Supercritical biharmonic elliptic problems in domains with small holes

Yuxia Guo [1] and **Juncheng Wei** *[2]

[1] Y. Guo - Department of Mathematics, Tsinghua University, Beijing, China. Email:yguo@math.tsinghua.edu.cn
[2] J. Wei - Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong. Email: wei@math.cuhk.edu.hk

Key words Supercritical Coron's problem, biharmonic, domains with holes, resonant exponents
MSC (2000) Primary: 35B40, 35B45; Secondary: 35J40, 92C40

Let $\mathcal{D}$ be a bounded, smooth domain in $\mathcal{R}^N$, $N \geq 5$, $P \in \mathcal{D}$. We consider the following biharmonic elliptic problem in $\Omega = \mathcal{D} \setminus B_\delta(P)$,

$$\begin{aligned} &\Delta^2 u = |u|^{p-1}u \qquad && \text{in } \Omega, \\ &u = \nabla u = 0 && \text{on } \partial\Omega \end{aligned}$$

with p supercritical, namely $p > \frac{N+4}{N-4}$. We find a sequence

$$p_1 < p_2 < p_3 < \cdots$$

such that if p is given, with $p \neq p_j$ for all j, then for all $\delta > 0$ sufficiently small, this problem is solvable.

1 Introduction and statement of the main results

In this paper we consider the following supercritical biharmonic problem

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \Omega\,, \tag{1.1}$$

$$u = \nabla u = 0 \quad \text{on } \partial\Omega\,, \tag{1.2}$$

where Ω is a smooth and bounded domain in $\mathcal{R}^N$ $(N \geq 5)$ and $p > \frac{N+4}{N-4}$.

A main characteristic of this problem is the role played by the critical exponent $p = \frac{N+4}{N-4}$ in the solvability question. When $1 < p < \frac{N+4}{N-4}$, a solution can be found as an extremal for the best constant in the compact embedding of $H_0^2(\Omega)$ into $L^{p+1}(\Omega)$, namely a minimizer of the variational problem

$$\inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}\,.$$

When $p \geq \frac{N+4}{N-4}$, this minimization procedure fails. The existence of a solution in general is quite difficult. Pucci and Serrin [18] showed that no solution exists in this case if the domain is strictly star-shaped. Bartsch, Weth and Willem [2] showed that in the case of $p = \frac{N+4}{N-4}$ and the domain Ω exhibits a small hole, a solution to (1.1) -(1.2) exists, generalizing earlier results of Coron [6]. When $p = \frac{N+4}{N-4}$ and the right hand side is replaced by $|u|^{p-1}u + f(x,u)$, where $f(x,u)$ is a lower order terms, there are many recent works on extending Brezis-Nirenberg's results to polyharmonic case, see Bernis-Grunau [3], Edmunds-Fortunato-Janelli [8], Gazzola-Grunau-Squassina [10], Ge [11], Grunau [12], and the references therein.

* Corresponding author:email:wei@math.cuhk.edu.hk, Phone:852-26097967, Fax: 852-26035154

In the corresponding second order case,

$$\Delta u + u^p = 0, u > 0 \quad \text{in } \Omega\,, \ u = 0 \quad \text{on } \partial\Omega\,, \tag{1.3}$$

when $p \geq \frac{N+2}{N-2}$, Pohozaev [17] discovered that no solution exists in this case if the domain is strictly star-shaped. In the classical paper [5], Brezis and Nirenberg considered the critical case $p = \frac{N+2}{N-2}$ and proved that compactness, and hence solvability, is restored by the addition of a suitable linear term. Coron [6] used a variational approach to prove that (1.3) is solvable for $p = \frac{N+2}{N-2}$ if Ω exhibits *a small hole*. Rey [19] established existence of multiple solutions if Ω exhibits several small holes. Bahri and Coron [1] established that solvability holds for $p = \frac{N+2}{N-2}$ whenever Ω has a non-trivial topology. Passaseo [16] constructed examples of domains having non-trivial topology while having no solutions to (1.3) for $N \geq 4$ and $p > \frac{N+1}{N-3}$.

In this paper we consider Problem (1.1)-(1.2) for exponents p above critical in a Coron's type domain: one exhibiting a small hole. Thus we assume in what follows that the domain Ω has the form

$$\Omega = \mathcal{D} \setminus B_\delta(Q) \tag{1.4}$$

where $\mathcal{D}$ is a bounded domain with smooth boundary, $B_\delta(Q) \subset \mathcal{D}$ and $\delta > 0$ is to be taken small. Thus we consider the problem of finding classical solutions of

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathcal{D} \setminus B_\delta(Q)\,, \tag{1.5}$$

$$u = \nabla u = 0 \quad \text{on } \partial\mathcal{D} \cup \partial B_\delta(Q)\,. \tag{1.6}$$

Our main result states that there is a sequence of *resonant exponents*,

$$\frac{N+4}{N-4} < p_j, j = 1, \ldots, \tag{1.7}$$

such that if p is supercritical and differs from all elements of this sequence then Problem (1.5)-(1.6) is solvable whenever δ is sufficiently small.

Theorem 1.1 *There exists a sequence of the form* (1.7) *such that if* $p > \frac{N+4}{N-4}$ *and* $p \neq p_j$ *for all* j*, then there is a* $\delta_0 > 0$ *such that for any* $\delta < \delta_0$*, Problem* (1.5)-(1.6) *possesses at least one solution.*

The corresponding second order elliptic problem

$$\Delta u + u^p = 0\,, \ u > 0 \quad \text{in } \mathcal{D} \setminus B_\delta(Q)\,, u = 0 \quad \text{on } \partial\mathcal{D} \cup \partial B_\delta(Q) \tag{1.8}$$

was studied recently by del Pino and the second author [9]. There it was proved that there exist resonant exponents $\frac{N+2}{N-2} < p_1 < p_2 < \ldots$ such that (1.8) admits a solution for δ small, provided that $p > \frac{N+2}{N-2}$ and $p \neq p_j$. Our proof essentially follows the same procedure of [9]. However some main difficulties arise as in [9] the resonant sequence is produced by the *principal eigenvalue* only while here the resonant sequences can be produced by *several eigenvalue sequences.*

In the background of our result is the problem

$$\Delta^2 w = w^p\,, \ w > 0 \quad \text{in } \mathcal{R}^N \setminus \bar{B}_1(0)\,, \tag{1.9}$$

$$w = 0 \ \text{ on } \partial B_1(0)\,, \quad \limsup_{|x|\to+\infty} |x|^{N-4} w(x) < +\infty\,. \tag{1.10}$$

In Section 2, we shall prove that problem (1.9)-(1.10) admits a unique radially symmetric solution $w(r)$ whenever $p > \frac{N+4}{N-4}$. (This is of independent interest.) The solutions we find have a profile similar to w suitably rescaled. More precisely, Let us observe that

$$w_\delta(x) = \delta^{-\frac{4}{p-1}} w(\delta^{-1}|x - Q|) \tag{1.11}$$

solves uniquely the same problem with $B_1(0)$ replaced with $B_\delta(Q)$. The idea is to consider w_δ as a first approximation for a solution of Problem (1.1)-(1.2), provided that $\delta > 0$ is chosen small enough. What we shall prove is that an actual solution of the problem, which differs little from w_δ does exist. To this end, it is necessary to understand in rather fine terms the linearized operator around w_δ.

An interesting question is whether the solutions we constructed are *positive*. This will depend on the domain $\mathcal{D}$ as for some domains the Green's function can become negative. (See for example [13].) We believe that the solutions are positive for domains with positive Green's functions. Another question is whether or not the sequence $\{p_j\}$ approaches $+\infty$ as in [9].

The result of Theorem 1.1 remains valid, with only minor modifications in the proof, for a problem of the form

$$\Delta^2 u - u^p - f(x,u) = 0\,,\ u > 0 \quad \text{in } \mathcal{D}\setminus B_\delta(Q)\,,$$

$$u = \nabla u = 0 \quad \text{on } \partial\mathcal{D}\cup\partial B_\delta(Q)\,.$$

where $f(x,u) \sim u^q$ for some $q \in [1, \frac{N+4}{N-4})$. We can also get existence of multiple solutions in a domain of the form

$$\mathcal{D}\setminus\bigcup_{j=1}^{m} B_\delta(Q_i)\,.$$

The question certainly opens on considering a non-spherical hole or, more generally, finding conditions which ensure solvability of rather general supercritical problems. A method beyond variational arguments or singular perturbations would be needed.

2 Existence and Uniqueness of Solution to (1.9)-(1.10)

In this section, we study the existence and uniqueness of solutions to the exterior domain problem (1.9)-(1.10).

Our main result in this section is the following:

Theorem 2.1 *Problem (1.9)-(1.10) admits a unique radially symmetric solution $w = w(r)$ (with least energy).*

Proof. We first prove *existence*: by Kelvin's transformation $\mathtt{w}(r) = r^{4-N}w(\frac{1}{r})$, the equations (1.9)-(1.10) are equivalent to the following problem in a ball B_1

$$\begin{cases} \Delta^2\mathtt{w} = r^\alpha \mathtt{w}^p \quad \mathtt{w} > 0 \quad \text{in } B_1, \\ \mathtt{w} = \nabla\mathtt{w} = 0 \quad \text{on } \partial B_1 \end{cases} \tag{2.1}$$

where $\alpha = p(N-4) - (N+4) > 0$.

We follow Ni's proof [15]. First we need the following radial lemma

Lemma 2.2 *Assume that $u \in H_0^2(B_1)$ and $u = u(r)$. Then we have*

$$|u(r)| \le \frac{C}{r^{(N-4)/2}}\|\Delta u\|_{L^2(B_1)}. \tag{2.2}$$

As a consequence, for $p < \frac{N+4+2\alpha}{N-4}$, the map $u \to r^\alpha u_+^{p+1}$ is a compact map from $H_0^2(B_1)$ to $L^1(B_1)$.

Proof: Since $u_r \in H_0^2(B_1)$, we obtain

$$|r^{N-1}u_r(r)| \le \int_0^r r^{N-1}|\Delta u| \le Cr^{\frac{N}{2}}\|\Delta u\|_{L^2(B_1)}. \tag{2.3}$$

Hence

$$|u(r)| \leq \int_r^1 |u_r(t)|dt \leq \frac{C}{r^{\frac{N-4}{2}}} \|\Delta u\|_{L^2(B_1)}.$$

Using (2.2), the rest of the proof is similar to the compactness lemma in [15]. □

Now we consider the following minimization problem

$$c_p = \inf_{u \in H^2_{0,r}(B_1), \int_{B_1} r^\alpha u_+^{p+1} = 1} \int_{B_1} |\Delta u|^2 \tag{2.4}$$

where $H^2_{0,r}(B_1) = H^2_0(B_1) \cap \{u = u(r)\}$. By Lemma 2.2, a standard argument shows that c_p can be attained by some $\tilde{u}$ which satisfies

$$\begin{cases} \Delta^2 \tilde{u} = r^\alpha \tilde{u}_+^p & \text{in } B_1, \\ \tilde{u} = \tilde{u}' = 0 & \text{on } \partial B_1. \end{cases} \tag{2.5}$$

Since the Green function Δ^2 under the Dirichlet boundary condition in B_1 is positive, (see [4] and [12]), we deduce that $\tilde{u} > 0$ and hence $\tilde{u}$ satisfies (2.1). This proves the existence with $\mathtt{w} = \tilde{u}$. (Note that $\mathtt{w}$ has the least energy.)

To prove the uniqueness of the least energy solution given in (2.4), we note first that c_p only depends on p and hence $\int_{B_1} r^\alpha \mathtt{w}^{p+1}$ also depends on p only. Now by Pohozaev's identity

$$\left(\frac{N+\alpha}{p+1} - \frac{N-4}{2}\right) \int_{B_1} r^\alpha \mathtt{w}^{p+1} = \frac{1}{2} \int_{\partial B_1} < x, \nu > |\Delta \mathtt{w}|^2 \tag{2.6}$$

which implies that $\mathtt{w}''(1)$ also depends on p only. Thus $\mathtt{w}''(1)$ is a fixed value (depending on p only).

Next we prove the *uniqueness*: Let us suppose $\phi = w_1 - w_2$, where w_1 and w_2 are two solutions to (1.9)-(1.10). By the above arguments, we may assume that $\phi(1) = \phi'(1) = \phi''(1) = 0$. Note that ϕ satisfies

$$\Delta^2 \phi = (w_2 + \phi)^p - w_2^p. \tag{2.7}$$

Now we use an idea of Swanson [20] to derive a contradiction. We first show that ϕ can not change sign. Note that ϕ can not have an infinite order of zeroes and ϕ is non-oscillatory at infinity. Suppose ϕ has k zeroes in $(1, +\infty)$. Then ϕ' has at least $k+1$ zeroes and $r^{N-1}\phi'$ has at least $k+2$ zeroes in $[0, +\infty)$. Hence $\Delta\phi$ has at least $k+1$ zeroes in $(0, +\infty)$ and $k+2$ zeroes in $[0, +\infty)$. This implies that $\Delta(\Delta\phi)$ have at least $k+1$ zeroes in $(0, +\infty)$. But $\Delta^2\phi$ and ϕ has the same number of zeroes. This gives a contradiction. Hence we may assume that $\phi > 0$.

But multiplying (2.7) by w_2 and integrating, we obtain

$$\int_{B_1^c} [(w_2 + \phi)^p w_2 - w_2^{p+1} - w_2^p \phi] = 0 \tag{2.8}$$

which is impossible since $\phi > 0$. Thus $w_1 = w_2$.

□

Remark 2.3 For the original problem, the energy functional becomes

$$\int_{\mathcal{R}^N \setminus B_1(0)} |\Delta w|^2 = \int_{B_1(0)} |\Delta \mathtt{w}|^2, \int_{\mathcal{R}^N \setminus B_1(0)} w^{p+1} = \int_{B_1(0)} r^\alpha \mathtt{w}^{p+1}. \tag{2.9}$$

Finally, we state the following important lemma.

Lemma 2.4 *Let φ_1 be the first positive eigenfunction of Δ^2 in B_1 under the Dirichlet boundary condition. Then the map*

$$p \to \mathtt{w}(p)\varphi_1^{-1} \tag{2.10}$$

is an analytic map.

Proof. Let φ_1 be a first positive eigenfunction of Δ^2 in $B_1(0)$ and consider the space C_3 of all radially symmetric continuous functions in $\bar{B}_1(0)$ for which $\|\varphi_1^{-1}u\|_\infty < +\infty$. Following Dancer's proof in [7], we obtain that if p_0 and u_0 are such that there exists a $\mu > 0$ for which $u_0 \geq \mu\phi$ then the map

$$(p,u) \in \mathcal{R} \times C_3 \mapsto (-\Delta)^{-2}(u^p) \in C_3$$

is analytic in a neighborhood of (p_0, u_0) (actually in a general domain). Dancer's proof applies with no significant changes to establish that the same is true for the map

$$(p,u) \in \mathcal{R} \times C_3 \mapsto (-\Delta)^{-2}(|x|^{p(N-4)-(N+4)}u^p) \in C_3\,.$$

The bottom line is the fact that the application $\gamma > 0 \mapsto |x|^\gamma$ defines a real analytic map into $C(\bar{B}_1(0))$. Indeed we can expand

$$|x|^\gamma = \sum_{k=0}^{\infty} \frac{|x|^{\gamma_0}\log^k|x|}{k!}(\gamma-\gamma_0)^k.$$

Taking into account that for sufficiently large k,

$$\sup_{|x|<1} |x|^{\gamma_0}|\log^k|x|| \leq \gamma_0^{-k}k^k e^{-k},$$

we see that the above power series is uniformly convergent on $|\gamma - \gamma_0|$ sufficiently small, thanks to Stirling's formula. This fact is also in the background of Dancer's proof to deal with the vanishing of u at the boundary in the proof of analyticity with respect to p. For analyticity with respect to u, we observe that

$$(u_0 + h)^p = u_0^p(1 + (h/u_0))^p$$

and a uniformly convergent Taylor's series can then be written for $\|h\|_{C_3}$ small. See Proposition 1 in [7] for the complete argument.

Now, $\mathtt{w} = \mathtt{w}(p)$ is the only solution of the problem

$$F(\mathtt{w},p) \equiv \mathtt{w} - (-\Delta)^{-2}(|x|^{p(N-4)-(N+4)}\mathtt{w}^p)) = 0.$$

From what has been said, for each $p_0 > 1$ the map $F(u,p)$ is analytic into C_3 in a neighborhood of $(\mathtt{w}(p_0), p_0)$. Besides, the map $F_u(\mathtt{w}(p_0), p_0)$ is an isomorphism of C_3 since the linearized equation

$$\Delta^2\psi - |x|^{p(N-2)-(N+2)}p\mathtt{w}^{p-1}\psi = 0 \quad \text{ in } B_1(0)\,,$$

$$\psi = \psi' = 0 \ \text{ on } \partial B_1(0)$$

admits only the trivial radial solution, as it follows from the previous uniqueness argument. (Note that we consider $\mathtt{w}(p)$ as the unique least energy solution.) From the implicit function theorem in analytic version we have that the map $p \mapsto \mathtt{w}(p)$ is analytic into C_3. The same is the true with $p \mapsto |x|^{p(N-4)-(N+4)}p\mathtt{w}^{p-1}$. □

3 The invertibility of linearized operator and conditions for non-resonance

In this section, we study the invertibility property for the linearized operator associated to w. That is, we consider the problem

$$\Delta^2\phi - pw^{p-1}\phi = h \quad \text{in } \mathcal{R}^N \setminus \bar{B}_1(0)\,, \tag{3.1}$$

$$\phi = \nabla\phi = 0 \ \text{ on } \partial B_1(0)\,, \quad \lim_{|x|\to+\infty} \phi(x) = 0\,, \tag{3.2}$$

In this section, we investigate under what conditions the homogeneous problem with $h = 0$ in (3.1)-(3.2) admits only the trivial solution.

Let us consider now Problem (3.1)-(3.2) for $h = 0$, and assume that we have a solution ϕ. The symmetry of the domain $\mathcal{R}^N \setminus B_1(0)$ allows us to expand ϕ into spherical harmonics. We write ϕ as

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\theta), \quad r > 0,\, \theta \in S^{N-1}$$

where Θ_k, $k \geq 0$ are the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the sphere S^{N-1}, normalized so that they constitute an orthonormal system in $L^2(S^{N-1})$. We take Θ_0 to be a positive constant, associated to the eigenvalue 0 and Θ_i, $1 \leq i \leq N$ is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = N-1$, $1 \leq i \leq N$. In general, λ_k denotes the eigenvalue associated to Θ_k, we repeat eigenvalues according to their multiplicity and we arrange them in an non-decreasing sequence. We recall that the set of eigenvalues is given by $\{j(N-2+j)\,|\,j \geq 0\}$.

The components ϕ_k then satisfy the differential equations

$$\left(\Delta - \frac{\lambda_k}{r^2}\right)^2 \phi_k = pw^{p-1}\phi_k, \phi_k = \phi_k(r) \quad r \in (1,\infty), \tag{3.3}$$

$$\phi_k(1) = \phi_k^{'}(1) = 0, \quad \phi_k(+\infty) = 0.$$

Let us consider first the radial mode $k = 0$, namely $\lambda_k = 0$. In this case, ϕ_0 satisfies

$$\Delta^2\phi_0 = pw^{p-1}\phi_0, \phi_0 = \phi_0(r),\ r > 1,\ \ \phi_0(1) = \phi_0^{'}(1) = 0 \tag{3.4}$$

We observe that the function

$$Z_1(r) = rw'(r) + \frac{4}{p-1}w$$

satisfies

$$\Delta^2 Z_1 = pw^{p-1}Z_1, \quad \text{for all } r > 1, \tag{3.5}$$

but $Z_1^{'}(1) = w^{''}(1) = \mathtt{w}^{''}(1) \neq 0$. See (2.6).

Multiplying (3.4) by Z_1 and (3.5) by ϕ_0, and integrating by parts, we obtain

$$Z_1^{'}(1)\Delta\phi_0(1) = 0, \tag{3.6}$$

and hence $\phi_0^{''}(1) = 0$. Similar to the uniqueness proof in Section 2, we obtain that $\phi_0 \equiv 0$.

Let us consider now mode 1, namely $k = 1, \ldots, N-1$, for which $\lambda_k = (N-1)$. In this case we also have an explicit solution whose derivative does not vanish at $r = 1$ but it does at $r = +\infty$. Simply $Z_1(r) = w'(r)$. Similar to (3.6), we may also assume that $\phi_1^{''}(1) = 0$.

It is remarkable to note that

$$\Delta - \frac{\lambda_k}{r^2} = r^k[\frac{\partial^2}{\partial r^2} + \frac{N+2k-1}{r}\frac{\partial}{\partial r}]r^{-k} \tag{3.7}$$

Thus equation (3.3) becomes

$$\left(\frac{\partial^2}{\partial r^2}+\frac{N+2k-1}{r}\frac{\partial}{\partial r}\right)^2\hat{\psi}_k = pw^{p-1}\hat{\psi}_k \tag{3.8}$$

where $r^k\hat{\psi}_k=\phi_k$.

When $k=1$, we have

$$\hat{\psi}_1(1)=\hat{\psi}_1^{'}(1)=\hat{\psi}_1^{''}(1)=0. \tag{3.9}$$

Using (3.8) and (3.9), similar arguments as those of case $k=0$ give $\hat{\psi}_1=0$ and hence again, the only possibility is that $\phi_k\equiv 0$ for all $k=1,\ldots,N$.

Let us consider now modes 2 or higher. Here unfortunately it is more complicated. Not only we do not have an explicit solution to the ODE to rely on, but it could be the case that a non-trivial solution exists. Let us assume this is the case for an arbitrary mode $k\geq N$.

Fixing each k, we consider the following new eigenvalue problem

$$\left(\Delta-\frac{\lambda_k}{r^2}\right)^2\phi_k=\nu w^{p-1}\phi_k, \quad r\in(1,\infty),\, \phi_k=\phi_k(r) \tag{3.10}$$

$$\phi_k(1)=\phi_k^{'}(1)=0, \quad \phi_k(+\infty)=0.$$

Using (3.7), (3.10) is equivalent to

$$\left(\frac{\partial^2}{\partial r^2}+\frac{N+2k-1}{r}\frac{\partial}{\partial r}\right)^2\tilde{\phi}=\nu w^{p-1}\tilde{\phi}, \text{ for } r>1,\ \tilde{\phi}(1)=\tilde{\phi}^{'}(1)=0 \tag{3.11}$$

where $\tilde{\phi}=r^{-k}\phi$.

By Kelvin's transform in dimension $N+2k$, (3.11) is equivalent to

$$\left(\frac{\partial^2}{\partial r^2}+\frac{N+2k-1}{r}\frac{\partial}{\partial r}\right)^2\hat{\phi}=\nu r^{\alpha}\mathtt{w}^{p-1}\hat{\phi}, \text{ for } r<1,\ \hat{\phi}(1)=\hat{\phi}^{'}(1)=0 \tag{3.12}$$

where $\hat{\phi}=r^{4-(N+2k)}\tilde{\phi}(\frac{1}{r})$.

The new eigenvalue problem (3.12) admits a variational structure: in fact it can be rewritten as

$$(\Delta_k)^2\hat{\phi}=\nu r^{\alpha}w^p\hat{\phi} \quad \text{in } B_1^l,\quad \hat{\phi}=\hat{\phi}(r)\in H_0^2(B_1^k) \tag{3.13}$$

where Δ_k is the Laplace operator in $\mathcal{R}^{N+2k}$ and B_1^k is the unit ball in $\mathcal{R}^{N+2k}$. By standard spectrum theory, (3.12) admits an infinite sequence of eigenvalues

$$\nu_{k,1}<\nu_{k,2}<\ldots<\nu_{k,m}<\ldots \tag{3.14}$$

Thus we have arrived at the following key result.

Lemma 3.1 *Assume that p is such that*

$$\nu_{k,m}(p)\neq p \quad \text{for all } k=2,3,\ldots,m=1,\ldots,\ldots \tag{3.15}$$

where $\nu_{k,m}(p)$ is an eigenvalue defined by (3.10). *Then Problem* (3.1)-(3.2) *with $h=0$ admits only the solution $\phi=0$.*

It remains to show that the set $\{\nu_{k,m}(p)=p\}$ is only finite. The key point is the following analyticity of $\nu_{k,m}$. But first we show that m and k are finite if p is also finite. That is we have

Lemma 3.2 *Suppose $p_0+\frac{1}{M}\leq p\leq p_0+M$ where $p_0=\frac{N+4}{N-4}$. Then if $\nu_{k,m}(p)=p$, it holds $k+m\leq K_M$, where K_M depends only on M.*

Proof. We show that $\mathtt{w} \leq C_M$ when $p \in [p_0 + \frac{1}{M}, p_0 + M]$. In fact, using a fixed test function, we first have

$$c_p \leq C_{M,1}, \text{ for } p \leq p_0 + M. \tag{3.16}$$

By the same proof as in Lemma 2.2,

$$\mathtt{w}(r) \leq C_M r^{\frac{4-N}{2}}$$

and hence

$$a(r) := r^\alpha \mathtt{w}^{p-1} \leq C_M r^\alpha r^{\frac{(p-1)(4-N)}{2}}.$$

Since $p > \frac{N+4}{N-4} + \frac{1}{M}$, it is easy to check that there exists $\epsilon_0 > 0$ such that

$$\int_{B_1} (a(r))^{\frac{N}{4}+\epsilon_0} \leq C_M. \tag{3.17}$$

Since $\mathtt{w}$ satisfies

$$\Delta^2 \mathtt{w} = a(r)\mathtt{w}, \mathtt{w} \in H_0^2(B_1), a(r) \in L^{\frac{N}{4}+\epsilon_0}(B_1)$$

by the regularity for biharmonic equations (see [21]), we obtain that $\mathtt{w} \leq C_M$.

Using variational arguments and comparison of eigenvalues, we have

$$\nu_{k,m} \geq \frac{1}{C_M} j_{k,m} \tag{3.18}$$

where $j_{k,m}$ is the $m-$th eigenvalue of

$$(\Delta_k)^2 \hat{\phi} = \nu \hat{\phi} \quad \text{in } B_1^k, \quad \hat{\phi} = \hat{\phi}(r) \in H_0^2(B_1^k). \tag{3.19}$$

Since $j_{k,m} \to +\infty$ as $k + m \to +\infty$, we deduce that $k + m \leq K_M$ if $\nu_{k,m}(p) = p$.

□

The next lemma shows that the map $\nu_{k,m}(p)$ is analytic in p.

Lemma 3.3 *The eigenvalues $\nu_{k,m}$ are simple and analytic in p.*

Proof. Let $\phi_{k,m}$ be the corresponding eigenfunction to (3.13) with $\nu = \nu_{k,m}$. We show that $\phi_{k,m}$ must be **simple** and unique. In fact, if there are two such $\phi_{k,m,1}, \phi_{k,m,2}$. We may combine them to obtain a new $\phi_{k,m}$ such that $\phi''_{k,m}(1) = 0$. The same argument leading $\phi_1 = 0$ shows that $\phi_{k,m} = 0$. This implies that the eigenvalues $\nu_{k,m}$, if exists, must be simple.

Since $\nu_{k,m}$ is simple and problem (3.13) is self-adjoint and the function $r^\alpha w^p$ is analytic in p, by standard theory in analytic perturbation of eigenvalues (see, e.g. Theorem 3.9 of [14]), we deduce that $\nu_{k,m}(p)$ is analytic in p.

□

Our next result will show that for p close to $\frac{N+4}{N-4}$, $\nu_{k,m}(p) \neq p$.

Lemma 3.4 *If p is close to $\frac{N+4}{N-4}$, then $\nu_{k,m}(p) - p \neq 0$.*

Proof. We shall prove first that as $p \to \frac{N+4}{N-4}$, after some rescaling the solution to (1.9)-(1.10) approaches a standard bubble in $\mathcal{R}^N$, i.e., solution to

$$\Delta^2 U = U^{\frac{N+4}{N-4}}, U > 0. \tag{3.20}$$

Note that all solutions to (3.20) are given by $U_{\lambda,a} = \lambda^{\frac{N-4}{2}} U_0(\lambda|x-a|)$ for some $\lambda > 0, a \in \mathcal{R}^N$, where $U_0 = c_N(\frac{1}{1+|x|^2})^{(N-4)/2}$.

To achieve this, we use an ODE argument. An alternative way of writing equation (1.9)-(1.10) and the eigenvalue problem (3.10) is by means of the so-called Emden-Fowler transformation,

$$\tilde{w}(s) = r^{\frac{4}{p-1}} w(r)\,, \quad \tilde{\psi}(s) = r^{\frac{4}{p-1}} \psi(r), \text{ where } r = e^s\,. \tag{3.21}$$

Then equation (1.9)-(1.10) is converted into

$$\tilde{w}^{(4)} + \alpha \tilde{w}^{(3)} + \beta \tilde{w}^{''} + \gamma \tilde{w}^{'} + \xi \tilde{w} = \tilde{w}^p\,, \quad \tilde{w}(0) = \tilde{w}^{'}(0) = \tilde{w}(\infty) = \tilde{w}^{'}(\infty) = 0,\ s \in [0,\infty) \tag{3.22}$$

where

$$\alpha = 2(N-1) + \frac{10+6p}{1-p},$$

$$\beta = \frac{11p^2+50p+35}{(1-p)^2} + \frac{(9+3p)2(N-1)}{1-p} + (N-1)(N-3),$$

$$\gamma = \frac{6p^3+70p^2+130p+50}{(1-p)^3} + \frac{2p^2+20p+26}{(1-p)^2} 2(N-1) + \frac{(6+2p)}{1-p}(N-1)(N-3),$$

$$\xi = \frac{4(3+p)(2+2p)(1+3p)}{(1-p)^4} + \frac{8(N-1)(3+p)(2+2p)}{(1-p)^3}$$
$$+ \frac{4(N-1)(N-3)(3+p)}{(1-p)^2} - \frac{4(N-1)(N-3)}{1-p}.$$

The eigenvalue problem (3.10) becomes

$$\tilde{\psi}^{(4)} + \alpha \tilde{\psi}^{(3)} + \beta \tilde{\psi}^{''} + \gamma \tilde{\psi}^{'} + (\xi - \lambda_k)\tilde{\psi} = \nu \tilde{w}^{p-1} \tilde{\psi}, \tag{3.23}$$

$$\tilde{\psi}(0) = \psi^{'}(0) = \tilde{\psi}(\infty) = \psi^{'}(\infty) = 0,\ s \in [0,\infty)\,. \tag{3.24}$$

We first note that since w is the least energy solution, by (2.9), we see that we have the following energy bound:

$$\int_{\mathcal{R}^N \setminus B_1} (w^{p+1} + |\Delta w|^2) \le C \tag{3.25}$$

where C is independent of p, for p close to $\frac{N+4}{N-4}$. Translating the bound (3.25) in terms of $\tilde{w}$, we obtain that

$$\int_0^\infty e^{(N - \frac{4(p+1)}{p-1})s} \tilde{w}^{p+1} ds \le C. \tag{3.26}$$

Since $N > \frac{4(p+1)}{p-1}$, we see that $\int_0^\infty \tilde{w}^{p+1} ds \le C$. This implies that

$$\|\tilde{w}\|_{L^\infty} \le C, \int_0^\infty \tilde{w}^{p+1} ds \le C. \tag{3.27}$$

By direct computation we see that as $p \to p_0, \alpha \to 0, \beta \to \beta_0 = -\frac{N^2-4N+8}{2}, \gamma \to \gamma_0 = \frac{N^3-N^2-4N-56}{8}, \xi \to \xi_0 = \frac{N^4}{4} - \frac{13N^3}{8} + \frac{7N^2}{4} + \frac{11N}{2} - 6$. Moreover, since $\tilde{w}$ is uniformly bounded, using (3.27), standard regularity theory shows there exists $R_\alpha \to +\infty$ such that

$$\tilde{w} = w_0(s - R_\alpha) + \textit{lower order terms}\,, \tag{3.28}$$

where w_0 is the unique homoclinic solution of the limiting equation,

$$w_0^{(4)} + \beta_0 w_0^{''} + \gamma_0 w_0^{'} + \xi_0 w_0^{\prime} = w_0^{p_0}, \quad w_0(0) = \max_{t \in R} w_0(t), \quad w_0(\infty) = 0.$$

So after a translation, the eigenvalue problem (3.23) becomes

$$\tilde{\psi}^{(4)} + \beta \tilde{\psi}^{''} + \gamma_0 \tilde{\psi}^{'} + (\xi_0 - \lambda_k)\tilde{\psi} = \nu w_0^{p_0 - 1}\tilde{\psi}, -\infty < s < +\infty, \tag{3.29}$$

$$\tilde{\psi}(-\infty) = \psi^{'}(-\infty) = \tilde{\psi}(\infty) = \psi^{'}(\infty) = 0, \ s \in (-\infty, \infty). \tag{3.30}$$

By Theorem 2.1 of [2], (see (2.13) of [2]), the following eigenvalue problem

$$(\Delta - \frac{\lambda}{r^2})^2 \phi = p_0 U_0^{p_0 - 1}\phi \tag{3.31}$$

is λ_0 or λ_1. Observe that $w_0(s) = r^{\frac{4}{p_0 - 1}} U_0(r), r = e^s$. Thus, the only λ_k satisfying

$$\tilde{\psi}^{(4)} + \alpha_0 \tilde{\psi}^{(3)} + \beta_0 \tilde{\psi}^{''} + \gamma_0 \tilde{\psi}^{'} + (\xi_0 - \lambda_k)\tilde{\psi} = p_0 w_0^{p-1}\tilde{\psi},$$

is λ_0 or λ_1. But as we already know, $\nu_{0,m}(p) \neq p$ and $\nu_{1,m}(p) \neq p$. Thus we conclude that for $k \geq 2$, $\nu_{k,m}(p) - p \neq 0$ as $p \to p_0$.

□

Combining Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have the following

Proposition 3.5 *For each pair (k, m) the set of numbers p for which $\nu_{k,m}(p) = p$ is finite (maybe empty). In particular, there exist countably many supercritical exponents $\{p_1, ..., p_j, ...\}$ with $p_j > \frac{N+4}{N-4}$ such that condition (3.15) holds if and only if $p \neq p_j$ for all $j = 1, 2, \ldots$.*

Proof. By Lemma 3.2, we have

$$\cup_{k,m=1}^{\infty} \left\{ \nu_{k,m}(p) = p \right\} \subset \cup_{M=1}^{\infty} \cup_{k+m \leq K_M} \left\{ \nu_{k,m}(p) = p, p \in [p_0 + \frac{1}{M}, p_0 + M] \right\}.$$

where each set of the right hand side contains only finite number of points (possibly empty), by Lemma 3.3. The proposition is thus proved.

□

Remark 3.6 We don't claim that $p_j \to +\infty$ as $j \to +\infty$. Neither do we claim the set $\{p_j\}$ is *nonempty*. These are not needed for our late construction. To prove these claims, one has to study the asymptotic behavior of w as $p \to +\infty$.

4 The operator $\Delta^2 - pw^{p-1}$ on $\mathcal{R}^N \setminus B_1(0)$

Let $p \neq p_j$ as in Proposition 3.5. In this section we solve the full problem (3.1)-(3.2), namely

$$\Delta^2 \phi - p w^{p-1}\phi = h \quad \text{in } \mathcal{R}^N \setminus \bar{B}_1(0),$$
$$\phi = \nabla \phi = 0 \ \text{ on } \partial B_1(0), \quad \lim_{|x| \to +\infty} \phi(x) = 0,$$

Our main result in this section concerns with solvability of this equation and estimates for the solution in appropriate norms. Following [9], let us fix a small number $\sigma > 0$ and consider the norms

$$\|\phi\|_* = \sum_{j \leq 3} \sup_{|x|>1} |x|^{N-j-\sigma} |\nabla^j \phi(x)| \tag{4.1}$$

and

$$\|h\|_{**} = \sup_{|x|>1} |x|^{N-\sigma} |h(x)|. \tag{4.2}$$

Proposition 4.1 *Assume that p satisfies condition* (3.15). *Then for any h with $\|h\|_{**} < +\infty$, Problem* (3.1)-(3.2) *has a unique solution $\phi = T(h)$ with $\|\phi\|_* < +\infty$. Besides, there exists a constant $C(p) > 0$ such that*

$$\|T(h)\|_* \le C\|h\|_{**}.$$

Proof. The proof makes use of duality via Kelvin's transform. Let $\tilde\phi, \tilde h$ be the Kelvin's transform of ϕ, h respectively, namely

$$\tilde\phi(x) = |x|^{4-N}\phi(\frac{x}{|x|^2}), \tilde h(x) = |x|^{-4-N}h(\frac{x}{|x|^2}), \tag{4.3}$$

we get the problem

$$\Delta^2\tilde\phi - |x|^{p(N-4)-(N+4)}p\mathtt{w}^{p-1}\tilde\phi = \tilde h \quad \text{in } B_1(0)\,,$$
$$\tilde\phi = \nabla\tilde\phi = 0 \quad \text{on } \partial B_1(0)\,.$$

Then we have

$$|\tilde h(x)| \le \|h\|_{**}\,|x|^{-4-\sigma}\,. \tag{4.4}$$

It follows in particular that, if σ is fixed small, $h \in L^q(B_1(0))$ for some $q > \frac{2N}{N+4}$, hence $h \in H^{-2}(B_1(0))$. From Lemma 3.1, it follows that only the trivial H_0^2-solution is present for 0 right hand side. Hence there exists a unique weak solution $\tilde\phi \in H_0^2(B(0,1)$ whose norm is controlled by a multiple of $\|h\|_{**}$. We can write

$$\tilde\phi(x) = \int_{B_1(0)} G(x,y)[|x|^{p(N-4)-(N+4)}p\mathtt{w}^{p-1}\tilde\phi + \tilde h] \tag{4.5}$$

where $G(x,y)$ is the Green's function of Δ^2 in $B_1(0)$. Note that $G(x,y) > 0$. (See [13].)

To obtain pointwise estimates, we use Maximum Principle. Let us now observe that

$$\Delta^2(|x|^{-\sigma}) = (N-2-\sigma)\sigma(2+\sigma)(N-4-\sigma)|x|^{-4-\sigma},$$

hence, fixed σ we can find a $\rho(p,N,\sigma) > 0$ such that for $|x| < \rho$,

$$\Delta^2(|x|^{-\sigma}) - p|x|^{p(N-4)-(N+4)}\mathtt{w}^{p-1}|x|^{-\sigma} \ge \frac{1}{2}(N-2-\sigma)\sigma(2+\sigma)(N-4-\sigma)r^{-4-\sigma}. \tag{4.6}$$

Since h is bounded by a σ-dependent multiple of $\|h\|_{**}$ on, say, $\frac{\rho}{2} < |x| < 1$, using (4.5), elliptic estimates yield that

$$\|\phi\|_{L^\infty(|x|\ge\rho)} + \|\Delta\phi\|_{L^\infty(|x|\ge\rho)} \le C\|h\|_{**}$$

with C depending on N, p, σ. Then from (4.4), (4.6) and maximum principle for $\Delta\phi$ in $|x| < \rho$, we deduce that

$$|\Delta\tilde\phi(x)| \le C|x|^{-2-\sigma}\|h\|_{**} \quad |x| < 1$$

and

$$|\tilde\phi(x)| \le C|x|^{-\sigma}\|h\|_{**} \quad |x| < 1.$$

Hence

$$\||x|^{N-4-\sigma}\phi\|_\infty = \||x|^{\sigma}\tilde\phi\|_\infty \le C\|h\|_{**}, \||x|^{N-2-\sigma}\Delta\phi\|_\infty \le C\|h\|_{**}$$

The desired conclusion for $\nabla\Delta\phi$ follows by scaling: consider a ball radius R centered at a point $\bar x$ with $|\bar x| = 2R$, for $R > 5$. Set

$$\hat\phi(y) = R^{N-4-\sigma}\phi(\bar x + Ry)$$

Then

$$\Delta^2\hat\phi - pR^4\hat w^{p-1}\hat\phi = R^{N-\sigma}\hat h\,, \quad y \in B(0,1).$$

Clearly in this ball

$$\|R^{N-\sigma}\hat{h}\|_\infty \le C\|h\|_{**}, \quad R^4\hat{w}^{p-1} = O(R^{-4}), \quad \|\hat{\phi}\|_\infty \le C\|h\|_{**}.$$

Elliptic estimates then imply

$$|\nabla\Delta\hat{\phi}(0)| \le C\|h\|_{**}$$

or

$$|\nabla\Delta\phi(\bar{x})| \le C\|h\|_{**}|\bar{x}|^{3-N+\sigma}.$$

Since $\bar{x}$ is arbitrary with $|\bar{x}| > 5$, the desired conclusions follow. This finishes the proof.

□

5 The operator $\Delta^2 - pw^{p-1}$ in $\delta^{-1}\mathcal{D} \setminus B_1(0)$

In this section and in what follows we shall assume that $Q = 0$, and consider the large expanded domain $\mathcal{D}_\delta = \delta^{-1}\mathcal{D}$. We shall carry out a gluing procedure that will permit to establish the same conclusion of Proposition 4.1 in this domain, provided that δ is taken sufficiently small. Thus we consider now the linear problem

$$\Delta^2\phi - pw^{p-1}\phi = h \quad \text{in } \mathcal{D}_\delta \setminus \bar{B}_1(0)\,, \tag{5.1}$$

$$\phi = \nabla\phi = 0 \quad \text{on } \partial B_1(0) \cup \partial\mathcal{D}_\delta. \tag{5.2}$$

We consider the same norms as in (4.1), (4.2) restricted to this domain.

Proposition 5.1 *Assume that p satisfies condition* (3.15). *Then there is a number δ_0 such that for all $\delta < \delta_0$ and any h with $\|h\|_{**} < +\infty$, Problem* (5.1)-(5.2) *has a unique solution $\phi = T_\delta(h)$ with $\|\phi\|_* < +\infty$. Besides, there exists a constant $C(p,\mathcal{D}) > 0$ such that*

$$\|T_\delta(h)\|_* \le C\|h\|_{**}.$$

P r o o f. We assume with no loss of generality that the domain $\mathcal{D}$ contains the ball $B_3(0)$. Let us consider a smooth, radial cut off $\eta(|y|)$ which equals one on $|y| < 2$ and vanishes identically for $|y| > 3$. We consider also a second cut-off $\zeta(|y|)$ which equals 1 on $|y| < 1$ and it is 0 for $|y| > 2$. In particular we have of course $\eta\zeta = \zeta$. Correspondingly, we also write

$$\eta_\delta(x) = \eta(\delta|x|)\,, \quad \zeta_\delta(x) = \zeta(\delta|x|)\,.$$

We look for a solution ϕ to Problem (5.1)-(5.2) in the form

$$\phi = \eta_\delta\varphi + \psi$$

where ϕ and ψ are required to satisfy the following system:

$$\begin{cases} \Delta^2\varphi - pw^{p-1}\varphi = p\zeta_\delta w^{p-1}\psi + \zeta_\delta h & \text{in } \mathcal{R}^N \setminus B_1(0) \\ \varphi = \nabla\varphi = 0 & \text{on } \partial B_1(0) \\ \varphi(x) \to 0 & \text{as } |x| \to +\infty\,, \end{cases} \tag{5.3}$$

$$\begin{cases} \Delta^2\psi - p(1-\zeta_\delta)w^{p-1}\psi = G(\varphi,\eta_\delta) + (1-\zeta_\delta)h & \text{in } \mathcal{D}_\delta \\ \psi = \nabla\psi = 0 & \text{on } \partial\mathcal{D}_\delta \cup \partial B_1(0), \end{cases} \tag{5.4}$$

where $G(\varphi,\eta_\delta) = -\Delta(\varphi\Delta\eta_\delta + 2\nabla\eta_\delta\nabla\varphi) - 2\nabla\eta_\delta\nabla(\Delta\varphi) - \Delta\varphi\Delta\eta_\delta$.

We shall solve equation (5.4) for ψ in terms of ϕ and h. To do so, let us consider the linear problem

$$\begin{cases} \Delta^2\psi - p(1-\zeta_\delta)w^{p-1}\psi = g & \text{in } \mathcal{D}_\delta \setminus B_1(0) \\ \psi = \nabla\psi = 0 & \text{on } \partial\mathcal{D}_\delta \cup \partial B_1(0). \end{cases} \tag{5.5}$$

for $g \in L^\infty(\mathcal{D}_\delta \cup \partial B_1(0))$. Scaling back δ by setting for any function ρ, $\tilde\rho(x) = \rho(\delta^{-1}x)$, we see that problem (5.5) is equivalent to

$$\begin{cases} (\Delta)^2\tilde\psi - p(1-\zeta_\delta)\delta^{-4}\tilde w^{p-1}\tilde\psi = \delta^{-4}\tilde g & \text{in } \mathcal{D} \setminus B_\delta(0) \\ \psi = \nabla\psi = 0 & \text{on } \partial\mathcal{D} \cup \partial B_\delta(0). \end{cases}$$

We see that

$$p(1-\zeta_\delta)\delta^{-4}\tilde w^{p-1} = o(\delta^2) < \lambda_1(\mathcal{D}) < \lambda_1(\mathcal{D} \setminus B_\delta(0)),$$

if δ is taken sufficiently small, by the decaying property of w. Hence this problem can be solved uniquely for $\tilde\psi$. In terms of ψ we get in addition the estimate

$$||\psi||_\infty \le C\delta^{-4}||g||_\infty,$$

where C does not depend on δ. The map $g \mapsto \psi$ defines of course a linear operator. Let us now go back to equation (5.4). Then this problem can be solved uniquely, as a linear operator of the pair (φ, h), which we simply call $\psi(\varphi, h)$. Setting

$$g = -\Delta(\varphi\Delta\eta_\delta + 2\nabla\eta_\delta\nabla\varphi) - 2\nabla\eta_\delta\nabla(\Delta\varphi) - \Delta\varphi\Delta\eta_\delta + (1-\zeta_\delta)h$$

By the definition of $||\varphi||_*$, we easily obtain that

$$||g||_\infty \le C\,[\,\delta^{N-\sigma}||\varphi||_* + \delta^{N-\sigma}||h||_{**}\,],$$

and hence

$$||\psi(\varphi,h)||_\infty \le C\,[\,\delta^{N-4-\sigma}||\varphi||_* + \delta^{N-4-\sigma}||h||_{**}\,]. \tag{5.6}$$

Let us replaced this ψ into equation (5.3). We have thus a solution of the full system if we solve the fixed point problem

$$\varphi \;=\; T(\,p\zeta_\delta w^{p-1}\,\psi(\varphi,h) \;+\; \zeta_\delta h\,) \tag{5.7}$$

where T is the linear operator defined by Proposition 4.1. We make now the observation that, assuming also $\sigma < (N-4)(p-1) - 8$,

$$|x|^{N-\sigma}w^{p-1}\,|\psi(\varphi,h)| \;\le\; |x|^{N-\sigma-(N-4)(p-1)}\,\delta^{N-4-\sigma}[||\varphi||_* + ||h||_{**}] \le$$

$$|x|^{N-2\sigma-8}\delta^{N-4-\sigma}[||\varphi||_* + ||h||_{**}] \le |x|^{-4}\delta^{\sigma}[||\varphi||_* + ||h||_{**}],$$

so that

$$||p\zeta_\delta w^{p-1}\,\psi(\varphi,h)||_{**} \;\le\; C\delta^{\sigma}[||\varphi||_* + ||h||_{**}].$$

From here and contraction mapping principle, we get then that if δ is chosen sufficiently small, then (5.7) can be solved uniquely in the form $\phi = T_\delta(h)$ where the bounds for T_δ are the same as those for T, independent of δ. This concludes the proof.

□

6 Proof of Theorem 1.1

Let us assume the validity of condition of condition (3.15) or, equivalently, that $p \neq p_j$ for all j, with p_j the sequence in (3.5). Problem (1.5)-(1.6) is, after setting $v(x) = \delta^{\frac{4}{p-1}} u(\delta x)$, equivalent to

$$\Delta^2 v = v^p \quad \text{in } \mathcal{D}_\delta \setminus \bar{B}_1(0)\,, \tag{6.1}$$

$$v = \nabla v = 0 \quad \text{on } \partial B_1(0) \cup \partial \mathcal{D}_\delta. \tag{6.2}$$

Let us consider the smooth cut-off function η_δ, introduced in the previous section, which equals 1 in $B(0, 2\delta^{-1})$ and 0 outside $B(0, 3\delta^{-1})$. We search for a solution v to problem (6.1)-(6.2) of the form

$$v = \eta_\delta w + \phi,$$

which is equivalent to the following problem for ϕ:

$$\Delta^2 \phi + p w^{p-1} \phi = N(\phi) + E \quad \text{in } \mathcal{D}_\delta \setminus \bar{B}_1(0)\,, \tag{6.3}$$

$$\phi = \nabla \phi = 0 \quad \text{on } \partial B_1(0) \cup \partial \mathcal{D}_\delta. \tag{6.4}$$

where

$$N(\phi) = N_1(\phi) + N_2(\phi)\,,$$
$$N_1(\phi) = -(\eta_\delta w + \phi)^p + (\eta_\delta w)^p + p(\eta_\delta w)^{p-1}\phi,$$
$$N_2(\phi) = p(1 - \eta_\delta^{p-1}) w^{p-1}\phi,$$

and

$$E = -\Delta^2(\eta_\delta w) - (\eta_\delta w)^p.$$

According to Proposition 5.1 we thus have a solution to (6.1)-(6.2) if ϕ solves the fixed point problem

$$\phi = T_\delta(N(\phi) + E)\,. \tag{6.5}$$

Let us estimate E. We have, explicitly,

$$-E = \eta_\delta(\eta_\delta^{p-1} - 1) w^p + \Delta\eta_\delta \Delta w + 2\nabla\eta_\delta \nabla(\nabla w) + \Delta(2\nabla\eta_\delta \nabla w + w\Delta\eta_\delta)$$

We clearly have, globally, $|E(x)| \leq C\delta^N$ and hence

$$\|E\|_{**} \leq C\delta^\sigma\,. \tag{6.6}$$

Let us measure now $N(\phi)$. We observe that

$$\|N_2(\phi)\|_{**} = \|p(1 - \eta_\delta^{p-1}) w^{p-1}\phi\|_{**} \leq C \sup_{|x| \geq \delta^{-1}} |x|^{N-\sigma} w(x)^{p-1} |\phi(x)|$$
$$\leq C\delta^4 \|\phi\|_*. \tag{6.7}$$

Next we shall now estimate $\|N_1(\phi)\|_{**}$. Let us assume first $p < 2$. Then we estimate

$$|N_1(\phi)| \leq C|\phi|^p,$$

$$|x|^{N-\sigma} |N_1(\phi)| \;\leq\; C|x|^{N-\sigma} |\phi(x)|^p \;\leq\; |x|^{N-\sigma} |x|^{-N-4} \|\phi\|_*^p \;\leq\; C\|\phi\|_*^p\,,$$

so that

$$\|N_1(\phi)\|_{**} \leq C\|\phi\|_*^p\,.$$

Let us assume now $p \geq 2$. In this case we have

$$|N_1(\phi)| \;\leq\; C\,(\,w^{p-2}\phi^2 + |\phi|^p\,).$$

Now, we directly check that

$$|x|^{N-\sigma} w^{p-2} \phi^2 \le C|x|^{(p-2)(4-N)-N+8+\sigma} \|\phi\|_*^2.$$

The power of $|x|$ in the last expression is always negative provided $p > \frac{N+4}{N-4}$. On the other hand,

$$|x|^{N-\sigma} |\phi|^p \le C\, |x|^{N-\sigma-p(N-4-\sigma)} \, \|\phi\|_*^p \le |x|^{-4+(p-1)\sigma} \, \|\phi\|_*^p \, .$$

We conclude from these estimates that, for any $p > \frac{N+4}{N-4}$,

$$\|N_1(\phi)\|_{**} \le C\, (\, \|\phi\|_*^p + \|\phi\|_*^2\,)\, . \tag{6.8}$$

Let us consider now the operator

$$\mathcal{T}(\phi) = T_\delta(N(\phi) + E)$$

defined in the region

$$\mathcal{B} = \{\phi \in C^2(\bar{\mathcal{D}}_\delta \setminus B_1(0)) \ / \ \|\phi\|_* \le \delta^{\frac{\sigma}{2}}\, \}\, .$$

Using estimates (6.6), (6.8), (6.7) we immediately get that $\mathcal{T}(\mathcal{B}) \subset \mathcal{B}$, provided that δ is sufficiently small. We observe that, in the bounded domain $\mathcal{D}_\delta \setminus B_1(0)$,

$$T_\delta = (\Delta^2 - pw^{p-1})^{-1}$$

applies boundedly C^1 into $C^{4,\alpha}$, hence compactly into C^1. It follows that the map $\mathcal{T}$ is actually compact on the closed, bounded set of C^2 given by $\mathcal{B}$. The existence of a fixed point of $\mathcal{T}$ on $\mathcal{B}$ thus follows from Schauder's theorem. This concludes the proof of the theorem. □

Acknowledgements The research of the first author is supported by NSFC (10571098) and the research of the second author is partially supported by an Earmarked Grant from RGC of Hong Kong. We thank the referee for carefully reading the manuscript and many critical suggestions.

References

[1] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, *Comm. Pure Appl. Math.* 41(1988), 255-294.

[2] T. Bartsch, T. Weth and M. Willem, A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator, *Cal. Var. PDE.* 18(2003), no.3, 253-268.

[3] F. Bernis and H. Grunau, Critical exponents and multiple critical dimensions for polyharmonic operators, *J. Diff. Eqns.* 117(1995), 469-486.

[4] T. Boggio, Sulle funzioni di Green d'ordine m, *Rend. Circ. Mat. Palermo* 20 (1995), 97-135.

[5] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* 36 (1983), 437–47.

[6] J.M. Coron, *Topologie et cas limite des injections de Sobolev*, C.R. Acad. Sc. Paris, 299, Series I (1984), 209-212.

[7] E.N. Dancer, Real analyticity and non-degeneracy, *Math. Ann.* 325 (2003), no. 2, 369–392.

[8] D.E. Edmunds, D. Fortunato and E. Jannelli, Critical exponents, critical dimensions and the biharmonic operators, *Arch. Rat. Mech. Anal.* 112(3) (1990), 269-289.

[9] M. del Pino and J. Wei, Supercritical elliptic problems in domains with small holes, *Ann. Inst. H. Poincare Anal. Non Lineaire*, to appear.

[10] F. Gazzola, H.-G. Grunau, M. Squassina, Existence and nonexistence results for critical growth biharmonic elliptic equations, *Cal. Var. PDE.* 18(2003), no.2, 117-113.

[11] Y. Ge, Positive solutions in semilinear critical problems for polyharmonic operators, *J. Math. Pures Appl.* 84(2005), 199-245.

[12] H.C. Grunau, Positive solutions to semilinear Dirichlet problems involving critical Sobolev exponents, *Cal. var. PDE* 3(1995), 243-252.

[13] H.C. Grunau and G. Sweeers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions, *Math. Ann* 307(1997), 589-626.

[14] T. Kato, *Perturbation theory for linear operators.* Second edition. Grundlehren der Mathematischen Wissenschaften, Band 132. Springer-Verlag, Berlin-New York, 1976.
[15] W.-M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, *Indiana Univ. Math. J.* 31 (1982), no. 6, 801–807.
[16] D. Passaseo, Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains,*J. Funct. Anal.* 114 (1993), no. 1, 97–105
[17] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet. Math. Dokl.* 6(1965), 1408-1411.
[18] P. Pucci and J. Serrin, A general variational idenity, *Indiana Univ. math. J.* 35(1986), no. 3, 681-703.
[19] O. Rey, On a variational problem with lack of compactness: the effect of small holes in the domain, *C. R. Acad. Sci. ParisSr. I Math.* 308 (1989), no. 12, 349–352.
[20] C. Swanson, The best Sobolev constant, *Appl. Anal.* 47(1992), 227-239.
[21] R. van der Vorst, Best constant for the embedding of the psace $H^2 \cap H_0^2(\Omega)$ into $L^{\frac{2N}{N-4}}(\Omega)$, *Diff. Int. Eqns.* 6(1993), 259-276.