NONRADIAL SOLUTIONS TO CRITICAL ELLIPTIC EQUATIONS OF CAFFARELLI-KOHN-NIRENBERG TYPE

MONICA MUSSO AND JUNCHENG WEI

ABSTRACT. We build an unbounded sequence of nonradial solutions for

$$\nabla(|x|^{-2a}\nabla u) + |x|^{-\frac{2N}{N-2}a} u^{\frac{N+2}{N-2}} = 0,\ u > 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

where $N \geq 5$ and $a < 0$. This answers a question of L. Veron.

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

The celebrated Caffarelli-Kohn-Nirenberg (CKN) inequalities ([2]) assert that there exists a constant $S = S(a, b)$ such that for all $u \in C_0^\infty(\mathbb{R}^N)$ it holds

$$S(\int_{\mathbb{R}^N} |x|^{-bq}|u|^q dx)^{\frac{2}{q}} \leq \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx \tag{1.1}$$

for $N \geq 3, -\infty < a < \frac{N-2}{2}, 0 \leq b - a \leq 1, q = \frac{2N}{N-2+2(b-a)}$. Associated with (1.1) is the Euler-Lagrangian equation

$$-\nabla(|x|^{-2a}\nabla u) = |x|^{-bq}u^{q-1} \text{ in } \mathbb{R}^N \tag{1.2}$$

which is called **CKN-type** equation throughout the paper.

There has been intensive research lately on the attainment and symmetries of extremal solutions of CKN inequalities. An interesting aspect of CKN inequalities is that it connects the classical Sobolev inequality ($a = b = 0$) with the Hardy inequality ($b = 1, a = 0$). In fact, when $a = b = 0$, inequality (1.1) is Sobolev inequality, and the constant $S(0, 0) := \mathbf{S}$ is attained. (1.2) becomes the well-known Yamabe problem

$$\Delta u + u^{\frac{N+2}{N-2}} = 0 \quad \text{in} \quad \mathbb{R}^N \tag{1.3}$$

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whose solutions are classified (Caffarelli-Gidas-Spruck [3]): all positive solutions are radially symmetric around some point and given by

$$U_{\varepsilon,\xi} = \varepsilon^{-\frac{n-2}{2}} U(\frac{x-\xi}{\varepsilon}) \quad \text{with} \quad \xi \in \mathbb{R}^N, \ \ \varepsilon > 0, \ \ U(y) = \alpha_N \left(\frac{1}{1+|y|^2} \right)^{\frac{N-2}{2}} \tag{1.4}$$

where $\alpha_N > 0$ is a generic constant.

In the parameter region

$$0 \le a < \frac{N-2}{2}, \ a \le b \le a+1, \tag{1.5}$$

Chou and Chu [5] proved that all solutions to (1.2) are radially symmetric, using the method of moving planes, and that these solutions also give rise to extremal solutions of CKN inequalities. On the other hand, in the parameter region

$$-\infty < a < 0, \ a \le b \le a+1, \tag{1.6}$$

some striking new phenomena are discovered by Catrina and Wang [4]: they showed that for $b = a+1$ or $b = a$, the best constant in (1.1) is $\mathbf{S}$ and is never achieved. Symmetry breaking extremal solutions are also found. This has initiated intensive studies on (1.1)-(1.2). We refer to Dolbeault-Esteban [12], Dolbeault-Esteban-Loss-Tarantello [13], del Pino-Dolbeault-Fillippas-Tertikas [9], Felli-Schneider [16], Lin-Wang [22] and the references therein.

In this paper, we are concerned with the case of $b = a$ and $q = \frac{2N}{N-2}$, namely the following nonlinear equation

$$\nabla(|x|^{-2a} \nabla u) + |x|^{-\frac{2N}{N-2}a} |u|^{\frac{4}{N-2}} u = 0, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}\,. \tag{1.7}$$

According to Catrina-Wang [4], the extremal solution to (1.1) does not exist. Since $a < 0$, the method of moving plane can not be applied. An interesting question is: if $a < 0$, are there any positive (radial or nonradial) solutions to (1.7)? When $a > 0$, all positive solutions are radially symmetric and unique (up to scaling). Another interesting question is: if $a > 0$, are there sign-changing solutions to (1.7)?

For both questions, our answers are affirmative.

Theorem 1.1. *Assume that $N \ge 5$.*
Part a. *If $a < 0$ or $a > N-2$, then for any sufficiently large integer k there is a finite energy solution to Problem (1.7) of the form*

$$u_k(x) = |x|^a \left[\sum_{j=1}^k \varepsilon_k^{-\frac{N-2}{2}} U\left(\varepsilon_k^{-1}(x-\xi_j^k)\right) + R_k(x) \right], \tag{1.8}$$

In (1.8) ε_k are positive numbers defined as

$$\varepsilon_k = c_{Nk} k^{-\frac{N-2}{N-4}}$$

where c_{Nk} is a positive constant, depending on N and k, such that $\lim_{k\to\infty} c_{Nk} = [a(a-(N-2))]^{\frac{1}{N-4}} A_N$, with A_N a positive constant depending only on N. Furthermore, the points ξ_j^k are *arranged in a regular polygon in* $\mathbb{R}^2 \times \{\bar{0}\}$ *as follows*

$$\xi_j^k = \sqrt{1-\varepsilon_k^2}\left(\cos(\frac{j-1}{k}\pi), \sin(\frac{j-1}{k}\pi), 0, \dots, 0\right) \in \mathbb{R}^N, \quad j = 1, \dots k.$$

Finally in (1.8) R_k *is a function that satisfies* $\|R_k\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \to 0$ *as* $k \to +\infty$*. Moreover,*

$$\frac{\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u_k|^2}{\left(\int_{\mathbb{R}^N} |x|^{-\frac{2N}{N-2}a}|u_k|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}} = k\boldsymbol{S} + O(1). \tag{1.9}$$

Here $O(1)$ *remains bounded as* $k \to \infty$*. As a consequence, problem (1.2) has infinitely many nonradial positive solutions.*

Part b. *Assume now that* $0 < a < N-2$ *and* $a \neq \frac{N-2}{2}$*. Then for any sufficiently large integer* k *there is a finite energy solution to Problem (1.7) of the form*

$$u_k(x) = |x|^a \left[\sum_{j=1}^{k} (-1)^{j+1} \varepsilon_k^{-\frac{N-2}{2}} U\left(\varepsilon_k^{-1}(x-\xi_j^k)\right) + R_k(x)\right], \tag{1.10}$$

where

$$\varepsilon_k = c_{Nk} k^{-\frac{N-2}{N-4}}$$

with c_{Nk} *positive numbers, depending on* N *and* k*, such that* $\lim_{k\to\infty} c_{Nk} = [a(N-2-a)]^{\frac{1}{N-4}} A_N$*, with* A_N *a positive constant depending only on* N*. The points* ξ_j^k *are arranged in a regular polygon in* $\mathbb{R}^2 \times \{\bar{0}\}$ *as follows*

$$\xi_j^k = \sqrt{1-\varepsilon_k^2}\left(\cos(\frac{j-1}{k}\pi), \sin(\frac{j-1}{k}\pi), 0, \dots, 0\right) \in \mathbb{R}^N, \quad j = 1, \dots k.$$

Finally in (1.10) R_k *is a function that satisfies* $\|R_k\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \to 0$ *as* $k \to +\infty$*. Moreover,*

$$\frac{\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u_k|^2}{\left(\int_{\mathbb{R}^N} |x|^{-\frac{2N}{N-2}a}|u_k|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}} = k\boldsymbol{S} + O(1). \tag{1.11}$$

Here $O(1)$ *remains bounded as* $k \to \infty$*.*

Problem (1.7) can also be regarded as a Hardy-type equation with critical Sobolev exponent. Define

$$v(x) = |x|^{\frac{\beta}{2}} u(x), \quad \text{where} \quad \beta := -2a. \tag{1.12}$$

A direct computation shows that u is a solution to (1.7) if and only if v solves

$$\Delta v - \gamma \frac{v}{|x|^2} + |v|^{\frac{4}{N-2}} v = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \tag{1.13}$$

We will use the notation

$$\gamma = \frac{\beta}{2}(\frac{\beta}{2} + N - 2) = a(a-(N-2)). \tag{1.14}$$

The condition $a \notin \{0, \frac{N-2}{2}, N-2\}$ implies that $\gamma > -(\frac{N-2}{2})^2$ and $\gamma \neq 0$.

In view of problem (1.13), Theorem 1.1 is equivalent to

Theorem 1.2. *Assume that $N \geq 5$.*

Part a. *If $\gamma > 0$, then for any sufficiently large integer k there is a finite energy solution to Problem (1.13) of the form*

$$u_k(x) = \left[\sum_{j=1}^{k} \varepsilon_k^{-\frac{N-2}{2}} U\left(\varepsilon_k^{-1}(x - \xi_j^k)\right) + R_k(x) \right], \tag{1.15}$$

where

$$\varepsilon_k = c_{Nk} k^{-\frac{N-2}{N-4}}$$

with c_{Nk} positive numbers, depending on N and k, such that $\lim_{k\to\infty} c_{Nk} = \gamma^{\frac{1}{N-4}} A_N$, with A_N a positive constant depending only on N. The points ξ_j^k are arranged in a regular polygon in $\mathbb{R}^2 \times \{\bar{0}\}$ as follows

$$\xi_j^k = \sqrt{1-\varepsilon_k^2} \left(\cos(\frac{j-1}{k}\pi), \sin(\frac{j-1}{k}\pi), 0, \dots, 0 \right) \in \mathbb{R}^N, \quad j = 1, \dots k.$$

Finally in (1.15) R_k is a function that satisfies $\|R_k\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \to 0$ as $k \to +\infty$. Moreover,

$$\frac{\int_{\mathbb{R}^N} (|\nabla u_k|^2 + \frac{\gamma}{|x|^2} u_k^2)}{\left(\int_{\mathbb{R}^N} |u_k|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}} = k\boldsymbol{S} + O(1). \tag{1.16}$$

Here $O(1)$ remains bounded as $k \to \infty$.

Part b. *Assume that $-\left(\frac{N-2}{2}\right)^2 < \gamma < 0$. Then for any sufficiently large integer k there is a finite energy solution to Problem (1.13) of the form*

$$u_k(x) = \left[\sum_{j=1}^{k} (-1)^{j+1} \varepsilon_k^{-\frac{N-2}{2}} U\left(\varepsilon_k^{-1}(x - \xi_j^k)\right) + R_k(x) \right], \tag{1.17}$$

where

$$\mu_k = c_{Nk} k^{-\frac{N-2}{N-4}}$$

with c_{Nk} positive numbers, depending on N and k, such that $\lim_{k\to\infty} c_{Nk} = (-\gamma)^{\frac{1}{N-4}} A_N$, with A_N a positive constant depending only on N. The points ξ_j^k are arranged in a regular polygon in $\mathbb{R}^2 \times \{\bar{0}\}$ as follows

$$\xi_j^k = \sqrt{1-\varepsilon_k^2} \left(\cos(\frac{j-1}{k}\pi), \sin(\frac{j-1}{k}\pi), 0, \dots, 0 \right) \in \mathbb{R}^N, \quad j = 1, \dots k.$$

Finally in (1.17) R_k is a function that satisfies $\|R_k\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \to 0$ as $k \to +\infty$. Moreover,

$$\frac{\int_{\mathbb{R}^N} (|\nabla u_k|^2 + \frac{\gamma}{|x|^2} u_k^2)}{\left(\int_{\mathbb{R}^N} |u_k|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{N}}} = k\boldsymbol{S} + O(1). \tag{1.18}$$

Here $O(1)$ remains bounded as $k \to \infty$.

Let us comment on previous works on (1.13). According to [20], L. Veron raised the following question: *For $\gamma \in \mathbb{R}$ and $\gamma \neq 0$, let $u \in C^{\infty}(\mathbb{R}^n \backslash \{0\})$ be a solution to Problem (1.13). Is it true that u must be radially symmetric about the origin?* L. Veron also pointed out that there may be solutions of a certain form as suggested in Section 4 of Bidaut-Veron and Veron [1]. The form of solutions suggested in [1] is invariant under Dihedral symmetry D_k. In [19], Jin-Li-Xu proved the following: *(i) for $\gamma \leq -\frac{(N-2)^2}{4}$, (1.13) has no smooth solutions; (ii) for $-\frac{(N-2)^2}{4} < \gamma < 0$, all solutions to (1.13) are radially symmetric; (iii) for $\gamma > -\frac{(N-2)^2}{4}$, problem (1.13) has infinitely many radial solutions; (iv) for $\gamma > \frac{N-2}{4}$, (1.13) has non-radial solutions. Moreover, the number of non-radial solutions goes to ∞ as $\gamma \to +\infty$.* The nonradial solutions in [19] are constructed by bifurcations. As commented in [19], these solutions are not the types of solutions suggested in [1], and *it is an interesting question to study the existence of solutions of the suggested form. The existence of nonradial solutions is also open when* $0 < \gamma < \frac{N-2}{4}$. Theorem 1.2 gives an affirmative answer to Veron's question and also fills the existence gap $0 < \gamma < \frac{N-2}{4}$ left in [19].

Problem (1.13) also arises in nonrelativistic molecular physics. The inverse square potentials describe the interaction between electric charges and dipole moments of molecules; see [23]. For mathematical analysis of such problems, we refer to Felli-Terracini [17], Azorero-Peral [18], Smets [25], Terracini [27] and the references therein.

The proof of Theorem 1.2 is by reduction method: we look for solutions of (1.13)

$$u_k(x) = \left[\sum_{j=1}^{k} \sigma_j \varepsilon_k^{-\frac{N-2}{2}} U\left(\varepsilon_k^{-1}(x - \xi_j^k)\right) \right] + \phi \tag{1.19}$$

where σ_j is either 1 or $(-1)^{j+1}$, and ϕ is small in suitable norm. Since the equation (1.13) is rotationally invariant, we look for solutions of (1.13) which are invariant under $\frac{2\pi}{k}$−rotation, for some integer $k \geq 2$ and also reflections. Therefore we put k bubbles at k equally distributed vortices lying on the unit circle. This leaves two parameters (ε_k, ξ_1^k) to be determined. The key idea is to use *k as parameter.* The reduction works if we employ the four fundamental invariances of (1.13): problem (1.13) is invariant under the Kelvin transform, scaling, reflections, and rotations.

The idea of using the number of bubbles as parameter was first used by Wei-Yan ([28]) in constructing infinitely many positive solutions to the prescribing scalar curvature problem. del Pino-Musso-Pacard-Pistoia ([10, 11]) also used this idea in constructing infinitely many sign-changing solutions to (1.3). See also Musso-Pacard-Wei [24], Wei-Yan [29] for the use of this idea in a different context. Here we use the same idea. Our main difficulty in this paper is the reduction part: we have to deal with the Hardy operator $\Delta + \frac{\gamma}{|x|^2}$ which has a singularity at the origin and we can not use Maximum Principle nor Green's representation formula nor weighted L^{∞} norms. Instead of using the weighted Sobolev L^{∞}-norms (as in [28]), here we use the L^p space.

We don't know if the restriction that $N \geq 5$ is only technical. Firstly, for the prescribed scalar curvature problem,

$$\Delta u + K(x) u^{\frac{N+2}{N-2}} = 0 \tag{1.20}$$

it has been proved that under some conditions on K, there is an energy bound for all solutions in the case of dimensions $N = 3, 4$. (Theorem 1.2 implies that the energy of solutions to (1.13) can be unbounded when $N \geq 5$.) We refer to Schoen-Zhang [26], Y.Y. Li [21], O. Druet [14]-[15] and the references therein. We don't know if similar result holds for (1.13) when $N = 3, 4$. Secondly, we do need $N \geq 5$ to define the scaling parameter in terms of k. See (2.4) below.

The paper is organized as follows: In Section 2 we define a first approximation of the solution we are looking for and we explain in details the scheme of the proof to get the result of Theorem 1.2. In Section 3 we give the proof of Theorem 1.2. Section 4 is devoted to estimate, in some appropriate norm, the error of approximation. Finally Sections 5 and 6 are devoted to prove some technical Lemmas.

2. Construction of a first approximation and scheme of the proof

We start with the construction of a first approximate solution to Problem (1.13). This first approaximate solution consists of $k-$bubbles located in a circe of appropriate size so that it is invariant under Kelvin's tarnsform, rotation by $\frac{2\pi}{k}$ and reflections.

Fix k to be an integer and denote by $R_k \in O(2) \times \{I_{N-2}\}$ the rotation of $\frac{2\pi}{k}$ in the (x_1, x_2)-plane. Set $\mathsf{e}_1 := (1, 0, \ldots, 0) \in \mathbb{R}^N$, let ε be a positive parameter and consider the regular polygon in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$ whose vertices are given by the orbit of the point

$$\xi_1 := \sqrt{1-\varepsilon^2}\, \mathsf{e}_1 \in \mathbb{R}^N , \tag{2.1}$$

under the action of the group generated by R_k, namely

$$\xi_j = R_k^{j-1} \xi_1, \quad j = 1, \ldots, k.$$

Define

$$V_{j\varepsilon}(x) = \varepsilon^{-\frac{n-2}{2}} U(\frac{x - \xi_j}{\varepsilon}) \quad \text{where} \quad U(y) = \alpha_N \left(\frac{1}{1+|y|^2} \right)^{\frac{N-2}{2}} \tag{2.2}$$

Observe that equation (1.13) in invariant under Kelvin transformation,

$$v(x) = |x|^{2-N} v(\frac{x}{|x|^2}),$$

so it is natural to look for solutions v to (1.13) in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ that are invariant under Kelvin transform.

Thanks to the choice of the point ξ_1 given by formula (2.1), we observe that each function $V_{j\varepsilon}$ is indeed invariant under Kelvin transform. Define the first approximate solution as

$$V_+[\varepsilon](x) = \sum_{j=1}^k V_{j\varepsilon}(x), \quad \text{and} \quad V_-[\varepsilon](x) = \sum_{j=1}^k (-1)^j V_{j\varepsilon}(x). \tag{2.3}$$

These new functions V_+ and V_- are also invariant under Kelvin transformation. The choice of ξ_1 and the Kelvin transformation invariance are borrowed from [10].

For simplicity of notation we will write $V_\pm[\varepsilon](x) = V_\pm(x)$.

In our construction, the parameter ε is not independent on k. In fact its dependence on k is at main order explicit, and changes from dimension to dimension. To be more precise, we assume that

$$\varepsilon = \frac{\mu}{k^{1+\frac{2}{N-4}}} \tag{2.4}$$

where μ is a positive parameter, uniformly bounded away from zero and from infinity as $k \to \infty$. In fact we assume that there exists a positive, small number δ, independent of k, such that

$$\delta < \mu < \delta^{-1} \quad \text{for all} \quad k \quad \text{large}. \tag{2.5}$$

This choice of ϵ first appeared in [28] for the prescribed scalar curvature problem.

To simplify the notation we will denote with V the function V_+ or the function V_-, depending if we are considering the case of positive or sign changing solutions.

The function E defined as

$$E(x) = \Delta V + |V|^{p-1}V - \frac{\gamma}{|x|^2}V \tag{2.6}$$

is the error of approximation. It is clear that a basic issue for our construction is to measure the size of this error function E, both in a region near the concentration points ξ_j and also far away. One of the main difficulties in this paper is the presence of the Hardy operator $\Delta + \frac{\gamma}{|x|^2}$. For reasons that will become clear later, it is convenient to do this measurement using the $L^{\frac{2N}{N+2}}$-norm.

We have the validity of the following

Proposition 2.1. *Let δ and η be two positive small numbers. There exist k_0 and C such that, for all $k \geq k_0$ and $\varepsilon = \frac{\mu}{k^{1+\frac{2}{N-4}}}$ satisfying (2.5), the following estimates hold true*

$$\|E\|_{L^{\frac{2N}{N+2}}(B(\xi_j,\frac{\eta}{k}))} \leq C \begin{cases} k^{-5} & \text{if} \quad N=5, \\ k^{-4}|\log k|^{\frac{N+2}{2N}} & \text{if} \quad N=6, \\ k^{-1-\frac{6}{N-4}} & \text{if} \quad N \geq 7. \end{cases} \tag{2.7}$$

for any $j = 1, \ldots, k$, and

$$\|E\|_{L^{\frac{2N}{N+2}}(\mathbb{R}^N \setminus \bigcup_{j=1}^k B(\xi_j,\frac{\eta}{k}))} \leq C \begin{cases} k^{-3-\frac{1}{2}} & \text{if} \quad N=5, \\ k^{-3}|\log k|^{\frac{N+2}{2N}} & \text{if} \quad N=6, \\ k^{-\frac{6}{N-4}} & \text{if} \quad N \geq 7. \end{cases} \tag{2.8}$$

where E is the error of approximation defined in (2.6).

We postpone the proof of this result to Section 4: Appendix 1.

We start observing the following facts: the function $V_\pm$ defined in (2.3) not only is invariant under Kelvin transformation, it is also invariant under the group of rotations $R_k \in O(2) \times \{I_{N-2}\}$. Furthermore it is even in the last $(N-2)$ coordinates, namely

$$V_\pm(x_1, x_2, -x_i, \ldots, x_N) = V_\pm(x_1, x_2, x_i, \ldots, x_N) \quad \text{for all} \quad i = 3, \ldots, N. \tag{2.9}$$

It is thus natural to work in a space of functions that respect all the above symmetries. Let

$$H = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \,:\, u(x) = |x|^{2-N} u(\frac{x}{|x|}),\ u(R_k x) = u(x), \quad u \quad \text{satisfies (2.9)}\}. \tag{2.10}$$

In particular the functions $V_\pm$ defined in (2.3) belong to H.

When $\gamma > 0$, we will look for solutions to (1.13) belonging to the space H of the form

$$v(x) = V_+(x) + \phi_+(x) \tag{2.11}$$

where $\phi_+ \in H$ is a lower order term.

On the other hand, when $-(\frac{N-2}{2})^2 < \gamma < 0$, our solution will look like

$$v(x) = V_-(x) + \phi_-(x) \tag{2.12}$$

where again $\phi_- \in H$ is a lower order term. As before, to simplify the notation we will denote with ϕ the function ϕ_+ or the function ϕ_-, depending if we are considering the case of positive or sign changing solutions.

Let $p = \frac{N+2}{N-2}$. In terms of ϕ, problem (1.13) takes the form

$$\Delta\phi - \frac{\gamma}{|x|^2}\phi + pV^{p-1}\phi + E + N(\phi) = 0, \quad \text{in} \quad \mathbb{R}^N, \quad \phi \in H. \tag{2.13}$$

In (2.13), the function E was introduced before in (2.6), and

$$N(\phi) = \left[|U+\phi|^{p-1}(U+\phi) - |U|^{p-1}U - p|U|^{p-1}\phi \right]. \tag{2.14}$$

We will solve Problem (2.13) using a gluing argument. The idea is to decompose the nonlinear problem (2.13) into $k+1$ sub-problems: the first k problems correspond to the k-bubble regions (inner part), and the last problem corresponds to the region outside the bubbles (outer part). This idea of decomposition principle has been used in a number of problems, e.g. the counter-example to De Giorgi conjecture by del Pino, Kowalcyzk and Wei [6], and also the construction of multiple end solutions to nonlinear Schrödinger equations by del Pino, Kowalczyk, Pacard and Wei [7]. Note that this is reduction procedure is different from [28].

For any $j = 1, \ldots, k$, let ζ_j be a cut-off function defined as follows. Let $\zeta(s)$ be a smooth function such that $\zeta(s) = 1$ for $s < 1$ and $\zeta(s) = 0$ for $s > 2$. Then we set

$$\zeta_j(y) = \begin{cases} \zeta(\, k\eta^{-1}|y|^{-2}|(y - \xi_j|y|)\,|\,) & \text{if } |y| > 1\,, \\[2ex] \zeta(\, k\eta^{-1}\,|y - \xi_j|\,) & \text{if } |y| \le 1\,, \end{cases} \tag{2.15}$$

for a certain $\eta > 0$ small and independent of k. Observe that

$$\zeta_j(y) = \zeta_j(y/|y|^2). \tag{2.16}$$

A function ϕ of the form

$$\phi \;=\; \sum_{j=1}^k \phi_j + \psi$$

is a solution of Problem (2.13) if we can solve the following coupled system of elliptic equations in $\bar\phi = (\phi_1, \ldots, \phi_k)$ and ψ:

$$\Delta\phi_j + p|V|^{p-1}\zeta_j\phi_j + \zeta_j[p|V|^{p-1}\psi - \frac{\gamma}{|x|^2}\phi_j + E + N(\phi_j + \Sigma_{i\neq j}\phi_i + \psi)] = 0, \quad j = 1, \ldots, k, \tag{2.17}$$

(2.18)
$$\Delta\psi - \frac{\gamma}{|x|^2}\psi + \left(p|V|^{p-1} - \frac{\gamma}{|x|^2}\right)\sum_j(1-\zeta_j)\phi_j + (1-\Sigma_{j=1}^k\zeta_j)\,(p|V|^{p-1}\psi + E + N(\Sigma_{j=1}^k\phi_j + \psi)) \;=\; 0.$$

To solve System (2.17)-(2.18) we will solve first problem (2.18) for given ϕ_j's of a special form that we describe next. Define

$$\mathcal{L}^{\frac{2N}{N-2}} = \{u \in L^{\frac{2N}{N-2}} \,:\, u(x) = |x|^{2-N}u(\frac{x}{|x|^2}),\, u(R_k x) = u(x), \quad u \quad \text{satisfies (2.9)}\} \tag{2.19}$$

The function ϕ_j will inherit the size of the measure of the error of approximation E defined in (2.6) in the *interior region* $B(\xi_j, \frac{\eta}{k})$, for some $\eta > 0$, small and independent of k. Thus, given the result in Proposition 2.1, we assume that $\phi_j \in H \bigcap \mathcal{L}^{\frac{2N}{N-2}}$ for any $j = 1, \ldots, k$, with

$$\| \left(\frac{\varepsilon}{\varepsilon^2 + |x-\xi_j|^2}\right)^{-\frac{N-2}{2}} \phi_j\|_\infty \leq \sigma, \tag{2.20}$$

for some fixed constant σ, independent of k and small. For further reference, we will use the notation

$$\|\phi\|_{j*} = \| \left(\frac{\varepsilon}{\varepsilon^2 + |x-\xi_j|^2}\right)^{-\frac{N-2}{2}} \phi\|_\infty. \tag{2.21}$$

A trivial observation is that

$$\|\phi\|_{\frac{2N}{N-2}}\| \leq C\|\phi\|_{j*},$$

for some explicit constant C.

The following result holds.

Lemma 2.2. *Let $\delta > 0$. There exists k_0 and C such that for all $k \geq k_0$ and all $\varepsilon = \frac{\mu}{k^{1+\frac{2}{N-4}}}$ with $\delta \leq \mu \leq \delta^{-1}$ for all k large, the following holds: Let $\phi_j \in H \bigcap \mathcal{L}^{\frac{2N}{N-2}}$, $j = 1, \ldots, k$ satisfy conditions (2.20). Then there exists a unique solution $\psi = \Psi(\phi_1) \in H \bigcap \mathcal{L}^{\frac{2N}{N-2}}$ to equation (2.18), such that*

$$\|\psi\|_{\frac{2N}{N-2}} \leq C[g(k) + \|\phi_1\|_{*1}^2] \quad \text{where} \quad g(k) = \begin{cases} k^{-3-\frac{1}{2}} & \text{if} \quad N = 5, \\ k^{-3}|\log k|^{\frac{N+2}{2N}} & \text{if} \quad N = 6, \\ k^{-\frac{6}{N-4}} & \text{if} \quad N \geq 7. \end{cases} \tag{2.22}$$

The operator Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1^1) - \Psi(\phi_1^2)\|_{\frac{2N}{N-2}} \;\leq\; C\;\|\phi_1^1 - \phi_1^2\|_{*1}.$$

Furthermore the function $\psi(\phi_1)$ depends continuously on the parameter μ, in the sense that the function $\mu \to \psi \in \mathcal{L}^{\frac{2N}{N-2}}$ is continuous in the natural topologies.

Moreover, if we define $\psi_1 := \zeta_1 \psi$ *(see (2.15) for the definition of* ζ_1*), then we get the finer estimate*

$$\|\psi_1\|_{\frac{2N}{N-2}} \leq C\left[k^{-1-\frac{2}{N-4}}\|\phi_1\|_{1*} + \|\phi_1\|_{1*}^2\right] \tag{2.23}$$

We postpone the proof of the above Lemma to Section 5: Appendix 2, Proof of Lemma 2.2.

Let us consider now the operator $\psi = \psi(\bar\phi)$ defined in the previous Lemma, that gives a solution to Problem (2.18) for $\bar\phi = (\phi_1, \dots, \phi_k)$ fixed. Our next interest is to solve Equations (2.17). We claim that, if we solve Problem (2.17) for $j = 1$, then automatically Problem (2.17) is solved for any $j = 1, \dots, k$. This is simply due to the invariance of Equation (2.17) under the rotation R_k of the angle $\frac{2\pi}{k}$ in the first two components in $\mathbb{R}^N$. In fact, one gets that if ϕ_1 is a solution to Equation (2.17) for $j = 1$, then a posteriori $\phi_j(x) := \phi_1(R_k^{j-1}x)$ is the solution to Equation (2.17), for $j = 2, \dots, k$.

We thus solve Equation (2.17) for $j = 1$.

Let us rewrite Equation (2.17) as follows

$$\Delta\phi_1 + p|V_{1\varepsilon}|^{p-1}\phi_1 + h(x) = 0, \tag{2.24}$$

where $V_{1\varepsilon}$ is defined in (2.2) and

$$h(x) = p(|V|^{p-1}\zeta_1 - |V_{1\varepsilon}|^{p-1})\phi_1 + \zeta_1[p|V|^{p-1}\psi - \frac{\gamma}{|x|^2}\phi_1 + E + N(\phi_1 + \Sigma_{i\neq 1}\phi_i + \psi)]. \tag{2.25}$$

Define

$$\mathcal{L}^{\frac{2N}{N+2}} = \{u \in L^{\frac{2N}{N+2}} \,:\, u(x) = |x|^{-2-N}u(\frac{x}{|x|^2}),\ u(R_kx) = u(x), \quad u \quad \text{satisfies (2.9)}\} \tag{2.26}$$

We make the following observation: if we assume that ϕ_1 is invariant under Kelvin transform, that it is invariant under the rotation R_k and it is even with respect to the last $(N-2)$ variables, then thanks to the properties of $\psi(\bar\phi)$ we find that the function $h(x)$ defined in (2.25) also satisfies

$$h(x) = |x|^{-2-N}h(\frac{x}{|x|^2}), \quad h(R_kx) = h(x) \qquad \text{for all} \qquad x \in \mathbb{R}^N \setminus \{0\}$$

and

$$h(x_1, x_2, x_3, \dots, x_i, \dots, x_N) = h(x_1, x_2, x_3, \dots, -x_i, \dots, x_N), \qquad \text{for all} \qquad i = 3, \dots, N.$$

Even more, if we assume that $\phi_1 \in \mathcal{L}^{\frac{2N}{N-2}}$ then $h \in \mathcal{L}^{\frac{2N}{N+2}}$.

We solve (2.17) by a finite dimensional reduction procedure. This consists of two steps. (See [8].)

Step 1: For a general function $f \in \mathcal{L}^{\frac{2N}{N+2}}$, we consider first the linear problem

$$\Delta\phi + pV_{1\varepsilon}^{p-1}\phi = f + c\,V_{1\varepsilon}^{p-1}\,Z_{1\varepsilon}, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \quad \int V_{1\varepsilon}^{p-1}\phi Z_{1\varepsilon} = 0 \tag{2.27}$$

where

$$Z_{1\varepsilon}(x) = \varepsilon^{-\frac{N-2}{2}}Z(\varepsilon^{-1}(x-\xi_1)), \quad \text{with} \quad Z(y) = \frac{N-2}{2}U(y) + \nabla U(y)\cdot y \tag{2.28}$$

(see (2.2) for the definition of U) and

$$c = -\frac{\int_{\mathbb{R}^N} f Z_{1\varepsilon}}{\int_{\mathbb{R}^N} V_{1\varepsilon}^{p-1} Z_{1\varepsilon}^2}.$$

We have the following result.

Lemma 2.3. *Let us assume that $f \in \mathcal{L}^{\frac{2N}{N+2}}$. Then Problem (2.27) has a unique solution $\phi = T(h) \in \mathcal{D}^{1,2}(\mathbb{R}^N) \bigcap \mathcal{L}^{\frac{2N}{N-2}}$ satisfying*

$$\|\phi\|_{1*} \le C \|f\|_{\frac{2N}{N+2}}, \tag{2.29}$$

for some positive constant C.

We use the above lemma to solve the corresponding projected version of (2.24)

$$\Delta \phi_1 + p|V_{1\varepsilon}|^{p-1}\phi_1 + h(x) = c V_{1\varepsilon}^{p-1} Z_{1\varepsilon}, \quad \int_{\mathbb{R}^N} \phi_1 V_{1\varepsilon}^{p-1} Z_{1\varepsilon} = 0 \tag{2.30}$$

where here h is the explicit function defined in (2.25) and

$$c := \frac{\int_{\mathbb{R}^N} h(x) Z_{1\varepsilon}}{\int_{\mathbb{R}^N} V_{1\varepsilon}^{p-1} Z_{1\varepsilon}^2}.$$

We can prove the following result

Lemma 2.4. *Let $\delta > 0$. There exists k_0 and C such that for all $k \ge k_0$ and all $\varepsilon = \frac{\mu}{k^{1+\frac{2}{N-4}}}$ with $\delta \le \mu \le \delta^{-1}$ for all k large, the following holds: there exists a unique solution $\phi_1 = \phi_1(\mu) \in \mathcal{L}^{\frac{2N}{N-2}}$ to equation (2.30), such that*

$$\|\phi_1\|_{1*} \le C \begin{cases} k^{-5} & \text{if} \quad N = 5, \\ k^{-4} |\log k|^{\frac{N+2}{2N}} & \text{if} \quad N = 6, \\ k^{-1-\frac{6}{N-4}} & \text{if} \quad N \ge 7. \end{cases}, \tag{2.31}$$

Furthermore the solution ϕ_1 depends continuously on μ, in the sense that the function $\mu \to \phi_1 \in \mathcal{L}^{\frac{2N}{N-2}}$ is continuous in the natural topologies.

To make our exposition clearer, we also postpone the proofs of Lemma 2.3 and Lemma 2.4 to Section 6 Appendix 3.

Step 2: Once (2.30) is solved, it is clear that $V + \phi$ becomes an exact solution to (1.13) if there exists a choice for the parameter μ so that

$$c = c[\mu] = 0. \tag{2.32}$$

This is done in Section 3, where we also conclude the proof of our results.

3. Proof of the results

This Section is devoted to the

Proof of Theorem 1.2. Let $\delta > 0$ be a small fixed number. Lemma 2.4 garantees the existence of a large integer k_0, such that for all $\varepsilon = \frac{\mu}{k^{1+\frac{2}{N-4}}}$ with $\delta < \mu < \delta^{-1}$, for all integers $k \geq k_0$ there exists $\phi_1 \in H$ and $c \in \mathbb{R}$ solution to the non linear Problem

$$\Delta \phi_1 + p V_{1\varepsilon}^{p-1} \phi_1 + h(x) = c\, V_{1\varepsilon}^{p-1} Z_{1\varepsilon}, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \qquad \int V_{1\varepsilon}^{p-1} \phi Z_{1\varepsilon} = 0$$

where we recall the expression of h given by (2.25)

$$h(x) = p(|V|^{p-1}\zeta_1 - |V_{1\varepsilon}|^{p-1})\phi_1 + \zeta_1 [p|V|^{p-1}\psi(\phi_1) - \frac{\gamma}{|x|^2}\phi_1 + E + N(\phi_1 + \Sigma_{i\neq 1}\phi_i + \psi)],$$

E is given by (2.6), $N(\phi)$ is defined in (2.14), $V_{1\varepsilon}$ is defined in (2.2), $Z_{1\varepsilon}$ is defined in (2.28) and ψ is the function whose existence is guaranteed by Lemma 2.2. Furthermore, the function ϕ_1 and the constant c depends continuously on μ. For further reference, we write

$$h(x) = \zeta_1 E + \mathcal{L}(\phi_1) + \mathcal{N}(\phi_1) \tag{3.1}$$

where

$$\mathcal{L}(\phi_1) = p(|V|^{p-1}\zeta_1 - |V_{1\varepsilon}|^{p-1})\phi_1 - \zeta_1 \frac{\gamma}{|x|^2}\phi_1 \tag{3.2}$$

and

$$\mathcal{N}(\phi_1) = \zeta_1 \left[p|V|^{p-1}\psi(\phi_1) + N(\phi_1 + \Sigma_{i\neq 1}\phi_i + \psi) \right]. \tag{3.3}$$

It is thus a trivial observation to say that the function

$$v = V + \phi$$

is a solution to our original problem (1.13) if we can chose μ so that

$$c = c[\mu] = 0. \tag{3.4}$$

At this point we need to distinguish the case of positive solutions $v_+ = V_+ + \phi_+$ from the case of sign changing solutions $v_- = V_- + \phi_-$. With obvious notation, define the continuous function $g_\pm(\mu)$

$$g_\pm(\mu) = \int_{\mathbb{R}^N} h_\pm Z_{1k}. \tag{3.5}$$

Observe that equation (3.4) is equivalent to find μ so that

$$g_\pm(\mu) = 0$$

Define the constant Γ_N^+ to be given by

$$\Gamma_N^+ = \lim_{k\to\infty} \frac{1}{k^{N-2}} \sum_{j\neq 1} \frac{1}{|\hat{\xi}_j - \hat{\xi}_1|^{N-2}}, \qquad N \geq 5, \tag{3.6}$$

where $\hat{\xi}_j = (e^{\frac{2\pi(j-1)i}{k}}, 0) \in \mathbb{C} \times \mathbb{R}^{N-2}$. We claim that Γ_N^+ is a positive number. Indeed, observe that $|\hat{\xi}_j - \hat{\xi}_1| = \frac{2\pi(j-1)}{k}\left(1 + O(\frac{1}{k})\right)$ as $k \to \infty$. Thus

$$\Gamma_N^+ = \frac{1}{(2\pi)^{N-2}} \sum_{j=1}^{\infty} \frac{1}{j^{N-2}} > 0.$$

Furthermore, we define the constant Γ_N^- to be given by

$$\Gamma_N^- = \lim_{k\to\infty} \frac{1}{k^{N-2}} \sum_{j\neq 1} (-1)^{j+1} \frac{1}{|\hat{\xi}_j - \hat{\xi}_1|^{N-2}}, \quad N \geq 5. \tag{3.7}$$

A simple analysis shows that

$$\Gamma_N^- = \frac{1}{(2\pi)^{N-2}} \sum_{j=1}^{\infty} \frac{(-1)^j}{j^{N-2}} = -\frac{1}{(2\pi)^{N-2}} \left(1 - \sum_{j=2}^{\infty} \frac{(-1)^j}{j^{N-2}}\right) < 0,$$

since $\left|\sum_{j=2}^{\infty} \frac{(-1)^j}{j^{N-2}}\right| \leq \sum_{j=2}^{\infty} \frac{1}{j^{N-2}} \leq \int_1^{\infty} \frac{1}{x^{N-2}}\, dx = \frac{1}{N-3} < 1$.

Then we claim that the expression of $g_{\pm}(\mu)$ can be explicitly computed as follows

$$g_+(\mu) = \frac{1}{k^{1+\frac{2}{N-4}}} \left[-\gamma\mu a_N + \mu^{N-3}\Gamma_N^+ b_N + \frac{1}{k^{\frac{4}{N(N-4)}}}\Theta_k(\mu)\right] \tag{3.8}$$

and

$$g_-(\mu) = \frac{1}{k^{1+\frac{2}{N-4}}} \left[-\gamma\mu a_N + \mu^{N-3}\Gamma_N^- b_N + \frac{1}{k^{\frac{4}{N(N-4)}}}\Theta_k(\mu)\right] \tag{3.9}$$

In (3.8) and (3.9) a_N and b_N are positive constants that depend on the dimension N. Furthermore $\Theta_k(\mu)$ denotes a generic continuous function of the variable μ, which is uniformly bounded as $k \to \infty$.

Observe now that if $\gamma > 0$, then the function $g_+(\mu)$ has a positive zero

$$\mu = \left(\frac{\gamma a_N}{b_N \Gamma_N^+}\right)^{\frac{1}{N-4}} + O\left(\frac{1}{k^{\frac{4}{N(N-4)}}}\right).$$

This fact proves the existence to

$$\Delta v - \gamma\frac{v}{|x|^2} + |v|^{\frac{4}{N-2}} v = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, v = V + \phi. \tag{3.10}$$

We claim that $v^- \equiv 0$, where v^- denotes the negative part of v. In fact, if $v^- \not\equiv 0$, then multiplying (1.13) by v^- and using Sobolev inequality yield

$$\int_{\mathbb{R}^N\setminus\{0\}} |\nabla v^-|^2 + \frac{\gamma}{|x|^2}(v^-)^2 \geq C > 0$$

where $C > 0$ is a generic constant. This is impossible due to the structure of $v = V + \phi$ and the estimate of ϕ. As a consequence, we obtain a positive solution of the form (1.15) predicted by Theorem 1.2, Part a.

On the other hand, if $-\left(\frac{N-2}{2}\right)^2 < \gamma < 0$, then the function $g_+(\mu)$ has a positive zero

$$\mu = \left(\frac{\gamma a_N}{b_N \Gamma_N^-}\right)^{\frac{1}{N-4}} + O(\frac{1}{k^{\frac{4}{N(N-4)}}}).$$

This fact proves the existence of the sign changing solution of the form (1.17) predicted by Theorem 1.2, Part b.

An adaptation of the arguments that we will use below to prove estimates (3.8) and (3.9) give the expansion of the energy (1.16) and (1.18). We omit the proof of this fact.

The rest of the Section will be devoted to prove (3.8). In exactly the same way one gets (3.9).

In the rest of the proof, with $\Theta_k(\mu)$ we denote a generic continuous function of the variable μ that is uniformly bounded as $k \to \infty$.

Expansion (3.8) is consequence of three facts:

(3.11)
$$\int_{\mathbb{R}^N} \zeta_1 E Z_{1\varepsilon} = \frac{1}{k^{2+\frac{4}{N-4}}} \left[-\gamma\mu(\int_{\mathbb{R}^N} U(y)Z(y)dy) + p\mu^{N-3}\Gamma_N(\int_{\mathbb{R}^N} U^{p-1}(y)Z(y)\,dy) + \frac{1}{k}\Theta_k(\mu)\right]$$

with

$$\int_{\mathbb{R}^N} U(y)Z(y)dy > 0 \quad \text{and} \quad \int_{\mathbb{R}^N} U^{p-1}(y)Z(y)\,dy > 0, \tag{3.12}$$

$$\int_{\mathbb{R}^N} \mathcal{L}(\phi_1) Z_{1\varepsilon} = \frac{1}{k^{2+\frac{6}{N-4}}}\Theta_k(\mu) \tag{3.13}$$

and

$$\int_{\mathbb{R}^N} \mathcal{N}(\phi_1) Z_{1\varepsilon} = \frac{1}{k^{2+\frac{8}{N-4}}}\Theta_k(\mu) \tag{3.14}$$

The rest of this section is devoted to give the proof of expansions (3.11)-(3.12)-(3.13)-(3.14).

The computation near the bubble part is similar to those in [28]. We need to assert the new contribution from the Hardy-operator.

Proof of (3.11) and (3.12). Given the definition of the cut off function ζ_1, we write

$$\int_{\mathbb{R}^N} \zeta_1 E Z_{1\varepsilon} = \int_{B(\xi_1, \frac{2\eta}{k})} \zeta_1 E Z_{1\varepsilon} = \int_{B(\xi_1,\frac{\eta}{k})} E Z_{1\varepsilon} + \int_{B(\xi_1,\frac{2\eta}{k})\setminus B(\xi_1,\frac{\eta}{k})} \zeta_1 E Z_{1\varepsilon} = A + B. \tag{3.15}$$

We start with A. Recall that

$$E = E_1(x) - E_2(x), \quad E_1(x) = \Delta V + V^p, \quad E_2(x) = \frac{\gamma}{|x|^2} V(x).$$

We have the validity of the following expansion

$$\int_{B(\xi_1,\frac{\eta}{k})} E_2 Z_{1\varepsilon}\,dx = \gamma\mu k^{-2-\frac{4}{N-4}} \left(\int_{\mathbb{R}^N} U(y)Z(y)\,dy\right)\left(1 + \frac{1}{k}\Theta_k(\mu)\right). \tag{3.16}$$

Indeed, we have by definition

$$\int_{B(\xi_1,\frac{2\eta}{k})} E_2 Z_{1\varepsilon}\,dx = \varepsilon^{-\frac{N-2}{2}} \sum_{j=1}^{k} \int_{B(\xi_1,\frac{2\eta}{k})} \frac{\gamma}{|x|^2} V_{j\varepsilon}(x) Z(\frac{x-\xi_1}{\varepsilon}) \tag{3.17}$$

The main term is the integral of $\frac{\gamma}{|x|^2}V_{1\varepsilon}Z_{1\varepsilon}$ over the ball $B(\xi_1,\frac{\eta}{k})$:

$$\begin{aligned}
\varepsilon^{-\frac{N-2}{2}} \int_{B(\xi_1,\frac{\eta}{k})} \frac{\gamma}{|x|^2} V_{1\varepsilon} Z(\frac{x-\xi_1}{\varepsilon}) &= \varepsilon^2 \int_{B(0,\frac{\eta}{\varepsilon k})} \frac{\gamma}{|\varepsilon y+\xi_1|^2} U(y)Z(y) \\
&= \varepsilon^2 [\int_{B(0,\frac{\eta}{\varepsilon k})} U(y)Z(y)\,dy + \int_{B(0,\frac{\eta}{\varepsilon k})} (\frac{1}{|\varepsilon y+\xi_1|^2} - 1) U(y)Z(y)\,dy] \\
&= \mu k^{-2-\frac{4}{N-4}} \int_{\mathbb{R}^N} U(y)Z(y)\,dy - \mu k^{-2-\frac{4}{N-4}} \int_{\mathbb{R}^N\setminus B(0,\frac{\eta}{\varepsilon k})} U(y)Z(y)\,dy \\
&\quad + \mu k^{-2-\frac{4}{N-4}} \varepsilon^2 \int_{B(0,\frac{\eta}{\varepsilon k})} |y|^2 U(y)Z(y)\,dy (1+o(1)\Theta_k(\mu))
\end{aligned} \tag{3.18}$$

where $o(1)\to 0$ as $k\to\infty$ and $\Theta_k(\mu)$ is a continuous function in the variable μ, uniformly bounded as $k\to\infty$. Observe now that

$$\varepsilon^2 \int_{B(0,\frac{\eta}{\varepsilon k})} |y|^2 U(y)Z(y)\,dy = C \begin{cases} \frac{\varepsilon}{k}\Theta_k(\mu) & \text{if} \quad N=5, \\ \varepsilon^2 |\log k| \Theta_k(\mu) & \text{if} \quad N=6, \\ \varepsilon^2 \Theta_k(\mu) & \text{if} \quad N>6 \end{cases}$$

and that

$$\int_{\mathbb{R}^N\setminus B(0,\frac{\eta}{\varepsilon k})} U(y)Z(y)\,dy = (\varepsilon k)^{N-4}\Theta_k(\mu).$$

Thus we conclude that

$$\varepsilon^{-\frac{N-2}{2}} \int_{B(\xi_1,\frac{\eta}{k})} \frac{\gamma}{|x|^2} V_{1\varepsilon} Z(\frac{x-\xi_1}{\varepsilon}) = \mu k^{-2-\frac{4}{N-4}} (\int_{\mathbb{R}^N} U(y)Z(y)\,dy) \left(1+\frac{1}{k}\Theta_k(\mu)\right) \tag{3.19}$$

Now we estimate the other terms in (3.17). For instance consider $j=2$. Performing the change of variables $\frac{x-\xi_1}{\varepsilon}=y$ we have

$$\left| \varepsilon^{-\frac{N-2}{2}} \int_{B(\xi_1,\frac{\eta}{k})} \frac{\gamma}{|x|^2} V_{2\varepsilon} Z(\frac{x-\xi_1}{\varepsilon}) \right| = \left| \varepsilon^2 \int_{B(0,\frac{\eta}{\varepsilon k})} U(y+\varepsilon^{-1}(\xi_1-\xi_2)) Z(y) \right|$$

$$\le C\varepsilon^2 (k\varepsilon)^{N-2} \int_0^{\frac{\eta}{\varepsilon k}} \frac{t^{N-1}}{(1+t)^{N-2}} \le C k^{-2-\frac{4}{N-4}} k^{-2},$$

where we used the fact that, if $|y|\le \frac{\eta}{\varepsilon k}$ then $|U(y+\varepsilon^{-1}(\xi_1-\xi_2))| \le C(\varepsilon k)^{N-2}$.

We thus conclude that

$$\varepsilon^{-\frac{N-2}{2}} \int_{B(\xi_1,\frac{\eta}{k})} \frac{\gamma}{|x|^2} V_{2\varepsilon} Z(\frac{x-\xi_1}{\varepsilon}) = k^{-2} k^{-2-\frac{4}{N-4}} \Theta_k(\mu)$$

and hence

$$\varepsilon^{-\frac{N-2}{2}}\int_{B(\xi_1,\frac{\eta}{k})}\frac{\gamma}{|x|^2}\left(\sum_{j\neq 1}V_{j\varepsilon}(x)\right)Z(\frac{x-\xi_1}{\varepsilon})=k^{-2-\frac{2}{N-4}}k^{-1}\Theta_k(\mu).$$

This last fact, together with (3.13) and (3.19), give the validity of (3.16).

Let us now evaluate $\int_{B(\xi_1,\frac{\eta}{k})}E_1Z_{1\varepsilon}$. Recall that

$$E_1(x)=[V^p-\sum_{j=1}^k V_j^p]$$

In the ball $B(\xi_1,\frac{\eta}{k})$, we perform the natural change of variables $\frac{x-\xi_1}{\varepsilon}=y$, that gives

$$V(x)=\varepsilon^{-\frac{N-2}{2}}\left[U(y)+\sum_{j\neq 1}U(y+\varepsilon^{-1}(\xi_1-\xi_j))\right].$$

Define $\tilde U(y)=\sum_{j\neq 1}U(y+\varepsilon^{-1}(\xi_1-\xi_j))$. Thus,

$$\begin{aligned}I&=\int_{B(0,\frac{\eta}{\varepsilon k})}\left([U+\tilde U]^p-[U^p+\sum_{j\neq 1}U^p(y+\varepsilon^{-1}(\xi_1-\xi_j))]\right)Z(y)\,dy\\&=p\left(\int_{B(0,\frac{\eta}{\varepsilon k})}U^{p-1}\tilde U Z(y)\,dy+\int_{B(0,\frac{\eta}{\varepsilon k})}[(U-s\tilde U)^{p-1}-U^{p-1}]\tilde U Z(y)\,dy\right.\\&\left.+\int_{B(0,\frac{\eta}{\varepsilon k})}\sum_{j\neq 1}U^p(y+\varepsilon^{-1}(\xi_1-\xi_j))Z(y)\,dy\right)=I_1+I_2+I_3\end{aligned}\tag{3.20}$$

We start with the observation that

$$\begin{aligned}I_1&=\varepsilon^{N-2}\left(\sum_{j\neq 1}\frac{1}{|\xi_1-\xi_j|^{N-2}}\right)\left[p\int_{B(0,\frac{\eta}{\varepsilon k})}U^{p-1}Z\right](1+(\varepsilon k)^2O(1))\\&=p\Gamma_N(\int_{\mathbb{R}^N}U^{p-1}Z)(\varepsilon k)^{N-2}+(\varepsilon k)^N\Theta_k(\mu)\\&=\frac{1}{k^{2+\frac{4}{N-4}}}\left[\mu^{N-3}p\Gamma_N^+(\int_{\mathbb{R}^N}U^{p-1}Z)+\frac{1}{k^{\frac{4}{N-4}}}\Theta_k(\mu)\right].\end{aligned}\tag{3.21}$$

Let us now evaluate I_2. We use the inequality $|(a+b)^s-a^s|\le C|b|^s$, for any $0<s<1$, to get first that, for $y\in B(0,\frac{\eta}{\varepsilon k})$, one has

$$\tilde U(y)\le Ck(\varepsilon k)^{N-2}.$$

Thus we get

$$I_2=\varepsilon^{N-2}k(\varepsilon k)^{N-2}(\int_0^{\frac{1}{\varepsilon k}}\frac{t^{N-1}}{(1+t)^{N+2}}\,dt)\Theta_k(\mu)=k^{-2-\frac{4}{N-4}}k^{-\frac{2}{N-4}}\Theta_k(\mu).$$

In an analogous way one gets

$$I_3 = k^{-2-\frac{4}{N-4}} k^{-\frac{2}{N-4}} \Theta_k(\mu).$$

With this we conclude that

$$\int_{B(\xi_1,\frac{\eta}{\varepsilon})} E_1 Z_{1k} = \frac{1}{k^{1+\frac{2}{N-4}}} \left[\mu^{N-3} p \Gamma_N \left(\int_{\mathbb{R}^N} U^{p-1} Z \right) + \frac{1}{k} \Theta_k(\mu) \right]. \tag{3.22}$$

Estimates (3.16) and (3.22) give the computation of the term A in (3.15).

We will next check that B is smaller than A in (3.15). We write

$$\int_{B(\xi_1,\frac{2\eta}{k})\setminus B(\xi_1,\frac{\eta}{k})} \zeta_1 E Z_{1\varepsilon} = \int_{B(\xi_1,\frac{2\eta}{k})\setminus B(\xi_1,\frac{\eta}{k})} \zeta_1 E_1 Z_{1\varepsilon} - \int_{B(\xi_1,\frac{2\eta}{k})\setminus B(\xi_1,\frac{\eta}{k})} \zeta_1 E_2 Z_{1\varepsilon}$$

Arguing as in the proof of estimate (3.16) (see in particular (3.18)), we have that

$$\left| \int_{B(\xi_1,\frac{2\eta}{k})\setminus B(\xi_1,\frac{\eta}{k})} \zeta_1 E_2 Z_{1\varepsilon} \right| \le C\varepsilon^2 \int_{\frac{\eta}{k\varepsilon}}^{\frac{2\eta}{k\varepsilon}} \frac{t^{N-1}}{(1+t^2)^{N-2}} \le C\varepsilon^2 (k\varepsilon)^{N-2}$$

thus we get

$$\int_{B(\xi_1,\frac{2\eta}{k})\setminus B(\xi_1,\frac{\eta}{k})} \zeta_1 E_2 Z_{1\varepsilon} = k^{-2-\frac{4}{N-2}} \varepsilon^2 \Theta_k(\mu). \tag{3.23}$$

On the other hand, arguing as in (3.20), (3.21) and (3.22) we have

$$\left| \int_{B(\xi_1,\frac{2\eta}{k})\setminus B(\xi_1,\frac{\eta}{k})} \zeta_1 E_1 Z_{1\varepsilon} \right| \le C(\varepsilon k)^{N-2} \int_{\frac{\eta}{k\varepsilon}}^{\frac{2\eta}{k\varepsilon}} \frac{t^{N-1}}{(1+t^2)^{\frac{N+2}{2}}} \le C\varepsilon^2 (k\varepsilon)^{N-2}$$

thus we get

$$\int_{B(\xi_1,\frac{2\eta}{k})\setminus B(\xi_1,\frac{\eta}{k})} \zeta_1 E_1 Z_{1\varepsilon} = k^{-2-\frac{4}{N-2}} \varepsilon^2 \Theta_k(\mu). \tag{3.24}$$

From (3.23) and (3.24) we conclude that

$$B = k^{-4-\frac{8}{N-4}} \Theta_k(\mu). \tag{3.25}$$

Estimate (3.11) thus follows from (3.15), (3.16), (3.22) and (3.25).

Finally, since (see (2.2) and (2.28))

$$Z(y) = \left(\frac{\partial}{\partial \mu} U_\mu(y) \right)_{\mu=1}, \quad \text{where} \quad U_\mu(y) = \alpha_N \left(\frac{\mu}{(\mu^2 + |y|^2)} \right)^{\frac{N-2}{2}}$$

we have that

$$\int_{\mathbb{R}^N} U(y) Z(y) = \left(\frac{\partial}{\partial \mu} \int_{\mathbb{R}^N} U_\mu^2 \right)_{\mu=1}, \quad \text{and} \quad \int_{\mathbb{R}^N} U^{p-1}(y) Z(y) = \left(\frac{\partial}{\partial \mu} \int_{\mathbb{R}^N} U_\mu^p \right)_{\mu=1}$$

Thus we have the validity of (3.12) because

$$\int_{\mathbb{R}^N} U_\mu^2 = \mu^2 \int_{\mathbb{R}^N} U^2, \quad \text{and} \quad \int_{\mathbb{R}^N} U_\mu^p = \mu^{\frac{N-2}{2}} \int_{\mathbb{R}^N} U^p.$$

Proof of (3.13). Referring to (3.2), we first observe that

$$\left|\int_{\mathbb{R}^N} p(V^{p-1}\zeta_1 - V_{1\varepsilon}^{p-1})\phi_1 Z_{1\varepsilon}\right| \le C\left|\int_{B(\xi_1,\frac{2\eta}{k})}(V^{p-1}\zeta_1 - V_{1\varepsilon}^{p-1})\phi_1 Z_{1\varepsilon}\right|$$

$$\le C\int_{B(\xi_1,\frac{2\eta}{k})} V_{1\varepsilon}^{p-1}\phi_1\left(\sum_{j>1}V_{j\varepsilon}\right) \le C\|\phi_1\|_{1*}\int_{B(\xi_1,\frac{2\eta}{k})} V_{1\varepsilon}^{p}\left(\sum_{j>1}V_{j\varepsilon}\right)$$

(performing the change of variables $\varepsilon y = x-\xi_1$)

$$= C\|\phi_1\|_{1*}\int_{B(0,\frac{2\eta}{\varepsilon k})} U^p\left(\sum_{j>1}\left(\frac{1}{(1+|y-\varepsilon^{-1}(\xi_j-\xi_1)|^2}\right)^{\frac{N-2}{2}}\right) \le C(\varepsilon k)^{N-2}\|\phi_1\|_{1*}$$

where we have used the fact that in the region we are considering we have $\sum_{j>1}\left(\frac{1}{(1+|y-\varepsilon^{-1}(\xi_j-\xi_1)|^2}\right)^{\frac{N-2}{2}} \le C(\varepsilon k)^{N-2}$. Collecting the above estimates, we conclude that

$$\int_{\mathbb{R}^N} p(V^{p-1}\zeta_1 - V_{1\varepsilon}^{p-1})\phi_1 Z_{1\varepsilon} = k^{-2-\frac{4}{N-4}-\frac{2}{(N-4)}}\Theta_k(\mu) \tag{3.26}$$

where $\Theta_k(\mu)$ denotes a continuous function of μ, which is uniformly bounded as $k\to\infty$. To conclude the estimate (3.13), we observe that

$$\left|\int_{\mathbb{R}^N}\zeta_1\frac{\gamma}{|x|^2}\zeta_1\phi_1 Z_{1\varepsilon}\right| \le C\|\zeta_1\frac{\gamma}{|x|^2}\zeta_1\phi_1\|_{\frac{2N}{N+2}}\|Z_{1\varepsilon}\|_{\frac{2N}{N-2}} \le C\|\zeta_1\frac{\gamma}{|x|^2}\zeta_1\phi_1\|_{\frac{2N}{N+2}}.$$

We next estimate $\|\frac{\gamma}{|x|^2}\zeta_1\phi_1\|_{\frac{2N}{N+2}}$. A direct use of Holder inequality gives

$$\|\frac{\gamma}{|x|^2}\zeta_1\phi_1\|_{\frac{2N}{N+2}} \le C\left(\int_{B(\xi_1,\frac{c}{k})}|\phi_1|^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}} \le C(k^{-N})^{\frac{2}{N}}\|\phi_1\|_{\frac{2N}{N-2}}.$$

Thus we conclude

$$\left|\int_{\mathbb{R}^N}\zeta_1\frac{\gamma}{|x|^2}\zeta_1\phi_1 Z_{1\varepsilon}\right| \le Ck^{-3-\frac{6}{N-4}} \tag{3.27}$$

Estimate (3.13) follows directly from (3.26) and (3.27).

Proof of (3.14). Since (2.23) holds, we get

$$\left|\int \zeta_1 p|V|^{p-1}\psi(\phi_1)Z_{1\varepsilon}\right| \le C\|\zeta_1 p|V|^{p-1}\psi(\phi_1)\|_{\frac{2N}{N+2}}\|Z_{1\varepsilon}\|_{\frac{2N}{N-2}} \le C\|\zeta_1 p|V|^{p-1}\|_{\frac{N}{2}}\|\zeta_1\psi(\phi_1)\|_{\frac{2N}{N-2}}$$

$$\le Ck^{-2-\frac{8}{N-4}}. \tag{3.28}$$

Referring to (3.3), we have that

$$|\zeta_1 N(\phi_1)| \le C\zeta_1 V_{1\varepsilon}^{p-2}\left(\sum_j|\phi_j|^2+|\psi|^2\right)$$

Thus, we get

$$\left|\int_{\mathbb{R}^N}\zeta_1 N(\phi_1)Z_{1,\varepsilon}\right| \le C\left|\int_{B(\xi_1,\frac{\eta}{k})}\zeta_1 V_{1\varepsilon}^{p-1}\left(\sum_j|\phi_j|^2+|\psi|^2\right)\right|$$
$$\le C\|V_{1\varepsilon}^{(p-1)}\|_{\frac{N}{2}}\left[\|\zeta_1\phi_1\|^2_{\frac{2N}{N-2}}+\sum_{j\neq 1}\|\zeta_1\phi_j\|^2_{\frac{2N}{N-2}}+\|\zeta_1\psi\|^2_{\frac{2N}{N-2}}\right].$$

Observe now that $\|V_{1\varepsilon}^{(p-1)}\|_{\frac{N}{2}}\le C$, and $\|\zeta_1\phi_1\|^2_{\frac{2N}{N-2}}\le C\|\phi_1\|^2_{1*}$, while for $j\neq 1$, one has

$$\|\zeta_1\phi_j\|_{\frac{2N}{N-2}}\le C\|\phi_j\|_{j*}\left(\int_{B(\xi_1,\frac{\eta}{k})}(\frac{\varepsilon}{\varepsilon^2+|x-\xi_j|^2})^N\right)^{\frac{N-2}{2}}\le C(\varepsilon k)^{\frac{N-2}{2}}\|\phi_j\|_{j*}\le Ck^{-1-\frac{2}{N-4}}\|\phi_1\|_{1*}.$$

Collecting the above estimates and using (2.23), we get

$$\int_{\mathbb{R}^N}\zeta_1 N(\phi_1)Z_{1,\varepsilon}=k^{-2-\frac{12}{N-4}}\Theta_k(\mu), \tag{3.29}$$

where $\Theta_k(\mu)$ is a continuous function of μ, uniformly bounded as $k\to\infty$.

Estimate (3.14) follows directly from (3.28) and (3.29).

This concludes the proof of the Theorem.

4. Appendix 1: Estimate of the error and Proof of Proposition 2.1

This section is devoted to estimate the error term E defined in (2.6). We write $E=E_1-E_2$, where

$$E_1(x)=\Delta V+|V|^{p-1}V,\quad \text{and}\quad E_2(x)=\frac{\gamma}{|x|^2}V. \tag{4.1}$$

Denote $q=\frac{2N}{N+2}$. Let $\eta>0$ be a small number and decompose the entire space $\mathbb{R}^N$ as follows

$$\mathbb{R}^N=\bigcup_{j=1}^k B(\xi_j,\frac{\eta}{k})\bigcup R$$

We estimate the error in each of the $k+1$-subsets.

Interior estimate. We first estimate the L^q norm of the error in each ball $B(\xi_j,\frac{\eta}{k})$.

Let us fix j, say $j=1$, and observe that, if we denote by

$$\tilde E(y)=\varepsilon^{\frac{N+2}{2}}E(\varepsilon y+\xi_1),\quad |y|\le\frac{\eta}{k\varepsilon}$$

we have

$$\|E\|_{L^q(|x-\xi_1|\le\frac{\eta}{k})}=\|\tilde E\|_{L^q(|y|\le\frac{\eta}{k\varepsilon})}$$

Let $\tilde E_i(y)=\varepsilon^{\frac{N+2}{2}}E_i(\varepsilon y+\xi_1)$, for $i=1,2$. In $|y|\le\frac{\eta}{k\varepsilon}$, we have, for some $0<s<1$,

$$\tilde E_1(y)=p\left(U(y)+s(\sum_{j\neq 1}U(y-\varepsilon^{-1}(\xi_j-\xi_1)))\right)^{p-1}\times$$
$$\times\left[\sum_{j\neq 1}U(y-\varepsilon^{-1}(\xi_j-\xi_1)))\right]-\sum_{j\neq 1}U^p(y-\varepsilon^{-1}(\xi_j-\xi_1))$$

where U is the basic cell in our construction, defined in (2.2). Notice that $\varepsilon^{-1}|\xi_i-\xi_j|\sim(\varepsilon k)^{-1}|j-i|$ so that

$$U(y-\varepsilon(\xi_j-\xi_1))\le C\frac{\varepsilon^{N-2}k^{N-2}}{|j-1|^{N-2}}\quad\text{for}\quad |y|\le\frac{\eta}{\varepsilon k}.$$

With this in mind, we can estimate in the region $|y|\le\frac{\eta}{\varepsilon k}$,

$$|\tilde E_1(y)|\le C\frac{(\varepsilon k)^{N-2}}{(1+|y|^2)^2}\tag{4.2}$$

and hence

$$\int_{|y|\le\frac{\eta}{\varepsilon k}}|\tilde E_1|^q\le C\begin{cases}(\varepsilon k)^{(N-2)q} & \text{if}\quad N=5,\\ (\varepsilon k)^{(N-2)q}|\log k| & \text{if}\quad N=6,\\ (\varepsilon k)^{(N-2)q+4q-N} & \text{if}\quad N\ge 7.\end{cases}$$

Thus we conclude that

$$\|E_1\|_{L^q(B(\xi_1,\frac{\eta}{k}))}=\|\tilde E_1\|_{L^q(B(0,\frac{\eta}{\varepsilon k}))}\le C\begin{cases}k^{-6} & \text{if}\quad N=5,\\ k^{-4}|\log k|^{\frac1q} & \text{if}\quad N=6,\\ k^{-1-\frac{6}{N-4}} & \text{if}\quad N\ge 7.\end{cases}\tag{4.3}$$

On the other hand, in the region $|y|\le\frac{\eta}{\varepsilon k}$, we have that

$$|\tilde E_2(y)|\le C\varepsilon^2U(y),$$

so

$$\|\varepsilon^2U\|_{L^q(|y|\le\frac{\eta}{\varepsilon k})}\le C\varepsilon^{2q}\int_0^{\frac{\eta}{\varepsilon k}}\frac{t^{N-1}}{(1+t)^{(N-2)q}}\le C\begin{cases}\varepsilon^{2q}(k\varepsilon)^{(N-2)q-N} & \text{if}\quad N=5,\\ \varepsilon^{2q}|\log\varepsilon| & \text{if}\quad N=6,\\ \varepsilon^{2q} & \text{if}\quad N\ge 7.\end{cases}$$

Thus we conclude

$$\|\tilde E_2\|_{L^q(B(\xi_1,\frac{\eta}{k}))}\le C\begin{cases}k^{-5} & \text{if}\quad N=5,\\ k^{-4}|\log k|^{\frac1q} & \text{if}\quad N=6,\\ k^{-2-\frac{4}{N-4}} & \text{if}\quad N\ge 7.\end{cases}\tag{4.4}$$

Collecting (4.3) and (4.4), we conclude that

$$\|E\|_{L^q(B(\xi_j,\frac{\eta}{k}))}\le C\begin{cases}k^{-5} & \text{if}\quad N=5,\\ k^{-4}|\log k|^{\frac1q} & \text{if}\quad N=6,\\ k^{-1-\frac{6}{N-4}} & \text{if}\quad N\ge 7.\end{cases}\tag{4.5}$$

Exterior region. We now turn to the exterior region

$$R = \mathbb{R}^N \setminus \bigcup_{j=1}^{k} B(\xi_j, \frac{\eta}{k}).$$

In this region, we have that $V_{j\varepsilon}(x) \le C \frac{\varepsilon^{\frac{N-2}{2}}}{|x-\xi_j|^{N-2}}$ and thus

$$|E_1(x)| \le C \sum_{j=1}^{k} \frac{\varepsilon^{\frac{N+2}{2}}}{|x-\xi_j|^{N+2}} \quad \text{and} \quad |E_2(x)| \le C \sum_{j=1}^{k} \frac{\varepsilon^{\frac{N-2}{2}}}{|x|^2|x-\xi_j|^{N-2}} \tag{4.6}$$

We start with the computation of the L^q-norm of E_1 in R. We have

$$\left(\int_R |E_1|^q\right)^{\frac{1}{q}} \le Ck \left(\int_{\mathbb{R}^N \setminus B(\xi_j, \frac{\eta}{k})} \frac{\varepsilon^{(\frac{N+2}{2})q}}{|x-\xi_j|^{(N+2)q}}\right)^{\frac{1}{q}} \le Ck\varepsilon^{(\frac{N+2}{2})} \left[\int_{B(\xi_j,1)\setminus B(\xi_j,\frac{\eta}{k})} \frac{1}{|x-\xi_j|^{(N+2)q}} + O(1)\right]^{\frac{1}{q}}.$$

Since

$$\int_{B(\xi_j,1)\setminus B(\xi_j,\frac{\eta}{k})} \frac{1}{|x-\xi_j|^{(N+2)q}} \le k^N,$$

we conclude that

$$\|E_1\|_{L^q(R)} \le Ck^{-\frac{6}{N-4}}, \quad \text{for any} \quad N \ge 5. \tag{4.7}$$

We now compute the L^q-norm in the region R of the part E_2. We separate R into a region close to 0 and the rest, we get

$$\|E_2\|_{L^q(R)} \le \left(\|E_2\|^q_{L^q(B(0,\frac{1}{2}))} + \|E_2\|^q_{L^q(R\setminus B(0,\frac{1}{2}))}\right)^{\frac{1}{q}}$$

Observe that E_2 has a singularity at 0, but

$$\|E_2\|^q_{L^q B(0,\frac{1}{2})} \le C(k\varepsilon^{\frac{N-2}{2}})^q \left(\int_{B(0,\frac{1}{2})} \frac{1}{|x|^{\frac{4N}{N+2}}}\right) \le C(k\varepsilon^{\frac{N-2}{2}})^q$$

thanks to $N > 2$. Thus, arguing as in the previous estimate (4.7), the size of $\|E_2\|_{L^q(R)}$ is given by the integral over a region close to $\partial B(\xi_j, \frac{\eta}{k})$. Indeed, we have

$$\|E_2\|^q_{L^q(R)} \le C(k\varepsilon^{(\frac{N-2}{2})})^q \left(\int_{\frac{\eta}{k}\le|x-\xi_j|\le 1} \frac{1}{|x-\xi_j|^{(N-2)q}}\right) \le C \begin{cases} k^{(-3-\frac{1}{2})q} & \text{if} \quad N=5, \\ k^{-3q}|\log k|^q & \text{if} \quad N=6, \\ k^{(-1-\frac{N-2}{N-4})q} & \text{if} \quad N \ge 7. \end{cases}$$

Thus we conclude that

$$\|E_2\|_{L^q(R)} \le C \begin{cases} k^{-3-\frac{1}{2}} & \text{if} \quad N=5, \\ k^{-3}|\log k| & \text{if} \quad N=6, \\ k^{-2-\frac{2}{N-4}} & \text{if} \quad N \ge 7. \end{cases} \tag{4.8}$$

Collecting together (4.7) and (4.8) we conclude that

$$\|E\|_{L^q(R)} \le C \begin{cases} k^{-3-\frac{1}{2}} & \text{if} \quad N=5, \\ k^{-3}|\log k|^{\frac{1}{q}} & \text{if} \quad N=6, \\ k^{-\frac{6}{N-4}} & \text{if} \quad N \ge 7. \end{cases} \tag{4.9}$$

This concludes the proof of Proposition 2.1.

5. Appendix 2: Proof of Lemma 2.2

Proof of Lemma 2.2. The result stated in Lemma 2.2 will be a consequence of a corresponding linear result and an application of the Contraction Mapping Principle. Thus let us first consider the linear problem

$$\Delta\psi - \frac{\gamma}{|x|^2}\psi = h \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \tag{5.1}$$

where h belongs to the space $\mathcal{L}^{\frac{2N}{N+2}}$ defined in (2.26). Hardy Inequality guarantees that if $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, then $\frac{u}{|x|} \in L^2(\mathbb{R}^N)$ and

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\, dx \le \int_{\mathbb{R}^N} |\nabla u|^2. \tag{5.2}$$

For any $\gamma > -(\frac{N-2}{2})^2$, we define the Hilbert space $\mathcal{D}_\gamma$ given by $\mathcal{D}^{1,2}(\mathbb{R}^N)$ equipped with the scalar product

$$(u,v)_\gamma = \int_{\mathbb{R}^N} [\nabla u \nabla v + \gamma \frac{u\,v}{|x|^2}].$$

We denote with $\|\cdot\|_\gamma$ the corresponding norm and with $\|\cdot\|$ the natural norm in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Inequality (5.2) gives that

$$(1+\frac{\gamma}{C_N})^{\frac{1}{2}}\|u\| \le \|u\|_\gamma$$

where $C_N = \left(\frac{N-2}{2}\right)^2$. Observe that

$$\|u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \le [S(1+\frac{\gamma}{C_N})]^{-1}\|u\|_\gamma \tag{5.3}$$

where S is the best Sobolev constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$.

Let us denote with T_γ the embedding $T_\gamma : \mathcal{D}_\gamma \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. Then the adjoint operator $T_\gamma^* : L^{\frac{2N}{N+2}}(\mathbb{R}^N) \to \mathcal{D}_\gamma$ defined as

$$\psi = T_\gamma^*(h) \quad \Leftrightarrow \quad \psi \quad \text{is the unique solution of} \quad \Delta\psi - \frac{\gamma}{|x|^2}\psi = h \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}$$

is a continuous operator and

$$\|T_\gamma^*(h)\|_\gamma \le C\|h\|_{L^{\frac{2N}{N+2}}}.$$

Observe furthermore that if $h(x) = |x|^{-N-2}h(\frac{x}{|x|^2})$ then the function $\tilde\psi(x) = |x|^{-N+2}\psi(\frac{x}{|x|^2})$ also satisfies $\Delta\tilde\psi - \frac{\gamma}{|x|^2}\tilde\psi = h$ in $\mathbb{R}^N \setminus \{0\}$. By uniqueness we get that $\psi(x) = |x|^{-N+2}\psi(\frac{x}{|x|^2})$. In a very similar way, one can show that if $h(R_k x) = h(x)$, where R_k denotes the rotation in the first two variables of the angle $\frac{2\pi}{k}$, also ψ is invariant under that rotation. And finally, if h is even in the last $(N-2)$ variable, we also get that ψ is. We thus conclude that, in case $h \in \mathcal{L}^{\frac{2N}{N+2}}$, then $\psi = T_\gamma^*(h) \in \mathcal{L}^{\frac{2N}{N-2}}$, thanks to (5.3).

Let us go back to problem (2.18): Problem (2.18) is equivalent to

(5.4)
$$\psi = -T_\gamma^* \left((p|V|^{p-1} - \frac{\gamma}{|x|^2}) \sum_j (1-\zeta_j)\phi_j + (1-\Sigma_{j=1}^k \zeta_j)\,(\,p|V|^{p-1}\psi + \,E + N(\Sigma_{j=1}^k \phi_j + \psi)\,)\right) := \mathcal{M}(\psi)$$

Observe furthermore that if $\psi \in \mathcal{L}^{\frac{2N}{N-2}}$, then $\mathcal{M}(\psi) \in \mathcal{L}^{\frac{2N}{N-2}}$.

We will see that the operator $\mathcal{M}$ is a contraction mapping in the set

$$X = \{\psi \in \mathcal{L}^{\frac{2N}{N-2}} \,:\, \|\psi\|_{\frac{2N}{N-2}} \le cg(k)\}$$

for some $c > 0$, where $g(k)$ is defined in (2.22).

Referring to (5.4), Holder inequality gives

$$p\int_{\mathbb{R}^N} |(1-\sum_j \zeta_j)V^{p-1}\psi|^{\frac{2N}{N+2}} \le \left(\int_{\mathbb{R}^N} |(1-\sum_j \zeta_j)V^{p-1}|^{\frac{N+2}{4}}\right)^{\frac{4}{N+2}} \left(\int_{\mathbb{R}^N} |\psi|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N+2}}$$

Arguing as in the argument to get (4.9), we see that

$$\left(\int_{\mathbb{R}^N} |(1-\sum_j \zeta_j)V^{p-1}|^{\frac{N+2}{4}}\right)^{\frac{4}{N+2}} \le Ck \left(\int_{|x-\xi_j|>\frac{\eta}{k}} \frac{\varepsilon^{\frac{N+2}{2}}}{|x-\xi_j|^{N+2}}\right)^{\frac{4}{N+2}} \le C\,\varepsilon^2\, k^{1+\frac{8}{N+2}} \le Ck^{-1-\frac{4}{N-4}+\frac{8}{N+2}}$$

We thus conclude that

$$\|p(1-\sum_j \zeta_j)V^{p-1}\psi\|_{\frac{2N}{N+2}} \le o(1)\|\psi\|_{\frac{2N}{N-2}} \tag{5.5}$$

with $o(1) \to 0$ as $k \to \infty$.

We next estimate the $\frac{2N}{N+2}$-norm of the term $pV^{p-1}\sum_j(1-\zeta_j)\phi_j$. A direct use of Holder inequality gives

$$\|pV^{p-1}\sum_j(1-\zeta_j)\phi_j\|_{\frac{2N}{N+2}} \le Ck\|pV^{p-1}(1-\zeta_1)\phi_1\|_{\frac{2N}{N+2}} \le Ck\|\phi_1\|_{1*}\|V^{p-1}(1-\zeta_1)V_{1\varepsilon}\|_{\frac{2N}{N+2}}$$

Arguing as in the estimate (4.7), we get

$$k\|V^{p-1}(1-\zeta_1)V_{1\varepsilon}\|_{\frac{2N}{N+2}} \le Ck^{-\frac{6}{N-4}}$$

from which we conclude that

$$\|pV^{p-1}\sum_j(1-\zeta_j)\phi_j\|_{\frac{2N}{N+2}} \le Ck^{-\frac{6}{N-4}}\|\phi_1\|_{1*}. \tag{5.6}$$

On the other hand, we write

$$|(1-\sum_j \zeta_j)N(\sum_j \phi_j+\psi)| \le C|(1-\sum_j \zeta_j)| \left(\sum_j |\tilde\phi_j|^p + |\psi|^p\right),$$

from which we easily get

$$\|(1-\sum_j \zeta_j)N(\sum_j \phi_j+\psi)\|_{\frac{2N}{N+2}} \le C\left[\sum_j \|(1-\sum_i \zeta_i)|\phi_j|^p\|_{\frac{2N}{N+2}} + \|(1-\sum_i \zeta_i)|\psi|^p\|_{\frac{2N}{N+2}}\right]$$

Let us fix $j=1$. Holder inequality gives

$$\int \left[(1-\sum_i \zeta_i)|\phi_1|^p\right]^{\frac{2N}{N+2}} \le C\|\phi_1\|_{1*}^{\frac{2N}{N+2}} \int_{\mathbb{R}^N\setminus B(\xi_1,\frac{\eta}{k})} V_{1\varepsilon}^{\frac{2N}{N-2}} \le C(\varepsilon k)^N \|\phi_1\|_{1*}^{\frac{2N}{N+2}}$$

from which we conclude that

$$\|(1-\sum_i \zeta_i)|\phi_1|^p\|_{\frac{2N}{N+2}} \le Ck^{-1-\frac{6}{N-4}}\|\phi_1\|_{1*}.$$

On the other hand, a direct use of Holder inequality gives

$$\|(1-\sum_i \zeta_i)|\psi|^p\|_{\frac{2N}{N+2}} \le C\|\psi\|^p_{\frac{2N}{N-2}}.$$

We thus conclude that

$$\|(1-\sum_j \zeta_j)N(\sum_j \tilde\phi_+\psi)\|_{\frac{2N}{N+2}} \le C\left[\frac{1}{k^{\frac{6}{N-4}}}\|\phi_1\|^p_{\frac{2N}{N-2}} + \|\psi\|^p_{\frac{2N}{N-2}}\right]. \tag{5.7}$$

Next we shall estimate $\|\frac{\gamma}{|x|^2}\left(\sum_{j=1}^k (1-\zeta_j)\phi_j\right)\|_{\frac{2N}{N+2}}$. We start with the observation

$$\|\frac{\gamma}{|x|^2}\left(\sum_{j=1}^k (1-\zeta_j)\phi_j\right)\|_{\frac{2N}{N+2}} \le Ck\|\frac{\gamma}{|x|^2}(1-\zeta_1)\phi_1\|_{\frac{2N}{N+2}} \le Ck\|\phi_1\|_{1*}\|\frac{\gamma}{|x|^2}(1-\zeta_1)V_{1\varepsilon}\|_{\frac{2N}{N+2}}$$

Arguing as in (4.8) we get that $\|\frac{\gamma}{|x|^2}(1-\zeta_1)V_{1\varepsilon}\|_{\frac{2N}{N+2}} \le Ck^{-3-\frac{2}{N-4}}$. Thus, we conclude that

$$\|\frac{\gamma}{|x|^2}\left(\sum_{j=1}^k (1-\zeta_j)\phi_j\right)\|_{\frac{2N}{N+2}} \le Ck^{-2-\frac{2}{N-4}}\|\phi_1\|_{\frac{2N}{N-2}} \tag{5.8}$$

From estimates (5.1), (5.6), (5.7) and (5.8) we conclude that $\mathcal{M}$ defined in (5.4) maps X into itself.

Next we will show that $\mathcal{M}$ is a contraction mapping. Observe that

$$|\mathcal{M}(\psi_1)-\mathcal{M}(\psi_2)| \le C\left[p|V|^{p-1}(1-\sum_i \zeta_i)|\psi_1-\psi_2| + (1-\sum_i \zeta_i)|N(\phi+\psi_1)-N(\phi+\psi_2)|\right]$$

Arguing as in (5.1), we easily get

$$\|p|V|^{p-1}(1-\sum_i \zeta_i)|\psi_1-\psi_2|\|_{\frac{2N}{N+2}} \leq Co(1)\|\psi_1-\psi_2\|_{\frac{2N}{N-2}} \tag{5.9}$$

with $o(1)\to 0$ as $k\to\infty$. On the other hand,

$$(1-\sum_i \zeta_i)|N(\phi+\psi_1)-N(\phi+\psi_2)| \leq C(1-\sum_i \zeta_i)\left[(|V+\phi+\psi_1|^p-|V+\phi+\psi_2|^p)+p\left(|V|^{p-1}|\psi_1-\psi_2|\right)\right]$$

Thanks to the assumptions on ϕ, we get that $|V+\phi+\psi_1|^p-|V+\phi+\psi_2|^p \leq C|V|^{p-1}|\psi_1-\psi_2|$, and then arguing again as in (5.1) we can conclude that

$$\|(1-\sum_i \zeta_i)|N(\phi+\psi_1)-N(\phi+\psi_2)|\|_{\frac{2N}{N+2}} \leq Co(1)\|\psi_1-\psi_2\|_{\frac{2N}{N-2}} \tag{5.10}$$

with $o(1)\to 0$ as $k\to\infty$. We thus get from (5.9) and (5.10) that $\mathcal{M}$ is a contraction in X.

Consider now the function $\psi_1 := \zeta_1\psi$. Then ψ_1 solves

$$\Delta\psi_1 - \frac{\gamma}{|x|^2}\psi_1 = H \quad \text{in} \quad \mathbb{R}^N\setminus\{0\}$$

where

$$H(x) = -p(1-\sum_j \zeta_j)|V|^{p-1}\psi_1 - (p|V|^{p-1}-\frac{\gamma}{|x|^2})\zeta_1\sum_j(1-\zeta_j)\phi_j$$

$$-\zeta_1(1-\sum_j\zeta_j)(E+N(\phi)) - \nabla\psi\nabla\zeta_1 - \Delta\zeta_1\psi. \tag{5.11}$$

Since ψ_1 solves the above equation, the previous argument gives

$$\|\psi_1\|_{\frac{2N}{N-2}} \leq C\|H\|_{\frac{2N}{N+2}} \tag{5.12}$$

To get our estimate (2.23), we just need to evaluate the $L^{\frac{2N}{N+2}}$ norm of the function h.

We start with the observation that, arguing as in (5.1), we get

$$\|p(1-\sum_j\zeta_j)|V|^{p-1}\psi_1\|_{\frac{2N}{N+2}} \leq Ck^{-1-\frac{4}{N-4}+\frac{8}{N+2}}\|\psi_1\|_{\frac{2N}{N+2}} \tag{5.13}$$

Let us now consider the term $(p|V|^{p-1}-\frac{\gamma}{|x|^2})\zeta_1\sum_j(1-\zeta_j)\phi_j$. Assume first that $j\neq 1$, then

$$\left||V|^{p-1}\zeta_1(1-\zeta_j)\phi_j\right| \leq C\|\phi_j\|_{*j}\left||V|^{p-1}\zeta_1(1-\zeta_j)V_{j\varepsilon}\right|$$

Thus we get, using Holder inequality,

$$\||V|^{p-1}\zeta_1(1-\zeta_j)\phi_j\|_{\frac{2N}{N+2}} \leq C\|\phi_1\|_{1*}\|V^{p-1}\zeta_1\|_{\frac{N}{2}}\|\zeta_1V_{j\varepsilon}\|_{\frac{2N}{N-2}}$$

and, taking into account that $\|V^{p-1}\zeta_1\|_{\frac{N}{2}}\leq C$ while $\|\zeta_1V_{j\varepsilon}\|_{\frac{2N}{N-2}} \leq C\frac{(\varepsilon k)^{N-2}}{|j-1|^{N-2}}$, we get

$$\||V|^{p-1}\zeta_1(1-\zeta_j)\phi_j\|_{\frac{2N}{N+2}} \leq C\frac{(\varepsilon k)^{N-2}}{|j-1|^{N-2}}\|\phi_1\|_{1*}$$

and hence

$$\||V|^{p-1}\zeta_1\sum_{j\neq 1}(1-\zeta_j)\phi_j\|_{\frac{2N}{N+2}} \leq Ck^{-1-\frac{2}{N-4}}\|\phi_1\|_{1*}$$

On the other hand, if $j=1$, we get

$$\left|p|V|^{p-1}\zeta_1(1-\zeta_1)\phi_1\right| \leq C\|\phi_1\|_{*1}\zeta_1(1-\zeta_1)V_{1\varepsilon}^p$$

and hence

$$\|p|V|^{p-1}\zeta_1(1-\zeta_1)\phi_1\|_{\frac{2N}{N+2}} \leq Ck^{-1-\frac{6}{N-4}}\|\phi_1\|_{*1}.$$

Now, arguing as in (5.8), we get that

$$\|\frac{\gamma}{|x|^2}\zeta_1\left(\sum_{j=1}^k(1-\zeta_j)\phi_j\right)\|_{\frac{2N}{N+2}} \leq Ck^{-2-\frac{2}{N-4}}\|\phi_1\|_{*1}$$

Collecting the above estimates we conclude that

$$\|(p|V|^{p-1}-\frac{\gamma}{|x|^2})\zeta_1\sum_j(1-\zeta_j)\phi_j\|_{\frac{2N}{N+2}} \leq Ck^{-1-\frac{2}{N-4}}\|\phi_1\|_{1*}. \tag{5.14}$$

Next we evaluate $\|\zeta_1(1-\sum_j\zeta_j)E\|_{\frac{2N}{N+2}}$. A first observation is that $\|\zeta_1(1-\sum_j\zeta_j)E\|_{\frac{2N}{N+2}} \leq C\|\zeta_1(1-\sum_j\zeta_j)E_1\|_{\frac{2N}{N+2}}$. Arguing as in (4.2), one has that in the region where $\zeta_1(1-\sum_j\zeta_j)\neq 0$, we have

$$|E_1(x)| \leq C\varepsilon^{-\frac{N+2}{2}}\frac{(\varepsilon k)^{N-2}}{(1+|\frac{x-\xi_1}{\varepsilon}|^2)} \leq C\varepsilon^{\frac{N+2}{2}}k^{N-2}$$

This gives immediately that

$$\|\zeta_1(1-\sum_j\zeta_j)E\|_{\frac{2N}{N+2}} \leq C\varepsilon^{\frac{N+2}{2}}k^{N-2}k^{-\frac{N+2}{2}} \leq Ck^{-4-\frac{6}{N-4}} \tag{5.15}$$

Next consider the nonlinear term $\zeta_1(1-\sum_j\zeta_j)N(\phi)$. In the region where $\zeta_1(1-\sum_j\zeta_j)\neq 0$, we have

$$|N(\phi)| \leq C\left[|V|^{p-2}\sum_j|\phi_j|^2+|\psi_1|^p\right] \leq C\left[V_{1\varepsilon}^p\|\phi_1\|_{1*}^2+|\psi_1|^p\right]$$

Thus

$$\|\zeta_1(1-\sum_j\zeta_j)N(\phi)\|_{\frac{2N}{N+2}} \leq C\left[\|\phi_1\|_{1*}^2+\|\psi_1\|_{\frac{2N}{N-2}}^p\right] \tag{5.16}$$

Using the inequality $\|fg\|_{\frac{2N}{N+2}} \leq C\|f\|_{\frac{N}{2}}\|g\|_{\frac{2N}{N-2}}$, we get

$$\|\Delta\zeta_1\psi\|_{\frac{2N}{N+2}} \leq C\|\Delta\zeta_1\|_{\frac{N}{2}}\|\psi_1\|_{\frac{2N}{N-2}} \leq Ck^{-2\frac{N-2}{N}}\|\psi_1\|_{\frac{2N}{N-2}} \tag{5.17}$$

Using now the inequality $\|fg\|_{\frac{2N}{N+2}} \leq \|f\|_2\|g\|_N$, we get

$$\|\nabla\zeta_1\nabla\psi\|_{\frac{2N}{N+2}} \leq \|\zeta_1\nabla\psi\|_2\|\nabla\zeta_1\|_N.$$

Now, a direct consequence of the definition of ψ_1 gives that $\|\zeta_1\nabla\psi\|_2 \le C\|\psi_1\|_{H^1} \le C'\|\psi_1\|_{\frac{2N}{N-2}}$, which in particular implies that

$$\|\nabla\zeta_1\nabla\psi\|_{\frac{2N}{N+2}} \le Ck^{-\frac{N-1}{N}}\|\psi_1\|_{\frac{2N}{N-2}} \tag{5.18}$$

Collecting estimates (5.13), (5.14), (5.15), (5.16), (5.17), (5.18), inequality (5.12) gives

$$\|\psi_1\|_{\frac{2N}{N-2}} \le C\left[o(1)\|\psi_1\|_{\frac{2N}{N-2}} + k^{-4-\frac{6}{N-4}} + k^{-1-\frac{2}{N-4}}\|\phi_1\|_{1*} + \|\phi_1\|_{1*}^2\right]$$

where $o(1)\to 0$ as $k\to\infty$. This gives the validity of estimate (2.23).

This concludes the proof of the result. □

6. Appendix 3

We start with the proof of

Proof of Lemma 2.3. Let us define $\tilde\phi(y) = \varepsilon^{\frac{N-2}{2}}\phi(\varepsilon y+\xi_1)$ and $\tilde h(y) = \varepsilon^{\frac{N+2}{2}}h(\varepsilon y+\xi_1)$ and consider the equivalent problem for $\tilde\phi$ and $\tilde h$ given by

$$\Delta\tilde\phi + pU^{p-1}\tilde\phi = \tilde h + cU^{p-1}(y)Z(y) \quad \text{in} \quad \mathbb{R}^N, \quad \int_{\mathbb{R}^N} U^{p-1}Z\tilde\phi = 0 \tag{6.1}$$

With no loss of generality we may assume that

$$\int_{\mathbb{R}^N}\tilde h U^{p-1}Z = 0. \tag{6.2}$$

The evenness of the function h in the last $(N-2)$ variables implies that

$$\int_{\mathbb{R}^N}\tilde h\frac{\partial U}{\partial y_j} = 0, \qquad \text{for all} \quad j=3,\dots,N.$$

We want to show that also $\int_{\mathbb{R}^N}\tilde h\frac{\partial U}{\partial y_j}=0$, for $j=1,2$. Consider the vector integral

$$I = \int_{\mathbb{R}^N}\tilde h\begin{bmatrix}\frac{\partial U}{\partial y_1}\\ \frac{\partial U}{\partial y_2}\end{bmatrix} = c_N\int_{\mathbb{R}^N}\frac{\tilde h(y)}{(1+|y|^2)^{\frac{N}{2}}}\bar y\,dy, \quad \text{where} \quad \bar y = \begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

Changing the variable $\bar y$ into $e^{\frac{2\pi}{k}i}\bar y$ and using the rotational symmetry of $\tilde h$, we get $e^{\frac{2\pi}{k}i}I = I$, thus $I=0$ since $k\ne 1$.

Let us consider the subspace

$$X = \{\tilde\phi\in\mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N}\tilde\phi Z = 0, \quad \int_{\mathbb{R}^N}\tilde\phi\frac{\partial U}{\partial y_j} = 0, \qquad \text{for all} \quad j=1,\dots,N\}$$

which is well defined thanks to the Sobolev's embedding $\mathcal{D}^{1,2}(\mathbb{R}^N)\hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$.

Since $\int_{\mathbb{R}^N}\tilde hZ = 0$, $\int_{\mathbb{R}^N}\tilde h\frac{\partial U}{\partial y_j} = 0$, for all $j=1,\dots,N$, finding weak solution to (6.1) corresponds to finding $\phi\in X$ such that

$$\int_{\mathbb{R}^N}\nabla\tilde\phi\nabla\Psi - p\int_{\mathbb{R}^N}U^{p-1}\tilde\phi\Psi + \int_{\mathbb{R}^N}\tilde h\Psi = 0 \qquad \text{for all} \qquad \Psi\in X.$$

Now, for $\tilde h \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$, let us denote by $\tilde\phi = A(\tilde h) \in H$ the unique solution of the problem

$$\int_{\mathbb{R}^N} \nabla\tilde\phi \nabla\psi + \int_{\mathbb{R}^N} \tilde h \psi = 0 \quad \text{for all} \quad \psi \in X, \tag{6.3}$$

given by Riesz's theorem. Then A defines a continuous linear map between $L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and X. Problem (6.1) can be formulated as

$$\tilde\phi - A(\, pU^{p-1}\tilde\phi\,) = A(\tilde h), \quad \tilde\phi \in X. \tag{6.4}$$

The map $\tilde\phi \in X \mapsto U^{p-1}\tilde\phi \in L^{\frac{N}{2}}(\mathbb{R}^N)$ is easily seen to be compact, thanks to local compactness of Sobolev's embeddings and the fact that $U^{p-1} = O(|y|^{-4})$.

Hence, Fredholm's alternative applies to problem (6.1): for $\tilde h = 0$, (6.1) reduces to $(\Delta + pU^{p-1})(\tilde\phi) = 0$ with $\tilde\phi \in X$. Elliptic regularity yields that $\tilde\phi$ is also bounded, and hence it is a linear combination of the functions Z and $\frac{\partial U}{\partial y_j}$ for $j = 1, \ldots, k$. Then, the definition of X implies that necessarily $\tilde\phi = 0$. We conclude that Problem (6.1) is uniquely solvable in X for any $\tilde h$. Besides,

$$\|\nabla\tilde\phi\|_{L^2(\mathbb{R}^N)} + \|\tilde\phi\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \;\leq\; C\,\|\tilde h\|_{\frac{2N}{N+2}}.$$

Arguing by uniqueness, as in the proof of Lemma 2.2, we find that $\tilde\phi$ satisfies the corresponding symmetries.

It remains to prove that ϕ satisfies estimate (2.29). In terms of $\tilde\phi$, this is equivalent to show that

$$\|(1 + |y|^{N-2})\tilde\phi\|_\infty \leq C\|\tilde h\|_{\frac{2N}{N+2}}. \tag{6.5}$$

Being $\tilde\phi$ a solution to (6.1), local elliptic estimates yield

$$\|\tilde\phi\|_{L^\infty(B_1)} \leq C\,\|\,\tilde h\|_{\frac{2N}{N+2})}.$$

Now, let us consider Kelvin's transform of $\tilde\phi$,

$$\hat\phi(y) = |y|^{2-N}\tilde\phi(|y|^{-2}y)$$

Then we check that $\hat\phi$ satisfies the equation

$$\Delta\hat\phi + p\gamma U^{p-1}(y)\hat\phi = \hat h \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{6.6}$$

where $\hat h(y) = |y|^{-N-2}\tilde h(|y|^{-2}y)$. We observe that

$$\|\hat h\|_{L^{\frac{2N}{N+2}}(B(0,2))} \;=\; \|\,\tilde h\|_{L^{\frac{2N}{N+2}}(|y|>\frac12)} \leq C\,\|\,\tilde h\|_{\frac{2N}{N+2}}.$$

and

$$\|\nabla\hat\phi\|_{L^2(\mathbb{R}^n)} + \|\hat\phi\|_{\frac{2N}{N-2}} = \|\nabla\tilde\phi\|_{L^2(\mathbb{R}^N)} + \|\tilde\phi\|_{\frac{2N}{N-2}}.$$

Then we get, from elliptic estimates applied to equation (6.6),

$$\|\hat\phi\|_{L^\infty(B_1)} \leq C\|\hat h\|_{L^{\frac{2N}{N+2}}(B_2)} \leq C\,\|\,\tilde h\|_{\frac{2N}{N+2}}.$$

But

$$\|\hat\phi\|_{L^\infty(B_1)} = \|\,|y|^{N-2}\tilde\phi\,\|_{L^\infty(\mathbb{R}^N\setminus B_1)},$$

Combining the above estimates, relation (2.29) follows.

The proof is complete. □

We have now the tools to prove Lemma 2.4.

Proof of Lemma 2.4. Let T be the linear operator defined by Lemma 2.3. Then we can set up Problem (2.30) as the fixed point problem

(6.7)
$$\phi_1 = T\left(p(|V|^{p-1}\zeta_1 - |V_{1\varepsilon}|^{p-1})\phi_1 + \zeta_1[p|V|^{p-1}\psi(\phi) - \frac{\gamma}{|x|^2}\phi_1 + E + N(\phi_1 + \Sigma_{i\neq 1}\phi_i + \psi)]\right) := \mathcal{F}(\phi_1).$$

Observe first that

$$\|\zeta_1 E\|_{\frac{2N}{N+2}} \leq Cf(k), \quad \text{where} \quad f(k) := \begin{cases} k^{-5} & \text{if} \quad N=5, \\ k^{-4}|\log k|^{\frac{N+2}{2N}} & \text{if} \quad N=6, \\ k^{-1-\frac{6}{N-4}} & \text{if} \quad N\geq 7, \end{cases} \tag{6.8}$$

as proved in Proposition 2.1. We show that the map $\mathcal{F}(\phi_1)$ is a Contraction Mapping in the ball

$$X = \{\phi \in \mathcal{L}^{\frac{2N}{N-2}} \,:\, \|\phi\|_{1*} \leq \alpha f(k)\}, \tag{6.9}$$

for some constant α large, but independent of k.

Let us consider first the term $p\zeta_1|V|^{p-1}\psi(\phi_1)$. Holder inequality and estimate (2.23) give

$$\|p\zeta_1|V|^{p-1}\psi(\phi_1)\|_{\frac{2N}{N+2}} \leq C\|\zeta_1|V|^{p-1}\|_{\frac{N}{2}}\|\zeta_1\psi\|_{\frac{2N}{N-2}} \leq C\left[k^{1-\frac{2}{N-4}}\|\phi_1\|_{1*} + \|\phi_1\|_{1*}^2\right] \tag{6.10}$$

where $o(1) \to 0$ as $k \to \infty$.

Consider now the term $p(|V|^{p-1}\zeta_1 - |V_{1\varepsilon}|^{p-1})\phi_1$. First we observe that

$$\int_{\mathbb{R}^N} |p(|V|^{p-1}\zeta_1 - |V_{1\varepsilon}|^{p-1})\phi_1|^{\frac{2N}{N+2}} \leq C\|\phi_1\|_{1*}^{\frac{2N}{N+2}} \int_{B(\xi_1,\frac{\eta}{k})} |V_{1\varepsilon}^{p-1}\left(\sum_{j>1} V_{j\varepsilon}\right)|^{\frac{2N}{N+2}}$$

Thus, we get

$$\|p(|V|^{p-1}\zeta_1 - |V_{1\varepsilon}|^{p-1})\phi_1\|_{\frac{2N}{N+2}} \leq C\|\phi_1\|_{1*}\|V_{1\varepsilon}\|_{\frac{N}{2}}\|\sum_{j>1} V_{j\varepsilon}\|_{\frac{2N}{N-2}} \leq C\|\phi_1\|_{1*}k^{-\frac{2}{N-4}}. \tag{6.11}$$

We next estimate $\|\frac{\gamma}{|x|^2}\zeta_1\phi_1\|_{\frac{2N}{N+2}}$. A direct use of Holder inequality gives

$$\|\frac{\gamma}{|x|^2}\zeta_1\phi_1\|_{\frac{2N}{N+2}} \leq C(\int_{B(\xi_1,\frac{c}{k})} |\phi_1|^{\frac{2N}{N+2}})^{\frac{N+2}{2N}} \leq C(k^{-N})^{\frac{2}{N}}\|\phi_1\|_{\frac{2N}{N-2}}. \tag{6.12}$$

Finally, we are left with $\|\zeta_1 N(\phi_1 + \sum_{j\neq 1}\phi_j + \psi)\|_{\frac{2N}{N+2}}$. We have

$$\left|N(\phi_1 + \sum_{j\neq 1}\phi_j + \psi)\right| \leq C\left[|\phi_1|^p + \sum_{j>1}|\phi_j|^p + \psi|\right].$$

Arguing as in (6.10)-(6.11), we easily get thanks to (2.23)

$$\|\zeta_1|V|^{p-1}|\phi+\psi|\|_{\frac{2N}{N+2}} \le C\left[\|\phi_1\|^p_{\frac{2N}{N-2}} + \sum_{j>1}\|\zeta_1\phi_j\|^p_{\frac{2N}{N-2}} + \|\zeta_1\psi\|^p_{\frac{2N}{N-2}}\right]$$

$$\le C\left[\|\phi_1\|^p_{1*} + o(1)\|\phi_1\|^p_{1*} + \|\phi_1\|^{2p}_{1*}\right], \tag{6.13}$$

where $o(1)\to 0$ as $k\to\infty$. Thus consequence of estimates (6.10)-(6.13) is that the map $\mathcal{F}$ defined in (6.7) maps X into X. Next we will show that $\mathcal{F}$ is a contraction mapping in X. This will conclude our proof.

Observe that

$$\left|\mathcal{F}(\phi_1^1)-\mathcal{F}(\phi_1^2)\right| \le C\left[(|V|^{p-1}\zeta_1 - |V_{1\varepsilon}|^{p-1})|\phi_1^1-\phi_1^2| + \zeta_1|V|^{p-1}|\psi(\phi^1)-\psi(\phi^2)|\right.$$

$$\left. + \frac{\gamma}{|x|^2}|\phi_1^1-\phi_1^2| + |N(\phi^1+\psi(\phi^1)) - N(\phi^2+\psi(\phi^2))|\right]$$

Arguing as in (6.10) we get

$$\|(|V|^{p-1}\zeta_1 - |V_{1\varepsilon}|^{p-1})|\phi_1^1-\phi_1^2| + \zeta_1|V|^{p-1}|\psi(\phi^1)-\psi(\phi^2)|\|_{\frac{2N}{N+2}}$$

$$\le Ck^{-1-\frac{4}{N(N-4)}}(\|\phi^1-\phi^2\|_{\frac{2N}{N-2}} + \|\psi(\phi^1)-\psi(\phi^2)\|_{\frac{2N}{N-2}}) \le o(1)\|\phi_1^1-\phi_1^2\|_{\frac{2N}{N-2}},$$

where $o(1)\to 0$ as $k\to\infty$. As in (6.12) we get

$$\|\frac{\gamma}{|x|^2}(\phi_1^1-\phi_1^2)\|_{\frac{2N}{N+2}} \le o(1)\|\phi_1^1-\phi_1^2\|_{\frac{2N}{N-2}},$$

where $o(1)\to 0$ as $k\to\infty$. Finally, denote by $f(t)=t^p$ and $\hat\phi^i = \phi^i+\psi(\phi^i)$. Then we have

$$\|N(\phi^1+\psi(\phi^1)) - N(\phi^2+\psi(\phi^2))\|_{\frac{2N}{N+2}} = \|\left(\int_0^1 \frac{d}{dt}f(V+\hat\phi^2+t(\hat\phi^1-\hat\phi^2))\right) - f'(V)(\hat\phi^1-\hat\phi^2)\|_{\frac{2N}{N+2}}$$

$$\le \|\int_0^1 [f'(V+\hat\phi^2+t(\hat\phi^1-\hat\phi^2)) - f'(V)](\hat\phi^1-\hat\phi^2)\|_{\frac{2N}{N+2}} \le C\|\hat\phi_1^1-\hat\phi_1^2\|_{\frac{2N}{N-2}} \sup_{\|z\|_{\frac{2N}{N-2}}\le r}\|f'(V+z)-f'(V)\|_{\frac{N}{2}},$$

If we choose the number α in the definition of the set X (6.9) small, but fixed independently of k, we can obtain that

$$\|N(\phi^1+\psi(\phi^1)) - N(\phi^2+\psi(\phi^2))\|_{\frac{2N}{N+2}} \le \frac{1}{2}\|\hat\phi_1^1-\hat\phi_1^2\|_{\frac{2N}{N-2}}$$

Thus we conclude that $\mathcal{F}$ is a contraction map in X.

$\square$

References

[1] M.F. Bidaut-Veron and L. Veron, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*, Invent. Math. 106(1991), 489-539.

[2] L. Caffarelli, R. Kohn and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. 53(1984), 259-275.

[3] L. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, in Comm. Pure Appl. Math. **42** (1989) 271-297.

[4] F. Catrina and Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal solutions*, Comm. Pure Appl. Math. 54(2)(2001), 229-258.

[5] K.S. Chou and C.W. Chu, *On a best constant in a weighted Sobolev-Hardy inequality*, J. London Math. Soc. (2) 48(1993), 137-151.

[6] M. del Pino, M. Kowalczyk, J. Wei, *On De Giorgi's Conjecture in Dimensions $N \geq 9$*, Annals of Mathematics, to appear.

[7] M. del Pino, M. Kowalczyk, F. Pacard, J. Wei, *The Toda system and multiple-end solutions of autonomous planar elliptic problems*, Advances in Math ematics 224(2010), no.4, 1462-1516.

[8] M. del Pino, P. Felmer, P., and M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron's problem*, Calc. Var. Partial Differential Equations 16, 113–145, 2003.

[9] M. del Pino, J. Dolbeault, S. Fillippas and A. Tertikas, *A logarithmic Hardy inequality*, J. Funct. Anal. 259(2010), 2045-2072.

[10] M. del Pino, M. Musso, F. Pacard and A. Pistoia, *Torus action on S^n and sign changing solutions for conformally invariant equations*, to appear in Annali della Scuola Normale di Pisa.

[11] M. del Pino, M. Musso, F. Pacard and A. Pistoia, *Large energy entire solutions for the Yamabe equation*, to appear in Journal of Differential Equations.

[12] J. Dolbeault and M. Esteban, *About existence, symmetry and symmetry breaking for extremal functions of some interpolation functional inequalities*, arxiv:1011.5465

[13] J. Dolbeault, M. Esteban, M. Loss and G. Tarantello, *On the symmetry of extremals for the Caffarelli-Kohn-Nirenberg inequalities*, Adv. Nonlinear Stud. 9(2009), 713-726.

[14] O. Druet, *From one bubble to several bubbles: the low-dimensional case,* J. Diff. Geom. 63(2003), 399-473.

[15] O. Druet, *Compactness for Yamabe metrics in low dimensions,* Int. Math. Res. Not. 23(2004), 1143-1191.

[16] V. Felli and M. Schneider, *Perturbation results of critical elliptic equations of Cafferelli-Kon-Nirenberg type*, J. Diff. Eqns. 191(2003), 121-142.

[17] V. Felli and S. Terracini, *Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity*, Comm. Part. Diff. Eqns. 31(2006), 469-495.

[18] J. Garcia Azorero, I. Peral *Hardy inequalities and some critical elliptic and parabolic problems*, J. Diff. Equations 144 (1998), no. 2, 441–476.

[19] Q. Jin, Y.Y. Li and H.Y. Xu, *Symmetry and asymmetry: the method of moving spheres*, Advances in Diff. Eqns. 13(2008), no.7-8, 601-640.

[20] Y.Y. Li, private communications.

[21] Y.Y. Li, *Prescribed scalar curvature on S^n and related problems. II. Existence and compactness,* Comm. Pure Appl. Math. 49(1996), 541–597 (1996).

[22] C.S. Lin and Z.Q. Wang, *Symmetry of extremal functions for the Caffarelli-Kohn-Nirenberg inequalities.* Proc. Amer. Math. Soc. 132(2006), 1685-1691.

[23] J. M. Levy-Leblond, *Elctron capture by polar molecules*, Phys. Rev. 153(1967), 1-4.

[24] M. Musso, F. Pacard and J. Wei, *Finite-energy sign changing solutions to nonlinar Schrodinger equations*, J. Europ. Math. Soc., to appear.

[25] D. Smets, *Nonlinear Schrodinger equations with Hardy potentials and critical nonlinearities*, Trans. Amer. Math. Soc. 357(2005), 2909-2938.

[26] R. Schoen and D. Zhang, *Prescribed scalar curvature problem on the $n-$sphere*, Calc. Var. PDE, 4(1996), 1-25.

[27] S. Terracini, *On positive entire solutions to a class of equations with singular coefficients and critical exponent*, Adv. Diff. Eqns. 1(1996), 241-264.

[28] J. Wei and S. Yan, *Infinitely many solutions for the prescribed scalar curvature problem on S^N*, Journal Functional Analysis 258(2010), 3048-3081.

[29] J. Wei and S. Yan, *Infinitely many positive solutions to a nonlinear Schrodinger equation in $\mathbb{R}^N$*, Cal. Var. PDE. 37(2010), 423-439.

M. Musso - Departamento de Matemática, Pontificia Universidad Catolica de Chile, Avda. Vicuña Mackenna 4860, Macul, Chile., Email : mmusso@mat.puc.cl

J. Wei - Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong. Email: wei@math.cuhk.edu.hk