REGULARIZATION OF POINT VORTICES PAIRS FOR THE EULER EQUATION IN DIMENSION TWO

DAOMIN CAO, ZHONGYUAN LIU, AND JUNCHENG WEI

Abstract. In this paper, we construct stationary classical solutions of the incompressible Euler equation approximating singular stationary solutions of this equation. This procedure is carried out by constructing solutions to the following elliptic problem

$$\begin{cases} -\varepsilon^2\Delta u = \sum_{i=1}^m \chi_{\Omega_i^+}\big(u - q - \frac{\kappa_i^+}{2\pi}\ln\frac{1}{\varepsilon}\big)_+^p - \sum_{j=1}^n \chi_{\Omega_j^-}\big(q - \frac{\kappa_j^-}{2\pi}\ln\frac{1}{\varepsilon} - u\big)_+^p, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

where $p>1$, $\Omega\subset\mathbb{R}^2$ is a bounded domain, Ω_i^+ and Ω_j^- are mutually disjoint subdomains of Ω and $\chi_{\Omega_i^+}$(resp. $\chi_{\Omega_j^-}$) are characteristic functions of Ω_i^+(resp. Ω_j^-), q is a harmonic function.

We show that if Ω is a simply-connected smooth domain, then for any given C^1-stable critical point of Kirchhoff-Routh function $\mathcal{W}(x_1^+,\cdots,x_m^+,x_1^-,\cdots,x_n^-)$ with $\kappa_i^+>0\,(i=1,\cdots,m)$ and $\kappa_j^->0\,(j=1,\cdots,n)$, there is a stationary classical solution approximating stationary $m+n$ points vortex solution of incompressible Euler equations with total vorticity $\sum_{i=1}^m\kappa_i^+ - \sum_{j=1}^n\kappa_j^-$. The case that $n=0$ can be dealt with in the same way as well by taking each Ω_j^- as empty set and set $\chi_{\Omega_j^-}\equiv 0$, $\kappa_j^-=0$.

1. Introduction and main results

The incompressible Euler equations

$$\begin{cases}\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P,\\ \nabla\cdot\mathbf{v}=0,\end{cases}\tag{1.1}$$

describe the evolution of the velocity $\mathbf{v}$ and the pressure P in an incompressible flow. In $\mathbb{R}^2$, the vorticity of the flow is defined by $\omega := \nabla\times\mathbf{v} = \partial_1 v_2 - \partial_2 v_1$, which satisfies the equation

$$\omega_t + \mathbf{v}\cdot\nabla\omega = 0.$$

The velocity $\mathbf{v}$ of an incompressible fluid in two dimensions admits a stream function ψ such that $\mathbf{v} = \mathbf{J}\nabla\psi = (\frac{\partial}{\partial x_2}\psi, -\frac{\partial}{\partial x_1}\psi)$, where $\mathbf{J}$ denotes the symplectic matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

By the definitions, ψ is a solution of the Poisson equation $-\Delta\psi=\omega$.

Suppose that ω is known, then the velocity $\mathbf{v}$ can be recovered by the following Biot-Savart law

$$\mathbf{v}=\omega*\frac{1}{2\pi}\frac{-\mathbf{J}x}{|x|^2}.$$

One special singular solution of Euler equations is given by $\omega=\sum_{i=1}^m\kappa_i\delta_{x_i(t)}$ ($\kappa_i\neq 0$ is called the strength of the i^{th} vortex x_i), which is related to

$$\mathbf{v}=-\sum_{i=1}^m\frac{\kappa_i}{2\pi}\frac{\mathbf{J}(x-x_i(t))}{|x-x_i(t)|^2}.$$

The positions of the vortices $x_i:\mathbb{R}\to\mathbb{R}^2$ satisfy the following Kirchhoff law

$$\kappa_i\frac{dx_i}{dt}=\mathbf{J}\nabla_{x_i}\mathcal{W},$$

where $\mathcal{W}$ is the so called Kirchhoff-Routh function defined by

$$\mathcal{W}(x_1,\cdots,x_m)=\frac{1}{2}\sum_{i\neq j}^m\frac{\kappa_i\kappa_j}{2\pi}\log\frac{1}{|x_i-x_j|}.$$

For a simply connected bounded domain $\Omega\subset\mathbb{R}^2$, let v_n be the normal component of the velocity $\mathbf{v}$ on $\partial\Omega$, that is $v_n(x)=\mathbf{v}(x)\cdot\nu(x)$, where $\nu(x)$ is the unit outward normal on $\partial\Omega$ at $x\in\partial\Omega$. Then by $\nabla\cdot\mathbf{v}=0$, $\int_{\partial\Omega}v_n=0$. It turns out that the Kirchhoff-Routh function for bounded domain Ω is associated with Green function and v_n. Suppose that $\mathbf{v}_0$ is the unique harmonic field whose normal component on the boundary $\partial\Omega$ is v_n. If Ω is simply-connected, then $\mathbf{v}_0$ can be represented by $\mathbf{v}_0=\mathbf{J}\nabla\psi_0$, and ψ_0 is determined up to a constant by

$$\begin{cases}-\Delta\psi_0=0, \ \text{in}\,\Omega,\\ -\dfrac{\partial\psi_0}{\partial\tau}=v_n, \ \text{on}\,\partial\Omega,\end{cases}\tag{1.2}$$

where $\frac{\partial\psi_0}{\partial\tau}$ denotes the tangential derivative on $\partial\Omega$. The Kirchhoff-Routh function associated to the vortex dynamics then is given by (see Lin [23])

$$\mathcal{W}(x_1,\cdots,x_m)=\frac{1}{2}\sum_{i\neq j}^m\kappa_i\kappa_jG(x_i,x_j)+\frac{1}{2}\sum_{i=1}^m\kappa_i^2H(x_i,x_i)+\sum_{i=1}^m\kappa_i\psi_0(x_i),\tag{1.3}$$

where G is the Green function of $-\Delta$ on Ω with 0 Dirichlet boundary condition and H is its regular part (the Robin function).

For m clockwise vortices motion (corresponding to $\kappa_i^+>0$) and n anti-clockwise vortices motion (corresponding to $-\kappa_j^-<0$), the Kirchhoff-Routh function associated to the vortex dynamics becomes

$$
\begin{aligned}
&\mathcal{W}(x_1^+,\cdots,x_m^+,x_1^-,\cdots,x_n^-)\\
&\quad=\frac{1}{2}\sum_{i,k=1,i\neq k}^{m}\kappa_i^+\kappa_k^+G(x_i^+,x_k^+)+\frac{1}{2}\sum_{j,l=1,j\neq l}^{n}\kappa_j^-\kappa_l^-G(x_j^-,x_l^-)\\
&\quad+\frac{1}{2}\sum_{i=1}^{m}(\kappa_i^+)^2H(x_i^+,x_i^+)+\frac{1}{2}\sum_{j=1}^{n}(\kappa_j^-)^2H(x_j^-,x_j^-)\\
&\quad-\sum_{i=1}^{m}\sum_{j=1}^{n}\kappa_i^+\kappa_j^-G(x_i^+,x_j^-)+\sum_{i=1}^{m}\kappa_i^+\psi_0(x_i^+)-\sum_{j=1}^{n}\kappa_j^-\psi_0(x_j^-).
\end{aligned}
\tag{1.4}
$$

It is known that critical points of the Kirchhoff-Routh function $\mathcal{W}$ give rise to stationary vortex points solutions of the Euler equations(see the Kirchhoff law). As for the existence and multiplicity of critical points of $\mathcal{W}$ given by (1.3), we refer to [5] and the references therein, where the case $\psi_0\equiv 0$ was studied. We expect that it is possible to obtain, at least by add assumptions on ψ_0, the multiplicity and even non-degeneracy of critical points for $\mathcal{W}$.

There exist huge literatures dealing with the stationary incompressible Euler equations, see [1, 2, 4], [6]-[11], [16]-[18], [19, 21, 22], [28]-[31] and references therein. Roughly speaking, there are two methods to construct stationary solutions of the Euler equation, which are the vorticity method and the stream-function method. The vorticity method was first established by Arnold (see [3]). Benjamin [6] developed a new approach based on a variational principle for the vorticity to study the existence of vortex rings in three dimensions which was adapted successfully by Burton [8] and Turkington [31].

The stream-function method consists in observing that if $\omega=\lambda f(\psi)$, that is, if ψ satisfies

$$
\begin{cases}
-\Delta\psi=\lambda f(\psi), & x\in\Omega,\\
u=\psi_0, & x\in\partial\Omega,
\end{cases}
$$

for some arbitrary function $f\in C^1(\mathbb{R})$, then $\mathbf{v}=\mathbf{J}\nabla\psi$ and $P=\lambda F(\psi)-\frac{1}{2}|\nabla\psi|^2$ is a stationary solution to the Euler equations, where $F(t)=\int_0^t f(s)ds$. Moreover, the velocity $\mathbf{v}$ is irrotational on the set where $f(\psi)=0$. f is called vorticity function and λ the vortex strength parameter.

Set $q=-\psi_0$ and $u=\psi-\psi_0$, then u satisfies the following Dirichlet boundary value problem

$$
\begin{cases}
-\Delta u=\lambda f(u-q), & x\in\Omega,\\
u=0, & x\in\partial\Omega.
\end{cases}
\tag{1.5}
$$

One of our motivation to study (1.5) is to justify the weak formulation for point vortex solutions of the incompressible Euler equations by approximating these solutions with classical solutions. A lot of work has been done in this respect, see [1, 2, 4, 7, 19, 25, 27, 28, 30, 31, 33, 34] and the references therein. Our work is also motivated by a recent paper of Smets and Van Schaftingen [30], which will be described in more detail later.

In [18] Elcrat and Miller, by a rearrangements of functions, have studied steady, inviscid flows in two dimensions which have concentrated regions of vorticity. In particular, they studied such flows which "desingularize" a configuration of point vortices in stable equilibrium with an irrotational flow, which generalized their earlier work for one vortex [16][17]. Saffman and Sheffield [29] have found an example of a steady flow in aerodynamics with a single point vortex which is stable for a certain range of the parameters. This has been generalized in [16], where some examples computationally of stable configurations of two point vortices were briefly discussed. Further examples of multiple point vortex configurations are given in [26], where a theorem on the existence of such configurations is also given.

It is worthwhile to point out that except [18] the above approximations can just give explanation for the formulation to single point vortex solutions. Smets and Van Schaftingen [30] investigated the following problem

$$\begin{cases} -\varepsilon^2\Delta u = \left(u - q - \frac{\kappa}{2\pi}\ln\frac{1}{\varepsilon}\right)_+^p, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $p > 1$. They gave the exact asymptotic behavior and expansion of the least energy solution by estimating the upper bounds on the energy. The solutions for (1.6) in [30] were obtained by finding a minimizer of the corresponding functional in a suitable function space, which gives approximation to a single point non-vanishing vortex.

Concerning regularization of pairs of vortices, Smets and Van Schaftingen [30] also studied the following problem

$$\begin{cases} -\varepsilon^2\Delta u = \left(u - q - \frac{\kappa^+}{2\pi}\ln\frac{1}{\varepsilon}\right)_+^p - \left(q - \frac{\kappa^-}{2\pi}\ln\frac{1}{\varepsilon} - u\right)_+^p, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

and obtained solution with least energy among sign-change solutions of (1.7). They also obtained the exact asymptotic behavior and expansion of such solutions by similar methods for (1.6).

In this paper, we try to establish, by a different method, existence of stationary solutions concentrating near C^1-stable critical points of $\mathcal{W}(x_1^+,\cdots,x_m^+,x_1^-,\cdots,x_n^-)$ with both clockwise and anti-clockwise point vortex. To achieve our goal, we consider the following semilinear elliptic problem

$$\begin{cases} -\varepsilon^2\Delta u = \displaystyle\sum_{i=1}^m \chi_{\Omega_i^+}\big(u - q - \frac{\kappa_i^+}{2\pi}\ln\frac{1}{\varepsilon}\big)_+^p - \sum_{j=1}^n \chi_{\Omega_j^-}\big(q - \frac{\kappa_j^-}{2\pi}\ln\frac{1}{\varepsilon} - u\big)_+^p, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $p > 1$, $q \in C^2(\Omega)$, $\Omega_i^+(i = 1,\cdots,m)$ and $\Omega_j^-(j = 1,\cdots,n)$ are mutually disjoint subdomains of Ω such that $x_{i,*}^+ \in \Omega_i^+$, and $x_{j,*}^- \in \Omega_j^-$.

To be more precise, we will consider an equivalent problem of (1.8) instead. Let $w = \frac{2\pi}{|\ln\varepsilon|}u$ and $\delta = \varepsilon(\frac{2\pi}{|\ln\varepsilon|})^{\frac{p-1}{2}}$, then (1.8) becomes

$$\begin{cases} -\delta^2\Delta w = \sum_{i=1}^{m}\chi_{\Omega_i^+}(w-\kappa_i^+ - \frac{2\pi\, q}{|\ln\varepsilon|})_+^p - \sum_{j=1}^{n}\chi_{\Omega_j^-}(\frac{2\pi\, q}{|\ln\varepsilon|} - \kappa_j^- - w)_+^p, & \text{in } \Omega,\\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

We will use a reduction argument to prove our main results. To this end, we need to construct an approximate solution for (1.9). For the problem studied in this paper, the corresponding "limit" problem in $\mathbb{R}^2$ has no bounded nontrivial solution. So, we will follow the method in [14, 15] to construct an approximate solution. Since there are two parameters δ, ε in (1.9) and two terms in nonlinearity, which causes some difficulty, we must take this influence into careful consideration in order to perform the reduction argument. Let us point out that, when studying pair of vortex solutions with both anti-clockwise and clockwise point vortex, except the difficulties mentioned above we have additional difficulties due to the interactions between the positive part and the negative part of those solutions. For example, one can easily see that in the expressions of $s_{1,\delta}^+,\cdots,s_{m,\delta}^+,s_{1,\delta}^-,\cdots,s_{n,\delta}^-$ and $a_{1,\delta}^+,\cdots,a_{m,\delta}^+,a_{1,\delta}^-,\cdots,a_{n,\delta}^-$ in Lemma 2.1 the effect of interactions has to be taken into careful consideration and the estimates are much more involved. The method used in the present paper could be applied to study related problems, for example, shallow water vortices, which was studied recently by De Valeriola and Van Schaftingen in [32].

Our first main results concerning (1.1) is the following:

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded simply-connected smooth domain. Let $v_n : \partial\Omega \to \mathbb{R}$ be such that $v_n \in L^s(\partial\Omega)$ for some $s > 1$ satisfying $\int_{\partial\Omega} v_n = 0$. Let $\kappa_i^+ > 0, \kappa_j^- > 0,\ i = 1,\cdots,m,\ j = 1,\cdots,n$. Then, for any given C^1-stable critical point $(x_{1,*}^+,\cdots,x_{m,*}^+,x_{1,*}^-,\cdots,x_{n,*}^-)$ of Kirchhoff-Routh function $\mathcal{W}$ defined by (1.4), there exists $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0,\varepsilon_0)$, (1.1) has a stationary solution $\mathbf{v}_\varepsilon$ with outward boundary flux given by v_n, such that its vorticities $\omega_\varepsilon = \sum_{i=1}^{m}\omega_{i,\varepsilon}^+ + \sum_{j=1}^{n}\omega_{j,\varepsilon}^-$ satisfies for small ε,*

$$supp(\omega_{i,\varepsilon}^+) \subset B(x_{i,\varepsilon}^+, C\varepsilon), \quad \text{for } i = 1,\cdots,m,$$

$$supp(\omega_{j,\varepsilon}^-) \subset B(x_{j,\varepsilon}^-, C\varepsilon), \quad \text{for } j = 1,\cdots,n,$$

where $x_{i,\varepsilon}^+ \in \Omega_i^+$ $(i = 1,\cdots,m)$, $x_{j,\varepsilon}^- \in \Omega_j^-(j = 1,\cdots,n)$ and $C > 0$ is a constant independent of ε.

Furthermore as $\varepsilon \to 0$,

$$(x_{1,\varepsilon}^+,\cdots,x_{m,\varepsilon}^+,x_{1,\varepsilon}^-,\cdots,x_{n,\varepsilon}^-) \to (x_{1,*}^+,\cdots,x_{m,*}^+,x_{1,*}^-,\cdots,x_{n,*}^-),$$

$$\int_{B(x_{i,\varepsilon}^+,C\varepsilon)}\omega_{i,\varepsilon}^+ \to \kappa_i^+,\ i = 1,\cdots,m,$$

$$\int_{B(x^-_{j,\varepsilon},C\varepsilon)}\omega^-_{i,\varepsilon}\to-\kappa^-_j,\ j=1,\cdots,n,$$

$$\int_\Omega\omega_\varepsilon\to\sum_{i=1}^m\kappa^+_i-\sum_{j=1}^n\kappa^-_j.$$

Remark 1.2. The case $m=n=1$, corresponding to pairs of vortices, was studied by Smets and Van Schaftingen [30] by minimizing the corresponding energy functional in the Nehari manifold. In their paper $\mathcal{W}(x^+_{1,\varepsilon},x^-_{1,\varepsilon})\to\sup_{x^+_1,x^-_1\in\Omega,x^+_1\neq x^-_1}\mathcal{W}(x^+_1,x^-_1)$ as $\varepsilon\to0$. Our result extends theirs to more general critical points (with additional assumption that the critical point is non-degenerate or stable in the sense of C^1). The method used here is constructive and different from theirs.

Remark 1.3. In the case that $m=n=1$, suppose that $(x^+_{1,*},\,x^-_{1,*})$ is a strict local maximum(or minimum) point of Kirchhoff-Routh function $\mathcal{W}(x^+_1,\,x^-_1)$ defined by (1.4), then the C^1 stability is not needed and statement of Theorem 1.1 still holds which can be proved similarly (see Remark 1.5). Thus we can re-obtain corresponding existence result in [30].

Theorem 1.1 is proved via the following result concerning problem (1.8).

Theorem 1.4. *Suppose $q\in C^2(\Omega)$. Then for any given $\kappa^+_i>0,\kappa^-_j>0,\ i=1,\cdots,m,\ j=1,\dots,n$ and for any given C^1-stable critical point $(x^+_{1,*},\cdots,x^+_{m,*},x^-_{1,*},\cdots,x^-_{n,*})$ of Kirchhoff-Routh function $\mathcal{W}$ defined by (1.4), there exists $\varepsilon_0>0$, such that for each $\varepsilon\in(0,\varepsilon_0)$, (1.8) has a solution u_ε, such that the set $\Omega^+_{\varepsilon,i}=\{x:u_\varepsilon(x)-\frac{\kappa^+_i}{2\pi}\ln\frac{1}{\varepsilon}-q(x)>0\}\subset\subset\Omega^+_i$, $i=1,\cdots,m$, $\Omega^-_{\varepsilon,j}=\{x:u_\varepsilon(x)-\frac{\kappa^-_j}{2\pi}\ln\frac{1}{\varepsilon}-q(x)>0\}\subset\subset\Omega^-_j$, $j=1,\cdots,n$ and as $\varepsilon\to0$, each $\Omega^+_{\varepsilon,i}(resp.\,\Omega^-_{\varepsilon,j})$ shrinks to $x^+_{i,*}\in\Omega(resp.\,x^+_{j,*}\in\Omega)$.*

Remark 1.5. For the case $m=n=1$, suppose that $(x^+_{1,*},\,x^-_{1,*})$ is a strict local maximum(or minimum) point of Kirchhoff-Routh function $\mathcal{W}(x^+_1,\,x^-_1)$ defined by (1.4), statement of Theorem 1.4 still holds, which can be proved by making corresponding modification of the proof of Theorem 1.4 in obtaining critical point of $K(Z)$ defined by (4.1)(see Propositions 2.3, 2.5 and 2.6 in [13] for detailed arguments).

Remark 1.6. (1.9) can be considered as a free boundary problem. Similar problems have been studied extensively. The reader can refer to [12, 14, 15, 22] for more results on this kind of problems.

By the same way for the proof of Theorem 1.1 but simpler, we can extend the result of least energy solution obtained via constrained minimization problem in [30] to non-least energy solutions and show that multi-point vortex solutions can be approximated by stationary classical solutions. Indeed we have the following result concerning problem (1.1):

Theorem 1.7. *Suppose that $\Omega\subset\mathbb{R}^2$ is a bounded simply-connected smooth domain. Let $v_n:\partial\Omega\to\mathbb{R}$ be such that $v_n\in L^s(\partial\Omega)$ for some $s>1$ satisfying $\int_{\partial\Omega}v_n=0$. Let $\kappa_i>$*

$0,\ i=1,\cdots,m$. *Then, for any given* C^1*- stable critical point* $(x_1^*,\cdots,x_m^*)$ *of Kirchhoff-Routh function* $\mathcal{W}(x_1,\cdots,x_m)$ *defined by* (1.3)*, there exists* $\varepsilon_0>0$*, such that for each* $\varepsilon\in(0,\varepsilon_0)$*,* (1.1) *has a stationary solution* $\mathbf{v}_\varepsilon$ *with outward boundary flux given by* v_n*, such that its vorticities* $\omega_\varepsilon=\sum_{i=1}^m \omega_{i,\varepsilon}$ *satisfies for small* ε*,*

$$supp(\omega_{i,\varepsilon})\subset B(x_{i,\varepsilon},C\varepsilon),\qquad for\ i=1,\cdots,m,$$

where $x_{i,\varepsilon}\in\Omega_i\,(i=1,\cdots,m)$*,* $C>0$ *is a constant independent of* ε*.*

Furthermore as $\varepsilon\to 0$*,*

$$\int_{B(x_{i,\varepsilon},C\varepsilon)}\omega_{i,\varepsilon}\to\kappa_i,\ \ i=1,\cdots,m,$$

$$\int_\Omega \omega_\varepsilon\to\sum_{i=1}^m\kappa_i,$$

$$(x_{1,\varepsilon},\cdots,x_{m,\varepsilon})\to(x_1^*,\cdots,x_m^*).$$

Remark 1.8. The case $m=1$, corresponding to a point vortex, was studied by Smets and Van Schaftingen [30] by minimizing the corresponding energy functional. In their paper $\mathcal{W}(x_{1,\varepsilon})\to\sup_{x\in\Omega}\mathcal{W}(x)$ as $\varepsilon\to 0$.

We end this section by outlining the organization of our paper. In section 2, we construct the approximate solution for (1.9). We will carry out a reduction argument in section 3 and the main results will be proved in section 4. We put some basic estimates used in sections 3 and 4 in the appendix.

2. Approximate solutions

In this section, as preliminary, we will construct approximate solutions for (1.9).

Let $R>0$ be a large constant, such that for any $x\in\Omega$, $\Omega\subset\subset B_R(x)$. For any given $a>0$, it is well-known that the following problem

$$\begin{cases}-\delta^2\Delta w=(w-a)_+^p, & \text{in } B_R(0),\\ w=0, & \text{on } \partial B_R(0),\end{cases}\tag{2.1}$$

has a unique (positive) solution $W_{\delta,a}$, which can be represented by

$$W_{\delta,a}(x)=\begin{cases}a+\delta^{2/(p-1)}s_\delta^{-2/(p-1)}\phi\big(\frac{|x|}{s_\delta}\big), & |x|\le s_\delta,\\ a\ln\frac{|x|}{R}/\ln\frac{s_\delta}{R}, & s_\delta\le|x|\le R,\end{cases}\tag{2.2}$$

where and henceforth $\phi(x)=\phi(|x|)$ is the unique solution of

$$-\Delta\phi=\phi^p,\quad \phi>0,\ \phi\in H_0^1\big(B_1(0)\big)$$

and $s_\delta \in (0, R)$ is determined by

$$\delta^{2/(p-1)} s_\delta^{-2/(p-1)} \phi'(1) = \frac{a}{\ln(s_\delta/R)}. \tag{2.3}$$

By (2.3) we can obtain

$$\frac{s_\delta}{\delta |\ln \delta|^{(p-1)/2}} \to \left(\frac{|\phi'(1)|}{a}\right)^{(p-1)/2} > 0, \quad \text{as } \delta \to 0.$$

Moreover, by Pohozaev identity, we can get

$$\int_{B_1(0)} \phi^{p+1} = \frac{\pi(p+1)}{2}|\phi'(1)|^2, \quad \int_{B_1(0)} \phi^p = 2\pi|\phi'(1)|.$$

For any $z \in \Omega$, define $W_{\delta,z,a}(x) = W_{\delta,a}(x-z)$. Because $W_{\delta,z,a}$ does not vanish on $\partial\Omega$, we need to make a projection as follows. Let $PW_{\delta,z,a}$ be the solution of

$$\begin{cases} -\delta^2 \Delta w = (W_{\delta,z,a} - a)_+^p, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases}$$

then

$$PW_{\delta,z,a} = W_{\delta,z,a} - \frac{a}{\ln \frac{R}{s_\delta}} g(x,z), \tag{2.4}$$

where $g(x,z)$ satisfies

$$\begin{cases} -\Delta g = 0, & \text{in } \Omega, \\ g = \ln \frac{R}{|x-z|}, & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that

$$g(x,z) = \ln R + 2\pi h(x,z),$$

where $h(x,z) = -H(x,z)$.

Let $Z = (Z_m^+, Z_n^-)$, where $Z_m^+ = (z_1^+, \cdots, z_m^+)$, $Z_n^+ = (z_1^-, \cdots, z_n^-)$. We will construct solutions for (1.9) of the form

$$\sum_{i=1}^m PW_{\delta,z_i^+,a_{\delta,i}^+} - \sum_{j=1}^n PW_{\delta,z_j^-,a_{\delta,j}^-} + \omega_\delta,$$

where $z_i^+, z_j^- \in \Omega$, $a_{\delta,i}^+ > 0, a_{\delta,j}^- > 0$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$, ω_δ is a perturbation term. To make ω_δ as small as possible, we need to choose $a_{\delta,i}^+$, $a_{\delta,j}^-$ properly. Indeed we will choose $a_{\delta,i}^+$, $a_{\delta,j}^-$ such that, together with $(s_1^+(Z), \cdots, s_m^+(Z), s_1^-(Z), \cdots, s_n^-(Z))$, the system in Lemma 2.1 to be given in the following will be satisfied.

In this paper, we always assume that $z_i^+, z_j^- \in \Omega$ satisfies

$$\begin{aligned} &d(z_i^+, \partial\Omega) \geq \varrho,\ d(z_j^-, \partial\Omega) \geq \varrho, \quad |z_i^+ - z_k^+| \geq \varrho^{\bar{L}}, \quad i, k = 1, \cdots, m,\ i \neq k \\ &|z_j^- - z_l^-| \geq \varrho^{\bar{L}}, \quad |z_i^+ - z_j^-| \geq \varrho^{\bar{L}}, \quad j, l = 1, \cdots, n,\ j \neq l, \end{aligned} \tag{2.5}$$

where $\varrho > 0$ is a fixed small constant and $\bar{L} > 0$ is a fixed large constant.

Lemma 2.1. *For $\delta > 0$ small, there exist $(s_{\delta,1}^+(Z), \cdots, s_{\delta,m}^+(Z), s_{\delta,1}^-(Z), \cdots, s_{\delta,n}^-(Z))$ and $(a_{\delta,1}^+(Z), \cdots, a_{\delta,m}^+(Z), a_{\delta,1}^-(Z), \cdots, a_{\delta,n}^-(Z))$ satisfying the following system*

$$\delta^{2/(p-1)}(s_i^+)^{-2/(p-1)}\phi'(1) = \frac{a_i^+}{\ln \frac{s_i^+}{R}}, \qquad i = 1, \cdots, m, \tag{2.6}$$

$$\delta^{2/(p-1)}(s_j^-)^{-2/(p-1)}\phi'(1) = \frac{a_j^-}{\ln \frac{s_j^-}{R}}, \qquad j = 1, \cdots, n, \tag{2.7}$$

$$a_i^+ = \kappa_i^+ + \frac{2\pi q(z_i^+)}{|\ln \varepsilon|} + \frac{g(z_i^+, z_i^+)}{\ln \frac{R}{s_i^+}} a_i^+ - \sum_{\alpha \neq i}^{m} \frac{\bar{G}(z_i^+, z_\alpha^+)}{\ln \frac{R}{s_\alpha^+}} a_\alpha^+ + \sum_{l=1}^{n} \frac{\bar{G}(z_i^+, z_l^-)}{\ln \frac{R}{s_l^-}} a_l^-, \quad i = 1, \cdots, m, \tag{2.8}$$

$$a_j^- = \kappa_j^- - \frac{2\pi q(z_j^-)}{|\ln \varepsilon|} + \frac{g(z_j^-, z_j^-)}{\ln \frac{R}{s_j^-}} a_j^- - \sum_{\beta \neq j}^{n} \frac{\bar{G}(z_\beta^-, z_j^-)}{\ln \frac{R}{s_\beta^-}} a_\beta^- + \sum_{k=1}^{m} \frac{\bar{G}(z_j^-, z_k^+)}{\ln \frac{R}{s_k^+}} a_k^+, \quad j = 1, \cdots, n, \tag{2.9}$$

where $\bar{G}(x, y) = \ln \frac{R}{|x-y|} - g(x, y)$ for $x \neq y$.

Proof. We will prove that if $\delta > 0$ is small enough then system (2.6)-(2.9) has a solution $(s_1^+, \cdots, s_m^+, s_1^-, \cdots, s_n^-, a_1^+, \cdots, a_m^+, a_1^-, \cdots, a_n^-)$ in $\mathcal{D}$ defined by

$$\mathcal{D} := \left[\frac{\delta}{|\ln \delta|}, \delta|\ln \delta|\right]^{m+n} \times \prod_{i=1}^{m} \left[\frac{1}{2}\kappa_i^+, \frac{3}{2}\kappa_i^+\right] \times \prod_{j=1}^{n} \left[\frac{1}{2}\kappa_j^-, \frac{3}{2}\kappa_j^-\right].$$

It is not difficult to see, for any fixed $(s_1^+, \cdots, s_m^+, s_1^-, \cdots, s_n^-)$ and $0 < \delta < \delta^*$ small enough, that the subsystem (2.8)-(2.9) has a solution $(a_1^+, \cdots, a_m^+, a_1^-, \cdots, a_n^-)$ depending on $(s_1^+, \cdots, s_m^+, s_1^-, \cdots, s_n^-)$ such that $\frac{1}{2}\kappa_i^+ \leq a_i^+ \leq \frac{3}{2}\kappa_i^+$, $\frac{1}{2}\kappa_j^- \leq a_j^- \leq \frac{3}{2}\kappa_j^-$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. For such $(a_1^+, \cdots, a_m^+, a_1^-, \cdots, a_n^-)$ define

$$\theta_i^+(s_1^+, \cdots, s_m^+, s_1^-, \cdots, s_n^-) = \frac{(s_i^+)^{\frac{2}{p-1}}}{\ln \frac{R}{s_i^+}} + \frac{\phi'(1)}{a_i^+}\delta^{\frac{2}{p-1}}, \ \ i = 1, \cdots, m,$$

$$\theta_j^-(s_1^+, \cdots, s_m^+, s_1^-, \cdots, s_n^-) = \frac{(s_j^-)^{\frac{2}{p-1}}}{\ln \frac{R}{s_j^-}} + \frac{\phi'(1)}{a_j^-}\delta^{\frac{2}{p-1}}, \ \ j = 1, \cdots, n,$$

then it is easy to verify that for $i,\ell=1,\cdots,m,\ j,k=1,\cdots,n,$

$$\begin{cases} \theta_i^+(s_1^+,\cdots,s_m^+,s_1^-,\cdots,s_n^-)>0, & s_i^+=\delta|\ln\delta|,\ s_\ell^+,\ s_j^-\in[\frac{\delta}{|\ln\delta|},\ \delta|\ln\delta|]\ \text{for}\ \ell\neq i,\\ \theta_i^+(s_1^+,\cdots,s_m^+,s_1^-,\cdots,s_n^-)<0, & s_i^+=\frac{\delta}{|\ln\delta|},\quad s_\ell^+,\ s_j^-\in[\frac{\delta}{|\ln\delta|},\ \delta|\ln\delta|]\ \text{for}\ \ell\neq i,\\ \theta_j^-(s_1^+,\cdots,s_m^+,s_1^-,\cdots,s_n^-)>0, & s_j^-=\delta|\ln\delta|,\ s_\ell^+,\ s_k^-\in[\frac{\delta}{|\ln\delta|},\ \delta|\ln\delta|]\ \text{for}\ k\neq j,\\ \theta_j^-(s_1^+,\cdots,s_m^+,s_1^-,\cdots,s_n^-)<0, & s_j^-=\frac{\delta}{|\ln\delta|},\quad s_\ell^+,\ s_k^-\in[\frac{\delta}{|\ln\delta|},\ \delta|\ln\delta|]\ \text{for}\ k\neq j.\end{cases}$$

By the Poincaré-Miranda Theorem in [20, 24], we can get $(s_{\delta,1}^+,\cdots,s_{\delta,m}^+,s_{\delta,1}^-,\cdots,s_{\delta,n}^-)$ such that $\theta_i^+(s_{\delta,1}^+,\cdots,s_{\delta,m}^+,s_{\delta,1}^-,\cdots,s_{\delta,n}^-)=0,\ \theta_j^-(s_{\delta,1}^+,\cdots,s_{\delta,m}^+,s_{\delta,1}^-,\cdots,s_{\delta,n}^-)=0$ for $i=1,\cdots,m,\ j=1,\cdots,n$. This completes our proof of Lemma 2.1.

□

In the sequel, to simplify notation, we will use $a_{\delta,i}^\pm,\ s_{\delta,i}^\pm$ to denote $a_{\delta,i}^\pm(Z),\ s_{\delta,i}^\pm(Z)$ for given $Z=(Z_m^+,Z_n^-)$. From now on we will always choose $(a_{\delta,1}^+,\cdots,a_{\delta,m}^+,a_{\delta,1}^-,\cdots,a_{\delta,n}^-)$ and $(s_{\delta,1}^+,\cdots,s_{\delta,m}^+,s_{\delta,1}^-,\cdots,s_{\delta,n}^-)$ such that (2.6)–(2.9) hold. For $(a_{\delta,1}^+,\cdots,a_{\delta,m}^+,a_{\delta,1}^-,\cdots,a_{\delta,n}^-)$ and $(s_{\delta,1}^+,\cdots,s_{\delta,m}^+,s_{\delta,1}^-,\cdots,s_{\delta,n}^-)$ chosen in such a way define

$$P_{\delta,Z,i}^+=PW_{\delta,z_i^+,a_{\delta,i}^+},\ P_{\delta,Z,j}^-=PW_{\delta,z_j^-,a_{\delta,j}^-}. \tag{2.10}$$

Remark 2.2. It is not difficult to obtain the following asymptotic expansions:

$$\frac{1}{\ln\frac{R}{s_{\delta,i}^+}}=\frac{1}{\ln\frac{R}{\varepsilon}}+O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}\right),\ i=1,\cdots,m, \tag{2.11}$$

$$a_{\delta,i}^+=\kappa_i^++\frac{2\pi q(z_i^+)}{|\ln\varepsilon|}+\frac{g(z_i^+,z_i^+)}{\ln\frac{R}{\varepsilon}}-\sum_{\alpha\neq i}^m\frac{\bar G(z_i^+,z_\alpha^+)}{\ln\frac{R}{\varepsilon}}+\sum_{l=1}^n\frac{\bar G(z_i^+,z_l^-)}{\ln\frac{R}{\varepsilon}}+O\Big(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}\Big),\ i=1,\cdots,m, \tag{2.12}$$

$$\begin{cases}\dfrac{\partial a_{\delta,i}^+}{\partial z_{k,h}^+}=O\left(\dfrac{1}{|\ln\varepsilon|}\right),\ \dfrac{\partial\, s_{\delta,i}^+}{\partial z_{k,h}^+}=O\left(\dfrac{\varepsilon}{|\ln\varepsilon|}\right),\ i,k=1,\cdots,m,\ h=1,2,\\[2ex] \dfrac{\partial\, a_{\delta,i}^+}{\partial\, z_{j,h}^-}=O\left(\dfrac{1}{|\ln\varepsilon|}\right),\ \dfrac{\partial s_{\delta,i}^+}{\partial\, z_{j,h}^-}=O\left(\dfrac{\varepsilon}{|\ln\varepsilon|}\right),\ i=1,\cdots,m,\ j=1,\cdots,n,\ h=1,2.\end{cases} \tag{2.13}$$

Moreover, $a_{\delta,j}^-$ and $s_{\delta,j}^-$ have similar expansions.

To simplify notations, set

$$P_{\delta,Z}^+=\sum_{\alpha=1}^m P_{\delta,Z,\alpha}^+,\ P_{\delta,Z}^-=\sum_{\beta=1}^n P_{\delta,Z,\beta}^-. \tag{2.14}$$

Then, for any fixed constant $L>0$, we have that for $x\in B_{Ls_{\delta,i}^+}(z_i^+)$,

$$
\begin{aligned}
&P^+_{\delta,Z,i}(x)-\kappa^+_i-\frac{2\pi q(x)}{|\ln\varepsilon|}=W_{\delta,z_i^+,a^+_{\delta,i}}(x)-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(x,z_i^+)-\kappa^+_i-\frac{2\pi q(x)}{|\ln\varepsilon|}\\
=&W_{\delta,z_i^+,a^+_{\delta,i}}(x)-\kappa^+_i-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(z_i^+,z_i^+)-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\Big(\big\langle Dg(z_i^+,z_i^+),x-z_i^+\big\rangle+O(|x-z_i^+|^2)\Big)\\
&-\frac{2\pi q(z_i^+)}{|\ln\varepsilon|}-\frac{2\pi}{|\ln\varepsilon|}\left(\big\langle Dq(z_i^+),x-z_i^+\big\rangle+O(|x-z_i^+|^2)\right)\\
=&W_{\delta,z_i^+,a^+_{\delta,i}}(x)-\kappa^+_i-\frac{2\pi q(z_i^+)}{|\ln\varepsilon|}-\frac{2\pi}{|\ln\varepsilon|}\big\langle Dq(z_i^+),x-z_i^+\big\rangle\\
&-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(z_i^+,z_i^+)-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\big\langle Dg(z_i^+,z_i^+),x-z_i^+\big\rangle+O\left(\frac{(s^+_{\delta,i})^2}{|\ln\varepsilon|}\right),
\end{aligned}
$$

and for $k\neq i$ and $x\in B_{Ls^+_{\delta,i}}(z_i^+)$, by (2.2)

$$
\begin{aligned}
&P^+_{\delta,Z,k}(x)=W_{\delta,z_k^+,a^+_{\delta,k}}(x)-\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}g(x,z_k^+)=\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\bar G(x,z_k^+)\\
=&\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\bar G(z_i^+,z_k^+)+\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\big\langle D\bar G(z_i^+,z_k^+),x-z_i^+\big\rangle+O\Big(\frac{(s^+_{\delta,i})^2}{|\ln\varepsilon|}\Big),
\end{aligned}
$$

and

$$
P^-_{\delta,Z,j}(x)=\frac{a^-_{\delta,j}}{\ln\frac{R}{s^-_{\delta,j}}}\bar G(z_i^+,z_j^-)+\frac{a^-_{\delta,j}}{\ln\frac{R}{s^-_{\delta,j}}}\big\langle D\bar G(z_i^+,z_j^-),x-z_i^+\big\rangle+O\Big(\frac{(s^+_{\delta,i})^2}{|\ln\varepsilon|}\Big).
$$

So, by using (2.8), we obtain

$$
\begin{aligned}
&P^+_{\delta,Z}(x)-P^-_{\delta,Z}(x)-\kappa^+_i-\frac{2\pi q(x)}{|\ln\varepsilon|}\\
=&W_{\delta,z_i^+,a^+_{\delta,i}}(x)-a^+_{\delta,i}-\frac{2\pi}{|\ln\varepsilon|}\big\langle Dq(z_i^+),x-z_i^+\big\rangle-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\big\langle Dg(z_i^+,z_i^+),x-z_i^+\big\rangle\\
&+\sum_{k\neq i}^{m}\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\big\langle D\bar G(z_i^+,z_k^+),x-z_i^+\big\rangle-\sum_{l=1}^{n}\frac{a^-_{\delta,l}}{\ln\frac{R}{s^-_{\delta,l}}}\big\langle D\bar G(z_i^+,z_l^-),x-z_i^+\big\rangle\\
&+O\left(\frac{(s^+_{\delta,i})^2}{|\ln\varepsilon|}\right),\quad x\in B_{Ls^+_{\delta,i}}(z_i^+).
\end{aligned}
\tag{2.15}
$$

Similarly, we have

$$
\begin{aligned}
&P^-_{\delta,Z}(x) - P^+_{\delta,Z}(x) - \kappa^-_j + \frac{2\pi q(x)}{|\ln\varepsilon|}\\
=&W_{\delta,z^-_j,a^-_{\delta,j}}(x) - a^-_{\delta,j} + \frac{2\pi}{|\ln\varepsilon|}\left\langle Dq(z^-_j), x - z^-_j\right\rangle - \frac{a^-_{\delta,j}}{\ln\frac{R}{s^-_{\delta,j}}}\left\langle Dg(z^-_j, z^-_j), x - z^-_j\right\rangle\\
&+\sum_{l\neq j}^{n}\frac{a^-_{\delta,l}}{\ln\frac{R}{s^-_{\delta,l}}}\left\langle D\bar{G}(z^-_j, z^-_l), x - z^-_j\right\rangle - \sum_{k=1}^{m}\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\left\langle D\bar{G}(z^-_j, z^+_k), x - z^-_j\right\rangle\\
&+O\left(\frac{(s^-_{\delta,j})^2}{|\ln\varepsilon|}\right), \quad x\in B_{Ls^-_{\delta,j}}(z^-_j).
\end{aligned}
\tag{2.16}
$$

We end this section by giving the following expansion which can be obtained by direct computation and will be used in the next two sections.

$$
\begin{aligned}
&\frac{\partial W_{\delta,z^\pm_i,a^\pm_{\delta,i}}(x)}{\partial z^\pm_{i,h}}\\
=&\begin{cases}
\frac{1}{\delta}\Big(\frac{a^\pm_{\delta,i}}{|\phi'(1)||\ln\frac{R}{s^\pm_{\delta,i}}|}\Big)^{(p+1)/2}\phi'\Big(\frac{|x-z^\pm_i|}{s^\pm_{\delta,i}}\Big)\frac{z^\pm_{i,h}-x_h}{|x-z^\pm_i|} + O\left(\frac{1}{|\ln\varepsilon|}\right), & x\in B_{s^\pm_{\delta,i}}(z^\pm_i),\\
-\frac{a^\pm_{\delta,i}}{\ln\frac{R}{s^\pm_{\delta,i}}}\frac{z^\pm_{i,h}-x_h}{|x-z^\pm_i|^2} + O\left(\frac{1}{|\ln\varepsilon|}\right), & x\in\Omega\setminus B_{s^\pm_{\delta,i}}(z^\pm_i).
\end{cases}
\end{aligned}
\tag{2.17}
$$

3. THE REDUCTION

In this section, letting $P^+_{\delta,Z}$ and $P^-_{\delta,Z}$ be given as in (2.14), we are to look for solutions of the form $P^+_{\delta,Z} - P^-_{\delta,Z} + \omega_{\delta,Z}$, where $\omega_{\delta,Z}$ is a small perturbation when δ is small. We will show the existence of $\omega_{\delta,Z}$ for given Z so that $P^+_{\delta,Z} - P^-_{\delta,Z} + \omega_{\delta,Z}$ solves (1.9) in $W^{2,p}(\Omega)\cap H^1_0(\Omega)\setminus H_*$, where H_* is a finite dimension subspace of $W^{2,p}(\Omega)\cap H^1_0(\Omega)$. In the next section, we will show that if Z is a critical point of

$$
\begin{aligned}
K(Z) = \frac{\delta^2}{2}\int_\Omega &|D(P^+_{\delta,Z} - P^-_{\delta,Z} + \omega_{\delta,Z})|^2\\
&-\sum_{i=1}^{m}\frac{1}{p+1}\int_{\Omega^+_i}\left(P^+_{\delta,Z} - P^-_{\delta,Z} + \omega_{\delta,Z} - \kappa^+_i - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)^{p+1}_+\\
&-\sum_{j=1}^{n}\frac{1}{p+1}\int_{\Omega^-_j}\left(\frac{2\pi q(x)}{|\ln\varepsilon|} - \kappa^-_j - P^+_{\delta,Z} + P^-_{\delta,Z} - \omega_{\delta,Z}\right)^{p+1}_+,
\end{aligned}
$$

then $P^+_{\delta,Z} - P^-_{\delta,Z} + \omega_{\delta,Z}$ is indeed a solution of (1.9). Finding critical points of $K(Z)$ is a problem of finite dimension, so in this sense, we are trying to reduce the problem to a finite dimension one in this section.

To this end we need to study the kernel of linearized problem of (1.9) at $P^+_{\delta,Z} - P^-_{\delta,Z}$. First let us study the kernel of the following problem in $\mathbb{R}^2$:

$$-\Delta v - p w_+^{p-1} v = 0, \quad v \in L^\infty(\mathbb{R}^2), \tag{3.1}$$

where

$$w(x) = \begin{cases} \phi(|x|), & |x| \le 1, \\ \phi'(1)\ln|x|, & |x| > 1, \end{cases}$$

is the solution of

$$-\Delta w = w_+^p, \quad \text{in } \mathbb{R}^2. \tag{3.2}$$

Since $\phi'(1) < 0$ and $\ln|x|$ is harmonic for $|x| > 1$, we see that $w \in C^1(\mathbb{R}^2)$. Moreover, since w_+ is Lip-continuous, by the Schauder estimate, $w \in C^{2,\alpha}$ for any $\alpha \in (0,1)$.

It is easy to see that for $i = 1, 2$, $\frac{\partial w}{\partial x_i}$ is a solution of (3.1). Moreover, from Dancer and Yan [15], we know that w is also non-degenerate, in the sense that the kernel of the operator $Lv := -\Delta v - p w_+^{p-1} v$, $v \in D^{1,2}(\mathbb{R}^2)$ is spanned by $\left\{\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}\right\}$.

Let $P^+_{\delta,Z,i}$, $P^-_{\delta,Z,j}$ be the functions defined in (2.10). Set

$$F_{\delta,Z} = \Bigg\{ u : u \in L^p(\Omega), \int_\Omega \frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}} u = 0, \int_\Omega \frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,h}} u = 0, \\ i = 1, \cdots, m, \quad j = 1, \cdots, n, \quad h = 1, 2 \Bigg\},$$

and

$$E_{\delta,Z} = \Bigg\{ u : \ u \in W^{2,p}(\Omega) \cap H^1_0(\Omega), \int_\Omega \Delta\left(\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\right) u = 0, \int_\Omega \Delta\left(\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,h}}\right) u = 0, \\ i = 1, \cdots, m, \quad j = 1, \cdots, n, \quad h = 1, 2 \Bigg\}.$$

For any $u \in L^p(\Omega)$, define $Q_\delta u$ as follows:

$$Q_\delta u = u - \sum_{i=1}^m \sum_{h=1}^2 b^+_{i,h} \left(-\delta^2 \Delta\Big(\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\Big)\right) - \sum_{j=1}^n \sum_{\bar h=1}^2 b^-_{j,\bar h} \left(-\delta^2 \Delta\Big(\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,\bar h}}\Big)\right),$$

where the constants $b^+_{i,h}$, $b^-_{j,\bar h}$ satisfy

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{h=1}^{2}b_{i,h}^{+}\left(-\delta^2\int_{\Omega}\Delta\Big(\frac{\partial P_{\delta,Z,i}^{+}}{\partial z_{i,h}^{+}}\Big)\frac{\partial P_{\delta,Z,k}^{+}}{\partial z_{k,\hat h}^{+}}\right)\\&+\sum_{j=1}^{n}\sum_{\bar h=1}^{2}b_{j,\bar h}^{-}\left(-\delta^2\int_{\Omega}\Delta\Big(\frac{\partial P_{\delta,Z,j}^{-}}{\partial z_{j,\bar h}^{-}}\Big)\frac{\partial P_{\delta,Z,k}^{+}}{\partial z_{k,\hat h}^{+}}\right)=\int_{\Omega}u\frac{\partial P_{\delta,Z,k}^{+}}{\partial z_{k,\hat h}^{+}},\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{h=1}^{2}b_{i,h}^{+}\left(-\delta^2\int_{\Omega}\Delta\Big(\frac{\partial P_{\delta,Z,i}^{+}}{\partial z_{i,h}^{+}}\Big)\frac{\partial P_{\delta,Z,l}^{-}}{\partial z_{l,\tilde h}^{-}}\right)\\&+\sum_{j=1}^{n}\sum_{\bar h=1}^{2}b_{j,\bar h}^{-}\left(-\delta^2\int_{\Omega}\Delta\Big(\frac{\partial P_{\delta,Z,j}^{-}}{\partial z_{j,\bar h}^{-}}\Big)\frac{\partial P_{\delta,Z,l}^{-}}{\partial z_{l,\tilde h}^{-}}\right)=\int_{\Omega}u\frac{\partial P_{\delta,Z,l}^{-}}{\partial z_{l,\tilde h}^{-}}.\end{aligned}\tag{3.4}$$

Since $\int_{\Omega}\frac{\partial P_{\delta,Z,k}^{+}}{\partial z_{k,\hat h}^{+}}Q_\delta u=0$, $\int_{\Omega}\frac{\partial P_{\delta,Z,l}^{-}}{\partial z_{l,\tilde h}^{-}}Q_\delta u=0$, the operator Q_δ can be regarded as a projection from $L^p(\Omega)$ to $F_{\delta,Z}$. In order to show that we can solve (3.3) and (3.4) to obtain $b_{i,h}^{+}$ and $b_{j,\bar h}^{-}$, we just need the following estimate (by (2.13) and (2.17))

$$\begin{aligned}&-\delta^2\int_{\Omega}\Delta\Big(\frac{\partial P_{\delta,Z,i}^{+}}{\partial z_{i,h}^{+}}\Big)\frac{\partial P_{\delta,Z,k}^{+}}{\partial z_{k,\hat h}^{+}}\\&=p\int_{\Omega}\big(W_{\delta,z_i^+,a_{\delta,i}^+}-a_{\delta,i}^{+}\big)_+^{p-1}\left(\frac{\partial W_{\delta,z_i^+,a_{\delta,i}^+}}{\partial z_{i,h}^{+}}-\frac{\partial a_{\delta,i}^{+}}{\partial z_{i,h}^{+}}\right)\frac{\partial P_{\delta,Z,k}^{+}}{\partial z_{k,\hat h}^{+}}\\&=\delta_{ikh\hat h}\frac{c}{|\ln\varepsilon|^{p+1}}+O\left(\frac{\varepsilon}{|\ln\varepsilon|^{p+1}}\right),\end{aligned}\tag{3.5}$$

where $c>0$ is a constant, $\delta_{ikh\hat h}=1$, if $i=k$ and $h=\hat h$; otherwise, $\delta_{ijh\hat h}=0$.

Similarly,

$$-\delta^2\int_{\Omega}\Delta\Big(\frac{\partial P_{\delta,Z,j}^{-}}{\partial z_{j,\bar h}^{-}}\Big)\frac{\partial P_{\delta,Z,l}^{-}}{\partial z_{l,\tilde h}^{-}}=\delta_{jl\bar h\tilde h}\frac{c}{|\ln\varepsilon|^{p+1}}+O\left(\frac{\varepsilon}{|\ln\varepsilon|^{p+1}}\right),\tag{3.6}$$

where $c>0$ is a constant, $\delta_{jl\bar h\tilde h}=1$, if $j=l$ and $\bar h=\tilde h$; otherwise, $\delta_{jl\bar h\tilde h}=0$.

Set

$$\begin{aligned}L_\delta u=-\delta^2\Delta u&-\sum_{i=1}^{m}p\chi_{\Omega_i^+}\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}-\kappa_i^{+}-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}u\\&-\sum_{j=1}^{n}p\chi_{\Omega_j^-}\left(P_{\delta,Z}^{-}-P_{\delta,Z}^{+}-\kappa_j^{-}+\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}u,\end{aligned}$$

and

$$B_{\delta,Z} = \Big(\cup_{i=1}^m B_{Ls^+_{\delta,i}}(z_i^+)\Big)\bigcup\Big(\cup_{j=1}^n B_{Ls^-_{\delta,j}}(z_j^-)\Big).$$

We have the following lemma.

Lemma 3.1. *There are constants $\rho_0 > 0$ and $\delta_0 > 0$, such that for any $\delta \in (0, \delta_0]$, Z satisfying* (2.5)*, $u \in E_{\delta,Z}$ with $Q_\delta L_\delta u = 0$ in $\Omega \setminus B_{\delta,Z}$ for some $L > 0$ large, then*

$$\|Q_\delta L_\delta u\|_{L^p(\Omega)} \geq \frac{\rho_0 \delta^{\frac{2}{p}}}{|\ln \delta|^{\frac{(p-1)^2}{p}}} \|u\|_{L^\infty(\Omega)}.$$

Proof. Set $s^\pm_{N,j} = s^\pm_{\delta_N,j}$. Henceforth, we will use $\|\cdot\|_p$ and $\|\cdot\|_\infty$ to denote $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$ respectively.

We argue by contradiction. Suppose that there are $\delta_N \to 0$, Z_N satisfying (2.5) and $u_N \in E_{\delta_N,Z_N}$ with $Q_{\delta_N} L_{\delta_N} u_N = 0$ in $\Omega \setminus B_{\delta_N,Z_N}$ and $\|u_N\|_\infty = 1$ such that

$$\|Q_{\delta_N} L_{\delta_N} u_N\|_p \leq \frac{1}{N} \frac{\delta_N^{\frac{2}{p}}}{|\ln \delta_N|^{\frac{(p-1)^2}{p}}}.$$

First, we estimate the $b^+_{i,h,N}$, $b^-_{j,\bar{h},N}$ corresponding to $L_{\delta_N} u_N$, which satisfy

$$\begin{aligned} Q_{\delta_N} L_{\delta_N} u_N = L_{\delta_N} u_N &- \sum_{i=1}^m \sum_{h=1}^2 b^+_{i,h,N} \left(-\delta_N^2 \Delta \frac{\partial P^+_{\delta_N,Z_N,i}}{\partial z^+_{i,h}} \right) \\ &- \sum_{j=1}^n \sum_{\bar{h}=1}^2 b^-_{j,\bar{h},N} \left(-\delta_N^2 \Delta \frac{\partial P^-_{\delta_N,Z_N,j}}{\partial z^-_{j,\bar{h}}} \right). \end{aligned} \tag{3.7}$$

For each fixed k, multiplying (3.7) by $\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat{h}}}$, noting that

$$\int_\Omega \big(Q_{\delta_N} L_{\delta_N} u_N\big) \frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat{h}}} = 0,$$

we obtain

$$\begin{aligned} &\int_\Omega u_N \, L_{\delta_N} \left(\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat{h}}} \right) = \int_\Omega \big(L_{\delta_N} u_N\big) \frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat{h}}} \\ &= \sum_{i=1}^m \sum_{h=1}^2 b^+_{i,h,N} \int_\Omega \left(-\delta_N^2 \Delta \frac{\partial P^+_{\delta_N,Z_N,i}}{\partial z^+_{i,h}} \right) \frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat{h}}} \\ &\quad + \sum_{j=1}^n \sum_{\bar{h}=1}^2 b^-_{j,\bar{h},N} \int_\Omega \left(-\delta_N^2 \Delta \frac{\partial P^-_{\delta_N,Z_N,j}}{\partial z^-_{j,\bar{h}}} \right) \frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat{h}}}. \end{aligned}$$

Using (2.15), (2.16) and Lemma A.1, we obtain

$$
\begin{aligned}
&\int_\Omega u_N\, L_{\delta_N}\left(\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}\right)\\
&=\int_\Omega\Bigg[-\delta_N^2\Delta\left(\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}\right)-\sum_{i=1}^m p\chi_{\Omega_i^+}\left(P^+_{\delta_N,Z_N}-P^-_{\delta_N,Z_N}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon_N|}\right)_+^{p-1}\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}\\
&\quad-\sum_{j=1}^n p\chi_{\Omega_j^-}\left(P^-_{\delta_N,Z_N}-P^+_{\delta_N,Z_N}-\kappa_j^-+\frac{2\pi q(x)}{|\ln\varepsilon_N|}\right)_+^{p-1}\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}\Bigg]u_N\\
&=p\int_\Omega\left(W_{\delta_N,z^+_{k,N},a^+_{\delta_N,k}}-a^+_{\delta_N,k}\right)_+^{p-1}\left(\frac{\partial W_{\delta_N,z^+_{k,N},a^+_{\delta_N,k}}}{\partial z^+_{k,\hat h}}-\frac{\partial a^+_{\delta_N,k}}{\partial z^+_{k,\hat h}}\right)u_N\\
&\quad-p\sum_{\alpha=1}^m\int_{\Omega_\alpha^+}\left(W_{\delta_N,z^+_{\alpha,N},a^+_{\delta_N,\alpha}}-a^+_{\delta_N,\alpha}+O\left(\frac{s^+_{N,\alpha}}{|\ln\varepsilon_N|}\right)\right)_+^{p-1}\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}u_N\\
&\quad-p\sum_{\beta=1}^n\int_{\Omega_\beta^-}\left(W_{\delta_N,z^-_{\beta,N},a^-_{\delta_N,\beta}}-a^-_{\delta_N,\beta}+O\left(\frac{s^-_{N,\beta}}{|\ln\varepsilon_N|}\right)\right)_+^{p-1}\frac{\partial P^+_{\delta_N,Z_N,k}}{\partial z^+_{k,\hat h}}u_N\\
&=O\left(\frac{\varepsilon_N^2}{|\ln\varepsilon_N|^p}\right).
\end{aligned}
$$

Using (3.5) and (3.6), we obtain that $b^+_{i,h,N}=O\left(\varepsilon_N^2|\ln\varepsilon_N|\right)$. Similarly, $b^-_{i,h,N}=O\left(\varepsilon_N^2|\ln\varepsilon_N|\right)$. Therefore,

$$
\begin{aligned}
&\sum_{i=1}^m\sum_{h=1}^2 b^+_{i,h,N}\left(-\delta_N^2\Delta\frac{\partial P^+_{\delta_N,Z_N,i}}{\partial z^+_{i,h}}\right)+\sum_{j=1}^n\sum_{\bar h=1}^2 b^-_{j,\bar h,N}\left(-\delta_N^2\Delta\frac{\partial P^-_{\delta_N,Z_N,j}}{\partial z^-_{j,\bar h}}\right)\\
&=p\sum_{i=1}^m\sum_{h=1}^2 b^+_{i,h,N}\left(W_{\delta_N,z^+_{i,N},a^+_{\delta_N,i}}-a^+_{\delta_N,i}\right)_+^{p-1}\left(\frac{\partial W_{\delta_N,z^+_{i,N},a^+_{\delta_N,i}}}{\partial z^+_{i,h}}-\frac{\partial a^+_{\delta_N,i}}{\partial z^+_{i,h}}\right)\\
&\quad+p\sum_{j=1}^n\sum_{\bar h=1}^2 b^-_{j,\bar h,N}\left(W_{\delta_N,z^-_{j,N},a^-_{\delta_N,j}}-a^-_{\delta_N,j}\right)_+^{p-1}\left(\frac{\partial W_{\delta_N,z^-_{j,N},a^-_{\delta_N,j}}}{\partial z^-_{j,\bar h}}-\frac{\partial a^-_{\delta_N,j}}{\partial z^-_{j,\bar h}}\right)\\
&=O\left(\sum_{i=1}^m\sum_{h=1}^2\frac{\varepsilon_N^{\frac{2}{p}-1}|b^+_{i,h,N}|}{|\ln\varepsilon_N|^p}\right)+O\left(\sum_{j=1}^n\sum_{\bar h=1}^2\frac{\varepsilon_N^{\frac{2}{p}-1}|b^-_{j,\bar h,N}|}{|\ln\varepsilon_N|^p}\right)\\
&=O\left(\frac{\varepsilon_N^{\frac{2}{p}+1}}{|\ln\varepsilon_N|^{p-1}}\right)\quad\text{ in } L^p(\Omega).
\end{aligned}
$$

Thus, we obtain

$$L_{\delta_N}u_N = Q_{\delta_N}L_{\delta_N}u_N + O\left(\frac{\varepsilon_N^{\frac{2}{p}+1}}{|\ln\varepsilon_N|^{p-1}}\right) = O\left(\frac{1}{N}\frac{\delta_N^{\frac{2}{p}}}{|\ln\delta_N|^{\frac{(p-1)^2}{p}}}\right).$$

For any fixed i, j, define

$$\tilde{u}_{i,N}^+(y) = u_N(s_{N,i}^+ y + z_{i,N}^+), \quad \tilde{u}_{j,N}^-(y) = u_N(s_{N,j}^- y + z_{j,N}^-).$$

Let

$$\begin{aligned}\tilde{L}_N^{\pm}u = \\ &-\Delta u - p\sum_{k=1}^{m}\frac{(s_{N,i}^{\pm})^2}{\delta_N^2}\chi_{\Omega_k^+}\left(P_{\delta_N,Z_N}^+(s_{N,i}^{\pm}y+z_{i,N}^{\pm}) - P_{\delta_N,Z_N}^-(s_{N,i}^{\pm}y+z_{i,N}^{\pm}) - \kappa_k^+ - \frac{2\pi q}{|\ln\varepsilon_N|}\right)_+^{p-1}u \\ &-p\sum_{l=1}^{n}\frac{(s_{N,i}^{\pm})^2}{\delta_N^2}\chi_{\Omega_l^-}\left(P_{\delta_N,Z_N}^-(s_{N,i}^{\pm}y+z_{i,N}^{\pm}) - P_{\delta_N,Z_N}^+(s_{N,i}^{\pm}y+z_{i,N}^{\pm}) - \kappa_l^- + \frac{2\pi q}{|\ln\varepsilon_N|}\right)_+^{p-1}u.\end{aligned}$$

Then

$$(s_{N,i}^{\pm})^{\frac{2}{p}} \times \frac{\delta_N^2}{(s_{N,i}^{\pm})^2}\|\tilde{L}_N^{\pm}\tilde{u}_{i,N}^{\pm}\|_p = \|L_{\delta_N}u_N\|_p.$$

Noting that

$$\left(\frac{\delta_N}{s_{N,i}^{\pm}}\right)^2 = O\left(\frac{1}{|\ln\delta_N|^{p-1}}\right),$$

we find that

$$L_{\delta_N}u_N = o\left(\frac{\delta_N^{\frac{2}{p}}}{|\ln\delta_N|^{\frac{(p-1)^2}{p}}}\right).$$

As a result,

$$\tilde{L}_N^{\pm}\tilde{u}_{i,N}^{\pm} = o(1), \quad \text{in } L^p(\Omega_N^{\pm}),$$

where $\Omega_N^{\pm} = \left\{y : s_{N,i}^{\pm}y + z_{i,N}^{\pm} \in \Omega\right\}$.

Since $\|\tilde{u}_{i,N}^{\pm}\|_\infty = 1$, by the regularity theory of elliptic equations, we may assume that

$$\tilde{u}_{i,N}^{\pm} \to u_i^{\pm}, \quad \text{in } C_{loc}^1(\mathbb{R}^2).$$

It is easy to see that

$$\begin{aligned}&\sum_{k=1}^{m}\frac{(s_{N,i}^+)^2}{\delta_N^2}\chi_{\Omega_k^+}\left(P_{\delta_N,Z_N}^+(s_{N,i}^+y+z_{i,N}^+) - P_{\delta_N,Z_N}^-(s_{N,i}^+y+z_{i,N}^+) - \kappa_k^+ - \frac{2\pi q}{|\ln\varepsilon_N|}\right)_+^{p-1} \\ &= \frac{(s_{N,i}^+)^2}{\delta_N^2}\left(W_{\delta_N,z_{i,N}^+,a_{\delta_N,i}^+} - a_{\delta_N,i}^+ + O\left(\frac{s_{N,i}^+}{|\ln\varepsilon_N|}\right)\right)_+^{p-1} + o(1) \\ &\to w_+^{p-1}.\end{aligned}$$

Similarly,

$$\sum_{l=1}^{n}\frac{(s_{N,j}^{-})^2}{\delta_N^2}\chi_{\Omega_l^-}\left(P_{\delta_N,Z_N}^{-}(s_{N,j}^{-}y+z_{j,N}^{-})-P_{\delta_N,Z_N}^{+}(s_{N,j}^{-}y+z_{j,N}^{-})-\kappa_l^{-}+\frac{2\pi q}{|\ln\varepsilon_N|}\right)_+^{p-1}$$
$$\to w_+^{p-1}.$$

Then, by Lemma A.1, $u_i^{\pm}$ satisfies

$$-\Delta u-pw_+^{p-1}u=0.$$

Now from Proposition 3.1 in [15], we have

$$u_i^{\pm}=c_1^{\pm}\frac{\partial w}{\partial x_1}+c_2^{\pm}\frac{\partial w}{\partial x_2}. \tag{3.8}$$

Taking limit of

$$\int_{\Omega}\Delta\Big(\frac{\partial P_{\delta_N,Z_N,i}^{\pm}}{\partial z_{i,h}^{\pm}}\Big)u_N=0,$$

we get

$$\int_{\mathbb{R}^2}\phi_+^{p-1}\frac{\partial\phi}{\partial z_h}u_i^{\pm}=0,$$

which, together with (3.8), gives $u_i^{\pm}\equiv 0$. Thus for any $L>0$,

$$\tilde u_{i,N}^{\pm}\to 0,\quad \text{in } C^1(B_L(0)),$$

which implies that $u_N=o(1)$ on $\partial B_{Ls_{N,i}^{\pm}}(z_{i,N}^{\pm})$.

On one hand, by our assumption, $Q_{\delta_N}L_{\delta_N}u_N=0$ in $\Omega\setminus B_{\delta_N,Z_N}$. On the other hand, by Lemma A.1, for $i=1,\cdots,m$, $j=1,\cdots,n$, we have

$$\left(P_{\delta_N,Z_N}^{+}-P_{\delta_N,Z_N}^{-}-\kappa_i^{+}-\frac{2\pi q(x)}{|\ln\varepsilon_N|}\right)_+=0,\quad x\in\Omega_i^{+}\setminus B_{Ls_{N,i}^{+}}(z_{i,N}^{+}),$$
$$\left(P_{\delta_N,Z_N}^{-}-P_{\delta_N,Z_N}^{+}-\kappa_j^{-}+\frac{2\pi q(x)}{|\ln\varepsilon_N|}\right)_+=0,\quad x\in\Omega_j^{-}\setminus B_{Ls_{N,j}^{-}}(z_{j,N}^{-}).$$

Thus,

$$-\Delta u_N=0,\quad \text{in } \Omega\setminus B_{\delta_N,Z_N}.$$

However, $u_N=0$ on $\partial\Omega$ and $u_N=o(1)$ on $\partial B_{\delta_N,Z_N}$. So by maximum principle $u_N=o(1)$. This is a contradiction since $\|u_N\|_\infty=1$.

□

Proposition 3.2. *$Q_\delta L_\delta$ is one to one and onto from $E_{\delta,Z}$ to $F_{\delta,Z}$.*

Proof. Suppose that $Q_\delta L_\delta u = 0$. Then, by Lemma 3.1, $u \equiv 0$. Thus, $Q_\delta L_\delta$ is one to one.

Next, we prove that $Q_\delta L_\delta$ is an onto map from $E_{\delta,Z}$ to $F_{\delta,Z}$. Denote

$$\tilde{E} = \left\{ u: \ u \in H_0^1(\Omega), \int_\Omega D\left(\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\right) Du = 0, \ \int_\Omega D\left(\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,h}}\right) Du = 0, \right.$$
$$\left. i = 1, \cdots, m, \ \ j = 1, \cdots, n, \ \ h = 1, 2 \right\}.$$

Note that $E_{\delta,Z} = \tilde{E} \cap W^{2,p}(\Omega)$. On one hand, for any $\tilde{h} \in F_{\delta,Z}$, by the Riesz representation theorem there is a unique $u \in H_0^1(\Omega)$ such that

$$\delta^2 \int_\Omega DuD\varphi = \int_\Omega \tilde{h}\varphi, \quad \forall\, \varphi \in H_0^1(\Omega). \tag{3.9}$$

On the other hand, from $\tilde{h} \in F_{\delta,Z}$, we find that $u \in \tilde{E}$. Moreover, by the L^p-estimate $u \in W^{2,p}(\Omega)$. So, $u \in E_{\delta,Z}$. Thus, $Q_\delta(-\delta^2\Delta) = -\delta^2\Delta$ is a one to one and onto map from $E_{\delta,Z}$ to $F_{\delta,Z}$. Meanwhile, $Q_\delta L_\delta u = h$ is equivalent to

$$\begin{aligned} u = p\delta^{-2}(-Q_\delta\Delta)^{-1} \Big[Q_\delta \Big(& \textstyle\sum_{i=1}^m \chi_{\Omega_i^+} \Big(P^+_{\delta,Z} - P^-_{\delta,Z} - \kappa_i^+ - \frac{2\pi}{|\ln\varepsilon|} q(x) \Big)_+^{p-1} u \\ & - \textstyle\sum_{j=1}^m \chi_{\Omega_j^-} \Big(\frac{2\pi}{|\ln\varepsilon|} q(x) - \kappa_j^- + P^-_{\delta,Z} - P^+_{\delta,Z} \Big)_+^{p-1} u \Big) \Big] \\ & + \delta^{-2}(-Q_\delta\Delta)^{-1} h, \quad u \in E_{\delta,Z}. \end{aligned} \tag{3.10}$$

It is easy to check that

$$\begin{aligned} \delta^{-2}(-Q_\delta\Delta)^{-1} \Big[Q_\delta \Big(& \textstyle\sum_{i=1}^m \chi_{\Omega_i^+} \Big(P^+_{\delta,Z} - P^-_{\delta,Z} - \kappa_i^+ - \frac{2\pi}{|\ln\varepsilon|} q(x) \Big)_+^{p-1} u \\ & - \textstyle\sum_{j=1}^n \chi_{\Omega_j^-} \Big(\frac{2\pi}{|\ln\varepsilon|} q(x) - \kappa_j^- + P^-_{\delta,Z} - P^+_{\delta,Z} \Big)_+^{p-1} u \Big) \Big] \end{aligned}$$

is a compact operator in $E_{\delta,Z}$. By the Fredholm alternative, (3.10) is solvable if and only if

$$\begin{aligned} u = p\delta^{-2}(-Q_\delta\Delta)^{-1} \Big[Q_\delta \Big(& \textstyle\sum_{i=1}^m \chi_{\Omega_i^+} \Big(P^+_{\delta,Z} - P^-_{\delta,Z} - \kappa_i^+ - \frac{2\pi}{|\ln\varepsilon|} q(x) \Big)_+^{p-1} u \\ & - \textstyle\sum_{j=1}^n \chi_{\Omega_j^-} \Big(\frac{2\pi}{|\ln\varepsilon|} q(x) - \kappa_j^- + P^-_{\delta,Z} - P^+_{\delta,Z} \Big)_+^{p-1} u \Big) \Big] \end{aligned}$$

has only trivial solutions, which is true since $Q_\delta L_\delta$ is one to one. Thus the result follows. □

Now consider the equation

$$Q_\delta L_\delta \omega = Q_\delta l_\delta^+ - Q_\delta l_\delta^- + Q_\delta R_\delta^+(\omega) - Q_\delta R_\delta^-(\omega), \tag{3.11}$$

where

$$l_\delta^+ = \sum_{i=1}^m \chi_{\Omega_i^+}\left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \sum_{i=1}^m \Big(W_{\delta,z_i^+,a_{\delta,i}^+} - a_{\delta,i}^+\Big)_+^p, \tag{3.12}$$

$$l_\delta^- = \sum_{j=1}^n \chi_{\Omega_j^-}\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \sum_{j=1}^n \Big(W_{\delta,z_j^-,a_{\delta,j}^-} - a_{\delta,j}^-\Big)_+^p, \tag{3.13}$$

and

$$\begin{aligned}R_\delta^+(\omega) = &\sum_{i=1}^m \chi_{\Omega_i^+}\Bigg[\left(P_{\delta,Z}^+ - P_{\delta,Z}^- + \omega - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\\ &- p\left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\omega\Bigg],\end{aligned} \tag{3.14}$$

$$\begin{aligned}R_\delta^-(\omega) = &\sum_{j=1}^n \chi_{\Omega_j^-}\Bigg[\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \omega - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\\ &+ p\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\omega\Bigg].\end{aligned} \tag{3.15}$$

Using Proposition 3.2, we can rewrite (3.11) as

$$\omega = G_\delta\omega =: (Q_\delta L_\delta)^{-1}Q_\delta\big(l_\delta^+ - l_\delta^- + R_\delta^+(\omega) - R_\delta^-(\omega)\big). \tag{3.16}$$

The next proposition will enable us to reduce the problem of finding a solution for (1.9) to a finite dimensional problem.

Proposition 3.3. *There is an $\delta_0 > 0$, such that for any $\delta \in (0, \delta_0]$ and Z satisfying* (2.5), (3.11) *has a unique solution $\omega_\delta \in E_{\delta,Z}$, with*

$$\|\omega_\delta\|_\infty = O\Big(\delta|\ln\delta|^{\frac{p-1}{2}}\Big).$$

Proof. It follows from Lemma A.1 that if L is large enough and δ is small then

$$\left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+ = 0, \quad x \in \Omega_i^+ \setminus B_{Ls_{\delta,i}^+}(z_i^+),\ i = 1, \cdots, m,$$

$$\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+ = 0, \quad x \in \Omega_j^- \setminus B_{Ls_{\delta,j}^-}(z_j^-),\ j = 1, \cdots, n.$$

Let

$$M = E_{\delta,Z} \cap \left\{ \|\omega\|_\infty \le \delta |\ln\delta|^{\frac{p-1}{2}} \right\}.$$

Then M is complete under L^∞ norm and G_δ is a map from $E_{\delta,Z}$ to $E_{\delta,Z}$. Next we show that G_δ is indeed a contraction map from M to M by the following two steps.

Step 1. G_δ is a map from M to M.

For any $\omega \in M$, similar to Lemma A.1, it is not difficult to show that for large $L > 0$ and δ small

$$\begin{aligned}
&\left(P^+_{\delta,Z} - P^-_{\delta,Z} + \omega - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|} \right)_+ = 0, \quad x \in \Omega_i^+ \setminus B_{Ls^+_{\delta,i}}(z_i^+),\\
&\left(P^-_{\delta,Z} - P^+_{\delta,Z} - \omega - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|} \right)_+ = 0, \quad x \in \Omega_j^- \setminus B_{Ls^-_{\delta,j}}(z_j^-).
\end{aligned} \tag{3.17}$$

Note also that for any $u \in L^\infty(\Omega)$,

$$Q_\delta u = u \quad \text{in } \Omega \setminus B_{\delta,Z}.$$

Therefore, using Lemma A.1, (3.12)–(3.15), we have for any $\omega \in M$,

$$Q_\delta(l^+_\delta - l^-_\delta) + Q_\delta(R^+_\delta(\omega) - R^-_\delta(\omega)) = l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega) = 0, \quad \text{in } \Omega \setminus B_{Z,\delta}.$$

So, we can apply Lemma 3.1 to obtain

$$\begin{aligned}
&\|(Q_\delta L_\delta)^{-1}\big(Q_\delta(l^+_\delta - l^-_\delta) + Q_\delta(R^+_\delta(\omega) - R^-_\delta(\omega))\big)\|_\infty\\
\le& \frac{C|\ln\delta|^{\frac{(p-1)^2}{p}}}{\delta^{\frac{2}{p}}} \|Q_\delta(l^+_\delta - l^-_\delta) + Q_\delta(R^+_\delta(\omega) - R^-_\delta(\omega))\|_p.
\end{aligned}$$

Thus, for any $\omega \in M$,

$$\begin{aligned}
\|G_\delta(\omega)\|_\infty &= \|(Q_\delta L_\delta)^{-1} Q_\delta\big(l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega)\big)\|_\infty\\
&\le \frac{C|\ln\delta|^{\frac{(p-1)^2}{p}}}{\delta^{\frac{2}{p}}} \|Q_\delta\big(l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega)\big)\|_p.
\end{aligned} \tag{3.18}$$

It follows from (3.3)–(3.6) that the constant $b^\pm_{k,\hat h}$, corresponding to $u \in L^\infty(\Omega)$, satisfies

$$|b^\pm_{k,\hat h}| \le C|\ln\delta|^{p+1} \left(\sum_{i,h} \int_\Omega \left| \frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}} \right| |u| + \sum_{j,\bar h} \int_\Omega \left| \frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,\bar h}} \right| |u| \right).$$

Since

$$l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega) = 0, \quad \text{in } \Omega \setminus B_{\delta,Z},$$

we deduce that the constant $b^\pm_{k,\hat h}$, corresponding to $l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega)$, satisfies

$$
\begin{aligned}
|b^{\pm}_{k,\hat h}| \leq & C|\ln\delta|^{p+1}\sum_{i,h}\left(\sum_{\alpha=1}^{m}\int_{B_{Ls^+_{\delta,\alpha}}(z^+_\alpha)}\Big|\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\Big||l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega)|\right)\\
& + C|\ln\delta|^{p+1}\sum_{j,\bar h}\left(\sum_{\beta=1}^{n}\int_{B_{Ls^-_{\delta,\beta}}(z^-_\beta)}\Big|\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,\bar h}}\Big||l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega)|\right)\\
\leq & C\varepsilon^{1-\frac{2}{p}}|\ln\varepsilon|^p\|l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega)\|_p.
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
& \|Q_\delta(l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega))\|_p\\
\leq & \|l^+_\delta - l^-_\delta + R^+_\delta(\omega) - R^-_\delta(\omega)\|_p + C\sum_{i,h}|b^+_{i,h}|\left\|-\delta^2\Delta\Big(\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\Big)\right\|_p\\
& + C\sum_{j,\bar h}|b^-_{j,\bar h}|\left\|-\delta^2\Delta\Big(\frac{\partial P^-_{\delta,Z,j}}{\partial z^-_{j,\bar h}}\Big)\right\|_p\\
\leq & C\big(\|l^+_\delta\|_p + \|l^-_\delta\|_p + \|R^+_\delta(\omega)\|_p + \|R^-_\delta(\omega)\|_p\big).
\end{aligned}
\tag{3.19}
$$

To estimate $\|G_\delta(\omega)\|_\infty$, by (3.18) it suffices to estimate each term in the right hand side of (3.19). From Lemma A.1 and (2.15), we have

$$
\begin{aligned}
\|l^+_\delta\|_p = & \left\|\sum_{i=1}^{m}\chi_{\Omega^+_i}\left(P^+_{\delta,Z} - P^-_{\delta,Z} - \kappa^+_i - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)^p_+ - \sum_{i=1}^{m}\left(W_{\delta,z^+_i,a^+_{\delta,i}} - a^+_{\delta,i}\right)^p_+\right\|_p\\
\leq & \sum_{i=1}^{m}\frac{Cs^+_{\delta,i}}{|\ln\varepsilon|}\left\|\big(W_{\delta,z^+_i,a^+_{\delta,i}} - a^+_{\delta,i}\big)^{p-1}_+\right\|_p\\
= & O\left(\frac{\delta^{1+\frac{2}{p}}}{|\ln\delta|^{\frac{p-1}{2}+\frac{1}{p}}}\right).
\end{aligned}
$$

For the estimate of $\|R_\delta^+(\omega)\|_p$, we have

$$
\begin{aligned}
\|R_\delta^+(\omega)\|_p =& \Big\| \sum_{i=1}^m \chi_{\Omega_i^+} \Big[\Big(P_{\delta,Z}^+ - P_{\delta,Z}^- + \omega - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|} \Big)_+^p \\
& - \Big(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|} \Big)_+^p \\
& - p\Big(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|} \Big)_+^{p-1} \omega \Big] \Big\|_p \\
\le & C\|\omega\|_\infty^2 \Big\| \sum_{i=1}^m \chi_{\Omega_i^+} \left(P_{\delta,Z}^+ - P_{\delta,Z}^- - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|} \right)_+^{p-2} \Big\|_p \\
=& O\left(\frac{\delta^{\frac{2}{p}}\|\omega\|_\infty^2}{|\ln\delta|^{p-3+\frac{1}{p}}} \right).
\end{aligned} \tag{3.20}
$$

Similarly,

$$
\|l_\delta^-\|_p = O\left(\frac{\delta^{1+\frac{2}{p}}}{|\ln\delta|^{\frac{p-1}{2}+\frac{1}{p}}} \right), \quad \|R_\delta^-(\omega)\|_p = O\left(\frac{\delta^{\frac{2}{p}}\|\omega\|_\infty^2}{|\ln\delta|^{p-3+\frac{1}{p}}} \right).
$$

Thus,

$$
\begin{aligned}
\|G_\delta(\omega)\|_\infty \le& \frac{C|\ln\delta|^{\frac{(p-1)^2}{p}}}{\delta^{\frac{2}{p}}} \Big(\|l_\delta^+\|_p + \|l_\delta^-\|_p + \|R_\delta^+(\omega)\|_p + \|R_\delta^-(\omega)\|_p \Big) \\
\le& C|\ln\delta|^{\frac{(p-1)^2}{p}} \left(\frac{\delta}{|\ln\delta|^{\frac{p-1}{2}+\frac{1}{p}}} + \frac{\|\omega\|_\infty^2}{|\ln\delta|^{p-3+\frac{1}{p}}} \right) \\
\le& \delta|\ln\delta|^{\frac{p-1}{2}}.
\end{aligned} \tag{3.21}
$$

Step 2. G_δ is a contraction map.
In fact, for any $\omega_i \in M$, $i = 1, 2$, we have

$$
G_\delta\omega_1 - G_\delta\omega_2 = (Q_\delta L_\delta)^{-1} Q_\delta \big[R_\delta^+(\omega_1) - R_\delta^+(\omega_2) - (R_\delta^-(\omega_1) - R_\delta^-(\omega_2)) \big].
$$

Noting that

$$
R_\delta^+(\omega_1) = R_\delta^+(\omega_2) = 0, \quad \text{in } \Omega \setminus \cup_{i=1}^m B_{Ls_{\delta,i}^+}(z_i^+),
$$

and

$$
R_\delta^-(\omega_1) = R_\delta^-(\omega_2) = 0, \quad \text{in } \Omega \setminus \cup_{j=1}^n B_{Ls_{\delta,j}^-}(z_j^-),
$$

we can deduce as in Step 1 that

$$\begin{aligned}\|G_\delta\omega_1-G_\delta\omega_2\|_\infty\le&\frac{C|\ln\delta|^{\frac{(p-1)^2}{p}}}{\delta^{\frac{2}{p}}}(\|R_\delta^+(\omega_1)-R_\delta^+(\omega_2)\|_p+\|R_\delta^-(\omega_1)-R_\delta^-(\omega_2)\|_p)\\\le&C|\ln\delta|^{p-1}\left(\frac{\|\omega_1\|_\infty}{|\ln\delta|^{p-2}}+\frac{\|\omega_2\|_\infty}{|\ln\delta|^{p-2}}\right)\|\omega_1-\omega_2\|_\infty\\\le&C\delta|\ln\delta|^{\frac{p+1}{2}}\|\omega_1-\omega_2\|_\infty\le\frac{1}{2}\|\omega_1-\omega_2\|_\infty.\end{aligned}$$

By Step 1 and Step 2, G_δ is a contraction map from M to M and thus there is a unique $\omega_\delta\in M$ such that $\omega_\delta=G_\delta\omega_\delta$. Moreover, from (3.21), $\|\omega_\delta\|_\infty\le\delta|\ln\delta|^{\frac{p-1}{2}}$.

□

4. Proof of The main results

In this section, we will give proofs for our main results Theorem 1.1 and Theorem 1.7. As the first step, we need to show Theorem 1.4. First we will choose Z, such that $P_{\delta,Z}^+-P_{\delta,Z}^-+\omega_\delta$ is a solution of (1.9), where ω_δ is the map obtained in Proposition 3.3.

Define

$$\begin{aligned}I(u)=\frac{\delta^2}{2}\int_\Omega|Du|^2-&\sum_{i=1}^m\frac{1}{p+1}\int_\Omega\chi_{\Omega_i^+}\left(u-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p+1}\\&-\sum_{j=1}^n\frac{1}{p+1}\int_\Omega\chi_{\Omega_j^-}\left(\frac{2\pi q(x)}{|\ln\varepsilon|}-\kappa_j^--u\right)_+^{p+1},\end{aligned}$$

and set

$$K(Z)=I\left(P_{\delta,Z}^+-P_{\delta,Z}^-+\omega_\delta\right).\tag{4.1}$$

It is well-known that if Z is a critical point of $K(Z)$, then $P_{\delta,Z}^+-P_{\delta,Z}^-+\omega_\delta$ is a solution of (1.9). We will prove that $K(Z)$ has a critical point. To do this we need two preliminary lemmas, which together with estimates in the Appendix, give expansions of $K(Z)$, $\frac{\partial K(Z)}{\partial z_{i,h}^+}$ and $\frac{\partial K(Z)}{\partial z_{j,\bar h}^-}$ respectively.

Lemma 4.1. *We have*

$$K(Z)=I\left(P_{\delta,Z}^+-P_{\delta,Z}^-\right)+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\right).$$

Proof. Recalling that $P^+_{\delta,Z}=\sum_{i=1}^m P^+_{\delta,Z,i}$ and $P^-_{\delta,Z}=\sum_{j=1}^n P^-_{\delta,Z,j}$, we have

$$
\begin{aligned}
K(Z)=I\big(P^+_{\delta,Z}-P^-_{\delta,Z}\big)&+\delta^2\int_\Omega D\big(P^+_{\delta,Z}-P^-_{\delta,Z}\big)D\omega_\delta+\frac{\delta^2}{2}\int_\Omega|D\omega_\delta|^2\\
&-\sum_{i=1}^m\frac{1}{p+1}\int_\Omega\chi_{\Omega_i^+}\Bigg[\Big(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^{p+1}\\
&-\Big(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^{p+1}\Bigg]\\
&-\sum_{j=1}^n\frac{1}{p+1}\int_\Omega\chi_{\Omega_j^-}\Bigg[\Big(P^-_{\delta,Z}-P^+_{\delta,Z}-\omega_\delta-\kappa_j^-+\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^{p+1}\\
&-\Big(P^-_{\delta,Z}-P^+_{\delta,Z}-\kappa_j^-+\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^{p+1}\Bigg].
\end{aligned}
$$

Using Proposition 3.3 and (3.17), we find

$$
\begin{aligned}
&\int_{\Omega_i^+}\Bigg[\Big(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^{p+1}-\Big(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^{p+1}\Bigg]\\
=&\int_{B_{Ls^+_{\delta,i}}(z_i^+)}\Bigg[\Big(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^{p+1}-\Big(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\Big)_+^{p+1}\Bigg]\\
=&O\left(\frac{(s^+_{\delta,i})^2\|\omega_\delta\|_\infty}{|\ln\varepsilon|^p}\right)=O\Big(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\Big).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\delta^2\int_\Omega DP^+_{\delta,Z}D\omega_\delta&=\sum_{i=1}^m\int_\Omega\Big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\Big)_+^p\omega_\delta\\
=&\sum_{i=1}^m\int_{B_{s^+_{\delta,k}}(z_k^+)}(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i})_+^p\omega_\delta=O\Big(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\Big).
\end{aligned}
$$

Next, we estimate $\delta^2\int_\Omega |D\omega_\delta|^2$. Note that

$$\begin{aligned}-\delta^2\Delta\omega_\delta =&\sum_{i=1}^m \chi_{\Omega_i^+}\left(P_{\delta,Z}^+ - P_{\delta,Z}^- + \omega_\delta - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \sum_{i=1}^m \left(W_{\delta,z_i^+,a_{\delta,i}^+} - a_{\delta,i}^+\right)_+^p\\
&- \sum_{j=1}^n \chi_{\Omega_j^-}\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \omega_\delta - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p + \sum_{j=1}^n \left(W_{\delta,z_j^-,a_{\delta,j}^-} - a_{\delta,j}^-\right)_+^p\\
&+ \sum_{i=1}^m\sum_{h=1}^2 b_{i,h}^+\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,i}^+}{\partial z_{i,h}^+}\right) + \sum_{j=1}^n\sum_{\bar h=1}^2 b_{j,\bar h}^-\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,j}^-}{\partial z_{j,\bar h}^-}\right).\end{aligned}$$

Hence, by (2.15)–(2.16), we have

$$\begin{aligned}\delta^2\int_\Omega |D\omega_\delta|^2 =&\sum_{i=1}^m\int_{\Omega_i^+}\left[\left(P_{\delta,Z}^+ - P_{\delta,Z}^- + \omega_\delta - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \left(W_{\delta,z_i^+,a_{\delta,i}^+} - a_{\delta,i}^+\right)_+^p\right]\omega_\delta\\
&- \sum_{j=1}^n\int_{\Omega_j^-}\left[\left(P_{\delta,Z}^- - P_{\delta,Z}^+ - \omega_\delta - \kappa_j^- + \frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p - \left(W_{\delta,z_j^-,a_{\delta,j}^-} - a_{\delta,j}^-\right)_+^p\right]\omega_\delta\\
&+ \sum_{i=1}^m\sum_{h=1}^2 b_{i,h}^+\int_\Omega\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,i}^+}{\partial z_{i,h}^+}\right)\omega_\delta + \sum_{j=1}^n\sum_{\bar h=1}^2 b_{j,\bar h}^-\int_\Omega\left(-\delta^2\Delta\frac{\partial P_{\delta,Z,j}^-}{\partial z_{j,\bar h}^-}\right)\omega_\delta\\
=&p\sum_{i=1}^m\int_{\Omega_i^+}\left(W_{\delta,z_i^+,a_{\delta,i}^+} - a_{\delta,i}^+\right)_+^{p-1}\left(\frac{s_{\delta,i}^+}{|\ln\varepsilon|} + \omega_\delta\right)\omega_\delta + O\left(\sum_{i=1}^m\sum_{h=1}^2\frac{\varepsilon|b_{i,h}^+|\|\omega_\delta\|_\infty}{|\ln\varepsilon|^p}\right)\\
&- p\sum_{j=1}^n\int_{\Omega_j^-}\left(W_{\delta,z_j^-,a_{\delta,j}^-} - a_{\delta,j}^-\right)_+^{p-1}\left(\frac{s_{\delta,j}^-}{|\ln\varepsilon|} + \omega_\delta\right)\omega_\delta\\
&+ O\left(\sum_{j=1}^n\sum_{\bar h=1}^2\frac{\varepsilon|b_{j,\bar h}^-|\|\omega_\delta\|_\infty}{|\ln\varepsilon|^p}\right)\\
=&O\left(\frac{\varepsilon^4}{|\ln\varepsilon|^{p-1}}\right).\end{aligned}$$

Other terms can be estimated as above. So our assertion follows.

□

Lemma 4.2. *We have*

$$\frac{\partial K(Z)}{\partial z_{i,h}^+} = \frac{\partial}{\partial z_{i,h}^+} I\left(P_{\delta,Z}^+ - P_{\delta,Z}^-\right) + O\Big(\frac{\varepsilon^3}{|\ln\varepsilon|^{p-1}}\Big),\ i = 1,\cdots,m,$$

$$\frac{\partial K(Z)}{\partial z_{j,\bar h}^-} = \frac{\partial}{\partial z_{j,\bar h}^-} I\left(P_{\delta,Z}^+ - P_{\delta,Z}^-\right) + O\Big(\frac{\varepsilon^3}{|\ln\varepsilon|^{p-1}}\Big),\ j = 1,\cdots,n.$$

Proof. We give her the proof of the first one only. The second one can be proved similarly.

By the definition, we have

$$
\begin{aligned}
\frac{\partial K(Z)}{\partial z^+_{i,h}} =& \left\langle I'\Big(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta\Big), \frac{\partial P^+_{\delta,Z}}{\partial z^+_{i,h}}-\frac{\partial P^-_{\delta,Z}}{\partial z^+_{i,h}}+\frac{\partial \omega_\delta}{\partial z^+_{i,h}}\right\rangle\\
=&\frac{\partial}{\partial z^+_{i,h}} I\Big(P^+_{\delta,Z}-P^-_{\delta,Z}\Big)+\left\langle I'\big(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta\big), \frac{\partial \omega_\delta}{\partial z^+_{i,h}}\right\rangle+\delta^2\int_\Omega D\omega_\delta\, D\left(\frac{\partial P^+_{\delta,Z}}{\partial z^+_{i,h}}-\frac{\partial P^-_{\delta,Z}}{\partial z^+_{i,h}}\right)\\
&-\sum_{k=1}^m\int_{\Omega^+_k}\left[\left(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa^+_k-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)^p_+-\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa^+_k-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)^p_+\right]\\
&\quad\times\left(\frac{\partial P^+_{\delta,Z}}{\partial z^+_{i,h}}-\frac{\partial P^-_{\delta,Z}}{\partial z^+_{i,h}}\right)\\
&-\sum_{l=1}^n\int_{\Omega^-_l}\left[\left(P^-_{\delta,Z}-P^+_{\delta,Z}-\omega_\delta-\kappa^-_l+\frac{2\pi q(x)}{|\ln\varepsilon|}\right)^p_+-\left(P^-_{\delta,Z}-P^+_{\delta,Z}-\kappa^-_l+\frac{2\pi q(x)}{|\ln\varepsilon|}\right)^p_+\right]\\
&\quad\times\left(\frac{\partial P^-_{\delta,Z}}{\partial z^+_{i,h}}-\frac{\partial P^+_{\delta,Z}}{\partial z^+_{i,h}}\right).
\end{aligned}
$$

Since $\omega_\delta\in E_{\delta,Z}$, we have

$$
\int_\Omega\Big(W_{\delta,z^\pm_k,a^\pm_{\delta,k}}-a^\pm_{\delta,k}\Big)^{p-1}_+\left(\frac{\partial W_{\delta,z^\pm_k,a^\pm_{\delta,k}}}{\partial z^\pm_{k,h}}-\frac{\partial a^\pm_{\delta,k}}{\partial z^\pm_{k,h}}\right)\omega_\delta=0.
$$

Differentiating the above relation with respect to $z^+_{i,h}$, we deduce

$$
\begin{aligned}
&\left\langle I'\big(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta\big), \frac{\partial \omega_\delta}{\partial z^+_{i,h}}\right\rangle\\
=&\sum_{\alpha=1}^m\sum_{\hat h=1}^2 b^+_{\alpha,\hat h}\int_\Omega\left(-\delta^2\Delta\frac{\partial P^+_{\delta,Z,\alpha}}{\partial z^+_{\alpha,\hat h}}\right)\frac{\partial\omega_\delta}{\partial z^+_{i,h}}+\sum_{\beta=1}^n\sum_{\tilde h=1}^2 b^-_{\beta,\tilde h}\int_\Omega\left(-\delta^2\Delta\frac{\partial P^-_{\delta,Z,\beta}}{\partial z^-_{\beta,\tilde h}}\right)\frac{\partial\omega_\delta}{\partial z^+_{i,h}}\\
=&\sum_{\alpha=1}^m\sum_{\hat h=1}^2 pb^+_{\alpha,\hat h}\int_\Omega\Big(W_{\delta,z^+_\alpha,a^+_{\delta,\alpha}}-a^+_{\delta,\alpha}\Big)^{p-1}_+\left(\frac{\partial W_{\delta,z^+_\alpha,a^+_{\delta,\alpha}}}{\partial z^+_{\alpha,\hat h}}-\frac{\partial a^+_{\delta,\alpha}}{\partial z^+_{\alpha,\hat h}}\right)\frac{\partial\omega_\delta}{\partial z^+_{i,h}}\\
&+\sum_{\beta=1}^n\sum_{\tilde h=1}^2 pb^-_{\beta,\tilde h}\int_\Omega\Big(W_{\delta,z^-_\beta,a^-_{\delta,\beta}}-a^-_{\delta,\beta}\Big)^{p-1}_+\left(\frac{\partial W_{\delta,z^-_\beta,a^-_{\delta,\beta}}}{\partial z^-_{\beta,\tilde h}}-\frac{\partial a^-_{\delta,\beta}}{\partial z^-_{\beta,\tilde h}}\right)\frac{\partial\omega_\delta}{\partial z^+_{i,h}}\\
=&O\left(\sum_{\alpha=1}^m\sum_{\hat h=1}^2\frac{\varepsilon|b^+_{\alpha,\hat h}|}{|\ln\varepsilon|^p}+\sum_{\beta=1}^n\sum_{\tilde h=1}^2\frac{\varepsilon|b^-_{\beta,\tilde h}|}{|\ln\varepsilon|^p}\right)=O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p-1}}\right).
\end{aligned}
$$

On the other hand, using (3.20) (for the definition of $R^+_\delta(\omega)$, see (3.14)), we obtain

$$\begin{aligned}
&\sum_{k=1}^m \int_{\Omega_k^+}\left[\left(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p-\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right]\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\\
=&\sum_{k=1}^m \int_{\Omega_k^+}\left[\left(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p-\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right.\\
&\left.-p\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\omega_\delta\right]\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\\
&+\sum_{k=1}^m p\int_{\Omega_k^+}\left[\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}-\left(W_{\delta,z_k^+,a^+_{\delta,k}}-a^+_{\delta,k}\right)_+^{p-1}\right]\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\omega_\delta\\
&+O\left(\frac{(s^+_{\delta,k})^2\|\omega_\delta\|_\infty}{|\ln\varepsilon|^p}\right)\\
=&\int_\Omega R^+_\delta(\omega_\delta)\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}+\sum_{k=1}^m p\int_{\Omega_k^+}\left[\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\right.\\
&\left.-\left(W_{\delta,z_k^+,a^+_{\delta,k}}-a^+_{\delta,k}\right)_+^{p-1}\right]\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\omega_\delta+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\right)\\
=&O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p-1}}\right).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
&\int_{\Omega_l^+}\left[\left(P^+_{\delta,Z}-P^-_{\delta,Z}+\omega_\delta-\kappa_l^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p-\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_l^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right]\frac{\partial P^-_{\delta,Z,i}}{\partial z^-_{i,h}}\\
&=p\int_{\Omega_l^+}\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_l^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^{p-1}\frac{\partial P^-_{\delta,Z,i}}{\partial z^-_{i,h}}\omega_\delta+O\left(\frac{\varepsilon^4}{|\ln\varepsilon|^{p-1}}\right)\\
&=O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^p}\right).
\end{aligned}$$

Other teams can be estimated as above. Thus, the estimate follows. □

Proof of Theorem 1.4. Recall that $Z=(Z_m^+,Z_n^-)$. Set

$$\begin{aligned}\Phi(Z_m^+,Z_n^-)=&\sum_{i=1}^m 4\pi^2\kappa_i^+ q(z_i^+)-\sum_{j=1}^n 4\pi^2\kappa_j^- q(z_j^-)+\sum_{i=1}^m \pi(\kappa_i^+)^2 g(z_i^+,z_i^+)\\&+\sum_{j=1}^n \pi(\kappa_j^-)^2 g(z_j^-,z_j^-)-\sum_{i\neq k}\pi\kappa_i^+\kappa_k^+\bar{G}(z_i^+,z_k^+)-\sum_{j\neq l}\pi\kappa_j^-\kappa_l^-\bar{G}(z_l^-,z_j^-)\\&+\sum_{i=1}^m\sum_{j=1}^n 2\pi\kappa_i^+\kappa_j^-\bar{G}(z_i^+,z_j^-).\end{aligned}$$

Note that the Kirchhoff–Routh function associated to the vortex dynamics now is

$$\begin{aligned}\mathcal{W}(Z_m^+,Z_n^-)=&\frac{1}{2}\sum_{i,k=1,i\neq k}^m \kappa_i^+\kappa_k^+ G(z_i^+,z_k^+)+\frac{1}{2}\sum_{j,l=1,j\neq l}^n \kappa_j^-\kappa_l^- G(z_j^-,z_l^-)\\&+\frac{1}{2}\sum_{i=1}^m(\kappa_i^+)^2 H(z_i^+,z_i^+)+\frac{1}{2}\sum_{j=1}^n(\kappa_j^-)^2 H(z_j^-,z_j^-)\\&-\sum_{i=1}^m\sum_{j=1}^n \kappa_i^+\kappa_j^- G(z_i^+,z_j^-)+\sum_{i=1}^m\kappa_i^+\psi_0(z_i^+)-\sum_{j=1}^n\kappa_j^-\psi_0(z_j^-).\end{aligned}$$

Recalling that $h(z_i,z_j)=-H(z_i,z_j)$, we get

$$\Phi(Z_m^+,Z_n^-)=-4\pi^2\mathcal{W}(Z_m^+,Z_n^-)+\pi\ln R\left(\sum_{i=1}^m(\kappa_i^+)^2+\sum_{j=1}^n(\kappa_j^-)^2\right).$$

Hence, $\Phi(Z_m^+,Z_n^-)$ and $\mathcal{W}(Z_m^+,Z_n^-)$ possess the same critical points.

By Lemma 4.1, 4.2 and Proposition A.2, A.3, we have

$$K(Z)=\frac{C\delta^2}{\ln\frac{R}{\varepsilon}}+\frac{\pi(p-1)\delta^2}{4(\ln\frac{R}{\varepsilon})^2}\left(\sum_{i=1}^m(\kappa_i^+)^2+\sum_{j=1}^n(\kappa_j^-)^2\right)+\frac{\delta^2}{|\ln\varepsilon|^2}\Phi(Z)+O\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right),\quad(4.2)$$

and

$$\frac{\partial K(Z)}{\partial z_{i,h}^\pm}=-\frac{4\pi^2\delta^2}{|\ln\varepsilon|^2}\frac{\partial\mathcal{W}(Z)}{\partial z_{i,h}^\pm}+O\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right).\quad(4.3)$$

Thus, suppose that $(x_{1,*}^+,\cdots,x_{m,*}^+,\,x_{1,*}^-,\cdots,x_{n,*}^-)$ is a C^1-stable critical point of Kirchhoff-Routh function $\mathcal{W}(Z)$, then $K(Z)$ has a critical point $(x_{1,\varepsilon}^+,\cdots,x_{m,\varepsilon}^+,\,x_{1,\varepsilon}^-,\cdots,x_{n,\varepsilon}^-)=(x_{1,*}^+,\cdots,x_{m,*}^+,\,x_{1,*}^-,\cdots,x_{n,*}^-)+o(1)$.

Thus we get a solution w_δ for (1.9). Let $u_\varepsilon=\frac{|\ln\varepsilon|}{2\pi}w_\delta$, $\delta=\varepsilon\left(\frac{|\ln\varepsilon|}{2\pi}\right)^{\frac{1-p}{2}}$, it is not difficult to check that u_ε has all the properties listed in Theorem 1.4 and thus the proof of Theorem 1.4 is complete. □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 1.4, we obtain that u_ε is a solution to (1.8).

Set

$$\begin{aligned}
&\mathbf{v}_\varepsilon = \mathbf{J}\nabla(u_\varepsilon - q), \qquad \omega_\varepsilon = \nabla\times\mathbf{v}_\varepsilon,\\
P_\varepsilon =& \sum_{i=1}^m \frac{1}{p+1}\chi_{\Omega_i^+}\left(u_\varepsilon - q - \frac{\kappa_i^+|\ln\varepsilon|}{2\pi}\right)_+^{p+1}\\
&+\sum_{j=1}^n \frac{1}{p+1}\chi_{\Omega_j^-}\left(q - \frac{\kappa_j^-|\ln\varepsilon|}{2\pi} - u_\varepsilon\right)_+^{p+1} - \frac{1}{2}|\nabla(u_\varepsilon - q)|^2.
\end{aligned}$$

Then $(\mathbf{v}_\varepsilon, P_\varepsilon)$ forms a stationary solution for problem (1.1).

From our proof of Theorem 1.4, we see that as $\varepsilon\to 0$

$$(x_{1,\varepsilon}^+,\cdots,x_{m,\varepsilon}^+,x_{1,\varepsilon}^-,\cdots,x_{n,\varepsilon}^-)\to(x_{1,*}^+,\cdots,x_{m,*}^+,x_{1,*}^-,\cdots,x_{n,*}^-).$$

So we can find a positive constant C independent of ε so small that for small ε, $B(x_{i,\varepsilon}^+, C\varepsilon)\subseteq\Omega_i^+$, $B(x_{j,\varepsilon}^-, C\varepsilon)\subseteq\Omega_j^-$ for $i=1,\cdots,m$ and $j=1,\cdots,n$.

We now verify that as $\varepsilon\to 0$

$$\int_{B(x_{i,\varepsilon}^+,C\varepsilon)}\omega_{i,\varepsilon}^+\to\kappa_i^+,\ i=1,\cdots,m, \tag{4.4}$$

$$\int_{B(x_{j,\varepsilon}^-,C\varepsilon)}\omega_{i,\varepsilon}^-\to-\kappa_j^-,\ j=1,\cdots,n, \tag{4.5}$$

and

$$\int_\Omega\omega_\varepsilon\to\sum_{j=1}^m\kappa_j^+ - \sum_{j=1}^n\kappa_j^-. \tag{4.6}$$

By direct calculations, we have

$$\begin{aligned}
\int_{B(x_{i,\varepsilon}^+,C\varepsilon)}\omega_{i,\varepsilon}^+ &= \frac{1}{\varepsilon^2}\int_\Omega\chi_{\Omega_i^+}\left(u_\varepsilon - q - \frac{\kappa_i^+|\ln\varepsilon|}{2\pi}\right)_+^p\\
&=\frac{|\ln\varepsilon|^p}{(2\pi)^p\varepsilon^2}\int_{\Omega_i^+}\left(w_\delta - \kappa_i^+ - \frac{2\pi q}{|\ln\varepsilon|}\right)_+^p\\
&=\frac{|\ln\varepsilon|^p}{(2\pi)^p\varepsilon^2}\int_{B_{Ls_{\delta,i}^+}(z_i^+)}\left(W_{\delta,z_i^+,a_{\delta,i}^+} - a_{\delta,i}^+ + O\Big(\frac{s_{\delta,i}^+}{|\ln\varepsilon|}\Big)\right)_+^p\\
&=\frac{(s_{\delta,i}^+)^2|\ln\varepsilon|^p}{(2\pi)^p\varepsilon^2}\left(\frac{\delta}{s_{\delta,i}^+}\right)^{\frac{2p}{p-1}}\int_{B_1(0)}\phi^p\\
&=\frac{a_{\delta,i}^+|\ln\varepsilon|}{\ln\frac{R}{s_{\delta,i}^+}} + o(1)\\
&\to\kappa_i^+\quad\text{as }\varepsilon\to 0.
\end{aligned}$$

Therefore, (4.4) follows. (4.5) can be proved exactly the same way. (4.6) follows directly from (4.4) and (4.5). Therefore we complete our proof. □

Proof of Theorem 1.7. Theorem 1.7 can be proved simply by exactly the same arguments above and taking $\kappa_j^- = 0$, $\Omega_j^- = \emptyset$ and $\chi_{\Omega_j^-} \equiv 0$ for $j = 1, \cdots, n$.

□

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Appendix A. Energy expansion

In this appendix, we give precise expansions of $I\left(P_{\delta,Z}^+ - P_{\delta,Z}^-\right)$ and $\frac{\partial}{\partial z_{i,h}^\pm} I\left(P_{\delta,Z}^+ - P_{\delta,Z}^-\right)$, which have been used in section 3 and section 4.

We always assume that $z_i^+, z_j^- \in \Omega$ satisfy

$$d(z_i^+, \partial\Omega) \geq \varrho,\ d(z_j^-, \partial\Omega) \geq \varrho, \quad |z_i^+ - z_k^+| \geq \varrho^{\bar{L}}, \quad i, k = 1, \cdots, m,\ i \neq k$$
$$|z_j^- - z_l^-| \geq \varrho^{\bar{L}}, \quad |z_i^+ - z_j^-| \geq \varrho^{\bar{L}}, \quad j, l = 1, \cdots, n,\ j \neq l,$$

where $\varrho > 0$ is a fixed small constant and $\bar{L} > 0$ is a fixed large constant.

Lemma A.1. *For $x \in \Omega_i^+ (i = 1, \cdots, m)$ and $x \in \Omega_j^- (j = 1, \cdots, n)$, we have when $\varepsilon > 0$ small*

$$P_{\delta,Z}^+(x) - P_{\delta,Z}^-(x) > \kappa_i^+ + \frac{2\pi q(x)}{|\ln\varepsilon|}, \quad x \in B_{s_{\delta,i}^+(1-Ts_{\delta,i}^+)}(z_i^+),$$
$$P_{\delta,Z}^-(x) - P_{\delta,Z}^+(x) > \kappa_j^- - \frac{2\pi q(x)}{|\ln\varepsilon|}, \quad x \in B_{s_{\delta,j}^-(1-Ts_{\delta,j}^-)}(z_j^-),$$

where $T > 0$ is a large constant; while

$$P_{\delta,Z}^+(x) - P_{\delta,Z}^-(x) < \kappa_i^+ + \frac{2\pi q(x)}{|\ln\varepsilon|}, \quad x \in \Omega_i^+ \setminus B_{s_{\delta,i}^+(1+(s_{\delta,i}^+)^\sigma)}(z_i^+),$$
$$P_{\delta,Z}^-(x) - P_{\delta,Z}^+(x) < \kappa_j^- - \frac{2\pi q(x)}{|\ln\varepsilon|}, \quad x \in \Omega_j^- \setminus B_{s_{\delta,j}^-(1+(s_{\delta,j}^-)^\sigma)}(z_j^-),$$

where $\sigma > 0$ is a small constant.

Proof. Suppose that $x \in B_{s_{\delta,i}^+(1-Ts_{\delta,i}^+)}(z_i^+)$. It follows from (2.15) and $\phi_1'(s) < 0$ that

$$\begin{aligned}
P_{\delta,Z}^+(x) - P_{\delta,Z}^-(x) - \kappa_i^+ - \frac{2\pi q(x)}{|\ln\varepsilon|} &= W_{\delta,z_i^+,a_{\delta,i}^+}(x) - a_{\delta,i}^+ + O\left(\frac{s_{\delta,i}^+}{|\ln\varepsilon|}\right) \\
=&\frac{a_{\delta,i}^+}{|\phi'(1)||\ln\frac{R}{s_{\delta,i}^+}|}\phi\Big(\frac{|x - z_i^+|}{s_{\delta,i}^+}\Big) + O\Big(\frac{\varepsilon}{|\ln\varepsilon|}\Big) > 0,
\end{aligned}$$

if $T>0$ is large. On the other hand, if $x\in\Omega_i^+\setminus B_{(s_{\delta,i}^+)^{\tilde\sigma}}(z_i^+)$, where $\tilde\sigma>\sigma>0$ is a fixed small constant, then

$$\begin{aligned}&P_{\delta,Z}^+(x)-P_{\delta,Z}^-(x)-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\\ \le&\sum_{i=1}^m a_{\delta,i}^+\ln\frac{R}{|x-z_i^+|}/\ln\frac{R}{s_{\delta,i}^+}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}+o(1)\\ \le& C\tilde\sigma-\kappa_i^++o(1)<0.\end{aligned}$$

Finally, if $x\in B_{(s_{\delta,i}^+)^{\tilde\sigma}}(z_i^+)\setminus B_{s_{\delta,i}^+(1+T(s_{\delta,i}^+)^{\tilde\sigma})}(z_i^+)$ for some i and if $T>0$ is large then

$$\begin{aligned}&P_{\delta,Z}^+(x)-P_{\delta,Z}^-(x)-\kappa_i^+-\tfrac{2\pi}{|\ln\varepsilon|}q(x)\\ &=W_{\delta,z_i^+,a_{\delta,i}^+}(x)-a_{\delta,i}^++O\left(\frac{(s_{\delta,i}^+)^{\tilde\sigma}}{\ln\frac{R}{s_{\delta,i}^+}}\right)\\ &=a_{\delta,i}^+\frac{\ln\frac{R}{|x-z_i^+|}}{\ln\frac{R}{s_{\delta,i}^+}}-a_{\delta,i}^++O\left(\frac{(s_{\delta,i}^+)^{\tilde\sigma}}{\ln\frac{R}{s_{\delta,i}^+}}\right)\\ &\le -a_{\delta,i}^+\frac{\ln(1+T(s_{\delta,i}^+)^{\tilde\sigma})}{\ln\frac{R}{s_{\delta,i}^+}}+O\left(\frac{(s_{\delta,i}^+)^{\tilde\sigma}}{\ln\frac{R}{s_{\delta,i}^+}}\right)\\ &<0.\end{aligned}$$

Note that by the choice of $\tilde\sigma$, $B_{s_{\delta,i}^+(1+(s_{\delta,i}^+)^{\sigma})}(z_i^+)\supset B_{s_{\delta,i}^+(1+T(s_{\delta,i}^+)^{\tilde\sigma})}(z_i^+)$ for small δ. We therefore derive our conclusion. □

Proposition A.2. *We have when $\varepsilon>0$ small*

$$\begin{aligned}I\left(P_{\delta,Z}^+-P_{\delta,Z}^-\right)=&\frac{C\delta^2}{\ln\frac{R}{\varepsilon}}+\frac{\pi(p-1)\delta^2}{4(\ln\frac{R}{\varepsilon})^2}\left(\sum_{i=1}^m(\kappa_i^+)^2+\sum_{j=1}^n(\kappa_j^-)^2\right)+\sum_{i=1}^m\frac{4\pi^2\delta^2\kappa_i^+q(z_i^+)}{|\ln\varepsilon||\ln\frac{R}{\varepsilon}|}\\ &-\sum_{j=1}^n\frac{4\pi^2\delta^2\kappa_j^-q(z_j^-)}{|\ln\varepsilon||\ln\frac{R}{\varepsilon}|}+\sum_{i=1}^m\frac{\pi\delta^2(\kappa_i^+)^2g(z_i^+,z_i^+)}{(\ln\frac{R}{\varepsilon})^2}+\sum_{j=1}^n\frac{\pi\delta^2(\kappa_j^-)^2g(z_j^-,z_j^-)}{(\ln\frac{R}{\varepsilon})^2}\\ &-\sum_{k\ne i}^m\frac{\pi\delta^2\kappa_i^+\kappa_k^+\bar G(z_k^+,z_i^+)}{(\ln\frac{R}{\varepsilon})^2}-\sum_{l\ne j}^n\frac{\pi\delta^2\kappa_j^-\kappa_l^-\bar G(z_l^-,z_j^-)}{(\ln\frac{R}{\varepsilon})^2}\\ &+\sum_{i=1}^m\sum_{j=1}^n\frac{2\pi\delta^2\kappa_i^+\kappa_j^-\bar G(z_i^+,z_j^-)}{(\ln\frac{R}{\varepsilon})^2}+O\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right),\end{aligned}$$

where C is a positive constant.

Proof. Taking advantage of (2.4), we have

$$\delta^2\int_\Omega \left|D(P^+_{\delta,Z}-P^-_{\delta,Z})\right|^2=\sum_{k=1}^m\sum_{i=1}^m\int_\Omega \big(W_{\delta,z_k^+,a^+_{\delta,k}}-a^+_{\delta,k}\big)^p_+P^+_{\delta,Z,i}$$
$$+\sum_{l=1}^n\sum_{j=1}^n\int_\Omega \big(W_{\delta,z_l^-,a^-_{\delta,l}}-a^-_{\delta,l}\big)^p_+P^-_{\delta,Z,j}-2\sum_{j=1}^n\sum_{i=1}^m\int_\Omega \big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\big)^p_+P^-_{\delta,Z,j}.$$

First, we estimate

$$\begin{aligned}
&\int_{B_{s^+_{\delta,i}}(z_i^+)}\Big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\Big)^p_+\left(W_{\delta,z_i^+,a^+_{\delta,i}}-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(x,z_i^+)\right)\\
=&\int_{B_{s^+_{\delta,i}}(z_i^+)}\big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\big)^{p+1}+a^+_{\delta,i}\int_{B_{s^+_{\delta,i}}(z_i^+)}\big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\big)^p\\
&-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\int_{B_{s^+_{\delta,i}}(z_i)}\big(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\big)^pg(x,z_i^+)\\
=&\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2(p+1)}{p-1}}(s^+_{\delta,i})^2\int_{B_1(0)}\phi^{p+1}+a^+_{\delta,i}\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2p}{p-1}}(s^+_{\delta,i})^2\int_{B_1(0)}\phi^p\\
&-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2p}{p-1}}g(z_i^+,z_i^+)(s^+_{\delta,i})^2\int_{B_1(0)}\phi^p+O\left(\frac{(s^+_{\delta,i})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\frac{\pi(p+1)}{2}\frac{\delta^2(a^+_{\delta,i})^2}{(\ln\frac{R}{s^+_{\delta,i}})^2}+\frac{2\pi\delta^2(a^+_{\delta,i})^2}{\ln\frac{R}{s^+_{\delta,i}}}-\frac{2\pi\delta^2(a^+_{\delta,i})^2}{(\ln\frac{R}{s^+_{\delta,i}})^2}g(z_i^+,z_i^+)+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}$$

Next, for $k\neq i$,

$$\begin{aligned}
&\int_{B_{s^+_{\delta,k}}(z_k^+)}\big(W_{\delta,z_k^+,a^+_{\delta,k}}-a^+_{\delta,k}\big)^p_+\left(W_{\delta,z_i^+,a^+_{\delta,i}}-\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}g(x,z_i^+)\right)\\
=&\Big(\frac{\delta}{s^+_{\delta,k}}\Big)^{\frac{2p}{p-1}}\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\int_{B_{s^+_{\delta,k}}(z_k^+)}\phi^p\Big(\frac{|x-z_k^+|}{s^+_{\delta,k}}\Big)\bar G(x,z_i^+)\\
=&\Big(\frac{\delta}{s^+_{\delta,k}}\Big)^{\frac{2p}{p-1}}\frac{a^+_{\delta,i}(s^+_{\delta,k})^2}{\ln\frac{R}{s^+_{\delta,i}}}\bar G(z_k^+,z_i^+)\int_{B_1(0)}\phi^p+O\left(\frac{(s^+_{\delta,k})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\frac{2\pi\delta^2a^+_{\delta,i}a^+_{\delta,k}}{|\ln\frac{R}{s^+_{\delta,i}}||\ln\frac{R}{s^+_{\delta,k}}|}\bar G(z_i^+,z_k^+)+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\int_{B_{s^+_{\delta,i}}(z^+_i)} \left(W_{\delta,z^+_i,a^+_{\delta,i}} - a^+_{\delta,i}\right)^p_+ \left(W_{\delta,z^-_j,a^-_{\delta,j}} - \frac{a^-_{\delta,j}}{\ln \frac{R}{s^-_{\delta,j}}} g(x,z^-_j)\right)\\
=&\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2p}{p-1}} \frac{a^-_{\delta,j}}{\ln \frac{R}{s^-_{\delta,j}}} \int_{B_{s^+_{\delta,i}}(z^+_i)} \phi^p\Big(\frac{|x-z^+_i|}{s^+_{\delta,i}}\Big)\bar{G}(x,z^-_j)\\
=&\Big(\frac{\delta}{s^+_{\delta,i}}\Big)^{\frac{2p}{p-1}} \frac{a^-_{\delta,j}(s^+_{\delta,i})^2}{\ln \frac{R}{s^-_{\delta,j}}} \bar{G}(z^-_j,z^+_i) \int_{B_1(0)} \phi^p + O\left(\frac{(s^+_{\delta,i})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\frac{2\pi\delta^2 a^+_{\delta,i}a^-_{\delta,j}}{|\ln \frac{R}{s^+_{\delta,i}}||\ln \frac{R}{s^-_{\delta,j}}|}\bar{G}(z^+_i,z^-_j) + O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}$$

By Lemma A.1 and (2.15),

$$\begin{aligned}
&\sum_{k=1}^m \int_\Omega \chi_{\Omega^+_k}\left(P^+_{\delta,Z} - P^-_{\delta,Z} - \kappa^+_k - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)^{p+1}_+\\
=&\sum_{k=1}^m \int_{B_{Ls^+_{\delta,k}}(z^+_k)} \left(P^+_{\delta,Z} - P^-_{\delta,Z} - \kappa^+_k - \frac{2\pi q(x)}{|\ln\varepsilon|}\right)^{p+1}_+\\
=&\sum_{k=1}^m \int_{B_{Ls^+_{\delta,k}}(z^+_k)} \left(W_{\delta,z^+_k,a^+_{\delta,k}} - a^+_{\delta,k} + O\Big(\frac{s^+_{\delta,k}}{|\ln\varepsilon|}\Big)\right)^{p+1}_+\\
=&\sum_{k=1}^m \left(\frac{\delta}{s^+_{\delta,k}}\right)^{\frac{2(p+1)}{p-1}} \int_{B_{s^+_{\delta,k}}(z^+_k)} \phi^{p+1}\Big(\frac{|x-z^+_k|}{s^+_{\delta,k}}\Big) + O\left(\frac{(s^+_{\delta,k})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\sum_{k=1}^m \left(\frac{\delta}{s^+_{\delta,k}}\right)^{\frac{2(p+1)}{p-1}} (s^+_{\delta,k})^2 \int_{B_1(0)} \phi^{p+1} + O\left(\frac{(s^+_{\delta,k})^3}{|\ln\varepsilon|^{p+1}}\right)\\
=&\sum_{k=1}^m \frac{\pi(p+1)}{2}\frac{\delta^2(a^+_{\delta,k})^2}{(\ln\frac{R}{s^+_{\delta,k}})^2} + O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}$$

Other terms can be estimated as above. So, we have proved

$$
\begin{aligned}
I\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}\right)=&\sum_{i=1}^{m}\left[\frac{\pi(p+1)}{4}\frac{\delta^2(a_{\delta,i}^{+})^2}{|\ln\frac{R}{s_{\delta,i}^{+}}|^2}+\frac{\pi\delta^2(a_{\delta,i}^{+})^2}{|\ln\frac{R}{s_{\delta,i}^{+}}|}-\frac{\pi g(z_i^{+},z_i^{+})\delta^2(a_{\delta,i}^{+})^2}{|\ln\frac{R}{s_{\delta,i}^{+}}|^2}\right]\\
&+\sum_{j=1}^{n}\left[\frac{\pi(p+1)}{4}\frac{\delta^2(a_{\delta,j}^{-})^2}{|\ln\frac{R}{s_{\delta,j}^{-}}|^2}+\frac{\pi\delta^2(a_{\delta,j}^{-})^2}{|\ln\frac{R}{s_{\delta,j}^{-}}|}-\frac{\pi g(z_j^{-},z_j^{-})\delta^2(a_{\delta,j}^{-})^2}{|\ln\frac{R}{s_{\delta,j}^{-}}|^2}\right]\\
&+\sum_{k\neq i}^{m}\frac{\pi\bar{G}(z_k^{+},z_i^{+})\delta^2a_{\delta,i}^{+}a_{\delta,k}^{+}}{|\ln\frac{R}{s_{\delta,i}^{+}}||\ln\frac{R}{s_{\delta,k}^{+}}|}+\sum_{l\neq j}^{n}\frac{\pi\bar{G}(z_l^{-},z_j^{-})\delta^2a_{\delta,l}^{-}a_{\delta,j}^{-}}{|\ln\frac{R}{s_{\delta,l}^{-}}||\ln\frac{R}{s_{\delta,j}^{-}}|}\\
&-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{2\pi\bar{G}(z_i^{+},z_j^{-})\delta^2a_{\delta,i}^{+}a_{\delta,j}^{-}}{|\ln\frac{R}{s_{\delta,i}^{+}}||\ln\frac{R}{s_{\delta,j}^{-}}|}-\frac{\pi\delta^2}{2}\left(\sum_{i=1}^{m}\frac{(a_{\delta,i}^{+})^2}{|\ln\frac{R}{s_{\delta,i}^{+}}|^2}\right)\\
&-\frac{\pi\delta^2}{2}\left(\sum_{j=1}^{n}\frac{(a_{\delta,j}^{-})^2}{|\ln\frac{R}{s_{\delta,j}^{-}}|^2}\right)+O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}
$$

Thus, the result follows from Remark 2.2. □

Proposition A.3. *We have*

$$
\begin{aligned}
\frac{\partial}{\partial z_{i,h}^{+}}I\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}\right)=&\frac{4\pi^2\delta^2\kappa_i^{+}}{|\ln\varepsilon||\ln\frac{R}{\varepsilon}|}\frac{\partial q(z_i^{+})}{\partial z_{i,h}^{+}}+\frac{2\pi\delta^2(\kappa_i^{+})^2}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial g(z_i^{+},z_i^{+})}{\partial z_{i,h}^{+}}\\
&-\sum_{k\neq i}^{m}\frac{2\pi\delta^2\kappa_i^{+}\kappa_k^{+}}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial\bar{G}(z_k^{+},z_i^{+})}{\partial z_{i,h}^{+}}+\sum_{l=1}^{n}\frac{2\pi\delta^2\kappa_i^{+}\kappa_l^{-}}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial\bar{G}(z_i^{+},z_l^{-})}{\partial z_{i,h}^{+}}+O\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z_{j,\bar{h}}^{-}}I\left(P_{\delta,Z}^{+}-P_{\delta,Z}^{-}\right)=&-\frac{4\pi^2\delta^2\kappa_j^{-}}{|\ln\varepsilon||\ln\frac{R}{\varepsilon}|}\frac{\partial q(z_j^{-})}{\partial z_{j,\bar{h}}^{-}}+\frac{2\pi\delta^2(\kappa_j^{-})^2}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial g(z_j^{-},z_j^{-})}{\partial z_{j,\bar{h}}^{-}}\\
&-\sum_{l\neq j}^{n}\frac{2\pi\delta^2\kappa_j^{-}\kappa_l^{-}}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial\bar{G}(z_l^{-},z_j^{-})}{\partial z_{j,\bar{h}}^{-}}+\sum_{k=1}^{m}\frac{2\pi\delta^2\kappa_j^{-}\kappa_k^{+}}{(\ln\frac{R}{\varepsilon})^2}\frac{\partial\bar{G}(z_j^{-},z_k^{+})}{\partial z_{j,\bar{h}}^{-}}+O\left(\frac{\delta^2\ln|\ln\varepsilon|}{|\ln\varepsilon|^3}\right).
\end{aligned}
$$

Proof. Direct computation yields that

$$
\begin{aligned}
&\frac{\partial}{\partial z_{i,h}^+} I\left(P_{\delta,Z}^+ - P_{\delta,Z}^-\right)\\
=&\sum_{k=1}^{m}\int_{B_{Ls_{\delta,k}^+}(z_k^+)}\left[\left(W_{\delta,z_k^+,a_{\delta,k}^+}-a_{\delta,k}^+\right)_+^p-\left(P_{\delta,Z}^+-P_{\delta,Z}^- -\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right]\frac{\partial P_{\delta,Z}^+}{\partial z_{i,h}^+}\\
&+\sum_{l=1}^{n}\int_{B_{Ls_{\delta,l}^-}(z_l^-)}\left[\left(W_{\delta,z_l^-,a_{\delta,l}^-}-a_{\delta,l}^-\right)_+^p-\left(P_{\delta,Z}^--P_{\delta,Z}^+ -\kappa_l^-+\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right]\frac{\partial P_{\delta,Z}^-}{\partial z_{i,h}^+}\\
&-\sum_{k=1}^{m}\int_{B_{Ls_{\delta,k}^+}(z_k^+)}\left[\left(W_{\delta,z_k^+,a_{\delta,k}^+}-a_{\delta,k}^+\right)_+^p-\left(P_{\delta,Z}^+-P_{\delta,Z}^- -\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right]\frac{\partial P_{\delta,Z}^-}{\partial z_{i,h}^+}\\
&-\sum_{l=1}^{n}\int_{B_{Ls_{\delta,l}^-}(z_l^-)}\left[\left(W_{\delta,z_l^-,a_{\delta,l}^-}-a_{\delta,l}^-\right)_+^p-\left(P_{\delta,Z}^--P_{\delta,Z}^+ -\kappa_l^++\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right]\frac{\partial P_{\delta,Z}^+}{\partial z_{i,h}^+}.
\end{aligned}
$$

For $k\neq i$, from (2.15), we have

$$
\begin{aligned}
&\int_{B_{Ls_{\delta,k}^+}(z_k^+)}\left[(W_{\delta,z_k^+,a_{\delta,k}^+}-a_{\delta,k}^+)_+^p-\left(P_{\delta,Z}^+-P_{\delta,Z}^- -\kappa_k^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)_+^p\right]\frac{\partial P_{\delta,Z,i}^+}{\partial z_{i,h}^+}\\
=&\int_{B_{Ls_{\delta,k}^+}(z_k^+)}\left[\left(W_{\delta,z_k^+,a_{\delta,k}^+}-a_{\delta,k}^+\right)^{p-1}\frac{s_{\delta,k}^+}{|\ln\varepsilon|}\right]\frac{C}{\ln\frac{R}{s_{\delta,i}^+}}\\
=&O\left(\frac{\varepsilon^3}{|\ln\varepsilon|^{p+1}}\right).
\end{aligned}
$$

Using (2.15), Lemma A.1 and Remark 2.2, we find that

$$
\begin{aligned}
&\int_{B_{Ls^+_{\delta,i}}(z_i^+)}\left[\left(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i}\right)^p_+-\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)^p_+\right]\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\\
=&\int_{B_{s^+_{\delta,i}(1+(s^+_{\delta,i})^\sigma)}(z_i)}\left[(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i})^p_+-\left(P^+_{\delta,Z}-P^-_{\delta,Z}-\kappa_i^+-\frac{2\pi q(x)}{|\ln\varepsilon|}\right)^p_+\right]\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\\
=&p\int_{B_{s^+_{\delta,i}}(z_i^+)}(W_{\delta,z_i^+,a^+_{\delta,i}}-a^+_{\delta,i})^{p-1}_+\Bigg[\frac{2\pi}{|\ln\varepsilon|}\langle Dq(z_i^+),x-z_i^+\rangle+\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\langle Dg(z_i^+,z_i^+),x-z_i^+\rangle\\
&-\sum_{k\neq i}^m\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\langle D\bar G(z_i^+,z_k^+),x-z_i^+\rangle+\sum_{l=1}^n\frac{a^-_{\delta,l}}{\ln\frac{R}{s^-_{\delta,l}}}\langle D\bar G(z_i^+,z_l^-),x-z_i^+\rangle\Bigg]\frac{\partial P^+_{\delta,Z,i}}{\partial z^+_{i,h}}\\
&+O\Big(\frac{\varepsilon^{2+\sigma}}{|\ln\varepsilon|^{p+1}}\Big)\\
=&-\frac{p\delta^2a^+_{\delta,i}}{|\phi'(1)||\ln\frac{R}{s^+_{\delta,i}}|}\Bigg(\frac{2\pi}{|\ln\varepsilon|}\frac{\partial q(z_i^+)}{\partial z^+_{i,h}}+\frac{a^+_{\delta,i}}{\ln\frac{R}{s^+_{\delta,i}}}\frac{\partial g(z_i^+,z_i^+)}{\partial z^+_{i,h}}-\sum_{k\neq i}^m\frac{a^+_{\delta,k}}{\ln\frac{R}{s^+_{\delta,k}}}\frac{\partial\bar G(z_i^+,z_k^+)}{\partial z^+_{i,h}}\\
&+\sum_{l=1}^n\frac{a^-_{\delta,l}}{\ln\frac{R}{s^-_{\delta,l}}}\frac{\partial\bar G(z_i^+,z_l^-)}{\partial z^+_{i,h}}\Bigg)\int_{B_1(0)}\phi^{p-1}(|x|)\phi'(|x|)\frac{x_h^2}{|x|}+O\Big(\frac{\varepsilon^{2+\sigma}}{|\ln\varepsilon|^{p+1}}\Big)\\
=&\frac{4\pi^2\delta^2a^+_{\delta,i}}{|\ln\varepsilon||\ln\frac{R}{s^+_{\delta,i}}|}\frac{\partial q(z_i^+)}{\partial z^+_{i,h}}+\frac{2\pi\delta^2(a^+_{\delta,i})^2}{(\ln\frac{R}{s^+_{\delta,i}})^2}\frac{\partial g(z_i^+,z_i^+)}{\partial z^+_{i,h}}-\sum_{k\neq i}^m\frac{2\pi\delta^2a^+_{\delta,i}a^+_{\delta,k}}{|\ln\frac{R}{s^+_{\delta,k}}||\ln\frac{R}{s^+_{\delta,i}}|}\frac{\partial\bar G(z_i^+,z_k^+)}{\partial z^+_{i,h}}\\
&+\sum_{l=1}^n\frac{2\pi\delta^2a^+_{\delta,i}a^-_{\delta,l}}{|\ln\frac{R}{s^-_{\delta,l}}||\ln\frac{R}{s^+_{\delta,i}}|}\frac{\partial\bar G(z_i^+,z_l^-)}{\partial z^+_{i,h}}+O\Big(\frac{\varepsilon^{2+\sigma}}{|\ln\varepsilon|^{p+1}}\Big),
\end{aligned}
$$

since

$$\int_{B_1(0)}\phi^{p-1}(|x|)\phi'(|x|)\frac{x_h^2}{|x|}=-\frac{2\pi}{p}|\phi'(1)|.$$

Other terms can be estimated similarly. Thus, the result follows. □

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Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100190, P.R. China
E-mail address: dmcao@amt.ac.cn

Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100190, P.R. China
E-mail address: liuzy@amss.ac.cn

Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong
E-mail address: wei@math.cuhk.edu.hk