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On Single Interior Spike Solutions of Gierer-Meinhardt System: Uniqueness and Spectrum Estimates

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We study the interior spike solutions to a steady state problem of the shadow system of the Gierer-Meinhardt system arising from biological pattern formation. We first show that at a nondegenerate peak point the interior spike solution is locally unique and then we establish the spectrum estimates of the associated linearized operator. We also prove that the corresponding solution to the shadow system is unstable. Furthermore, the metastability of such solutions is analyzed.

1 Introduction

In 1957, Turing [25] proposed a mathematical model for morphogenesis, which describes the development of complex organisms from a single shell. He speculated that localized peaks in concentration of a chemical substance, known as an inducer or morphogen, could be responsible for a group of cells developing differently from the surrounding cells. He then demonstrated, with linear analysis, how a nonlinear reaction diffusion system could possibly generate such isolated peaks. Later in 1972, Gierer and Meinhardt [6] demonstrated the existence of such solution numerically for the following (so-called Gierer-Meinhardt system)

$$(GM) \qquad \begin{cases} \frac{\partial A}{\partial t} = \epsilon^2 \Delta A - A + \frac{A^p}{H^q}, \ x \in \Omega, t > 0, \\ \tau \frac{\partial H}{\partial t} = D_h \Delta H - H + \frac{A^r}{H^s}, x \ \in \Omega, t > 0, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0, \ x \in \partial\Omega. \end{cases}$$

Here, the unknowns $A = A(x,t)$ and $H = H(x,t)$ represent the respective concentrations at point $x \in \Omega \subset R^N$ and at time t of the biochemical called an activator and an inhibitor; $\epsilon > 0, \tau > 0$ are all positive constants; $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^N; Ω is a smooth bounded domain in R^N; $\nu(x)$ is the unit outer normal at $x \in \partial\Omega$. The exponents (p,q,r,s) are assumed to satisfy the condition

$$(A) \qquad p>1, q>0, r>0, s\geq 0, \text{ and } \gamma_0 := \frac{qr}{(p-1)(s+1)} > 1.$$

Gierer-Meinhardt system was used in [6] to model head formation in the hydra. *Hydra*, an animal of a few millimeters in length, is made up of approximately 100,000 cells of about fifteen different types. It consists of a "head" region located at one end along its length. Typical experiments on *hydra* involve removing part of the "head" region and transplanting it to other parts of the body column. Then, a new "head" will form if and only if the transplanted area is sufficiently far from the (old) head. These observations have led to the assumption of the existence of two chemical substances-a *slowly* diffusing (i.e. $\epsilon << 1$) activator A and a *fast* diffusing (i.e. $D_h >> 1$) inhibitor H.

The numerical studied of [6] and more recent those of [11] have revealed that in the limit $\epsilon \to 0$, the (GM) system seems to have stable stationary solutions with the property that the activator concentration is localized around a finite number of points in $\overline{\Omega}$. Moreover, as $\epsilon \to 0$, the pattern exhibits a "spike layer phenomenon" by which we mean that the activator concentration is localized in narrower and narrower regions around some points and eventually shrinks to a certain number of points as $\epsilon \to 0$, whereas the maximum value of the activator concentration diverges to $+\infty$.

If we let $D_h \to \infty$ and suppose that the quantity $-H + A^r/H^s$ remains bounded, then $\Delta H \to 0$, $\frac{\partial H}{\partial \nu} = 0$ on $\partial\Omega$, we find that $H(x) \to \xi$, a constant. To derive the equation for ξ, we integrate both sides of the equation for H over Ω and observe that $\int_\Omega \Delta H dx = 0$ due to the boundary condition. Hence in the limit of $D_h \to \infty$, we obtain the following so-called *shadow system* of (GM)

$$\begin{cases} \frac{\partial A}{\partial t} = \epsilon^2 \Delta A - A + \frac{A^p}{\xi^q} \text{ in } \Omega, \ \frac{\partial A}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ \tau|\Omega| \frac{d\xi}{dt} = -|\Omega|\xi + \frac{1}{\xi^s}\int_\Omega A^r dx, \end{cases} \tag{1.1}$$

(For the derivation of (1.1) and more details on the Gierer-Meinhardt system as well as its properties, see the review article [19].)

Let us consider the stationary solution to the shadow system (1.1). Set $A(x) = \xi^{q/(p-1)}u(x)$. Then it is easy to see that u satisfies

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

Throughout this paper, we always assume that

$$1 < p < (\frac{N+2}{N-2})_+ (= \frac{N+2}{N-2} \text{ when } N \geq 3; = +\infty \text{ when } N = 1,2).$$

(Equation (1.2) is also known as the steady state problem for a chemotactic aggregation model with logarithmic sensitivity by Keller-Segel [15], see e.g. [18].)

It is known that equation (1.2) has both boundary and interior spike layer solutions. See [19] for an overview of (1.2). For boundary spike solutions, please see [2], [3], [8], [10], [16], [18], [20], [21], [22], [31], [32], [33], and the references therein. (When $p = \frac{N+2}{N-2}$, please see [7] for the latest results and references.)

The existence of interior spike solutions depends highly on the geometry of the domain. In [30] and [29], the author first constructed a single interior spike solution. To state the result, we need to introduce some notations. Let

$$d\mu_{P_0}(z) = \lim_{\epsilon \to 0} \frac{e^{-\frac{2|z-P_0|}{\epsilon}} dz}{\int_{\partial\Omega} e^{-\frac{2|z-P_0|}{\epsilon}} dz}. \tag{1.3}$$

It is easy to see that the support of $d\mu_{P_0}(z)$ is contained in $\bar{B}_{d(P_0,\partial\Omega)}(P_0) \cap \partial\Omega$.

A point P_0 is called "nondegenerate peak point" if the followings hold: there exists $a \in R^N$ such that

$$(H1) \qquad \int_{\partial\Omega} e^{<z-P_0,a>}(z-P_0) d\mu_{P_0}(z) = 0 \tag{1.4}$$

and

$$(H2) \qquad (\int_{\partial\Omega} e^{<z-P_0,a>}(z-P_0)_i(z-P_0)_j d\mu_{P_0}(z)) := G(P_0) \text{ is nonsingular.} \tag{1.5}$$

Such a vector a is unique. Moreover, $G(P_0)$ is a positive definite matrix. A geometric characterization of a nondegenerate peak point P_0 is the following:

$$P_0 \in \text{ interior (convex hull of support}(d\mu_{P_0}(z)).$$

For a proof of the above facts, see Theorem 5.1 of [30].

In [29] and [30], the author proved the following theorem.

Theorem A. Assume that P_0 is a nondegenerate peak point in Ω. Then for $\epsilon << 1$, equation (1.2) has a solution u_ϵ satisfying the following properties:

(1) u_ϵ has a unique local maximum point P_ϵ and $P_\epsilon \to P_0$ as $\epsilon \to 0$,

(2) $u_\epsilon \leq Ce^{-\beta|x-P_\epsilon|/\epsilon}$ for some constants $C > 0, \beta > 0$ and $u_\epsilon(P_\epsilon) \to w(0)$ as $\epsilon \to 0$, where w is the unique solution of the following problem

$$\begin{cases} \Delta w - w + w^p = 0, w > 0 \text{ in } R^N, \\ w(0) = \max_{y \in R^N} w(y), w(y) \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{1.6}$$

Remark: For the existence of multiple interior spike solutions for equations similar to (1.2) (e.g. the Cahn-Hilliard equation), see [1], [9], [14] and [34].

As far we know, all the previous results on spike layer solutions are mainly concerned with *existence*. In this paper, we study various properties of u_ϵ constructed in Theorem A. We first establish the local uniqueness of u_ϵ (Theorem 1.1). Then we obtain the spectrum estimates of the associated linearized operator (Theorem 1.3). Finally we study the stability, instability and metastability of interior spike solutions for the corresponding shadow system (1.1) (Corollaries 1.1 and 1.2). (Here say a solution is (linearly) stable if all the eigenvalues of the associated linearized operator have negative real part, it is (linearly) unstable if the associated linearized operator has one eigenvalue with positive real part, and it is metastable if those eigenvalues of the associated linearized operator with positive real parts are exponentially small.) The results and techniques in this paper may be useful in studying properties of multiple boundary and multiple interior spike solutions, and also in studying the nucleation phenomen in the Cahn-Hilliard equation.

Our first result shows that u_ϵ is locally unique and that P_ϵ approaches P_0 along the direction a (defined by (H1) and (H2)).

Theorem 1.1 *Suppose that P_0 is a nondegenerate peak point. Then for $\epsilon << 1$, if there are two families of single interior spike solutions $u_{\epsilon,1}$ and $u_{\epsilon,2}$ of (1.2) such that $P_\epsilon^1 \to P_0, P_\epsilon^2 \to P_0$ where $u_{\epsilon,1}(P_\epsilon^1) = \max_{P\in\bar\Omega} u_\epsilon(P), u_{\epsilon,2}(P_\epsilon^2) = \max_{P\in\bar\Omega} u_{\epsilon,2}(P)$, then $P_\epsilon^1 = P_\epsilon^2, u_{\epsilon,1} = u_{\epsilon,2}$. Moreover, $P_\epsilon^1 = P_\epsilon^2 = P_0 + \epsilon(\frac{1}{2}d(P_0,\partial\Omega)a + o(1))$ as $\epsilon \to 0$.*

The second result concerns the eigenvalue estimates associated with the linearized operator at u_ϵ: $L_\epsilon = \epsilon^2\Delta - 1 + pu_\epsilon^{p-1}$. (Here the domain of L_ϵ is $H^2(\Omega)$.) We first note the following result.

Lemma 1.2 *The following eigenvalue problem*

$$\Delta\phi - \phi + pw^{p-1}\phi = \mu\phi \text{ in } R^N, \phi \in H^1(R^N) \tag{1.7}$$

admits the following set of eigenvalues:

$$\mu_1 > 0, \mu_2 = ... = \mu_{N+1} = 0, \mu_{N+2} < 0, ... \tag{1.8}$$

Moreover, the eigenfunction corresponding to μ_1 is radial and of constant sign.

Proof This follows from Theorem 2.12 of [18] and Lemma 4.2 of [21]. □

Next, as in [29] and [30], we introduce the following definition: for each $P \in \Omega$, let $w_{\epsilon,P}$ be the unique solution of

$$\epsilon^2\Delta u - u + w^p(\frac{x-P}{\epsilon}) = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{1.9}$$

Let $\varphi_{\epsilon,P}(x) = w((x-P)/\epsilon) - w_{\epsilon,P}(x)$. (It was proved in [29] and [30] that $-\epsilon\log[-\varphi_{\epsilon,P}(P)] \to 2d(P,\partial\Omega)$ as $\epsilon \to 0$.)

Our second result is about the eigenvalue estimates.

Theorem 1.3 *The following eigenvalue problem*

$$\epsilon^2\Delta\phi - \phi + pu_\epsilon^{p-1}\phi = \tau^\epsilon\phi \quad in\ \Omega,\ \frac{\partial\phi}{\partial\nu} = 0 \quad on\ \partial\Omega \tag{1.10}$$

admits the following set of eigenvalues:

$\tau_1^\epsilon = \mu_1 + o(1), \tau_j^\epsilon = (c_0 + o(1))\varphi_{\epsilon,P_0}(P_0)\lambda_{j-1}, j = 2, ..., N+1, \tau_l^\epsilon = \mu_l + o(1),\ l \geq N+2,$

where $\lambda_j, j = 1, ..., N$ are the eigenvalues of $G(P_0)$ and

$$c_0 = 2d^{-2}(P_0, \partial\Omega)\frac{\int_{R^N} pw^{p-1}w' u_*'(r)}{\int_{R^N}(\frac{\partial w}{\partial y_1})^2 dy} < 0, \tag{1.11}$$

where $u_*(r)$ is the unique radial solution of the following problem

$$\Delta u - u = 0, u(0) = 1, u = u(r) \quad in\ R^N. \tag{1.12}$$

Furthermore, the eigenfunction (suitably normalized) corresponding to $\tau_j^\epsilon, j = 2, ..., N+1$ is given by the following:

$$\phi_j^\epsilon = \sum_{l=1}^{N}(a_{j-1,l} + o(1))\epsilon\frac{\partial w_{\epsilon,P}}{\partial P_l}|_{P=P_\epsilon}, \tag{1.13}$$

where $\vec{a}_j = (a_{j,1}, ..., a_{j,N})^t$ is the eigenvector corresponding to λ_j, namely

$$G(P_0)\vec{a}_j = \lambda_j\vec{a}_j, j = 1, ..., N.$$

Remark: In [26], M. J. Ward used matched asymptotic expansions to analyze single interior spike solutions for equation (1.2). He obtained similar results to Theorem 1.3.

From Theorem 1.3, we see that except the eigenvalues near 0, all the eigenvalues of L_ϵ are small perturbations of those for $L_0 := \Delta - 1 + pw^{p-1}$. Moreover, u_ϵ is an isolated solution with Morse index $N+1$ (note that $G(P_0)$ is a positive definite matrix). Furthermore, L_ϵ is invertible. Hence u_ϵ is also nondegenerate.

Finally we study the stability and metastability of interior spike solutions to the shadow system (1.1). Let u_ϵ be the solution in Theorem A. Then it is easy to see that $(A_\epsilon, \xi_\epsilon)$ defined by the following

$$A_\epsilon = \xi_\epsilon^{q/(p-1)}u_\epsilon, \xi_\epsilon = (\frac{1}{|\Omega|}\int_\Omega u_\epsilon^r dx)^{-(p-1)/(qr-(p-1)(s+1))} \tag{1.14}$$

is a solution pair of the stationary problem to the shadow system (1.1). A direct application of Theorem 1.3 is the following corollary.

Corollary 1.1 *For $\epsilon << 1$, $(A_\epsilon, \xi_\epsilon)$ is unstable with respect to the shadow system (1.1).*

Remark: The fact that $(A_\epsilon, \xi_\epsilon)$ is unstable follows easily from the fact that $\tau_j^\epsilon, 2 \leq j \leq N+1$, in Theorem 1.3 are positive (by (1.11)). Note that τ_j^ϵ is exponentially small.

Therefore it is interesting to study the metastability of $(A_\epsilon, \xi_\epsilon)$. (Some numerical results and formal analysis are done in [27]. In one dimensional case, Ni, Takagi and Yanagida have some partial results, see [23].) We have the following partial results

Corollary 1.2 *Assume that either $r = 2, 1 < p \leq 1 + \frac{4}{N}$ or $r = p+1, 1 < p < (\frac{N+2}{N-2})_+$. Then, for $\epsilon << 1$ and τ small, $(A_\epsilon, \xi_\epsilon)$ is metastable with respect to the shadow system (1.1).*

Remark: In the original Gierer-Meinhardt system, we have $r = 2, p = 2, q = 1, s = 0$. Thus it satisfies the assumption in Corollary 1.2. As far as we know, Corollary 1.2 is the first result on the metastability of interior spike solutions of (1.1).

In the rest of this section, we introduce the basic idea behind the proofs of Theorems 1.1 and 1.3 and Corollaries 1.1 and 1.2.

To prove Theorems 1.1 and 1.3, we introduce the following energy functional

$$J_\epsilon(u) = \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \frac{1}{p+1}\int_\Omega u^{p+1}, u \in H^1(\Omega).$$

It is known that u_ϵ is a solution of (1.2) if and only if u_ϵ is a critical point of J_ϵ. The basic idea of our proof is to use the classical Liapunov-Schmidt procedure to reduce problem (1.2) into a finite-dimensional problem. (See [9], [10] and [29] for similar treatments.) We need to introduce some notations first.

Let $P \in \Omega$. Set

$$\Omega_{\epsilon,P} = \{y | \epsilon y + P \in \Omega\}, H^2_\nu(\Omega_{\epsilon,P}) = \{u \in H^2(\Omega_{\epsilon,P}) | \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_{\epsilon,P}\},$$

$$S_\epsilon(u) = \Delta u - u + u^p, u \in H^2_\nu(\Omega_{\epsilon,P}),\ \partial_j = \frac{\partial}{\partial P_j},$$

$$\mathcal{K}_{\epsilon,P} = span\ \{\partial_j w_{\epsilon,P} | j = 1, ..., N\} \subset H^2_\nu(\Omega_{\epsilon,P}), \mathcal{K}^\perp_{\epsilon,P} = \{u \in H^2_\nu(\Omega_{\epsilon,P}) | \int_\Omega u\partial_j w_{\epsilon,P} = 0, j = 1, ..., N\},$$

and

$$\mathcal{C}_{\epsilon,P} = span\ \{\partial_j w_{\epsilon,P} | j = 1, ..., N\} \subset L^2(\Omega_{\epsilon,P}), \mathcal{C}^\perp_{\epsilon,P} = \{u \in L^2(\Omega_{\epsilon,P}) | \int_\Omega u\partial_j w_{\epsilon,P} = 0, j = 1, ..., N\}.$$

Let $Q^0_\epsilon := P_0 + \epsilon\frac{1}{2}d(P_0, \partial\Omega)a$ and $\Lambda := B_{\beta_0\epsilon}(Q^0_\epsilon)$, where β_0 is sufficiently small. (In Section 2, we will show that for any single interior peak solution u_ϵ with the unique local maximum point P_ϵ near P_0, then we have $P_\epsilon \in \Lambda$.) For each $P \in \Lambda$ we can find a unique $v_{\epsilon,P} \in \mathcal{K}^\perp_{\epsilon,P}$ such that

$$S_\epsilon(w_{\epsilon,P} + v_{\epsilon,P}) \in \mathcal{C}_{\epsilon,P}.$$

We then define

$$K_\epsilon(P) := \epsilon^{-N} J_\epsilon(w_{\epsilon,P} + v_{\epsilon,P}) : \Lambda \to R. \tag{1.15}$$

An important observation is the following fact:

Proposition: $u_\epsilon = w_{\epsilon,Q_\epsilon} + v_{\epsilon,Q_\epsilon}, Q_\epsilon \in \Lambda$ is a critical point of J_ϵ if and only if Q_ϵ is a critical point of K_ϵ in Λ.

By the above Proposition, the local uniqueness problem is reduced to counting the number of critical points of K_ϵ in Λ. Thus we need to compute $\partial_j K_\epsilon(P)$ and $\partial^2_{ij} K_\epsilon(P)$. By some lengthy computations, we can relate $(\partial^2_{ij} K_\epsilon(P))$ with the matrix $G(P_0)$. By some degree arguments, we prove Theorem 1.1.

Theorem 1.3 is proved by using the results for $(\partial^2_{ij} K_\epsilon(P))$.

To prove Corollaries 1.1 and 1.2, we just need to analyze the following eigenvalue problem

$$\begin{cases} \epsilon^2 \Delta \phi_\epsilon - \phi_\epsilon + p\frac{A_\epsilon^{p-1}}{\xi_\epsilon^q}\phi_\epsilon - q\frac{A_\epsilon^p}{\xi_\epsilon^{q+1}}\eta = \alpha_\epsilon \phi_\epsilon, \ \frac{\partial \phi_\epsilon}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ \frac{r}{\tau|\Omega|}\int_\Omega \frac{A_\epsilon^{r-1}\phi_\epsilon}{\xi_\epsilon^s}dx - \frac{1+s}{\tau}\eta = \alpha_\epsilon \eta. \end{cases} \tag{1.16}$$

By using (1.14), it is easy to see that the eigenvalues of problem (1.16) in $H^2_\nu(\Omega_{\epsilon,P_\epsilon}) \times L^\infty(\Omega)$ are the same as the eigenvalues of the following eigenvalue problem

$$\epsilon^2 \Delta \phi - \phi + p u_\epsilon^{p-1}\phi - \frac{qr}{s+1+\tau\alpha_\epsilon}\frac{\int_\Omega u_\epsilon^{r-1}\phi}{\int_\Omega u_\epsilon^r}u_\epsilon^p = \alpha_\epsilon \phi, \phi \in H^2_\nu(\Omega_{\epsilon,P_\epsilon}). \tag{1.17}$$

Let α_ϵ be an eigenvalue of (1.17). Then the following holds. (The proof of it is routine and thus we leave it to Appendix.)

Lemma A. *(1) $\alpha_\epsilon = o(1)$ if and only if $\alpha_\epsilon = (1+o(1))\tau_j^\epsilon$ for some $j = 2, ..., N+1$, where τ_j^ϵ is given by Theorem 1.3.*

(2) If $\alpha_\epsilon \to \alpha_0 \neq 0$. Then α_0 is an eigenvalue of the following eigenvalue problem

$$\Delta \phi - \phi + p w^{p-1}\phi - \frac{qr}{s+1+\tau\alpha_0}\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r}w^p = \alpha_0 \phi, \phi \in H^2(R^N). \tag{1.18}$$

Since $\tau_{j+1}^\epsilon = (c_0 + o(1))\varphi_{\epsilon,P_0}(P_0)\lambda_j$ for some $j = 1, ..., N$ and $c_0 < 0, \varphi_{\epsilon,P_0}(P_0) < 0$, we conclude by Lemma A that (1.16) has positive eigenvalues $\alpha_\epsilon = (c_0 + o(1))\varphi_{\epsilon,P_0}(P_0)\lambda_j, j = 1, ..., N$. Thus $(A_\epsilon, \xi_\epsilon)$ is always unstable. This proves Corollary 1.1.

Corollary 1.2 follows from the following Theorem and Lemma A.

Theorem 1.4 *Suppose (r,p) satisfy the assumption in Corollary 1.2 and τ is small. Let $\alpha_0 \neq 0$ be an eigenvalue of the problem (1.18). Then we have $Re(\alpha_0) \leq -c_1$ for some $c_1 > 0$.*

Finally we remark that the results of Theorems 1.1 and 1.3 hold true if in equation (1.2) we replace the term u^p by general $f(u)$ where $f'(0) = f(0) = 0$ and f satisfies the conditions in Section 1 of [29].

The paper is organized as follows: In Section 2, we use the classical Liapunov-Schmidt reduction approach to transform the problem (1.2) into a finite dimensional problem and by computing the derivatives of $K_\epsilon(P)$ we prove Theorem 1.1 in Section 3. In Section 4, we prove Theorem 1.3. We finish the proof of Theorem 1.4 in Section 5 and thus completes the proof of Corollary 1.2. Some remarks are made in Section 6. Appendix contains the proof of Lemma A.

2 Proof of Theorem 1.1: Preliminaries

This section is divided into two parts. In the first part, we analyze the location of the local maximum point of the single interior spike solution. In the second part, we use the Liapunov-Schmidt reduction method to reduce the problem into a finite dimensional problem.

Let u_ϵ be a solution of (1.2) with a single local maximum point P_ϵ and $P_\epsilon \to P_0$. By Theorem 1.1 of [29],

Lemma 2.1 *We have*

$$\lim_{\epsilon\to 0} \frac{\int_{\partial\Omega} e^{-\frac{2|z-P_\epsilon|}{\epsilon}}(z-P_0)dz}{\int_{\partial\Omega} e^{-\frac{2|z-P_\epsilon|}{\epsilon}}dz} = 0. \tag{2.1}$$

The following important lemma gives the speed of P_ϵ approaching P_0.

Lemma 2.2 $P_\epsilon = P_0 + \epsilon(\frac{1}{2}d(P_0,\partial\Omega)a + o(1))$ *where a is given by (H1) and (H2).*

Proof Our main tool is the identity (2.1). We first prove that $|P_\epsilon - P_0| = O(\epsilon)$. Suppose not, we have $\frac{|P_\epsilon - P_0|}{\epsilon} \to +\infty$.

Note that

$$e^{-\frac{2|z-P_\epsilon|}{\epsilon}} = e^{-\frac{2|z-P_0|}{\epsilon}+2\langle \frac{z-P_0}{|z-P_0|}, \frac{P_\epsilon-P_0}{|P_\epsilon-P_0|}\rangle \frac{|P_\epsilon-P_0|}{\epsilon}+o(\frac{|P_\epsilon-P_0|}{\epsilon})}.$$

Let $\frac{P_\epsilon-P_0}{|P_\epsilon-P_0|} = e_\epsilon \in S^{N-1}$ (i.e. $|e_\epsilon| = 1$) and $\Gamma_1 = \{z \in \partial\Omega | \langle \frac{z-P_0}{|z-P_0|}, e_\epsilon\rangle > 1-2\delta\}$ and $\Gamma_2 = \partial\Omega\backslash\Gamma_1$, where $\delta > 0$ is a very small but fixed number. Then on Γ_1,

$$\int_{\Gamma_1} e^{-\frac{2|z-P_\epsilon|}{\epsilon}} \frac{\langle z-P_0, e_\epsilon\rangle}{|z-P_0|}dz$$

$$\geq (1-2\delta)e^{2(1-\delta)\frac{|P_\epsilon-P_0|}{\epsilon}} \int_{\{z\in\partial\Omega|\langle\frac{z-P_0}{|z-P_0|},e_\epsilon\rangle>1-\delta\}} e^{-\frac{2|z-P_0|}{\epsilon}}dz$$

$$\geq Ce^{2(1-\delta)\frac{|P_\epsilon-P_0|}{\epsilon}} \int_{\partial\Omega} e^{-\frac{2|z-P_0|}{\epsilon}}dz, \tag{2.2}$$

since (assuming that $e_\epsilon \to e_0$)

$$\lim_{\epsilon\to 0}\frac{\int_{\{z\in\partial\Omega|\langle\frac{z-P_0}{|z-P_0|},e_\epsilon\rangle>1-\delta\}}e^{-\frac{2|z-P_0|}{\epsilon}}dz}{\int_{\partial\Omega}e^{-\frac{2|z-P_0|}{\epsilon}}dz}=\int_{\{z\in\partial\Omega|\langle\frac{z-P_0}{|z-P_0|},e_0\rangle>1-\delta\}}d\mu_{P_0}(z)>0$$

(because $P_0\in interior(cov(support(d\mu_{P_0})))$.)

On the other hand, on Γ_2, we have

$$\int_{\Gamma_2}e^{-\frac{2|z-P_\epsilon|}{\epsilon}}\frac{\langle z-P_0,e_\epsilon\rangle}{|z-P_0|}dz=O(e^{(1-2\delta)\frac{|P_\epsilon-P_0|}{\epsilon}}\int_{\partial\Omega}e^{-\frac{2|z-P_0|}{\epsilon}}dz)=o(\int_{\Gamma_1}e^{-\frac{2|z-P_\epsilon|}{\epsilon}}\langle z-P_0,e_\epsilon\rangle dz).\ \text{(by (2.2))}$$

Thus

$$\frac{\int_{\partial\Omega}e^{-\frac{2|z-P_\epsilon|}{\epsilon}}\frac{\langle z-P_0,e_\epsilon\rangle}{|z-P_0|}dz}{\int_{\partial\Omega}e^{-\frac{2|z-P_\epsilon|}{\epsilon}}dz}=\frac{\int_{\Gamma_1}e^{-\frac{2|z-P_\epsilon|}{\epsilon}}\frac{\langle z-P_0,e_\epsilon\rangle}{|z-P_0|}dz}{\int_{\Gamma_1}e^{-\frac{2|z-P_\epsilon|}{\epsilon}}dz}+o(1)\geq C,$$

a contradiction to (2.1)! Hence $P_\epsilon-P_0=O(\epsilon)$.

Let $P_\epsilon-P_0=\epsilon b_\epsilon$ and $b_\epsilon\to b_0$. Since

$$\int_{\partial\Omega}e^{-\frac{2|z-P_\epsilon|}{\epsilon}}(z-P_0)dz=\int_{\partial\Omega}e^{-\frac{2|z-P_0|}{\epsilon}+2<\frac{z-P_0}{|z-P_0|},b_\epsilon>}(z-P_0)dz+o(\int_{\partial\Omega}e^{-\frac{2|z-P_0|}{\epsilon}}dz),$$

we have by (2.1)

$$\int_{\partial\Omega}e^{<z-P_0,\frac{2}{|z-P_0|}b_0>}(z-P_0)d\mu_{P_0}(z)=0.$$

By the proof of Theorem 5.1 in [30], there exists a unique vector a such that

$$\int_{\partial\Omega}e^{<z-P_0,a>}(z-P_0)d\mu_{P_0}(z)=0.$$

Thus we obtain

$$\frac{2}{d(P_0,\partial\Omega)}b_0=a,b_0=\frac{1}{2}d(P_0,\partial\Omega)a.$$

□

In the rest of this section, we will describe the so-called Liapunov-Schmidt procedure. Most of the material is from Sections 4,5 and 6 in [29]. See also Sections 3 and 5 in [9].

We first introduce some notations.

Let $\Omega_{\epsilon,P},H^2_\nu(\Omega_{\epsilon,P}),\mathcal{K}_{\epsilon,P},\mathcal{K}^\perp_{\epsilon,P},\mathcal{C}^\perp_{\epsilon,P},S_\epsilon(u)$ be defined in Section 1.

For any $u,v\in H^1(\Omega)$, we define

$$<u,v>_\epsilon=\epsilon^{-N}\int_\Omega(\epsilon^2\nabla u\cdot\nabla v+u\cdot v),\quad \|u\|_\epsilon=<u,v>_\epsilon^{\frac{1}{2}}.$$

Set

$$\tilde{L}_{\epsilon,P}(\phi)=\Delta\phi-\phi+pw^{p-1}_{\epsilon,P}\phi,\quad L_{\epsilon,P}=\pi_{\epsilon,P}\circ\tilde{L}_{\epsilon,P},$$

where $\pi_{\epsilon,P}$ is the projection from $L^2(\Omega_{\epsilon,P})$ into $\mathcal{C}^\perp_{\epsilon,P}$.

We recall the following result in [29] (see Propositions 6.1, 6.2 and 6.3 in [29]).

Lemma 2.3 *For $\epsilon << 1$, $L_{\epsilon,P} : \mathcal{K}^{\perp}_{\epsilon,P} \to \mathcal{C}^{\perp}_{\epsilon,P}$ is one-to-one and onto. Moreover, the inverse of $L_{\epsilon,P}$ exists and bounded.*

Next, we have

Lemma 2.4 *For any $Q \in \Omega$, there exists a unique $v_{\epsilon,Q} \in \mathcal{K}^{\perp}_{\epsilon,Q}$ such that*

$$S_\epsilon(w_{\epsilon,Q} + v_{\epsilon,Q}) \in \mathcal{C}_{\epsilon,Q}.$$

Moreover, $v_{\epsilon,Q}$ is C^1 in Q and

$$\|v_{\epsilon,Q}\|_\epsilon \leq C|\varphi_{\epsilon,Q}(Q)|^{\frac{1+\sigma}{2}}, \tag{2.3}$$

$$\|\frac{\partial v_{\epsilon,Q}}{\partial Q_j}\|_\epsilon \leq C\epsilon^{-2}|\varphi_{\epsilon,Q}(Q)|^{\frac{1+\sigma}{2}}, \tag{2.4}$$

where $\sigma = \min(1, p-1)$.

Proof The existence of $v_{\epsilon,Q} \in \mathcal{K}^{\perp}_{\epsilon,Q}$ such that $S_\epsilon(w_{\epsilon,Q}+v_{\epsilon,Q}) \in \mathcal{C}_{\epsilon,Q}$ follows from Section 6 in [29]. The C^1-smoothness of $v_{\epsilon,Q}$ in Q follows from Lemma 3.5 in [9]. For estimate (2.3), please see Proposition 6.6 of [29]. It remains to estimate $\frac{\partial v_{\epsilon,Q}}{\partial Q_j}$ and prove (2.4). Set

$$\frac{\partial v_{\epsilon,Q}}{\partial Q_j} = \sum_{k=1}^{N} \alpha^{\epsilon}_{jk} \frac{\partial w_{\epsilon,Q}}{\partial Q_k} + v^{\perp}_{\epsilon,Q}, v^{\perp}_{\epsilon,Q} \in \mathcal{K}^{\perp}_{\epsilon,Q}.$$

We first note that by (2.3)

$$\int_{\Omega_{\epsilon,Q}} \frac{\partial v_{\epsilon,Q}}{\partial Q_j}\frac{\partial w_{\epsilon,Q}}{\partial Q_l} = -\int_{\Omega_{\epsilon,Q}} v_{\epsilon,Q}\frac{\partial^2 w_{\epsilon,Q}}{\partial Q_j \partial Q_l} = O(\epsilon^{-2}|\varphi_{\epsilon,Q}(Q)|^{(1+\sigma)/2}).$$

Hence

$$\sum_{k=1}^{N} \alpha^{\epsilon}_{jk} \int_{\Omega_{\epsilon,Q}} \frac{\partial w_{\epsilon,Q}}{\partial Q_k}\frac{\partial w_{\epsilon,Q}}{\partial Q_l} = O(\epsilon^{-2}|\varphi_{\epsilon,Q}(Q)|^{(1+\sigma)/2}).$$

Since $\int_{\Omega_{\epsilon,Q}} \frac{\partial w_{\epsilon,Q}}{\partial Q_k}\frac{\partial w_{\epsilon,Q}}{\partial Q_l} = \epsilon^{-2}(\Gamma + o(1))\delta_{kl}$, where $\Gamma > 0$, we obtain $\alpha^{\epsilon}_{jk} = O(|\varphi_{\epsilon,Q}(Q)|^{(1+\sigma)/2})$.

Next we observe that

$$S_\epsilon(w_{\epsilon,Q} + v_{\epsilon,Q}) = \sum_{i=1}^{N} \beta^{\epsilon}_i(Q)\frac{\partial w_{\epsilon,Q}}{\partial Q_i}$$

where $\beta^{\epsilon}_i(Q) \in C^1$ and $\beta^{\epsilon}_i(Q) = O(\epsilon^{-1}|\varphi_{\epsilon,Q}(Q)|)$.

Differentiating the above equation by $\frac{\partial}{\partial Q_j}$, we have,

$$S^{'}_\epsilon(w_{\epsilon,Q} + v_{\epsilon,Q})(\frac{\partial w_{\epsilon,Q}}{\partial Q_j} + \frac{\partial v_{\epsilon,Q}}{\partial Q_j}) - \sum_{i=1}^{N} \beta^{\epsilon}_i(Q)\frac{\partial^2 w_{\epsilon,Q}}{\partial Q_i \partial Q_j} \in \mathcal{C}_{\epsilon,Q}.$$

Hence

$$S_\epsilon^{'}(w_{\epsilon,Q}+v_{\epsilon,Q})(v_{\epsilon,Q}^{\perp})+S_\epsilon^{'}(w_{\epsilon,Q}+v_{\epsilon,Q})(\frac{\partial w_{\epsilon,Q}}{\partial Q_j}+\sum_{i=1}^N \alpha_{ij}^\epsilon \frac{\partial w_{\epsilon,Q}}{\partial Q_i})$$

$$-\sum_{i=1}^N \beta_i^\epsilon(Q)\frac{\partial^2 w_{\epsilon,Q}}{\partial Q_i \partial Q_j}\in \mathcal{C}_{\epsilon,Q}.$$

It is easy to see that

$$\pi_{\epsilon,Q}\circ S_\epsilon^{'}(w_{\epsilon,Q}+v_{\epsilon,Q}):\mathcal{K}_{\epsilon,Q}^{\perp}\to \mathcal{C}_{\epsilon,Q}^{\perp}$$

is invertible for ϵ sufficiently small. Hence (since $v_{\epsilon,Q}^{\perp}\in \mathcal{K}_{\epsilon,Q}^{\perp}$)

$$\|v_{\epsilon,Q}^{\perp}\|_{H^2(\Omega_{\epsilon,Q})}\le C\|\pi_{\epsilon,Q}\circ S_\epsilon^{'}(w_{\epsilon,Q}+v_{\epsilon,Q})(\frac{\partial w_{\epsilon,Q}}{\partial Q_j}+\sum_{k=1}^N \alpha_{jk}^\epsilon \frac{\partial w_{\epsilon,Q}}{\partial Q_j}))\|\|_{L^2(\Omega_{\epsilon,Q})}$$

$$+C\sum_{i=1}^N |\beta_i^\epsilon(Q)|\|\frac{\partial^2 w_{\epsilon,Q}}{\partial Q_j \partial Q_i})\|_{L^2(\Omega_{\epsilon,Q})}$$

$$\le C\epsilon^{-2}|\varphi_{\epsilon,Q}(Q)|^{(1+\sigma)/2}.$$

(2.4) is thus proved. □

Let $K_\epsilon(P)$ be defined by (1.15). Then we have (see Section 5 of [29])

Lemma 2.5 *$u_\epsilon = w_{\epsilon,P}+v_{\epsilon,P}$ is a solutions of (1.2) if and only of P is a critical point of $K_\epsilon(P)$.*

Let u_ϵ be a single interior spike solution with the unique local maximum $P_\epsilon \to P_0$. By Lemma 2.4, we have $u_\epsilon = w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon}$ for some $Q_\epsilon \in \Lambda$ and $v_\epsilon \in \mathcal{K}_{\epsilon,Q_\epsilon}^{\perp}$.

The next lemma relates P_ϵ and Q_ϵ.

Lemma 2.6 $Q_\epsilon = P_\epsilon + o(\epsilon)$.

Proof In fact, by Lemma 2.4, we have the decomposition of $u_\epsilon = w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon}$ with $Q_\epsilon \to P_0$. Moreover similar arguments as in the proof of Theorem 1.1 of [29] show that

$$\lim_{\epsilon\to 0}\frac{\int_{\partial\Omega} e^{\frac{2|z-Q_\epsilon|}{\epsilon}}(z-Q_\epsilon)dz}{\int_{\partial\Omega} e^{-\frac{2|z-Q_\epsilon|}{\epsilon}}dz}=0.$$

By Lemma 2.1, we have

$$Q_\epsilon = P_0+\epsilon(\frac{1}{2}d(P_0,\partial\Omega)a+o(1)).$$

Hence $Q_\epsilon = P_\epsilon + o(\epsilon)$. □

3 Proof of Theorem 1.1: Final proof

In this section, we complete the proof of Theorem 1.1.

Let $Q_\epsilon^0 = P_0 + \epsilon \frac{1}{2} d(P_0, \partial\Omega) a$. Then by Lemma 2.6, all critical points of $K_\epsilon(P)$ near P_0 are in $B_{\beta_0 \epsilon}(Q_\epsilon^0)$ for β_0 small.

By Lemma 2.5, to prove Theorem 1.1, we just need to show that $K_\epsilon(P)$ has only one critical point in $B_{\beta_0 \epsilon}(Q_\epsilon^0)$.

Let us define

$$\tilde{K}_\epsilon(P) = -\frac{\gamma}{\epsilon^2 d^2(P_0, \partial\Omega)} \varphi_{\epsilon, P_0}(P_0) \int_{\partial\Omega} e^{<z-P_0, a>} < z - P_0, P - Q_\epsilon^0 >^2 d\mu_{P_0}(z).$$

where $\gamma := \int_{R^N} p w^{p-1} w' u_*'(r)$ and $u_*'(r)$ is the unique radial solution of (1.12).

By assumptions (H1) and (H2), it is easy to see that $\tilde{K}_\epsilon(P)$ has only one critical point in $B_{\beta_0 \epsilon}(Q_\epsilon^0)$.

We now compute $\nabla K_\epsilon(P)$. The computation is very complicated. The reader can refer to Section 7 in [29]. We need one key estimate.

Let $w_{\epsilon,P}$ be defined by (1.9). Recall that $\varphi_{\epsilon,P}(x) = w((x-P)/\epsilon) - w_{\epsilon,P}(x)$.

The following key lemma gives the estimates we need. (See Section 9 in [29].)

Lemma 3.1 *Suppose that P_0 is a nondegenerate peak point. Let $P_\epsilon = P_0 + \epsilon b_\epsilon$, then we have*

$$\varphi_{\epsilon, P_\epsilon}(x) = (1+o(1)) \varphi_{\epsilon, P_0}(P_0) \times \int_{\partial\Omega} e^{\langle \frac{z-P_0}{|z-P_0|}, y \rangle} e^{\langle \frac{z-P_0}{|z-P_0|}, 2b_\epsilon \rangle} d\mu_{P_0}(z)$$

where $d\mu_{P_0}(z)$ is defined by (1.3).

By using Lemma 3.1, we obtain

Lemma 3.2 *For $P \in B_{\beta_0 \epsilon}(Q_\epsilon^0)$, we have*

$$\nabla K_\epsilon(P) = \nabla \tilde{K}_\epsilon(P) + O(\beta_0 |P - Q_\epsilon^0| \epsilon^{-2} \varphi_{\epsilon, P_0}(P_0)). \tag{3.1}$$

Proof Observe that

$$\nabla_j K_\epsilon(P) =< w_{\epsilon,P} + v_{\epsilon,P}, \partial_j (w_{\epsilon,P} + v_{\epsilon,P}) >_\epsilon - \int_{\Omega_{\epsilon,P}} (w_{\epsilon,P} + v_{\epsilon,P})^p \partial_j (w_{\epsilon,P} + v_{\epsilon,P})$$

$$= \int_{\Omega_{\epsilon,P}} [w^p - (w_{\epsilon,P})^p](\partial_j w_{\epsilon,P}) + O(|\varphi_{\epsilon,P}(P)|^{1+\sigma}) \text{ (by Lemma 2.4)}$$

$$= \int_{\Omega_{\epsilon,P}} p w^{p-1} \varphi_{\epsilon,P}(x) (-\epsilon^{-1} \frac{\partial w}{\partial y_j}) + O(|\varphi_{\epsilon,P}(P)|^{1+\sigma})$$

$$= (-\epsilon^{-1}(1+O(\beta_0)) \varphi_{\epsilon,P_0}(P_0)) \int_{\Omega_{\epsilon,P}} p w^{p-1} \frac{\partial w}{\partial y_j} \Big(\int_{\partial\Omega} e^{\langle \frac{z-P_0}{|z-P_0|}, y \rangle} e^{\langle z-P_0, \frac{2}{d(P_0,\partial\Omega)} \frac{(P-P_0)}{\epsilon} \rangle} d\mu_{P_0}(z) \Big) dy$$

(by Lemma 3.1)

$$= (-\epsilon^{-2}\varphi_{\epsilon,P_0}(P_0))(2\gamma+O(\beta_0))\times\int_{\partial\Omega} e^{<z-P_0,a>}(\frac{z-P_0}{|z-P_0|})_j < \frac{z-P_0}{|z-P_0|}, P-Q_\epsilon^0 > d\mu_{P_0}(z).$$

□

The next lemma shows that each critical point of $K_\epsilon(P)$ is nondegenerate.

Lemma 3.3 *Let Q_ϵ be a critical point of $K_\epsilon(P)$ over $B_{\beta_0\epsilon}(Q_\epsilon^0)$. Then we have*

$$\frac{\partial^2 K_\epsilon(P)}{\partial P_i \partial P_j}|_{P=Q_\epsilon}$$

$$= \frac{-2\gamma}{\epsilon^2 d^2(P_0,\partial\Omega)}\varphi_{\epsilon,P_0}(P_0)(\int_{\partial\Omega} e^{<z-P_0,a>}(z-P_0)_i(z-P_0)_j d\mu_{P_0}(z) + O(\beta_0)). \tag{3.2}$$

Proof In fact, we have

$$\frac{\partial^2 K_\epsilon(P)}{\partial P_i \partial P_j}|_{P=Q_\epsilon}$$

$$=< \frac{\partial(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_i}, \frac{\partial(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_j} >_\epsilon |_{P=Q_\epsilon} + < w_{\epsilon,P}+v_{\epsilon,P}, \frac{\partial^2(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_i \partial P_j} >_\epsilon |_{P=Q_\epsilon}$$

$$-\int_{\Omega_{\epsilon,Q_\epsilon}} p(w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon})^{p-1}\frac{\partial(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_i}|_{P=Q_\epsilon}\frac{\partial(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_j}|_{P=Q_\epsilon}$$

$$-\int_{\Omega_{\epsilon,Q_\epsilon}} (w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon})^{p}\frac{\partial^2(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_i \partial P_j}|_{P=Q_\epsilon}$$

$$=< \frac{\partial(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_j}, \frac{\partial(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_j} >_\epsilon |_{P=Q_\epsilon}$$

$$-\int_{\Omega_{\epsilon,Q_\epsilon}} p(w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon})^{p-1}\frac{\partial(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_j}|_{P=Q_\epsilon}\frac{\partial(w_{\epsilon,P}+v_{\epsilon,P})}{\partial P_j}|_{P=Q_\epsilon}$$

$$=< \frac{\partial w_{\epsilon,P}}{\partial P_i}, \frac{\partial w_{\epsilon,P}}{\partial P_j} >_\epsilon |_{P=Q_\epsilon} - \int_{\Omega_{\epsilon,Q_\epsilon}} p(w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon})^{p-1}\frac{\partial w_{\epsilon,P}}{\partial P_i}|_{P=Q_\epsilon}\frac{\partial w_{\epsilon,P}}{\partial P_j}|_{P=Q_\epsilon}$$

$$+ < \frac{\partial w_{\epsilon,P}}{\partial P_i}, \frac{\partial v_{\epsilon,P}}{\partial P_j} >_\epsilon |_{P=Q_\epsilon} - \int_{\Omega_{\epsilon,Q_\epsilon}} p(w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon})^{p-1}\frac{\partial w_{\epsilon,P}}{\partial P_i}|_{P=Q_\epsilon}\frac{\partial v_{\epsilon,P}}{\partial P_j}|_{P=Q_\epsilon}$$

$$+ < \frac{\partial w_{\epsilon,P}}{\partial P_j}, \frac{\partial v_{\epsilon,P}}{\partial P_i} >_\epsilon |_{P=Q_\epsilon} - \int_{\Omega_{\epsilon,Q_\epsilon}} p(w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon})^{p-1}\frac{\partial w_{\epsilon,P}}{\partial P_j}|_{P=Q_\epsilon}\frac{\partial v_{\epsilon,P}}{\partial P_i}|_{P=Q_\epsilon}$$

$$+ < \frac{\partial v_{\epsilon,P}}{\partial P_i}, \frac{\partial v_{\epsilon,P}}{\partial P_j} >_\epsilon |_{P=Q_\epsilon} - \int_{\Omega_{\epsilon,Q_\epsilon}} p(w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon})^{p-1}\frac{\partial v_{\epsilon,P}}{\partial P_i}|_{P=Q_\epsilon}\frac{\partial v_{\epsilon,P}}{\partial P_j}|_{P=Q_\epsilon}$$

$$= I_1 + I_2 + I_3 + I_4,$$

where $I_i, i = 1,2,3,4$ are defined at the last equality.

We first note that by Lemma 2.4,

$$I_4 = O(\epsilon^{-4}|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma}) = o(\epsilon^{-2}|\varphi_{\epsilon,P_0}(P_0)|).$$

Next,

$$I_2 = \int_{\Omega_{\epsilon,Q_\epsilon}} [pw^{p-1}\frac{\partial w}{\partial P_i} - p(w_{\epsilon,P}+v_{\epsilon,P})^{p-1}\frac{\partial w_{\epsilon,P}}{\partial P_i}]\frac{\partial v_{\epsilon,P}}{\partial P_j}|_{P=Q_\epsilon}$$
$$= O(\epsilon^{-4}|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma}) = o(\epsilon^{-2}|\varphi_{\epsilon,P_0}(P_0)|)$$

by Lemma 2.4.

Similarly

$$I_3 = O(\epsilon^{-4}|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma}) = o(\epsilon^{-2}|\varphi_{\epsilon,P_0}(P_0)|).$$

It remains to estimate I_1:

$$I_1 = \int_{\Omega_{\epsilon,Q_\epsilon}} [pw^{p-1}\frac{\partial w}{\partial P_j}|_{P=Q_\epsilon} - p(w_{\epsilon,Q_\epsilon}+v_{\epsilon,Q_\epsilon})^{p-1}\frac{\partial w_{\epsilon,P}}{\partial P_j}|_{P=Q_\epsilon}]\frac{\partial w_{\epsilon,P}}{\partial P_i}|_{P=Q_\epsilon}$$

$$= \int_{\Omega_{\epsilon,Q_\epsilon}} [pw^{p-1}\frac{\partial w}{\partial P_j}|_{P=Q_\epsilon} - p(w_{\epsilon,Q_\epsilon})^{p-1}\frac{\partial w_{\epsilon,P}}{\partial P_j}|_{P=Q_\epsilon}]\frac{\partial w_{\epsilon,P}}{\partial P_i}|_{P=Q_\epsilon} + O(\epsilon^{-2}|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

$$= \int_{\Omega_{\epsilon,Q_\epsilon}} \frac{\partial}{\partial P_j}|_{P=Q_\epsilon}[w^p - (w_{\epsilon,Q_\epsilon})^p]\frac{\partial w_{\epsilon,P}}{\partial P_i}|_{P=Q_\epsilon} + O(\epsilon^{-2}|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

$$= \int_{\Omega_{\epsilon,Q_\epsilon}} \frac{\partial}{\partial P_j}|_{P=Q_\epsilon}[pw^{p-1}\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon+\epsilon y)]\frac{\partial w_{\epsilon,P}}{\partial P_i}|_{P=Q_\epsilon} + O(\epsilon^{-2}|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

$$= (-2\epsilon^{-2})\varphi_{\epsilon,P_0}(P_0)(1+o(1))$$

$$\times \int_{R^N} pw^{p-1}\int_{\partial\Omega} e^{<\frac{z-P_0}{|z-P_0|},y>}e^{<\frac{z-P_0}{|z-P_0|},\frac{2(Q_\epsilon-P_0)}{\epsilon}>}d\mu_{P_0}(z)(\frac{z-P_0}{|z-P_0|})_j\frac{\partial w}{\partial y_i} + O(\epsilon^{-2}|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

(by Lemma 3.1)

$$= (-2\epsilon^{-2})\varphi_{\epsilon,P_0}(P_0)(1+O(\beta_0))$$

$$\times \int_{R^N} pw^{p-1}\int_{\partial\Omega} e^{<\frac{z-P_0}{|z-P_0|},y>}e^{<z-P_0,a>}(\frac{z-P_0}{|z-P_0|})_j d\mu_{P_0}(z)\frac{\partial w}{\partial y_i}dy + O(\epsilon^{-2}|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

$$= \frac{-2\gamma}{\epsilon^2 d^2(P_0,\partial\Omega)}\varphi_{\epsilon,P_0}(P_0)(\int_{\partial\Omega} e^{<z-P_0,a>}(z-P_0)_i(z-P_0)_j d\mu_{P_0}(z) + O(\beta_0)).$$

Combining the estimates for $I_i, i=1,2,3,4$, we obtain (3.2). □

The proof of Theorem 1.1 is completed by the following lemma.

Lemma 3.4 *There exists a unique critical point of $K_\epsilon(P)$ over $B_{\beta_0\epsilon}(Q_\epsilon^0)$.*

Proof As we already know, $K_\epsilon(P)$ has a critical point $Q_\epsilon = P_0 + \epsilon(\frac{1}{2}d(P_0,\partial\Omega)a + o(1))$ and any other critical point of $K_\epsilon(P)$ is in $B_{\beta_0\epsilon}(Q_\epsilon^0)$.

We now show that Q_ϵ is unique.

First, by Lemma 3.3, there are only finite number of critical points of $K_\epsilon(P)$ in $B_{\beta_0\epsilon}(Q_\epsilon^0)$. Let k_ϵ be the number of critical points. At each critical point, we have by Lemma 3.3,

$$deg(\nabla K_\epsilon, B_{\beta_i\epsilon}(Q_\epsilon^i), 0) = (-1)^N$$

where $\beta_i > 0$ are small constants so that $B_{\beta_i\epsilon}(Q_\epsilon^i)$ contains only one critical point (i.e. Q_ϵ^i) of $\nabla K_\epsilon(P)$, since $G(P_0)$ has N positive eigenvalues.

Hence by the additivity of the degree we have

$$deg(\nabla K_\epsilon, B_{\beta_0\epsilon}(P_0), 0) = k_\epsilon(-1)^N.$$

On the other hand, since $\tilde{K}_\epsilon(P)$ has only one critical point in $B_{\beta_0\epsilon}(Q_\epsilon^0)$ and by Lemma 3.2, $\nabla K_\epsilon(P) = \nabla\tilde{K}_\epsilon(P) + O(\beta_0\epsilon^{-2}\varphi_{\epsilon,P_0}(P_0))$, by a continuity argument (note that $\nabla K_\epsilon(P) \neq 0$ and $\nabla\tilde{K}_\epsilon(P) \neq 0$ on $\partial B_{\beta_0\epsilon}(Q_\epsilon^0)$), we obtain

$$deg(\nabla K_\epsilon, B_{\beta_0\epsilon}(Q_\epsilon^0), 0) = deg(\nabla\tilde{K}_\epsilon(P), B_{\beta_0\epsilon}(Q_\epsilon^0), 0) = (-1)^N.$$

Hence $k_\epsilon = 1$.

□

4 Eigenvalue Estimates: Proofs of Theorem 1.3

In this section, we shall study eigenvalue estimates for $L_\epsilon := \epsilon^2\Delta - 1 + p(u_\epsilon)^{p-1}$ and finish the proof of Theorem 1.3.

Proof of Theorem 1.3: Let $u_\epsilon = w_{\epsilon,Q_\epsilon} + v_{\epsilon,Q_\epsilon}$. By Section 3, u_ϵ is unique. Let $(\tau^\epsilon, \phi_\epsilon)$ be a pair such that

$$L_\epsilon\phi_\epsilon = \tau^\epsilon\phi_\epsilon \text{ in } \Omega, \frac{\partial\phi_\epsilon}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{4.1}$$

We may normalize ϕ_ϵ such that $\|\phi_\epsilon\|_\epsilon = 1$.

If $|\tau_\epsilon| \geq \alpha > 0$, then as $\epsilon \to 0$, $\tau_\epsilon \to \mu_j$ for some $j \notin \{2, ..., N+1\}$.

We now assume that $\tau_\epsilon \to 0$ as $\epsilon \to 0$. Then after a scaling and limiting process (see [20], [21] and [24]), we have $\tilde{\phi}_\epsilon(y) = \phi_\epsilon(Q_\epsilon + \epsilon y) \to \phi_0$, where ϕ_0 is a solution of

$$\Delta v - v + pw^{p-1}v = 0 \text{ in } R^N, v \in H^1(R^N).$$

By Lemma 4.2 of [21], there exists s_j such that $\phi_0 = \sum_{j=1}^N s_j \frac{\partial w}{\partial y_j}$.

This suggests that we should decompose ϕ_ϵ as $\phi_\epsilon = \sum_{j=1}^N s_j^\epsilon \epsilon\partial_j w_{\epsilon,Q_\epsilon} + \bar{\phi}_\epsilon$, where $\bar{\phi}_\epsilon \in \mathcal{K}_{\epsilon,Q_\epsilon}^\perp$ and $|s_j^\epsilon| \leq C$. Since $\|\phi_\epsilon\|_\epsilon = 1$, we have $\|\bar{\phi}_\epsilon\|_\epsilon \leq C$ and $\bar{\phi}_\epsilon$ satisfies

$$(L_\epsilon - \tau^\epsilon)\bar{\phi}_\epsilon + \sum_{j=1}^N s_j^\epsilon[p(u_\epsilon)^{p-1}\epsilon\partial_j w_{\epsilon,Q_\epsilon} - pw^{p-1}\epsilon\partial_j w] = \tau^\epsilon \sum_{j=1}^N s_j^\epsilon \epsilon\partial_j w_{\epsilon,Q_\epsilon}. \tag{4.2}$$

Since $\tau^\epsilon \to 0$, then by the same argument as in Proposition 6.3 of [29] we have that

$\pi_{\epsilon,Q_\epsilon} \circ (L_\epsilon - \tau^\epsilon) : \mathcal{K}^\perp_{\epsilon,Q_\epsilon} \to \mathcal{C}^\perp_{\epsilon,Q_\epsilon}$ is invertible. Since $\bar{\phi}_\epsilon \in \mathcal{K}^\perp_{\epsilon,Q_\epsilon}$, we have

$$\|\bar{\phi}_\epsilon\|_{H^2(\Omega_{\epsilon,Q_\epsilon})} = O(\| \sum_{j=1}^N s_j^\epsilon [p(u_\epsilon)^{p-1} \epsilon \partial_j w_{\epsilon,Q_\epsilon} - pw^{p-1} \epsilon \partial_j w] \|_{L^2(\Omega_{\epsilon,Q_\epsilon})})$$

$$= O((|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{(1+\sigma)/2}) \sum_{j=1}^N |s_j^\epsilon|).$$

Multiplying (4.2) by $\epsilon \partial_k (w_{\epsilon,Q_\epsilon})$ we obtain

$$\sum_{j=1}^N s_j^\epsilon \int_{\Omega_{\epsilon,Q_\epsilon}} [p(u_\epsilon)^{p-1} \epsilon \partial_j w_{\epsilon,Q_\epsilon} - pw^{p-1} \epsilon \partial_j w] \epsilon \partial_k w_{\epsilon,Q_\epsilon}$$

$$= \tau^\epsilon \sum_{j=1}^N \int_{\Omega_{\epsilon,Q_\epsilon}} s_j^\epsilon \epsilon \partial_j w_{\epsilon,Q_\epsilon} \epsilon \partial_k w_{\epsilon,Q_\epsilon}$$

$$+ \int_{\Omega_{\epsilon,Q_\epsilon}} [p(u_\epsilon)^{p-1} \bar{\phi}_\epsilon \epsilon \partial_k (w_{\epsilon,Q_\epsilon}) - pw^{p-1} \bar{\phi}_\epsilon \epsilon \partial_k w]. \tag{4.3}$$

We first estimate the right-hand side of (4.3). By some simple computations as in the proof of the estimate I_1 in Lemma 3.3, we conclude that

$$RHS \quad of \ (4.3)$$

$$= \tau^\epsilon \sum_{j=1}^N s_j^\epsilon (A\delta_{jk} + o(1)) + O(\sum_{j=1}^N |s_j^\epsilon| |\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{(1+\sigma)}) + O(\tau^\epsilon (\sum_{j=1}^N |s_j^\epsilon| |\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{(1+\sigma)/2}))$$

where $A = \int_{R^N} (\frac{\partial w}{\partial y_1})^2 dy$.

For the left hand side of (4.3), we have by Estimate C in Appendix A

$$LHS \quad of \quad (4.3)$$

$$= \sum_{j=1}^N s_j^\epsilon (\int_{\Omega_{\epsilon,Q_\epsilon}} [p(w_{\epsilon,Q_\epsilon})^{p-1} \epsilon \partial_j w_{\epsilon,Q_\epsilon} - pw^{p-1} \epsilon \partial_j w] \epsilon \partial_k w_{\epsilon,Q_\epsilon}) + O(|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma}))$$

$$= \int_{\partial\Omega} e^{<z-P_0,a>} < \frac{z-P_0}{|z-P_0|}, s^\epsilon > (\frac{z-P_0}{|z-P_0|})_k \varphi_{\epsilon,P_0}(P_0)(2\gamma + o(1))$$

where $s^\epsilon = (s_1^\epsilon, ..., s_N^\epsilon)$.

Hence we have

$$\tau^\epsilon = O(\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)) = O(\varphi_{\epsilon,P_0}(P_0))$$

and $\tau^\epsilon / \varphi_{\epsilon,P_0}(P_0) \to \tau_0, s^\epsilon \to s$ where (τ_0, s) satisfies

$$(2\gamma) G(P_0) s = A d^2(P_0, \partial\Omega) \tau_0 s.$$

Hence $\frac{Ad^2(P_0,\partial\Omega)}{2\gamma} \tau_0$ is an eigenvalue of $G(P_0)$. Therefore $\tau^\epsilon / \varphi_{\epsilon,P_0}(P_0) \to \tau_j, s^\epsilon \to \vec{a}_j$

where

$$\tau_j = \frac{2\gamma}{Ad^2(P_0, \partial\Omega)}\lambda_j, G(P_0)\vec{a}_j = \lambda_j \vec{a}_j, j = 1, ..., N.$$

On the other hand, we can construct a solution $(\phi_{\epsilon,j}, \tau_{\epsilon,j})$ of (4.1) by putting

$$\phi_{\epsilon,j} = \partial_j(w_{\epsilon,Q_\epsilon} + v_{\epsilon,Q_\epsilon}) + \epsilon^2\Psi_{\epsilon,j}, \Psi_{\epsilon,j} \in \mathcal{K}^{\perp}_{\epsilon,Q_\epsilon},$$

$$\tau_{\epsilon,j} = \varphi_{\epsilon,P_0}(P_0)(\tau_j + \sigma_j), \sigma_j = o(1).$$

By using Lemma 2.3 and the implicit function theorem, we can conclude that there exists $(\phi_{\epsilon,j}, \tau_{\epsilon,j})$ satisfying equation (4.1). This finishes the proof of Theorem 1.3.

5 Proof of Theorem 1.4

In this section, we prove Theorem 1.4 and thus finish the proof of Corollary 1.2.

We first remark that if (1.18) has an eigenvalue $\alpha_0 \neq 0$ and $Re(\alpha_0) \geq 0$, then we have $|s + 1 + \tau\alpha_0| \geq s + 1$ and thus $|\alpha_0| \leq C$ where C is independent of τ. Hence, to prove Theorem 1.4 for τ small, it is enough to show that all nonzero eigenvalues α_0 of the following eigenvalue problem

$$\Delta\phi - \phi + pw^{p-1}\phi - \frac{qr}{s+1}\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p = \alpha_0\phi, \phi \in H^2(R^N) \tag{5.1}$$

have negative real part.

To this end, we first introduce some notations and make some preparations. Set

$$L\phi := L_0\phi - \gamma_0(p-1)\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p, \phi \in H^2(R^N)$$

where $\gamma_0 = \frac{qr}{(s+1)(p-1)} > 1$ and $L_0 := \Delta - 1 + pw^{p-1}$. Note that L is not selfadjoint if $r \neq p+1$.

Let

$$X_0 := kernel(L_0) = span\{\frac{\partial w}{\partial y_j} | j = 1, ..., N\}.$$

Then

$$L_0 w = (p-1)w^p, L_0(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) = w \tag{5.2}$$

and

$$\int_{R^N}(L_0^{-1}w)w = \int_{R^N} w(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) = (\frac{1}{p-1} - \frac{N}{4})\int_{R^N} w^2, \tag{5.3}$$

$$\begin{aligned}\int_{R^N}(L_0^{-1}w)w^p &= \int_{R^N} w^p(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) \\ &= \int_{R^N}(L_0^{-1}w)\frac{1}{p-1}L_0 w = \frac{1}{p-1}\int_{R^N} w^2.\end{aligned} \tag{5.4}$$

Proof of Theorem 1.4:

We divide the proof into three cases.

Case 1: $r = 2, 1 < p < 1 + \frac{4}{N}$.

Since L is not self-adjoint, we introduce a new operator as follows:

$$L_1\phi := L_0\phi - (p-1)\frac{\int_{R^N} w\phi}{\int_{R^N} w^2}w^p - (p-1)\frac{\int_{R^N} w^p\phi}{\int_{R^N} w^2}w + (p-1)\frac{\int_{R^N} w^{p+1}\int_{R^N} w\phi}{(\int_{R^N} w^2)^2}w. \quad (5.5)$$

We have the following important lemma.

Lemma 5.1 *(1) L_1 is selfadjoint and the kernel of L_1 (denoted by X_1) = span $\{w, \frac{\partial w}{\partial y_j}, j = 1, ..., N\}$. (2) There exists a positive constant $a_1 > 0$ such that*

$$L_1(\phi,\phi)$$

$$:= \int_{R^N}(|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2) + \frac{2(p-1)\int_{R^N} w\phi\int_{R^N} w^p\phi}{\int_{R^N} w^2} - (p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}(\int_{R^N} w\phi)^2$$

$$\geq a_1 d^2_{L^2(R^N)}(\phi, X_1)$$

for all $\phi \in H^1(R^N)$, where $d_{L^2(R^N)}$ means the distance in L^2-norm.

Proof By (5.5), L_1 is selfadjoint. Next we compute the kernel of L_1. It is easy to see that $w, \frac{\partial w}{\partial y_j}, j = 1, ..., N, \in kernel(L_1)$. On the other hand, if $\phi \in kernel(L_1)$, then by (5.2)

$$L_0\phi = c_1(\phi)w + c_2(\phi)w^p = c_1(\phi)L_0(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) + c_2(\phi)L_0(\frac{w}{p-1})$$

where

$$c_1(\phi) = (p-1)\frac{\int_{R^N} w^p\phi}{\int_{R^N} w^2} - (p-1)\frac{\int_{R^N} w^{p+1}\int_{R^N} w\phi}{(\int_{R^N} w^2)^2}, c_2(\phi) = (p-1)\frac{\int_{R^N} w\phi}{\int_{R^N} w^2}.$$

Hence

$$\phi - c_1(\phi)(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) - c_2(\phi)\frac{1}{p-1}w \in kernel(L_0). \quad (5.6)$$

Note that

$$c_1(\phi) = (p-1)c_1(\phi)\frac{\int_{R^N} w^p(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{\int_{R^N} w^2} - (p-1)c_1(\phi)\frac{\int_{R^N} w^{p+1}\int_{R^N} w(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{(\int_{R^N} w^2)^2}$$

$$= c_1(\phi) - c_1(\phi)(\frac{1}{p-1} - \frac{N}{4})\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}$$

by (5.3) and (5.4). This implies that $c_1(\phi) = 0$. By (5.6) and Lemma 1.2, this proves (1).

It remains to prove (2). Suppose (2) is not true, then by (1) there exists (α, ϕ) such that (i) α is real and positive, (ii) $\phi \perp w$, $\phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \ldots, N$, and (iii) $L_1\phi = \alpha\phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0-\alpha)\phi=(p-1)\frac{\int_{R^N}w^p\phi}{\int_{R^N}w^2}w. \tag{5.7}$$

We first claim that $\int_{R^N}w^p\phi\neq 0$. In fact if $\int_{R^N}w^p\phi=0$, then $\alpha>0$ is an eigenvalue of L_0. By Lemma 1.2, $\alpha=\mu_1$ and ϕ has constant sign. This contradicts with the fact that $\phi\perp w$. Therefore $\alpha\neq\mu_1,0$, and hence $L_0-\alpha$ is invertible in $X_0^{\perp}$. So (5.7) implies

$$\phi=(p-1)\frac{\int_{R^N}w^p\phi}{\int_{R^N}w^2}(L_0-\alpha)^{-1}w.$$

Thus

$$\int_{R^N}w^p\phi=(p-1)\frac{\int_{R^N}w^p\phi}{\int_{R^N}w^2}\int_{R^N}((L_0-\alpha)^{-1}w)w^p,$$

$$\int_{R^N}w^2=(p-1)\int_{R^N}((L_0-\alpha)^{-1}w)w^p,$$

$$\int_{R^N}w^2=\int_{R^N}((L_0-\alpha)^{-1}w)((L_0-\alpha)w+\alpha w),$$

$$0=\int_{R^N}((L_0-\alpha)^{-1}w)w. \tag{5.8}$$

Let $h_1(\alpha)=\int_{R^N}((L_0-\alpha)^{-1}w)w$, then $h_1(0)=\int_{R^N}(L_0^{-1}w)w=\int_{R^N}(\frac{1}{p-1}w+\frac{1}{2}x\cdot\nabla w)w=(\frac{1}{p-1}-\frac{N}{4})\int_{R^N}w^2>0$ since $1<p<1+\frac{4}{N}$. Moreover $h_1'(\alpha)=\int_{R^N}((L_0-\alpha)^{-2}w)w=\int_{R^N}((L_0-\alpha)^{-1}w)^2>0$. This implies $h_1(\alpha)>0$ for all $\alpha\in(0,\mu_1)$. Clearly, also $h_1(\alpha)<0$ for $\alpha\in(\mu_1,\infty)$ (since $\lim_{\alpha\to+\infty}h_1(\alpha)=0$). This is a contradiction to (5.8)!

□

We now finish the proof of Theorem 1.4 in Case 1. Let $\alpha_0=\alpha_R+i\alpha_I$ and $\phi=\phi_R+i\phi_I$. Since $\alpha_0\neq 0$, we can choose $\phi\perp kernel(L_0)$. Then we obtain two equations

$$L_0\phi_R-(p-1)\gamma_0\frac{\int_{R^N}w\phi_R}{\int_{R^N}w^2}w^p=\alpha_R\phi_R-\alpha_I\phi_I, \tag{5.9}$$

$$L_0\phi_I-(p-1)\gamma_0\frac{\int_{R^N}w\phi_I}{\int_{R^N}w^2}w^p=\alpha_R\phi_I+\alpha_I\phi_R. \tag{5.10}$$

Multiplying (5.9) by ϕ_R and (5.10) by ϕ_I and adding them together, we obtain

$$-\alpha_R\int_{R^N}(\phi_R^2+\phi_I^2)=L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)$$

$$+(p-1)(\gamma_0-2)\frac{\int_{R^N}w\phi_R\int_{R^N}w^p\phi_R+\int_{R^N}w\phi_I\int_{R^N}w^p\phi_I}{\int_{R^N}w^2}$$

$$+(p-1)\frac{\int_{R^N}w^{p+1}}{(\int_{R^N}w^2)^2}[(\int_{R^N}w\phi_R)^2+(\int_{R^N}w\phi_I)^2].$$

Multiplying (5.9) by w and (5.10) by w we obtain

$$(p-1)\int_{R^N} w^p\phi_R - \gamma_0(p-1)\frac{\int_{R^N} w\phi_R}{\int_{R^N} w^2}\int_{R^N} w^{p+1} = \alpha_R\int_{R^N} w\phi_R - \alpha_I\int_{R^N} w\phi_I, \quad (5.11)$$

$$(p-1)\int_{R^N} w^p\phi_I - \gamma_0(p-1)\frac{\int_{R^N} w\phi_I}{\int_{R^N} w^2}\int_{R^N} w^{p+1} = \alpha_R\int_{R^N} w\phi_I + \alpha_I\int_{R^N} w\phi_R. \quad (5.12)$$

Multiplying (5.11) by $\int_{R^N} w\phi_R$ and (5.12) by $\int_{R^N} w\phi_I$ and adding them together, we obtain

$$(p-1)\int_{R^N} w\phi_R\int_{R^N} w^p\phi_R + (p-1)\int_{R^N} w\phi_I\int_{R^N} w^p\phi_I$$

$$= (\alpha_R + \gamma_0(p-1)\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2})((\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2).$$

Therefore we have

$$-\alpha_R\int_{R^N}(\phi_R^2+\phi_I^2) = L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$

$$+(p-1)(\gamma_0-2)(\frac{1}{p-1}\alpha_R + \gamma_0\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2})\frac{(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2}{\int_{R^N} w^2}$$

$$+(p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}[(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2].$$

Set

$$\phi_R = c_R w + \phi_R^\perp, \phi_R^\perp \perp X_1, \phi_I = c_I w + \phi_I^\perp, \phi_I^\perp \perp X_1.$$

Then

$$\int_{R^N} w\phi_R = c_R\int_{R^N} w^2, \int_{R^N} w\phi_I = c_I\int_{R^N} w^2,$$

$$d^2_{L^2(R^N)}(\phi_R, X_1) = \|\phi_R^\perp\|^2_{L^2}, d^2_{L^2(R^N)}(\phi_I, X_1) = \|\phi_I^\perp\|^2_{L^2}.$$

By some simple computations we have

$$L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$

$$+(\gamma_0-1)\alpha_R(c_R^2+c_I^2)\int_{R^N} w^2 + (p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{R^N} w^{p+1} + \alpha_R(\|\phi_R^\perp\|^2_{L^2} + \|\phi_I^\perp\|^2_{L^2}) = 0.$$

By Lemma 5.1 (2)

$$(\gamma_0-1)\alpha_R(c_R^2+c_I^2)\int_{R^N} w^2$$

$$+(p-1)(\gamma_0-1)^2(c_R^2+c_I^2)\int_{R^N} w^{p+1} + (\alpha_R + a_1)(\|\phi_R^\perp\|^2_{L^2} + \|\phi_I^\perp\|^2_{L^2}) \le 0.$$

Since $\gamma_0 > 1$, we must have $\alpha_R < 0$, which proves Theorem 1.4 in Case 1.

Case 2: $r = 2, p = 1 + \frac{4}{N}$.

In this case we have

$$\int_{R^N}(L_0^{-1}w)w = \int_{R^N} w(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) = 0. \tag{5.13}$$

Set

$$w_0 = \frac{1}{p-1}w + \frac{1}{2}x\nabla w. \tag{5.14}$$

We will follow the proof in Case 1. We just need to take care of w_0. We first have the following lemma which is similar to Lemma 5.1. The proof is omitted.

Lemma 5.2 *(1) The kernel of L_1 is given by $X_1 = span\ \{w, w_0, \frac{\partial w}{\partial y_j}, j = 1, ..., N\}$. (2) There exists a positive constant $a_2 > 0$ such that*

$$L_1(\phi,\phi) = \int_{R^N}(|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2)$$

$$+\frac{2(p-1)\int_{R^N} w\phi \int_{R^N} w^p\phi}{\int_{R^N} w^2} - (p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}(\int_{R^N} w\phi)^2$$

$$\geq a_2 d^2_{L^2(R^N)}(\phi, X_1), \quad \forall\phi \in H^1(R^N).$$

Now we can finish the proof of Theorem 1.4 in Case 2.

Suppose that $\alpha_0 \neq 0$ is an eigenvalue of L. Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp kernel(L_0)$. Then similar to Case 1, we obtain two equations (5.9) and (5.10). We now decompose

$$\phi_R = c_R w + b_R w_0 + \phi_R^\perp, \phi_R^\perp \perp X_1, \phi_I = c_I w + b_I w_0 + \phi_I^\perp, \phi_I^\perp \perp X_1.$$

Similar to Case 1, we obtain

$$L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I)$$

$$+(\gamma_0 - 1)\alpha_R(c_R^2 + c_I^2)\int_{R^N} w^2 + (p-1)(\gamma_0-1)^2(c_R^2 + c_I^2)\int_{R^N} w^{p+1}$$

$$+\alpha_R(b_R^2(\int_{R^N} w_0^2)^2 + b_I^2(\int_{R^N} w_0^2)^2 + \|\phi_R^\perp\|^2_{L^2} + \|\phi_I^\perp\|^2_{L^2}) \leq 0$$

By Lemma 5.2 (2)

$$(\gamma_0 - 1)\alpha_R(c_R^2 + c_I^2)\int_{R^N} w^2 + (p-1)(\gamma_0-1)^2(c_R^2 + c_I^2)\int_{R^N} w^{p+1}$$

$$+\alpha_R(b_R^2(\int_{R^N} w_0^2)^2 + b_I^2(\int_{R^N} w_0^2)^2) + (\alpha_R + a_2)(\|\phi_R^\perp\|^2_{L^2} + \|\phi_I^\perp\|^2_{L^2}) \leq 0$$

If $\alpha_R \geq 0$, then necessarily we have

$$c_R = c_I = 0, \phi_R^\perp = 0, \phi_I^\perp = 0.$$

Hence $\phi_R = b_R w_0, \phi_I = b_I w_0$. This implies that

$$b_R L_0 w_0 = (b_R - b_I) w_0, b_I L_0 w_0 = (b_R + b_I) w_0,$$

which is impossible unless $b_R = b_I = 0$. A contradiction !

Case 3: $r = p+1, 1 < p < (\frac{N+2}{N-2})_+$.

Let $r = p+1$. L becomes

$$L = L_0 - \frac{qr}{s+1} \frac{\int_{R^N} w^p \cdot}{\int_{R^N} w^{p+1}} w^p.$$

We will follow the proof of Case 1. We need to define another operator.

$$L_3 \phi := L_0 \phi - (p-1) \frac{\int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} w^p. \tag{5.15}$$

We have the following lemma.

Lemma 5.3 *(1) L_3 is selfadjoint and the kernel of L_3 (denoted by X_3) consists of $w, \frac{\partial w}{\partial y_j}, j = 1, ..., N$. (2) There exists a positive constant $a_3 > 0$ such that*

$$L_3(\phi, \phi) := \int_{R^N} (|\nabla \phi|^2 + \phi^2 - p w^{p-1} \phi^2) + \frac{(p-1)(\int_{R^N} w^p \phi)^2}{\int_{R^N} w^{p+1}}$$

$$\geq a_3 d^2_{L^2(R^N)}(\phi, X_3), \ \forall \phi \in H^1(R^N).$$

Proof The proof of (1) is similar to that of Lemma 5.1. We omit the details. It remains to prove (2). Suppose (2) is not true, then by (1) there exists (α, ϕ) such that (i) α is real and positive, (ii) $\phi \perp w, \phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \ldots, N$, and (iii) $L_3 \phi = \alpha \phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0 - \alpha) \phi = \frac{(p-1) \int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} w^p. \tag{5.16}$$

Similar to the proof of Lemma 5.1, we have that $\int_{R^N} w^p \phi \neq 0, \alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^\perp$. So (5.16) implies

$$\phi = \frac{(p-1) \int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} (L_0 - \alpha)^{-1} w^p.$$

Thus

$$\int_{R^N} w^p \phi = (p-1) \frac{\int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} \int_{R^N} ((L_0 - \alpha)^{-1} w^p) w^p,$$

$$\int_{R^N} w^{p+1} = (p-1) \int_{R^N} ((L_0 - \alpha)^{-1} w^p) w^p. \tag{5.17}$$

Let $h_3(\alpha) = (p-1) \int_{R^N} ((L_0 - \alpha)^{-1} w^p) w^p - \int_{R^N} w^{p+1}$, then $h_3(0) = (p-1) \int_{R^N} (L_0^{-1} w^p) w^p -$

$\int_{R^N} w^{p+1} = 0$. Moreover $h_3'(\alpha) = (p-1)\int_{R^N}((L_0-\alpha)^{-2}w^p)w^p = (p-1)\int_{R^N}((L_0-\alpha)^{-1}w^p)^2 > 0$. This implies $h_3(\alpha) > 0$ for all $\alpha \in (0,\mu_1)$. Clearly, also $h_3(\alpha) < 0$ for $\alpha \in (\mu_1,\infty)$. A contradiction to (5.17)!

□

We now finish the proof of Theorem 1.4 in Case 3.

Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp kernel(L_0)$. Then similarly we obtain two equations

$$L_0\phi_R - (p-1)\gamma_0 \frac{\int_{R^N} w^p\phi_R}{\int_{R^N} w^{p+1}} w^p = \alpha_R\phi_R - \alpha_I\phi_I, \tag{5.18}$$

$$L_0\phi_I - (p-1)\gamma_0 \frac{\int_{R^N} w^p\phi_I}{\int_{R^N} w^{p+1}} w^p = \alpha_R\phi_I + \alpha_I\phi_R. \tag{5.19}$$

Multiplying (5.18) by ϕ_R and (5.19) by ϕ_I and adding them together, we obtain

$$-\alpha_R \int_{R^N} (\phi_R^2 + \phi_I^2) = L_3(\phi_R,\phi_R) + L_3(\phi_I,\phi_I)$$

$$+(p-1)(\gamma_0 - 1)\frac{(\int_{R^N} w^p\phi_R)^2 + (\int_{R^N} w^p\phi_I)^2}{\int_{R^N} w^{p+1}}.$$

By Lemma 5.3 (2)

$$\alpha_R \int_{R^N} (\phi_R^2 + \phi_I^2) + a_2 d_{L^2}^2(\phi, X_1) + (p-1)(\gamma_0 - 1)\frac{(\int_{R^N} w^p\phi_R)^2 + (\int_{R^N} w^p\phi_I)^2}{\int_{R^N} w^{p+1}} \leq 0$$

which implies $\alpha_R < 0$ since $\gamma_0 > 1$.

Theorem 1.4 in Case 3 is thus proved.

6 Conclusions and Remarks

In this paper, we have established the uniqueness and metastability of single interior spike solutions to the shadow system of Gierer-Meinhardt system. Both the uniqueness and stability result depend on the geometry of the domain (in contrast to the 1-D case [23]).

For the uniqueness, a key fact is the estimate of the speed of the spike P_ϵ approaching the limit P_0 (Lemma 2.2). It turns out at a nondegenerate peak point, we must have $P_\epsilon - P_0 = O(\epsilon)$. (This is not true in degenerate peak case.) For the stability, it can be reduced to the study of a nonlocal eigenvalue problem. In Theorem 1.4, this nonlocal eigenvalue problem is analyzed in the two cases $r = 2$ and $r = p+1$. It is well-known that for the scalar equation, there will always be one eigenvalue bounded above from zero, which eliminates the possibility of metastablity. For the non-local system, we have established that the non-local term pushes this positive eigenvalue into the left half plane, and thus making the single interior peak metastable.

It remains a difficult question whether or not uniqueness holds at a degenerate peak point. (Single interior peak solutions exist at degenerate peak points, see [4].) For the stability, it would be an interesting question to see what happens when $2 < r < p+1$. (It is clear that when r is close to 2 or $p+1$, Theorem 1.4 still holds true.)

Although the analysis in this paper was carried out for the Gierer-Meinhardt system, the results can certainly be generalized to a much wide class of non-local reaction diffusion systems that have localized spike solutions. We remark that even for the (GM) system, some important and interesting questions have not been solved, such as the dynamics of interior spikes, the stability of boundary spikes, the *strong coupling* case (i.e. $D_h < +\infty$), the role of τ on the stability, the stability of multiple-spike solutions, etc. Some recent progress has been made in [5], [13] and [12].

Appendix

In this appendix, we prove Lemma A in Section 1.

We first note that (2) can be proved by an easy perturbation argument. We just need to prove (1).

Proof of (1) of Lemma A: The proof is similar to that in Section 4. Let $(\alpha_\epsilon, \phi_\epsilon)$ satisfy

$$\epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + pu_\epsilon^{p-1}\phi_\epsilon - \frac{qr}{s+1+\tau\alpha_\epsilon}\frac{\int_\Omega u_\epsilon^{r-1}\phi_\epsilon}{\int_\Omega u_\epsilon^r}u_\epsilon^p = \alpha_\epsilon\phi_\epsilon, \phi_\epsilon \in H_\nu^2(\Omega_{\epsilon,P_\epsilon}), \tag{6.1}$$

where $\alpha_\epsilon \to 0$ and $\|\phi_\epsilon\|_\epsilon = 1$. Then we have

$$L_\epsilon(\phi_\epsilon) - \eta(\phi_\epsilon)u_\epsilon^p = \alpha_\epsilon\phi_\epsilon,$$

where

$$\eta(\phi_\epsilon) = \frac{qr}{s+1+\tau\alpha_\epsilon}\int_\Omega u_\epsilon^{r-1}\phi_\epsilon/(\int_\Omega u_\epsilon^r).$$

Let $\bar{\phi}_\epsilon = \phi_\epsilon - \frac{1}{p-1}\eta(\phi_\epsilon)u_\epsilon$. Then by a simple computation we have

$$L_\epsilon(\bar{\phi}_\epsilon) = \alpha_\epsilon(\bar{\phi}_\epsilon + c_\epsilon u_\epsilon\int_\Omega u_\epsilon^{r-1}\bar{\phi}_\epsilon), \tag{6.2}$$

where

$$c_\epsilon = \frac{qr}{(p-1)(s+1+\tau\alpha_\epsilon) - qr}\frac{1}{\int_\Omega u_\epsilon^r}.$$

Let ϕ_{j+1}^ϵ be the eigenfunction of τ_{j+1}^ϵ in Theorem 1.3. Set

$$\mathcal{C}_\epsilon := span\{\phi_{j+1}^\epsilon, j = 1, ..., N\} \subset L^2(\Omega_{\epsilon,P_\epsilon}),$$

$$\mathcal{K}_\epsilon := span\{\phi_{j+1}^\epsilon, j = 1, ..., N\} \subset H_\nu^2(\Omega_{\epsilon,P_\epsilon}).$$

We decompose $\bar{\phi}_\epsilon$ as

$$\bar{\phi}_\epsilon = \sum_{j=1}^N b_j^\epsilon \phi_{j+1}^\epsilon + \bar{\phi}_\epsilon^\perp,$$

where $b_j^\epsilon \to b_j^0$, and $\bar{\phi}_\epsilon^\perp \perp \mathcal{K}_\epsilon$. We first obtain that $\bar{\phi}_\epsilon^\perp$ satisfies

$$L_\epsilon(\bar{\phi}_\epsilon^\perp) = \sum_{j=1}^N (\alpha_\epsilon - \tau_{j+1}^\epsilon) b_j^\epsilon \phi_{j+1}^\epsilon + \alpha_\epsilon \bar{\phi}_\epsilon^\perp + \alpha_\epsilon u_\epsilon c_\epsilon \int_\Omega u_\epsilon \bar{\phi}_\epsilon^\perp + \alpha_\epsilon c_\epsilon (\sum_{j=1}^N b_j^\epsilon \int_\Omega u_\epsilon \phi_{j+1}^\epsilon) u_\epsilon. \quad (6.3)$$

By Lemma 7.1 of [29],

$$\int_\Omega u_\epsilon \phi_{j+1}^\epsilon = O(\epsilon^N \varphi_{\epsilon,P_\epsilon}(P_\epsilon)), \qquad c_\epsilon \int_\Omega u_\epsilon \phi_{j+1}^\epsilon = O(\varphi_{\epsilon,P_\epsilon}(P_\epsilon)).$$

Then by a similar argument as in the proof of Propositions 6.1 and 6.2 in [29], we obtain that

$$L_\epsilon : \mathcal{K}_\epsilon^\perp \to \mathcal{C}_\epsilon^\perp$$

is an invertible map if $\epsilon > 0$ is small enough. Hence

$$\|\bar{\phi}_\epsilon^\perp\|_\epsilon = O(\varphi_{\epsilon,P_\epsilon}(P_\epsilon)|\alpha_\epsilon| \sum_{j=1}^N |b_j^\epsilon|).$$

Next we multiple both sides of (6.3) by ϕ_k^ϵ and integrate over Ω, we obtain

$$\sum_{j=1}^N (\alpha_\epsilon - \tau_{j+1}^\epsilon) b_j^\epsilon \int_\Omega \phi_k^\epsilon \phi_{j+1}^\epsilon = O(\epsilon^N \varphi_{\epsilon,P_\epsilon}(P_\epsilon)|\alpha_\epsilon| \sum_{j=1}^N |b_j^\epsilon|).$$

Hence we have

$$(\alpha_\epsilon - \tau_{j+1}^\epsilon) b_j^\epsilon = O(\varphi_{\epsilon,P_\epsilon}(P_\epsilon)(|\alpha_\epsilon| + \sum_{j=1}^N |\tau_{j+1}^\epsilon|) \sum_{j=1}^N |b_j^\epsilon|), \; j = 1, \ldots, N.$$

This shows that

$$\alpha_\epsilon - \tau_{j+1}^\epsilon = o(1)(\sum_{j=1}^N |\tau_{j+1}^\epsilon|)$$

for some $j = 1, ..., N$. By Theorem 1.3, this proves (1) of Lemma A.

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References

[1] P. Bates and G. Fusco, Equilibria with many nuclei for the Cahn-Hilliard equation, preprint.

[2] P. Bates, E. N. Dancer and J. Shi, Multi-spike stationary solutions of the Cahn-Hillirad equation in higher-dimensional and instability, preprint.

[3] Del Pino, Felmer and J. Wei, On the role of mean curvature in some singularly perturbed Neumann problems, *SIAM J. Math. Analy.*, to appear.

[4] Del Pino, Felmer and J. Wei, On the role of the distance function in some singularly perturbed Neumann problems, *Comm. PDE*, to appear.

[5] S.-I. Ei and J. Wei, Dynamics of interior spikes for Gierer-Meinhartd system, preprint.

[6] A. Gierer and H. Meinhardt, A theory of biological pattern formation. *Kybernetik (Berlin)* 12(1972), 30-39.

[7] N. Ghoussoub and C. Gui, Multi-peak solutions for a semilinear Neumann problem involving the critical Sobolev expoenent, *Math. Zeit*, to appear.

[8] C. Gui, Multi-peak solutions for a semilinear Neumann problem, *Duke Math. J.* 84 (1996), 739-769.

[9] C. Gui and J. Wei, Multiple interior peak solutions for some singular perturbation problems, *J. Diff. Eqns.*, to appear.

[10] C. Gui, J. Wei and M. Winter, Multiple boundary peak solutions for some singularly perturbed Neumann problems, *Analyse Non Linearie, Annoles de l'Institut H. Poincare*, to appear.

[11] D. M. Holloway, Reaction-diffusion theory of localized structures with application to vertebrate organogenesis, *PhD thesis*, University of British Columbia, 1995.

[12] D. Iron and M. J. Ward, A metastable spike solution for a non-local reaction-diffusion model, *SIAM J. Appl. Math.*, submitted.

[13] D. Iron, M. J. Ward and J. Wei, On the effect of strong coupling on the stability of multiple spike solutions, preprint.

[14] M. Kowalczyk, Multiple spike layers in the shadow Gierer-Meinhardt system: existence of equilibria and approximate invariant manifold, preprint.

[15] K. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.* 26 (1970), 399-415.

[16] Y.-Y. Li, On a singularly perturbed equation with Neumann boundary condition, *Comm. in Part. Diff. Eqns.* 23(1998), 487-545.

[17] C.-S. Lin and W.-M. Ni, On the diffusion coefficient of a semilinear Neumann problem, *Calculus of variations and partial differential equations (Trento, 1986)* 160–174, Lecture Notes in Math., 1340, Springer, Berlin-New York, 1988.

[18] C.-S. Lin, W.-M. Ni and I. Takagi, Large amplitude stationary solutions to a chemotaxis systems, *J. Diff. Eqns* 72 (1988), 1-27.

[19] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, *Notices of Amer. Math. soc.* 45(1998), 9-18.

[20] W.-M. Ni and I. Takagi, On the shape of least energy solution to a semilinear Neumann problem, *Comm. Pure Appl. Math.* 41(1991), 819-851.

[21] W.-M. Ni and I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, *Duke Math. J.* 70 (1993), 247-281.

[22] W.-M. Ni and I. Takagi, Point-condensation generated by a reaction-diffusion system in axially symmetric domains, *Japan J. Industrial Appl. Math.* 12(1995), 327-365.

[23] W.-M. Ni, I. Takagi and E. Yanagida, preprint.

[24] W.-M. Ni and J. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, *Comm. Pure Appl. Math.* 48 (1995), 731-768.

[25] A. M. Turing, The chemical basis of morphogenesis, *Phil. Trans. Roy. Soc. Lond. B* 237 (1952), 37-72.

[26] M. J. Ward, An asymptotic analysis of localized solutions for some reaction-diffusion models in multi-dimensional domains, *Studies in Appl. Math.* 97 (1996), 103-126.

[27] M. J. Ward, private communications.

[28] J. Wei, On the construction of single-peaked solutions to a singularly perturbed elliptic Dirichlet problem, *J. Diff. Eqns* 129 (2) 1996, 315-333.

[29] J. Wei, On the interior spike layer solutions for some singular perturbation problems, *Proc. Royal Soc. Edinburgh, Section A (Mathematics)* 128(1998), 849-874.

[30] J. Wei, On the interior spike layer solutions of singularly perturbed semilinear Neumann problem, *Tohoku Math. J.* 50(1998), 159-178.

[31] J. Wei, On the boundary spike layer solutions of singularly perturbed semilinear Neumann problem, *J. Diff. Eqns.*, 134 (1997), 104-133.

[32] J. Wei and M. Winter, Stationary solutions of Cahn-Hilliard equations, *Analyse Non Linearie, Annoles de l'Institut H. Poincare* 15(1998), 459-492.

[33] J. Wei and M. Winter, Multiple boundary spike solutions for a wide class of singular perturbation problems, *J. London Math. Soc.*, to appear.

[34] J. Wei and M. Winter, Stationary solutions of Cahn-Hilliard equations: Interior Spikes, *J. Diff. Eqns.* 148(1998), 231-267.