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SKYRMIONS IN GROSS-PITAEVSKII FUNCTIONALS*

Dedicated to Professor Wu Wenjun on the occasion of his 90th birthday

Fanghua Lin
Courant Institute of Mathematical Sciences, New York University, New York, USA
E-mail: linf@cims.nyu.edu
Taichia Lin
Department of Mathematics, National Taiwan University, Taipei, Taiwan 106;
Taida Institute of Mathematical Sciences (TIMS), Taipei, Taiwan, China
E-mail : tclin@math.ntu.edu.tw
Juncheng Wei
Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China
E-mail: wei@math.cuhk.edu.hk

Abstract In Bose-Einstein condensates (BECs), skyrmions can be characterized by pairs of linking vortex rings coming from two-component wave functions. Here we construct skyrmions by studying critical points of Gross-Pitaevskii functionals with two-component wave functions. Using localized energy method, we rigorously prove the existence, and describe the configurations of skyrmions in such BECs.

Key words skyrmions; Bose-Einstein condensate; linked vortex rings; localized energy method

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1 Introduction

Vortex rings formed in nature with various scales and composed of vortices whose core is an one-dimensional close loop in three space dimensions have fascinated scientists and mathematicians for a long time. They can also be observed in the trapped Bose-Einstein condensate (BEC) represented by one-component wave functions (cf. [1]). In a double condensate (a binary mixture of BECs with two different hyperfine states) described by two-component wave functions (cf. [13]), the skyrmion may be formed with a pair of linking vortex rings (cf. [7]). The skyrmion can be depicted as a quantized vortex ring in one component close to the core of which is confined the second component carrying quantized circulation around the ring.

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The GP functional with two-component wave functions is a standard model to describe a double condensate. In [3], GP functionals can be mapped onto a version of the nonlinear sigma model having a similar form to the Skyrme model. Conventionally, the Skyrme model gives skyrmions which are topologically non-trivial maps from three-dimensional space to a target manifold in order to represent baryons in nuclear physics (cf. [5], [9], [10], [17]). It would be natural to believe that skyrmions can be found in GP functionals with two-component wave functions. Here we want to prove rigorously the existence of configuration of skyrmions by studying critical points of GP functionals with two-component wave functions.

Physically, the two-component GP functional can be written as

$$E[\Psi_1,\Psi_2]=\int_{\mathbb{R}^3}\sum_{i=1}^{2}\left(\frac{\hbar}{2M}\,|\nabla\Psi_i|^2+V_i\,|\Psi_i|^2\right)+\sum_{i,j=1}^{2}U_{ij}\,|\Psi_i|^2\,|\Psi_j|^2, \tag{1.1}$$

under the following constraints:

$$\int_{\mathbb{R}^3}|\Psi_i|^2=N_i,\quad i=1,2, \tag{1.2}$$

where $\hbar$ is the Planck constant, M is the atomic mass, and V_i is the i-th trap potential. The coefficients U_{ij}'s are determined by all mutual s-wave scattering lengths. Due to Feshbach resonance, U_{ij}'s can be tuned over a very large range by adjusting the externally applied magnetic field (cf. [8]). Besides, Ψ_i is the complex-valued wave function of the i-th component BEC, and N_i is a positive constant denoting the number of atoms of the i-th component BEC. By numerical simulations, a configuration with the topology of a skyrmion, i.e., a topological soliton of the $S^3\to S^3$ map (cf. [14]) can be imprinted in a double condensate (cf. [15]), where S^3 is the unit sphere in $\mathbb{R}^4$. Furthermore, stable skyrmions may exist in a homogeneous two-component BEC under the condition that phase separation occurs due to strong inter-component repulsion without the effect of trap potentials (cf. [2]). This motivates us to replace $\mathbb{R}^3$ by S^3 and to set $V_i\equiv 0$ for $i=1,2$ in (1.1) and (1.2).

Mathematically, we may compactify $\mathbb{R}^3$ into S^3 if (Ψ_1,Ψ_2) approaches a constant vector at infinity of $\mathbb{R}^3$. Hence we may replace $\mathbb{R}^3$ by S^3 in (1.1) and (1.2), respectively. Let $V_i\equiv 0$ for $i=1,2$ and choose suitable scales on U_{ij}'s and N_i's. Then we may transform the functional (1.1) and the condition (1.2) (up to constants) into

$$E_{\Lambda,\beta}(u,v)=\int_{S^3}|\nabla u|^2+|\nabla v|^2+\frac{\Lambda}{2}(1-|u|^2-|v|^2)^2+4\beta|u|^2|v|^2, \tag{1.3}$$

for $u,v\in H^1(S^3;\mathbb{C})$ satisfying

$$\int_{S^3}|u|^2=c_{1,\Lambda}|S^3|,\quad \int_{S^3}|v|^2=c_{2,\Lambda}|S^3|, \tag{1.4}$$

where $|S^3|=2\pi^2$, Λ and β are large parameters, and $c_{j,\Lambda}$'s are positive constants such that $c_{j,\Lambda}\to c_j$ as $\Lambda\uparrow\infty$, $0<c_1,c_2<1$, and $c_1+c_2=1$. It is evident that the large parameter Λ forces the vector (u,v) to be close to S^3 in order to get finite energy, and another large parameter β may provide strong inter-component repulsion to fulfill the condition of phase separation in the physical literature (cf. [18]). In this paper, we study critical points of the functional (1.3) with the constraint (1.4) in order to represent skyrmions in double condensates.

2 Problems and Results

For simplicity, we first assume $(u,v)\in S^3$ and

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (\sin\lambda)\mathrm{e}^{im\phi} \\ (\cos\lambda)\mathrm{e}^{in\theta} \end{pmatrix}, \tag{2.1}$$

where $\lambda=\lambda(r)$, $\lambda(0)=0$, $\lambda(\frac{\pi}{2})=\frac{\pi}{2}$, $m,n\in\mathbb{Z}$, (r,ϕ,θ) are standard Hopf (toroidal) coordinates of S^3 defined by

$$\begin{aligned} x_1 &= \cos r\,\cos\theta, \quad x_2=\cos r\,\sin\theta, \\ x_3 &= \sin r\,\cos\phi, \quad x_4=\sin r\,\sin\phi, \end{aligned} \tag{2.2}$$

for $(x_1,\cdots,x_4)\in S^3=\{(x_1,\cdots,x_4):\sum\limits_{j=1}^{4}x_j^2=1\}$, where $r\in[0,\frac{\pi}{2}]$, θ and $\phi\in[0,2\pi]$. For each fixed value of $r\in[0,\frac{\pi}{2}]$, the θ and ϕ coordinates sweep out a two-dimensional torus. Taken together, these tori almost fill S^3. The exceptions occur at the endpoints $r=0$ and $r=\frac{\pi}{2}$, where the stack of tori collapses to the circles $\Gamma_1=\{(x_1,x_2,0,0):x_1^2+x_2^2=1\}$ and $\Gamma_2=\{(0,0,x_3,x_4):x_3^2+x_4^2=1\}$, respectively. It is obvious that Γ_1 and Γ_2 are linking circles in S^3. The coordinates r, θ and ϕ are everywhere orthogonal to each other. Thus, the metric on S^3 may be written as

$$\mathrm{d}s^2=\mathrm{d}r^2+\cos^2 r\mathrm{d}\theta^2+\sin^2 r\mathrm{d}\phi^2.$$

Besides, the volume form is given by

$$\mathrm{d}V=\sin r\cos r\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi.$$

Consequently,

$$\int_{S^3}|\nabla w|^2=\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\Big[(\sin 2r)|\partial_r w|^2+\frac{\sin 2r}{\cos^2 r}|\partial_\theta w|^2+\frac{\sin 2r}{\sin^2 r}|\partial_\phi w|^2\Big]\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi, \tag{2.3}$$

and

$$\int_{S^3}|w|^2=\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}(\sin 2r)|w|^2\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi, \tag{2.4}$$

for $w\in H^1(S^3;\mathbb{C})$.

By (2.1), (2.3) and (2.4), the energy functional (1.3) can be reduced to

$$\mathcal{E}_\beta(\lambda)=2\pi^2\int_0^{\frac{\pi}{2}}\Big[(\sin 2r)|\lambda'|^2+\frac{m^2\sin 2r}{\sin^2 r}\sin^2\lambda+\frac{n^2\sin 2r}{\cos^2 r}\cos^2\lambda+\beta(\sin 2r)\sin^2 2\lambda\Big]\mathrm{d}r, \tag{2.5}$$

under the constraint

$$\int_0^{\frac{\pi}{2}}(\sin 2r)\sin^2\lambda\mathrm{d}r=c_1\in(0,1), \tag{2.6}$$

which may come from (1.4) and (2.1). Let $\varepsilon = 1/\sqrt{\beta}$. Then the energy functional can be written as $\mathcal{E}_\beta = 2\pi^2\,\varepsilon^{-2}\,E_\varepsilon$, where $0<\varepsilon\ll 1$ is a small parameter, and

$$E_\varepsilon(\lambda)=\int_0^{\frac{\pi}{2}}\Big\{\varepsilon^2\Big[(\sin 2r)|\lambda'|^2+\frac{m^2\sin 2r}{\sin^2 r}\sin^2\lambda+\frac{n^2\sin 2r}{\cos^2 r}\cos^2\lambda\Big]+(\sin 2r)\sin^2 2\lambda\Big\}\mathrm{d}r. \tag{2.7}$$

To find critical points of E_ε under the constraint (2.6), we study solutions (λ,μ)'s of the following problem:

$$\begin{cases} -\varepsilon^2\Big[\lambda''+\dfrac{2\cos 2r}{\sin 2r}\lambda'-\dfrac{2}{\sin^2 2r}(m^2\cos^2 r-n^2\sin^2 r)\sin 2\lambda\Big]+\sin 4\lambda \\ =\mu\varepsilon\sin 2\lambda,\quad 0<r<\dfrac{\pi}{2}, \\ \lambda(0)=0,\quad \lambda(\dfrac{\pi}{2})=\dfrac{\pi}{2}, \end{cases} \tag{2.8}$$

where μ is the associated Lagrange multiplier. Note that the conditions $\lambda(0)=0$ and $\lambda(\frac{\pi}{2})=\frac{\pi}{2}$ are crucial to let (u,v) (defined in (2.1)) form a smooth map from S^3 to S^3 with topological charge mn. Here topological charge means how many times the domain sphere S^3 are wrapped on the image sphere S^3. Actually, we may find solutions of the equation in (2.8) satisfying another conditions e.g. $\lambda(0)=0$ and $\lambda(\frac{\pi}{2})=0$ or π but the corresponding map (u,v) may become multi-valued and lose smoothness at $r=\pi/2$, i.e., the circle Γ_2.

In this paper, we show the following result.

For each $\mu\in\mathbb{R}$, the problem (2.8) has a solution $\lambda=\lambda_\varepsilon(r)$ satisfying

$$\lambda_\varepsilon(r)\to\begin{cases}0, & \forall\, 0\le r<t_0,\\ \dfrac{\pi}{2}, & \forall\, t_0<r\le\dfrac{\pi}{2},\end{cases} \tag{2.9}$$

and

$$E_\varepsilon\left(\lambda_\varepsilon\right)=O(\varepsilon), \tag{2.10}$$

as $\varepsilon\to 0+$, where $0<t_0<\frac{\pi}{2}$ depends on μ. As $\varepsilon>0$ sufficiently small, the profile of λ_ε having a sharp interface near t_0 can be sketched in Figure 1 as follows:

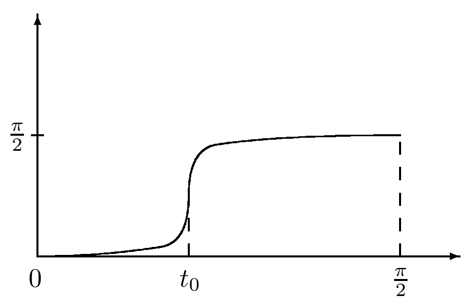

Figure 1

Moreover, we may choose a suitable μ to fulfill the condition (2.6), and the associated solution can be proved as a local minimizer of the energy functional (2.7) under the constraint (2.6). This may give the linear stability of the solution.

To find critical points of $E_{\Lambda,\beta}$, we assume

$$\begin{pmatrix} u \\ v \end{pmatrix} = \rho \begin{pmatrix} (\cos\lambda)\mathrm{e}^{i\phi} \\ (\sin\lambda)\mathrm{e}^{i\theta} \end{pmatrix}, \tag{2.11}$$

where $\rho=\rho(r)$ and $\lambda=\lambda(r)$ satisfy the following boundary conditions:

$$\begin{cases} \rho(0)=\rho(\frac{\pi}{2})=0, \\ \lambda(0)=0, \quad \lambda(\frac{\pi}{2})=\frac{\pi}{2}. \end{cases} \tag{2.12}$$

Here (r,ϕ,θ) are standard Hopf coordinates of S^3. It is remarkable that

$$\begin{pmatrix} (\cos\lambda)\mathrm{e}^{i\phi} \\ (\sin\lambda)\mathrm{e}^{i\theta} \end{pmatrix} = \begin{pmatrix} (\sin\widetilde{\lambda})\mathrm{e}^{i\phi} \\ (\cos\widetilde{\lambda})\mathrm{e}^{i\theta} \end{pmatrix}, \quad \widetilde{\lambda}=\frac{\pi}{2}-\lambda$$

has the same form as (2.1) with $m=n=1$. Then, by (1.3), (2.3), (2.4), and (2.11), the energy functional $E_{\Lambda,\beta}(u,v)$ can be written as

$$\begin{aligned} \mathcal{E}_{\Lambda,\beta} &= \mathcal{E}_{\Lambda,\beta}(\rho,\lambda) \\ &= 2\pi^2\int_0^{\frac{\pi}{2}}[(\sin 2r)\,\rho^2|\lambda'|^2+\beta(\sin 2r)\rho^4\,\sin^2 2\lambda]\mathrm{d}r \\ &\quad +2\pi^2\int_0^{\frac{\pi}{2}}\Big[(\sin 2r)\,|\rho'|^2+\Big(\frac{\sin 2r}{\sin^2 r}\,\cos^2\lambda+\frac{\sin 2r}{\cos^2 r}\,\sin^2\lambda\Big)\rho^2+(\sin 2r)\frac{\Lambda}{2}(1-\rho^2)^2\Big]\mathrm{d}r. \end{aligned} \tag{2.13}$$

Besides, the constraint (1.4) becomes

$$\int_0^{\frac{\pi}{2}}(\sin 2r)\,\rho^2\,\cos^2\lambda\mathrm{d}r=c_{1,\Lambda}, \quad \int_0^{\frac{\pi}{2}}(\sin 2r)\,\rho^2\,\sin^2\lambda\mathrm{d}r=c_{2,\Lambda}, \tag{2.14}$$

where $c_{j,\Lambda}\to c_j$ as $\Lambda\to\infty$, $0<c_1,c_2<1$, and $c_1+c_2=1$.

Let $\delta=\sqrt{1/\Lambda}$ and $\varepsilon=\sqrt{1/\beta}$, where Λ and β are large parameters tending to infinity. Then the functional $\mathcal{E}_{\Lambda,\beta}=2\pi^2\,\varepsilon^{-2}\,\widetilde{E}_{\delta,\varepsilon}$, where

$$\begin{aligned} \widetilde{E}_{\delta,\varepsilon}(\lambda,\rho) &= \int_0^{\frac{\pi}{2}}[\varepsilon^2\,(\sin 2r)\rho^2|\lambda'|^2+(\sin 2r)\rho^4\sin^2 2\lambda]\mathrm{d}r \\ &\quad +\int_0^{\frac{\pi}{2}}\Big\{\epsilon^2(\sin 2r)|\rho'|^2+\epsilon^2\Big[\frac{\sin 2r}{\sin^2 r}\,\cos^2\lambda+\frac{\sin 2r}{\cos^2 r}\,\sin^2\lambda\Big]\rho^2 \\ &\quad +\frac{\epsilon^2}{2\delta^2}(\sin 2r)\,(1-\rho^2)^2\Big\}\mathrm{d}r, \end{aligned} \tag{2.15}$$

and the constraint (2.14) becomes

$$\int_0^{\frac{\pi}{2}}(\sin 2r)\,\rho^2\,\cos^2\lambda\mathrm{d}r=c_{1,\delta}, \quad \int_0^{\frac{\pi}{2}}(\sin 2r)\,\rho^2\,\sin^2\lambda\mathrm{d}r=c_{2,\delta}, \tag{2.16}$$

where $c_{j,\delta}\to c_j$ as $\delta\to 0$, $0<c_1,c_2<1$, and $c_1+c_2=1$. Without loss of generality, we assume $\rho\to 1$ almost everywhere as $\delta\to 0$. Actually, such a hypothesis will be removed later. Then two conditions of (2.16) can be reduced to one condition as follows:

$$\int_0^{\frac{\pi}{2}}(\sin 2r)\,\rho^2\,\sin^2\lambda\mathrm{d}r=c_{2,\delta}, \tag{2.17}$$

where $c_{2,\delta} \to c_2$ as $\delta \to 0$. Critical points of $\widetilde{E}_{\delta,\varepsilon}$ subject to (2.17) satisfy

$$-\frac{\varepsilon^2}{\sin 2r}(\rho^2\lambda'\sin 2r)' + \rho^4 \sin 4\lambda - \varepsilon^2 \left(\frac{2\cos 2r}{\sin^2 2r}\right)\rho^2 \sin 2\lambda = \varepsilon\mu\rho^2\sin 2\lambda, \quad \forall\, 0 < r < \pi/2, \tag{2.18}$$

and

$$\begin{aligned}&-\frac{\delta^2}{\sin 2r}(\rho'\sin 2r)' + (\rho^2-1)\,\rho + \delta^2\left(|\lambda'|^2 + \frac{\cos^2\lambda}{\sin^2 r} + \frac{\sin^2\lambda}{\cos^2 r}\right)\rho + \frac{2\delta^2}{\varepsilon^2}\,\rho^3\sin^2 2\lambda \\ &= \delta\mu\rho\sin^2\lambda, \quad \forall\, 0 < r < \pi/2,\end{aligned} \tag{2.19}$$

with the conditions of (2.12), where μ is the Lagrange multiplier. Under the assumption

$$0 < \varepsilon^2 \ll \delta \ll \varepsilon \ll 1, \tag{2.20}$$

we may show the following result.

For each $\mu \in \mathbb{R}$, there exists a solution $(\lambda, \rho) = (\lambda_{\delta,\varepsilon,\mu}, \rho_{\delta,\varepsilon,\mu})$ to (2.18)–(2.19) such that

$$\lambda_{\delta,\varepsilon,\mu}(r) \to \begin{cases} 0, & \forall\, 0 \le r < t_0, \\ \dfrac{\pi}{2}, & \forall\, t_0 < r \le \frac{\pi}{2}, \end{cases} \tag{2.21}$$

$$\rho_{\delta,\varepsilon,\mu}(r) \to \begin{cases} 1, & \forall\, 0 < r < \dfrac{\pi}{2}, \\ 0, & \text{if } r = 0, \dfrac{\pi}{2}, \end{cases} \tag{2.22}$$

and

$$\widetilde{E}_{\delta,\varepsilon}\,(\lambda_{\delta,\varepsilon,\mu}, \rho_{\delta,\varepsilon,\mu}) = O(\varepsilon) + O\Big(\varepsilon^2 \log\frac{1}{\delta}\Big), \tag{2.23}$$

as $\varepsilon \to 0+$, where $0 < t_0 < \frac{\pi}{2}$ depends on μ. Moreover, we may find a suitable μ such that the condition (2.17) is fulfilled. When $\varepsilon > 0$ is sufficiently small, the graph of $\lambda_{\delta,\varepsilon,\mu}$ has a sharp interface near t_0. Besides, the profile of $\rho_{\delta,\varepsilon,\mu}$ gives linking vortex rings around $r = 0, \frac{\pi}{2}$, i.e., the circles $\Gamma_j, j = 1, 2$. Therefore, by (2.11), we may obtain skyrmions of GP functionals. We point out that, on one hand, one may regard $\widetilde{E}_{\delta,\varepsilon}$ as an approximation to E_ε when $0 < \delta \ll \varepsilon \ll 1$. On the other hand, by (2.23), it is evident that $\widetilde{E}_{\delta,\varepsilon}$ is of $O(\varepsilon)$ which is same as E_ε in (2.10) if $\delta \gg \varepsilon^2 > 0$ holds. This provides one of the reasons for the technical condition (2.20) in the sense that certain restrictions may needed in order to accommodate phase-seperations and vortex-confinements. We use this technical assumption mainly for the purpose of simplifications of some proofs. We refer to Section 7 of the paper for details.

The rest of paper is organized as follows: In Section 3, we introduced the heteroclinic solution of Sine-Gordon equation. The heteroclinic solution can be used to approach solutions of (2.8) with (2.9) in Section 4. We study the spectrum of linearized operator and the local minimizer of E_ε in Section 5 and 6, respectively. In Section 7, we find solutions of (2.18)–(2.19) with (2.21)–(2.23).

3 Heteroclinic Solution

Let w denote the unique heteroclinic solution of Sine-Gordon equation given by

$$\begin{cases} -w'' + \sin 4w = 0 \quad \text{in} \quad \mathbb{R}, \\ w(-\infty) = 0, \quad w(+\infty) = \frac{\pi}{2}. \end{cases} \tag{3.1}$$

Note that the solution w can be written as

$$w(x) = \frac{\pi}{4} + \frac{1}{2}\arcsin\left[\tanh(2x)\right], \quad \forall\, x \in \mathbb{R}. \tag{3.2}$$

The following lemma plays an important role in our study.

Lemma 3.1 The eigenvalue problem

$$\begin{cases} -\phi'' + 4(\cos 4w)\phi = \lambda\phi \quad \text{in} \quad \mathbb{R}, \\ \phi(\pm\infty) = 0 \end{cases} \tag{3.3}$$

has the following set of eigenvalues

$$\lambda_1 = 0, \quad \phi_1 = w'; \quad \lambda_2 > 0, \tag{3.4}$$

where λ_1 is the first eigenvalue, ϕ_1 is the first eigenfunction and λ_2 is the second eigenvalue.

Proof Using (3.2), the eigenvalue problem (3.3) becomes

$$-\phi'' + 4(1 - 2(\tanh(2x))^2)\phi = \lambda\phi,\ \phi \in H^1(\mathbb{R}). \tag{3.5}$$

Letting $y = 2x$, (3.5) becomes

$$-\phi'' - (-1 + 2(\cosh(y))^{-2})\phi = \lambda\phi. \tag{3.6}$$

In fact, (3.6) can be written as

$$-\phi'' - (-1 + w_0^2)\phi = \lambda\phi, \phi \in H^1(\mathbb{R}), \tag{3.7}$$

where $w_0 = \sqrt{2}(\cosh y)^{-1}$ is the unique ODE solution of

$$w_0'' - w_0 + w_0^3 = 0, w_0 > 0.$$

It is well-known that the eigenvalues of (3.7) are given by $\lambda_1 = 0, \phi_1 = cw_0 = c\sqrt{2}\,\mathrm{sech}\, y; \lambda_2 > 0$. See Lemma 4.1 of [19]. This proves the lemma.

As a consequence, we have

$$\int_{\mathbb{R}} \phi'^2 + 4\int_{\mathbb{R}} (\cos 4w)\phi^2 \geq 0,\ \forall \phi \in H^1(\mathbb{R}). \tag{3.8}$$

It is also easy to see that

$$\begin{cases} w(x) = O(\mathrm{e}^{-c_1|x|}) & \text{as} \quad x \to -\infty, \\ w(x) = \dfrac{\pi}{2} + O(\mathrm{e}^{-c_1|x|}) & \text{as} \quad x \to +\infty, \end{cases} \tag{3.9}$$

where c_1 is a positive constant. Fix $t \in (0, \frac{\pi}{2})$, we define

$$w_t(x) = \begin{cases} 0, & 0 < x < t - 2\delta_0, \\ w\Big(\dfrac{x-t}{\varepsilon}\Big), & t - \delta_0 < x < t + \delta_0, \\ \dfrac{\pi}{2}, & t + 2\delta_0 < x < \dfrac{\pi}{2}, \end{cases} \tag{3.10}$$

where $\delta_0 > 0$ is a small constant independent of ε. Because of (3.9), we may use smooth cut-off functions to define $w_t(x)$ for $x \in [t - 2\delta_0, t - \delta_0] \cup [t + \delta_0, t + 2\delta_0]$ such that

$$w_t(x) = w\Big(\frac{x-t}{\varepsilon}\Big) + O\Big(\mathrm{e}^{-\frac{\delta_0}{\varepsilon}}\mathrm{e}^{-\frac{c_2|x-t|}{\varepsilon}}\Big), \tag{3.11}$$

where c_2 is a positive constant.

4 Solutions of (2.8)

Let $\mu > 0$ be a fixed number. We shall use localized energy method to find solutions of (2.8) with the following asymptotic behavior

$$\lambda(r) = w_{t_\varepsilon,\varepsilon}(r) + \phi_\varepsilon(r), \quad \|\phi_\varepsilon\|_{L^\infty} = O(\varepsilon).$$

For references on localized energy method, we refer to Section 2.3 of [21].

To this end, we divide our proof into two steps:

Step I For each $t \in (0, \frac{\pi}{2})$, there exists a unique function $\phi_{\varepsilon,t}$ and a unique number $c_\varepsilon(t)$ such that $\lambda(r) = w_t(r) + \phi_{\varepsilon,t}(r)$ satisfying

$$\begin{cases} -\varepsilon^2\left[\lambda'' + \dfrac{2\cos 2r}{\sin 2r}\lambda' - \dfrac{2}{\sin^2 2r}(m^2\cos^2 r - n^2\sin^2 r)\sin 2\lambda\right] \\ + \sin 4\lambda - \varepsilon\mu\sin 2\lambda = c_\varepsilon(t) w'\Big(\dfrac{r-t}{\varepsilon}\Big), \\ \displaystyle\int_0^{\frac{\pi}{2}} w'\Big(\frac{r-t}{\varepsilon}\Big)\phi_{\varepsilon,t}(r)\mathrm{d}r = 0, \quad \phi_{\varepsilon,t}(0) = \phi_{\varepsilon,t}\Big(\frac{\pi}{2}\Big) = 0. \end{cases} \tag{4.1}$$

Step II There exists a constant $t_\varepsilon \in (0, \frac{\pi}{2})$ such that

$$c_\varepsilon(t_\varepsilon) = 0.$$

The proof of Step I relies on the following Lemma.

Lemma 4.1 Consider the following linearized problem

$$\begin{cases} -\varepsilon^2\left[\phi'' + \dfrac{2\cos 2r}{\sin 2r}\phi' - \dfrac{4}{\sin^2 2r}(m^2\cos^2 r - n^2\sin^2 r)(\cos 2w_t)\phi\right] \\ +4(\cos 4w_t)\phi - 2\varepsilon\mu(\cos 2w_t)\phi = h, \\ \displaystyle\int_0^{\frac{\pi}{2}} w'\Big(\frac{r-t}{\varepsilon}\Big)\phi(r)\mathrm{d}r = 0, \quad \phi(0) = 0, \quad \phi\Big(\frac{\pi}{2}\Big) = 0. \end{cases} \tag{4.2}$$

Then

$$\|\phi\|_{L^\infty(0,\frac{\pi}{2})} \le c\|h\|_{L^\infty(0,\frac{\pi}{2})}. \tag{4.3}$$

Furthermore,

$$|\phi(r)| \le C\|h\|_* \mathrm{e}^{-\sigma|\frac{r-t}{\varepsilon}|}, \quad \forall\, r \in \Big(0, \frac{\pi}{2}\Big), \tag{4.4}$$

where $\sigma \in (0,1)$ is a small number, C is a positive constant independent of ε, and $\|\cdot\|_*$ is defined by

$$\|h\|_* = \sup_{r\in(0,\frac{\pi}{2})} \mathrm{e}^{\sigma|\frac{r-t}{\varepsilon}|}|h(r)|, \quad \forall\, h \in L^\infty\Big(0, \frac{\pi}{2}\Big).$$

Proof First, we prove (4.3) by contradiction. Suppose that $\|h\|_{L^\infty(0,\frac{\pi}{2})} = o_\varepsilon(1)$ and $\|\phi\|_{L^\infty(0,\frac{\pi}{2})} = 1$, where $o_\varepsilon(1)$ is a small quantity tending to zero as ε goes to zero. Let $r_\varepsilon \in (0, \frac{\pi}{2})$ such that $\phi(r_\varepsilon) = \|\phi\|_{L^\infty(0,\frac{\pi}{2})} = 1$. If r_ε is close to zero, then

$$\begin{cases} \phi''(r_\varepsilon) < 0, \quad (m^2 \cos^2 r_\varepsilon - n^2 \sin^2 r_\varepsilon)(\cos 2w_t(r_\varepsilon))\phi(r_\varepsilon)) > 0, \\ \phi'(r_\varepsilon) = 0. \end{cases} \tag{4.5}$$

Consequently, by (4.5) and the equation of (4.2), we have

$$4(\cos 4w_t(r_\varepsilon))\phi(r_\varepsilon) - 2\varepsilon\mu(\cos 2w_t(r_\varepsilon))\phi(r_\varepsilon) \le h(r_\varepsilon) = o_\varepsilon(1),$$

which is impossible. Similarly, if r_ε is close to $\frac{\pi}{2}$, we may also get a contradiction. Hence by (3.11), r_ε must be close to t. In fact, the same argument as above may show that

$$|r_\varepsilon - t| \le c\,\varepsilon, \tag{4.6}$$

where c is a positive constant. Let $r_\varepsilon = t + \varepsilon y_\varepsilon$. Then (4.6) implies $|y_\varepsilon| \le c$ so due to notation convenience, we may assume $y_\varepsilon \to y_0$ as $\varepsilon \to 0+$.

Now, we rescale the variable by setting $r = t + \varepsilon y$ and $\tilde{\phi}_\varepsilon(y) := \phi(t + \varepsilon y)$. Then by (4.2), we obtain $\tilde{\phi}_\varepsilon(y) \to \phi_0(y)$ as $\varepsilon \to 0+$, where ϕ_0 satisfies

$$-\phi_0'' + 4(\cos 4w)\phi_0 = 0 \quad \text{in } \mathbb{R}, \tag{4.7}$$

and

$$\int_{\mathbb{R}} \phi_0 w' \mathrm{d}y = 0. \tag{4.8}$$

By (4.7) and Lemma 3.1, we have $\phi_0(y) = c_*\, w'(y)$ and hence by (4.8), $c_* = 0$, i.e., $\phi_0 \equiv 0$. However, $1 = \phi(r_\varepsilon) = \tilde{\phi}_\varepsilon(y_\varepsilon) \to \phi_0(y_0)$, i.e., $\phi_0(y_0) = 1$. Therefore, we get a contradiction and complete the proof of (4.3).

To prove (4.4), we notice that the function $\|h\|_*\, \mathrm{e}^{-\sigma|\frac{r-t}{\varepsilon}|}$ is a supersolution of (4.2) for $|r - t| \ge \varepsilon R$, provided $\sigma > 0$ sufficiently small, where R is a positive constant independent of ε. Here we have used the fact that

$$|h(r)| \le \|h\|_* \mathrm{e}^{-\sigma|\frac{r-t}{\varepsilon}|}, \quad \forall\, r \in \Big(0, \frac{\pi}{2}\Big).$$

Moreover, $C\|h\|_*\, \mathrm{e}^{-\sigma|\frac{r-t}{\varepsilon}|}$ is a supersolution of (4.2) for $0 < r < \frac{\pi}{2}$, where C is a positive constant independent of ε. Then (4.4) follows from comparison principle.

Let us define

$$\begin{cases} \|\phi\|_* = \sup\limits_{r\in(0,\frac{\pi}{2})} \mathrm{e}^{\sigma|\frac{r-t}{\varepsilon}|}|\phi(r)|, \\ \|h\|_{**} = \sup\limits_{r\in(0,\frac{\pi}{2})} \mathrm{e}^{\sigma|\frac{r-t}{\varepsilon}|}|h(r)|. \end{cases} \tag{4.9}$$

Then, by Lemma 4.1 and a contraction mapping principle (see our earlier papers Phy. D, JMP), we have

Proposition 4.2 For each $t \in (0, \frac{\pi}{2})$, there exists $(\phi_{\varepsilon,t}, c_\varepsilon(t))$ a unique solution of (4.1) such that

$$\|\phi_{\varepsilon,t}\|_* \le K\,\varepsilon, \tag{4.10}$$

where K is a positive constant independent of ε. Moreover, the map $t \to \phi_{\varepsilon,t}$ is of C^2.

Now we proceed to Step II. We first expand $c_\varepsilon(t)$ as follows:

Lemma 4.3 As $\varepsilon \to 0+$, we have

$$\Big(\int_{\mathbb{R}}(w'(y))^2\mathrm{d}y\Big)c_\varepsilon(t) = -2\cot(2t)\varepsilon\int_{\mathbb{R}}(w'(y))^2\mathrm{d}y - \varepsilon\mu + O(\varepsilon^2). \tag{4.11}$$

The proof of Lemma 4.3 is simple: we just multiply (4.1) by $w'(y)$ and integrate it over $\mathbb{R}$. Using $r = t + \varepsilon\, y$ and integrate by parts, we may obtain (4.11).

By Proposition 4.2 and Lemma 4.3, we may derive the following main result of this section.

Theorem 4.4 For each $\mu \in \mathbb{R}$, there exists a solution $u_{\varepsilon,\mu}$ to (2.8) with the following properties

$$u_{\varepsilon,\mu}(r) = w\Big(\frac{r - t_{\varepsilon,\mu}}{\varepsilon}\Big) + O(\varepsilon \mathrm{e}^{-\sigma|\frac{r-t_{\varepsilon,\mu}}{\varepsilon}|}), \tag{4.12}$$

and

$$E_\varepsilon\,(u_{\varepsilon,\mu}) = O(\varepsilon),$$

where

$$t_{\varepsilon,\mu} = t_0 + O(\varepsilon), \tag{4.13}$$

and $t_0 \in (0, \pi/2)$ satisfies

$$2\cot(2t_0)\int_{\mathbb{R}}(w'(y))^2\mathrm{d}y = -\mu. \tag{4.14}$$

5 Spectrum Estimates

In this section, we estimate the spectrum of the following linearized problem

$$\begin{cases} -\varepsilon^2\Big[\phi'' + \dfrac{2\cos 2r}{\sin 2r}\phi' - \dfrac{4}{\sin^2 2r}(m^2\cos^2 r - n^2\sin^2 r)(\cos 2u_{\varepsilon,\mu})\phi\Big] \\ +4(\cos 4u_{\varepsilon,\mu})\phi - 2\varepsilon\mu(\cos 2u_{\varepsilon,\mu})\phi = \lambda_\varepsilon\phi, \quad \forall r \in (0, \pi/2), \\ \phi(0) = 0, \quad \phi\Big(\dfrac{\pi}{2}\Big) = 0, \end{cases} \tag{5.1}$$

where $u_{\varepsilon,\mu}$ is the solution (defined in Theorem 4.4) of (2.8) satisfying (4.12) and (4.14). Our main result is the following.

Theorem 5.1 For ε sufficiently small, $\lambda_{\varepsilon,j}, j=1,2$, the first and the second eigenvalues of (5.1) satisfy

$$\lambda_{\varepsilon,1} = -4\varepsilon^2 \csc^2(2t_0) + o(\varepsilon^2), \qquad \lambda_{\varepsilon,2} \geq \delta_0 > 0,$$

where t_0 and δ_0 are positive constants.

Proof Without loss of generality, we may assume $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$ for $j=1,2$. Then, by (4.12) and (5.1), λ_0's satisfy

$$-\phi_0'' + 4(\cos 4w)\phi_0 = \lambda_0\, \phi_0 \quad \text{in } \mathbb{R}, \tag{5.2}$$

where $\phi_0(y) = \lim\limits_{\varepsilon\to 0} \phi(t_{\varepsilon,\mu} + \varepsilon y)$ for $y \in \mathbb{R}$. Hence (5.2) and Lemma 3.1 imply that either $\lambda_0 = 0$ having the associated eigenfunction $\phi_0 = cw'$ or $\lambda_0 \geq 2\delta_0 > 0$, where $\delta_0 > 0$ and c are suitable constants.

To complete the proof, we only need to concentrate on the eigenvalues λ_ε's with $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us decompose

$$\phi(r) = w'(y) + \phi^\perp(r), \quad \forall r = t_{\varepsilon,\mu} + \varepsilon\, y \in (0, \pi/2), \tag{5.3}$$

where $\phi^\perp$ satisfies

$$\int_0^{\frac{\pi}{2}} \phi^\perp(r) w'\Big(\frac{r - t_{\varepsilon,\mu}}{\varepsilon}\Big) \mathrm{d}r = 0. \tag{5.4}$$

Then (5.1) and (5.3) give

$$\begin{aligned} &-\varepsilon^2\Big[{\phi^\perp}'' + \frac{2\cos 2r}{\sin 2r}{\phi^\perp}' - \frac{4}{\sin^2 2r}(m^2\cos^2 r - n^2 \sin^2 r)(\cos 2u_{\varepsilon,\mu})\phi^\perp\Big] \\ &+4(\cos 4u_{\varepsilon,\mu})\phi^\perp - 2\varepsilon\mu(\cos 2u_{\varepsilon,\mu})\phi^\perp - \lambda_\varepsilon \phi^\perp = E_\varepsilon, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} E_\varepsilon = {}& w''' + \varepsilon\frac{2\cos 2r}{\sin 2r}w'' - \frac{4\varepsilon^2}{\sin^2 2r}(m^2\cos^2 r - n^2\sin^2 r)(\cos 2u_{\varepsilon,\mu})w' \\ &-4(\cos 4u_{\varepsilon,\mu})w' + 2\varepsilon\mu(\cos 2u_{\varepsilon,\mu})w' + \lambda_\varepsilon w'. \end{aligned} \tag{5.6}$$

Setting $r = t_{\varepsilon,\mu} + \varepsilon\, y$ and using (3.1) and (4.12), it is easy to get the following estimate

$$E_\varepsilon = O\left((\varepsilon + |\lambda_\varepsilon|)\mathrm{e}^{-2\sigma|\frac{r-t_{\varepsilon,\mu}}{\varepsilon}|}\right), \tag{5.7}$$

where σ is a positive constant independent of ε. By the same proof as in Lemma 4.1, we have

$$\phi^\perp = O\left((\varepsilon + |\lambda_\varepsilon|)\mathrm{e}^{-\sigma|\frac{r-t_{\varepsilon,\mu}}{\varepsilon}|}\right). \tag{5.8}$$

Now we expand $\phi_{\varepsilon,t_{\varepsilon,\mu}}(r) = u_{\varepsilon,\mu}(r) - w(\frac{r-t_{\varepsilon,\mu}}{\varepsilon})$. By Theorem 4.4, it is easy to see that

$$\phi_{\varepsilon,t_{\varepsilon,\mu}}(r) = \varepsilon\phi_1\Big(\frac{r-t_{\varepsilon,\mu}}{\varepsilon}\Big) + O(\varepsilon^2),$$

where $\phi_1 = \phi_1(y)$ satisfies

$$\begin{cases} -\phi_1'' + 4(\cos 4w)\phi_1 - 2(\cot 2t_0)w' - \mu \sin 2w = 0, \quad \forall\, y \in \mathbb{R}, \\ \phi_1(\pm\infty) = 0. \end{cases} \tag{5.9}$$

Note that $\sin 2w$ and w' are even functions. So ϕ_1 is also even. Consequently,

$$\int_{\mathbb{R}} \phi_1' w' \mathrm{d}y = 0, \tag{5.10}$$

$$\int_{\mathbb{R}} (\cos 2w)\phi_1 w' \mathrm{d}y = 0. \tag{5.11}$$

We may multiply (4.1) (with $t = t_{\varepsilon,\mu}$) by w' and integrate to y-variable. Then by (5.10) and (5.11), we obtain

$$\begin{aligned} & -2\varepsilon \int_{\mathbb{R}} (\cot 2r){w'}^2 \mathrm{d}y - 2\varepsilon^2 \int_{\mathbb{R}} (\cot 2r)\phi_1' w' \mathrm{d}y \\ & + \int_{\mathbb{R}} \frac{2\varepsilon^2}{\sin^2 2r}(m^2 \cos^2 r - n^2 \sin^2 r)(\sin 2w) w' \mathrm{d}y \\ & -\varepsilon\mu \int_{\mathbb{R}} (\sin (2w + 2\varepsilon\phi_1)) w' \mathrm{d}y + o(\varepsilon^2) = 0, \end{aligned} \tag{5.12}$$

where $r = t_{\varepsilon,\mu} + \varepsilon\, y$. Here we have used the fact that $c_\varepsilon(t_{\varepsilon,\mu}) = 0$. Note that

$$\int_{\mathbb{R}} (\sin 2w) w' \mathrm{d}y = -\frac{1}{2}\cos 2w\Big|_{-\infty}^{+\infty} = \frac{1}{2}\cos 0 - \frac{1}{2}\cos 2\cdot\frac{\pi}{2} = 1. \tag{5.13}$$

Hence (5.10)–(5.13) give

$$-2\,(\cot 2t_{\varepsilon,\mu}) \int_{\mathbb{R}} {w'}^2 \mathrm{d}y - \mu + \frac{2\varepsilon}{\sin^2(2t_{\varepsilon,\mu})}(m^2 \cos^2\, t_{\varepsilon,\mu} - n^2 \sin^2\, t_{\varepsilon,\mu}) + o(\varepsilon) = 0. \tag{5.14}$$

Let $t_{\varepsilon,\mu} = t_0 + \varepsilon t_1 + o(\varepsilon)$. Then by (4.14) and Taylor expansion on (5.14), we have

$$(4 \csc^2 2t_0) t_1 \int_{\mathbb{R}} {w'}^2 \mathrm{d}y = \frac{2}{\sin^2 2t_0}(m^2 \cos^2 t_0 - n^2 \sin^2 t_0). \tag{5.15}$$

It is clear to see that

$$\begin{aligned} & 2(\cot 2(t_{\varepsilon,\mu} + \varepsilon y))(w' + \varepsilon\phi_1') \\ = {} & 2(\cot 2t_{\varepsilon,\mu})w' + 2\varepsilon(\cot 2t_0)\phi_1' - 4\varepsilon(\csc^2 2t_0) y w' + o(\varepsilon) \\ = {} & 2(\cot 2t_0)w' - 4\varepsilon(\csc^2 2t_0)x_1 w' + 2\varepsilon(\cot 2t_0)\phi_1' - 4\varepsilon(\csc^2 2x_0) y w' + o(\varepsilon). \end{aligned} \tag{5.16}$$

Let $\phi_{\varepsilon,t_{\varepsilon,\mu}}(r) = \varepsilon\phi_1(y) + \varepsilon^2\phi_2(y) + O(\varepsilon^3)$, where $y = \frac{r - t_{\varepsilon,\mu}}{\varepsilon}$. Then by (4.1) with $t = t_{\varepsilon,\mu}$, (5.9) and (5.16), ϕ_2 satisfies

$$\begin{cases} -\phi_2'' + 4(\cos 4w)\phi_2 + 4(\csc^2 2t_0) t_1 w' + 4(\csc^2 2t_0) y w' \\ + \dfrac{2}{\sin^2 2t_0}(m^2 \cos^2 t_0 - n^2 \sin^2 t_0) \sin 2w \\ -8(\sin 4w)\phi_1^2 - 2\mu(\cos 2w)\phi_1 - 2(\cot 2t_0)\phi_1' = 0 \quad \text{in } \mathbb{R}. \end{cases} \tag{5.17}$$

Here we have used the fact that $c_\varepsilon(t_{\varepsilon,\mu}) = 0$. Since w' solves $\phi'' = 4(\cos 4w)\phi$ in $\mathbb{R}$, then we may assume

$$\int_{\mathbb{R}} \phi_2 w' \mathrm{d}y = 0. \tag{5.18}$$

Similarly, we may expand

$$\phi^{\perp}(r) = \varepsilon\phi_1^{\perp}(y) + \varepsilon^2\phi_2^{\perp}(y) + o(\varepsilon^2), \tag{5.19}$$

and

$$\lambda_\varepsilon = \varepsilon\,\lambda_1 + \varepsilon^2\,\lambda_0 + o(\varepsilon^2), \tag{5.20}$$

where λ_j's are constants and $\phi_j^{\perp}$'s are functions independent of ε such that

$$\int_{\mathbb{R}} \phi_j^{\perp} w' \mathrm{d}y = 0, \quad j = 1, 2. \tag{5.21}$$

Here $\phi_1^{\perp} = \phi_1^{\perp}(y)$ satisfies

$$\begin{cases} -{\phi_1^{\perp}}'' + 4(\cos 4w)\phi_1^{\perp} = 2(\cot 2t_0)w'' + 2\mu(\cos 2w)w' + 16(\sin 4w)w'\phi_1 + \lambda_1\,w' & \text{in } \mathbb{R}, \\ \phi_1^{\perp}(\pm\infty) = 0. \end{cases} \tag{5.22}$$

Since $\int_{\mathbb{R}} \phi_1^{\perp} w'\mathrm{d}y = 0$ and w' solves $-\phi'' + 4(\cos 4w)\phi = 0$ in $\mathbb{R}$, then by (5.22), we have $\lambda_1 = 0$. Consequently, (5.21) becomes

$$\lambda_\varepsilon = \varepsilon^2\,\lambda_0 + o(\varepsilon^2), \tag{5.23}$$

and (5.22) becomes

$$\begin{cases} -{\phi_1^{\perp}}'' + 4(\cos 4w)\phi_1^{\perp} = 2(\cot 2t_0)w'' + 2\mu(\cos 2w)w' + 16(\sin 4w)w'\phi_1 & \text{in } \mathbb{R}, \\ \phi_1^{\perp}(\pm\infty) = 0. \end{cases} \tag{5.24}$$

By (5.9), it is easy to check that $\phi_1'(y)$ satisfies (5.24). Thus $\phi_1^{\perp}$ can be written as

$$\phi_1^{\perp} = \phi_1' + cw', \tag{5.25}$$

where $c = -\frac{\int_{\mathbb{R}} \phi_1' w' \mathrm{d}y}{\int_{\mathbb{R}} w'^2 \mathrm{d}y}$. Since w' and ϕ_1 are even functions, then $\int_{\mathbb{R}} \phi_1' w'\mathrm{d}y = 0$, i.e., $c = 0$. Consequently, (5.25) becomes

$$\phi_1^{\perp} = \phi_1'. \tag{5.26}$$

Substituting (5.19) and (5.23) into (5.5), we have

$$-{\phi_2^{\perp}}'' + 4(\cos 4w)\phi_2^{\perp} = E_\varepsilon + E_{\varepsilon,2}, \tag{5.27}$$

where

$$\begin{aligned} -E_{\varepsilon,2} &= \varepsilon^{-1}\Big[-({\phi_1^{\perp}}'' + 2\varepsilon(\cot 2r){\phi_1^{\perp}}' - \frac{4\varepsilon^2}{\sin^2 2r}(m^2\cos^2 r - n^2\sin^2 r)(\cos 2u_{\varepsilon,\mu})\phi_1^{\perp}) \\ &\qquad +4(\cos 4(w+\varepsilon\phi_1))\phi_1^{\perp} - 2\varepsilon\mu(\cos 2(w+\varepsilon\phi_1))\phi_1^{\perp}\Big] + o(1) \\ &= -2(\cot 2t_0){\phi_1^{\perp}}' - 16(\sin 4w)\phi_1\phi_1^{\perp} - 2\mu(\cos 2w)\phi_1^{\perp} + o(1), \end{aligned} \tag{5.28}$$

$$\begin{aligned} E_\varepsilon &= -4(\csc^2 2t_0)(t_1+y)w'' - \frac{4}{\sin^2 2t_0}(m^2\cos^2 t_0 - n^2\sin^2 t_0)(\cos 2w)w' \\ &\qquad +32(\cos 4w)\phi_1^2 w' - 4\mu(\sin 2w)\phi_1 w' + \lambda_0 w' + 16(\sin 4w)\phi_2 w' + o(1). \end{aligned} \tag{5.29}$$

Multiply (5.27) by w' and integrate it over $\mathbb{R}$. Then we obtain

$$\begin{aligned}&\lambda_0\int_{\mathbb{R}} {w'}^2-4\mu\int_{\mathbb{R}}(\sin 2w)\phi_1 {w'}^2+32\int_{\mathbb{R}}(\cos 4w)\phi_1^2{w'}^2+16\int_{\mathbb{R}}(\sin 4w)\phi_2{w'}^2\\&-4\csc^2 2t_0\int_{\mathbb{R}} yw''w'-\frac{4}{\sin^2 2x_0}(m^2\cos^2 t_0-n^2\sin^2 t_0)\int_{\mathbb{R}}(\cos 2w){w'}^2\\&+2\cot 2t_0\int_{\mathbb{R}} w'{\phi_1^{\perp}}'+16\int_{\mathbb{R}}(\sin 4w)\phi_1\phi_1^{\perp}w'+2\mu\int_{\mathbb{R}}(\cos 2w)\phi_1^{\perp}w'=0.\end{aligned}\tag{5.30}$$

Here we have used integrating by parts. Since $\cos 2w$ is odd and w' is even, then

$$\int_{\mathbb{R}}(\cos 2w){w'}^2=\int_{\mathbb{R}}(\sin 2w)w''=0.\tag{5.31}$$

Using integration by part, we obtain

$$\int_{\mathbb{R}} yw''w'=-\frac{1}{2}\int_{\mathbb{R}}{w'}^2.\tag{5.32}$$

By (5.26) and integration by part, we have

$$\begin{aligned}16\int(\sin 4w)\phi_1\phi_1^{\perp}w'+32\int(\cos 4w)\phi_1^2{w'}^2&=16\int(\sin 4w)\phi_1\phi_1'w'+32\int(\cos 4w)\phi_1^2{w'}^2\\&=8\int(\sin 4w)(\phi_1^2)'w'+32\int(\cos 4w)\phi_1^2{w'}^2\\&=-8\int(\sin 4w)\phi_1^2w''.\end{aligned}\tag{5.33}$$

Since $w''=\sin 4w$ in $\mathbb{R}$, then

$$-w^{(4)}+4(\cos 4w)w''-16(\sin 4w){w'}^2=0\quad\text{in}\quad\mathbb{R}.\tag{5.34}$$

Multiplying (5.34) by ϕ_2, we may use (5.17) and integration by part to get

$$\begin{aligned}-16\int_{\mathbb{R}}(\sin 4w){w'}^2\phi_2&=4(\csc^2 2t_0)\int_{\mathbb{R}} yw'w''+\frac{2}{\sin^2 2t_0}(m^2\cos^2 t_0-n^2\sin^2 t_0)\int_{\mathbb{R}}(\sin 2w)w''\\&\quad-8\int_{\mathbb{R}}(\sin 4w)\phi_1^2w''-2\mu\int_{\mathbb{R}}(\cos 2w)\phi_1w''-2(\cot 2t_0)\int_{\mathbb{R}}\phi_1'w''.\end{aligned}\tag{5.35}$$

Substituting (5.31)–(5.33) and (5.35) into (5.30), we obtain

$$\begin{aligned}&\lambda_0\int_{\mathbb{R}}{w'}^2-4\mu\int_{\mathbb{R}}(\sin 2w)\phi_1{w'}^2+4(\csc^2 2t_0)\int_{\mathbb{R}}{w'}^2+2(\cot 2t_0)\int_{\mathbb{R}} w'{\phi_1^{\perp}}'\\&+2\mu\int_{\mathbb{R}}(\cos 2w)\phi_1^{\perp}w'+2(\cot 2t_0)\int_{\mathbb{R}}\phi_1'w''+2\mu\int_{\mathbb{R}}(\cos 2w)\phi_1w''=0.\end{aligned}\tag{5.36}$$

On the other hand, using integration by part, it is obvious that

$$2\mu\int_{\mathbb{R}}(\cos 2w)\phi_1w''=4\mu\int_{\mathbb{R}}(\sin 2w)\phi_1{w'}^2-2\mu\int_{\mathbb{R}}(\cos 2w)\phi_1'w'.\tag{5.37}$$

Thus, by (5.26), (5.36) and (5.37), we have

$$\begin{aligned}\lambda_0\int_{\mathbb{R}}{w'}^2&=-4(\csc^2 2t_0)\int_{\mathbb{R}}{w'}^2-2(\cot 2t_0)\int_{\mathbb{R}} w'\phi_1''-2(\cot 2t_0)\int_{\mathbb{R}}\phi_1'w''\\&=-4(\csc^2 2t_0)\int_{\mathbb{R}}{w'}^2-2(\cot 2t_0)\int_{\mathbb{R}}(\phi_1'w')'=-4(\csc^2 2t_0)\int_{\mathbb{R}}{w'}^2,\end{aligned}$$

i.e., $\lambda_0=-4\csc^2 2t_0$. Therefore we may complete the proof of Theorem 5.1.

6 Local Minimizers of E_ε

Let $\mu \in \mathbb{R}$ and $u_{\varepsilon,\mu}$ be the solution constructed in Section 4 (see Theorem 4.4). We first have

Lemma 6.1 For ε sufficiently small, $u_{\varepsilon,\mu}$ is locally unique and nondegenerate. As a result, $u_{\varepsilon,\mu}$ is continuous in μ.

Proof Since the spectrum of the linearized problem (5.1) with respect to $u_{\varepsilon,\mu}$ is non-zero, then the uniqueness follows from the same proof in [20]. Moreover, $u_{\varepsilon,\mu}$ is locally unique, i.e., if there exists another solution $\hat{u}_{\varepsilon,\mu} \sim w\big(\frac{x-\hat{t}_{\varepsilon,\mu}}{\varepsilon}\big)$, $\hat{t}_{\varepsilon,\mu} = t_0 + o(1)$, then

$$\hat{u}_{\varepsilon,\mu} \equiv u_{\varepsilon,\mu}.$$

The continuity follows from the uniqueness.

By (4.12), (4.13), and (4.14), we may obtain

$$\begin{aligned}
\rho(\mu) :=& \int_0^{\frac{\pi}{2}} (\sin 2r)\, \sin^2 u_{\varepsilon,\mu} \mathrm{d}r \\
=& \int_0^{t_{\varepsilon,\mu}} (\sin 2r) \sin^2 u_{\varepsilon,\mu} \mathrm{d}r + \int_{t_{\varepsilon,\mu}}^{\frac{\pi}{2}} (\sin 2r)(\sin^2 u_{\varepsilon,\mu} - 1)\mathrm{d}r + \int_{t_{\varepsilon,\mu}}^{\frac{\pi}{2}} \sin 2r \mathrm{d}r \\
=& \int_0^{t_{\varepsilon,\mu}} (\sin 2r) \sin^2 u_{\varepsilon,\mu} \mathrm{d}r + \int_{t_{\varepsilon,\mu}}^{\frac{\pi}{2}} (\sin 2r)(\sin^2 u_{\varepsilon,\mu} - 1)\mathrm{d}r + \Big(-\frac{1}{2}\cos 2r\Big)\Big|_{t_{\varepsilon,\mu}}^{\frac{\pi}{2}} \\
=& \frac{1}{2}(1+\cos 2t_0) + O(\varepsilon),
\end{aligned}$$

i.e.,

$$\rho(\mu) = \frac{1}{2}(1+\cos 2t_0) + O(\varepsilon), \tag{6.1}$$

where $2(\cot 2t_0)\int_{\mathbb{R}} w'^2 = -\mu$. Due to the continuity of $u_{\varepsilon,\mu}$ in μ, $\rho(\mu)$ is continuous in μ. Furthermore, by Mean-Value Theorem, there exists $\mu_\varepsilon \in \mathbb{R}$ such that $\rho(\mu_\varepsilon) = c_1 \in (0,1)$, i.e., (2.6) holds, provided $\lambda = u_{\varepsilon,\mu_\varepsilon}$ and

$$\frac{1}{2}(1+\cos 2t_0) = c_1. \tag{6.2}$$

Hence $u_{\varepsilon,\mu_\varepsilon}$ is a critical point of the energy functional $E_\varepsilon(\cdot)$ under the constraint (2.6).

Now, we want to show that $u_{\varepsilon,\mu_\varepsilon}$ is a local minimizer of the energy functional $E_\varepsilon(\cdot)$ under the constraint (2.6). We consider the associated quadratic form as follows:

$$\begin{aligned}
Q[\psi] :=& E''_\varepsilon(u_{\varepsilon,\mu_\varepsilon})[\psi] \\
=& \int_0^{\frac{\pi}{2}} \Big[\varepsilon^2(\sin 2r)|\psi'|^2 + \varepsilon^2\frac{m^2 \sin 2r}{\sin^2 r}(\cos 2u_{\varepsilon,\mu_\varepsilon})\psi^2 \\
&+\varepsilon^2\frac{n^2\sin 2r}{\cos^2 r}(-\cos 2u_{\varepsilon,\mu_\varepsilon})\psi^2 + (\sin 2r)(4\cos 4u_{\varepsilon,\mu_\varepsilon})\psi^2\Big]\mathrm{d}r,
\end{aligned} \tag{6.3}$$

for $\psi \in H_0^1((0,\pi/2))$ with the following constraint

$$\int_0^{\frac{\pi}{2}} (\sin 2r)(\sin 2u_{\varepsilon,\mu_\varepsilon})\psi \mathrm{d}r = 0. \tag{6.4}$$

Let

$$\tilde{Q}[\psi] = Q[\psi] - 2\varepsilon\mu_\varepsilon \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})\psi^2 \mathrm{d}r, \tag{6.5}$$

and let

$$\psi = c_1\psi_1(r) + \psi_2(r), \quad \psi_j \in H_0^1((0,\pi/2)), \quad j = 1,2, \tag{6.6}$$

such that

$$\int_0^{\frac{\pi}{2}} (\sin 2r)\psi_1\psi_2 \mathrm{d}r = 0, \tag{6.7}$$

where $c_1 \in \mathbb{R}$ is a constant and ψ_1 is the eigenfunction corresponding to the first eigenvalue $\lambda_{\varepsilon,1}$ defined in Theorem 5.1. Then using (5.1), (6.5), (6.7) and integration by parts, we have

$$\tilde{Q}[\psi] = c_1^2\tilde{Q}[\psi_1] + \tilde{Q}[\psi_2] = c_1^2\lambda_{\varepsilon,1} \int_0^{\frac{\pi}{2}} (\sin 2r)\, \psi_1^2 \mathrm{d}r + \tilde{Q}[\psi_2]. \tag{6.8}$$

On the other hand, (6.4) and (6.6) imply

$$c_1 \int_0^{\frac{\pi}{2}} (\sin 2r)(\sin 2u_{\varepsilon,\mu_\varepsilon})\psi_1 \mathrm{d}r + \int_0^{\frac{\pi}{2}} (\sin 2r)(\sin 2u_{\varepsilon,\mu_\varepsilon})\psi_2 \mathrm{d}r = 0. \tag{6.9}$$

From the proof of Theorem 5.1, we obtain

$$\psi_1(r) = w'(y) + O(\varepsilon), \quad u_{\varepsilon,\mu_\varepsilon} = w(y) + O(\varepsilon), \quad r = t_{\varepsilon,\mu_\varepsilon} + \varepsilon\, y. \tag{6.10}$$

Hence (6.9) and (6.10) give

$$c_1 = O\Big(\varepsilon^{-1} \int_0^{\frac{\pi}{2}} (\sin 2r)|\psi_2| \mathrm{d}r\Big). \tag{6.11}$$

Moreover, by (6.10), (6.11) and Hölder inequality, we obtain

$$\left| c_1^2\varepsilon^2 \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_1^2 \mathrm{d}r \right| \le C\, \varepsilon \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r, \tag{6.12}$$

where C is a positive constant independent of ε. Besides, (6.6) and (6.10) imply

$$\begin{aligned}
&\varepsilon\mu_\varepsilon \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})\psi^2 \mathrm{d}r \\
&= \varepsilon\mu_\varepsilon \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})(c_1^2\psi_1^2 + 2c_1\psi_1\psi_2 + \psi_2^2) \mathrm{d}r \\
&= 2\varepsilon\mu_\varepsilon\, c_1 \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})\psi_1\psi_2 \mathrm{d}r + \varepsilon\mu_\varepsilon \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})(c_1^2\psi_1^2 + \psi_2^2) \mathrm{d}r \\
&= 2\varepsilon\mu_\varepsilon\, c_1 \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})\psi_1\psi_2 \mathrm{d}r + O\Big(\varepsilon \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r\Big),
\end{aligned}$$

i.e.,

$$\begin{aligned}
\varepsilon\mu_\varepsilon \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})\psi^2 \mathrm{d}r = 2\varepsilon\mu_\varepsilon\, c_1 \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})\psi_1\psi_2 \mathrm{d}r \\
+ O\Big(\varepsilon \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r\Big).
\end{aligned} \tag{6.13}$$

Here we have used (6.11) and the fact that

$$\int_{\mathbb{R}} (\cos 2w) {w'}^2 \mathrm{d}y = 0.$$

By (6.10), (6.11) and Hölder inequality, we obtain

$$\begin{aligned}
&\Big|\varepsilon\mu_\varepsilon\, c_1 \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})\psi_1\psi_2 \mathrm{d}r\Big| \\
&\le |\mu_\varepsilon|\, |\varepsilon\, c_1| \Big(\int_0^{\frac{\pi}{2}} (\sin 2r)\psi_1^2 \mathrm{d}r\Big)^{\frac12} \Big(\int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r\Big)^{\frac12} \\
&\le C\Big(\int_0^{\frac{\pi}{2}} (\sin 2r)\psi_1^2 \mathrm{d}r\Big)^{\frac12} \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r \\
&\le C\sqrt{\varepsilon} \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r,
\end{aligned}$$

i.e.,

$$\Big|\varepsilon\mu_\varepsilon\, c_1 \int_0^{\frac{\pi}{2}} (\sin 2r)(\cos 2u_{\varepsilon,\mu_\varepsilon})\psi_1\psi_2 \mathrm{d}r\Big| \le C\sqrt{\varepsilon} \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r, \tag{6.14}$$

where C is a positive constant independent of ε. Thus, by (6.5), (6.8), (6.13), and (6.14), we have

$$Q[\psi] \ge c_1^2 \lambda_{\varepsilon,1} \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_1^2 \mathrm{d}r + \tilde{Q}[\psi_2] - C\sqrt{\varepsilon} \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r. \tag{6.15}$$

Consequently, (6.12), (6.15) and Theorem 5.1 imply

$$Q[\psi] \ge \big(\delta_0 - C\sqrt{\varepsilon}\big) \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r \ge C^{-1}\frac{\delta_0}{2} \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r, \tag{6.16}$$

provided $\varepsilon > 0$ is sufficiently small. Since $\phi = c_1\phi_1 + \phi_2$, then by (6.7) and (6.12), we obtain

$$\int_0^{\frac{\pi}{2}} (\sin 2r)\psi^2 \mathrm{d}r = \int_0^{\frac{\pi}{2}} (\sin 2r)(c_1\psi_1 + \psi_2)^2 \mathrm{d}r \le C_\varepsilon \int_0^{\frac{\pi}{2}} (\sin 2r)\psi_2^2 \mathrm{d}r.$$

So (6.16) becomes

$$Q[\psi] \ge C_\varepsilon^{-1} \int_0^{\frac{\pi}{2}} (\sin 2r)\psi^2 \mathrm{d}r,$$

where C_ε is a positive constant which may depend on ε. We may summarize what have been proved as follows:

Theorem 6.2 There exists $u_{\varepsilon,\mu_\varepsilon}$ a local minimizer of $E_\varepsilon[\cdot]$ under the constraint (2.6).

7 Critical Points of $\widetilde{E}_{\delta,\varepsilon}$

In this section, we study critical points of the functional $\widetilde{E}_{\delta,\varepsilon}$ (defined in (2.15)) by solving equations (2.18) and (2.19). Now, we want to simplify these equations. Let $S_0 = S_0(t)$ be the unique solution of

$$\begin{cases} S_0'' + \dfrac{1}{t}S_0' - \dfrac{S_0}{t^2} + S_0 - S_0^3 = 0, \quad \forall t > 0, \\ S_0(0) = 0, \quad S_0(+\infty) = 1. \end{cases} \tag{7.1}$$

It is well known that

$$S_0(t) = t + O(t^3) \quad \text{for } t > 0 \text{ small}, \tag{7.2}$$

$$S_0(t) = 1 - \frac{1}{2t^2} + O\Big(\frac{1}{t^4}\Big) \quad \text{for } t \text{ large}. \tag{7.3}$$

One may refer to [4] and [6] for the solution S_0. Let (ρ, λ) be a solution of (2.18) and (2.19), where

$$\rho = s\,(r)\,\hat{\rho},$$

and s is a smooth function defined by

$$s(r) = \begin{cases} S_0\Big(\dfrac{r}{\delta}\Big) & \text{if} \quad 0 \le r \le \varepsilon, \\ S_0\Big(\dfrac{\frac{\pi}{2} - r}{\delta}\Big) & \text{if} \quad \dfrac{\pi}{2} - \varepsilon \le r \le \dfrac{\pi}{2}, \\ 1 & \text{if} \quad 2\varepsilon \le r \le \dfrac{\pi}{2} - 2\varepsilon, \\ \eta_1(r) & \text{if} \quad \varepsilon < r < 2\varepsilon \text{ or } \dfrac{\pi}{2} - 2\varepsilon < r < \dfrac{\pi}{2} - \varepsilon. \end{cases} \tag{7.4}$$

Here we assume that

$$0 < \varepsilon^2 \ll \delta \ll \varepsilon \ll 1, \tag{7.5}$$

and $\eta_1(r) \sim 1$ as $\delta \to 0$. It is clear that $s(0) = s(\frac{\pi}{2}) = 0$ so $\rho(0) = \rho(\frac{\pi}{2}) = 0$. Then (2.18) and (2.19) become

$$\begin{aligned} S_1[\lambda, \hat{\rho}] := &-\varepsilon^2 \Big[\lambda'' + \Big(\frac{2\cos 2r}{\sin 2r} + \frac{2s'}{s} + \frac{2\hat{\rho}'}{\hat{\rho}}\Big)\lambda'\Big] \\ &+ s^2 \hat{\rho}^2 \sin 4\lambda - \varepsilon^2 \left(\frac{2\cos 2r}{\sin^2 2r}\right) \sin 2\lambda - \varepsilon\mu \sin 2\lambda = 0, \end{aligned} \tag{7.6}$$

and

$$\begin{aligned} S_2[\lambda, \hat{\rho}] := &-\delta^2 \Big(\hat{\rho}'' + \frac{2s'}{s}\hat{\rho}' + \frac{2\cos 2r}{\sin 2r}\hat{\rho}'\Big) - \delta^2 \Big(\frac{2\cos 2r}{\sin 2r} - \frac{1}{r}\Big)\frac{s'}{s}\hat{\rho} \\ &+ (1 - \hat{\rho})\, s^2 + (\hat{\rho}^3 - 1)\, s^2 + \delta^2 \hat{\rho} \left[|\lambda'|^2 + \Big(\frac{\cos^2 \lambda}{\sin^2 r} - \frac{1}{r^2}\Big) + \frac{\sin^2 \lambda}{\cos^2 r}\right] \\ &+ \frac{2\delta^2}{\varepsilon^2}\, s^2 \hat{\rho}^3 \sin 2\lambda - \delta\mu\, \hat{\rho}\, \sin^2 \lambda = 0. \end{aligned} \tag{7.7}$$

Here we have used the fact

$$\begin{aligned} &\rho' = s'\hat{\rho} + s\hat{\rho}', \\ &\frac{\rho'}{\rho} = \frac{s'}{s} + \frac{\hat{\rho}'}{\hat{\rho}}, \\ &\rho'' = s''\hat{\rho} + 2s'\hat{\rho}' + s\hat{\rho}''. \end{aligned}$$

To fulfill (2.12), we require the boundary conditions as follows:

$$\begin{cases} \lambda(0) = 0, \quad \lambda\Big(\dfrac{\pi}{2}\Big) = \dfrac{\pi}{2}, \\ \hat{\rho}'(0) = 0, \quad \hat{\rho}\Big(\dfrac{\pi}{2}\Big) = 1. \end{cases} \tag{7.8}$$

As for (4.14) and (6.2), we may assume that

$$2(\cot 2t_0) \int_{\mathbb{R}} {w'}^2(y)\mathrm{d}y = -\mu, \tag{7.9}$$

and t_0 is the unique solution of

$$\frac{1}{2}(1+\cos 2t_0) = c_2. \tag{7.10}$$

We need the following lemma.

Lemma 7.1 The linear problem

$$\begin{cases} \phi'' + \dfrac{1}{t}\phi' - \dfrac{1}{t^2}\,\phi + \phi - 3S_0^2\phi = 0, \quad \forall\, t > 0, \\ \phi(0) = 0, \quad |\phi| \le C\,t, \quad \forall\, t > 0 \end{cases} \tag{7.11}$$

admits only zero solution, where C is a positive constant independent of t. Furthermore, the linear problem

$$\begin{cases} \phi'' + \dfrac{1}{t}\phi' - \dfrac{1}{t^2}\,\phi - 2S_0^2\phi = 0, \quad \forall\, t > 0, \\ \phi(0) = 0, \quad |\phi| \le C\,t, \quad \forall\, t > 0 \end{cases} \tag{7.12}$$

also admits only zero solution.

Proof Setting $\phi = t\psi$, then ψ satisfies

$$\begin{cases} \psi'' + \dfrac{3}{t}\psi' + (1-3S_0^2)\psi = 0, \quad \forall\, t > 0, \\ \psi'(0) = 0, \quad |\psi| \le C, \quad \forall\, t > 0. \end{cases} \tag{7.13}$$

Since $S_0(t) \to 1$ as $t \to +\infty$, we may use comparison principle on (7.13) to derive that

$$|\psi| \le C\mathrm{e}^{-t} \quad \text{for } t \text{ large},$$

which in turn implies that

$$\begin{cases} |\phi| \le C, \quad \forall\, t > 0, \\ |\phi| \le C\mathrm{e}^{-t/2} \quad \text{for } t \text{ large}. \end{cases} \tag{7.14}$$

Hence by (7.11), (7.14) and the result of [11] and [12], we obtain $\phi \equiv 0$. Similarly, letting ϕ satisfy (7.12) and $\psi = t\phi$, then ψ satisfies

$$\begin{cases} \psi'' + \dfrac{3}{t}\psi' - 2S_0^2\psi = 0, \quad \forall\, t > 0, \\ \psi'(0) = 0, \quad |\psi| \le C, \quad \forall\, t > 0. \end{cases} \tag{7.15}$$

By Maximum Principle, we conclude that (7.12) also has only zero solution. Therefore, we may complete the proof.

For $t > 0$, we define norms

$$\|\phi\|_{*,\varepsilon} = \sup_{r\in(0,\frac{\pi}{2})} \mathrm{e}^{\sigma|\frac{r-t}{\varepsilon}|}(|\phi(r)| + \varepsilon|\phi'(r)|), \tag{7.16}$$

$$\|\psi\|_{*,\delta}=\sup_{r\in(0,\frac{\pi}{2})}(|\psi(r)|+\delta|\psi'(r)|), \tag{7.17}$$

where σ is a small constant, and

$$\|h\|_{**,\varepsilon}=\sup_{r\in(0,\frac{\pi}{2})}\mathrm{e}^{\sigma|\frac{r-t}{\varepsilon}|}|h(r)|, \tag{7.18}$$

$$\|h\|_{**}=\sup_{r\in(0,\frac{\pi}{2})}|h(r)|. \tag{7.19}$$

As for the proof in Section 4, we may choose

$$\begin{cases}\lambda(r)=w_t(r)+\phi(r),\\ \hat{\rho}(r)=1+\psi(r),\end{cases} \tag{7.20}$$

for $t\in(t_0-\delta_1,t_0+\delta_1)$ and $r\in(0,\pi/2)$, where w_t is defined in (3.10) and δ_1 is a positive constant independent of ε and δ.

Now we follow three steps.

Step I For each $t\in(t_0-\delta_1,t_0+\delta_1)$, we find a unique pair $(\phi,\psi)=(\phi_t,\psi_t)$ such that

$$S_1[w_t+\phi_t,1+\psi_t]=c_\varepsilon(t)w'\Big(\frac{r-t}{\varepsilon}\Big), \tag{7.21}$$

$$S_2[w_t+\phi_t,1+\psi_t]=0, \tag{7.22}$$

with

$$\|\phi_t\|_{*,\varepsilon}\le C\varepsilon, \tag{7.23}$$

$$\|\psi_t\|_{*,\delta}\le C\Big(\frac{\delta^2}{\varepsilon^2}+\delta\Big). \tag{7.24}$$

Step II There exists $t_\varepsilon=t_0+O(\varepsilon)$ such that

$$c_\varepsilon(t_\varepsilon)=0. \tag{7.25}$$

Step III We show that as $\varepsilon\to0$ and $\delta/\varepsilon\to0$,

$$\int_0^{\frac{\pi}{2}}(\sin 2r)\rho^2\sin^2\lambda\,\mathrm{d}r\to c_2. \tag{7.26}$$

As in Section 4, the proof of Step I relies on the following lemma.

Lemma 7.2 Consider the following linearized equations

$$\begin{aligned}L_1[\phi,\psi]:=&-\varepsilon^2\Big[\phi''+\Big(\frac{2\cos 2r}{\sin 2r}+\frac{2s'}{s}\Big)\phi'\Big]-2\varepsilon^2\psi' w_t'+2(\sin 4w_t)\,s^2\psi\\ &+4(\cos 4w_t)\,s^2\phi-\frac{4\varepsilon^2\cos 2r}{\sin^2 2r}(\cos 2w_t)\phi-2\varepsilon\mu(\cos 2w_t)\phi=h_1,\end{aligned} \tag{7.27}$$

and

$$\begin{aligned}L_2[\phi,\psi]:=&-\delta^2\Big[\psi''+\Big(\frac{2s'}{s}+\frac{2\cos 2r}{\sin 2r}\Big)\psi'\Big]-\delta^2\Big(\frac{2\cos 2r}{\sin 2r}-\frac{1}{r}\Big)\frac{s'}{s}\psi\\ &+2s^2\psi+\delta^2\psi\Big[(w_t')^2+\Big(\frac{\cos^2 w_t}{\sin^2 r}-\frac{1}{r^2}\Big)+\frac{\sin^2 w_t}{\cos^2 r}\Big]\\ &+\frac{6\delta^2}{\varepsilon^2}(\sin 2w_t)\,s^2\,\psi-\delta\mu(\sin^2 w_t)\psi+2\delta^2\,w_t'\phi'\end{aligned} \tag{7.28}$$

$$+\Big[-\delta^2\frac{\sin 2w_t}{\sin^2 r}+\delta^2\frac{\sin 2w_t}{\cos^2 r}+4\frac{\delta^2}{\varepsilon^2}(\cos 2w_t)\,s^2-\delta\mu\,(\sin 2w_t)\Big]\phi=h_2, \tag{7.29}$$

where
$$\int_0^{\frac{\pi}{2}} w'\Big(\frac{r-t}{\delta}\Big)\,\phi(r)\mathrm{d}r = 0, \quad \phi(0) = \phi\left(\frac{\pi}{2}\right) = 0,$$
and
$$\psi(0) = \psi\left(\frac{\pi}{2}\right) = 0.$$
For any h_1 and $h_2 \not\equiv 0$, there exists (ϕ, ψ) a unique solution to (7.27) and (7.28) such that
$$\|\phi\|_{*,\varepsilon} \le C\|h_1\|_{**,\varepsilon} + C\tfrac{\varepsilon}{\delta}\|h_2\|_{**}, \tag{7.30}$$
$$\|\psi\|_{*,\delta} \le C\|h_2\|_{**} + C\tfrac{\delta^2}{\varepsilon^2}\|h_1\|_{**,\varepsilon}, \tag{7.31}$$
provided the assumption (7.5) holds, where C is a positive constant independent of ε and δ.

Proof Let
$$\begin{cases} \tilde h_1 = h_1 + 2\varepsilon^2 w_t'\,\psi' - 2(\sin 4w_t)\, s^2\psi, \\ \tilde h_2 = h_2 - 2\delta^2 w_t'\,\phi' - \left[-\delta^2\dfrac{\sin 2w_t}{\sin^2 r} + \delta^2\dfrac{\sin 2w_t}{\cos^2 r} + 4\dfrac{\delta^2}{\varepsilon^2}(\cos 2w_t)\, s^2 - \delta\mu\,(\sin 2w_t)\right]\phi. \end{cases} \tag{7.32}$$
Firstly, we may follow the proof of Lemma 4.1 to get
$$\|\phi\|_{L^\infty} \le C\|\tilde h_1\|_{L^\infty}. \tag{7.33}$$
Next, we prove that
$$\|\psi\|_{L^\infty} \le C\|\tilde h_2\|_{L^\infty}. \tag{7.34}$$
Suppose (7.34) fails. Then we may assume that $\|\tilde h_2\|_{L^\infty} = o(1)$ but $\|\psi\|_{L^\infty} = \psi(r_\delta) = 1$, where $0 < r_\delta < \frac{\pi}{2}$. If $r_\delta \le \frac{\pi}{2} - 2\varepsilon$ and $\frac{r_\delta}{\delta} \to +\infty$ as $\delta \to 0$, then $s^2(r_\delta) \to 1$ as $\delta \to 0$. Hence we may consider the equation (7.28) at $r = r_\delta$ and obtain that
$$\psi(r_\delta) \le C\|\tilde h_2\|_{L^\infty} = o(1),$$
which contradicts with $\psi(r_\delta) = 1$. Here we have used the facts that
$$\left|\Big(\frac{2\cos 2r}{\sin 2r} - \frac{1}{r}\Big)\frac{s'}{s}\right| \le \frac{C}{\delta}, \tag{7.35}$$
$$\left|\frac{\cos^2 w_t}{\sin^2 r} - \frac{1}{r^2}\right| \le C, \quad \forall\, r \in \Big(0, \frac{\pi}{2}\Big), \tag{7.36}$$
where C is a positive constant independent of ε and δ. Similarly, for the case that $r_\delta > \frac{\pi}{2} - 2\varepsilon$ and $\frac{\frac{\pi}{2} - r_\delta}{\delta} \to +\infty$, we may also get a contradiction. On the other hand, suppose $\frac{r_\delta}{\delta} \to r_0 > 0$ as $\delta \to 0$ (up to a subsequence). Let $\widetilde\psi_\delta(t) = \psi(\frac{r}{\delta})$ and $t = \frac{r}{\delta}$. Then $\widetilde\psi_\delta$'s approach to a solution of
$$\begin{cases} \psi'' + \dfrac{3}{t}\psi' - 2S_0^2\,\psi = 0, \quad \forall\, t > 0, \\ \psi \le 1, \quad \psi(r_0) = 1, \quad \psi'(0) = 0. \end{cases}$$
Thus, by the proof of Lemma 7.1, we have $\psi \equiv 0$ which gives a contradiction. Similarly, we may also get a contradiction, provided $\frac{\frac{\pi}{2} - r_\delta}{\delta} \to r_1 > 0$ as $\delta \to 0$ (up to a subsequence). Therefore, (7.34) is proved.

Now we prove the gradient estimate

$$\|\delta\psi'\|_{L^\infty} \le C\|\tilde{h}_2\|_{L^\infty}. \tag{7.37}$$

It is clear that (7.28) can be regarded as a linear second-order problem on S^3 given by

$$-\delta^2\triangle_{S^3}\psi + \Big[2s^2 + O\Big(\frac{\delta^2}{\varepsilon^2}+\delta\Big)\Big]\,\psi = \tilde{h}_2 \quad \text{in } S^3. \tag{7.38}$$

Then by (7.34) and the standard L^p-estimate for (7.38), we obtain (7.37). Similarly, we can prove

$$\|\varepsilon\phi'\|_{L^\infty} \le C\|\tilde{h}_1\|_{L^\infty}. \tag{7.39}$$

By comparison principle, we have

$$\Big|\mathrm{e}^{\sigma\frac{|r-t|}{\varepsilon}}\phi(r)\Big| \,\le\, C\|\tilde{h}_1\|_{**,\varepsilon}, \quad \forall\, r\in\Big(0,\frac{\pi}{2}\Big), \tag{7.40}$$

provided $\sigma>0$ is sufficiently small. Here as for the proof of Lemma 4.1, we have used the fact that the function $C\|\tilde{h}_1\|_{**,\varepsilon}\,\mathrm{e}^{-\sigma|\frac{r-t}{\varepsilon}|}$ is a supersolution of (7.27) for $0<r<\frac{\pi}{2}$, where C is a positive constant independent of ε.

To obtain a gradient estimate as in (7.40), we use the transformation

$$\hat{\phi} = \mathrm{e}^{\sigma\frac{|x-t|}{\varepsilon}}\,\phi. \tag{7.41}$$

Then $\hat{\phi}$ satisfies

$$-\varepsilon^2\triangle_{S^3}\hat{\phi} + \big[4(\cos 4w_t) - \sigma^2 + O(\varepsilon)\big]\,\hat{\phi} = \mathrm{e}^{\sigma\frac{|r-t|}{\varepsilon}}\tilde{h}_1 \quad \text{in } S^3 \quad \text{with } |r-t|\ge c\varepsilon. \tag{7.42}$$

Hence by (7.39) and elliptic regularity estimates of (7.42), we obtain

$$\Big|\mathrm{e}^{\sigma\frac{|r-t|}{\varepsilon}}\varepsilon\phi'\Big| \le C\|\tilde{h}_1\|_{**,\varepsilon}, \quad \forall\, r\in\Big(0,\frac{\pi}{2}\Big). \tag{7.43}$$

Here we have used the fact that

$$\varepsilon\hat{\phi}' = \mathrm{e}^{\sigma\frac{|r-t|}{\varepsilon}}\varepsilon\phi' + O\big(\sigma\,\mathrm{e}^{\sigma\frac{|r-t|}{\varepsilon}}\big)\,\phi.$$

Thus (7.34), (7.37), (7.40), and (7.43) may give

$$\|\phi\|_{*,\varepsilon} \le C\|\tilde{h}_1\|_{**,\varepsilon}, \quad \|\psi\|_{*,\delta} \le C\|\tilde{h}_2\|_{**}. \tag{7.44}$$

By (7.5) and (7.16)–(7.19), it is easy to get that

$$\begin{cases} \big\|\varepsilon^2\psi'\,w_t'\big\|_{**,\varepsilon} + \big\|(\sin 4w_t)\,s^2\psi\big\|_{**,\varepsilon} \le C\dfrac{\varepsilon}{\delta}\,\|\psi\|_{*,\delta}, \\ \Big\|2\delta^2\,w_t'\phi' + \Big[-\delta^2\dfrac{\sin 2w_t}{\sin^2 r} + \delta^2\dfrac{\sin 2w_t}{\cos^2 r} + 4\dfrac{\delta^2}{\varepsilon^2}\,(\cos 2w_t)\,s^2 - \delta\mu\,(\sin 2w_t)\Big]\,\phi\Big\|_{**} \\ \le C\dfrac{\delta^2}{\varepsilon^2}\,\|\phi\|_{*,\varepsilon}, \end{cases} \tag{7.45}$$

provided $\sigma>0$ is sufficiently small, where $o(1)$ is a small quantity tending to zero as ε goes to zero. Here we have used the assumption (7.5). Hence (7.44) and (7.45) imply

$$\|\phi\|_{*,\varepsilon} \le C\|\tilde{h}_1\|_{**,\varepsilon} \le C\|h_1\|_{**,\varepsilon} + C\frac{\varepsilon}{\delta}\|\psi\|_{*,\delta} \le C\|h_1\|_{**,\varepsilon} + C\frac{\varepsilon}{\delta}\|h_2\|_{**} + C\frac{\delta}{\varepsilon}\|\phi\|_{*,\varepsilon}.$$

Consequently,

$$\|\phi\|_{*,\varepsilon} \le C\|h_1\|_{**,\varepsilon} + C\frac{\varepsilon}{\delta}\|h_2\|_{**}. \tag{7.46}$$

Here we have used $\delta \ll \varepsilon$ from the assumption (2.20). Similarly, we obtain

$$\|\psi\|_{*,\delta} \le C\|h_2\|_{**} + C\frac{\delta^2}{\varepsilon^2}\|h_1\|_{**,\varepsilon}. \tag{7.47}$$

Therefore, by (7.46) and (7.47), we may complete the proof of Lemma 7.2.

To finish Step I, we expand

$$S_1[w_t+\phi, 1+\psi] = S_1[w_t, 1] + L_1[\phi,\psi] + N_1[\phi,\psi], \tag{7.48}$$

where $L_1[\phi,\psi]$ is given by (7.27) and $N_1[\phi,\psi]$ is the higher-order term which can be estimated as follows:

$$N_1[\phi,\psi] = O\left(|\phi|^2 + |\sin 4w_t|\,|\psi|^2 + |\phi|\,|\psi| + \varepsilon^2|\psi'||\psi||w_t'|\right). \tag{7.49}$$

We calculate

$$\begin{aligned}
&S_1[w_t,1]\\
&= -\varepsilon^2\Big(w_t'' + \Big(\frac{2\cos 2r}{\sin 2r} + \frac{2s'}{s}\Big)w_t'\Big) + (\sin 4w_t)\, s^2 - \varepsilon^2\left(\frac{2\cos 2r}{\sin^2 2r}\right)\sin 2w_t - \varepsilon\mu\sin 2w_t\\
&= -\varepsilon^2\Big(\frac{2\cos 2r}{\sin 2r} + \frac{2s'}{s}\Big)w_t' + (\sin 4w_t)\,(s^2-1) - \varepsilon^2\left(\frac{2\cos 2r}{\sin^2 2r}\right)\sin 2w_t - \varepsilon\mu\sin 2w_t.
\end{aligned} \tag{7.50}$$

Note that $w_t(r) = 0$ for $0 < r < t - 2\delta_0$ and $w_t(r) = \pi/2$ for $t + 2\delta_0 < r < \pi/2$. It is easy to see that

$$\|S_1[w_t,1]\|_{**,\varepsilon} \le C\,\varepsilon. \tag{7.51}$$

Similarly, we expand

$$S_2[w_{t_\varepsilon}+\phi, 1+\psi] = S_2[w_t,1] + L_2[\phi,\psi] + N_2[\phi,\psi],$$

where $L_2[\phi,\psi]$ is given by (7.28) and $N_2[\phi,\psi]$ is the higher-order term:

$$\begin{aligned}
N_2[\phi,\psi] = {}& O\bigg(|\psi|^2 + \frac{\delta^2}{\epsilon^2}\left(|\phi|\,|\psi| + \phi^2\right) + \delta^2|\psi|\,|\phi'|^2 + \delta^2|w_t'|\,|\phi'|\,|\psi|\bigg)\\
&+O\bigg(\delta^2\Big(\frac{\phi^2}{\sin^2 r} + \frac{\phi^2}{\cos^2 r}\Big)\bigg).
\end{aligned} \tag{7.52}$$

Suppose $\phi(0) = \phi(\pi/2) = 0$ and $\phi \in C^1([0,\pi/2])$. Then

$$\left|\frac{\phi(r)}{\sin r}\right| + \left|\frac{\phi(r)}{\cos r}\right| \le \frac{C}{\varepsilon}\,\|\phi\|_{*,\varepsilon}, \quad \forall 0 < r < \frac{\pi}{2}.$$

Consequently, (7.52) becomes

$$N_2[\phi,\psi] = O\bigg(|\psi|^2 + \frac{\delta^2}{\varepsilon^2}\left(|\phi|\,|\psi| + \phi^2\right) + \delta^2|\psi|\,|\phi'|^2 + \delta^2|w_t'|\,|\phi'|\,|\psi| + \frac{\delta^2}{\varepsilon^2}\,\|\phi\|^2_{*,\varepsilon}\bigg). \tag{7.53}$$

We estimate $S_2[w_t,1]$ as follows:

$$S_2[w_t,1]=-\delta^2\Big(\frac{2\cos 2r}{\sin 2r}-\frac{1}{r}\Big)\frac{s'}{s}+\delta^2\left[(w_t')^2+\Big(\frac{\cos^2 w_t}{\sin^2 r}-\frac{1}{r^2}\Big)+\frac{\sin^2 w_t}{\cos^2 r}\right]$$
$$+\frac{2\delta^2}{\varepsilon^2}\,s^2\sin 2w_t-\delta\mu\sin^2 w_t. \tag{7.54}$$

Noting that

$$(w_t')^2=O\Big(\frac{1}{\varepsilon^2}\Big),$$

and $w_t'(r)=0$ for $0<r<t-2\delta_0$ and $t+2\delta<r<\frac{\pi}{2}$. It is easy to see that

$$\|S_2[w_t,1]\|_{**}\le C\Big(\frac{\delta^2}{\varepsilon^2}+\delta\Big)\le C\frac{\delta^2}{\varepsilon^2}. \tag{7.55}$$

Here we have used the assumption (7.5). Set

$$\mathcal{B}=\left\{(\phi,\psi)\in\big(C^1([0,\pi/2])\big)^2:\|\phi\|_{*,\varepsilon}\le C\frac{\delta}{\varepsilon},\ \|\psi\|_{*,\delta}\le\Big(\frac{\delta}{\varepsilon}\Big)^{1+\sigma},\ \phi(0)=\phi(\pi/2)=0\right\},$$

where $0<\sigma<\frac{1}{2}$ is a small constant. Let us denote the map from (h_1,h_2) to (ϕ,ψ) be $\mathcal{T}=(\mathcal{T}_1,\mathcal{T}_2)$. Namely, $\phi=\mathcal{T}_1(h_1,h_2),\psi=\mathcal{T}_2(h_1,h_2)$. By Lemma 7.2, we have

$$\|\mathcal{T}_1(h_1,h_2)\|_{*,\varepsilon}\le C\|h_1\|_{**,\varepsilon}+C\frac{\varepsilon}{\delta}\|h_2\|_{**},\quad \|\mathcal{T}_2(h_1,h_2)\|_{*,\delta}\le C\|h_2\|_{**}+C\frac{\delta^2}{\varepsilon^2}\|h_1\|_{**,\varepsilon}. \tag{7.56}$$

It is easy to see that

$$S_1[w_t+\phi,1+\psi]=0,\quad S_2[w_t+\phi,1+\psi]=0$$

is equivalent to

$$(\phi,\psi)=\mathcal{T}(-S_1[w_t,1]-N_1,-S_2[w_t,1]-N_2):=\mathcal{G}(\phi,\psi). \tag{7.57}$$

Then by (7.49), (7.53), and (7.56), we obtain that

$$\begin{aligned}&\|\mathcal{T}_1(-S_1[w_t,1]-N_1,-S_2[w_t,1]-N_2)\|_{*,\varepsilon}\\ \le{}& C\|S_1[w_t,1]+N_1\|_{**,\varepsilon}+C\frac{\varepsilon}{\delta}(\|S_2[w_t,1]\|_{**}+\|N_2\|_{**})\\ \le{}& C\varepsilon+C\Big(\frac{\delta}{\varepsilon}\Big)^2+C\frac{\varepsilon}{\delta}\|\psi\|_{**}^2\le C\varepsilon+C\Big(\frac{\delta}{\varepsilon}\Big)^2+C\Big(\frac{\delta}{\varepsilon}\Big)^{1+2\sigma}\le C\frac{\delta}{\varepsilon}.\end{aligned} \tag{7.58}$$

Here we have used $\varepsilon^2\ll\delta$ from the assumption (2.20). Similarly we have

$$\begin{aligned}\|\mathcal{T}_2(-S_1[w_t,1]-N_1,-S_2[w_t,1]-N_2)\|_{*,\delta}&\le C\|S_2[w_t,1]+N_2\|_{**}+C\frac{\delta^2}{\varepsilon^2}\|S_1[w_t,1]+N_1\|_{**,\varepsilon}\\ &\le C\Big(\frac{\delta}{\varepsilon}\Big)^2+C\Big(\frac{\delta}{\varepsilon}\Big)^{1+\sigma}+C\frac{\delta^2}{\varepsilon}\\ &\le C\Big(\frac{\delta}{\varepsilon}\Big)^{1+\sigma}.\end{aligned} \tag{7.59}$$

Here we have used $0<\sigma<1/2$ and $\frac{\delta^2}{\varepsilon}\ll\big(\frac{\delta}{\varepsilon}\big)^{1+\sigma}$ from $\delta\ll\varepsilon\ll 1$ as another part of (2.20). Thus the map $\mathcal{G}$ is a map from $\mathcal{B}$ to $\mathcal{B}$. Similarly, we can show that $\mathcal{G}$ is a contraction map.

Then as for the proof of Proposition 4.2 of Section 4, we may prove Step I using contraction mapping principle.

For Step II, we can use the same argument as Lemma 4.3 to get

$$\Big(\int_{\mathbb{R}}(w'(y))^2\mathrm{d}y\Big)c_\varepsilon(t) = -2(\cot 2t)\varepsilon\int_{\mathbb{R}}(w'(y))^2\mathrm{d}y + \varepsilon\mu + O(\varepsilon^2), \tag{7.60}$$

and hence there exists t_ε such that

$$c_\varepsilon(t_\varepsilon) = 0. \tag{7.61}$$

Thus we have obtained the following theorem.

Theorem 7.3 Under the condition (7.5), there exists a solution $(\lambda_{\delta,\varepsilon,\mu}, \rho_{\delta,\varepsilon,\mu})$ to (2.18)–(2.19) with the following properties

$$\lambda_{\delta,\varepsilon,\mu}(r) = w\Big(\frac{r-t_{\varepsilon,\mu}}{\varepsilon}\Big) + O\big(\varepsilon\, \mathrm{e}^{-\sigma|\frac{r-t_{\varepsilon,\mu}}{\varepsilon}|}\big), \tag{7.62}$$

$$\rho_{\delta,\varepsilon,\mu}(r) = s\Big(\frac{r}{\delta}\Big)\Big(1+O\Big(\frac{\delta}{\varepsilon}\Big)\Big), \tag{7.63}$$

and

$$\widetilde{E}_{\delta,\varepsilon,\mu}(\lambda_{\delta,\varepsilon,\mu}, \rho_{\delta,\varepsilon,\mu}) = O(\varepsilon) + O\Big(\varepsilon^2\log\frac{1}{\delta}\Big),$$

for each $\mu \in \mathbb{R}$, where $t_{\varepsilon,\mu} = t_0 + O(\varepsilon)$ and t_0 satisfies

$$2(\cot 2t_0)\int_{\mathbb{R}}(w'(y))^2\mathrm{d}y = \mu.$$

For Step III, we can use (7.62) and (7.63) to compute

$$\int_0^{\frac{\pi}{2}}(\sin 2r)\,\rho_{\delta,\varepsilon,\mu}\sin^2\lambda_{\delta,\varepsilon,\mu}\mathrm{d}r \to \int_{t_0}^{\frac{\pi}{2}}\sin 2r\mathrm{d}r = \frac{1}{2}(1+\cos 2t_0),$$

as $\varepsilon, \delta \to 0$. Therefore, we may choose t_0 suitably such that

$$\frac{1}{2}(1+\cos 2t_0) = c_2 \in (0,1), \quad 0 < t_0 < \frac{\pi}{2},$$

and then we complete the proof of Step III.

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