HAUSDORFF DIMENSION OF RUPTURES FOR SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION WITH SINGULAR NONLINEARITY

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Abstract. We consider the following semilinear elliptic equation with singular nonlinearity:

$$\Delta u - \frac{1}{u^\alpha} + h(x) = 0 \text{ in } \Omega$$

where $\alpha > 1, h(x) \in C^1(\Omega)$ and Ω is an open subset in $\mathbb{R}^n, n \geq 2$. Let u be a non-negative **finite energy stationary** solution and $\Sigma = \Big\{ x \in \Omega : \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u|$ exists, and is equal to $0 \Big\}$ be the **rupture** set of u. We show that the Hausdorff dimension of Σ is less than or equal to $\frac{(n-2)\alpha + (n+2)}{\alpha+1}$.

1. Introduction

Let Ω be an open subset in $\mathbb{R}^n$ $(n \geq 2)$. In this paper we consider partial regularity for nonnegative solutions of the following equation

$$\Delta u - \frac{1}{u^\alpha} + h(x) = 0 \quad \text{in } \Omega \tag{1.1}$$

where $\alpha > 1$, $h \in C^1(\Omega)$ such that $\|h\|_{L^\infty(\Omega)} \leq a$ and $\|\nabla h\|_{L^\infty(\Omega)} \leq b$ for some constants $a, b > 0$. In particular, we are concerned with the Hausdorff dimension of the zero set:

$$\Sigma = \Big\{ x \in \Omega : \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \text{ exists, and is equal to } 0 \Big\}. \tag{1.2}$$

Problem (1.1) arises in the study of steady states of thin films. Equations of the type

$$u_t = -\nabla \cdot (f(u) \nabla \Delta u) - \nabla \cdot (g(u) \nabla u) \tag{1.3}$$

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The zero set Σ defined in (1.2) is the liquid/solid interface and is sometimes called set of **ruptures**. The coefficient $f(u)$ reflects surface tension effects- a typical choice is $f(u) = u^3$. The coefficient of the

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second-order term can reflect additional forces such as gravity $g(u) = u^3$, van der Waals interactions $g(u) = u^m, m < 0$. For backgrounds on (1.3), we refer to [BP1, BP2, LP1, LP2, LP3, WB] and the references therein.

In general, let us assume that $f(u) = u^p, g(u) = u^m$, where $p, m \in R$. Then a steady-state equation for (1.3) with Neumann boundary condition becomes

$$\Delta u + \frac{u^q}{q} - C = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $q = m - p + 1$ and C is some constant. (Here we have assumed that $q \neq 0$. If $q = 0$, we have to replace $\frac{u^q}{q}$ by $\log u$.) For thin films under van der Waals forces, we have $f(u) = u^3$, $g(u) = u^m$, $q = m - 2 < -2$. The one-dimensional steady-state problem of (1.3) has been studied thoroughly in [LP1, LP3] and the references therein. Numerical work on two-dimensional van der Waals driven rupture in (1.3) suggested that the rupture can occur in points [BBD, HLU] or rings [WB, YD, YH].

The main result of our paper is to give an estimate on the Hausdorff dimension of the rupture set Σ. Roughly speaking, we prove that the Hausdorff dimension of Σ is less than or equal to $((n-2)\alpha + (n+2))/(\alpha+1)$.

We begin with some definitions. We call u a nonnegative **finite energy** solution of (1.1) in Ω if $u \geq 0$ in Ω, u satisfies (1.1) pointwisely in $\Omega \backslash \Sigma$, and the energy of u

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{1-\alpha}\int_\Omega u^{1-\alpha} dx - \int_\Omega h(x) u dx$$

is finite.

We also say that such a **finite energy** solution u is **stationary** if, in addition, it satisfies

$$\int_\Omega \left[\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial \phi^j}{\partial x_i} - \frac{1}{2}|\nabla u|^2 \frac{\partial \phi^i}{\partial x_i} - \frac{1}{1-\alpha} u^{1-\alpha}\frac{\partial \phi^i}{\partial x_i} + u\frac{\partial h}{\partial x_i}\phi^i + uh\frac{\partial \phi^i}{\partial x_i}\right] dx = 0 \tag{1.5}$$

for all regular vector field ϕ with compact supports in Ω (summation over i and j is understood).

For **finite energy** solutions $u \in H^1(\Omega)$ and $\int_\Omega u^{1-\alpha}(x) dx < \infty$ the identity (1.5) is obtained by assuming that the functional $E(u)$ is stationary with respect to domain variations, that is,

$$\left.\frac{d}{dt}E(u_t)\right|_{t=0} = 0$$

where $u_t(x) = u(x + t\phi(x))$, $\phi(x) = (\phi^1(x), \phi^2(x), \ldots, \phi^n(x))$. (The identity (1.5) can also be obtained by multiplying (1.1) by $\phi \cdot \nabla u$ and integrating it by parts in Ω (if it can be integrated by parts)). Examples of stationary solution include minimizers of the energy functional $E(u)$ (if they exist). The concept of **stationary** solutions was introduced in [Ev].

Let us define

$$\mathcal{E}_\alpha = \{u \in H^1(\Omega) : \ u \geq 0 \ \text{ in } \Omega, \int_\Omega u^{1-\alpha}(x)dx < \infty\}.$$

Let $u \in \mathcal{E}_\alpha$ be a finite energy solution of (1.1). We easily see that away from Σ, the classical regularity theory ensures that u is regular. Therefore Σ is the set of singularities of u^{-1}. Moreover, by the definition, Σ is a relatively closed subset of Ω.

Our partial regularity result is the following theorem.

Theorem 1.1. *Let $\alpha > 1$ be given. If $u \in \mathcal{E}_\alpha$ is a* **finite energy** *solution of (1.1), which is* **stationary***, then u is smooth outside a closed rupture set of u with locally finite Hausdorff μ-dimensional measure, where $\mu = \frac{(n-2)\alpha+(n+2)}{(\alpha+1)}$. In other words, the Hausdorff dimension of Σ is less than or equal to μ.*

In a recent paper [JL], Jiang and Lin studied the weak solution of (1.1) in the sense that $u \in H^1_{loc}(\Omega)$, $u \in L^1(\Omega)$ and $u^{-\alpha} \in L^1(\Omega)$. Using an important Poincaré type inequality, they found that $H^s(\Sigma) = 0$, where $s = n - 2 + \frac{4}{\alpha+2}$. On the other hand, here we assume that $u^{1-\alpha} \in L^1(\Omega)$ which is weaker than $u^{-\alpha} \in L^1(\Omega)$. But the Hausdorff dimension of Σ obtained in this paper is larger than that obtained in [JL].

We will first establish a monotonicity inequality for the nonnegative finite energy stationary solutions $u \in \mathcal{E}_\alpha$ of (1.1). Then, using such monotonicity of the energy of u, we obtain the measure estimate of the singular set Σ of u^{-1}. This estimate on Σ may have potential applications on the estimates for ruptures of thin films. For example, if $n = 2, \alpha > 3$, Theorem 1.1 implies that there are no finite energy stationary solutions with ring ruptures.

We don't know if $\mu = ((n-2)\alpha + (n+2))/(\alpha+1)$ is the optimal.

About the applicability of Theorem 1.1, we see that under the flow (1.3) and the fact that the pressure is constant (i.e., $\frac{1}{|\Omega|}\int_\Omega u(x,t) \equiv \mathit{constant}$), the energy of $u(x,t)$ is decreasing with respect to t. Thus, if we start with a finite energy initial data, then the limit of $u(x,t)$ as $t \to +\infty$ (if exists) is also of finite energy. We also believe only local minimizers of $E(u)$ are stable with respect to the flow (1.3). Our theorem gives estimates on Hausdorff dimension of ruptures of stable attractors.

A different kind of problem

$$\Delta u + k(x)\frac{1}{u^\alpha} = 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.6}$$

was studied in [CR, De, GHW, Go, GL] and the references therein, where $k(x) > 0$. The regularity of ∇u is obtained. Problem (1.6) is fundamentally different from (1.1): the sign of nonlinearity makes the Maximum Principle applicable to (1.6) which allow the use of e.g. a super-sub solutions scheme. In fact the following problem

$$\Delta u + \frac{1}{u^\alpha} - h(x) = 0 \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

possesses a (unique) positive solution in case that h is, for example, positive.

An interesting problem is to construct solutions with ruptures to (1.4). In this regard, we remark that when Ω is the unit ball B of $\mathbb{R}^n$, the problem

$$\Delta u - \frac{1}{u^\alpha} + h(x) = 0 \ \text{ in } B, \ \ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial B \tag{1.7}$$

has been studied for $h(x) = h(|x|)$ in [DH]. They showed that (1.7) has a nonnegative radial solution $u \in C^0(B)$ satisfying

$$c_1 r^{2/(\alpha+1)} \leq u(r) \leq c_2, \ \ c_1, c_2 > 0.$$

It is unknown if the solution constructed in [DH] has ruptures. It appears that it is a quite difficult problem in constructing rupture solutions in higher dimension. Partial progress has been done in [GW].

Our results here are in the same spirit of those in [Ev, Scn, Pa] where the Hausdorff dimensions of the **blow up** set of harmonic maps or some nonlinear elliptic problems are studied. The proof of Theorem 1.1 is divided into three steps:

Step 1. We show that if $u \in \mathcal{E}_\alpha$, then $u \in L^\infty_{loc}(\Omega)$.

Step 2. Fix $x_0 \in \Omega$ such that $B(x_0, 2r_0) \subset \Omega$. We show that there exists a constant $C = C(a, b, \|u\|_{L^\infty(B(x_0,2r_0))}, n)$ such that the following functional

$$E_u(x_0, r) \equiv -\frac{\alpha+1}{2(\alpha-1)} r^{-\mu} \int_{B(x_0,r)} u^{1-\alpha} dx + \frac{1}{4}\frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(x_0,r)} u^2 ds\Big]$$
$$-\frac{1}{4} r^{-\mu-1} \int_{\partial B(x_0,r)} u^2 ds + C\int_0^r \xi^{n-\mu-1} d\xi$$

is an increasing function of $r \in (0, r_0)$.

These are done in Section 2.

Step 3. Using the monotonicity formula, we show that there exists $\epsilon^* > 0$ such that for $x_0 \in \Sigma$,

$$\underline{\lim}_{r\to 0^+} r^{-\mu} \int_{B(x_0,r)} \Big[|\nabla u|^2 + u^{1-\alpha}\Big] dx \geq \epsilon^*,$$

which concludes the proof of Theorem 1.1.

This is done in Section 3.

Finally, we remark the negative power $u^{-\alpha}$ can be considered as **negative supercritical** in $R^n, n \geq 2$. In fact, it is known ([ACW]) that $u^{-\alpha}$ is subcritical if $\alpha < 3$ and supercritical if $\alpha > 3$. (A naive reason is as follows: the critical Sobolev exponent is $\frac{n+2}{n-2}$ which equals -3 if $n = 1$.) Thus formally $u^{-\alpha}$ is supercritical for $n \geq 2$. Our results give estimates on the singular set for negative supercritical problem, which is new.

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2. A Monotonicity Inequality

In this section we shall obtain a key monotonicity inequality for **finite energy stationary** solutions $u \in \mathcal{E}_\alpha$ of (1.1). To do this, we first recall the following result.

Lemma 2.1. *Let $f \geq 0$ in Ω and $g \in L^q(\Omega)$ for some $q > n/2$. Let u be a nonnegative solution of the equation*

$$\Delta u = f + g \quad in\ \Omega.$$

Then for any $B_{2R} \subset \Omega$, and $\|u\|_{H^1(B_{2R})} < \infty$, we have

$$\sup_{B_R} u \leq c(n,q)(R^{-\frac{n}{2}}\|u\|_{L^2(B_{2R})} + R^{2-\frac{n}{q}}\|g\|_{L^q(B_{2R})}).$$

Proof. Similar to the proof of Lemma 4.1 of [JL]. □

Lemma 2.1 (with $f = u^{-\alpha}$, $g = h$) implies that if $u \in \mathcal{E}_\alpha$ is a nonnegative solution of (1.1), then $u \in L^\infty_{loc}(\Omega)$.

Now we establish the key monotonicity inequality for **finite energy stationary** solutions $u \in \mathcal{E}_\alpha$ of (1.1). We follow the notation in [Ev, Pa].

Fix $x_0 \in \Omega$ such that $B(x_0, 2r_0) \subset \Omega$, where $0 < r_0 \leq R$ and R is given in Lemma 2.1. Let $r, m > 0$ be such that $r + m < r_0$. Set $\phi(x) = \xi(|x - x_0|)(x - x_0)$, where

$$\xi(|x-x_0|) \equiv \begin{cases} 1 & \text{for } |x-x_0| \leq r, \\ 1 + \frac{r - |x-x_0|}{m} & \text{for } r \leq |x-x_0| \leq r+m, \\ 0 & \text{for } |x-x_0| \geq r+m. \end{cases}$$

We derive from (1.5), letting $m \to 0^+$, that the following identity holds

$$
\begin{aligned}
&\frac{n}{\alpha-1}\int_{B(x_0,r)} u^{1-\alpha}dx - \frac{n-2}{2}\int_{B(x_0,r)} |\nabla u|^2 dx + \frac{r}{2}\int_{\partial B(x_0,r)} |\nabla u|^2 ds \\
&\quad +n\int_{B(x_0,r)} hudx - r\int_{\partial B(x_0,r)} huds - \frac{r}{\alpha-1}\int_{\partial B(x_0,r)} u^{1-\alpha}ds \\
&\quad + \int_{B(x_0,r)} u < x-x_0, \nabla h >= r\int_{\partial B(x_0,r)} (u_r)^2 ds, \qquad (2.1)
\end{aligned}
$$

where $u_r = \frac{\partial u}{\partial r}$. (Another equivalent derivation of (2.1) is by multiplying (1.1) with $(x-x_0)\cdot \nabla u$ and integrating over $B(x_0,r)$.)

On the other hand, multiplying (1.1) by u and integrating over $B(x_0,r)$ we find, for almost every $0 < r < r_0$

$$
\int_{B(x_0,r)} |\nabla u|^2 dx = \int_{\partial B(x_0,r)} uu_r ds - \int_{B(x_0,r)} u^{1-\alpha}dx + \int_{B(x_0,r)} h(x)udx. \qquad (2.2)
$$

Taking the derivative of (2.2) with respect to r, we get

$$
\int_{\partial B(x_0,r)} |\nabla u|^2 ds = \frac{d}{dr}\Big[\int_{\partial B(x_0,r)} uu_r ds\Big] - \int_{\partial B(x_0,r)} u^{1-\alpha}ds + \int_{\partial B(x_0,r)} huds. \qquad (2.3)
$$

Substituting $\int_{B(x_0,r)} |\nabla u|^2 dx$ of (2.2) and $\int_{\partial B(x_0,r)} |\nabla u|^2 ds$ of (2.3) into (2.1), we finally obtain

$$
\begin{aligned}
&\Big(\frac{n}{\alpha-1} + \frac{n-2}{2}\Big)\int_{B(x_0,r)} u^{1-\alpha}dx - \Big(\frac{1}{2} + \frac{1}{\alpha-1}\Big)r\int_{\partial B(x_0,r)} u^{1-\alpha}ds \\
&\quad +\Big(n - \frac{n-2}{2}\Big)\int_{B(x_0,r)} hudx - \frac{r}{2}\int_{\partial B(x_0,r)} huds + \frac{r}{2}\frac{d}{dr}\Big[\int_{\partial B(x_0,r)} uu_r ds\Big] \\
&\quad -\frac{(n-2)}{2}\int_{\partial B(x_0,r)} uu_r ds + \int_{B(x_0,r)} u < x-x_0, \nabla h > \\
&\quad = r\int_{\partial B(x_0,r)} (u_r)^2 ds. \qquad (2.4)
\end{aligned}
$$

Rewriting (2.4), we have

$$
\begin{aligned}
&-\frac{(\alpha+1)}{2(\alpha-1)}\frac{d}{dr}\Big[r^{-\mu}\int_{B(x_0,r)} u^{1-\alpha}dx\Big] + \frac{1}{2}r^{-\mu}\frac{d}{dr}\Big[\int_{\partial B(x_0,r)} uu_r ds\Big] \\
&\quad +\frac{(n+2)}{2}r^{-(\mu+1)}\int_{B(x_0,r)} hudx - \frac{1}{2}r^{-\mu}\int_{\partial B(x_0,r)} huds \\
&\quad +r^{-(\mu+1)}\int_{B(x_0,r)} u < x-x_0, \nabla h > dx \\
&\quad = r^{-\mu}\int_{\partial B(x_0,r)} \Big[(u_r)^2 + \frac{(n-2)}{2}r^{-1}uu_r\Big]ds, \qquad (2.5)
\end{aligned}
$$

where $\mu = \frac{(n-2)\alpha+(n+2)}{\alpha+1}$. In the following, we denote C for positive constants which may vary from line to line.

Since $u \in H^1(\Omega)$, it follows from Lemma 2.1 that there exists $C > 0$ such that $\|u\|_{L^\infty(B(x_0,r_0))} \le C$. This and the facts that $|\nabla h| \le b$ and $\Big| < x - x_0, \nabla h > \Big| \le r|\nabla h|$ imply that

$$r^{-(\mu+1)}\Big|\int_{B(x_0,r)} u < x - x_0, \nabla h > dx\Big| \le Cr^{n-\mu}, \tag{2.6}$$

where $C = C(b, \|u\|_{L^\infty(B(x_0,r_0))}, n)$. On the other hand, we also know that

$$\Big|\frac{(n+2)}{2}r^{-(\mu+1)}\int_{B(x_0,r)} hudx\Big| \le Cr^{n-\mu-1}, \tag{2.7}$$

and

$$\Big|\frac{1}{2}r^{-\mu}\int_{\partial B(x_0,r)} huds\Big| \le Cr^{n-\mu-1}, \tag{2.8}$$

where $C = C(a, \|u\|_{L^\infty(B(x_0,r_0))}, n)$.

Substituting (2.6), (2.7) and (2.8) into (2.5), we obtain

$$-\frac{\alpha+1}{2(\alpha-1)}\frac{d}{dr}\Big[r^{-\mu}\int_{B(x_0,r)} u^{1-\alpha}dx\Big] + \frac{1}{2}r^{-\mu}\frac{d}{dr}\Big[\int_{\partial B(x_0,r)} uu_r ds\Big] + Cr^{n-\mu-1}$$

$$\ge r^{-\mu}\int_{\partial B(x_0,r)}\Big[|u_r|^2 + \frac{(n-2)}{2}r^{-1}uu_r\Big]ds, \tag{2.9}$$

where $C = C(a, b, \|u\|_{L^\infty(B(x_0,r_0))}, n)$.

Using the identity

$$\frac{d}{dr}\Big[\int_{\partial B(x_0,r)} u^2 ds\Big] = 2\int_{\partial B(x_0,r)} uu_r ds + (n-1)\int_{\partial B(x_0,r)} u^2 r^{-1} ds \tag{2.10}$$

we have that

$$\frac{1}{2}\frac{d^2}{dr^2}\Big[r^{-\mu}\int_{\partial B(x_0,r)} u^2 ds\Big] - \frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(x_0,r)} uu_r ds\Big]$$

$$= (n-\mu-1)r^{-\mu}\int_{\partial B(x_0,r)}\Big[\frac{(n-2-\mu)}{2}r^{-2}u^2 + r^{-1}uu_r\Big]ds, \tag{2.11}$$

Note that

$$r^{-\mu}\frac{d}{dr}\Big[\int_{\partial B(x_0,r)} uu_r ds\Big] = \frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(x_0,r)} uu_r ds\Big] + \mu r^{-\mu-1}\int_{\partial B(x_0,r)} uu_r ds \tag{2.12}$$

Substituting (2.12) and (2.11) into (2.9), we obtain that

$$
\begin{aligned}
&-\frac{\alpha+1}{2(\alpha-1)}\frac{d}{dr}\Big[r^{-\mu}\int_{B(x_0,r)}u^{1-\alpha}dx\Big]\\
&\quad+\frac{1}{4}\frac{d^2}{dr^2}\Big[r^{-\mu}\int_{\partial B(x_0,r)}u^2ds\Big]+Cr^{n-\mu-1}\\
&\geq r^{-\mu}\int_{\partial B(x_0,r)}\Big[(u_r)^2+\frac{2n-2\mu-3}{2}r^{-1}uu_r\\
&\quad+\frac{1}{4}(n-\mu-1)(n-\mu-2)r^{-2}u^2\Big]ds
\end{aligned}
$$

which yields that

$$
\begin{aligned}
(2.13)\qquad &-\frac{\alpha+1}{2(\alpha-1)}\frac{d}{dr}\Big[r^{-\mu}\int_{B(x_0,r)}u^{1-\alpha}dx\Big]+\frac{1}{4}\frac{d^2}{dr^2}\Big[r^{-\mu}\int_{\partial B(x_0,r)}u^2ds\Big]\\
&-\frac{1}{4}\frac{d}{dr}\Big[r^{-\mu-1}\int_{\partial B(x_0,r)}u^2ds\Big]+Cr^{n-\mu-1}\\
&\geq r^{-\mu}\int_{\partial B(x_0,r)}\Big[(u_r)^2+(n-\mu-2)r^{-1}uu_r+\frac{1}{4}(n-\mu-2)^2r^{-2}u^2\Big]ds\\
&=r^{-\mu}\int_{\partial B(x_0,r)}\Big(u_r+\frac{(n-\mu-2)}{2}r^{-1}u\Big)^2ds\geq 0.
\end{aligned}
$$

Since $n-\mu-1=\frac{\alpha-3}{\alpha+1}>-1$ for $\alpha>1$, we conclude from (2.13) that

$$
\begin{aligned}
E_u(x_0,r)\equiv &-\frac{\alpha+1}{2(\alpha-1)}r^{-\mu}\int_{B(x_0,r)}u^{1-\alpha}dx+\frac{1}{4}\frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(x_0,r)}u^2ds\Big]\\
(2.14)\qquad &-\frac{1}{4}r^{-\mu-1}\int_{\partial B(x_0,r)}u^2ds+C\int_0^r\xi^{n-\mu-1}d\xi
\end{aligned}
$$

is an increasing function of r for $r\in(0,r_0)$. (Note that $C=C(a,b,\|u\|_{L^\infty(B(x_0,r_0))},n)$.)

Next we obtain another formulation of $E_u(x_0,r)$. First we have

$$
(2.15)\quad \frac{d}{dr}\Big[r^{-\mu}\int_{\partial B(x_0,r)}u^2ds\Big]=(n-\mu-1)r^{-\mu-1}\int_{\partial B(x_0,r)}u^2ds+2r^{-\mu}\int_{\partial B(x_0,r)}uu_rds.
$$

Then by (2.2)

$$
(2.16)\qquad \int_{\partial B(x_0,r)}uu_rds=\int_{B(x_0,r)}|\nabla u|^2dx+\int_{B(x_0,r)}u^{1-\alpha}dx-\int_{B(x_0,r)}h(x)udx.
$$

Substituting (2.15) and (2.16) into (2.14), we obtain an equivalent formulation of $E_u(x_0,r)$:

$$
E_u(x_0,r)=-\frac{1}{(\alpha-1)}r^{-\mu}\int_{B(x_0,r)}u^{1-\alpha}dx+\frac{1}{2}r^{-\mu}\int_{B(x_0,r)}|\nabla u|^2dx
$$

$$-\frac{1}{(\alpha+1)}r^{-\mu-1}\int_{\partial B(x_0,r)} u^2 ds - \frac{1}{2}r^{-\mu}\int_{B(x_0,r)} hudx + C\int_0^r \xi^{n-\mu-1}d\xi. \tag{2.17}$$

All the derivatives in the above expressions are to be understood in the sense of distributions. Now we obtain the following lemmas.

Lemma 2.2. *If $u \in \mathcal{E}_\alpha$ is a nonnegative* **finite energy stationary** *solution of (1.1), then $E_u(x_0, r)$, defined at (2.14), is an increasing function of r for $r \in (0, r_0)$, where $B(x_0, 2r_0) \subset \Omega$.*

Lemma 2.3. *$E_u(x_0, r)$ is a continuous function of $x_0 \in \Omega$ and $r > 0$.*

Proof. The proof is similar to that of Lemma 2 of [Pa]. □

3. Hausdorff Dimension Estimate

In this section we will prove Theorem 1.1. For any fixed $\epsilon > 0$ sufficiently small and $u \in \mathcal{E}_\alpha$ being a finite energy stationary solution of (1.1), by Lemma 2.2, one easily sees that if $x_0 \in \Sigma$, there are two cases for x_0:

(*i*) $\lim_{r\to 0^+} E_u(x_0, r) \geq -\epsilon$,

(*ii*) $\lim_{r\to 0^+} E_u(x_0, r) < -\epsilon$.

For the first case, we have the following lemma.

Lemma 3.1. *There exists $\epsilon^* > 0$ such that if $\lim_{r\to 0^+} E_u(x_0, r) \geq -\epsilon^*$, then*

$$\underline{\lim}_{r\to 0^+} r^{-\mu}\int_{B(x_0,r)} |\nabla u|^2 dx \geq \epsilon^*. \tag{3.1}$$

Proof. The monotonicity of $E_u(x_0, r)$ on r implies that, if $\lim_{r\to 0^+} E_u(x_0, r) \geq -\epsilon$, for some $\epsilon > 0$, there exists $0 < r_0 < R$ such that for $0 < r < r_0$,

$$E_u(x_0, r) \geq -\epsilon.$$

It follows from the second formulation (2.17) of $E_u(x_0, r)$ that

$$\begin{aligned}
&-\frac{1}{(\alpha-1)}r^{-\mu}\int_{B(x_0,r)} u^{1-\alpha}dx + \frac{1}{2}r^{-\mu}\int_{B(x_0,r)} |\nabla u|^2 dx \\
&-\frac{1}{(\alpha+1)}r^{-\mu-1}\int_{\partial B(x_0,r)} u^2 ds - \frac{1}{2}r^{-\mu}\int_{B(x_0,r)} hudx + C\int_0^r \xi^{n-\mu-1}d\xi \\
&\geq -\epsilon.
\end{aligned}$$

Suppose that $\underline{\lim}_{r\to 0^+} r^{-\mu}\int_{B(x_0,r)}|\nabla u|^2dx<\epsilon$. Then

$$\begin{aligned}
&-\frac{1}{(\alpha-1)}\overline{\lim}_{r\to 0^+}\left[r^{-\mu}\int_{B(x_0,r)}u^{1-\alpha}dx\right]\\
&+\frac{1}{2}\underline{\lim}_{r\to 0^+}\left[r^{-\mu}\int_{B(x_0,r)}|\nabla u|^2dx\right]\\
&-\frac{1}{(\alpha+1)}\overline{\lim}_{r\to 0}\left[r^{-\mu-1}\int_{\partial B(x_0,r)}u^2ds\right]\\
&\geq -\epsilon
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{\alpha-1}\overline{\lim}_{r\to 0^+}\left[r^{-\mu}\int_{B(x_0,r)}u^{1-\alpha}dx\right]\\
&+\frac{1}{\alpha+1}\overline{\lim}_{r\to 0}\left[r^{-\mu-1}\int_{\partial B(x_0,r)}u^2ds\right]\leq\frac{3\epsilon}{2}.
\end{aligned}$$

This shows that there exists $0<r_1<r_0$ such that for $0<r<r_1$,

$$r^{-\mu}\int_{B(x_0,r)}u^{1-\alpha}dx\leq 2(\alpha-1)\epsilon, \tag{3.2}$$

$$r^{-\mu-1}\int_{\partial B(x_0,r)}u^2ds\leq 2(\alpha+1)\epsilon. \tag{3.3}$$

It follows from (3.3) that for $0<r<r_1$,

$$\int_{B(x_0,r)}u^2dx\leq C\epsilon r^{\mu+2}, \tag{3.4}$$

where $C=C(\alpha,n)$. Thus, we derive from (3.2), (3.4) and the Hölder inequality that

$$\begin{aligned}
Cr^n &= \int_{B(x_0,r)}u^{-2(\alpha-1)/(\alpha+1)}u^{2(\alpha-1)/(\alpha+1)}dx\\
&\leq \left(\int_{B(x_0,r)}u^{1-\alpha}dx\right)^{2/(\alpha+1)}\left(\int_{B(x_0,r)}u^2dx\right)^{(\alpha-1)/(\alpha+1)}\\
&\leq C\epsilon r^{\frac{2\mu+(\mu+2)(\alpha-1)}{\alpha+1}}\\
&= C\epsilon r^n,
\end{aligned}$$

which is a contradiction if we choose $\epsilon>0$ sufficiently small. Thus, we conclude that the $\epsilon^*>0$ as mentioned in the lemma exists and

$$\underline{\lim}_{r\to 0^+}r^{-\mu}\int_{B(x_0,r)}|\nabla u|^2dx\geq\epsilon^*,$$

which finishes the proof of the lemma. □

To study the second case, we need the following Poincaré type inequality from [JL]. (See Theorem 2.1 of [JL]. Note that $n-2<\mu<n$.)

Lemma 3.2. *Let B_ℓ be any ball in $\mathbb{R}^n$ with radius ℓ, and $T\subset B_\ell$ be a $\mathcal{H}^\mu$-measurable set, such that*

$$\mathcal{H}^\mu(T)\geq \theta_1\ell^\mu, \tag{3.5}$$

and that for any $x\in\mathbb{R}^n$, and $r>0$,

$$\mathcal{H}^\mu(T\cap B_r(x))\leq \theta_2 r^\mu \tag{3.6}$$

holds. Then for any $u\in H^1(B_\ell)$ such that $T\subset\Sigma$, where Σ is defined in (1.2), we have

$$\int_{B_\ell}u^2\leq c(n,\mu)\frac{\theta_2^2}{\theta_1^2}\ell^2\int_{B_\ell}|\nabla u|^2. \tag{3.7}$$

The following lemma plays an important role.

Lemma 3.3. *Let $T\subset\Sigma$ be as in Lemma 3.2 and (3.5), (3.6) hold. Let $u\in\mathcal{E}_\alpha$ be a* **finite energy stationary** *solution of (1.1) and $x_0\in T$. Then, for $0<2r<d(x_0,\partial\Omega)$ sufficiently small,*

$$r^{-\mu-1}\int_{\partial B(x_0,r)}u^2ds\leq Cr^{-\mu}\int_{B(x_0,r)}\Big[|\nabla u|^2+u^{1-\alpha}-hu\Big]dx \tag{3.8}$$

where $C=C(n,\alpha)>0$.

Proof. Without loss of generality, we assume that $x_0=0$. Define $F(r)=\int_{B(0,r)}u^2dx$. Then $F(r)\to 0$ as $r\to 0$. We use polar coordinates (t,θ) on $B(0,r)$. Let $G(t)=\int_{\partial B(0,1)}u^2(t,\theta)d\theta$ for $0<t<r$. Then

$$\begin{aligned}\frac{dG}{dt}(t) &= 2\int_{\partial B(0,1)}u(t,\theta)u_t(t,\theta)d\theta\\ &= 2t^{1-n}\int_{\partial B(0,t)}uu_tds\\ &= 2t^{1-n}\int_{B(0,t)}\Big[|\nabla u|^2+u(u^{-\alpha}-h)\Big]dx\\ &\geq 0,\end{aligned}$$

where we are using (2.2) and the fact that

$$\int_{B(0,t)}[u^{1-\alpha}-hu]dx>0, \tag{3.9}$$

for $t > 0$ sufficiently small. To obtain (3.9), we use the Young's inequality to see that

$$|B(0,t)| \leq d_1 \int_{B(0,t)} u dx + d_2 \int_{B(0,t)} u^{1-\alpha} dx, \tag{3.10}$$

where $d_1 = d_1(\alpha) > 0$, $d_2 = d_2(\alpha) > 0$. Since $\lim_{t\to 0^+} \frac{\int_{B(0,t)} u dx}{|B(0,t)|} = 0$, for any $0 < \epsilon < \frac{1}{10 d_1}$, there is $t_0 = t_0(\epsilon) > 0$ such that for $0 < t < t_0$,

$$\int_{B(0,t)} u dx \leq \epsilon |B(0,t)|. \tag{3.11}$$

This and (3.10) imply that

$$\int_{B(0,t)} u^{1-\alpha} dx \geq \frac{1}{2d_2} |B(0,t)|.$$

This and (3.11) imply (3.9) (noting that $\|h\|_{L^\infty(\Omega)} \leq a$). Note that the derivative u_t in the computations is to be understood in the sense of distribution. Since $\int_{B(0,t)} |\nabla u|^2 dx$, $\int_{B(0,t)} u^{1-\alpha} dx$ and $\int_{B(0,t)} h(x)u(x)dx$ are continuous functions of $0 < t < r$ (see Lemma 2.3), we have that $\frac{dG}{dt}(t)$ is a continuous function of $0 < t < r$. Thus, $G \in C^1(0,r)$ is an increasing function. This also implies that $\frac{d^2}{dt^2} F(t)$ is a continuous function for $0 < t < r$.

Now we consider the function $F(r)$. By making a Taylor expansion of $F(r)$ at r, we obtain that

$$\begin{aligned}
0 &= \int_{B(0,r)} u^2 dx - \Big(\int_{\partial B(0,r)} u^2 ds \Big) r + \Big[\int_0^1 \eta \Big((n-1) \xi_\eta^{n-2} \int_{\partial B(0,1)} u^2(\xi_\eta, \theta) d\theta \\
&\quad + 2 \int_{\partial B(0,\xi_\eta)} u u_{\xi_\eta} ds \Big) d\eta \Big] r^2 \\
&= \int_{B(0,r)} u^2 dx - \Big(\int_{\partial B(0,r)} u^2 ds \Big) r + \Big[\int_0^1 \eta \Big((n-1) \xi_\eta^{n-2} G(\xi_\eta) \\
&\quad + 2 \int_{B(0,\xi_\eta)} (|\nabla u|^2 + u(u^{-\alpha} - h)) dx \Big) d\eta \Big] r^2,
\end{aligned}$$

where $\xi_\eta = \eta r$ and $0 < \eta < 1$. Since $G(t)$ is an increasing function, we obtain that

$$\begin{aligned} r\Big[\int_{\partial B(0,r)} u^2 ds\Big] &\le \int_{B(0,r)} u^2dx + r^2\Big[\int_0^1 \eta\Big((n-1)\eta^{n-2}r^{n-2}G(r) \\ &\qquad +2\int_{B(0,r)}(|\nabla u|^2 + u(u^{-\alpha}-h))dx\Big)d\eta\Big] \\ &\le \int_{B(0,r)} u^2dx + \frac{(n-1)}{n}r^n\int_{\partial B(0,1)} u^2(r,\theta)d\theta \\ &\qquad +r^2\int_{B(0,r)}(|\nabla u|^2 + u(u^{-\alpha}-h))dx \\ &= \int_{B(0,r)} u^2dx + \frac{(n-1)}{n}r\int_{\partial B(0,r)} u^2ds \\ &\qquad +r^2\int_{B(0,r)}(|\nabla u|^2 + u(u^{-\alpha}-h))dx, \end{aligned}$$

where we use the fact that for r sufficiently small,

$$\int_{B(0,r)}[u^{1-\alpha}-hu]dx \ge \int_{B(0,\xi_\eta)}[u^{1-\alpha}-hu]dx. \tag{3.12}$$

We explain a little on the proof of (3.12). It follows from (3.9) that for r sufficiently small, $J(r) := \int_{B(0,r)}[u^{1-\alpha}-hu]dx > 0$. This, the continuity of $\int_{B(0,r)} u^{1-\alpha}dx$; $\int_{B(0,r)} hudx$ and $\lim_{r\to 0^+} J(r) = 0$ imply that $J(r)$ is increasing. Therefore, (3.12) holds. Thus, we see

$$\frac{1}{n}r\Big[\int_{\partial B(0,r)} u^2ds\Big] \le \int_{B(0,r)} u^2dx + r^2\int_{B(0,r)}\Big(|\nabla u|^2 + u^{1-\alpha} - hu\Big)dx.$$

This and Lemma 3.2 imply that

$$r^{-\mu-1}\int_{\partial B(0,r)} u^2ds \le Cr^{-\mu}\int_{B(0,r)}\Big[|\nabla u|^2 + u^{1-\alpha} - hu\Big]dx, \tag{3.13}$$

where $C = C(n)$. This completes the proof. □

Now we consider the second case:

$$\lim_{r\to 0^+} E_u(x_0, r) < -\epsilon^*$$

for $\epsilon^* > 0$ being given in Lemma 3.1.

Lemma 3.4. *If $x_0 \in \Sigma$ and $\lim_{r\to 0^+} E_u(x_0,r) < -\epsilon^*$, then*

$$\underline{\lim}_{r\to 0^+}\Big[\frac{1}{(\alpha-1)}r^{-\mu}\int_{B(x_0,r)} u^{1-\alpha}dx + \frac{1}{(\alpha+1)}r^{-\mu-1}\int_{\partial B(x_0,r)} u^2ds\Big] \ge \epsilon^*/2. \tag{3.14}$$

Proof. This follows from the second formulation (2.17) of $E_u(x_0,r)$. □

Finally we are in the position to prove Theorem 1.1.

We shall show $\mathcal{H}^{\mu}(\Sigma)=0$. We prove it by contradiction. Suppose $\mathcal{H}^{\mu}(\Sigma)>0$ (possibly with infinite measure). Then since Σ is a Souslin set, Theorem 5.6 and its proof in [Fa] say that, there is a closed subset $T\subset\Sigma$, with $0<\mathcal{H}^{\mu}(T)<\infty$, and for some constant $\theta>0$,

$$\mathcal{H}^{\mu}(T\cap B_r(x))\leq\theta r^{\mu}$$

holds for any $x\in\mathbb{R}^n$, $r>0$.

Let $\epsilon^*>0$ be given in Lemma 3.1 and $x_0\in T$. Then either

$$\lim_{r\to 0+}E_u(x_0,r)\geq-\epsilon^*$$

or

$$\lim_{r\to 0+}E_u(x_0,r)<-\epsilon^*.$$

In the first case, we have by Lemma 3.1

$$\underline{\lim}_{r\to 0+}r^{-\mu}\int_{B(x_0,r)}|\nabla u|^2dx\geq\epsilon^*.$$

In the second case, we have (3.14). By Lemma 3.3 we have

$$\underline{\lim}_{r\to 0+}r^{-\mu}\int_{B(x_0,r)}\left[|\nabla u|^2+u^{1-\alpha}\right]dx\geq C\epsilon^*$$

for some constant $C=C(n,\alpha)>0$, since

$$|r^{-\mu}\int_{B(x_0,r)}hudx|\leq Cr^{n-\mu}$$

and $n-\mu>0$.

In conclusion, we have proved that there exists $\epsilon^*>0$ such that if $x_0\in T$, then

$$\underline{\lim}_{r\to 0+}r^{-\mu}\int_{B(x_0,r)}\left[|\nabla u|^2+u^{1-\alpha}\right]dx\geq C\epsilon^*, \tag{3.15}$$

for some $C=C(n,\alpha)>0$. This implies that there exists $\delta_0>0$ sufficiently small such that for $0<r<\delta_0$,

$$r^{-\mu}\int_{B(x_0,r)}\left[|\nabla u|^2+u^{1-\alpha}\right]dx\geq\frac{C}{2}\epsilon^*. \tag{3.16}$$

Then for any $0<\delta<\frac{\delta_0}{10}$ and for any U open, such that $T\subset U$,

$$\left\{B_r(x):x\in T,0<r<\frac{1}{2}\delta,B_r(x)\subset U\ \text{ and }\ r^{-\mu}\int_{B(x,r)}\left[|\nabla u|^2+u^{1-\alpha}\right]dx\geq\frac{C}{2}\epsilon^*\right\}$$

is a finite covering of T. Hence, by Vitali covering lemma, there is a pairwise disjoint subcollection $\{B_{r_k}(x_k)\}_{k=1}^{\infty}$, such that $T\subset\cup_{k=1}^{\infty}B_{5r_k}(x_k)$. Hence, it follows from (3.16) that

$$\begin{aligned}\mathcal{H}^{\mu}_{5\delta}(T) &\leq C(\mu)\Sigma_{k=1}^{\infty}(5r_k)^{\mu}\\ &\leq C(n,\mu,\theta)\Sigma_{k=1}^{\infty}\int_{B_{r_k}(x_k)}\left[|\nabla u|^2+u^{1-\alpha}\right]dx\\ &\leq C(n,\mu,\theta)\int_U\left[|\nabla u|^2+u^{1-\alpha}\right]dx.\end{aligned}$$

Since $\mathcal{H}^{\mu}(T)<\infty$, we can choose U with arbitrary small $\mathcal{H}^n$-measure so that the right hand side of the inequality can be arbitrarily small. Thus we have $\mathcal{H}^{\mu}_{5\delta}(T)=0$. Letting $\delta\to 0$, we conclude $\mathcal{H}^{\mu}(T)=0$, which gives the contradiction and completes the proof of Theorem 1.1. □

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