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VANISHING ESTIMATES FOR LIOUVILLE EQUATION WITH QUANTIZED SINGULARITIES

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ABSTRACT. In this article we continue with the research initiated in our previous work on singular Liouville equations with quantized singularity. The main goal of this article is to prove that as long as the bubbling solutions violate the spherical Harnack inequality near a singular source, the first derivatives of coefficient functions must tend to zero.

1. INTRODUCTION

In this article we study bubbling solutions of

$$\Delta u + \mathrm{H}(x)e^{u} = 4\pi\alpha\delta_0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \tag{1.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^2$ that contains the origin, $\alpha > -1$ is a constant and δ_0 is the Dirac mass at 0, H is a positive and smooth function. Since

$$\Delta(\frac{1}{2\pi}\log|x|) = \delta_0,$$

One can use the logarithmic function $2\pi\alpha\log|x|$ to remove the singular source from the equation: let $u_1(x) = u(x) - 2\alpha\log|x|$, then u_1 satisfies

$$\Delta u_1 + |x|^{2\alpha}\mathrm{H}(x)e^{u_1} = 0, \quad \text{in} \quad \Omega. \tag{1.2}$$

If a sequence of solutions $\{u^k\}_{k=1}^{\infty}$ of (1.2) satisfies

$$\lim_{k\to\infty} u^k(x_k) = \infty, \qquad \text{for some } \bar{x} \in B_\tau \text{ and } x_k \to \bar{x}.$$

we say u^k is a sequence of bubbling solutions or blowup solutions, $\bar{x}$ is called a blowup point. For many reasons in applications it is most interesting to consider $\alpha \in \mathbb{N}$ (the set of natural numbers) and when 0 is the only blowup point of u^k. Our set-up of bubbling solutions is as follows: Let u_k be a sequence of solutions of

$$\Delta \mathrm{u}_k(x) + |x|^{2N}\mathrm{H}_k(x)e^{\mathrm{u}_k} = 0, \quad \text{in} \quad B_\tau \tag{1.3}$$

for some $\tau > 0$ independent of k. B_τ is the ball centered at the origin with radius τ. In addition we postulate the usual assumptions on u_k and H_k: For a positive

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constant C independent of k, the following holds:

$$\begin{cases} \|\mathrm{H}_k\|_{C^3(\bar{B}_\tau)} \le C, \quad \frac{1}{C} \le \mathrm{H}_k(x) \le C, \quad x \in \bar{B}_\tau, \\ \int_{B_\tau} \mathrm{H}_k e^{\mathrm{u}_k} \le C, \\ |\mathrm{u}_k(x) - \mathrm{u}_k(y)| \le C, \quad \forall x, y \in \partial B_\tau, \end{cases} \tag{1.4}$$

and since we study the asymptotic behavior of blowup solutions around the singular source, we assume that there is no blowup point except at the origin:

$$\max_{K \subset\subset B_\tau \setminus \{0\}} \mathrm{u}_k \le C(K). \tag{1.5}$$

Also, we use the value of u_k on ∂B_τ to define a harmonic function $\phi_k(x)$:

$$\begin{cases} \Delta \phi_k(x) = 0, \quad \text{in} \quad B_\tau, \\ \phi_k(x) = \mathrm{u}_k(x) - \frac{1}{2\pi\tau} \int_{\partial B_\tau} \mathrm{u}_k dS, \quad x \in \partial B_\tau. \end{cases} \tag{1.6}$$

Clearly the mean value property of harmonic functions implies $\phi_k(0) = 0$ and the finite oscillation of u_k on ∂B_τ means that all derivatives of ϕ_k are uniformly bounded in $B_{\tau/2}$. In this article we consider the case that:

$$\max_{x \in B_\tau} \mathrm{u}_k(x) + 2(1+N) \log |x| \to \infty, \tag{1.7}$$

which is equivalent to saying that the spherical Harnack inequality does not hold for u_k. It is also mentioned in literature (see [14, 19]) that 0 is called an non-simple blowup point. The main result of this article is

Theorem 1.1. *Let $\{\mathrm{u}_k\}$ be a sequence of solutions of (1.3) such that (1.4),(1.5) and (1.7) hold. Then*

$$\nabla(\log \mathrm{H}_k + \phi_k)(0) = o(1), \quad \text{as } k \to \infty,$$

where ϕ_k is defined in (1.6).

When bubbling solutions satisfy (1.7), they are called non-simple blowup solutions (see [14]). Theorem 1.1 is a compliment of Theorem 1.1 of [19], which asserts that under certain conditions (see Theorem A below) $\nabla(\log \mathrm{H}_k + \phi_i^k)(0)$ tends to zero. Theorem 1.1 removes the restrictions in [19]. In other words, the combination of Theorem A and Theorem 1.1 proves that $\nabla(\log \mathrm{H}_k + \phi_k)(0) \to 0$ as long as the non-simple blowup situation occurs. Besides the advancement of analytical understanding, this conclusion is particularly important in application. Theorem 1.1 can be applied to situations beyond single equations. For certain systems of equations such as Toda systems, the bubble accumulations can be described by a sequence of bubbling solutions with quantized singular source. Theorem 1.1 is very useful to rule out complicated bubbling accumulation pictures in Toda systems.

It remains an open question whether or not $\nabla(\log \mathrm{H}_k + \phi_k)(0)$ tends to 0 if the bubbling solutions satisfy the spherical Harnack inequality around the origin. We tend to believe one can construct a sequence of bubbling solutions that satisfy spherical Harnack inequality with nonzero first derivatives of the coefficients functions. In particular the works of Del Pino-Esposito-Musso [8, 9] on two dimensional Euler flows seem to suggest that for simple blowup solutions with quantized singular sources, the first derivatives of the coefficient functions may not tend to zero at singular sources. Instead the $(N+1)$-th derivatives should vanish. Our result, together with [8, 9], demonstrates a striking contrast between simple and non-simple bubblings.

The non-simple bubbling situation and vanishing theorems have profound impact to problems in geometry and physics. For example for the following mean field equation defined on a Riemann surface (M,g):

$$\Delta_g u+\rho\Big(\frac{h(x)e^{u(x)}}{\int_M he^u}-\frac{1}{Vol_g(M)}\Big)=4\pi\sum_j\alpha_j\Big(\delta_{p_j}-\frac{1}{Vol_g(M)}\Big), \tag{1.8}$$

the solution u represents a conformal metric with prescribed conic singularities (see [10, 17, 18]) . In particular if the singular source is quantized, the Liouville equation has close ties with Algebraic geometry, integrable system, number theory and complex Monge-Ampere equations (see [7]). In Physics the understanding of non-simple blowup phenomenon would be extremely useful for the study of mean field limits of point vortices in the Euler flow [4, 5] and models in the Chern-Simons-Higgs theory [13] and in the electroweak theory [1], etc. It is also remarkable that non-simple bubbling solutions also occur in systems. In [12], the non-simple blowup solutions are studied for singular Liouville systems. Finally we remark that when the blowup point is a location of a singular source, whether or not this point has to be a critical point of coefficient functions has intrigued people for years. Our previous result [19] is the first result for singular Liouville equation, the second author proved a surprising vanishing theorem for singular Toda systems in [22].

The organization of this paper is as follows. In section two we review a few fundamental tools for the proof of the main theorem and invoke several key estimates established in our pervious work [19]. Then in section three we use a sequence of global solutions to approximate our blowup solutions. The point-wise estimates proved in this section are more precise than what is established in [19] and are important for our argument. In section four we prove a crucial estimate on the difference between blowup solution and the global solutions as the first term in the approximation. As a consequence of section four, we move to section five to complete the proof of the main theorem. The proof in section five is similar to the proof of uniqueness theorems for bubbling solutions in [20], [2], [15], etc.

Notation: We will use $B(x_0,r)$ to denote a ball centered at x_0 with radius r. If x_0 is the origin we use B_r. C represents a positive constant that may change from place to place.

2. Preliminary discussions

For simple notation we set

$$u_k(x)=\mathrm{u}_k(x)-\phi_k(x),\quad \text{and} \tag{2.1}$$

$$h_k(x)=\mathrm{H}_k(x)e^{\phi_k(x)}. \tag{2.2}$$

to write the equation of u_k as

$$\Delta u_k(x)+|x|^{2N}h_k(x)e^{u_k}=0, \quad \text{in} \quad B_\tau \tag{2.3}$$

Without loss of generality we assume

$$\lim_{k\to\infty} h_k(0)=1. \tag{2.4}$$

Obviously (1.7) is equivalent to

$$\max_{x\in B_\tau} u_k(x)+2(1+N)\log|x|\to\infty, \tag{2.5}$$

It is well known [14, 3] that u_k exhibits a non-simple blowup profile. It is established in [14, 3] that there are $N+1$ local maximum points of u_k: p_0^k,....,p_N^k and they are evenly distributed on $\mathbb{S}^1$ after scaling according to their magnitude: Suppose along a subsequence

$$\lim_{k\to\infty} p_0^k/|p_0^k|=e^{i\theta_0},$$

then

$$\lim_{k\to\infty}\frac{p_l^k}{|p_0^k|}=e^{i(\theta_0+\frac{2\pi l}{N+1})}, \quad l=1,...,N.$$

For many reasons it is convenient to denote $|p_0^k|$ as δ_k and define μ_k as follows:

$$\delta_k=|p_0^k| \quad \text{and} \quad \mu_k=u_k(p_0^k)+2(1+N)\log\delta_k. \tag{2.6}$$

Since p_l^k's are evenly distributed around ∂B_{δ_k}, standard results for Liouville equations around a regular blowup point can be applied to have $u_k(p_l^k)=u_k(p_0^k)+o(1)$. Also, (1.7) gives $\mu_k\to\infty$. The interested readers may look into [14, 3] for more detailed information.

In our previous work [19] we prove the following vanishing type estimates for the first derivatives of the coefficient function $\log h_k$:

Theorem A: Let u_k, ϕ_k, h_k, δ_k, μ_k be defined by (2.3), (1.6), (2.2), (2.6) respectively. Then

$$|\nabla\log h_k(0)|=O(\delta_k)+O(\delta_k^{-1}e^{-\mu_k}\mu_k). \tag{2.7}$$

Here we observe that if $\mu_k e^{-\mu_k}=o(\delta_k)$, we already have $\nabla h_k(0)=o(1)$, which is equivalent to $\nabla(\mathrm{H}_k e^{\phi_k})(0)=o(1)$. Thus throughout the paper we assume

$$\delta_k\le C\mu_k e^{-\mu_k}. \tag{2.8}$$

Finally we shall use E to denote a frequently appearing error term of the size $O(\delta_k^2)+O(\mu_k e^{-\mu_k})$. Because of (2.8),

$$E=O(\mu_k e^{-\mu_k}).$$

3. Approximating bubbling solutions by global solutions

First we recall that $|p_0^k| = \delta_k$, so we write p_0^k as $p_0^k = \delta_k e^{i\theta_k}$ and define v_k as

$$v_k(y) = u_k(\delta_k y e^{i\theta_k}) + 2(N+1)\log\delta_k, \quad |y| < \tau\delta_k^{-1}. \tag{3.1}$$

If we write out each component, (3.1) is

$$v_k(y_1, y_2) = u_k(\delta_k(y_1\cos\theta_k - y_2\sin\theta_k), \delta_k(y_1\sin\theta_k + y_2\cos\theta_k)) + 2(1+N)\log\delta_k.$$

Then it is standard to verify that v_k solves

$$\Delta v_k(y) + |y|^{2N}\mathfrak{h}_k(\delta_k y)e^{v_k(y)} = 0, \quad |y| < \tau/\delta_k, \tag{3.2}$$

where

$$\mathfrak{h}_k(x) = h_k(xe^{i\theta_k}), \quad |x| < \tau. \tag{3.3}$$

Thus the image of p_0^k after scaling is $Q_1^k = e_1 = (1,0)$. Let Q_1^k, Q_2^k,...,Q_N^k be the images of p_i^k $(i = 1,...,N)$ after the scaling:

$$Q_l^k = \frac{p_l^k}{\delta_k}e^{-i\theta_k}, \quad l = 1,...,N.$$

It is established by Kuo-Lin in [14] and independently by Bartolucci-Tarantello in [3] that

$$\lim_{k\to\infty} Q_l^k = \lim_{k\to\infty} p_l^k/\delta_k = e^{\frac{2l\pi i}{N+1}}, \quad l = 0,....,N. \tag{3.4}$$

Then in our previous work [19] we obtained (see (3.13) in [19])

$$Q_l^k - e^{\frac{2\pi li}{N+1}} = E. \tag{3.5}$$

Choosing $3\varepsilon > 0$ small and independent of k, we can make disks centered at Q_l^k with radius 3ε (denoted as $B(Q_l^k, 3\varepsilon)$) mutually disjoint. Let

$$\mu_k = \max_{B(Q_0^k,\varepsilon)} v_k. \tag{3.6}$$

Since Q_l^k are evenly distributed around ∂B_1, it is easy to use standard estimates for single Liouville equations ([21, 11, 6]) to obtain

$$\max_{B(Q_l^k,\varepsilon)} v_k = \mu_k + o(1), \quad l = 1,...,N.$$

Let

$$V_k(x) = \log\frac{e^{\mu_k}}{(1 + \frac{e^{\mu_k}\mathfrak{h}_k(\delta_k e_1)}{8(1+N)^2}|y^{N+1} - e_1|^2)^2}. \tag{3.7}$$

Clearly V_k is a solution of

$$\Delta V_k + \mathfrak{h}_k(\delta_k e_1)|y|^{2N}e^{V_k} = 0, \quad \text{in} \quad \mathbb{R}^2, \quad V_k(e_1) = \mu_k. \tag{3.8}$$

This expression is based on the classification theorem of Prajapat-Tarantello [16].

The estimate of $v_k(x)-V_k(x)$ is important for the main theorem of this article. For convenience we use

$$\beta_l=\frac{2\pi l}{N+1}, \quad \text{so } e_1=e^{i\beta_0}=Q_0^k, \quad e^{i\beta_l}=Q_l^k+E, \text{ for } l=1,...,N.$$

Proposition 3.1. *Let $l=0,...,N$ and δ be small so that $B(e^{i\beta_l},\delta)\cap B(e^{i\beta_s},\delta)=\emptyset$ for $l\neq s$. In each $B(e^{i\beta_l},\delta)$*

$$|v_k(x)-V_k(x)|\le\begin{cases} C\mu_k e^{-\mu_k/2}, & |x-e^{i\beta_l}|\le Ce^{-\mu_k/2},\\ C\frac{\mu_k e^{-\mu_k}}{|x-e^{i\beta_l}|}+O(\mu_k^2e^{-\mu_k}), & Ce^{-\mu_k/2}\le|x-e^{i\beta_l}|\le\delta.\end{cases} \tag{3.9}$$

Remark 3.1. *Once (3.9) is established. We shall use a re-scaled version of Proposition 3.1. Let $\varepsilon_k=e^{-\frac{1}{2}\mu_k}$, we have*

$$|v_k(e^{i\beta_l}+\varepsilon_k y)-V_k(e^{i\beta_l}+\varepsilon_k y)|\le C\mu_k^2\varepsilon_k(1+|y|)^{-1}, \quad 0<|y|<\delta_0\varepsilon_k^{-1}. \tag{3.10}$$

Proof of Proposition 3.1: The main idea of the proof is as follows. First from the Green's representation of v_k we obtain a rather precise estimate of v_k in B_3 away from bubbling disks. On the other hand around each Q_m^k we invoke a standard pointwise estimate in [6, 21, 11] for Liouville equation around a blowup point, which provides a precise description of v_k in a neighborhood of Q_m^k. The comparison of these two estimates gives an accurate estimate of the maximum of v_k around each local maximum point.

Fixing the neighborhood of one Q_m^k, we first cite a result of Gluck [11] (Appendix B of [19]) to write v_k as

$$v_k(y)=\log\frac{e^{\mu_{k,m}}}{(1+e^{\mu_{k,m}}\frac{|\tilde{Q}_m^k|^{2N}\mathfrak{h}_k(\delta_k\tilde{Q}_m^k)}{8}|y-\tilde{Q}_m^k|^2)^2}+\phi_m^k(y)+O(\mu_k^2e^{-\mu_k}) \tag{3.11}$$

where $\mu_{k,m}=v_k(\tilde{Q}_m^k)$, ϕ_m^k is the harmonic function taking 0 at Q_m^k that makes $v_k-\phi_m^k=constant$ on $\partial B(Q_m^k,\delta)$. $\tilde{Q}_m^k$ is where $v_k-\phi_m^k$ takes its local maximum in a neighborhood of Q_m^k. The difference between $\tilde{Q}_m^k$ and Q_m^k is $O(e^{-\mu_k})$. First we claim that

$$\mu_{k,m}-\mu_k=E. \tag{3.12}$$

From the Green's representation formula for v_k, we have, for y away from bubbling areas and $|y|\sim 1$,

$$\begin{aligned} v_k(y)&=v_k|_{\partial\Omega_k}+\int_{\Omega_k}G(y,\eta)\mathfrak{h}_k(\eta)|\eta|^{2N}e^{v_k}d\eta,\\ &=v_k|_{\partial\Omega_k}+\sum_{l=0}^N G(y,Q_l^k)\int_{B(Q_l^k,\varepsilon)}|\eta|^{2N}\mathfrak{h}_k(\delta_k\eta)e^{v_k}d\eta\\ &+\sum_l\int_{B(Q_l^k,\varepsilon)}(G(y,\eta)-G(y,Q_l^k))|\eta|^{2N}\mathfrak{h}_k(\delta_k\eta)e^{v_k}d\eta+E,\\ &=v_k|_{\partial\Omega_k}+8\pi\sum_l G(y,Q_l^k)+E\end{aligned}$$

where $\Omega_k = B(0, \tau\delta_k^{-1})$. Note that we use two standard estimates. First the integration outside bubbling disk is E because

$$v_k(x) \le -\mu_k - (4(N+1) - o(1))\log|x| + C, \quad 3 < |x| < \tau\delta_k^{-1}.$$

Second, in the evaluation of the integral terms above we use standard bubble expansion formula (see Gluck [11], for example) and symmetry properties. This part is mentioned in Lemma 2.1 and Appendix B of [19]. It is important to point out that the second estimate does not depend on m. In particular if we consider y located at $|y - Q_m^k| = \varepsilon$, the expression of v_k can be written as

$$\begin{aligned} v_k(y) &= v_k|_{\partial\Omega_k} - 4\log|y - Q_m^k| + \phi_m^k \\ &\quad - 4\sum_{l=0, l\neq m}^{N} \log|Q_m^k - Q_l^k| + 8\pi\sum_{l=0}^{N} H(Q_m^k, Q_l^k) + E, \end{aligned} \tag{3.13}$$

where

$$\phi_m^k = \sum_{l=0, l\neq m}^{N} (-4)\log\frac{|y - Q_l^k|}{|Q_m^k - Q_l^k|} + 8\pi\sum_{l=0}^{N}(H(y, Q_l^k) - H(Q_m^k, Q_l^k)) \tag{3.14}$$

is the harmonic function that takes 0 at Q_m^k and eliminates the oscillation of v_k on $\partial B(Q_m^k, \varepsilon)$. On the other hand from (3.11) we have

$$v_k(y) = -\mu_{k,m} - 2\log\frac{|\tilde{Q}_m^k|^{2N}\mathfrak{h}_k(\delta_k\tilde{Q}_m^k)}{8} - 4\log|y - Q_m^k| + \phi_m^k + O(\mu_k^2 e^{-\mu_k}). \tag{3.15}$$

Comparing (3.15) and (3.13) on $|y - Q_m^k| = \varepsilon$ we have

$$\begin{aligned} & -\mu_{m,k} - 2\log\frac{|\tilde{Q}_m^k|^{2N}\mathfrak{h}_k(\delta_k\tilde{Q}_m^k)}{8} \\ &= -4\sum_{l=0, l\neq m}^{N} \log|Q_m^k - Q_l^k| + 8\pi\sum_{l=0}^{N} H(Q_m^k, Q_l^k) + v_k|_{\partial\Omega_k} + O(\mu_k^2 e^{-\mu_k}). \end{aligned} \tag{3.16}$$

To evaluate terms in (3.16) we observe that (see (3.5))

$$\begin{aligned} |\tilde{Q}_m^k|^{2N} &= 1 + E, \qquad \mathfrak{h}_k(\delta_k\tilde{Q}_m^k) = 1 + E, \\ Q_m^k &= e^{i\beta_m} + E, \qquad \tilde{Q}_m^k = Q_m^k + O(e^{-\mu_k}), \end{aligned}$$

and by the expression of $H_k(y, \eta)$:

$$\begin{aligned} H_k(y, \eta) &= \frac{1}{2\pi}\log\left(\frac{|\eta|}{\tau\delta_k^{-1}}\left|\frac{\tau^2\delta_k^{-2}\eta}{|\eta|^2} - y\right|\right) \\ &= \frac{1}{2\pi}\log(\tau\delta_k^{-1}) + \frac{1}{2\pi}\log\left|\frac{\eta}{|\eta|} - \frac{|\eta|}{\tau^2}\delta_k^2 y\right| \end{aligned}$$

we have

$$H_k(Q_m^k, Q_l^k) = \frac{1}{2\pi}\log(\tau\delta_k^{-1}) + E.$$

Thus two terms in (3.16) are

$$8\pi\sum_{l=0}^{N} H(Q_m^k, Q_l^k) = 4(N+1)\log(\tau\delta_k^{-1}) + E \tag{3.17}$$

$$\sum_{l=0,l\neq m}^{N} \log|Q_m^k - Q_l^k| = \sum_{l=0,l\neq m}^{N} \log|e^{i\beta_m} - e^{i\beta_l}| + E \\ = \log(N+1) + E. \tag{3.18}$$

Using (3.17) and (3.18) in (3.16) we have

$$v_k|_{\partial\Omega_k} = -\mu_{m,k} - 2\log\frac{\mathfrak{h}_k(\delta_k e_1)}{8} + 4\log(1+N) - 4(1+N)\log(\tau\delta_k^{-1}) \\ + O(\mu_k^2 e^{-\mu_k}), \quad m = 0,1,...,N. \tag{3.19}$$

The value $v_k|_{\partial\Omega_k}$ is independent of m. In particular $\mu_{0,k} = \mu_k$. Thus the comparison of $\mu_{m,k}$ in (3.19) proves (3.12). Next we observe that around Q_l^k

$$V_k(y) = \log\frac{e^{\mu_k}}{(1+\frac{|\tilde{Q}_l^k|^{2N}\mathfrak{h}_k(\delta_k e_1)e^{\mu_k}}{8}|y-\tilde{Q}_l^k|^2)^2} + \tilde{\phi}_l^k(y) + O(\mu_k^2 e^{-\mu_k}), \tag{3.20}$$

where $y \in B(e^{i\beta_l}, \delta_0)$, $\tilde{Q}_l^k = e^{i\beta_l} + O(e^{-\mu_k})$,

$$\tilde{\phi}_l^k(x) = \sum_{m=0,m\neq l}^{N} (-4)\log\frac{|y-e^{i\beta_m}|}{|e^{i\beta_m} - e^{i\beta_l}|}, \tag{3.21}$$

δ_0 is a small positive number independent of k. The way to prove (3.20), by direct computation from the expression of V_k, is as follows: It is easy to see that

$$V_k(y) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|y-\eta|\mathfrak{h}_k(\delta_k e_1)|\eta|^{2N} e^{V_k(\eta)} d\eta + C, \quad y \in \mathbb{R}^2.$$

Then $\mathfrak{h}_k(\delta_k e_1)e^{V_k}|y|^{2N}$ weakly converges to $8\pi\delta_{e^{i\beta_l}}$ in a small neighborhood of $e^{i\beta_l}$. For $y \in \partial B(e^{i\beta_l}, \delta)$ we have

$$V_k(y) = -\sum_{l=0}^{N} 4\log|y-e^{i\beta_l}| + C_k + O(\mu_k e^{-\mu_k}).$$

From there we know that harmonic function around $e^{i\beta_l}$ that equals 0 at $e^{i\beta_l}$ is ϕ_l^k in (3.21). On the other hand the equation of v_k around $e^{i\beta_l}$ is (3.8). The standard expansion (see [11]) for blowup solution leads to (3.20).

Since $Q_m^k - e^{i\beta_m} = E$, we can replace $\tilde{\phi}_l^k$ by ϕ_l^k and have

$$V_k(x) = \log\frac{e^{\mu_k}}{(1+\frac{\mathfrak{h}_k(\delta_k e_1)e^{\mu_k}}{8}|y-e^{i\beta_l}|^2)^2} + \phi_l^k + O(\mu_k^2 e^{-\mu_k})$$

in $B(Q_l^k, e^{-\mu_k/2})$. Thus in the region $B(Q_l^k, e^{-\mu_k/2})$, the comparison between v_k and V_k boils down to the evaluation of:

$$\log\frac{e^{\mu_{l,k}}}{(1+\frac{\mathfrak{h}_k(\delta_k e_1)e^{\mu_{l,k}}}{8}|y-e^{i\beta_l}-p_k|^2)^2} - \log\frac{e^{\mu_k}}{(1+\frac{\mathfrak{h}_k(\delta_k e_1)e^{\mu_k}}{8}|y-e^{i\beta_l}|^2)^2}, \tag{3.22}$$

for $|p_k| = E$. By elementary computation we see that the difference between the two terms in (3.22) is $O(\mu_k e^{-\mu_k/2})$ if $|y-e^{i\beta_l}| \le Ce^{-\mu_k/2}$. On the other hand, for

$Ce^{-\mu_k/2} < |y - e^{i\beta_l}| < \varepsilon/2$, the comparison of expressions of v_k and U_k gives the difference upper bound as

$$O(e^{-\mu_k})|y - e^{i\beta_l}|^{-1} + O(\mu_k^2 e^{-\mu_k}), \quad Ce^{-\mu_k/2} \le |y - e^{i\beta_l}| < \varepsilon.$$

Moreover

$$v_k - V_k = O(\mu_k^2 e^{-\mu_k}) \quad \text{on} \quad \partial B(Q_l^k, \varepsilon), \quad l = 0, ..., N. \tag{3.23}$$

Also we observe from the expression of V_k that

$$V_k(x) = -\mu_k - 2\log\frac{\mathfrak{h}_k(\delta_k e_1)}{8} + 4\log(N+1) - 4(1+N)\log(\tau\delta_k^{-1}) + E, \tag{3.24}$$

for $x \in \partial\Omega_k$, thus

$$v_k - V_k = O(\mu_k^2 e^{-\mu_k}) \quad \text{on} \quad \partial\Omega_k.$$

Then the closeness of v_k and V_k on $\Omega_k \setminus (\cup_l B(Q_l^k, \varepsilon))$ can be obtained by a standard maximum principle argument: If we use w_k to denote $v_k - V_k$:

$$w_k(z) = (v_k - V_k)(z),$$

then it is easy to see w_k satisfies

$$|\Delta w_k(z)| \le Ce^{-\mu_k}|z|^{-4-2N}, \quad \Omega_k \setminus B_2, \quad |w_k| \le C\mu_k^2 e^{-\mu_k} \text{ on } \partial B_2 \cup \partial\Omega_k,$$

Then $|w_k|$ can be majorized by $Q(\mu_k^2 e^{-\mu_k} - e^{-\mu_k} r^{-1-2N})$ for a large $Q > 1$, which yields the smallness of $v_k - V_k$ on $\Omega_k \setminus B_2$ as a consequence. Proposition 3.1 is established. □

4. FIRST CRUCIAL BOUND FOR $v_k - V_k$

In this section we establish the first major estimate of $v_k - V_k$. The main result in this section is

Proposition 4.1. *Let $w_k = v_k - V_k$, then*

$$|w_k(y)| \le C\delta_k, \quad y \in \Omega_k := B(0, \tau\delta_k^{-1}).$$

Proof of Proposition 4.1:

First we recall the equation for v_k is (3.2), $v_k =$ constant on $\partial B(0, \tau\delta_k^{-1})$. Moreover $v_k(e_1) = \mu_k$. Recall that V_k defined in (3.7) satisfies

$$\Delta V_k + \mathfrak{h}_k(\delta_k e_1)|y|^{2N} e^{V_k} = 0, \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |y|^{2N} e^{V_k} < \infty,$$

V_k has its local maximums at $e^{i\beta_l}$ for $l = 0, ..., N$ and $V_k(e_1) = \mu_k$. For $|y| \sim \delta_k^{-1}$,

$$V_k(y) = -\mu_k - 4(N+1)\log\delta_k^{-1} + C + O(\delta_k^{N+1}) + O(e^{-\mu_k}).$$

Let $\Omega_k = B(0, \tau\delta_k^{-1})$, we shall derive a precise, point-wise estimate of w_k in $B_3 \setminus \cup_{l=1}^N B(Q_l^k, \lambda)$ where $\lambda > 0$ is a small number independent of k. Here we note that among $N+1$ local maximum points, we already have e_1 as a common local maximum point for both v_k and V_k and we shall prove that w_k is very small in B_3 if we exclude all bubbling disks except the one around e_1. Before we carry out more specific computation we emphasize the importance of

$$w_k(e_1) = |\nabla w_k(e_1)| = 0. \tag{4.1}$$

Now we write the equation of w_k as

$$\Delta w_k + \mathfrak{h}_k(\delta_k y)|y|^{2N} e^{\xi_k} w_k = (\mathfrak{h}_k(\delta_k e_1) - \mathfrak{h}_k(\delta_k y))|y|^{2N} e^{V_k} \tag{4.2}$$

in Ω_k, where ξ_k is obtained from the mean value theorem:

$$e^{\xi_k(x)} = \begin{cases} \frac{e^{v_k(x)} - e^{V_k(x)}}{v_k(x) - V_k(x)}, & \text{if} \quad v_k(x) \neq V_k(x), \\ \\ e^{V_k(x)}, & \text{if} \quad v_k(x) = V_k(x). \end{cases}$$

An equivalent form is

$$e^{\xi_k(x)} = \int_0^1 \frac{d}{dt} e^{t v_k(x) + (1-t) V_k(x)} dt = e^{V_k(x)} \big(1 + \frac{1}{2} w_k(x) + O(w_k(x)^2)\big). \tag{4.3}$$

For convenience we write the equation for w_k as

$$\Delta w_k + \mathfrak{h}_k(\delta_k y)|y|^{2N} e^{\xi_k} w_k = \delta_k \nabla \mathfrak{h}_k(\delta_k e_1) \cdot (e_1 - y)|y|^{2N} e^{V_k} + E_1 \tag{4.4}$$

where

$$E_1 = O(\delta_k^2)|y - e_1|^2 |y|^{2N} e^{V_k}, \quad y \in \Omega_k.$$

Let $M_k = \max_{x \in \bar{\Omega}_k} |w_k(x)|$. We shall get a contradiction by assuming $M_k/\delta_k \to \infty$ at this moment. Set

$$\tilde{w}_k(y) = w_k(y)/M_k, \quad x \in \Omega_k.$$

Clearly $\max_{x \in \Omega_k} |\tilde{w}_k(x)| = 1$. The equation for $\tilde{w}_k$ is

$$\Delta \tilde{w}_k(y) + |y|^{2N} \mathfrak{h}_k(\delta_k e_1) e^{\xi_k} \tilde{w}_k(y) = \frac{\delta_k}{M_k} \nabla \mathfrak{h}_k(\delta_k e_1) \cdot (e_1 - y)|y|^{2N} e^{V_k} + \tilde{E}_1, \tag{4.5}$$

in Ω_k, where

$$\tilde{E}_1 = o(\delta_k)|y - e_1|^2 |y|^{2N} e^{V_k}, \quad y \in \Omega_k. \tag{4.6}$$

Now we give a more precise estimate of e^{ξ_k}. By Proposition 3.1

$$\xi_k(y) = V_k(y) + \begin{cases} O(\mu_k e^{-\mu_k/2}), & |y - e_1| \leq e^{-\mu_k/2}, \\ \\ O(\mu_k^2 e^{-\mu_k})|y - e_1|^{-1}, & e^{-\mu_k/2} \leq |y - e_1| \leq \delta_0. \end{cases} \tag{4.7}$$

Since V_k is not exactly symmetric around e_1, we shall replace the re-scaled version of V_k around e_1 by a radial function. Let U_k be solutions of

$$\Delta U_k + \mathfrak{h}_k(\delta_k e_1) e^{U_k} = 0, \quad \text{in} \quad \mathbb{R}^2, \quad U_k(0) = \max_{\mathbb{R}^2} U_k = 0.$$

Then we have

$$U_k(z) = \log \frac{1}{(1 + \frac{\mathfrak{h}_k(\delta_k e_1)}{8}|z|^2)^2}$$

and

$$V_k(e_1 + \varepsilon_k z) + 2 \log \varepsilon_k = U_k(z) + O(\varepsilon_k)|z| + O(\mu_k^2 \varepsilon_k^2). \tag{4.8}$$

Also we observe that

$$\log |e_1 + \varepsilon_k y| = O(\varepsilon_k)|y|. \tag{4.9}$$

Thus, the combination of (4.7), (4.8) and (4.9) gives

$$\begin{aligned} &2N\log|e_1+\varepsilon_k z|+\xi_k(e_1+\varepsilon_k z)+2\log\varepsilon_k-U_k(z)\\ =&O(\mu_k^2\varepsilon_k)(1+|z|)\quad 0\le|z|<\delta_0\varepsilon_k^{-1}. \end{aligned} \tag{4.10}$$

Since we shall use the re-scaled version, based on (4.10) we have

$$\varepsilon_k^2|e_1+\varepsilon_k z|^{2N}e^{\xi_k(e_1+\varepsilon_k z)}=e^{U_k(z)}+O(\mu_k^2\varepsilon_k)(1+|z|)^{-3} \tag{4.11}$$

Here we note that the estimate in (4.10) is not optimal.

The first key estimate is

Lemma 4.1.

$$\tilde{w}_k(y)=o(1),\quad \nabla\tilde{w}_k=o(1)\quad in\quad B(e_1,\delta)\setminus B(e_1,\delta/8) \tag{4.12}$$

where $B(e_1,3\delta)$ does not include other blowup points.

Proof of Lemma 4.1:

If (4.12) is not true, we have, without loss of generality that $\tilde{w}_k\to c>0$. Note that $\tilde{w}_k$ tends to a global harmonic function with removable singularity. So $\tilde{w}_k$ tends to constant. Here we assume $c>0$ but the argument for $c<0$ is the same. Let

$$W_k(z)=\tilde{w}_k(e_1+\varepsilon_k z),\quad \varepsilon_k=e^{-\frac{1}{2}\mu_k},$$

then if we use W to denote the limit of W_k, we have

$$\Delta W+e^U W=0,\quad \mathbb{R}^2,\quad |W|\le 1,$$

and U is a solution of $\Delta U+e^U=0$ in $\mathbb{R}^2$ with $\int_{\mathbb{R}^2}e^U<\infty$. Since 0 is the local maximum of U,

$$U(x)=\log\frac{1}{(1+\frac{1}{8}|x|^2)^2}.$$

Here we further claim that $W\equiv 0$ in $\mathbb{R}^2$ because $W(0)=|\nabla W(0)|=0$, a fact well known based on the classification of the kernel of the linearized operator. Going back to W_k, we have

$$W_k(x)=o(1),\quad |x|\le R_k \text{ for some }\quad R_k\to\infty.$$

Based on the expression of $\tilde{w}_k$, (4.8) and (4.11) we write the equation of W_k as

$$\Delta W_k(z)+\mathfrak{h}_k(\delta_k e_1)e^{U_k(z)}W_k(z)=-\frac{\delta_k}{M_k}\nabla\mathfrak{h}_k(\delta_k e_1)\cdot z\varepsilon_k e^{U_k(z)}+E_2^k, \tag{4.13}$$

for $|z|<\delta_0\varepsilon_k^{-1}$ where

$$E_2^k(z)=o(1)\mu_k^2\varepsilon_k(1+|z|)^{-3}.$$

Let

$$g_0^k(r)=\frac{1}{2\pi}\int_0^{2\pi}W_k(r,\theta)d\theta. \tag{4.14}$$

Then clearly $g_0^k(r)\to c>0$ for $r\sim\varepsilon_k^{-1}$. The equation for g_0^k is

$$\frac{d^2}{dr^2}g_0^k(r)+\frac{1}{r}\frac{d}{dr}g_0^k(r)+\mathfrak{h}_k(\delta_k e_1)e^{U_k(r)}g_0^k(r)=\tilde{E}_0^k(r)$$
$$g_0^k(0)=\frac{d}{dr}g_0^k(0)=0.$$

where $\tilde{E}_0^k(r)$ has the same upper bound as that of $E_2^k(r)$:

$$|\tilde{E}_0^k(r)|\le C\mu_k^2\varepsilon_k(1+r)^{-3}.$$

For the homogeneous equation, the two fundamental solutions are known: g_{01}, g_{02}, where

$$g_{01}=\frac{1-c_1r^2}{1+c_1r^2},\quad c_1=\frac{\mathfrak{h}_k(\delta_k e_1)}{8}.$$

By the standard reduction of order process, $g_{02}(r)=O(\log r)$ for $r>1$. Then it is easy to obtain, assuming $|W_k(z)|\le 1$, that

$$|g_0(r)|\le C|g_{01}(r)|\int_0^r s|\tilde{E}_0^k(s)g_{02}(s)|ds+C|g_{02}(r)|\int_0^r s|g_{01}(s)\tilde{E}_0^k(s)|ds.$$

After evaluation we have

$$|g_0(r)|\le C\mu_k^2\varepsilon_k\log(2+r).\quad 0<r<\delta_0\varepsilon_k^{-1}.$$

Clearly this is a contradiction to (4.14). We have proved $c=0$, which means $\tilde{w}_k=o(1)$ in $B(e_1,\delta_0)\setminus B(e_1,\delta_0/8)$. Then it is easy to use the equation for $\tilde{w}_k$ and standard Harnack inequality to prove $\nabla\tilde{w}_k=o(1)$ in the same region. Lemma 4.1 is established. $\Box$

Remark 4.1. *From Lemma 4.1 one obtains easily that $w_k=o(1)$ in $B(e_1,\varepsilon)$ for $\varepsilon>0$ small. Indeed, using the same notation W_k in the proof of Lemma 4.1 we already have $W_k=o(1)$ in B_{R_k} for some $R_k\to\infty$. Then by the smallness for $W_k(y)$ for $|y|\sim\varepsilon_k^{-1}$, it is easy to majorize W_k in $B(0,\varepsilon\varepsilon_k^{-1})\setminus B_{R_k}$ based on the fast decay of e^{U_k}. This part is omitted because it is similar to the last part of the proof of Proposition 3.1.*

The smallness of $\tilde{w}_k$ around e_1 can be used to obtain the following second key estimate:

Lemma 4.2.

$$\tilde{w}_k=o(1)\quad in\quad B(e^{i\beta_l},\delta)\quad l=1,..,N. \tag{4.15}$$

Proof of Lemma 4.2: We abuse the notation W_k by defining it as

$$W_k(z)=\tilde{w}_k(e^{i\beta_l}+\varepsilon_k z),\quad |z|<\delta_0\varepsilon_k^{-1}.$$

First because of the smallness of δ_k (see 2.8), which implies that $\varepsilon_k^{-1}|Q_l^k-e^{i\beta_l}|\to 0$. So the scaling around $e^{i\beta_l}$ or Q_l^k does not affect the limit function.

$$|e^{i\beta_l}+\varepsilon_k z|^{2N}\mathfrak{h}_k(\delta_k e_1)e^{\xi_k(e^{i\beta_l}+\varepsilon_k z)}\to e^{U(z)}$$

where $U(z)$ is a solution of

$$\Delta U + e^U = 0, \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^U < \infty.$$

Here we recall that $\lim_{k\to\infty} \mathfrak{h}_k(\delta_k e_1) = 1$. Since W_k converges to a solution of the linearized equation:

$$\Delta W + e^U W = 0, \quad \text{in} \quad \mathbb{R}^2.$$

W can be written as a linear combination of three functions:

$$W(x) = c_0\phi_0 + c_1\phi_1 + c_2\phi_2,$$

where

$$\phi_0 = \frac{1-\frac{1}{8}|x|^2}{1+\frac{1}{8}|x|^2}$$

$$\phi_1 = \frac{x_1}{1+\frac{1}{8}|x|^2}, \quad \phi_2 = \frac{x_2}{1+\frac{1}{8}|x|^2}.$$

First we claim that $c_0 = 0$. Let $\Omega_{l,k} = B(0, \delta_0\varepsilon_k^{-1})$,

$$H_l^k(z) = |e^{i\beta_l} + \varepsilon_k z|^{2N}\mathfrak{h}_k(\delta_k e_1),$$

$$U_k(z) = \xi_k(e^{i\beta_l} + \varepsilon_k z) + 2\log\varepsilon_k, \quad \text{clearly} \quad U_k \to U.$$

Here we note that U is radial. Based on (4.5) we write the equation for W_k as

$$\Delta W_k(z) + H_l^k(z)e^{U_k}W_k = E_l^k(z) \tag{4.16}$$

where

$$E_l^k(z) = o(1)(1+|z|)^{-4}, \quad |z| < \delta_0\varepsilon_k^{-1}.$$

Integrating both sides of (4.16), we have

$$\int_{\partial\Omega_{l,k}} \partial_\nu W_k + \int_{\Omega_{l,k}} H_l^k e^{U_k} W_k = \int_{\Omega_{l,k}} E_l^k dz.$$

Based on the estimate of $\nabla\tilde{w}_k$ away from bubbling disks, we see that the first term and the third term above are both tending to 0, the second term tends to Λc_0 for some $\Lambda > 0$. Thus $c_0 = 0$.

To prove $c_1 = c_2 = 0$, we consider the Pohozaev identity of W_k:

$$\begin{aligned} &\int_{\partial\Omega_{l,k}} \left((\partial_\nu W_k)^2 - \frac{1}{2}|\nabla W_k|^2 + \frac{1}{2}H_l^k e^{U_k} W_k\right)\delta_0\varepsilon_k^{-1} \\ &-\frac{1}{2}\int_{\Omega_{l,k}} W_k^2(2H_k e^{U_k} + z_i\partial_i(H_l^k e^{U_k})) - \int_{\Omega_{l,k}} E_l^k z_i\partial_i W_k = 0. \end{aligned} \tag{4.17}$$

It is easy to see that the first term and the third term are $o(1)$. To evaluate the second term, we first observe that the integration outside B_{R_k} for any $R_k \to \infty$ is $o(1)$. So we only need to evaluate

$$\int_{B(0,R_k)} W_k^2(2H_k e^{U_k} + z_i\partial_i(H_l^k e^{U_k})).$$

Direct computation shows that

$$2H_ke^{U_k}+z_i\partial_i(H_l^ke^{U_k})\to 2\frac{1-c|z|^2}{(1+c|z|^2)^3}\quad \text{in}\quad C^2_{loc}(\mathbb{R}^2),$$

$$W^2(z)=\frac{c_1^2z_1^2}{(1+c|z|^2)^2}+\frac{c_2^2z_2^2}{(1+c|z|^2)^2}+\frac{2c_1c_2z_1z_2}{(1+c|z|^2)^2},\quad \text{in}\quad C^2_{loc}(\mathbb{R}^2),\quad c=1/8.$$

Using these two facts and direct computation we have

$$\int_{B(0,R_k)}W_k^2(2H_ke^{U_k}+z_i\partial_i(H_l^ke^{U_k}))=D(c_1^2+c_2^2)+o(1)$$

for some $D\neq 0$. Thus $c_1=c_2=0$. Lemma 4.2 is established. □

We have proved that $\tilde{w}_k=o(1)$ in B_3, which also immediately implies $\tilde{w}_k=o(1)$ in B_R for any fixed $R>>1$. Outside B_R, a crude estimate of v_k is

$$v_k(y)\le -\mu_k-4(N+1)\log|y|+C,\quad 3<|y|<\tau\delta_k^{-1}.$$

Using this and the Green's representation of w_k we can first observe that the oscillation on each ∂B_r is $o(1)$ ($R<r<\tau\delta_k^{-1}/2$) and then by the Green's representation of $\tilde{w}_k$ and fast decay rate of e^{V_k} we obtain $\tilde{w}_k=o(1)$ in $B(0,\tau\delta_k^{-1})$. A contradiction to $\max|\tilde{w}_k|=1$. Proposition 4.1 is established. □.

5. Proof of Theorem 1.1

Let $\hat{w}_k=w_k/\delta_k$. Then the equation for $\hat{w}_k$ is

$$\Delta\hat{w}_k+|y|^{2N}e^{\xi_k}\hat{w}_k=\nabla\mathfrak{h}_k(0)\cdot(e_1-y)|y|^{2N}e^{V_k}+O(\delta_k)e^{V_k}|y-e_1|^2, \tag{5.1}$$

in Ω_k. By Proposition 4.1, $|\hat{w}_k(y)|\le C$. Before we carry out the remaining part of the proof we observe that $\hat{w}_k$ converges to a harmonic function in $\mathbb{R}^2$ minus finite singular points. Since $\hat{w}_k$ is bounded, all these singularities are removable. Thus $\hat{w}_k$ converges to a constant. Based on the information around e_1, we shall prove that this constant is 0. However, looking at the right hand side the equation,

$$\nabla\mathfrak{h}_k(0)\cdot(e_1-y)|y|^{2N}e^{V_k}\rightharpoonup\sum_{l=1}^N 8\pi\nabla\mathfrak{h}_k(0)\cdot(e_1-e^{i\beta_l})\delta_{e^{i\beta_l}}.$$

If $\nabla\mathfrak{h}_k(0)\neq 0$ we would get a contradiction by comparing the Pohozaev identities of v_k and V_k.

Now we use the notation W_k again and use Proposition 4.1 to rewrite the equation for W_k. Let

$$W_k(z)=\hat{w}_k(e_1+\varepsilon_kz),\quad |z|<\delta_0\varepsilon_k^{-1}$$

for $\delta_0>0$ small. Then from Proposition 4.1 we have

$$\mathfrak{h}_k(\delta_ky)=\mathfrak{h}_k(\delta_ke_1)+\delta_k\nabla\mathfrak{h}_k(\delta_ke_1)(y-e_1)+O(\delta_k^2)|y-e_1|^2, \tag{5.2}$$

$$|y|^{2N}=|e_1+\varepsilon_kz|^{2N}=1+O(\varepsilon_k)|z|, \tag{5.3}$$

$$V_k(e_1+\varepsilon_kz)+2\log\varepsilon_k=U_k(z)+O(\varepsilon_k)|z|+O(\varepsilon_k^2)(\log(1+|z|))^2 \tag{5.4}$$

and

$$\xi_k(e_1+\varepsilon_k z)+2\log\varepsilon_k = U_k(z)+O(\varepsilon_k)(1+|z|). \tag{5.5}$$

Using (5.2),(5.3),(5.4) and (5.5) in (5.1) we write the equation of W_k as

$$\Delta W_k+\mathfrak{h}_k(\delta_k e_1)e^{U_k(z)}W_k = -\varepsilon_k\nabla\mathfrak{h}_k(0)\cdot z e^{U_k(z)}+E_w, \quad 0<|z|<\delta_0\varepsilon_k^{-1} \tag{5.6}$$

where

$$E_w = O(\delta_k)(1+|z|)^{-4}+O(\varepsilon_k)(1+|z|)^{-3}W_k(z), \quad |z|<\delta_0\varepsilon_k^{-1}. \tag{5.7}$$

At this moment we use $|W_k(z)|\le C$ and a rough estimate of E_w is

$$E_w(z) = O(\varepsilon_k)(1+|z|)^{-3}, \quad |z|<\delta_0\varepsilon_k^{-1}. \tag{5.8}$$

Since $\hat{w}_k$ obviously converges to a global harmonic function with removable singularity, we have $\hat{w}_k\to\bar{c}$ for some $\bar{c}\in\mathbb{R}$. Then we claim that

Lemma 5.1. $\bar{c}=0$.

Proof of Lemma 5.1:
If $\bar{c}_1\neq 0$, we use $W_k(z)=\bar{c}+o(1)$ on $B(0,\delta_0\varepsilon_k^{-1})\setminus B(0,\frac{1}{2}\delta_0\varepsilon_k^{-1})$ and consider the projection of W_k on 1:

$$g_0(r)=\frac{1}{2\pi}\int_0^{2\pi}W_k(re^{i\theta})d\theta.$$

If we use F_0 to denote the projection to 1 of the right hand side we have, using the rough estimate of E_w in (5.8)

$$g_0''(r)+\frac{1}{r}g_0'(r)+\mathfrak{h}_k(\delta_k e_1)e^{U_k(r)}g_0(r)=F_0, \quad 0<r<\delta_0\varepsilon_k^{-1}$$

where

$$F_0(r)=O(\varepsilon_k)(1+|z|)^{-3}.$$

In addition we also have

$$\lim_{k\to\infty} g_0(\delta_0\varepsilon_k^{-1})=\bar{c}_1+o(1).$$

Here the $O(\delta_k)(1+|z|)^{-4}$ is absorbed because of the smallness of δ. For simplicity we omit k in some notations. By the same argument as in Lemma 4.1, we have

$$g_0(r)=O(\varepsilon_k)(\log(2+r))^2, \quad 0<r<\delta_0\varepsilon_k^{-1}.$$

Thus $\bar{c}_1=0$. Lemma 5.1 is established. □

Based on Lemma 5.1 and standard Harnack inequality for elliptic equations we have

$$\tilde{w}_k(x)=o(1),\ \nabla\tilde{w}_k(x)=o(1),\ x\in B_3\setminus(\cup_{l=1}^N(B(e^{i\beta_l},\delta_0)\setminus B(e^{i\beta_l},\delta_0/8))). \tag{5.9}$$

Equation (5.9) is equivalent to $w_k=o(\delta_k)$ and $\nabla w_k=o(\delta_k)$ in the same region.

Proof of Theorem 1.1:

For $s=1,...,N$ we consider the Pohozaev identity around Q_s^k. Let $\Omega_{s,k}=B(Q_s^k,r)$ for small $r>0$. For v_k we have

$$\int_{\Omega_{s,k}}\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k y))e^{v_k}-\int_{\partial\Omega_{s,k}}e^{v_k}|y|^{2N}\mathfrak{h}_k(\delta_k y)(\xi\cdot\nu)$$
$$=\int_{\partial\Omega_{s,k}}(\partial_\nu v_k\partial_\xi v_k-\frac{1}{2}|\nabla v_k|^2(\xi\cdot\nu))dS. \tag{5.10}$$

where ξ is an arbitrary unit vector. Correspondingly the Pohozaev identity for V_k is

$$\int_{\Omega_{s,k}}\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k e_1))e^{V_k}-\int_{\partial\Omega_{s,k}}e^{V_k}|y|^{2N}\mathfrak{h}_k(\delta_k e_1)(\xi\cdot\nu)$$
$$=\int_{\partial\Omega_{s,k}}(\partial_\nu V_k\partial_\xi V_k-\frac{1}{2}|\nabla V_k|^2(\xi\cdot\nu))dS. \tag{5.11}$$

Using $w_k=v_k-V_k$ and $|w_k(y)|\le C\delta_k$ we have

$$\begin{aligned}&\int_{\partial\Omega_{s,k}}(\partial_\nu v_k\partial_\xi v_k-\frac{1}{2}|\nabla v_k|^2(\xi\cdot\nu))dS\\=&\int_{\partial\Omega_{s,k}}(\partial_\nu V_k\partial_\xi V_k-\frac{1}{2}|\nabla V_k|^2(\xi\cdot\nu))dS\\&+\int_{\partial\Omega_{s,k}}(\partial_\nu V_k\partial_\xi w_k+\partial_\nu w_k\partial_\xi V_k-(\nabla V_k\cdot\nabla w_k)(\xi\cdot\nu))dS+O(\delta_k^2).\end{aligned}$$

If we just use crude estimate: $\nabla w_k=o(\delta_k)$, then

$$\int_{\partial\Omega_{s,k}}(\partial_\nu v_k\partial_\xi v_k-\frac{1}{2}|\nabla v_k|^2(\xi\cdot\nu))dS-\int_{\partial\Omega_{s,k}}(\partial_\nu V_k\partial_\xi V_k-\frac{1}{2}|\nabla V_k|^2(\xi\cdot\nu))dS=o(\delta_k).$$

The difference on the second terms is minor:

$$\int_{\partial\Omega_{s,k}}e^{v_k}|y|^{2N}\mathfrak{h}_k(\delta_k y)(\xi\cdot\nu)-\int_{\partial\Omega_{s,k}}e^{V_k}|y|^{2N}\mathfrak{h}_k(\delta_k e_1)(\xi\cdot\nu)=O(\delta_k\varepsilon_k^2).$$

To evaluate the first term, we use

$$\begin{aligned}&\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k y))e^{v_k}\\=&\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k e_1)+|y|^{2N}\delta_k\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1)+O(\delta_k^2))e^{V_k}(1+w_k+O(\delta_k^2))\\=&\partial_\xi(|y|^{2N})\mathfrak{h}_k(\delta_k e_1)e^{V_k}+\delta_k\partial_\xi(|y|^{2N}\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1))e^{V_k}\\&+\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k e_1))e^{V_k}w_k+O(\delta_k^2)e^{V_k}.\end{aligned}\tag{5.12}$$

For the third term on the right hand side of (5.12) we use the equation for w_k:

$$\Delta w_k+\mathfrak{h}_k(\delta_k e_1)e^{V_k}|y|^{2N}w_k=-\delta_k\nabla\mathfrak{h}_k(\delta_k e_1)\cdot(y-e_1)|y|^{2N}e^{V_k}+O(\delta_k^2)e^{V_k}|y|^{2N}.$$

From integration by parts we have

$$\begin{aligned}
&\int_{\Omega_{s,k}} \partial_\xi(|y|^{2N})\mathfrak{h}_k(\delta_k e_1)e^{V_k}w_k\\
=&2N\int_{\Omega_{s,k}} |y|^{2N-2}y_\xi \mathfrak{h}_k(\delta_k e_1)e^{V_k}w_k\\
=&2N\int_{\Omega_{s,k}} \frac{y_\xi}{|y|^2}(-\Delta w_k-\delta_k\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1)|y|^{2N}e^{V_k}+O(\delta_k^2)e^{V_k})\\
=&-2N\delta_k\int_{\Omega_{s,k}} \frac{y_\xi}{|y|^2}\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1)|y|^{2N}e^{V_k}\\
&+2N\int_{\partial\Omega_{s,k}}(\partial_\nu(\frac{y_\xi}{|y|^2})w_k-\partial_\nu w_k\frac{y_\xi}{|y|^2})+O(\delta_k^2)\\
=&-16N\delta_k\pi(e^{i\beta_s}\cdot\xi)\nabla\mathfrak{h}_k(\delta_k e_1)(e^{i\beta_s}-e_1)+o(\delta_k),
\end{aligned}\tag{5.13}$$

where we have used $\nabla w_k, w_k = o(\delta_k)$ on $\partial\Omega_{s,k}$. For the second term on the right hand side of (5.12), we have

$$\begin{aligned}
&\int_{\Omega_{s,k}} \delta_k\partial_\xi(|y|^{2N}\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1))e^{V_k}\\
=&2N\delta_k\int_{\Omega_{s,k}} y_\xi|y|^{2N-2}\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1)e^{V_k}+\delta_k\int_{\Omega_{s,k}}|y|^{2N}\partial_\xi\mathfrak{h}_k(\delta_k e_1)e^{V_k}\\
=&16N\pi\delta_k(e^{i\beta_s}\cdot\xi)\nabla\mathfrak{h}_k(\delta_k e_1)(e^{i\beta_s}-e_1)+8\pi\delta_k\partial_\xi\mathfrak{h}_k(\delta_k e_1)+o(\delta_k).
\end{aligned}\tag{5.14}$$

Using (5.13) and (5.14) in the difference between (5.10) and (5.11), we have

$$\delta_k\partial_\xi\mathfrak{h}_k(\delta_k e_1)=o(\delta_k).$$

Thus $\nabla\mathfrak{h}_k(\delta_k e_1)=o(1)$. Theorem 1.1 is established. □

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