$$(m,n) \in \{(1,0),(2,0),(3,0),(0,1),(0,2),(0,3),(1,1),(1,2)\}.$$

Now we shall state the corresponding properties for $K_{m,n}(x,y)$ in the following two lemmas. The first result is as follows

Lemma 3.1. *Let $K_{m,0}(x,y)$ be defined in (3.2) with $m=1,2,3$. Then for r small we have*

$$r|K_{m,0}(x,y)| \le \begin{cases} C, & \text{for} \quad m=1,\\ C\left(\min\left\{\log\frac{1+\sqrt{|y|}}{\sqrt{|y|}},\log\frac{1}{\varepsilon}\right\}+e^{-\frac{C}{\varepsilon^2}y}\right), & \text{for} \quad m=2,\\ C\left(\min\left\{\frac{1}{\varepsilon},\frac{1}{\sqrt{|y|}}\right\}+\frac{1}{\varepsilon}\max\left\{1,\log\frac{\varepsilon^2}{|y|}\right\}e^{-\frac{C}{\varepsilon^2}|y|}\right), & \text{for} \quad m=3.\end{cases}$$

While for r large we have

$$r|K_{m,0}(x,y)| \le \begin{cases} C, & \text{for} \quad m=1,\\ \frac{C}{r^{\frac{1}{2}}}, & \text{for} \quad m=2,\\ \frac{C}{\varepsilon^{\frac{1}{2}}r^{\frac{1}{2}}} & \text{for} \quad m=3.\end{cases}$$

Proof. By definition we have

$$K_{m,0}(x,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\xi_1^m e^{ix\xi_1+iy\xi_2}}{\xi_1^4+\xi_1^2+\xi_2^2+\varepsilon^2\xi_1^2\xi_2^2+\varepsilon^4\xi_2^4}d\xi_1 d\xi_2. \tag{3.3}$$

Concerning the denominator, we write it as

$$\xi_1^4+\xi_1^2+\xi_2^2+\varepsilon^2\xi_1^2\xi_2^2+\varepsilon^4\xi_2^4=\varepsilon^4(\xi_2^2+a(\xi_1))(\xi_2^2+b(\xi_1)),$$

where $a(\xi_1)$, $b(\xi_1)$ are functions of ξ_1 defined as:

$$a(\xi_1)=\frac{1+\varepsilon^2\xi_1^2-D(\xi_1)}{2\varepsilon^4},\quad b(\xi_1)=\frac{1+\varepsilon^2\xi_1^2+D(\xi_1)}{2\varepsilon^4},$$

and

$$D(x):=\sqrt{(1+\varepsilon^2 x)^2-4\varepsilon^4(x^2+x^4)}=\sqrt{1+2\varepsilon^2x^2-4\varepsilon^4x^2-3\varepsilon^4x^4}.$$

The function $D(x)$ can be decomposed by

$$D(x)=\sqrt{-3\varepsilon^4(x^2-c_\varepsilon^2)(x^2+d_\varepsilon^2)},$$

where

$$\begin{aligned} c_\varepsilon^2 =& \frac{1-2\varepsilon^2+2\sqrt{1-\varepsilon^2+\varepsilon^4}}{3\varepsilon^2}=\frac{1}{\varepsilon^2}-1+O(\varepsilon^2) \text{ as } \varepsilon\to 0,\\ d_\varepsilon^2 =& \frac{-1+2\varepsilon^2+2\sqrt{1-\varepsilon^2+\varepsilon^4}}{3\varepsilon^2}=\frac{1}{3\varepsilon^2}+\frac{1}{3}+O(\varepsilon^2) \text{ as } \varepsilon\to 0.\end{aligned} \tag{3.4}$$

Then we get that $D(x)$ admits two roots

$$c_\varepsilon=\frac{1}{\varepsilon}-\frac{\varepsilon}{2}+O(\varepsilon^2) \quad \text{and} \quad -c_\varepsilon=-\frac{1}{\varepsilon}+\frac{\varepsilon}{2}+O(\varepsilon^2) \quad \text{as} \quad \varepsilon\to 0.$$

Around the root points c_ε and $-c_\varepsilon$, we have

$$D(x)=\begin{cases}\sqrt{6\varepsilon^4c_\varepsilon(c_\varepsilon^2+d_\varepsilon^2)}\left((c_\varepsilon-x)^{\frac{1}{2}}+O((x-c_\varepsilon)^{\frac{3}{2}})\right) & \text{as} \quad x\to c_\varepsilon,\\ \sqrt{6\varepsilon^4c_\varepsilon(c_\varepsilon^2+d_\varepsilon^2)}\left((x+c_\varepsilon)^{\frac{1}{2}}+O((x+c_\varepsilon)^{\frac{3}{2}})\right) & \text{as} \quad x\to -c_\varepsilon.\end{cases} \tag{3.5}$$

It is known that $D(\xi)$ is positive when $|\xi| < c_\varepsilon$ and negative for $|\xi| > c_\varepsilon$. We write the integral (3.3) as

$$K_{m,0}(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\xi_1^m}{D(\xi_1)} \left(\frac{e^{ix\xi_1+iy\xi_2}}{\xi_2^2+a(\xi_1)} - \frac{e^{ix\xi_1+iy\xi_2}}{\xi_2^2+b(\xi_1)} \right) d\xi_2 d\xi_1. \tag{3.6}$$

We shall first study $K_{m,0}(x,y)$ for the case $y \neq 0$. Without loss of generality, we may assume that $y > 0$. From the above discussion on $D(x)$ we see that both $a(\xi_1)$ and $b(\xi_1)$ turns to be imaginary number if $|\xi_1| > c_\varepsilon$. In our calculations, for convenience we always assume that the real parts of $\sqrt{a(\xi_1)}$ and $\sqrt{b(\xi_1)}$ are positive. Then using the Residue Theorem we could write (3.6) as

$$K_{m,0}(x,y) = \pi \int_{\mathbb{R}} \frac{\xi^m}{D(\xi)} \left(\frac{e^{ix\xi-\sqrt{a(\xi)}y}}{\sqrt{a(\xi)}} - \frac{e^{ix\xi-\sqrt{b(\xi)}y}}{\sqrt{b(\xi)}} \right) d\xi. \tag{3.7}$$

To estimate the above integral (3.7), we write

$$\begin{aligned}
&\frac{e^{ix\xi-\sqrt{a(\xi)}y}}{\sqrt{a(\xi)}} - \frac{e^{ix\xi-\sqrt{b(\xi)}y}}{\sqrt{b(\xi)}} \\
&= \frac{e^{ix\xi-\sqrt{a(\xi)}y}}{\sqrt{a(\xi)}} - \frac{e^{ix\xi-\sqrt{a(\xi)}y}}{\sqrt{b(\xi)}} + \frac{e^{ix\xi-\sqrt{a(\xi)}y}}{\sqrt{b(\xi)}} - \frac{e^{ix\xi-\sqrt{b(\xi)}y}}{\sqrt{b(\xi)}} \\
&= e^{ix\xi-\sqrt{a(\xi)}y} \left(\frac{b(\xi)-a(\xi)}{\sqrt{a(\xi)b(\xi)}(\sqrt{a(\xi)}+\sqrt{b(\xi)})} + \frac{1-e^{(\sqrt{a(\xi)}-\sqrt{b(\xi)})y}}{\sqrt{b(\xi)}} \right).
\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) and we decompose $K_{m,0}(x,y)$ into two parts

$$\begin{aligned}
K_{m,0}(x,y) &= \int_{\mathbb{R}} \frac{\pi\xi^m}{D(\xi)} \frac{e^{ix\xi-\sqrt{a(\xi)}y}(b(\xi)-a(\xi))}{\sqrt{a(\xi)b(\xi)}(\sqrt{a(\xi)}+\sqrt{b(\xi)})} d\xi \\
&\quad + \int_{\mathbb{R}} \frac{\pi\xi^m}{D(\xi)} \frac{e^{ix\xi-\sqrt{a(\xi)}y}}{\sqrt{b(\xi)}} \left(1-e^{(\sqrt{a(\xi)}-\sqrt{b(\xi)})y}\right) d\xi \\
&=: \pi(I_1+I_2).
\end{aligned}$$

Concerning the integrand in I_1, we could write it as

$$\begin{aligned}
\frac{\xi^m}{D(\xi)} \frac{e^{ix\xi-\sqrt{a(\xi)}y}(b(\xi)-a(\xi))}{\sqrt{a(\xi)b(\xi)}(\sqrt{a(\xi)}+\sqrt{b(\xi)})} &= \frac{\xi^m e^{ix\xi-\sqrt{a(\xi)}y}}{\varepsilon^4\sqrt{b(\xi)}(\sqrt{a(\xi)b(\xi)}+a(\xi))} \\
= e^{ix\xi-\sqrt{a(\xi)}y} &\frac{\xi^m\sqrt{1+\varepsilon^2\xi^2-D(\xi)}}{\sqrt{2}\varepsilon^2\xi^2(1+\xi^2)+\frac{\sqrt{2}}{2}|\xi|\sqrt{1+\xi^2}(1+\varepsilon^2\xi^2-D(\xi))}.
\end{aligned}$$

We set

$$M_m(\xi) := \frac{\xi^m\sqrt{1+\varepsilon^2\xi^2-D(\xi)}}{\sqrt{2}\varepsilon^2\xi^2(1+\xi^2)+\frac{\sqrt{2}}{2}|\xi|\sqrt{1+\xi^2}(1+\varepsilon^2\xi^2-D(\xi))}. \tag{3.9}$$

It is not difficult to check that

$$M_m(\xi) = sgn(\xi)\xi^{m-1} + O(\xi^m) \quad \text{as} \quad |\xi| \to 0. \tag{3.10}$$

Therefore we can see that $M_m(\xi)$ are all bounded at 0 and $M_2(\xi)$, $M_3(\xi) = 0$ at $\xi = 0$.

Next we introduce a cut-off function $\chi_\varepsilon(\xi)$, defined by

$$\chi_\varepsilon(\xi)=\begin{cases}1, & \text{if} \quad |\xi|\in\left[\frac{3}{4}c_\varepsilon,\frac{3}{2}c_\varepsilon\right],\\ \\ 0, & \text{if} \quad |\xi|\in[0,+\infty)\setminus\left[\frac{1}{2}c_\varepsilon,2c_\varepsilon\right].\end{cases} \tag{3.11}$$

We decompose I_1 as

$$\begin{aligned}I_1&=\int_{\mathbb{R}}(1-\chi_\varepsilon(\xi))e^{ix\xi-\sqrt{a(\xi)}y}M_m(\xi)d\xi+\int_{\mathbb{R}}\chi_\varepsilon(\xi)e^{ix\xi-\sqrt{a(\xi)}y}M_m(\xi)d\xi\\&=I_{11}+I_{12}.\end{aligned} \tag{3.12}$$

From (3.10) we can see that the integrand function of I_{11} is discontinuous at $\xi=0$ for $m=1$, continuous for $m=2$ and differentiable for $m=3$. Using integration by parts, we have

$$\begin{aligned}I_{11}=&\lim_{\xi\to0^-}\frac{e^{ix\xi-\sqrt{a(\xi)}y}}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}M_m(\xi)-\lim_{\xi\to0^+}\frac{e^{ix\xi-\sqrt{a(\xi)}y}}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}M_m(\xi)\\&-\int_{\mathbb{R}}e^{ix\xi-\sqrt{a(\xi)}y}\partial_\xi\left(\frac{(1-\chi_\varepsilon(\xi))M_m(\xi)}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}\right)d\xi\\=&I_{111}+I_{112}+I_{113}.\end{aligned} \tag{3.13}$$

For the right hand side of (3.13), we notice that the boundary terms vanish for $m=2,3$ and equals to $\frac{2ix}{x^2+y^2}$ for $m=1$. Then we have

$$|I_{111}|+|I_{112}|\begin{cases}\leq\frac{C}{r}, & \text{for} \quad m=1,\\ \\ =0, & \text{for} \quad m=2,3.\end{cases} \tag{3.14}$$

Concerning the integrand term of I_{113}, we have

$$\begin{aligned}\partial_\xi\left(\frac{(1-\chi_\varepsilon(\xi))M_m(\xi)}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}\right)=&\frac{\frac{2a''(\xi)a(\xi)-(a'(\xi))^2}{4(a(\xi))^{\frac{3}{2}}}}{\left(ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y\right)^2}y(1-\chi_\varepsilon(\xi))M_m(\xi)-\frac{\chi_\varepsilon'(\xi)M_m(\xi)}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}\\&+\frac{1-\chi_\varepsilon(\xi)}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}M_m'(\xi)\\=&I_{1131}+I_{1132}+I_{1133}.\end{aligned}$$

By direct computation we could derive the following estimate

$$|I_{1131}|+|I_{1132}|+|I_{1133}|\leq C\frac{|\xi|^{m-1}\min\left\{\varepsilon|\xi|,\varepsilon^2\xi^2\right\}}{\varepsilon^2(1+\xi^4)r}. \tag{3.15}$$

By (3.15) we get

$$
\begin{aligned}
|I_{113}| &\le \frac{C}{r}\int_{-2c_\varepsilon}^{2c_\varepsilon}\frac{e^{-\sqrt{a(\xi)}y}}{(1+|\xi|)^{3-m}}d\xi+\frac{C}{r}\int_{\mathbb{R}\setminus[-2c_\varepsilon,2c_\varepsilon]}\frac{1}{\varepsilon|\xi|^{4-m}}e^{-\frac{C|\xi|}{\varepsilon}y}d\xi \\
&\le \begin{cases}\frac{C}{r}, & \text{if} \quad m=1,\\ C\left(\min\left\{\log\frac{1+\sqrt{y}}{\sqrt{y}},\log\frac{1}{\varepsilon}\right\}+e^{-\frac{C}{\varepsilon^2}y}\right)\frac{1}{r}, & \text{if} \quad m=2,\\ C\left(\min\left\{\frac{1}{\sqrt{y}},\frac{1}{\varepsilon}\right\}+\frac{1}{\varepsilon}\max\left\{1,\log\frac{\varepsilon^2}{y}\right\}e^{-\frac{C}{\varepsilon^2}y}\right)\frac{1}{r}, & \text{if} \quad m=3,\end{cases}
\end{aligned}
\tag{3.16}
$$

where we used that

$$
\int_0^{2c_\varepsilon}\frac{1}{1+\xi}e^{-C\xi^2y}d\xi\le C\int_0^{2c_\varepsilon\sqrt{y}}\frac{1}{\xi+\sqrt{y}}e^{-C\xi^2}d\xi\le C\min\left\{\log\frac{1+\sqrt{y}}{\sqrt{y}},\log\frac{1}{\varepsilon}\right\},
$$

$$
\int_0^{2c_\varepsilon}e^{-C\xi^2y}d\xi\le\frac{C}{\sqrt{y}}\int_0^{Cc_\varepsilon\sqrt{y}}e^{-\xi^2}d\xi\le C\min\left\{\frac{1}{\sqrt{y}},\frac{1}{\varepsilon}\right\},
$$

and

$$
\begin{aligned}
\int_{2c_\varepsilon}^{\infty}\frac{1}{\xi}e^{-C\frac{\xi}{\varepsilon}y}d\xi &= Ce^{-\frac{C}{\varepsilon^2}y}\int_{\frac{2Cc_\varepsilon y}{\varepsilon}}^{\infty}\frac{1}{\xi}e^{-\xi}d\xi \\
&\le Ce^{-\frac{C}{\varepsilon^2}y}\left(\int_{\frac{2Cc_\varepsilon y}{\varepsilon}}^{\max\{1,\frac{2Cc_\varepsilon y}{\varepsilon}\}}\frac{1}{\xi}d\xi+\int_{\max\{1,\frac{2Cc_\varepsilon y}{\varepsilon}\}}^{\infty}e^{-\xi}d\xi\right) \\
&\le C\max\left\{1,\log\frac{\varepsilon^2}{y}\right\}e^{-\frac{C}{\varepsilon^2}y}.
\end{aligned}
$$

The estimations (3.16) and (3.14) give the asymptotic behavior for I_{11} when r is small, i.e.,

$$
|I_{11}|\le\begin{cases}\frac{C}{r}, & \text{if} \quad m=1,\\ C\left(\min\left\{\log\frac{1+\sqrt{y}}{\sqrt{y}},\log\frac{1}{\varepsilon}\right\}+e^{-\frac{C}{\varepsilon^2}y}\right)\frac{1}{r}, & \text{if} \quad m=2,\\ C\left(\min\left\{\frac{1}{\sqrt{y}},\frac{1}{\varepsilon}\right\}+\frac{1}{\varepsilon}\left\{1,\log\frac{\varepsilon^2}{y}\right\}e^{-\frac{C}{\varepsilon^2}y}\right)\frac{1}{r}, & \text{if} \quad m=3.\end{cases}
\tag{3.17}
$$

While for $r\to\infty$, we could use integration by parts once more after (3.13) for $m=2$ and two more times for $m=3$, due to (3.10). Then at least we can get that

$$
|I_{11}|\le\frac{C}{r^{\frac{3}{2}}},\quad \text{as} \quad r\to\infty \quad \text{for } m\ge 2.
\tag{3.18}
$$

Concerning the second term I_{12} on the right handside of (3.12), when $|x|\le Cy$ is small, we have

$$
I_{12}=-\int_{\mathbb{R}}e^{ix\xi-\sqrt{a(\xi)}y}\partial_\xi\left(\frac{1}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}\chi_\varepsilon(\xi)M_m(\xi)\right)d\xi.
$$

As (3.15) we get that

$$
\left|\partial_\xi\left(\frac{1}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}\chi_\varepsilon(\xi)M_m(\xi)\right)\right|\le\frac{C}{r}\varepsilon^{4-m}\left(1+\frac{1}{\varepsilon^{\frac{1}{2}}\sqrt{|\xi-c_\varepsilon|}}\right)e^{-\frac{C}{\varepsilon^2}y}.
\tag{3.19}
$$

Therefore,

$$|I_{12}| \le C\frac{\varepsilon^{3-m}}{r}e^{-\frac{C}{\varepsilon^2}y} \le C\frac{\varepsilon^{4-m}}{r^{\frac{3}{2}}}. \tag{3.20}$$

While if $|x| \ge Cy$, then we have

$$\int_{\mathbb{R}} \chi_\varepsilon(\xi)e^{ix\xi-\sqrt{a(\xi)}y}M_m(\xi)d\xi = -\frac{1}{ix}\int_{\mathbb{R}} e^{ix\xi}\partial_\xi\left(\chi_\varepsilon(\xi)e^{-\sqrt{a(\xi)}y}M_m(\xi)\right)d\xi.$$

By direct computation, we have

$$\begin{aligned}\partial_\xi\left(\chi_\varepsilon(\xi)e^{-\sqrt{a(\xi)}y}M_m(\xi)\right) &= \chi_\varepsilon'(\xi)e^{-\sqrt{a(\xi)}y}M_m(\xi) + \chi_\varepsilon(\xi)e^{-\sqrt{a(\xi)}y}M_m'(\xi)\\ &\quad - \chi_\varepsilon(\xi)\frac{a'(\xi)}{2\sqrt{a(\xi)}}ye^{-\sqrt{a(\xi)}y}M_m(\xi)\\ &= I_{121}+I_{122}+I_{123}.\end{aligned}$$

From the expression of $M_m(\xi)$, we could obtain the following estimation

$$|I_{121}|+|I_{122}|+|I_{123}| \le \frac{C\varepsilon^{3-m}}{r}\left(1+\frac{1}{\varepsilon^{\frac{1}{2}}\sqrt{|\xi-c_\varepsilon|}}\right)e^{-\frac{C}{\varepsilon^2}y}.$$

Then

$$\begin{aligned}|I_{12}| &\le C\left(\int_{-2c_\varepsilon}^{-\frac{1}{2}c_\varepsilon}+\int_{\frac{1}{2}c_\varepsilon}^{2c_\varepsilon}\right)\frac{\varepsilon^{3-m}}{r}\left(1+\frac{1}{\varepsilon^{\frac{1}{2}}\sqrt{|\xi-c_\varepsilon|}}\right)e^{-\frac{C}{\varepsilon^2}y}d\xi\\ &\le \frac{C}{r}\varepsilon^{2-m}e^{-\frac{C}{\varepsilon^2}y}.\end{aligned} \tag{3.21}$$

For r large, without loss of generality we only give the explanation for the situation $|x| \ge Cy$, while the case $|x| \le Cy$ can be handled similarly. The crucial point is to obtain the most singular part in $\partial_\xi(\chi_\varepsilon(\xi)e^{-\sqrt{a(\xi)}y}M_m(\xi))$. Using (3.5), we could write

$$I_{121}+I_{122}+I_{123} = \begin{cases} l_\varepsilon\frac{1}{\sqrt{c_\varepsilon-\xi}}(1+c_1\sqrt{c_\varepsilon-\xi}+O(c_\varepsilon-\xi)), \text{ around } c_\varepsilon,\\ l_\varepsilon\frac{1}{\sqrt{c_\varepsilon+\xi}}(1+c_1\sqrt{c_\varepsilon+\xi}+O(c_\varepsilon+\xi)), \text{ around } -c_\varepsilon,\end{cases} \tag{3.22}$$

where

$$\begin{aligned} l_\varepsilon = &+\frac{c_\varepsilon^{m+1}(1-3\varepsilon^2c_\varepsilon^2-2\varepsilon^2)y}{2\sqrt{3\varepsilon^4c_\varepsilon(c_\varepsilon^2+d_\varepsilon^2)}(\sqrt{2}\varepsilon^2c_\varepsilon^2(1+c_\varepsilon^2)+\frac{\sqrt{2}}{2}(1+\varepsilon^2c_\varepsilon^2)c_\varepsilon\sqrt{1+c_\varepsilon^2})}e^{-\frac{\sqrt{1+\varepsilon^2c_\varepsilon^2}}{\sqrt{2}\varepsilon^2}y}\\ &-\frac{c_\varepsilon^{m+1}(1-2\varepsilon^2-3\varepsilon^2c_\varepsilon^2)}{\sqrt{6c_\varepsilon(1+\varepsilon^2c_\varepsilon^2)(c_\varepsilon^2+d_\varepsilon^2)}(\sqrt{2}\varepsilon^2c_\varepsilon^2(1+c_\varepsilon^2)+\frac{\sqrt{2}}{2}c_\varepsilon(1+\varepsilon^2c_\varepsilon^2)\sqrt{1+c_\varepsilon^2})}e^{-\frac{\sqrt{1+\varepsilon^2c_\varepsilon^2}}{\sqrt{2}\varepsilon^2}y}\\ &+\frac{c_\varepsilon^{m+2}\sqrt{(1+c_\varepsilon^2)(1+\varepsilon^2c_\varepsilon^2)}(1-2\varepsilon^2-3\varepsilon^2c_\varepsilon^2)}{\sqrt{3c_\varepsilon(c_\varepsilon^2+d_\varepsilon^2)}(\sqrt{2}\varepsilon^2c_\varepsilon^2(1+c_\varepsilon^2)+\frac{\sqrt{2}}{2}c_\varepsilon(1+\varepsilon^2c_\varepsilon^2)\sqrt{1+c_\varepsilon^2})^2}e^{-\frac{\sqrt{1+\varepsilon^2c_\varepsilon^2}}{\sqrt{2}\varepsilon^2}y}.\end{aligned}$$

Around the zero point c_ε and $-c_\varepsilon$, except the most singular terms $\frac{l_\varepsilon}{\sqrt{c_\varepsilon-\xi}}$ and $\frac{l_\varepsilon}{\sqrt{c_\varepsilon+\xi}}$, the left terms are continuous at c_ε and one can get better decay after using integration by parts once more. For the most singular term we have

$$\int_{\mathbb{R}} e^{ix\xi}\left(l_\varepsilon\frac{1}{\sqrt{c_\varepsilon-\xi}}+l_\varepsilon\frac{1}{\sqrt{c_\varepsilon+\xi}}\right)d\xi=\frac{l_\varepsilon}{\sqrt{x}}\left(e^{ixc_\varepsilon}\int_{\mathbb{R}}\frac{e^{it}}{\sqrt{-t}}dt+e^{-ixc_\varepsilon}\int_{\mathbb{R}}\frac{e^{it}}{\sqrt{t}}dt\right). \tag{3.23}$$

Using the asymptotic expansion (3.4) we can check that

$$l_\varepsilon=O\left(\varepsilon^{\frac{1}{2}-m}ye^{-\frac{C}{\varepsilon^2}y}+\varepsilon^{\frac{5}{2}-m}e^{-\frac{C}{\varepsilon^2}y}\right). \tag{3.24}$$

Using (3.22), (3.23) and (3.24), we derive that

$$|I_{12}|\le\begin{cases}\frac{C}{r}, & \text{if} \quad m=1,\\ \frac{C}{r^{\frac{3}{2}}}, & \text{if} \quad m=2,\\ \frac{C}{\varepsilon^{\frac{1}{2}}r^{\frac{3}{2}}}, & \text{if} \quad m=3.\end{cases} \tag{3.25}$$

Using (3.17), (3.18), (3.20), (3.21) and (3.25) we get that for r small that

$$|I_1|\le\begin{cases}\frac{C}{r}, & \text{if} \quad m=1,\\ C\left(\min\left\{\log\frac{1+\sqrt{y}}{\sqrt{y}},\log\frac{1}{\varepsilon}\right\}+e^{-\frac{C}{\varepsilon^2}y}\right)\frac{1}{r}, & \text{if} \quad m=2,\\ C\left(\min\left\{\frac{1}{\sqrt{y}},\frac{1}{\varepsilon}\right\}+\frac{1}{\varepsilon}\max\left\{1,\log\frac{\varepsilon^2}{y}\right\}e^{-\frac{C}{\varepsilon^2}y}\right)\frac{1}{r}, & \text{if} \quad m=3.\end{cases} \tag{3.26}$$

While for r large that

$$|I_1|\le\begin{cases}\frac{C}{r}, & \text{if} \quad m=1,\\ \frac{C}{r^{\frac{3}{2}}}, & \text{if} \quad m=2,\\ \frac{C}{\varepsilon^{\frac{1}{2}}r^{\frac{3}{2}}}, & \text{if} \quad m=3.\end{cases} \tag{3.27}$$

Next we consider I_2. We set

$$z:=\left(\sqrt{a(\xi)}-\sqrt{b(\xi)}\right)y=-\frac{D(\xi)}{\varepsilon^4(\sqrt{a(\xi)}+\sqrt{b(\xi)})}y.$$

As I_1 we write

$$I_2=\int_{\mathbb{R}}\frac{y\xi^m e^{ix\xi-\sqrt{a(\xi)}y}}{\varepsilon^4(\sqrt{a(\xi)b(\xi)}+b(\xi))}\frac{e^z-1}{z}\left((1-\chi_\varepsilon(\xi))+\chi_\varepsilon(\xi)\right)d\xi=\pi(I_{21}+I_{22}).$$

Concerning I_{21}, using integration by parts we get

$$I_{21}=-\int_{\mathbb{R}}e^{ix\xi-\sqrt{a(\xi)}y}\partial_\xi\left(\frac{1}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}\frac{y\xi^m(1-\chi_\varepsilon(\xi))}{\varepsilon^4(b(\xi)+\sqrt{a(\xi)b(\xi)})}\eta(z)\right)d\xi, \tag{3.28}$$

where $\eta(z)=\frac{e^z-1}{z}$. As (3.15) and (3.19) we could obtain the following estimation,

$$\left|\partial_\xi\left(\frac{1}{ix\xi-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}\frac{y\xi^m(1-\chi_\varepsilon(\xi))}{\varepsilon^4(b(\xi)+\sqrt{a(\xi)b(\xi)})}\eta(z)\right)\right|$$
$$\leq\begin{cases}C\dfrac{\varepsilon(1+\xi)^m+(1+\xi)^{m-1}}{r}y, & \text{if}\quad |\xi|\leq 2c_\varepsilon,\\ C\dfrac{1}{\varepsilon^2\xi^{3-m}r}y, & \text{if}\quad |\xi|\geq 2c_\varepsilon.\end{cases}$$

Then we have

$$\begin{aligned}|I_{21}|\leq&\int_{-2c_\varepsilon}^{2c_\varepsilon}\frac{C}{r}\left(\varepsilon(1+\xi)^{m-2}+(1+\xi)^{m-3}\right)e^{-C\xi^2y}d\xi\\&+\int_{\mathbb{R}\setminus[-2c_\varepsilon,2c_\varepsilon]}\frac{C}{\varepsilon\xi^{4-m}r}e^{-C\frac{|\xi|}{\varepsilon}y}d\xi\\ \leq&\begin{cases}\frac{C}{r}, & \text{if}\quad m=1,\\ C\left(\min\left\{\log\frac{1+\sqrt{y}}{\sqrt{y}},\log\frac{1}{\varepsilon}\right\}+e^{-\frac{C}{\varepsilon^2}y}\right)\frac{1}{r}, & \text{if}\quad m=2,\\ C\left(\min\left\{\frac{1}{\sqrt{y}},\frac{1}{\varepsilon}\right\}+\frac{1}{\varepsilon}\max\left\{1,\log\frac{\varepsilon^2}{y}\right\}e^{-\frac{C}{\varepsilon^2}y}\right)\frac{1}{r}, & \text{if}\quad m=3.\end{cases}\end{aligned}\tag{3.29}$$

The above estimate (3.29) gives the control on the asymptotic behavior for I_{21} when r is small. While for $r\to\infty$, we could use integration by parts once more after (3.28) for $m=2$, and two more times for $m=3$, then there is no boundary terms due to that the integrand is C^1 continuous at 0 for $m=2$ and C^2 continuous at 0 for $m=3$. After differentiation we could obtain that the integrand function has better decay and one can at least show that

$$|I_{21}|\leq\frac{C}{r^{\frac{3}{2}}},\quad\text{as}\quad r\to\infty\quad\text{for } m\geq 2.\tag{3.30}$$

Concerning I_{22}, if $|y|\geq C|x|$ we have

$$I_{22}=-\int_{\mathbb{R}}e^{ix\xi-\sqrt{a(\xi)}y}\partial_\xi\left(\frac{1}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}\frac{y\xi^m\eta(z)\chi_\varepsilon(\xi)}{\varepsilon^4(\sqrt{a(\xi)b(\xi)}+b(\xi))}\right)d\xi.$$

It is not difficult to check that

$$\left|\partial_\xi\left(\frac{1}{ix-\frac{a'(\xi)}{2\sqrt{a(\xi)}}y}\frac{y\xi^m\eta(z)\chi_\varepsilon(\xi)}{\varepsilon^4(\sqrt{a(\xi)b(\xi)}+b(\xi))}\right)\right|\leq\frac{C}{r}\left(\frac{\varepsilon^{-\frac{1}{2}-m}y^2}{\sqrt{|c_\varepsilon-\xi|}}+\frac{\varepsilon^{\frac{3}{2}-m}y}{\sqrt{|c_\varepsilon-\xi|}}\right)e^{-\frac{C}{\varepsilon^2}y}.$$

As a consequence, we derive that

$$\begin{aligned}|I_{22}|&\leq\frac{C}{r}\left(\int_{-2c_\varepsilon}^{-\frac{1}{2}c_\varepsilon}+\int_{\frac{1}{2}c_\varepsilon}^{2c_\varepsilon}\right)\frac{y\varepsilon^{\frac{3}{2}-m}+y^2\varepsilon^{-\frac{1}{2}-m}}{\sqrt{|c_\varepsilon-\xi|}}e^{-\frac{C}{\varepsilon^2}y}d\xi\\&=\frac{C}{r}(\varepsilon^{1-m}y+\varepsilon^{-1-m}y^2)e^{-\frac{C}{\varepsilon^2}y}\leq\frac{C}{r}\varepsilon^{3-m}e^{-\frac{C}{\varepsilon^2}y}.\end{aligned}\tag{3.31}$$

While for $|x| \geq C|y|$ we use integration by parts to get that

$$I_{22} = -\int_{\mathbb{R}} \frac{1}{ix} e^{ix\xi} \partial_\xi \left(\frac{y\xi^m e^{-\sqrt{a(\xi)}y}}{\varepsilon^4(\sqrt{a(\xi)b(\xi)} + b(\xi))} \eta(z)\chi_\varepsilon(\xi) \right) d\xi. \tag{3.32}$$

The support of the integrand is $[-2c_\varepsilon, -\frac{1}{2}c_\varepsilon] \cup [\frac{1}{2}c_\varepsilon, 2c_\varepsilon]$. In the support of this function one can check

$$\begin{aligned} &\left| \partial_\xi \left(\frac{y\xi^m e^{-\sqrt{a(\xi)}y}}{\varepsilon^4(\sqrt{a(\xi)b(\xi)} + b(\xi))} \eta(z)\chi_\varepsilon(\xi) \right) \right| \\ &\leq C \left(\varepsilon^{1-m} y + \varepsilon^{-1-m} y^2 + \frac{y}{\varepsilon^{-\frac{1}{2}+m}\sqrt{|\xi - c_\varepsilon|}} + \frac{y^2}{\varepsilon^{\frac{3}{2}+m}\sqrt{|\xi - c_\varepsilon|}} \right) e^{-\frac{C}{\varepsilon^2}y}. \end{aligned}$$

Then

$$\begin{aligned} |I_{22}| &\leq \frac{C}{r} \left(\int_{-2c_\varepsilon}^{-\frac{1}{2}c_\varepsilon} + \int_{\frac{1}{2}c_\varepsilon}^{2c_\varepsilon} \right) \left(\frac{y}{\varepsilon^{-\frac{1}{2}+m}\sqrt{|\xi - c_\varepsilon|}} + \frac{y^2}{\varepsilon^{\frac{3}{2}+m}\sqrt{|\xi - c_\varepsilon|}} \right) e^{-\frac{C}{\varepsilon^2}y} d\xi \\ &\leq \frac{C}{r} \min\{\varepsilon^{-2-m}y^2, \varepsilon^{-m}y\} e^{-\frac{C}{\varepsilon^2}y} \leq \begin{cases} \frac{C}{r}, & \text{if} \quad m = 1, \\ \frac{C}{r}, & \text{if} \quad m = 2, \\ \frac{C}{\varepsilon r} e^{-\frac{C}{\varepsilon^2}y}, & \text{if} \quad m = 3. \end{cases} \end{aligned} \tag{3.33}$$

While for r large, in order to get a better decay on the integration, we notice that the most singular term behave as $\frac{1}{\sqrt{c_\varepsilon - \xi}}$ and $\frac{1}{\sqrt{c_\varepsilon + \xi}}$. Following almost the same argument as we did from (3.21) to (3.25) we conclude that

$$|I_{22}| \leq \begin{cases} \frac{C}{r}, & \text{if} \quad m = 1, \\ \frac{C}{r^{\frac{3}{2}}}, & \text{if} \quad m = 2, \\ \frac{C}{\varepsilon^{\frac{1}{2}} r^{\frac{3}{2}}} e^{-\frac{C}{\varepsilon^2}y}, & \text{if} \quad m = 3. \end{cases} \tag{3.34}$$

Combined with (3.29), (3.30), (3.33) we get for r small that

$$|I_2| \leq \begin{cases} \frac{C}{r}, & \text{if} \quad m = 1, \\ C\left(\min\left\{ \log \frac{1+\sqrt{y}}{\sqrt{y}}, \log \frac{1}{\varepsilon} \right\} + e^{-\frac{C}{\varepsilon^2}y} \right) \frac{1}{r}, & \text{if} \quad m = 2, \\ C\left(\min\left\{ \frac{1}{\sqrt{y}}, \frac{1}{\varepsilon} \right\} + \frac{1}{\varepsilon} \max\left\{ 1, \log \frac{\varepsilon^2}{y} \right\} e^{-\frac{C}{\varepsilon^2}y} \right) \frac{1}{r}, & \text{if} \quad m = 3. \end{cases} \tag{3.35}$$

While for r large it holds that

$$|I_2| \leq \begin{cases} \frac{C}{r}, & \text{if} \quad m = 1, \\ \frac{C}{r^{\frac{3}{2}}}, & \text{if} \quad m = 2, \\ \frac{C}{\varepsilon^{\frac{1}{2}} r^{\frac{3}{2}}} e^{-\frac{C}{\varepsilon^2}y}, & \text{if} \quad m = 3. \end{cases} \tag{3.36}$$

It remains to consider the case where $y = 0$. In this case, we derive that

$$K_{m,0}(x, 0) = \pi \int_{\mathbb{R}} e^{ix\xi} M_m(\xi) d\xi.$$

Since the integrand decays at infinity for $m = 1$ and $m = 2$ and the estimations for $K_{m,0}(x, 0)$ ($m = 1$ and $m = 2$) are similar to the case $y \neq 0$. While for $m = 3$, one can easily see that as $|\xi| \to \infty$ we have

$$M_3(\xi) = \begin{cases} \dfrac{\sqrt{1-\sqrt{3}i}}{\sqrt{2}+\frac{\sqrt{2}}{2}(1-\sqrt{3}i)}\frac{1}{\varepsilon}\left(1+O(\varepsilon^{-2}\xi^{-2})\right), & \text{as} \quad \xi \to \infty, \\ -\dfrac{\sqrt{1-\sqrt{3}i}}{\sqrt{2}+\frac{\sqrt{2}}{2}(1-\sqrt{3}i)}\frac{1}{\varepsilon}\left(1+O(\varepsilon^{-2}\xi^{-2})\right), & \text{as} \quad \xi \to -\infty. \end{cases} \tag{3.37}$$

Then we write

$$\begin{aligned} K_{3,0} &= \pi \int_{\mathbb{R}} e^{ix\xi}\left(M_3(\xi) - sgn(\xi)\frac{\sqrt{1-\sqrt{3}i}}{\sqrt{2}+\frac{\sqrt{2}}{2}(1-\sqrt{3}i)}\frac{1}{\varepsilon}\right)d\xi \\ &\quad + \pi \int_{\mathbb{R}} sgn(\xi)\frac{\sqrt{1-\sqrt{3}i}}{\sqrt{2}+\frac{\sqrt{2}}{2}(1-\sqrt{3}i)}\frac{1}{\varepsilon}e^{ix\xi}d\xi \\ &= \pi \int_{\mathbb{R}} e^{ix\xi}\widetilde{M}_3(\xi)d\xi + \frac{i}{\varepsilon x}\frac{2\sqrt{1-\sqrt{3}i}}{\sqrt{2}+\frac{\sqrt{2}}{2}(1-\sqrt{3}i)}, \end{aligned}$$

where we used that

$$\pi \int_{\mathbb{R}} e^{ix\xi} sgn(\xi)d\xi = -\frac{2}{ix}.$$

and

$$\widetilde{M}_3(\xi) = M_3(\xi) - sgn(\xi)\frac{\sqrt{1-\sqrt{3}i}}{\sqrt{2}+\frac{\sqrt{2}}{2}(1-\sqrt{3}i)}\frac{1}{\varepsilon}.$$

With (3.37), we see that $\widetilde{M}_3'(\xi)$ decays at infinity like $O(\varepsilon^{-2}\xi^{-3})$. Then following the arguments of deriving I_1, we could get

$$r|K_{3,0}(x,0)| \leq \frac{C}{\varepsilon r}.$$

Then we finish the whole proof. □

In the next lemma, we shall investigate the behavior of $K_{m,n}(x, y)$ with $(m, n) = (0, 1), (0, 2), (0, 3), (1, 1), (1, 2)$.

Lemma 3.2. *Let $K_{m,n}(x, y)$ be defined in* (3.2). *Then for r small we have*

$$r|K_{m,n}(x,y)| \leq \begin{cases} C\min\left\{\log\frac{1+\sqrt{|y|}}{\sqrt{|y|}}, \log\frac{1}{\varepsilon}\right\}, & \text{for} \quad (m,n) = (0,1), \\ C\varepsilon^{-2}\left(\log\frac{1}{\varepsilon}+1\right)e^{-\frac{C}{\varepsilon^2}|y|} + \frac{C}{\sqrt{|y|}}\varepsilon^{-1}, & \text{for} \quad (m,n) = (0,2), \\ C\varepsilon^{-4}\left(\log\frac{1}{\varepsilon}+\max\left\{1,\log\frac{\varepsilon^2}{|y|}\right\}\right)e^{-\frac{C}{\varepsilon^2}|y|} + \frac{C}{\sqrt{|y|}}\varepsilon^{-3}, & \text{for} \quad (m,n) = (0,3), \\ C\left(\min\left\{\frac{1}{\sqrt{|y|}}, \frac{1}{\varepsilon}\right\} + \frac{1}{\varepsilon}e^{-\frac{C}{\varepsilon^2}|y|}\right), & \text{for} \quad (m,n) = (1,1), \\ C\varepsilon^{-2}\left(\min\left\{\frac{1}{\sqrt{|y|}}, \frac{1}{\varepsilon}\right\} + \frac{1}{\varepsilon}\max\left\{1,\log\frac{\varepsilon^2}{|y|}\right\}e^{-\frac{C}{\varepsilon^2}|y|}\right), & \text{for} \quad (m,n) = (1,2). \end{cases}$$

While for r large we have

$$r|K_{m,n}(x,y)| \le \begin{cases} C\log\frac{1}{\varepsilon}, & \text{for} \quad (m,n)=(0,1),\\ C\varepsilon^{-\frac{3}{2}}r^{-\frac{1}{2}}, & \text{for} \quad (m,n)=(0,2),\\ C\varepsilon^{-\frac{7}{2}}r^{-\frac{1}{2}}, & \text{for} \quad (m,n)=(0,3),\\ C\varepsilon^{-\frac{1}{2}}r^{-\frac{1}{2}}, & \text{for} \quad (m,n)=(1,1),\\ C\varepsilon^{-\frac{5}{2}}r^{-\frac{1}{2}}, & \text{for} \quad (m,n)=(1,2). \end{cases}$$

Proof. The proof of this lemma goes almost the same as Lemma 3.1. Here we only sketch the proof for the case $(m,n)=(1,2)$. First, we study the integration for $y>0$. Using Residue's Theorem we get

$$\begin{aligned} K_{1,2}(x,y) &= \pi\int_{\mathbb{R}}\frac{\xi e^{ix\xi}}{D(\xi)}\left(\sqrt{b(\xi)}e^{-\sqrt{b(\xi)}y}-\sqrt{a(\xi)}e^{-\sqrt{a(\xi)}y}\right)d\xi\\ &= \pi\int_{\mathbb{R}}\frac{\xi(\sqrt{b(\xi)}-\sqrt{a(\xi)})e^{ix\xi-\sqrt{b(\xi)}y}}{D(\xi)}d\xi\\ &\quad+\pi\int_{\mathbb{R}}\frac{\xi\sqrt{a(\xi)}e^{ix\xi-\sqrt{a(\xi)}y}}{D(\xi)}\left(e^{(\sqrt{a(\xi)}-\sqrt{b(\xi)})y}-1\right)d\xi\\ &= \pi(I_3+I_4). \end{aligned}$$

In the following, we shall estimate I_3 and I_4 respectively. Concerning I_3 first, we write

$$I_3=\int_{\mathbb{R}}\frac{\xi}{\varepsilon^4(\sqrt{b(\xi)}+\sqrt{b(\xi)})}e^{ix\xi-\sqrt{b(\xi)}y}\left((1-\chi_\varepsilon(x))+\chi_\varepsilon(x)\right)d\xi=I_{31}+I_{32},$$

where $\chi_\varepsilon(x)$ is introduced in (3.11). Concerning I_{31}, using integration by parts we get

$$I_{31}=-\int_{\mathbb{R}}e^{ix\xi-\sqrt{b(\xi)}y}\partial_\xi\left(\frac{1}{ix-\frac{b'(\xi)}{2\sqrt{b(\xi)}}y}\frac{(1-\chi_\varepsilon(\xi))\xi}{\varepsilon^4(\sqrt{a(\xi)}+\sqrt{b(\xi)})}\right)d\xi$$

By direct computation, we have

$$\begin{aligned} &\partial_\xi\left(\frac{1}{ix-\frac{b'(\xi)}{2\sqrt{b(\xi)}}y}\frac{(1-\chi_\varepsilon(\xi))\xi}{\varepsilon^4(\sqrt{a(\xi)}+\sqrt{b(\xi)})}\right)\\ &=\frac{\frac{2b''(\xi)b(\xi)-(b'(\xi))^2}{4(b(\xi))^{\frac{3}{2}}}y}{\left(ix-\frac{b'(\xi)}{2\sqrt{b(\xi)}}y\right)^2}\frac{(1-\chi_\varepsilon(\xi))\xi}{\varepsilon^4(\sqrt{a(\xi)}+\sqrt{b(\xi)})}+\frac{1}{ix-\frac{b'(\xi)}{2\sqrt{b(\xi)}}y}\frac{(1-\chi_\varepsilon(\xi))-\chi_\varepsilon'(\xi)\xi}{\varepsilon^4(\sqrt{a(\xi)}+\sqrt{b(\xi)})}\\ &\quad+\frac{1}{ix-\frac{b'(\xi)}{2\sqrt{b(\xi)}}y}\frac{(1-\chi_\varepsilon(\xi))\xi(\sqrt{a(\xi)}+\sqrt{b(\xi)})'}{\varepsilon^4(\sqrt{a(\xi)}+\sqrt{b(\xi)})^2}\\ &=I_{311}+I_{312}+I_{313}. \end{aligned}$$

One can check that have

$$|I_{311}| + |I_{312}| + |I_{313}| \leq \begin{cases} \frac{C}{\varepsilon^2 r} & \text{for } |\xi| \leq 2c_\varepsilon, \\ \\ \frac{C}{\varepsilon^3 |\xi| r} & \text{for } |\xi| \geq 2c_\varepsilon. \end{cases}$$

Then

$$\begin{aligned} |I_{31}| &\leq \frac{C}{\varepsilon^2 r} \int_{-2c_\varepsilon}^{2c_\varepsilon} e^{-\sqrt{b(\xi)}y} d\xi + \frac{C}{\varepsilon^2 r} \int_{\mathbb{R}\setminus[-2c_\varepsilon, 2c_\varepsilon]} \frac{1}{\varepsilon|\xi|} e^{-\sqrt{b(\xi)}y} d\xi \\ &\leq \frac{C}{\varepsilon^2 r} \int_{-2c_\varepsilon}^{2c_\varepsilon} e^{-\frac{C}{\varepsilon^2}y} d\xi + \frac{C}{\varepsilon^2 r} \int_{\mathbb{R}\setminus[-2c_\varepsilon, 2c_\varepsilon]} \frac{1}{\varepsilon|\xi|} e^{-C\frac{|\xi|}{\varepsilon}y} d\xi \\ &\leq \frac{C}{\varepsilon^3 r} \max\left\{1, \log\frac{\varepsilon^2}{y}\right\} e^{-\frac{C}{\varepsilon^2}y}. \end{aligned} \tag{3.38}$$

For I_{31}, the estimate (3.38) gives the asymptotic behavior for r small. While for r large, we could apply the integration by parts to derive a better decay for I_{31} as r tends to ∞, and there is no boundary term arising due to the integrand function vanishes at 0. At least one can show that $|I_{31}| \leq C\varepsilon^{-\frac{5}{2}} r^{-\frac{3}{2}}$.

For I_{32}, when $|x| \leq Cy$ we have

$$I_{32} = -\int_{\mathbb{R}} e^{ix\xi - \sqrt{b(\xi)}} \partial_\xi \left(\frac{1}{ix - \frac{b'(\xi)}{2\sqrt{b(\xi)}}y} \frac{\chi_\varepsilon(\xi)\xi}{\varepsilon^4(\sqrt{a(\xi)} + \sqrt{b(\xi)})} \right) d\xi.$$

For the term inside the bracket we could bound it by

$$\left| \partial_\xi \left(\frac{1}{ix - \frac{b'(\xi)}{2\sqrt{b(\xi)}}y} \frac{\chi_\varepsilon(\xi)\xi}{\varepsilon^4(\sqrt{a(\xi)} + \sqrt{b(\xi)})} \right) \right| \leq \frac{C}{r}\varepsilon^{-2} \left(1 + \frac{1}{\varepsilon^{\frac{1}{2}}\sqrt{|c_\varepsilon - \xi|}} \right) e^{-\frac{C}{\varepsilon^2}y}.$$

Based on the above estimation we gain that

$$|I_{32}| \leq \frac{C}{\varepsilon^3 r} e^{-\frac{C}{\varepsilon^2}y}. \tag{3.39}$$

While if $|x| \geq Cy$, then by integration by parts

$$I_{32} = -\frac{1}{ix} \int_{\mathbb{R}} e^{ix\xi} \partial_\xi \left(e^{-\sqrt{b(\xi)}y} \frac{\chi_\varepsilon(\xi)\xi}{\varepsilon^4(\sqrt{a(\xi)} + \sqrt{b(\xi)})} \right) d\xi. \tag{3.40}$$

By direct computation, we have

$$\begin{aligned} &\left| \partial_\xi \left(e^{-\sqrt{b(\xi)}y} \frac{\chi_\varepsilon(\xi)\xi}{\varepsilon^4(\sqrt{a(\xi)} + \sqrt{b(\xi)})} \right) \right| \\ &\leq \frac{C}{r} \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^{\frac{5}{2}}\sqrt{|c_\varepsilon - \xi|}} + \frac{1}{\varepsilon^4}y + \frac{1}{\varepsilon^{\frac{9}{2}}\sqrt{|c_\varepsilon - \xi|}}y \right) e^{-\frac{C}{\varepsilon^2}y}. \end{aligned}$$

As a consequence

$$
\begin{aligned}
|I_{32}| &\leq \frac{C}{\varepsilon^2 r} e^{-\frac{C}{\varepsilon^2} y} \left(\int_{-2c_\varepsilon}^{-\frac{1}{2}c_\varepsilon} + \int_{\frac{1}{2}c_\varepsilon}^{2c_\varepsilon} \right) \left(1 + \frac{1}{\varepsilon^{\frac{1}{2}} \sqrt{|c_\varepsilon - \xi|}} + \frac{1}{\varepsilon^2} y + \frac{1}{\varepsilon^{\frac{5}{2}} \sqrt{|c_\varepsilon - \xi|}} y \right) d\xi \\
&\leq \frac{C}{\varepsilon^3 r} e^{-\frac{C}{\varepsilon^2} y}.
\end{aligned} \tag{3.41}
$$

While for r large, we could follow the computations performed from (3.22) to (3.25) to derive that

$$
|I_{32}| \leq \frac{C}{\varepsilon^{\frac{5}{2}} r^{\frac{3}{2}}} e^{-\frac{C}{\varepsilon^2} y}. \tag{3.42}
$$

Using (3.38)-(3.42) we get that

$$
|I_3| \leq \begin{cases} \frac{C}{\varepsilon^3 r} \max\left\{1, \log \frac{\varepsilon^2}{y}\right\} e^{-\frac{C}{\varepsilon^2} y}, & \text{as} \quad r \to 0, \\ \frac{C}{\varepsilon^{\frac{5}{2}} r^{\frac{3}{2}}} e^{-\frac{C}{\varepsilon^2} y}, & \text{as} \quad r \to \infty. \end{cases} \tag{3.43}
$$

To study I_4, we write it as

$$
I_4 = -\int_{\mathbb{R}} \frac{y\xi \sqrt{a(\xi)} e^{ix\xi - \sqrt{a(\xi)} y}}{\varepsilon^4 (\sqrt{a(\xi)} + \sqrt{b(\xi)})} \eta(z)((1 - \chi_\varepsilon(\xi)) + \chi_\varepsilon(\xi)) d\xi = -\pi(I_{41} + I_{42}).
$$

As we did for I_2 of last lemma and I_3 from above, we could see that

$$
|I_{41}| \leq \begin{cases} C\left(\min\left\{\frac{1}{\sqrt{y}}, \frac{1}{\varepsilon}\right\} + \frac{1}{\varepsilon} \max\left\{1, \log \frac{\varepsilon^2}{y}\right\} e^{-\frac{C}{\varepsilon^2} y}\right) \frac{1}{\varepsilon^2 r}, & \text{as} \quad r \to 0, \\ \frac{C}{\varepsilon^{\frac{5}{2}} r^{\frac{3}{2}}}, & \text{as} \quad r \to \infty, \end{cases} \tag{3.44}
$$

and

$$
|I_{42}| \leq \begin{cases} C\left(\min\left\{\frac{1}{\sqrt{y}}, \frac{1}{\varepsilon}\right\} + \frac{1}{\varepsilon} \max\left\{1, \log \frac{\varepsilon^2}{y}\right\} e^{-\frac{C}{\varepsilon^2} y}\right) \frac{1}{\varepsilon^2 r}, & \text{as} \quad r \to 0, \\ \frac{C}{\varepsilon^{\frac{5}{2}} r^{\frac{3}{2}}} e^{-\frac{C}{\varepsilon^2} y}, & \text{as} \quad r \to \infty. \end{cases} \tag{3.45}
$$

Combining (3.44) and (3.45) we get

$$
|I_4| \leq \begin{cases} C\left(\min\left\{\frac{1}{\sqrt{y}}, \frac{1}{\varepsilon}\right\} + \frac{1}{\varepsilon} \max\left\{1, \log \frac{\varepsilon^2}{y}\right\} e^{-\frac{C}{\varepsilon^2} y}\right) \frac{1}{\varepsilon^2 r}, & \text{as} \quad r \to 0, \\ \frac{C}{\varepsilon^{\frac{5}{2}} r^{\frac{3}{2}}}, & \text{as} \quad r \to \infty. \end{cases}
$$

It remains to consider the case where $y = 0$. In this case, the process is as lemma 3.1, we have to analyze the following integral

$$
\int_{\mathbb{R}} \frac{\xi}{\varepsilon^4 (\sqrt{a(\xi)} + \sqrt{b(\xi)})} e^{ix\xi} d\xi.
$$

It is not difficult to check that as $|\xi| \to \infty$ we have

$$
\frac{\xi}{\varepsilon^4 (\sqrt{a(\xi)} + \sqrt{b(\xi)})} = \begin{cases} \frac{(1 + O(\varepsilon^{-2}\xi^{-2}))}{\varepsilon^3 \left(\sqrt{\frac{1-\sqrt{3}i}{2}} + \sqrt{\frac{1+\sqrt{3}i}{2}}\right)}, & \text{as } \xi \to \infty, \\ \\ -\frac{(1 + O(\varepsilon^{-2}\xi^{-2}))}{\varepsilon^3 \left(\sqrt{\frac{1-\sqrt{3}i}{2}} + \sqrt{\frac{1+\sqrt{3}i}{2}}\right)}, & \text{as } \xi \to -\infty. \end{cases}
$$

Then we write

$$
\begin{aligned}
K_{1,2}(x,0) &= \pi \int_{\mathbb{R}} e^{ix\xi} \left(\frac{\xi}{\varepsilon^4(\sqrt{a(\xi)}+\sqrt{b(\xi)})} - \frac{sgn(\xi)}{\varepsilon^3\left(\sqrt{\frac{1-\sqrt{3}i}{2}}+\sqrt{\frac{1+\sqrt{3}i}{2}}\right)} \right) d\xi \\
&= \pi \int_{\mathbb{R}} e^{ix\xi} \frac{\xi}{\varepsilon^4(\sqrt{a(\xi)}+\sqrt{b(\xi)})} d\xi - \frac{1}{x} \frac{2i}{\varepsilon^3\left(\sqrt{\frac{1-\sqrt{3}i}{2}}+\sqrt{\frac{1+\sqrt{3}i}{2}}\right)}.
\end{aligned}
$$

Then we can proceed the arguments as I_3 to treat the first one, and the integration by parts process works due to that the integrand function decays enough after taking differentiation. Then we get that

$$
\varepsilon^3 |rK_{1,2}(x,0)| \leq \begin{cases} C, & \text{as} \quad r \to 0, \\ \frac{C\varepsilon^{\frac{1}{2}}}{r^{\frac{1}{2}}}, & \text{as} \quad r \to \infty. \end{cases}
$$

□

With Lemmas 3.1 and 3.2 we state the main result of this section

Theorem 3.3. *Let $G(x,y)$ be the Green kernel of the linear operator* (3.1). *Then for r small, we have*

$$
r|\partial_x^m \partial_y^n G(x,y)| \leq \begin{cases}
C, & \text{for} \quad (m,n)=(1,0) \\
C\left(\min\left\{\log\frac{1+\sqrt{|y|}}{\sqrt{|y|}}, \log\frac{1}{\varepsilon}\right\} + e^{-\frac{C}{\varepsilon^2}|y|}\right), & \text{for} \quad (m,n)=(2,0), \\
C\left(\min\left\{\frac{1}{\sqrt{|y|}}, \frac{1}{\varepsilon}\right\} + \frac{1}{\varepsilon}\max\left\{1, \log\frac{\varepsilon^2}{|y|}\right\} e^{-\frac{C}{\varepsilon^2}|y|}\right), & \text{for} \quad (m,n)=(3,0), \\
C\min\left\{\log\frac{1+\sqrt{|y|}}{\sqrt{|y|}}, \log\frac{1}{\varepsilon}\right\}, & \text{for} \quad (m,n)=(0,1), \\
C\varepsilon^{-2}\left(\log\frac{1}{\varepsilon}+1\right)e^{-\frac{C}{\varepsilon^2}|y|} + \frac{C}{\sqrt{|y|}}\varepsilon^{-1}, & \text{for} \quad (m,n)=(0,2), \\
C\varepsilon^{-4}\left(\log\frac{1}{\varepsilon}+\max\left\{1, \log\frac{\varepsilon^2}{|y|}\right\}\right)e^{-\frac{C}{\varepsilon^2}|y|} + \frac{C}{\sqrt{|y|}}\varepsilon^{-3}, & \text{for} \quad (m,n)=(0,3), \\
C\left(\min\left\{\frac{1}{\sqrt{|y|}}, \frac{1}{\varepsilon}\right\} + \frac{1}{\varepsilon}e^{-\frac{C}{\varepsilon^2}|y|}\right), & \text{for} \quad (m,n)=(1,1), \\
C\varepsilon^{-2}\left(\min\left\{\frac{1}{\sqrt{|y|}}, \frac{1}{\varepsilon}\right\} + \frac{1}{\varepsilon}\max\left\{1, \log\frac{\varepsilon^2}{|y|}\right\} e^{-\frac{C}{\varepsilon^2}|y|}\right), & \text{for} \quad (m,n)=(1,2).
\end{cases}
$$

While for r large, we have

$$
r|\partial_x^m \partial_y^n G(x,y)| \leq \begin{cases}
C, & \text{for} \quad (m,n)=(1,0), \\
\frac{C}{r^{\frac{1}{2}}}, & \text{for} \quad (m,n)=(2,0), \\
\frac{C}{\varepsilon^{\frac{1}{2}} r^{\frac{1}{2}}} & \text{for} \quad (m,n)=(3,0), \\
C\log\frac{1}{\varepsilon}, & \text{for} \quad (m,n)=(0,1), \\
C\varepsilon^{-\frac{3}{2}} r^{-\frac{1}{2}}, & \text{for} \quad (m,n)=(0,2), \\
C\varepsilon^{-\frac{7}{2}} r^{-\frac{1}{2}}, & \text{for} \quad (m,n)=(0,3), \\
C\varepsilon^{-\frac{1}{2}} r^{-\frac{1}{2}}, & \text{for} \quad (m,n)=(1,1), \\
C\varepsilon^{-\frac{5}{2}} r^{-\frac{1}{2}}, & \text{for} \quad (m,n)=(1,2).
\end{cases}
$$

In addition, we have the following estimation on the integration $\int_{B_r(0)} |\partial_x^m \partial_y^n G(x,y)| dxdy$, *denoted by* $\mathcal{I}_{m,n}$

$$
\mathcal{I}_{m,n} \le \begin{cases} Cr, & \text{for } (m,n) = (1,0), \\ Cr^{\frac{1}{2}}, & \text{for } (m,n) = (2,0), \\ C\left(r^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} (\log(1+r))^2 r^{\frac{1}{4}}\right), & \text{for } (m,n) = (3,0), \\ Cr(\log r)^2, & \text{for } (m,n) = (0,1), \\ C\left(\varepsilon^{-1} r^{\frac{1}{2}} + \varepsilon^{-\frac{1}{2}} r^{\frac{1}{4}} \log r\right), & \text{for } (m,n) = (0,2), \\ C\left(\varepsilon^{-3} r^{\frac{1}{2}} + \varepsilon^{-\frac{5}{2}} \log \frac{1}{\varepsilon} (\log(1+r))^2 r^{\frac{1}{4}}\right), & \text{for } (m,n) = (0,3), \\ Cr^{\frac{1}{2}}, & \text{for } (m,n) = (1,1), \\ C\left(\varepsilon^{-2} r^{\frac{1}{2}} + \varepsilon^{-\frac{3}{2}} (\log(1+r))^2 r^{\frac{1}{4}}\right), & \text{for } (m,n) = (1,2). \end{cases}
$$

Proof. The estimation on the asymptotic behavior and decay estimate follows easily by Lemmas 3.1 and 3.2. It remains to prove the estimation on the integral $\mathcal{I}_{m,n}$.

Since the proof for each case is almost similar, we shall only give the details for $(0,3)$ and $(1,2)$. For the former one, we have

$$
\begin{aligned}
\mathcal{I}_{0,3} \le\ & C\varepsilon^{-4} \int_0^r \int_0^r \frac{1}{\sqrt{x^2+y^2}} \left(\log \frac{1}{\varepsilon} + \log \frac{1}{|y|}\right) e^{-\frac{C}{\varepsilon^2}|y|} dxdy \\
& + C\varepsilon^{-3} \int_{B_r(0)} \frac{1}{\sqrt{x^2+y^2}\sqrt{|y|}} dxdy \\
\le\ & C\varepsilon^{-4} \int_0^r \left(\int_0^{\frac{r}{|y|}} \frac{1}{\sqrt{x^2+1}} dx\right) \left(\log \frac{1}{\varepsilon} + \log \frac{1}{|y|}\right) e^{-\frac{C}{\varepsilon^2}|y|} dy \\
& + C\varepsilon^{-3} \int_0^r \int_0^{2\pi} \frac{1}{\rho^{\frac{3}{2}}} \frac{1}{\sqrt{|\sin\theta|}} d\theta d\rho \\
\le\ & C\varepsilon^{-4} \int_0^r \left(\log \frac{1}{\varepsilon} + \log \frac{1}{|y|}\right) \log\left(1 + \frac{r}{|y|}\right) e^{-\frac{C}{\varepsilon^2}|y|} dy + C\varepsilon^{-3} r^{\frac{1}{2}} \\
\le\ & \left(\varepsilon^{-3} r^{\frac{1}{2}} + \varepsilon^{-\frac{5}{2}} \log \frac{1}{\varepsilon} (\log(1+r))^2 r^{\frac{1}{4}}\right).
\end{aligned} \tag{3.46}
$$

While for the latter one, we have

$$
\begin{aligned}
\mathcal{I}_{1,2} \le\ & \frac{C}{\varepsilon^2} \int_{B_r(0)} \frac{1}{\sqrt{x^2+y^2}} \left(\min\left\{\frac{1}{\sqrt{|y|}}, \frac{1}{\varepsilon}\right\} + \frac{1}{\varepsilon} \max\left\{1, \log \frac{\varepsilon^2}{|y|}\right\} e^{-\frac{C}{\varepsilon^2}|y|}\right) dxdy \\
\le\ & C\varepsilon^{-3} \int_0^r \left(\int_0^{\frac{r}{|y|}} \frac{1}{\sqrt{x^2+1}} dx\right) \left(1 + \log \frac{1}{|y|}\right) e^{-\frac{C}{\varepsilon^2}|y|} dxdy \\
& + C\varepsilon^{-2} \int_0^r \int_0^{2\pi} \frac{1}{\rho^{\frac{3}{2}}} \frac{1}{\sqrt{|\sin\theta|}} d\theta d\rho \\
\le\ & C\varepsilon^{-3+\frac{3}{2}} \int_0^r \left(1 + \log \frac{1}{|y|}\right) \log\left(1 + \frac{r}{|y|}\right) \frac{1}{|y|^{\frac{3}{4}}} dy + C\varepsilon^{-2} r^{\frac{1}{2}} \\
\le\ & C\left(\varepsilon^{-2} r^{\frac{1}{2}} + \varepsilon^{-\frac{3}{2}} (\log(1+r))^2 r^{\frac{1}{4}}\right).
\end{aligned} \tag{3.47}
$$

Hence we finish the proof. □

4. THE LINEARIZED KP-I OPERATOR

In this section, we study the mapping properties of the modified linearized KP-I operator

$$\begin{aligned}&\partial_x^4\phi-(2\sqrt{2}-\varepsilon^2)\partial_x^2\phi-6(\sqrt{2}-\varepsilon^2)\partial_x(\partial_x q\partial_x\phi)-2\partial_y^2\phi+2\varepsilon^2\partial_x^2\partial_y^2\phi+\varepsilon^4\partial_y^4\phi\\&=\partial_x h_1+\partial_y h_2.\end{aligned}\tag{4.1}$$

The functions ϕ, h_1 and h_2 are in suitable functional spaces defined as follows respectively:

$$\mathcal{H}_1:=\left\{f\mid f\in\mathcal{H}_{ox},\ \|f\|_a<+\infty\right\},\tag{4.2}$$

$$\mathcal{H}_2:=\left\{f\mid f\in\mathcal{H}_e,\ \|f\|_b^2:=\int_{\mathbb{R}^2}(|f|^2+|\partial_x f|^2)dxdy<+\infty\right\},\tag{4.3}$$

and

$$\mathcal{H}_3:=\left\{f\mid f\in\mathcal{H}_{oxy},\ \|f\|_c^2:=\int_{\mathbb{R}^2}(|f|^2+|\partial_y f|^2)dxdy<+\infty\right\},\tag{4.4}$$

where

$$\|f\|_a^2=\int_{\mathbb{R}^2}\left(|\partial_x^4\phi|^2+\varepsilon^4|\partial_x^2\partial_y^2\phi|^2+\varepsilon^8|\partial_y^4\phi|^2+|\nabla^2\phi|^2+|\nabla\phi|^2\right)dxdy,$$

$\mathcal{H}_o$ and $\mathcal{H}_e$ are given as

$$\mathcal{H}_{ox}=\left\{f\mid f(x,y)=-f(-x,y)=f(x,-y)\right\},$$

$$\mathcal{H}_e=\left\{f\mid f(x,y)=f(-x,y)=f(x,-y)\right\},$$

and

$$\mathcal{H}_{oxy}=\left\{f\mid f(x,y)=-f(-x,y)=-f(x,-y)\right\}.$$

The first result of this section is about the a-priori estimate

Proposition 4.1. *Let $\phi\in\mathcal{H}_1$ be a solution of* (4.1) *with $h_1\in\mathcal{H}_2$ and $h_2\in\mathcal{H}_3$. Then there exist positive constants ε_0 and C such that for all $\varepsilon\in(0,\varepsilon_0)$, it holds that*

$$\|\phi\|_a\leq C\left(\|h_1\|_b+\|h_2\|_c\right).$$

Proof. Consider the differential operator

$$L\phi=\partial_x^2\phi-(2\sqrt{2}-\varepsilon^2)\phi-6\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{\frac{5}{2}}(\partial_x q\phi)-2\partial_x^{-2}\partial_y^2\phi.$$

It is known from [30, Theorem 2] that it admits only negative eigenvalue λ_1, the associated eigenfunction is denoted by ϕ_0, i.e.,

$$\partial_x^2\phi_0-(2\sqrt{2}-\varepsilon^2)\phi_0-6\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{\frac{5}{2}}(\partial_x q\phi_0)-2\partial_x^{-2}\partial_y^2\phi_0+\lambda_1\phi_0=0.$$

We notice that as $\varepsilon\to 0$, the negative eigenvalue λ_1 has a limit value which corresponds to the unique negative eigenvalue of the linearized equation of (2.33). It is known that $\int_{-\infty}^{\infty}\phi_0 dx=0$ for any y and we can define

$$\phi_1:=\partial_x^{-1}\phi_0=\int_{-\infty}^{x}\phi_0 dx.$$

Then we see that ϕ_1 is a function odd in x, even in y, and

$$\partial_x^4\phi_1 - (2\sqrt{2}-\varepsilon^2)\partial_x^2\phi_1 - 6\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{\frac{5}{2}}\partial_x(\partial_x q\partial_x\phi_1) - 2\partial_y^2\phi_1 = -\lambda_1\partial_x^2\phi_1.$$

We decompose ϕ into $\phi = c\phi_1 + \phi_2$, with

$$c = \frac{\int_{\mathbb{R}^2}\partial_x\phi\partial_x\phi_1 dx}{\int_{\mathbb{R}^2}\partial_x\phi_1\partial_x\phi_1 dx},$$

and ϕ_2 belonging to the complement of ϕ_1 in the space $\mathcal{H}_1$ under the following product

$$(f,g) = \int_{\mathbb{R}^2}\left(\partial_x^2 f\partial_x^2 g + (2\sqrt{2}-\varepsilon^2)\partial_x f\partial_x g + 6\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{\frac{5}{2}}\partial_x q\partial_x f\partial_x g + 2\partial_y f\partial_y g\right)dxdy.$$

For any function ψ in the complement space of ϕ_1 in $\mathcal{H}_1$, we see that

$$(\psi,\psi) \geq \lambda_2\|\partial_x\psi\|^2_{L^2(\mathbb{R}^2)},$$

where λ_2 refers to the smallest positive eigenvalue of L and it is not difficult to see that λ_2 has a uniform bound for any ε. In addition, for such ψ, we can choose $\lambda_* \in (0,1)$ such that

$$\lambda_2(1-\lambda_*) - 24\lambda_* \geq 2\lambda_*. \tag{4.5}$$

As a consequence

$$\begin{aligned}(\psi,\psi) &\geq (1-\lambda_*)(\psi,\psi) - 6\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{\frac{5}{2}}\lambda_*\max_{(x,y)\in\mathbb{R}^2}|\partial_x q|\int_{\mathbb{R}^2}(\partial_x\psi)^2 dxdy \\ &\quad + 2\lambda_*\int_{\mathbb{R}^2}(\partial_y\psi)^2 dxdy \\ &\geq (\lambda_2(1-\lambda_*) - 24\lambda_*)\int_{\mathbb{R}^2}(\partial_x\psi)^2 dxdy + 2\lambda_*\int_{\mathbb{R}}(\partial_y\psi)^2 dxdy \\ &= 2\lambda_*\int_{\mathbb{R}^2}|\nabla\psi|^2 dxdy,\end{aligned} \tag{4.6}$$

where we used $\max_{(x,y)\in\mathbb{R}^2} 6\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{\frac{5}{2}}|\partial_x q(x,y)| \leq 24$. We write (4.1) as

$$\begin{aligned}&\partial_x^4\phi_2 - (2\sqrt{2}-\varepsilon^2)\partial_x^2\phi_2 - 6(\sqrt{2}-\varepsilon^2)\partial_x(\partial_x q\partial_x\phi_2) - 2\partial_y^2\phi_2 + 2\varepsilon^2\partial_x^2\partial_y^2\phi_2 + \varepsilon^4\partial_y^4\phi_2 \\ &= \partial_x h_1 + \partial_y h_2 + c\lambda_1\partial_x^2\phi_1 + c\left(-6\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{\frac{5}{2}} + 6\sqrt{2} - 6\varepsilon^2\right)\partial_x(\partial_x q\partial_x\phi_1) \\ &\quad - 2c\varepsilon^2\partial_x^2\partial_y^2\phi_1 - c\varepsilon^4\partial_y^4\phi_1.\end{aligned} \tag{4.7}$$

For convenience, we set

$$\Lambda_\varepsilon = \left(6\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{\frac{5}{2}} - 6\sqrt{2} + 6\varepsilon^2\right).$$

It is not difficult to see that $\Lambda_\varepsilon = O(\varepsilon^2)$ and negative when ε is small enough. Multiplying (4.7) by $c\phi_1$ and using integration by parts we gain

$$\begin{aligned}
&-c\int_{\mathbb{R}^2}\left(\Lambda_\varepsilon\partial_x q\partial_x\phi_1\partial_x\phi_2 - 2\varepsilon^2\partial_x\partial_y\phi_1\partial_x\partial_y\phi_2 - \varepsilon^4\partial_y^2\phi_1\partial_y^2\phi_2\right)dxdy\\
&= -c\int_{\mathbb{R}^2}h_1\partial_x\phi_1 dxdy - c\int_{\mathbb{R}^2}h_2\partial_y\phi_1 dxdy - c^2\lambda_1\|\partial_x\phi_1\|^2_{L^2(\mathbb{R}^2)} - c^2\varepsilon^4\|\partial_y^2\phi_1\|^2_{L^2(\mathbb{R}^2)}\\
&\quad - 2c^2\varepsilon^2\|\partial_x\partial_y\phi_1\|^2_{L^2(\mathbb{R}^2)} + c^2\Lambda_\varepsilon\int_{\mathbb{R}^2}\partial_x q|\partial_x\phi_1|^2 dxdy.
\end{aligned}\tag{4.8}$$

From (4.8) and Young's inequality we get

$$\begin{aligned}
&c^2(-\lambda_1-\delta_\varepsilon)\|\partial_x\phi_1\|^2_{L^2(\mathbb{R}^2)}\\
&\le -\frac{\lambda_*}{32\lambda_1}\left(\varepsilon^2\|\partial_x\phi_2\|^2_{L^2(\mathbb{R}^2)} + 2\varepsilon^2\|\partial_x\partial_y\phi_2\|^2_{L^2(\mathbb{R}^2)} + \varepsilon^4\|\partial_y^2\phi_2\|^2_{L^2(\mathbb{R}^2)}\right)\\
&\quad - \frac{2}{\lambda_1}\|h_1\|^2_{L^2(\mathbb{R}^2)} - \frac{2}{\lambda_1}\frac{\|\partial_y\phi_1\|^2_{L^2(\mathbb{R}^2)}}{\|\partial_x\phi_1\|^2_{L^2(\mathbb{R}^2)}}\|h_2\|^2_{L^2(\mathbb{R}^2)},
\end{aligned}\tag{4.9}$$

where λ_* is given in (4.5), δ_ε depends on ε and can be arbitrarily small as $\varepsilon \to 0$. In (4.9) we have also used $\delta(\varepsilon)\|\partial_x\phi_1\|_{L^2(\mathbb{R}^2)}$ to control the terms $\varepsilon^2\|\partial_x\partial_y\phi_1\|^2_{L^2(\mathbb{R}^2)}$ and $\varepsilon^4\|\partial_y^2\phi_1\|^2_{L^2(\mathbb{R}^2)}$, due to that ϕ_1 is a concrete function. By the same reason, we see that the coefficient before $\|h_2\|_{L^2(\mathbb{R}^2)}$ is a specified constant.

Multiplying (4.7) by ϕ_2 and using (4.6) we get

$$\begin{aligned}
&2\lambda_*\|\nabla\phi_2\|^2_{L^2(\mathbb{R}^2)} + 2\varepsilon^2\|\partial_x\partial_y\phi_2\|^2_{L^2(\mathbb{R}^2)} + \varepsilon^4\|\partial_y^2\phi_2\|^2_{L^2(\mathbb{R}^2)}\\
&\le \int_{\mathbb{R}^2}\left((\partial_x^2\phi_2)^2 + (2\sqrt{2}-\varepsilon^2)(\partial_x\phi_2)^2 + 6\sqrt{2}\left(\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}\right)^{\frac{5}{2}}\partial_x q(\partial_x\phi_2)^2 + 2(\partial_y\phi_2)^2\right)dxdy\\
&\quad + \int_{\mathbb{R}^2}\left(2\varepsilon^2(\partial_x\partial_y\phi_2)^2 + \varepsilon^4(\partial_y^2\phi_2)^2\right)dxdy\\
&\le \Lambda_\varepsilon\int_{\mathbb{R}}\partial_x q|\partial_x\phi_2|^2 dxdy + \frac{1}{\lambda_*}\|h_1\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{\lambda_*}\|h_2\|^2_{L^2(\mathbb{R}^2)}\\
&\quad + \frac{\lambda_*}{2}\|\nabla\phi_2\|^2_{L^2(\mathbb{R}^2)} + c^2\left(\frac{\lambda_1^2}{\lambda_*} + \tilde{\delta}_\varepsilon\right)\|\partial_x\phi_1\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\varepsilon^2\|\partial_x\phi_2\|^2_{L^2(\mathbb{R}^2)}\\
&\quad + \varepsilon^2\|\partial_x\partial_y\phi_2\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\varepsilon^4\|\partial_y^2\phi_2\|^2_{L^2(\mathbb{R}^2)},
\end{aligned}\tag{4.10}$$

where $\lambda_* > 0$ is given in (4.5), and $\tilde{\delta}_\varepsilon$ can be arbitrary small as $\varepsilon \to 0$. Taking ε such that

$$\left(2 - \Lambda_\varepsilon\max_{(x,y)\in\mathbb{R}^2}|\partial_x q|\right)\varepsilon^2 < \lambda_*,$$

we get

$$\begin{aligned}
&\lambda_*\|\nabla\phi_2\|^2_{L^2(\mathbb{R}^2)} + \varepsilon^2\|\partial_x\phi_2\|^2_{L^2(\mathbb{R}^2)} + 2\varepsilon^2\|\partial_x\partial_y\phi_2\|^2_{L^2(\mathbb{R}^2)} + \varepsilon^4\|\partial_y^2\phi_2\|^2_{L^2(\mathbb{R}^2)}\\
&\le \frac{2}{\lambda_*}\|h_1\|^2_{L^2(\mathbb{R}^2)} + \frac{2}{\lambda_*}\|h_2\|^2_{L^2(\mathbb{R}^2)} + 2c^2\left(\frac{\lambda_1^2}{\lambda_*} + \tilde{\delta}_\varepsilon\right)\|\partial_x\phi_1\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}\tag{4.11}$$

Using (4.9) and (4.11) we derive that

$$\begin{aligned}&\frac{c^2\lambda_1^2}{\lambda_*}\|\partial_x\phi_1\|^2_{L^2(\mathbb{R}^2)}+\lambda_*\|\nabla\phi_2\|^2_{L^2(\mathbb{R}^2)}\\&\le\left(\frac{8}{\lambda_*}\frac{\|\partial_y\phi_1\|^2_{L^2(\mathbb{R}^2)}}{\|\partial_x\phi_1\|^2_{L^2(\mathbb{R}^2)}}+\frac{2}{\lambda_*}\right)\|h_2\|^2_{L^2(\mathbb{R}^2)}+\frac{10}{\lambda_*}\|h_1\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}\tag{4.12}$$

This implies that

$$\|\nabla\phi\|^2_{L^2(\mathbb{R})}\le C\left(\|h_1\|^2_{L^2(\mathbb{R}^2)}+\|h_2\|^2_{L^2(\mathbb{R}^2)}\right).\tag{4.13}$$

On the other hand, for the specific function ϕ_1,

$$\int_{\mathbb{R}^2}(\partial_x^2\phi_1)^2dxdy\le C\int_{\mathbb{R}^2}(\partial_x\phi_1)^2dxdy.\tag{4.14}$$

Together with (4.10) we have

$$\int_{\mathbb{R}^2}(\partial_x^2\phi_2)^2dxdy\le C\left(\|h_1\|^2_{L^2(\mathbb{R}^2)}+\|h_2\|^2_{L^2(\mathbb{R}^2)}\right).\tag{4.15}$$

As a consequence of (4.13), (4.14) and (4.15) we get that

$$\|\partial_x(q\partial_x\phi)\|_{L^2(\mathbb{R}^2)}\le C\left(\|h_1\|_{L^2(\mathbb{R}^2)}+\|h_2\|_{L^2(\mathbb{R}^2)}\right).\tag{4.16}$$

Next, we write (4.1) as

$$\begin{aligned}&\partial_x^4\phi-(2\sqrt{2}-\varepsilon^2)\partial_x^2\phi-2\partial_y^2\phi+2\varepsilon^2\partial_x^2\partial_y^2\phi+\varepsilon^4\partial_y^4\phi\\&=\partial_xh_1+\partial_yh_2+6(\sqrt{2}-\varepsilon^2)\partial_x(\partial_xq\partial_x\phi).\end{aligned}\tag{4.17}$$

Denote the right handside by h, we get from (4.16) that

$$\|h\|_{L^2(\mathbb{R}^2)}\le C\int_{\mathbb{R}^2}(|\partial_xh_1|^2+|\partial_yh_2|^2+|h_1|^2+|h_2|^2)dxdy=C(\|h_1\|_b+\|h_2\|_c).$$

Multiplying (4.17) by ϕ, after integration by parts and using the Hölder inequality we have

$$\int_{\mathbb{R}^2}\left(|\partial_x^2\phi|^2+|\nabla\phi|^2+\varepsilon^2|\partial_x\partial_y\phi|^2+\varepsilon^4|\partial_y^2\phi|^2\right)dxdy\le C\|h\|^2_{L^2(\mathbb{R}^2)}.$$

To prove the original conclusion, it is enough to show that each term in the definition of $\|\cdot\|_a$ is bounded by $\|h\|_{L^2(\mathbb{R}^2)}$. We only give explanation for the term $\|\partial_x^4\phi\|^2_{L^2(\mathbb{R}^2)}$, while the other terms can be handled similarly. Taking Fourier transformation on both sides of (4.17) we have

$$\hat\phi(\xi_1,\xi_2)=\frac{1}{\xi_1^4+(2\sqrt{2}-\varepsilon^2)\xi_1^2+2\xi_2^2+2\varepsilon^2\xi_1^2\xi_2^2+\varepsilon^4\xi_2^4}\hat h(\xi_1,\xi_2).$$

It is known that

$$\begin{aligned}\|\partial_x^4\phi\|_{L^2(\mathbb{R}^2)}=\|\xi_1^4\hat\phi\|_{L^2(\mathbb{R}^2)}&=\left\|\frac{\xi_1^4}{\xi_1^4+2\sqrt{2}\xi_1^2+2\xi_2^2+2\varepsilon^2\xi_1^2\xi_2^2+\varepsilon^4\xi_2^4}\hat h(\xi_1,\xi_2)\right\|_{L^2(\mathbb{R}^2)}\\&\le C\|\hat h\|_{L^2(\mathbb{R}^2)}=C\|h\|_{L^2(\mathbb{R}^2)}\le C\left(\|h_1\|_b+\|h_2\|_c\right).\end{aligned}$$

Then we prove the conclusion. □

After establishing the L^2 theory for equation (4.1), we consider the (4.1) in a suitable weighted Sobolev space, which helps us to study the nonlinear problem (2.30). Now, we introduce the weighted Sobolev space for ϕ, h_1 and h_2:

$$\phi \in \mathcal{F}_1 := \{f \mid f \in \mathcal{H}_1,\ \|f\|_* < +\infty\}, \tag{4.18}$$

$$h_1 \in \mathcal{F}_2 := \{f \mid f \in \mathcal{H}_2,\ \|f\|_{**} < +\infty\}, \tag{4.19}$$

and

$$h_2 \in \mathcal{F}_3 := \{f \mid f \in \mathcal{H}_3,\ \|f\|_{***} < +\infty\}, \tag{4.20}$$

where

$$\begin{aligned}
\|f\|_* = {} & \|f\|_a + \|(1+r)^{1-\delta} f\|_{L^\infty(\mathbb{R}^2)} + \frac{1}{\log\frac{1}{\varepsilon}} \|(1+r) f\|_{L^\infty(\mathbb{R}^2)} \\
& + \|(1+r)^{\frac{3}{2}-\delta}\partial_x f\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^{\frac{1}{2}} \|(1+r)^{\frac{3}{2}}\partial_x f\|_{L^\infty(\mathbb{R}^2)} \\
& + \|(1+r)^{\frac{3}{2}}\partial_x^2 f\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^{\frac{1}{2}} \|(1+r)^{\frac{3}{2}}\partial_x^3 f\|_{L^\infty(\mathbb{R}^2)} \\
& + \varepsilon^{\frac{1}{2}} \|(1+r)^{\frac{3}{2}}\partial_x^4 f\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^{\frac{1}{2}} \|(1+r)^{\frac{3}{2}-\delta}\partial_y f\|_{L^\infty(\mathbb{R}^2)} \\
& + \varepsilon^{\frac{3}{2}} \|(1+r)^{\frac{3}{2}}\partial_y f\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^{\frac{3}{2}} \|(1+r)^{\frac{3}{2}}\partial_y^2 f\|_{L^\infty(\mathbb{R}^2)} \\
& + \varepsilon^{\frac{7}{2}} \|(1+r)^{\frac{3}{2}}\partial_y^3 f\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^{\frac{11}{2}} \|(1+r)^{\frac{3}{2}}\partial_y^4 f\|_{L^\infty(\mathbb{R}^2)} \\
& + \varepsilon^{\frac{1}{2}} \|(1+r)^{\frac{3}{2}}\partial_x\partial_y f\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^{\frac{1}{2}} \|(1+r)^{\frac{3}{2}}\partial_x^2\partial_y f\|_{L^\infty(\mathbb{R}^2)} \\
& + \varepsilon^{\frac{3}{2}} \|(1+r)^{\frac{3}{2}}\partial_x\partial_y^2 f\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^{\frac{5}{2}} \|(1+r)^{\frac{3}{2}}\partial_x^2\partial_y^2 f\|_{L^\infty(\mathbb{R}^2)} \\
& + \varepsilon^{\frac{3}{2}} \|(1+r)^{\frac{3}{2}-\delta}\partial_x^{-1}\partial_y^2 f\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^{\frac{7}{2}} \|(1+r)^{\frac{3}{2}-\delta} \|\partial_x^{-1}\partial_y^3 f\|_{L^\infty(\mathbb{R}^2)} \\
& + \varepsilon^{\frac{11}{2}} \|(1+r)^{\frac{3}{2}-\delta}\partial_x^{-1}\partial_y^4 f\|_{L^\infty(\mathbb{R}^2)},
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
\|f\|_{**} = {} & \|f\|_b + \|(1+r)^{5/2-\delta} h\|_{L^\infty(\mathbb{R}^2)} + \|(1+r)^{5/2-\delta}\partial_x h\|_{L^\infty(\mathbb{R}^2)} \\
& + \|(1+r)^{5/2-\delta}\partial_x^2 h\|_{L^\infty(\mathbb{R}^2)},
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\|f\|_{***} = {} & \|f\|_c + \|(1+r)^{3-\delta} h\|_{L^\infty(\mathbb{R}^2)} + \|(1+r)^{3-\delta}\partial_y h\|_{L^\infty(\mathbb{R}^2)} \\
& + \|(1+r)^{3-\delta}\partial_{xy} h\|_{L^\infty(\mathbb{R}^2)}.
\end{aligned} \tag{4.23}$$

Here δ is slightly greater than 0. We remark that for any function $f \in \mathcal{F}_3$, we could define $\partial_x^{-1} f = -\int_x^\infty f(s,y)ds$ and it is easy to check that

$$\|(1+r)^{2-\delta}\partial_x^{-1} f\|_{L^\infty(\mathbb{R}^2)} + \|(1+r)^{2-\delta}\partial_x^{-1}\partial_y f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{***}. \tag{4.24}$$

We shall use (4.24) in the following proposition frequently.

The second result of this section is as follows

Proposition 4.2. *For each $h_1 \in \mathcal{F}_2$ and $h_2 \in \mathcal{F}_3$. Then there exist positive constants ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the linear equation* (4.1) *admits a solution $\phi \in \mathcal{F}_1$, with*

$$\|\phi\|_* \le C\left(\|h_1\|_{**} + \|h_2\|_{***}\right). \tag{4.25}$$

Proof. From Proposition 4.1 and Lax-Milgram Theorem we can get a solution $\phi \in \mathcal{H}_1$ to equation (4.1) provided $h_1 \in \mathcal{H}_2$ and $h_2 \in \mathcal{H}_3$, the smooth property follows by the Sobolev inequality and Schauder estimate for the biharmonic equation, we refer the readers to [5, 6, 25, 35, 36]. Then we shall focus on the decay estimate in the following. To prove (4.25), we write equation (4.1) as

$$\begin{aligned}&\partial_x^4\phi-(2\sqrt{2}-\varepsilon^2)\partial_x^2\phi-2\partial_y^2\phi+2\varepsilon^2\partial_x^2\partial_y^2\phi+\varepsilon^4\partial_y^4\phi\\&=\partial_x\left(6(\sqrt{2}-\varepsilon^2)\partial_xq\partial_x\phi+h_1\right)+\partial_yh_2.\end{aligned}\tag{4.26}$$

In addition, it holds that

$$\|\phi\|_a\le C\left(\|h_1\|_b+\|h_2\|_c\right).$$

As a consequence, we have

$$\|\nabla\phi\|_{L^2(\mathbb{R}^2)}+\|\nabla^2\phi\|_{L^2(\mathbb{R}^2)}+\|\partial_y\partial_x^2\phi\|_{L^2(\mathbb{R}^2)}+\|\partial_x^3\phi\|_{L^2(\mathbb{R}^2)}\le C\left(\|h_1\|_b+\|h_2\|_c\right).\tag{4.27}$$

Using Gagliardo-Nirenberg interpolation inequality (see [1] for the proof of this inequality)

$$\|D^ju\|_{L^{p_1}(\mathbb{R}^n)}\le C\|D^mu\|^{\alpha}_{L^r(\mathbb{R}^n)}\|u\|^{1-\alpha}_{L^{p_2}(\mathbb{R}^n)},\tag{4.28}$$

where $\frac{1}{p_1}=\frac{j}{n}+\left(\frac{1}{r}-\frac{m}{n}\right)\alpha+\frac{1-\alpha}{p_2}$. Using (4.28) and (4.27) we conclude that

$$\|\partial_x\phi\|_{L^p(\mathbb{R}^2)}+\|\partial_x^2\phi\|_{L^p(\mathbb{R}^2)}\le C\|\phi\|_a\le C\left(\|h_1\|_b+\|h_2\|_c\right),\ \forall p\in[2,+\infty).\tag{4.29}$$

Denote the Green kernel of the linear operator defined on the left hand side of (4.26) by $G(x,y)$. In the following we shall derive the a-priori estimate (4.25) by analyzing $\|\phi\|_*$ term by term. The proof is quite long, we shall divide our discussion into the following five steps

Step 1. We start with the estimation on $\partial_x^m\phi(x,y)$ for $m=0,1,\cdots,4$. Using Green representation we have

$$\begin{aligned}\phi(x,y)=&-6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_xG(x-s,y-t)(\partial_xq\partial_x\phi)(s,t)dsdt\\&-\int_{\mathbb{R}^2}\partial_xG(x-s,y-t)h_1(s,t)dsdt\\&-\int_{\mathbb{R}^2}\partial_yG(x-s,y-t)h_2(s,t)dsdt.\end{aligned}\tag{4.30}$$

We shall consider the asymptotic behavior of $\phi(x,y)$ when $r=\sqrt{x^2+y^2}$ is large. For the first term on the right hand side of (4.30) we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\partial_xG(x-s,y-t)(\partial_xq\partial_x\phi)(s,t)dsdt\right|\\&\le\int_{B_{r/2}(x,y)}\left|\partial_xG(x-s,y-t)(\partial_xq\partial_x\phi)(s,t)\right|dsdt\\&\quad+\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}\left|\partial_xG(x-s,y-t)(\partial_xq\partial_x\phi)(s,t)\right|dsdt\\&\le\frac{C}{r^2}\left(\int_{B_{r/2}(x,y)}|\partial_xG(x-s,y-t)|^{\frac{6}{5}}dsdt\right)^{\frac{5}{6}}\left(\int_{B_{r/2}(x,y)}|\partial_x\phi(s,t)|^6dsdt\right)^{\frac{1}{6}}\\&\quad+\frac{C}{r}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_xq(s,t)|^2dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x\phi(s,t)|^2dsdt\right)^{\frac{1}{2}}\\&\le\frac{C}{r}\|\phi\|_a\le\frac{C}{r}\left(\|h_1\|_b+\|h_2\|_c\right),\end{aligned}\tag{4.31}$$

where we used the fact that $|\partial_x G(x,y)| \leq \frac{C}{r}$ (see Theorem 3.3) and (4.29). While for the second term on the right hand side of (4.30), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \partial_x G(x-s,y-t)h_1(s,t)dsdt\right| \\
&\leq C\|h_1\|_{**}\int_{\mathbb{R}^2} \frac{1}{\sqrt{(x-s)^2+(y-t)^2}}\frac{1}{(1+s^2+t^2)^{5/2-\delta}}dsdt \qquad (4.32)\\
&\leq \frac{C}{r}\|h_1\|_{**}.
\end{aligned}$$

Similarly, one can show that

$$\left|\int_{\mathbb{R}^2} \partial_y G(x-s,y-t)h_2(s,t)dsdt\right| \leq \frac{C\log\frac{1}{\varepsilon}}{r}\|h_2\|_{***}. \qquad (4.33)$$

Using (4.31), (4.32) and (4.33) we get that

$$\|(1+r)\phi\|_{L^\infty(\mathbb{R}^2)} \leq C\log\frac{1}{\varepsilon}\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \qquad (4.34)$$

If we do not pursue the good decay of ϕ, we could replace (4.33) by the following one

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} G(x-s,y-t)\partial_y h_2(s,t)dsdt\right| \\
&\leq \left|\int_{\mathbb{R}^2} \partial_x G(x-s,y-t)\partial_x^{-1}\partial_y h_2(s,t)dsdt\right| \\
&\leq \frac{C}{r^{2-\delta}}\|h_2\|_{***}\int_{B_{r/2}(x,y)} |\partial_x G(x-s,y-t)|dsdt \qquad (4.35)\\
&\quad + C\|h_2\|_{***}\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)} \frac{1}{\sqrt{(x-s)^2+(y-t)^2}(1+\sqrt{s^2+t^2})^{2-\delta}}dsdt \\
&\leq \frac{C}{r^{1-\delta}}\|h_2\|_{***}.
\end{aligned}$$

Then by (4.31), (4.32) and (4.35) we get

$$\|(1+r)^{1-\delta}\phi\|_{L^\infty(\mathbb{R}^2)} \leq C\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \qquad (4.36)$$

Next we study the asymptotic behavior of $\partial_x\phi$, we get that

$$\begin{aligned}
\partial_x\phi(x,y) = &-6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2} \partial_x^2 G(x-s,y-t)(\partial_x q\partial_x\phi)(s,t)dsdt \\
&-\int_{\mathbb{R}^2} \partial_x^2 G(x-s,y-t)h_1(s,t)dsdt \qquad (4.37)\\
&-\int_{\mathbb{R}^2} \partial_x\partial_y G(x-s,y-t)h_2(s,t)dsdt.
\end{aligned}$$

As (4.31)-(4.33), we could use (4.29) and Theorem 3.3 derive that

$$\left|\int_{\mathbb{R}^2} \partial_x^2 G(x-s,y-t)(\partial_x q\partial_x\phi)(s,t)dsdt\right| \leq Cr^{-\frac{3}{2}}\|\phi\|_a, \qquad (4.38)$$

where we have used the following inequality

$$\begin{aligned}\int_{B_{r/2}(0)}|\partial_x^2 G(s,t)|^{\frac{6}{5}}dsdt &\leq \int_{B_1(0)}|\partial_x^2 G(s,t)|^{\frac{6}{5}}dsdt+\int_{B_{r/2}(0)\setminus B_1(0)}|\partial_x^2 G(s,t)|^{\frac{6}{5}}dsdt\\ &\leq C\int_{B_1(0)}\left(\log\frac{1}{\sqrt{|t|}}\frac{1}{\sqrt{s^2+t^2}}\right)^{\frac{6}{5}}dsdt\\ &\quad + C\int_{B_{r/2}(0)\setminus B_1(0)}\left(\frac{1}{r^{3/2}}\right)^{\frac{6}{5}}dsdt\leq Cr^{\frac{1}{5}}.\end{aligned}$$

For the second and third terms on the right hand side of (4.37), we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\partial_x^2 G(x-s,y-t)h_1(s,t)dsdt\right|\\ &\leq \frac{C}{(1+r)^{\frac{5}{2}-\delta}}\|h_1\|_{**}\int_{B_{r/2}(x,y)}|\partial_x^2 G(x-s,y-t)|dsdt\\ &\quad+\frac{C}{r^{\frac{3}{2}}}\|h_1\|_{**}\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}\frac{1}{\left(1+\sqrt{s^2+t^2}\right)^{5/2-\delta}}dsdt\\ &\leq Cr^{-\frac{3}{2}}\|h_1\|_{**},\end{aligned}\tag{4.39}$$

where we used Theorem 3.3. Similar as (4.39) we get

$$\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)h_2(s,t)dsdt\right|\leq C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_2\|_{***}.\tag{4.40}$$

Using (4.38)-(4.40) we have

$$\|(1+r)^{\frac{3}{2}}\partial_x\phi\|_{L^\infty(R^2)}\leq C\left(\|h_1\|_{**}+\varepsilon^{-\frac{1}{2}}\|h_2\|_{***}\right).\tag{4.41}$$

As (4.36), if we do not pursue the good decay of $\partial_x\phi$, we could replace (4.40) by the following one

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\partial_x G(x-s,y-t)\partial_y h_2(s,t)dsdt\right|\\ &=\left|\int_{\mathbb{R}^2}\partial_x^2 G(x-s,y-t)\partial_x^{-1}\partial_y h_2(s,t)dsdt\right|\\ &\leq\frac{C}{(1+r)^{2-\delta}}\|h_2\|_{**}\int_{B_{r/2}(x,y)}|\partial_x^2 G(x-s,y-t)|dsdt\\ &\quad+\frac{C}{r^{3/2}}\|h_2\|_{**}\int_{B_{r/2}(0)}\frac{1}{(1+\sqrt{s^2+t^2})^{2-\delta}}dsdt\\ &\quad+C\|h_2\|_{**}\int_{\mathbb{R}^2\setminus(B_{r/2}(x,y)\cup B_{r/2}(0))}\frac{\left(\sqrt{(x-s)^2+(y-s)^2}\right)^{-\frac{3}{2}}}{\left(1+\sqrt{s^2+t^2}\right)^{2-\delta}}dsdt\\ &\leq Cr^{-\frac{3}{2}+\delta}\|h_2\|_{**},\end{aligned}\tag{4.42}$$

where we used (4.24). Then combined with (4.38) and (4.39) we get

$$\|(1+r)^{\frac{3}{2}-\delta}\partial_x\phi(x,y)\|_{L^\infty(\mathbb{R}^2)}\leq C(\|h_1\|_{**}+\|h_2\|_{***}).\tag{4.43}$$

To study the term $\partial_x^2\phi$, we notice that

$$\begin{aligned}\partial_x^2\phi=&(6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_x^2G(x-s,y-t)\partial_x(\partial_x q\partial_x\phi)(s,t)dsdt\\&+\int_{\mathbb{R}^2}\partial_x^2G(x-s,y-t)\partial_x h_1(s,t)dsdt\\&+\int_{\mathbb{R}^2}\partial_x^2G(x-s,y-t)\partial_y h_2(s,t)dsdt.\end{aligned}\tag{4.44}$$

For the first term on the right hand side of (4.44), we notice that

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\partial_x^2G(x-s,y-t)\partial_x(\partial_x q\partial_x\phi)(s,t)dsdt\right|\\&\le\frac{C}{r^3}\left(\int_{B_{r/2}(x,y)}|\partial_x^2G(x-s,y-t)|^{\frac{6}{5}}dsdt\right)^{\frac{5}{6}}\left(\int_{B_{r/2}(x,y)}|\partial_x\phi|^6dsdt\right)^{\frac{1}{6}}\\&\quad+\frac{C}{r^2}\left(\int_{B_{r/2}(x,y)}|\partial_x^2G(x-s,y-t)|^{\frac{6}{5}}dsdt\right)^{\frac{5}{6}}\left(\int_{B_{r/2}(x,y)}|\partial_x^2\phi|^6dsdt\right)^{\frac{1}{6}}\\&\quad+\frac{C}{r^{3/2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x q|^2dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^2\phi|^2dsdt\right)^{\frac{1}{2}}\\&\quad+\frac{C}{r^{3/2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^2 q|^2dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x\phi|^2dsdt\right)^{\frac{1}{2}}\le\frac{C}{r^{\frac{3}{2}}}\|\phi\|_a.\end{aligned}\tag{4.45}$$

While for the second and third terms on the right hand side of (4.44), following the argument as we did in (4.39) we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\partial_x^2G(x-s,y-t)\partial_x h_1(s,t)dsdt\right|&\le Cr^{-\frac{3}{2}}\|h_1\|_{**},\\\left|\int_{\mathbb{R}^2}\partial_x^2G(x-s,y-t)\partial_y h_2(s,t)dsdt\right|&\le Cr^{-\frac{3}{2}}\|h_2\|_{***}.\end{aligned}\tag{4.46}$$

Therefore, from (4.45) and (4.46) we get that

$$\|(1+r)^{\frac{3}{2}}\partial_x^2\phi\|_{L^\infty(R^2)}\le C(\|h_1\|_{**}+\|h_2\|_{**}).\tag{4.47}$$

For $\partial_x^3\phi(x,y)$ we have

$$\begin{aligned}\partial_x^3\phi(x,y)=&(6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_x^3G(x-s,y-t)\partial_x(\partial_x q\partial_x\phi)(s,t)dsdt\\&+\int_{\mathbb{R}^2}\partial_x^3G(x-s,y-t)(\partial_x h_1(s,t)+\partial_y h_2(s,t))dsdt.\end{aligned}\tag{4.48}$$

For the first term on the right hand side of (4.48), we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\partial_x^3 G(x-s,y-t)\partial_x(\partial_x q\partial_x\phi)(s,t)dsdt\right| \\
&\le \frac{C}{r^3}\left(\int_{B_{r/2}(x,y)}|\partial_x^3 G(x-s,y-t)|^{\frac{6}{5}}dsdt\right)^{\frac{5}{6}}\left(\int_{B_{r/2}(x,y)}|\partial_x\phi|^6 dsdt\right)^{\frac{1}{6}} \\
&\quad+\frac{C}{r^2}\left(\int_{B_{r/2}(x,y)}|\partial_x^3 G(x-s,y-t)|^{\frac{6}{5}}dsdt\right)^{\frac{5}{6}}\left(\int_{B_{r/2}(x,y)}|\partial_x^2\phi|^6 dsdt\right)^{\frac{1}{6}} \\
&\quad+\frac{C\varepsilon^{-\frac{1}{2}}}{r^{3/2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x q|^2 dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^2\phi|^2 dsdt\right)^{\frac{1}{2}} \\
&\quad+\frac{C\varepsilon^{-\frac{1}{2}}}{r^{3/2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^2 q|^2 dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x\phi|^2 dsdt\right)^{\frac{1}{2}} \\
&\le C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|\phi\|_a + C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right),
\end{aligned}
\tag{4.49}
$$

where we used

$$
\begin{aligned}
\int_{B_{r/2}(0)}|\partial_x^3 G(s,t)|^{\frac{6}{5}}dsdt &\le \int_{B_1(0)}|\partial_x^3 G(s,t)|^{\frac{6}{5}}dsdt + C\varepsilon^{-\frac{3}{5}}\int_{B_{r/2}(0)\setminus B_1(0)}\frac{1}{(\sqrt{s^2+t^2})^{\frac{9}{5}}}dsdt \\
&\le C + C\varepsilon^{-\frac{3}{5}}r^{\frac{1}{10}}.
\end{aligned}
$$

For the second term on the right hand side of (4.48), as (4.39) we have that

$$
\left|\int_{\mathbb{R}^2}\partial_x^3 G(x-s,y-t)\partial_x h_1(s,t)dsdt\right| \le C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_1\|_{**}, \tag{4.50}
$$

and

$$
\left|\int_{\mathbb{R}^2}\partial_x^3 G(x-s,y-t)\partial_y h_2(s,t)dsdt\right| \le C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_2\|_{***}. \tag{4.51}
$$

Hence from (4.49) to (4.51) we get that

$$
\|(1+r)^{\frac{3}{2}}\partial_x^3\phi\|_{L^\infty(R^2)} \le C\varepsilon^{-\frac{1}{2}}(\|h_1\|_{**}+\|h_2\|_{**}). \tag{4.52}
$$

Concerning $\partial_x^4\phi(x,y)$, using Green's representation formula and integration by parts we get that

$$
\begin{aligned}
\partial_x^4\phi(x,y) =& -6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_x^3 G(x-s,y-t)\partial_x^2(\partial_x q\partial_x\phi)(s,t)dsdt \\
&-\int_{\mathbb{R}^2}\partial_x^3 G(x-s,y-t)\partial_x^2 h_1(s,t)dsdt \\
&-\int_{\mathbb{R}^2}\partial_x^3 G(x-s,y-t)\partial_{xy} h_2(s,t)dsdt.
\end{aligned}
\tag{4.53}
$$

For the first term on the right hand side of (4.53), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\partial_x^3 G(x-s,y-t)\partial_x^2(\partial_x q\partial_x\phi)(s,t)dsdt\right| \\
&\leq C\varepsilon^{-\frac{1}{2}}r^{-\frac{7}{2}}(\|h_1\|_{**}+\|h_2\|_{***})\int_{B_{r/2}(x,y)}|\partial_x^3 G(x-s,y-t)|dsdt \\
&\quad+\frac{C\varepsilon^{-\frac{1}{2}}}{r^{3/2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x q|^2dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^3\phi|^2dsdt\right)^{\frac{1}{2}} \\
&\quad+\frac{C\varepsilon^{-\frac{1}{2}}}{r^{3/2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^2 q|^2dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^2\phi|^2dsdt\right)^{\frac{1}{2}} \\
&\quad+\frac{C\varepsilon^{-\frac{1}{2}}}{r^{3/2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^3 q|^2dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x\phi|^2dsdt\right)^{\frac{1}{2}} \\
&\leq C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right)+C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|\phi\|_a \\
&\leq C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right),
\end{aligned} \tag{4.54}$$

where we have used (4.41), (4.47), (4.52) and Theorem 3.3. For the other two terms on the right hand side of (4.53) we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\partial_x^3 G(x-s,y-t)\partial_x^2 h_1(s,t)dsdt\right| \\
&\leq \frac{C}{(1+r)^{\frac{5}{2}-\delta}}\|h_1\|_{**}\int_{B_{r/2}(x,y)}|\partial_x^3 G(x-s,y-t)|dsdt \\
&\quad+\frac{C\varepsilon^{-\frac{1}{2}}}{r^{\frac{3}{2}}}\|h_1\|_{**}\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}\frac{1}{\left(1+\sqrt{s^2+t^2}\right)^{5/2-\delta}}dsdt \\
&\leq C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_1\|_{**}.
\end{aligned} \tag{4.55}$$

Similarly

$$\left|\int_{\mathbb{R}^2}\partial_x^3 G(x-s,y-t)\partial_x\partial_y h_2(s,t)dsdt\right|\leq C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_2\|_{***}. \tag{4.56}$$

Hence, we have

$$\|(1+r)^{\frac{3}{2}}\partial_x^4\phi\|_{L^\infty(R^2)}\leq C\varepsilon^{-\frac{1}{2}}(\|h_1\|_{**}+\|h_2\|_{**}). \tag{4.57}$$

Step 2. In this step, we study the terms $\partial_y^n\phi(x,y)$ for $n=1,2,3$. For $\partial_y\phi(x,y)$ we notice that

$$\begin{aligned}
\partial_y\phi(x,y)= & -6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)(\partial_x q\partial_x\phi)(s,t)dsdt \\
& -\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)h_1(s,t)dsdt \\
& -\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)h_2(s,t)dsdt.
\end{aligned} \tag{4.58}$$

For the first term on the right hand side of (4.58) we use (4.36) to derive that

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)(\partial_x q\partial_x\phi)(s,t)dsdt\right|\\&\le\frac{C}{(1+r)^{7/2-\delta}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right)\int_{B_{r/2}(x,y)}|\partial_x\partial_y G(x-s,y-t)|dsdt\\&\quad+\frac{C\varepsilon^{-\frac{1}{2}}}{r^{\frac{3}{2}}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x q|^2dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x\phi|^2dsdt\right)^{\frac{1}{2}}\\&\le C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right).\end{aligned}\tag{4.59}$$

For the other two terms on the right hand side of (4.58) we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)h_1(s,t)dsdt\right|\\&\le\frac{C}{(1+r)^{\frac{5}{2}-\delta}}\|h_1\|_{**}\int_{B_{r/2}(x,y)}|\partial_x\partial_y G(x-s,y-t)|dsdt\\&\quad+\frac{C\varepsilon^{-\frac{1}{2}}}{r^{\frac{3}{2}}}\|h_1\|_{**}\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}\frac{1}{\left(1+\sqrt{s^2+t^2}\right)^{5/2-\delta}}dsdt\\&\le C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_1\|_{**}.\end{aligned}\tag{4.60}$$

Similarly

$$\left|\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)h_2(s,t)dsdt\right|\le C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\|h_2\|_{***}.\tag{4.61}$$

Then we derive that

$$\|(1+r)^{\frac{3}{2}}\partial_y\phi(x,y)\|_{L^\infty(\mathbb{R}^2)}\le C(\varepsilon^{-\frac{1}{2}}\|h_1\|_{**}+\varepsilon^{-\frac{3}{2}}\|h_2\|_{***}).\tag{4.62}$$

As (4.42), we could replace (4.61) by the following one

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\partial_y G(x-s,y-t)\partial_y h_2(s,t)dsdt\right|\\&=\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x^{-1}\partial_y h_2(s,t)dsdt\right|\\&\le\frac{C}{(1+r)^{2-\delta}}\|h_2\|_{**}\int_{B_{r/2}(x,y)}|\partial_x\partial_y G(x-s,y-t)|dsdt\\&\quad+\frac{C}{\varepsilon^{\frac{1}{2}}r^{3/2}}\|h_2\|_{**}\int_{B_{r/2}(0)}\frac{1}{(1+\sqrt{s^2+t^2})^{2-\delta}}dsdt\\&\quad+C\varepsilon^{-\frac{1}{2}}\|h_2\|_{**}\int_{\mathbb{R}^2\setminus(B_{r/2}(x,y)\cup B_{r/2}(0))}\frac{\left(\sqrt{(x-s)^2+(y-s)^2}\right)^{-\frac{3}{2}}}{\left(1+\sqrt{s^2+t^2}\right)^{2-\delta}}dsdt\\&\le C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}+\delta}\|h_2\|_{**}.\end{aligned}\tag{4.63}$$

Then combined with (4.60) and (4.61) we could get a estimation on $\partial_y\phi(x,y)$ with a weaker decay and better coefficient

$$\|(1+r)^{\frac{3}{2}-\delta}\partial_y\phi(x,y)\|_{L^\infty(\mathbb{R}^2)}\le C\varepsilon^{-\frac{1}{2}}(\|h_1\|_{**}+\|h_2\|_{***}).\tag{4.64}$$

For $\partial_y^2\phi(x,y)$, by Green representation formula we have

$$\begin{aligned}\partial_y^2\phi(x,y) &= 6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_x(\partial_x q\partial_x\phi)(s,t)dsdt \\ &\quad+\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_x h_1(s,t)dsdt \\ &\quad+\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_y h_2(s,t)dsdt.\end{aligned} \tag{4.65}$$

Using Theorem 3.3, (4.41) and (4.47), as (4.59)-(4.61) we get

$$\left|\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_x(\partial_x q\partial_x\phi)(s,t)dsdt\right| \le C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}(\|h_1\|_{**}+\|h_2\|_{***}), \tag{4.66}$$

$$\left|\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_x h_1(s,t)dsdt\right| \le C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\|h_1\|_{**}, \tag{4.67}$$

and

$$\left|\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_y h_2(s,t)dsdt\right| \le C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\|h_2\|_{***}. \tag{4.68}$$

Combining (4.66)-(4.68) we get

$$\|(1+r)^{\frac{3}{2}}\partial_y^2\phi(x,y)\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{-\frac{3}{2}}(\|h_1\|_{**}+\|h_2\|_{***}). \tag{4.69}$$

Concerning $\partial_y^3\phi(x,y)$, we have

$$\begin{aligned}\partial_y^3\phi(x,y) &= 6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_y^3 G(x-s,y-t)\partial_x(\partial_x q\partial_x\phi)(s,t)dsdt \\ &\quad+\int_{\mathbb{R}^2}\partial_y^3 G(x-s,y-t)\partial_x h_1(s,t)dsdt \\ &\quad+\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_y h_2(s,t)dsdt.\end{aligned} \tag{4.70}$$

We could use Theorem 3.3, and follow the same argument as (4.66)-(4.69) to obtain that

$$\|(1+r)^{\frac{3}{2}}\partial_y^3\phi(x,y)\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{-\frac{7}{2}}(\|h_1\|_{**}+\|h_2\|_{***}). \tag{4.71}$$

Step 3. Now we consider $\partial_x\partial_y\phi$, $\partial_x^2\partial_y\phi$, $\partial_x\partial_y^2\phi$ and $\partial_x^2\partial_y^2\phi$. By representation formula we have

$$\begin{aligned}\partial_x\partial_y\phi(x,y) &= 6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x(\partial_x q\partial_x\phi)(s,t)dsdt \\ &\quad+\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x h_1(s,t)dsdt \\ &\quad+\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_y h_2(s,t)dsdt.\end{aligned} \tag{4.72}$$

For the first term on the right hand side of (4.72), by Theorem 3.3, (4.43) and (4.47), we get

$$\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x(\partial_x q\partial_x\phi)(s,t)dsdt\right| \le \frac{C\varepsilon^{-\frac{1}{2}}}{r^{\frac{3}{2}}}(\|h_1\|_{**}+\|h_2\|_{***}). \tag{4.73}$$

For the other two terms on the right hand side of (4.72), we have

$$\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x h_1(s,t)dsdt\right| \le C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_1\|_{**}, \tag{4.74}$$

and

$$\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_y h_2(s,t)dsdt\right| \leq C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_2\|_{***}. \tag{4.75}$$

The above three inequalities (4.78), (4.79) and (4.80) give that

$$\|(1+r)^{\frac{3}{2}}\partial_x\partial_y\phi(x,y)\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-\frac{1}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \tag{4.76}$$

For the term $\partial_x^2\partial_y\phi$, we have

$$\begin{aligned}\partial_x^2\partial_y\phi(x,y) = & -6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x^2(\partial_x q\partial_x\phi)(s,t)dsdt\\ & -\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x^2 h_1(s,t)dsdt\\ & -\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x\partial_y h_2(s,t)dsdt.\end{aligned} \tag{4.77}$$

For the first term on the right hand side of (4.77), by Theorem 3.3, (4.41), (4.47) and (4.52), following almost the same arguments in (4.54) we get

$$\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x^2(\partial_x q\partial_x\phi)(s,t)dsdt\right| \leq \frac{C\varepsilon^{-\frac{1}{2}}}{r^{\frac{3}{2}}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right), \tag{4.78}$$

For the other two terms on the right hand side of (4.77), as (4.60)-(4.61) we have

$$\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x^2 h_1(s,t)dsdt\right| \leq C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_1\|_{**}, \tag{4.79}$$

and

$$\left|\int_{\mathbb{R}^2}\partial_x\partial_y G(x-s,y-t)\partial_x\partial_y h_2(s,t)dsdt\right| \leq C\varepsilon^{-\frac{1}{2}}r^{-\frac{3}{2}}\|h_2\|_{***}. \tag{4.80}$$

The above three inequalities give that

$$\|(1+r)^{\frac{3}{2}}\partial_x^2\partial_y\phi(x,y)\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-\frac{1}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \tag{4.81}$$

Concerning $\partial_x\partial_y^2\phi$, using integration by parts we have

$$\begin{aligned}\partial_x\partial_y^2\phi = & -6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_x^2(\partial_x q\partial_x\phi)(s,t)dsdt\\ & -\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_x^2 h_1(s,t)dsdt\\ & -\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_x\partial_y h_2(s,t)dsdt.\end{aligned} \tag{4.82}$$

Using (4.41), (4.47), (4.52) and Theorem 3.3, we get

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2} \partial_y^2 G(x-s,y-t)\partial_x^2(\partial_x q \partial_x \phi)(s,t)dsdt\right| \\
\leq\ & \frac{C}{r^{7/2}}\varepsilon^{-\frac{1}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right)\int_{B_{r/2}(x,y)}|\partial_y^2 G(x-s,y-t)|dsdt \\
&+\frac{C}{r^{3/2}}\varepsilon^{-\frac{3}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x q|^2 dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^3 \phi|^2 dsdt\right)^{\frac{1}{2}} \\
&+\frac{C}{r^{3/2}}\varepsilon^{-\frac{3}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^2 q|^2 dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^2 \phi|^2 dsdt\right)^{\frac{1}{2}} \\
&+\frac{C}{r^{3/2}}\varepsilon^{-\frac{3}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x^3 q|^2 dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x \phi|^2 dsdt\right)^{\frac{1}{2}} \\
\leq\ & C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right),
\end{aligned} \tag{4.83}
$$

and as (4.60)-(4.61) we get that

$$
\left|\int_{\mathbb{R}^2} \partial_y^2 G(x-s,y-t)\partial_x^2 h_1(s,t)dsdt\right| \leq C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\|h_1\|_{**}, \tag{4.84}
$$

$$
\left|\int_{\mathbb{R}^2} \partial_y^2 G(x-s,y-t)\partial_x\partial_y h_2(s,t)dsdt\right| \leq C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\|h_2\|_{***}. \tag{4.85}
$$

Combining (4.83), (4.84) and (4.85) we have

$$
\|(1+r)^{\frac{3}{2}}\partial_x\partial_y^2\phi(x,y)\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \tag{4.86}
$$

Following almost the same arguments from (4.82) to (4.85) we have

$$
\|(1+r)^{\frac{3}{2}}\partial_x^2\partial_y^2\phi(x,y)\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-\frac{5}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \tag{4.87}
$$

Step 4. In this step we estimate $\partial_x^{-1}\partial_y^2\phi$ and $\partial_x^{-1}\partial_y^3\phi$, for the former one, using Green's representation formula, we have

$$
\begin{aligned}
\partial_x^{-1}\partial_y^2\phi =\ & -6(\sqrt{2}-\varepsilon^2)\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)(\partial_x q\partial_x\phi)(s,t)dsdt \\
&-\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)h_1(s,t)dsdt \\
&-\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)\partial_x^{-1}\partial_y h_2(s,t)dsdt.
\end{aligned} \tag{4.88}
$$

For the first one on the right hand side of (4.88), we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\partial_y^2 G(x-s,y-t)(\partial_x q\partial_x\phi)(s,t)dsdt\right| \\
\leq\ & Cr^{-7/2+\delta}\left(\|h_1\|_{**}+\|h_2\|_{***}\right)\int_{B_{r/2}(x,y)}|\partial_y^2 G(x-s,y-t)|dsdt \\
&+C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x q|^2 dsdt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|\partial_x\phi|^2 dsdt\right)^{\frac{1}{2}} \\
\leq\ & C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right).
\end{aligned} \tag{4.89}
$$

For the second term on the right hand side of (4.88) we get that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2} \partial_y^2 G(x-s,y-t)h_1(s,t)dsdt\right| \\
&\leq Cr^{-\frac{5}{2}+\delta}\|h_1\|_{**}\int_{B_{r/2}(x,y)}|\partial_y^2 G(x-s,y-t)|dsdt \\
&\quad + C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\int_{\mathbb{R}^2\setminus B_{r/2}(x,y)}|h_1(s,t)|dsdt \\
&\leq C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\|h_1\|_{**},
\end{aligned}
\tag{4.90}
$$

while for the third term on the right hand side of (4.88), by (4.24) we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2} \partial_y^2 G(x-s,y-t)\partial_x^{-1}\partial_y h_2(s,t)dsdt\right| \\
&\leq Cr^{-2+\delta}\|h_2\|_{***}\int_{B_{r/2}(x,y)}|\partial_y^2 G(x-s,y-t)|dsdt \\
&\quad + C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}}\|h_2\|_{***}\int_{B_{r/2}(0)}\frac{1}{(1+\sqrt{s^2+t^2})^{2-\delta}}dsdt \\
&\quad + C\varepsilon^{-\frac{3}{2}}\|h_2\|_{***}\int_{\mathbb{R}^2\setminus(B_{r/2}(x,y)\cup B_{r/2}(0))}\frac{C(1+\sqrt{s^2+t^2})^{-2+\delta}}{(\sqrt{(x-s)^2+(y-t)^2})^{3/2}}dsdt \\
&\leq C\varepsilon^{-\frac{3}{2}}r^{-\frac{3}{2}+\delta}\|h_2\|_{**}.
\end{aligned}
\tag{4.91}
$$

Then from (4.89)-(4.91) we get that

$$
\|(1+r)^{\frac{3}{2}-\delta}\partial_x^{-1}\partial_y^2\phi\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \tag{4.92}
$$

Similarly, we could derive that

$$
\|(1+r)^{\frac{3}{2}-\delta}\partial_x^{-1}\partial_y^3\phi\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-\frac{7}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \tag{4.93}
$$

Step 5. In the final step we estimate the terms $\partial_y^4\phi$ and $\partial_x^{-1}\partial_y^4\phi$. For the former one, using (4.47), (4.57), (4.69), (4.87) and equation (4.26), we get

$$
\varepsilon^4\|(1+r)^{\frac{3}{2}}\partial_y^4\phi\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \tag{4.94}
$$

Concerning the last term $\partial_x^{-1}\partial_y^4\phi$, we take ∂_x^{-1} on both sides of (4.26), and we get that

$$
\varepsilon^4\|(1+r)^{\frac{3}{2}-\delta}\partial_x^{-1}\partial_y^4\phi\|_{L^\infty(\mathbb{R}^2)} \leq C\varepsilon^{-\frac{3}{2}}\left(\|h_1\|_{**}+\|h_2\|_{***}\right). \tag{4.95}
$$

Then we finish the whole proof. □

5. The nonlinear system and proof of Theorem 1.1

In section 2, we have seen the leading terms of both equations (2.2) and (2.3) can be reduced to the following single equation

$$
f_1 = \frac{\sqrt{2}}{2}\partial_x g_1 - \frac{g_1^2}{2}. \tag{5.1}
$$

Once g_1 is given, then we are able to solve $O(1)$ terms in both (2.2) and (2.3) by finding f_1 through (5.1). To derive a solution to the original GP equation (1.2), we need to solve the equations (2.30) and (2.12):

$$\begin{aligned}&\partial_x^4 g_1-(2\sqrt{2}-\varepsilon^2)\partial_x^2 g_1-3(\sqrt{2}-\varepsilon^2)\partial_x((\partial_x g_1)^2)-2\partial_y^2 g_1+2\varepsilon^2\partial_x^2\partial_y^2 g_1+\varepsilon^4\partial_y^4 g_1\\&=P_1+P_2+P_3,\end{aligned}\tag{5.2}$$

and

$$(\sqrt{2}-\varepsilon^2)\partial_x f_2+2g_1 f_2=\partial_y^2 g_1+\partial_x f_1-(f_1+\varepsilon^2 f_2)^2 g_1,\tag{5.3}$$

where $P_1,\ P_2$ and P_3 are given in (2.27)-(2.29).

We look for solutions of (5.2) such that $g_1=q+\phi$, with q defined in (2.32), then equation (5.2) could be written as

$$\begin{aligned}&\partial_x^4\phi-(2\sqrt{2}-\varepsilon^2)\partial_x^2\phi-6(\sqrt{2}-\varepsilon^2)\partial_x(q\phi)-2\partial_y^2\phi+2\varepsilon^2\partial_x^2\partial_y^2\phi+\varepsilon^4\partial_y^4\phi\\&=-\partial_x^4 q+(2\sqrt{2}-\varepsilon^2)\partial_x^2 q+3(\sqrt{2}-\varepsilon^2)\partial_x((\partial_x q)^2)+2\partial_y^2 q-2\varepsilon^2\partial_x^2\partial_y^2 q-\varepsilon^4\partial_y^4 q\\&\quad+3(\sqrt{2}-\varepsilon^2)\partial_x((\partial_x\phi)^2)+P_1+P_2+P_3.\end{aligned}\tag{5.4}$$

We set

$$\Gamma_q=-\partial_x^4 q+(2\sqrt{2}-\varepsilon^2)\partial_x^2 q+3(\sqrt{2}-\varepsilon^2)\partial_x((\partial_x q)^2)+2\partial_y^2 q-2\varepsilon^2\partial_x^2\partial_y^2 q-\varepsilon^4\partial_y^4 q.\tag{5.5}$$

It is not difficult to see that $\Gamma_q\in\mathcal{H}_{ox}$ and

$$|\Gamma_q|+|\partial_x\Gamma_q|\le C\varepsilon^2(1+r)^{-5}.\tag{5.6}$$

As a consequence,

$$\partial_x^{-1}\Gamma_q=-\int_x^\infty\Gamma_q(s,y)ds\quad\text{is well defined and we have }\|\partial_x^{-1}\Gamma_q\|_{**}\le C\varepsilon^2.\tag{5.7}$$

The estimation (5.7) suggests that we look for solutions of (5.4) in the following space

$$\mathcal{F}_\phi=\{\phi\in\mathcal{F}_1\mid\|\phi\|_*\le C\varepsilon^2\},\tag{5.8}$$

with C sufficiently large. On the other hand, it is not difficult to check that $\|(\partial_x\phi)^2\|_{**}\le C\varepsilon^2$. Therefore we can absorb Γ_q and $3(\sqrt{2}-\varepsilon^2)\partial_x((\partial_x\phi)^2)$ into P_1.

In the following, we shall analyze (5.3). For a given $g_1=q+\phi$ with $\phi\in\mathcal{F}_\phi$, we solve f_2 of (5.3) in the following space

$$\mathcal{F}_{f_2}:=\{f\mid f\in\mathcal{H}_e,\ \|f\|_{****}<C\},\tag{5.9}$$

where

$$\begin{aligned}\|f\|_{****}=&\|(1+r)^{\frac{3}{2}-\delta}f\|_{L^\infty(\mathbb{R}^2)}+\|(1+r)^{\frac{3}{2}-\delta}\partial_x f\|_{L^\infty(\mathbb{R}^2)}\\&+\|(1+r)^{\frac{3}{2}-\delta}\partial_x^2 f\|_{L^\infty(\mathbb{R}^2)}+\varepsilon\|(1+r)^{\frac{3}{2}-\delta}\partial_x^3 f\|_{L^\infty(\mathbb{R}^2)}\\&+\varepsilon^2\|(1+r)^{\frac{3}{2}-\delta}\partial_y f\|_{L^\infty(\mathbb{R}^2)}+\varepsilon^2\|(1+r)^{\frac{3}{2}-\delta}\partial_x\partial_y f\|_{L^\infty(\mathbb{R}^2)}\\&+\varepsilon^4\|(1+r)^{\frac{3}{2}-\delta}\partial_y^2 f\|_{L^\infty(\mathbb{R}^2)}+\varepsilon^4\|(1+r)^{\frac{3}{2}-\delta}\partial_x\partial_y^2 f\|_{L^\infty(\mathbb{R}^2)}.\end{aligned}\tag{5.10}$$

Lemma 5.1. *Let $\phi \in \mathcal{F}_\phi$. Then for sufficiently small ε there exists a solution $f_{2,\phi} \in \mathcal{F}_{f_2}$ of* (5.3). *In addition, for any two functions ϕ_1, $\phi_2 \in \mathcal{F}_\phi$, we have the corresponding solutions f_{2,ϕ_1} and f_{2,ϕ_2} satisfying*

$$\|f_{2,\phi_1} - f_{2,\phi_2}\|_{****} \le C\varepsilon^{-\frac{3}{2}} \|\phi_1 - \phi_2\|_*. \tag{5.11}$$

Proof. We recall that f_1 is determined by g_1 by (5.1). For each given g_1, we shall solve (5.3) by a perturbation argument. To this aim, for each fixed $y \in \mathbb{R}$, let us consider the non-homogeneous equation

$$\left(\sqrt{2} - \varepsilon^2\right) \partial_x F_{\mathfrak{h}} + 2qF_{\mathfrak{h}} = \mathfrak{h}. \tag{5.12}$$

It is known the homogeneous equation

$$\left(\sqrt{2} - \varepsilon^2\right) \partial_x F_0 + 2qF_0 = 0$$

has a solution of the form

$$F_0\left(x, y\right) = \exp\left(-\int_{-\infty}^{x} \frac{2q\left(s, y\right)}{\sqrt{2} - \varepsilon^2} ds\right).$$

Using the expression

$$q(x, y) = -\left(\frac{2\sqrt{2}}{2\sqrt{2} - \varepsilon^2}\right)^2 \frac{\sqrt{8 - 2\sqrt{2}\varepsilon^2}x}{\frac{2\sqrt{2}-\varepsilon^2}{2\sqrt{2}}x^2 + \frac{(2\sqrt{2}-\varepsilon^2)^2}{4\sqrt{2}}y^2 + \frac{3}{2\sqrt{2}}},$$

we derive that

$$\int_{-\infty}^{x} q\left(s, y\right) ds \quad \text{is close to} \quad -D_\varepsilon \log\left(\frac{2\sqrt{2} - \varepsilon^2}{2\sqrt{2}}x^2 + \frac{(2\sqrt{2} - \varepsilon^2)^2}{4\sqrt{2}}y^2 + \frac{3}{2\sqrt{2}}\right),$$

where

$$D_\varepsilon = \left(\frac{2\sqrt{2}}{2\sqrt{2} - \varepsilon^2}\right)^3 \sqrt{2 - \frac{\sqrt{2}}{2}\varepsilon^2}.$$

As a consequence,

$$F_0 \quad \text{is close to} \quad \left(x^2 + \sqrt{2}y^2 + \frac{3}{2\sqrt{2}}\right)^2. \tag{5.13}$$

In particular, F_0 is not decaying at infinity and even in x. Applying the variation of parameter formula, we can write the solution $F_{\mathfrak{h}}$ of (5.12) as the following form

$$F_{\mathfrak{h}} = F_0\left(x, y\right) \int_{-\infty}^{x} \frac{\mathfrak{h}\left(s, y\right)}{F_0\left(s, y\right)} ds,$$

where the integral makes sense due to the boundedness of $\mathfrak{h}$ and the decay property of F_0^{-1}, see (5.13).

For a given $\phi \in \mathcal{F}_\phi$ and $f_2 \in \mathcal{F}_{f_2}$, we write equation (5.3) as

$$\begin{aligned}(\sqrt{2} - \varepsilon^2)\partial_x f_2 + 2qf_2 = &-2\phi f_2 + \partial_y^2(q + \phi) + \partial_x\left(\frac{\sqrt{2}}{2}\partial_x(q + \phi) - \frac{1}{2}(q + \phi)^2\right) \\ &- \left[\left(\frac{\sqrt{2}}{2}\partial_x(q + \phi) - \frac{1}{2}(q + \phi)^2\right) + \varepsilon^2 f_2\right]^2 (q + \phi).\end{aligned} \tag{5.14}$$

Denoting the right hand side of (5.14) by E_ϕ and we define the map $f_2 \to F_{E_\phi}$ by $\mathcal{M}(f_2)$.

We shall first derive a solution to (5.14) in the following space

$$\widetilde{\mathcal{F}}_{f_2} := \{ f \mid f \in \mathcal{H}_e, \ \|f\|_{*****} < C \}, \tag{5.15}$$

where

$$\begin{aligned} \|f\|_{*****} = & \|(1+r)^{\frac{3}{2}-\delta} f\|_{L^\infty(\mathbb{R}^2)} + \|(1+r)^{\frac{3}{2}-\delta}\partial_x f\|_{L^\infty(\mathbb{R}^2)} \\ & + \varepsilon^2 \|(1+r)^{\frac{3}{2}-\delta}\partial_y f\|_{L^\infty(\mathbb{R}^2)} + \varepsilon^4 \|(1+r)^{\frac{3}{2}-\delta}\partial_y^2 f\|_{L^\infty(\mathbb{R}^2)}. \end{aligned} \tag{5.16}$$

We claim that

$$\mathcal{M}(f_2) \in \widetilde{\mathcal{F}}_{f_2}. \tag{5.17}$$

Let us analyze the terms defined in $\|\cdot\|_{*****}$ term by term. At first, we write

$$\begin{aligned} \mathcal{M}(f_2) = & - F_0(x,y) \int_x^\infty \frac{E_\phi(s,y)}{F_0(s,y)} ds \\ = & \, \partial_x^{-1} E_\phi(x,y) - F_0(x,y) \int_x^\infty \frac{\partial_x^{-1} E_\phi(s,y)}{F_0^2(s,y)} \partial_x F_0(s,y) ds. \end{aligned} \tag{5.18}$$

For $f_2 \in \mathcal{F}_{f_2}$. By the definition of E_ϕ, $\|\phi\|_* \le C\varepsilon^2$ and $\|f_2\|_{*****} \le C$, it is not difficult to check that

$$\begin{aligned} |\partial_x^{-1} E_\phi(x,y)| \le & \, C(1+r)^{-\frac{3}{2}+\delta} \left(1 + \log\frac{1}{\varepsilon} \|f\|_{*****} \|\phi\|_* + \varepsilon^{-\frac{3}{2}} \|\phi\|_* \right) \\ \le & \, C(1+r)^{-\frac{3}{2}+\delta} (1 + \varepsilon^{-\frac{3}{2}} \|\phi\|_*). \end{aligned} \tag{5.19}$$

Together with (5.18) we get that

$$\|(1+r)^{\frac{3}{2}-\delta} \mathcal{M}(f_2)\|_{L^\infty(\mathbb{R}^2)} \le C(1 + \varepsilon^{-\frac{3}{2}} \|\phi\|_*). \tag{5.20}$$

Concerning the term $\partial_x f_2$, by (5.18) we have

$$\partial_x \mathcal{M}(f_2) = -\partial_x F_0(x,y) \int_x^\infty \frac{E_\phi(s,y)}{F_0(s,y)} ds + E_\phi(x,y). \tag{5.21}$$

For the right hand side of (5.14) we have

$$|E_\phi(x,y)| \le C(1+r)^{-\frac{3}{2}} \left(1 + \|f_2\|_{*****} \|\phi\|_* + \varepsilon^{-\frac{3}{2}} \|\phi\|_* \right). \tag{5.22}$$

Then it is not difficult to verify that

$$\|(1+r)^{\frac{3}{2}-\delta} \partial_x \mathcal{M}(f_2)\|_{L^\infty(\mathbb{R}^2)} \le C(1 + \varepsilon^{-\frac{3}{2}} \|\phi\|_*). \tag{5.23}$$

Next, we study the terms $\partial_y \mathcal{M}(f)$ and $\partial_y^2 \mathcal{M}(f)$, by direct computation

$$\begin{aligned} \partial_y \mathcal{M}(f_2) = & - \partial_y F_0(x,y) \int_x^\infty \frac{E_\phi(s,y)}{F_0(s,y)} ds + F_0(x,y) \int_x^\infty \frac{E_\phi(s,y) \partial_y F_0(s,y)}{F_0^2(s,y)} ds \\ & - F_0(x,y) \int_x^\infty \frac{\partial_y E_\phi(s,y)}{F_0(s,y)} ds. \end{aligned} \tag{5.24}$$

Using (5.22) we can estimate the first two terms of (5.24) by

$$\begin{aligned}&\left|\partial_y F_0(x,y)\int_x^\infty \frac{E_\phi(s,y)}{F_0(s,y)}ds\right| + \left|F_0(x,y)\int_x^\infty \frac{E_\phi(s,y)\partial_y F_0(s,y)}{F_0^2(s,y)}ds\right|\\ &\leq C(1+r)^{-\frac{3}{2}}\left(1+\|f_2\|_{*****}\|\phi\|_* + \varepsilon^{-\frac{3}{2}}\|\phi\|_*\right).\end{aligned} \tag{5.25}$$

While for the last term on the right hand side of (5.24), as (5.18) we write it as

$$\begin{aligned}F_0(x,y)\int_x^\infty \frac{\partial_y E_\phi(s,y)}{F_0(s,y)}ds = {}& F_0(x,y)\int_x^\infty \frac{\partial_x^{-1}\partial_y E_\phi(s,y)}{F_0^2(s,y)}\partial_x F_0(s,y)ds\\ &- \partial_x^{-1}\partial_y E_\phi(x,y).\end{aligned} \tag{5.26}$$

Using the condition $\|\phi\|_* \leq C\varepsilon^2$ and $\|f_2\|_{*****} \leq C$ we get that

$$|\partial_x^{-1}\partial_y E_\phi(x,y)| \leq C(1+r)^{-\frac{3}{2}+\delta}\left(1+\varepsilon^{-2}\log\frac{1}{\varepsilon}\|f\|_{****}\|\phi\|_* + \varepsilon^{-\frac{7}{2}}\|\phi\|_*\right). \tag{5.27}$$

Together with (5.24), (5.25) and (5.26) we have

$$\|(1+r)^{\frac{3}{2}-\delta}\partial_y \mathcal{M}(f_2)\|_{L^\infty(\mathbb{R}^2)} \leq C(1+\varepsilon^{-\frac{7}{2}}\|\phi\|_*). \tag{5.28}$$

Concerning $\partial_y^2\mathcal{M}(f_2)$,

$$\begin{aligned}\partial_y^2\mathcal{M}(f_2) = {}& -\partial_y^2 F_0(x,y)\int_x^\infty \frac{E_\phi(s,y)}{F_0(s,y)}ds + 2\partial_y F_0(x,y)\int_x^\infty \frac{E_\phi(s,y)\partial_y F_0(s,y)}{F_0^2(s,y)}ds\\ &- 2\partial_y F_0(x,y)\int_x^\infty \frac{\partial_y E_\phi(s,y)}{F_0(s,y)}ds + 2F_0(x,y)\int_x^\infty \frac{\partial_y E_\phi(x,y)\partial_y F_0(s,y)}{F_0^2(s,y)}ds\\ &+ F_0(x,y)\int_x^\infty \frac{E_\phi(s,y)(\partial_y^2 F_0(s,y)F_0(s,y) - 2(\partial_y F_0(s,y))^2)}{F_0^3(s,y)}ds\\ &- F_0(x,y)\int_x^\infty \frac{\partial_y^2 E_\phi(s,y)}{F_0(s,y)}ds.\end{aligned} \tag{5.29}$$

For the previous five terms on the right hand side of (5.29), we could use (5.22) and (5.27) to bound them by

$$C(1+r)^{-\frac{3}{2}+\delta}\left(1+\varepsilon^{-2}\log\frac{1}{\varepsilon}\|f\|_{*****}\|\phi\|_* + \varepsilon^{-\frac{7}{2}}\|\phi\|_*\right). \tag{5.30}$$

While for the sixth term, we write it as

$$\begin{aligned}F_0(x,y)\int_x^\infty \frac{\partial_y^2 E_\phi(s,y)}{F_0(s,y)}ds = {}& F_0(x,y)\int_x^\infty \frac{\partial_x^{-1}\partial_y^2 E_\phi(s,y)}{F_0^2(s,y)}\partial_x F_0(s,y)ds\\ &- \partial_x^{-1}\partial_y^2 E_\phi(x,y).\end{aligned} \tag{5.31}$$

Using the condition $\|\phi\|_* \leq C\varepsilon^2$ and $\|f_2\|_{*****} \leq C$,

$$|\partial_x^{-1}\partial_y^2 E_\phi(x,y)| \leq C(1+r)^{-\frac{3}{2}+\delta}\left(1+\varepsilon^{-4}\log\frac{1}{\varepsilon}\|f\|_{*****}\|\phi\|_* + \varepsilon^{-\frac{11}{2}}\|\phi\|_*\right). \tag{5.32}$$

Combined with (5.29)-(5.32) we get that

$$\|(1+r)^{\frac{3}{2}-\delta}\partial_y^2 \mathcal{M}(f_2)\|_{L^\infty(\mathbb{R}^2)} \leq C(1+\varepsilon^{-\frac{11}{2}}\|\phi\|_*). \tag{5.33}$$

Using (5.20), (5.23), (5.28), (5.33) we get

$$\|\mathcal{M}(f_2)\|_{*****} \leq C(1+\varepsilon^{-\frac{3}{2}}\|\phi\|_*). \tag{5.34}$$

It proves (5.17).

Next we shall show that $\mathcal{M}$ is a contraction map in $\widetilde{\mathcal{F}}_{f_2}$. For any $f_{2,1}, f_{2,2} \in \widetilde{\mathcal{F}}_{f_2}$, we have

$$\mathcal{M}(f_{2,1}) - \mathcal{M}(f_{2,2}) = -F_0(x,y)\int_x^\infty \frac{H(f_{2,1}, f_{2,2})}{F_0(s,y)}ds,$$

where

$$H(f_{2,1}, f_{2,2}) = -2\phi(f_{2,1}-f_{2,2}) - 2\varepsilon^2 f_1 g_1(f_{2,1}-f_{2,2}) - \varepsilon^4 g_1(f_{2,1}-f_{2,2})(f_{2,1}+f_{2,2}).$$

By the condition $\|\phi\|_* \leq C\varepsilon^2$, we could show that

$$\begin{aligned}
\|(1+r)^{\frac{3}{2}-\delta}\partial_x^{-1}H(f_{2,1}, f_{2,2})\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^{\frac{3}{2}}\|f_{2,1}-f_{2,2}\|_{*****}, \\
\|(1+r)^{\frac{3}{2}-\delta}H(f_{2,1}, f_{2,2})\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^{\frac{3}{2}}\|f_{2,1}-f_{2,2}\|_{*****}, \\
\|(1+r)^{\frac{3}{2}-\delta}\partial_y H(f_{2,1}, f_{2,2})\|_{L^\infty(\mathbb{R}^2)} &\leq C\|f_{2,1}-f_{2,2}\|_{*****}, \\
\|(1+r)^{\frac{3}{2}-\delta}\partial_y^2 H(f_{2,1}, f_{2,2})\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^{-2}\|f_{2,1}-f_{2,2}\|_{*****}.
\end{aligned} \tag{5.35}$$

From the above inequality (5.35) one can prove that

$$\|\mathcal{M}(f_{2,1}) - \mathcal{M}(f_{2,2})\|_{*****} \leq C\varepsilon^{\frac{3}{2}}\|f_{2,1}-f_{2,2}\|_{*****}. \tag{5.36}$$

Then using the contraction mapping principle we deduce the existence of solution $f_2 \in \widetilde{\mathcal{F}}_{f_2}$.

In order to show that $\mathcal{M}(f_2) \in \mathcal{F}_{f_2}$, we have to obtain the corresponding estimation on the following terms $\partial_x^2\mathcal{M}(f_2)$, $\partial_x^3\mathcal{M}(f_2)$, $\partial_x\partial_y\mathcal{M}(f_2)$ and $\partial_x\partial_y^2\mathcal{M}(f_2)$. Since the argument is almost similar, we shall use $\partial_x\partial_y\mathcal{M}(f_2)$ as an example to explain how to derive the estimation. We have already proved the existence of a solution to (5.14) in $\widetilde{\mathcal{F}}_{f_2}$, we denote the solution by f_2 and it satisfies (5.14). We differentiate (5.14) with respect to y, we get that

$$\begin{aligned}
(\sqrt{2}-\varepsilon^2)\partial_x\partial_y f_2 = &-2q\partial_y f_2 - 2\partial_y q f_2 - 2\partial_y\phi f_2 - 2\phi\partial_y f_2 + \partial_y^3(q+\phi) \\
&+ \partial_x\partial_y\left(\frac{\sqrt{2}}{2}\partial_x(q+\phi) - \frac{1}{2}(q+\phi)^2\right) \\
&- \partial_y\left(\left[\left(\frac{\sqrt{2}}{2}\partial_x(q+\phi) - \frac{1}{2}(q+\phi)^2\right) + \varepsilon^2 f_2\right]^2(q+\phi)\right).
\end{aligned} \tag{5.37}$$

Using $\phi \in \mathcal{F}_\phi$ and $f_2 \in \widetilde{F}(f_2)$, we see the right hand side of (5.37) is bounded by

$$C(1+r)^{-\frac{3}{2}+\delta}\left(1+\varepsilon^{-2}\|f_2\|_{*****} + \varepsilon^{-2}\|\phi\|_*\|f_2\|_{*****} + \varepsilon^{-\frac{7}{2}}\|\phi\|_*\right). \tag{5.38}$$

As a consequence,

$$\varepsilon^2\|(1+r)^{\frac{3}{2}-\delta}\partial_x\partial_y f_2\|_{L^\infty(\mathbb{R}^2)} \leq C(1+\|f_2\|_{*****} + \varepsilon^{-\frac{3}{2}}\|\phi\|_*). \tag{5.39}$$

Similarly, we can show that

$$\begin{aligned}
\|(1+r)^{\frac{3}{2}-\delta}\partial_x^2 f_2\|_{L^\infty(\mathbb{R}^2)} &\le C(1+\|f_2\|_{*****}+\varepsilon^{-\frac{3}{2}}\|\phi\|_*),\\
\varepsilon\|(1+r)^{\frac{3}{2}-\delta}\partial_x^3 f_2\|_{L^\infty(\mathbb{R}^2)} &\le C(1+\|f_2\|_{*****}+\varepsilon^{-\frac{3}{2}}\|\phi\|_*),\\
\varepsilon^4\|(1+r)^{\frac{3}{2}-\delta}\partial_x\partial_y^2 f_2\|_{L^\infty(\mathbb{R}^2)} &\le C(1+\|f_2\|_{*****}+\varepsilon^{-\frac{3}{2}}\|\phi\|_*).
\end{aligned} \tag{5.40}$$

From (5.34), (5.39) and (5.40) we have seen the solution $f_2 \in \mathcal{F}_{f_2}$.

It remains to prove (5.11), we shall first show that

$$\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****} \le C\varepsilon^{-\frac{3}{2}}\|\phi_1-\phi_2\|_*. \tag{5.41}$$

We notice that the function $f_{2,\phi_1}-f_{2,\phi_2}$ satisfies

$$\begin{aligned}
&(\sqrt{2}-\varepsilon^2)\partial_x(f_{2,\phi_1}-f_{2,\phi_2})+2q(f_{2,\phi_1}-f_{2,\phi_2})\\
&=-2\phi_1(f_{2,\phi_1}-f_{2,\phi_2})-2(\phi_1-\phi_2)f_{2,\phi_2}+\partial_y^2(\phi_1-\phi_2)\\
&\quad+\frac{\sqrt{2}}{2}\partial_x^2(\phi_1-\phi_2)-\partial_x(q\phi_1-q\phi_2)-\frac{1}{2}\partial_x(\phi_1^2-\phi_2^2)\\
&\quad-\left(\frac{\sqrt{2}}{2}\partial_x(q+\phi_1)-\frac{1}{2}(q+\phi_1)^2\right)^2(q+\phi_1)\\
&\quad+\left(\frac{\sqrt{2}}{2}\partial_x(q+\phi_2)-\frac{1}{2}(q+\phi_2)^2\right)^2(q+\phi_2)\\
&\quad-\varepsilon^2(\sqrt{2}\partial_x(q+\phi_1)-(q+\phi_1)^2)(q+\phi_1)(f_{2,\phi_1}-f_{2,\phi_2})\\
&\quad+\varepsilon^2\left(\frac{\sqrt{2}}{2}\partial_x(q+\phi_1)^2-(q+\phi_1)^3-\frac{\sqrt{2}}{2}\partial_x(q+\phi_2)^2+(q+\phi_2)^3\right)f_{2,\phi_2}\\
&\quad-\varepsilon^4(q+\phi_1)(f_{2,\phi_1}^2-f_{2,\phi_2}^2)-\varepsilon^4 f_{2,\phi_2}^2(\phi_1-\phi_2)\\
&=\Gamma_1(f_{2,\phi_1},f_{2,\phi_2})+\Gamma_2(\phi_1,\phi_2),
\end{aligned} \tag{5.42}$$

where

$$\begin{aligned}
\Gamma_1(f_{2,\phi_1},f_{2,\phi_2})=&-2\phi_1(f_{2,\phi_1}-f_{2,\phi_2})-\varepsilon^4(q+\phi_1)(f_{2,\phi_1}^2-f_{2,\phi_2}^2)\\
&-\varepsilon^2(\sqrt{2}\partial_x(q+\phi_1)-(q+\phi_1)^2)(q+\phi_1)(f_{2,\phi_1}-f_{2,\phi_2}),
\end{aligned}$$

and $\Gamma_2(\phi_1,\phi_2)$ collects the left terms in (5.42). As (5.35) we get that

$$\begin{aligned}
\|(1+r)^{\frac{3}{2}-\delta}\partial_x^{-1}\Gamma_1(f_{2,\phi_1},f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\le C\varepsilon^{\frac{3}{2}}\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****},\\
\|(1+r)^{\frac{3}{2}-\delta}\Gamma_1(f_{2,\phi_1},f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\le C\varepsilon^{\frac{3}{2}}\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****},\\
\|(1+r)^{\frac{3}{2}-\delta}\partial_y\Gamma_1(f_{2,\phi_1},f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\le C\log\frac{1}{\varepsilon}\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****},\\
\|(1+r)^{\frac{3}{2}-\delta}\partial_y^2\Gamma_1(f_{2,\phi_1},f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\le C\varepsilon^{-2}\log\frac{1}{\varepsilon}\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****},
\end{aligned} \tag{5.43}$$

and

$$\begin{aligned}\|(1+r)^{\frac{3}{2}-\delta}\partial_x^{-1}\Gamma_2(\phi_1,\phi_2)\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^{-\frac{3}{2}}\|\phi_1-\phi_2\|_*,\\ \|(1+r)^{\frac{3}{2}-\delta}\Gamma_2(\phi_1,\phi_2)\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^{-\frac{3}{2}}\|\phi_1-\phi_2\|_*,\\ \|(1+r)^{\frac{3}{2}-\delta}\partial_y\Gamma_2(\phi_1,\phi_2)\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^{-\frac{7}{2}}\|\phi_1-\phi_2\|_*,\\ \|(1+r)^{\frac{3}{2}-\delta}\partial_y^2\Gamma_2(\phi_1,\phi_2)\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^{-\frac{11}{2}}\|\phi_1-\phi_2\|_*,\end{aligned}\tag{5.44}$$

Using (5.43) and (5.44) we could argue as from (5.18) to (5.34) to conclude that

$$\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****}\leq C\varepsilon^{\frac{3}{2}}\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****}+C\varepsilon^{-\frac{3}{2}}\|\phi_1-\phi_2\|_*,\tag{5.45}$$

and it implies that

$$\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****}\leq C\varepsilon^{-\frac{3}{2}}\|\phi_1-\phi_2\|_*.\tag{5.46}$$

Next we use equation (5.42) and argue as (5.37)-(5.40) to derive that

$$\begin{aligned}\|(1+r)^{\frac{3}{2}-\delta}\partial_x^2(f_{2,\phi_1}-f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\leq C(\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****}+\varepsilon^{-\frac{3}{2}}\|\phi_1-\phi_2\|_*),\\ \varepsilon\|(1+r)^{\frac{3}{2}-\delta}\partial_x^3(f_{2,\phi_1}-f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\leq C(\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****}+\varepsilon^{-\frac{3}{2}}\|\phi_1-\phi_2\|_*),\\ \varepsilon^2\|(1+r)^{\frac{3}{2}-\delta}\partial_y\partial_x(f_{2,\phi_1}-f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\leq C(\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****}+\varepsilon^{-\frac{3}{2}}\|\phi_1-\phi_2\|_*),\\ \varepsilon^4\|(1+r)^{\frac{3}{2}-\delta}\partial_y^2\partial_x(f_{2,\phi_1}-f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\leq C(\|f_{2,\phi_1}-f_{2,\phi_2}\|_{*****}+\varepsilon^{-\frac{3}{2}}\|\phi_1-\phi_2\|_*).\end{aligned}\tag{5.47}$$

Together with (5.46) we get (5.11) holds. □

Now we are able to give the proof to Theorem 1.1.

Proof of Theorem 1.1. With Lemma 5.1 we have already seen that for a given function ϕ, we could find a solution to (5.3). Therefore, the original problem is equivalent to finding a solution to (5.4). We write (5.4) as

$$\begin{aligned}&\partial_x^4\phi-(2\sqrt{2}-\varepsilon^2)\partial_x^2\phi-6(\sqrt{2}-\varepsilon^2)\partial_x(q\phi)-2\partial_y^2\phi+2\varepsilon^2\partial_x^2\partial_y^2\phi+\varepsilon^4\partial_y^4\phi\\ &=\hat{P}_1+P_2+P_3,\end{aligned}\tag{5.48}$$

where

$$\hat{P}_1=P_1+\Gamma_q+3(\sqrt{2}-\varepsilon^2)\partial_x((\partial_x\phi)^2).$$

We shall find a solution of (5.48) in (5.8) via fixed point argument.

For any $\phi\in\mathcal{F}_\phi$, we first claim that

$$\|\hat{P}_1\|_{**}\leq C\varepsilon^2.\tag{5.49}$$

To prove the above claim, we have to check all the terms in $\hat{P}_1$ are bounded by $C\varepsilon^2$ in the norm of $\|\cdot\|_{**}$. For example, we use $\partial_x(\partial_x\phi)^2$ and $\varepsilon^2\partial_x(\partial_y g_1)^2$ as examples to explain, while the other terms can be handled similarly, we leave the details to the interested reader. For $\partial_x(\partial_x\phi)^2$,

$$\begin{aligned}\|(1+r)^{\frac{5}{2}-\delta}(\partial_x\phi)^2\|_{L^\infty(\mathbb{R}^2)} &\leq C\|(1+r)^{\frac{3}{2}-\delta}\partial_x\phi\|^2_{L^\infty(\mathbb{R}^2)}\leq C\|\phi\|_*^2,\\ \|(1+r)^{\frac{5}{2}-\delta}\partial_x(\partial_x\phi)^2\|_{L^\infty(\mathbb{R}^2)} &\leq C\|\phi\|_*\|(1+r)^{\frac{3}{2}}\partial_x^2\phi\|_{L^\infty(\mathbb{R}^2)}\leq C\|\phi\|_*^2,\\ \|(1+r)^{\frac{5}{2}-\delta}\partial_x^2(\partial_x\phi)^2\|_{L^\infty(\mathbb{R}^2)} &\leq C\|\phi\|_*\|(1+r)^{\frac{3}{2}}\partial_x^3\phi\|_{L^\infty(\mathbb{R}^2)}+C\|\phi\|_*^2\\ &\leq C\varepsilon^{-\frac{1}{2}}\|\phi\|_*^2.\end{aligned}\tag{5.50}$$

Thus, we have

$$\|(\partial_x\phi)^2\|_{**} \leq C\varepsilon^2. \tag{5.51}$$

While for the later one, we can check that

$$\begin{aligned}
\varepsilon^2\|(1+r)^{\frac{5}{2}-\delta}(\partial_y g_1)^2)\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^2 + C\varepsilon^{\frac{3}{2}}\|\phi\|_* + C\varepsilon\|\phi\|_*^2,\\
\varepsilon^2\|(1+r)^{\frac{5}{2}-\delta}\partial_x(\partial_y g_1)^2)\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^2 + C\varepsilon^{\frac{3}{2}}\|\phi\|_* + C\varepsilon\|\phi\|_*^2,\\
\varepsilon^2\|(1+r)^{\frac{5}{2}-\delta}\partial_x^2(\partial_y g_1)^2\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^2 + C\varepsilon^{\frac{3}{2}}\|\phi\|_* + C\varepsilon\|\phi\|_*^2.
\end{aligned} \tag{5.52}$$

It implies that

$$\|(\partial_y g_1)^2\|_{**} \leq C\varepsilon^2. \tag{5.53}$$

As (5.49), we can prove that

$$\|P_2\|_{***} \leq C\varepsilon^2. \tag{5.54}$$

Concerning P_3, we could show that each term decays faster than r^{-3}, then $\partial_x^{-1}P_3$ is well defined. Subsequently we could absorb P_3 into $\hat{P}_1$ and prove that $\|\partial_x^{-1}P_3\|_{**} \leq C\varepsilon^2$. For example, we shall use the term $g_1\partial_y^2(g_1^2)$ to illustrate this point. It is easy to see that

$$\begin{aligned}
\varepsilon^2\|(1+r)^{\frac{7}{2}-\delta}g_1\partial_y^2(g_1^2)\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^2 + C\varepsilon^{\frac{1}{2}}\|\phi\|_* + C\varepsilon^{-1}\|\phi\|_*^2 + C\varepsilon^{-2}\|\phi\|_*^3,\\
\varepsilon^2\|(1+r)^{\frac{7}{2}-\delta}\partial_x(g_1\partial_y^2(g_1^2))\|_{L^\infty(\mathbb{R}^2)} &\leq C\varepsilon^2 + C\varepsilon^{\frac{1}{2}}\|\phi\|_* + C\varepsilon^{-1}\|\phi\|_*^2 + C\varepsilon^{-2}\|\phi\|_*^3.
\end{aligned} \tag{5.55}$$

Then one can easily verify that

$$\|\partial_x^{-1}(g_1\partial_y^2(g_1^2))\|_{**} \leq C\varepsilon^2. \tag{5.56}$$

Repeating the same process for the other terms in P_3 we derive that

$$\|\partial_x^{-1}(P_3)\|_{**} \leq C\varepsilon^2. \tag{5.57}$$

With (5.49), (5.54) and (5.57) and Proposition 4.2, we find out a solution (denoted by $\mathcal{N}(\phi)$) to (5.48) such that

$$\|\mathcal{N}(\phi)\|_* \leq C(\|\hat{P}_1\|_{**} + \|P_2\|_{***} + \|\partial_x^{-1}P_3\|_{**}) \leq C\varepsilon^2. \tag{5.58}$$

Therefore we could define a map $\phi \to \mathcal{N}(\phi)$ which sends $\mathcal{F}_\phi$ to itself provided C is large enough.

Next, we shall prove the map $\phi \to \mathcal{N}(\phi)$ is a contraction map. We claim that for any two functions $\phi_1, \phi_2 \in \mathcal{F}_\phi$,

$$\begin{aligned}
\|\hat{P}_1(\phi_1) - \hat{P}_1(\phi_2)\|_{**} &\leq C\varepsilon^{\frac{1}{2}}\|\phi_1 - \phi_2\|_*,\\
\|P_2(\phi_1) - P_2(\phi_2)\|_{***} &\leq C\varepsilon^{\frac{1}{2}}\|\phi_1 - \phi_2\|_*,\\
\|\partial_x^{-1}P_3(\phi_1) - \partial_x^{-1}P_3(\phi_2)\|_{**} &\leq C\varepsilon^{\frac{1}{2}}\|\phi_1 - \phi_2\|_*.
\end{aligned} \tag{5.59}$$

We shall pick out one term from $\hat{P}_1$, P_2 and P_3 (see (2.27)-(2.29)) respectively as examples to prove (5.59). To stress the dependence of f_1, f_2, g_1 on ϕ, we replace them by $f_{1,\phi}, f_{2,\phi}, g_{1,\phi}$. For $\hat{P}_1$, we take $\varepsilon^4\partial_x(f_1 f_2)$ into consideration, we have

$$
\begin{aligned}
&\varepsilon^4(f_{1,\phi_1}f_{2,\phi_1} - f_{1,\phi_2}f_{2,\phi_2})\\
&= \varepsilon^4(f_{1,\phi_1} - f_{1,\phi_2})f_{2,\phi_1} + \varepsilon^4 f_{1,\phi_2}(f_{2,\phi_1} - f_{2,\phi_2})\\
&= \varepsilon^4\left(\left(\frac{\sqrt{2}}{2}\partial_x(q+\phi_1) - \frac{1}{2}(q+\phi_1)^2\right) - \left(\frac{\sqrt{2}}{2}\partial_x(q+\phi_2) - \frac{1}{2}(q+\phi_2)^2\right)\right) f_{2,\phi_1}\\
&\quad + \varepsilon^4 f_{1,\phi_2}(f_{2,\phi_1} - f_{2,\phi_2})\\
&\le C\varepsilon^4(1+r)^{-\frac{5}{2}+\delta}\|(1+r)^{\frac{3}{2}-\delta}\partial_x(\phi_1-\phi_2)\|_{L^\infty(\mathbb{R}^2)}\|(1+r)^{\frac{3}{2}-\delta}f_{2,\phi_1}\|_{L^\infty(\mathbb{R}^2)}\\
&\quad + C\varepsilon^4(1+r)^{-\frac{5}{2}+\delta}\|(1+r)^{\frac{3}{2}-\delta}(\phi_1-\phi_2)\|_{L^\infty(\mathbb{R}^2)}\|(1+r)^{\frac{3}{2}-\delta}f_{2,\phi_1}\|_{L^\infty(\mathbb{R}^2)}\\
&\quad + C\varepsilon^4(1+r)^{-\frac{5}{2}+\delta}\|(1+r)^{\frac{3}{2}-\delta}(\phi_1^2-\phi_2^2)\|_{L^\infty(\mathbb{R}^2)}\|(1+r)^{\frac{3}{2}-\delta}f_{2,\phi_1}\|_{L^\infty(\mathbb{R}^2)}\\
&\quad + C\varepsilon^4(1+r)^{-\frac{5}{2}+\delta}(1+\|\phi\|_*)\|(1+r)^{\frac{3}{2}-\delta}(f_{2,\phi_1}-f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)}\\
&\le C(1+r)^{-\frac{5}{2}+\delta}\varepsilon^{\frac{1}{2}}\|\phi_1-\phi_2\|_*,
\end{aligned}
\tag{5.60}
$$

where we used Lemma 5.1. Similarly, one can also show that

$$
\begin{aligned}
&\|(1+r)^{\frac{5}{2}-\delta}\partial_x(f_{1,\phi_1}f_{2,\phi_1} - f_{1,\phi_2}f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{\frac{1}{2}}\|\phi_1-\phi_2\|_*,\\
&\|(1+r)^{\frac{5}{2}-\delta}\partial_x^2(f_{1,\phi_1}f_{2,\phi_1} - f_{1,\phi_2}f_{2,\phi_2})\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{\frac{1}{2}}\|\phi_1-\phi_2\|_*.
\end{aligned}
\tag{5.61}
$$

Using (5.60) and (5.61) we have

$$
\varepsilon^4\|f_{1,\phi_1}f_{2,\phi_1} - f_{1,\phi_2}f_{2,\phi_2}\|_{**} \le C\varepsilon^{\frac{1}{2}}\|\phi_1-\phi_2\|_*. \tag{5.62}
$$

For P_2, we consider $\varepsilon^2\partial_y(\partial_x g_1\partial_y g_1)$, by direct computation, we have

$$
\begin{aligned}
&|\partial_x g_{1,\phi_1}\partial_y g_{1,\phi_1} - \partial_x g_{1,\phi_2}\partial_y g_{1,\phi_2}|\\
&= |\partial_x(q+\phi_1)\partial_y(q+\phi_1) - \partial_x(q+\phi_2)\partial_y(q+\phi_2)|\\
&\le |\partial_x q\partial_y(\phi_1-\phi_2)| + |\partial_y q\partial_x(\phi_1-\phi_2)| + |\partial_x\phi_1\partial_y\phi_1 - \partial_x\phi_2\partial_y\phi_2|\\
&\le C\varepsilon^{-\frac{1}{2}}(1+r)^{-3+\delta}\|\phi_1-\phi_2\|_*(1+\|\phi_1\|_*+\|\phi_2\|_*)\\
&\le C\varepsilon^{-\frac{1}{2}}(1+r)^{-3+\delta}\|\phi_1-\phi_2\|_*.
\end{aligned}
\tag{5.63}
$$

In a similar way as (5.63), one can check that

$$
\begin{aligned}
\varepsilon^2\|(1+r)^{3-\delta}\partial_y(\partial_x g_{1,\phi_1}\partial_y g_{1,\phi_1} - \partial_x g_{1,\phi_2}\partial_y g_{1,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\le C\varepsilon^{\frac{1}{2}}\|\phi_1-\phi_2\|_*,\\
\varepsilon^2\|(1+r)^{3-\delta}\partial_x\partial_y(\partial_x g_{1,\phi_1}\partial_y g_{1,\phi_1} - \partial_x g_{1,\phi_2}\partial_y g_{1,\phi_2})\|_{L^\infty(\mathbb{R}^2)} &\le C\varepsilon^{\frac{1}{2}}\|\phi_1-\phi_2\|_*.
\end{aligned}
\tag{5.64}
$$

As a consequence of (5.63) and (5.64) we get

$$
\varepsilon^2\|\partial_x g_{1,\phi_1}\partial_y g_{1,\phi_1} - \partial_x g_{1,\phi_2}\partial_y g_{1,\phi_2}\|_{***} \le C\varepsilon^{\frac{1}{2}}\|\phi_1-\phi_2\|_*. \tag{5.65}
$$

While for P_3, we study the term $\varepsilon^4\partial_y g_1\partial_y(g_1^2)$,

$$
\begin{aligned}
&\|(1+r)^{\frac{7}{2}-\delta}\left(\partial_y g_{1,\phi_1}\partial_y(g_{1,\phi_1}^2)-\partial_y g_{1,\phi_2}\partial_y(g_{1,\phi_2}^2)\right)\|_{L^\infty(\mathbb{R}^2)}\\
&=\|(1+r)^{\frac{7}{2}-\delta}(\partial_y(q+\phi_1)\partial_y((q+\phi_1)^2)-\partial_y(q+\phi_2)\partial_y((q+\phi_2)^2))\|_{L^\infty(\mathbb{R}^2)}\\
&\leq C\|(1+r)^{\frac{7}{2}-\delta}(\partial_y q)^2(\phi_1-\phi_2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{7}{2}-\delta}q\partial_y q\partial_y(\phi_1-\phi_2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{7}{2}-\delta}\partial_y q(\partial_y\phi_1\phi_1-\partial_y\phi_2\phi_2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{7}{2}-\delta}q((\partial_y\phi_1)^2-(\partial_y\phi_2)^2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{7}{2}-\delta}(\phi_1(\partial_y\phi_1)^2-\phi_2(\partial_y\phi_2)^2)\|_{L^\infty(\mathbb{R}^2)}\\
&\leq C\varepsilon^{-\frac{1}{2}}\|\phi_1-\phi_2\|_*+C\varepsilon^{-1}\|\phi_1-\phi_2\|_*^2+C\varepsilon^{-1}\|\phi_1-\phi_2\|_*^3,
\end{aligned}
\tag{5.66}
$$

and

$$
\begin{aligned}
&\|(1+r)^{\frac{5}{2}-\delta}\partial_x\left(\partial_y g_{1,\phi_1}\partial_y(g_{1,\phi_1}^2)-\partial_y g_{1,\phi_2}\partial_y(g_{1,\phi_2}^2)\right)\|_{L^\infty(\mathbb{R}^2)}\\
&=\|(1+r)^{\frac{5}{2}-\delta}\partial_x(\partial_y(q+\phi_1)\partial_y((q+\phi_1)^2)-\partial_y(q+\phi_2)\partial_y((q+\phi_2)^2))\|_{L^\infty(\mathbb{R}^2)}\\
&\leq C\|(1+r)^{\frac{5}{2}-\delta}\partial_x((\partial_y q)^2)(\phi_1-\phi_2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{5}{2}-\delta}(\partial_y q)^2\partial_x(\phi_1-\phi_2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{5}{2}-\delta}\partial_x(q\partial_y q)\partial_y(\phi_1-\phi_2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{5}{2}-\delta}q\partial_y q\partial_x\partial_y(\phi_1-\phi_2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{5}{2}-\delta}\partial_x(\partial_y q)(\partial_y\phi_1\phi_1-\partial_y\phi_2\phi_2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{5}{2}-\delta}\partial_y q\partial_x(\partial_y\phi_1\phi_1-\partial_y\phi_2\phi_2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{5}{2}-\delta}\partial_x q((\partial_y\phi_1)^2-(\partial_y\phi_2)^2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{5}{2}-\delta}q\partial_x((\partial_y\phi_1)^2-(\partial_y\phi_2)^2)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad+C\|(1+r)^{\frac{5}{2}-\delta}\partial_x(\phi_1(\partial_y\phi_1)^2-\phi_2(\partial_y\phi_2)^2)\|_{L^\infty(\mathbb{R}^2)}\\
&\leq C\varepsilon^{-\frac{1}{2}}\|\phi_1-\phi_2\|_*+C\varepsilon^{-1}\|\phi_1-\phi_2\|_*^2+C\varepsilon^{-1}\|\phi_1-\phi_2\|_*^3.
\end{aligned}
\tag{5.67}
$$

From (5.66) ad (5.67) we derive that

$$
\varepsilon^4\|\partial_x^{-1}\left(\partial_y g_{1,\phi_1}\partial_y((g_{1,\phi_1}^2))-\partial_y g_{1,\phi_2}\partial_y((g_{1,\phi_2}^2))\right)\|_{**}\leq C\varepsilon^{\frac{1}{2}}\|\phi_1-\phi_2\|_*. \tag{5.68}
$$

The other terms in $\hat{P}_1$, P_2 and P_3 can be treated in a similar way and we leave the computations to the interested reader.

Using the claim (5.59) and Proposition 4.2, we derive that

$$
\|\mathcal{N}(\phi_1)-\mathcal{N}(\phi_2)\|_*\leq C\varepsilon^{\frac{1}{2}}\|\phi_1-\phi_2\|_*. \tag{5.69}
$$

Then we deduce the existence of a solution ϕ to (5.48) by contraction principle. Thereby the original existence result is established. □

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