SOLUTIONS WITH INTERIOR BUBBLE AND BOUNDARY LAYER FOR AN ELLIPTIC PROBLEM

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Dedicated to Professor E.N. Dancer on the occasion of his 60^{th} birthday

Abstract We study positive solutions of the equation $\varepsilon^2\Delta u - u + u^{\frac{n+2}{n-2}} = 0$, where $n = 3, 4, 5$, and $\varepsilon > 0$ is small, with Neumann boundary condition in a smooth bounded domain $\Omega \subset R^n$. We prove that, along some sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$, there exists a solution with an interior bubble at an innermost part of the domain and a boundary layer on the boundary $\partial\Omega$.

1. INTRODUCTION AND STATEMENT OF THE RESULT

In recent years, there have been many works devoted to the study of the following singularly perturbed Neumann problem:

$$(1.1) \qquad \varepsilon^2\Delta u - u + u^p = 0, \quad u > 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega$$

where Ω is a smooth bounded domain in R^n, $p > 1$ and $\varepsilon > 0$ is small. Problem (1.1) arises in the study of many reaction-diffusion systems in chemistry or biology, see[12] and the references therein for backgrounds and progress up to 2004.

When $p < \frac{n+2}{n-2}$, it is known that there are many solutions with point condensations in the interior or on the boundary: for example, Gui and Wei [7] proved that given any two positive integers l_1, l_2, there are solutions to (1.1) with l_1 interior spikes and l_2 boundary spikes. Lin, Ni and Wei [8] showed that there are at least $\frac{C}{\varepsilon^n(|\ln\varepsilon|^n)}$ number of interior spikes solutions. When $p = \frac{n+2}{n-2}$, it is known that non-constant solutions exist for ε small enough [2], and the least energy solution blows up, as $\varepsilon \to 0$, at a point which maximizes the mean curvature of the boundary[1]. Higher energy solutions have also been exhibited, blowing up at one [13] or several (separated) boundary points [5][19].

However, the question of existence of **interior blow-up solutions** is still open. It is proved in [4], [6] and [14] that there are no interior bubble solutions.

In another direction, Malchiodi and Montenegro [10] proved that there exists solution concentrating on the whole boundary along some sequence $\{\varepsilon_j\} \to 0$. This boundary layer solution exists for any $p > 1$ and for any smooth bounded domain Ω.

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When $\Omega = B_1(0), p = \frac{n+2}{n-2}$, Wei and Yan [18] have built up an interior bubble solution on the top of the boundary layer solution, at least when the dimension $n = 3, 4, 5$. The solutions constructed in [18] are radially symmetric. In this paper, we consider general smooth bounded domain case and establish the same result.

Namely, we consider the following equation

$$
\begin{cases} \varepsilon^2 \Delta u - u + n(n-2) u^{\frac{n+2}{n-2}} = 0 & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}
$$

Let $d_x = \operatorname{dist}(x, \partial\Omega)$, where dist is the distant function in the general meaning, and $\Omega_\lambda = \{y| \quad \dfrac{y}{\lambda} \in \Omega\}$. Let $d_\Omega = \max_{x\in\Omega} d_x$. For given positive numbers γ and $\sigma < \frac{\gamma}{100}$ small enough, let

$$
M_\gamma = \{a \in \Omega| \quad d_a > \gamma\}, \qquad \lambda \in \Lambda := (e^{\frac{a_n\gamma - 2\sigma}{\varepsilon}}, e^{\frac{a_n d_\Omega + 2\sigma}{\varepsilon}}) \tag{1.3}
$$

where $a_n = 2\beta$ if $n = 3, 5$, $a_4 = \beta$ and $1 - \sigma < \beta < 1 + \sigma$.

In [10], it is proved that along $\varepsilon_j \to 0$ (1.2) has a boundary layer solution W_{ε_j} which is uniformly bounded and concentrating on $\partial\Omega$. Asymptotically, it can be proved (Lemma 2.2)

$$
C_1 \exp\{-\frac{(1+\sigma)d_x}{\varepsilon_j}\} \le W_{\varepsilon_j}(x) \le C_2 \exp\{-\frac{(1-\sigma)d_x}{\varepsilon_j}\} \tag{1.4}
$$

where C_1, C_2 are positive numbers.

From now on, we always consider $a \in M_\gamma$ and the sequence ε_j as in [10]. We omit index j for simplicity.

By suitable rescaling, (1.2) becomes

$$
\begin{cases} \Delta u - (\lambda\varepsilon)^{-2} u + n(n-2) u^{\frac{n+2}{n-2}} = 0 & \text{in} \quad \Omega_\lambda, \\ u > 0 & \text{in} \quad \Omega_\lambda, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega_\lambda. \end{cases} \tag{1.5}
$$

We set

$$
S_\lambda[u] := \Delta u - (\lambda\varepsilon)^{-2} u + n(n-2) u_+^{\frac{n+2}{n-2}} \tag{1.6}
$$

and

$$
J_\lambda[u] := \frac{1}{2} \int_{\Omega_\lambda} \left(|\nabla u|^2 + (\lambda\varepsilon)^{-2} u^2\right) - \frac{(n-2)^2}{2} \int_{\Omega_\lambda} u_+^{\frac{2n}{n-2}}. \tag{1.7}
$$

We recall that, according to [3], the functions

$$
U_{a,\lambda} = (\frac{\lambda}{1 + \lambda^2 |x-a|^2})^{\frac{n-2}{2}}, \qquad \lambda > 0, \quad a \in R^n \tag{1.8}
$$

are the only solutions to the problem

$$
\Delta u + n(n-2) u^{\frac{n+2}{n-2}} = 0, \qquad u > 0 \qquad \text{in} \quad R^n.
$$

The main result in this paper is:

Theorem 1.1. *Let $n = 3, 4, 5$ and $\Omega \subseteq R^n$ be a smooth bounded domain. Then there exist a sequence $\varepsilon_j \to 0$ and a sequence of solution u_{ε_j} of (1.2) with following properties:*
(1) u_{ε_j} has a local maximum point a_{ε_j}, such that

$$a_{\epsilon_j} \to a_0 \in \Omega, \quad d_{a_{\varepsilon_j}} \to d_\Omega \qquad as \quad j \to \infty;$$

(2) $u_{\varepsilon_j} = \varepsilon_j^{\frac{n-2}{2}} U_{a_{\varepsilon_j}, \lambda_{\varepsilon_j}}(x) + W_{\varepsilon_j} + \circ(1)$ where $\lambda_{\varepsilon_j} \sim \exp\{\frac{a_n + \circ(1)}{\varepsilon_j} d_{a_{\varepsilon_j}}\}$. As a consequence, $u_{\varepsilon_j}(a_{\varepsilon_j}) \sim (\lambda_{\varepsilon_j}\varepsilon_j)^{\frac{n-2}{2}}$ and $u_{\varepsilon_j} \to 0$ for any $x \in M_\gamma \backslash B_\delta(a_{\varepsilon_j})$, where δ is any small positive number, and u_{ε_j} blows up at a_0.

The paper is organized as follows: In section 2, we construct suitable approximate solution W and study its properties. In section 3, we solve the linear problem at W in a finite-codimensional space. Then in section 4, we are able to solve the nonlinear problem in that space. In section 5, we study the remaining finite-dimensional problem and prove Theorem 1.1. The proof of Lemma 2.3 may be found in Appendix.

Throughout the paper, the letter C will denote various constant independent of ε and λ.

2. Approximate solution

In this section, we construct suitable approximate solutions. Let W_ϵ be the solution of (1.2) constructed in [10]. First, we need to study the properties of W_ϵ. Following Remark 5.2 on page 138 in[10], we have the following lemma:

Lemma 2.1. *Consider the following eigenvalue problem :*

$$\varepsilon^2 \Delta\psi - \psi + n(n-2) W_\varepsilon^{\frac{4}{n-2}} \psi = \mu\psi, \qquad \psi \in H^1(\Omega).$$

If $(\psi_\varepsilon, \mu_\varepsilon), \psi_\varepsilon \not\equiv 0$ is a solution of the above problem, then we have $|\mu_\varepsilon| \geq C\varepsilon^{n-1}$.

From Lemma 2.1, we obtain

Corollary 2.1. *The linearized operator $L_\varepsilon(\psi) := \varepsilon^2\Delta\psi - \psi + n(n-2)W_\varepsilon^{\frac{4}{n-2}}\psi$ is an invertible operator from $H^2(\Omega)$ to $L^2(\Omega)$. Furthermore, we have*

$$\|\psi\|_{L^\infty(\Omega)} \leq C\varepsilon^{-\alpha} \|\varepsilon^2\Delta\psi - \psi + n(n-2)W_\varepsilon^{\frac{4}{n-2}}\psi\|_{L^\infty(\Omega)}$$

where $\alpha > n+1$ is a fixed constant.

Proof. Using Lemma 2.1, we have

$$\|\psi\|_{L^2(\Omega)} \leq C\varepsilon^{1-n} \|\varepsilon^2\Delta\psi - \psi + n(n-2)W_\varepsilon^{\frac{4}{n-2}}\psi\|_{L^2(\Omega)}.$$

Observe that

$$\|\Delta\psi\|_{L^2(\Omega)} \leq \varepsilon^{-2} \|\varepsilon^2\Delta\psi - \psi + n(n-2)W_\varepsilon^{\frac{4}{n-2}}\psi\|_{L^2(\Omega)} + C\varepsilon^{-2}\|\psi\|_{L^2(\Omega)}.$$

Hence

$$\|\psi\|_{H^2(\Omega)} \leq C\|\Delta\psi\|_{L^2(\Omega)} + C\|\psi\|_{L^2(\Omega)} \leq C\varepsilon^{-(1+n)} \|\varepsilon^2\Delta\psi - \psi + n(n-2)W_\varepsilon^{\frac{4}{n-2}}\psi\|_{L^2(\Omega)}.$$

By a bootstrapping argument, we get the desired result. □

The decay rate of W_ε can also be estimated.

Lemma 2.2. *It holds*

$$C_1 \exp\{-\frac{(1+\sigma)d_x}{\varepsilon_j}\} \le W_{\varepsilon_j}(x) \le C_2 \exp\{-\frac{(1-\sigma)d_x}{\varepsilon_j}\}. \tag{2.1}$$

Proof. Let $h_{\sigma,\varepsilon}$ be the unique solution of

$$\epsilon^2 \Delta h_{\sigma,\varepsilon} - (1-\sigma)^2 h_\sigma = 0 \text{ in } \Omega, \ h_{\sigma,\varepsilon} = 1 \text{ on } \partial\Omega. \tag{2.2}$$

By the vanishing visocity method ([9]), we have

$$h_{\sigma,\varepsilon} \sim e^{-\frac{(1-\sigma+o(1))d_x}{\epsilon}}. \tag{2.3}$$

Since $W_\varepsilon \ge C$ on $\partial\Omega$, and $\epsilon^2\Delta W_\varepsilon - W_\varepsilon \le 0$, by comparison principle, we obtain

$$Ch_{0,\varepsilon}(x) \le W_\varepsilon(x). \tag{2.4}$$

On the other hand, from ([10]), we see that

$$W_\varepsilon - w(\frac{d_x}{\epsilon})\eta(x) = \phi_\varepsilon = o(1)$$

where $\eta(x) = 1$ for $d_x \le \delta, \eta(x) = 0$ for $d_x > 2\delta$ and w is the solution of the problem

$$-w'' + w = w^{\frac{n+2}{n-2}}, \qquad w > 0 \quad \text{in} \quad R^1, \qquad w(x) \to 0 \quad \text{as} \quad |x| \to +\infty.$$

Now ϕ_ε satisfies

$$\epsilon^2 \Delta\phi_\varepsilon - \phi_\varepsilon + o(\phi_\varepsilon) + O(e^{\frac{d_x}{\epsilon}}) = 0$$

in Ω and $\phi_\varepsilon = o(1)$ on $\partial\Omega$. By comparison principle again, we have

$$|\phi_\varepsilon| \le \sigma^{-1} C h_{\sigma,\varepsilon}. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain the lemma. □

Next we consider a linear Neumann problem which can be viewed as a projection of $U_{a,\lambda}$

$$\begin{cases} \Delta V_{a,\lambda} - \varepsilon^{-2} V_{a,\lambda} + n(n-2) U_{a,\lambda}^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial V_{a,\lambda}}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.6}$$

where $U_{a,\lambda}$ is defined as in (1.8).

Define

$$\begin{aligned} W_1(y) &:= \lambda^{-\frac{n-2}{2}} V_{a,\lambda}(\frac{y}{\lambda}), \\ W_2(y) &:= (\lambda\varepsilon)^{-\frac{n-2}{2}} W_\varepsilon(\frac{y}{\lambda}), \qquad W := W_1 + W_2. \end{aligned} \tag{2.7}$$

By maximum principle, $\quad 0 \le W_1 \le U_{\xi,1} \quad$ where $\xi = \lambda a$.

When $n = 3$, let

$$V_{a,\lambda}(x) = U_{a,\lambda}(x) - \frac{1}{\lambda^{\frac{1}{2}}|x-a|}(1 - e^{-\frac{|x-a|}{\varepsilon}}) + \varphi_{a,\lambda}(x). \tag{2.8}$$

Then $\varphi_{a,\lambda}(x)$ satisfies

$$\begin{cases} \varepsilon^2 \Delta\varphi_{a,\lambda} - \varphi_{a,\lambda} - U_{a,\lambda} + \dfrac{1}{\lambda^{\frac{1}{2}}|x-a|} = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial\varphi_{a,\lambda}}{\partial\nu} + \dfrac{\partial}{\partial\nu}\left(U_{a,\lambda} - \dfrac{1}{\lambda^{\frac{1}{2}}|x-a|}(1 - e^{-\frac{|x-a|}{\varepsilon}})\right) = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

By the estimates in [16], we get

$$|\varphi_{a,\lambda}| = O\left(\frac{1}{\varepsilon^2\lambda^{\frac{3}{2}}(1+\lambda|x-a|)} + \frac{1}{\varepsilon\lambda^{\frac{1}{2}}}e^{-\max\{\frac{|x-a|}{\varepsilon},\quad \frac{d_a}{\varepsilon}\}}\right). \tag{2.9}$$

Using (2.7) and (2.9), we have

$$\begin{aligned} rlW_1(y) =& U_{\xi,1}(y) - \frac{1}{|y-\xi|}(1-e^{-\frac{|y-\xi|}{\lambda\varepsilon}}) \\ &+ O\left(\frac{1}{\varepsilon^2\lambda^2(1+|y-\xi|)} + \frac{1}{\varepsilon\lambda}e^{-\max\{\frac{|y-\xi|}{\lambda\varepsilon},\quad \frac{d_a}{\varepsilon}\}}\right). \end{aligned} \tag{2.10}$$

If $|y-\xi| \geq \sigma d_a\lambda$, then using (2.10) we have

$$|W_1(y)| = O\left(\frac{1}{\lambda^2} + \frac{1}{\varepsilon\lambda}e^{-\max\{\frac{|y-\xi|}{\lambda\varepsilon},\quad \frac{d_a}{\varepsilon}\}}\right). \tag{2.11}$$

Moreover, by similar computation if $|y-\xi| \geq \sigma d_a\lambda$,

$$|\partial_\lambda W_1(y)| = O\left(\frac{1}{\lambda^3} + \frac{1}{\varepsilon\lambda^2}e^{-\max\{\frac{|y-\xi|}{\lambda\varepsilon},\quad \frac{d_a}{\varepsilon}\}}\right),$$

$$|\partial_\lambda^2 W_1(y)| = O\left(\frac{1}{\lambda^4} + \frac{1}{\varepsilon\lambda^3}e^{-\max\{\frac{|y-\xi|}{\lambda\varepsilon},\quad \frac{d_a}{\varepsilon}\}}\right).$$

When $n = 4, 5$,

$$W_1(y) = U_{\xi,1}(y) - \lambda^{-\frac{n-2}{2}}(\varphi_2(\frac{y}{\lambda}) + \varphi_3(\frac{y}{\lambda})), \tag{2.12}$$

where $\varphi_2(x)$ satisfies

$$\begin{cases} \Delta\varphi_2 - \varepsilon^{-2}\varphi_2 + \varepsilon^{-2}U_{a,\lambda} &= 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial\varphi_2}{\partial\nu} &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

and $\varphi_3(x)$ satisfies

$$\begin{cases} \Delta\varphi_3 - \varepsilon^{-2}\varphi_3 &= 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial\varphi_3}{\partial\nu} &= \dfrac{\partial U_{a,\lambda}}{\partial\nu} \quad \text{on} \quad \partial\Omega. \end{cases}$$

Similar to estimates in Lemma A.1 of [17] we have

$$|\varphi_2| \leq \frac{C}{\lambda^{\frac{6-n}{2}}\varepsilon^3(1+\lambda|x-a|^{n-4})}, \quad |\varphi_3| = O(\lambda^{-\frac{n-2}{2}}).$$

Hence

$$W_1(y) = U_{\xi,1}(y) - \tilde{\varphi_2}(y) - \tilde{\varphi_3}(y)$$

where

$$|\tilde{\varphi_2}(y)| \leq \frac{C}{\lambda^2\varepsilon^3(1+|y-\xi|^{n-4})}, \quad |\tilde{\varphi_3}(y)| = O(\frac{1}{\lambda^{n-2}}).$$

If $|y-\xi| \geq \sigma d_a\lambda$, we get

$$|W_1(y)| \leq C\varepsilon^{-3}\lambda^{-n+2}. \tag{2.13}$$

By similar computation if $|y-\xi| \geq \sigma d_a\lambda$,

$$|\partial_\lambda W_1(y)| \leq C\varepsilon^{-3}\lambda^{-n+1}, \quad |\partial_\lambda^2 W_1(y)| \leq C\varepsilon^{-3}\lambda^{-n}. \tag{2.14}$$

Next we define two Sobolev norms. Let

$$\|\phi\|_* = \sup_{y\in\Omega_\lambda} (1+|y-\xi|)^{\frac{n-2+\sigma}{2}}|\phi(y)|, \quad \|f\|_{**} = \sup_{y\in\Omega_\lambda} (1+|y-\xi|)^{\frac{n+2+\sigma}{2}}|f(y)|.$$

In the Appendix, we shall prove

Lemma 2.3. *There hold*

$$\|S_\lambda[W]\|_{**} \le C\lambda^{-\frac{\beta_n+\sigma}{2}}, \qquad \|\partial_\lambda S_\lambda[W]\|_{**} \le C\lambda^{-\frac{\beta_n+\sigma+2}{2}},$$
$$\|\partial_\lambda^2 S_\lambda[W]\|_{**} \le C\lambda^{-\frac{\beta_n+\sigma+4}{2}} \tag{2.15}$$

where $\beta_n = 1$ if $n=3$ and $\beta_n = 2$ if $n = 4,5$. Furthermore, when $n=3$, we have

$$\begin{aligned} J_\lambda[W] =& J_\lambda[W_2] + \int_{R^3} U_{0,1}^6 + \frac{3}{2\lambda\varepsilon}\int_{R^3} U_{0,1}^5 - (B_3+o(1))(\lambda\varepsilon)^{-\frac{1}{2}}e^{-\frac{\beta d_a}{\varepsilon}} \\ &+ \varepsilon^{-1}\lambda^{-1}e^{\frac{-d_a}{\varepsilon}}E_1. \end{aligned} \tag{2.16}$$

When $n=4,5$,

$$\begin{aligned} J_\lambda[W] =& J_\lambda[W_2] + (n-2)\int_{R^n} U_{0,1}^{\frac{2n}{n-2}} + (A_n+o(1))(\lambda\varepsilon)^{-2}(\ln\lambda)^m \\ &- (B_n+o(1))(\lambda\varepsilon)^{-\frac{n-2}{2}}e^{-\frac{\beta d_a}{\varepsilon}} + \varepsilon^{-1}\lambda^{-2}E_2 \end{aligned} \tag{2.17}$$

with $E_1 = O(1)$, $E_2 = o(1)$ and

$$\partial_\lambda E_1 = O(\lambda^{-1}), \quad \partial_\lambda^2 E_1 = O(\lambda^{-2}), \quad \partial_\lambda E_2 = o(\lambda^{-1}), \quad \partial_\lambda^2 E_2 = o(\lambda^{-2})$$

where $1-\sigma < \beta < 1+\sigma, A_n, B_n$ are positive numbers and $m=1$ if $n=4$, $m=0$ if $n=5$.

3. Finite-dimensional reduction: a linear problem

Following the general strategy as in [16]-[17], we first consider the linearized problem at W, and we solve it in a finite-codimensional subspace. Namely, we equip $H^1(\Omega_\lambda)$ with the scalar product

$$(u,v)_\lambda = \int_{\Omega_\lambda} \nabla u\cdot\nabla v + (\lambda\varepsilon)^{-2}uv.$$

Let $\eta(r)$ be a smooth cut-off function such that $\eta(r)=1$ for $r\le \frac{1}{8}d_a\lambda$ and $\eta(r)=0$ for $r\ge\frac{1}{4}d_a\lambda$ where $r=|y-\xi|$, then $|D\eta|\le C\lambda^{-1}$ and $|D^2\eta|\le C\lambda^{-2}$. Define

$$Y_0 = \frac{\partial(\eta W)}{\partial\lambda}|_{\lambda=1}, \quad Y_i = \frac{\partial(\eta W)}{\partial\xi_i}, \quad 1\le i\le n.$$

Setting

$$Z_0 = -\Delta Y_0 + (\lambda\varepsilon)^{-2}Y_0, \quad Z_i = -\Delta Y_i + (\lambda\varepsilon)^{-2}Y_i, \quad 1\le i\le n,$$

then

$$\begin{aligned} |Z_0| &\le C\Big((\lambda\varepsilon)^{-3}(1+|y-\xi|^{-(n-2)}) + (\lambda\varepsilon)^{-\frac{n+4}{2}}\Big), \\ |Z_i| &\le C\Big((1+|y-\xi|)^{-(n+3)} + (\lambda\varepsilon)^{-\frac{n+4}{2}}\Big). \end{aligned}$$

Now we consider the following problem: $h \in L^\infty(\Omega_\lambda)$ being given, find a function ϕ satisfying

$$
\begin{cases}
\Delta\phi - (\lambda\varepsilon)^{-2}\phi + n(n+2)W^{\frac{4}{n-2}}\phi = h + \sum_{i=0}^{n} c_i Z_i & \text{in} \quad \Omega_\lambda, \\
\frac{\partial\phi}{\partial\nu} = 0 & \text{on} \quad \partial\Omega_\lambda, \\
\langle Z_i, \phi\rangle = 0 & 0 \le i \le n
\end{cases}
\tag{3.1}
$$

for some c_i's.

In this section, we shall prove

Proposition 3.1. *There exist $\varepsilon_0 > 0$ and a constant $C > 0$, independent of ε and d_a, λ satisfying (1.3), such that for all $\varepsilon \le \varepsilon_0$ and $h \in L^\infty(\Omega_\lambda)$, problem (3.1) has a unique solution $\phi = L_\lambda(h)$ and*

$$\|L_\lambda(h)\|_* \le C\varepsilon^{-\alpha}\|h\|_{**}, \qquad |c_i| \le C\varepsilon^{-\alpha}\|h\|_{**}$$

where α is defined as Corollary 2.1. Moreover, the map $L_\lambda(h)$ is C^2 with respect to λ and the L^∞_*-norm, and

$$\|\partial_\lambda L_\lambda(h)\|_* \le C\varepsilon^{-\alpha}\lambda^{-1}\|h\|_{**}, \quad \|\partial_\lambda^2 L_\lambda(h)\|_* \le C\varepsilon^{-\alpha}\lambda^{-2}\|h\|_{**}. \tag{3.2}$$

First we state two lemmas, whose proof is similar to Appendix A.3 of [17].

Lemma 3.1. *The Green function $G(x,y)$ of*

$$
\begin{cases}
\Delta G(x,y) - (\lambda\varepsilon)^{-2}G(x,y) + \delta_y = 0 & \text{in} \quad \Omega_\lambda, \\
\frac{\partial G}{\partial\nu} = 0 & \text{on} \quad \partial\Omega_\lambda
\end{cases}
$$

has the following decay property

$$|G(x,y)| \le C|x-y|^{-(n-2)}.$$

Lemma 3.2. *Let u satisfy*

$$
\begin{cases}
\Delta u - (\lambda\varepsilon)^{-2}u = f & \text{in} \quad \Omega_\lambda, \\
\frac{\partial u}{\partial\nu} = 0 & \text{on} \quad \partial\Omega_\lambda,
\end{cases}
$$

then

$$|u(y)| \le C\int_{\Omega_\lambda} \frac{|f(x)|}{|x-y|^{n-2}}dx.$$

Moreover,

$$\|u\|_* \le C\|f\|_{**}.$$

Proof of Proposition 3.1: We argue by contradiction. Suppose there exist sequences of $\varepsilon_j, \lambda_j, \phi_{\varepsilon_j}, h_{\varepsilon_j}$ such that $\|\phi_{\varepsilon_j}\|_* = 1, \|h_{\varepsilon_j}\|_{**} = o(\varepsilon^\alpha)$. For simplicity, we omit the index j.

Multiplying the first equation in (3.1) by Y_j and integrating on Ω_λ, then for $j = 1, \dots, n$.

$$\sum_i c_i\langle Z_i, Y_j\rangle = \langle \Delta Y_j - (\lambda\varepsilon)^{-2}Y_j + n(n-2)W^{\frac{4}{n-2}}Y_j, \phi_\varepsilon\rangle - \langle h_\varepsilon, Y_j\rangle. \tag{3.3}$$

On the one hand,

$$
\begin{cases}
\langle Z_0, Y_0\rangle = \|Y_0\|_\varepsilon^2 = \gamma_0 + o(1), & \\
\langle Z_i, Y_i\rangle = \|Y_i\|_\varepsilon^2 = \gamma_i + o(1), & 1 \le i \le n, \\
\langle Z_i, Y_j\rangle = o(1), & i \ne j.
\end{cases}
\tag{3.4}
$$

where $\gamma_i, 0 \leq i \leq n$ are positive numbers.

On the other hand, compute directly,

$$\langle \Delta Y_j - (\lambda\varepsilon)^{-2} Y_j + n(n-2) W^{\frac{4}{n-2}} Y_j, \phi_\varepsilon \rangle = o(\varepsilon^\alpha \|\phi_\varepsilon\|) = o(\varepsilon^\alpha), \tag{3.5}$$

$$\begin{aligned} |\langle h_\varepsilon, Y_j \rangle| \leq & C\|h_\varepsilon\|_{**} \int_{\Omega_\lambda} (1+|y-\xi|)^{-\frac{n+2+\sigma}{2}} \Big((1+|y-\xi|)^{-\frac{n-2}{2}} \\ & + (\lambda\varepsilon)^{-\frac{n-2}{2}} \Big) = O(\|h_\varepsilon\|_{**}) = o(\varepsilon^\alpha). \end{aligned} \tag{3.6}$$

Hence using (3.3) – (3.6), we get

$$c_i = o(\varepsilon^\alpha) \quad \text{as} \quad \varepsilon \to 0. \tag{3.7}$$

Since $\|\phi_\varepsilon\|_* = 1$, elliptic theory shows that along some subsequence the functions $\phi_\varepsilon(y) = \phi_\varepsilon(y-\xi)$ converge uniformly in any compact subset of R^n to a nontrivial solution of

$$-\Delta\phi = n(n-2) U_{0,1}^{\frac{4}{n-2}} \phi.$$

Since $|\phi| \leq C(1+|y|)^{-\frac{n-2+\sigma}{2}}$, a bootstrap argument leads to $|\phi| \leq C(1+|y|)^{2-n}$. As a consequence of [15], ϕ can be written as

$$\phi = \alpha_0 \Big(\frac{n-2}{2} U_{0,1} + y \cdot \nabla U_{0,1}\Big) + \sum_{i=1}^{n} \alpha_i \frac{\partial U_{0,1}}{\partial y_i}. \tag{3.8}$$

According to Lebsgue's Dominate Convergence Theorem, $\langle Z_i, \phi_\varepsilon \rangle = 0$ yields

$$\begin{cases} \int_{R^n} -\Delta(\frac{n-2}{2} U_{0,1} + y \cdot \nabla U_{0,1})\phi & = 0, \\ \int_{R^n} -\Delta \frac{\partial U_{0,1}}{\partial y_i} \phi & = 0, \qquad 1 \leq i \leq n, \\ \int_{R^n} \nabla \frac{\partial U_{0,1}}{\partial y_i} \cdot \nabla \frac{\partial U_{0,1}}{\partial y_j} & = 0, \qquad i \neq j. \end{cases}$$

Using (3.4), (3.7) and (3.8) we know $\alpha_i' s$ solve a homogeneous quasidiagonal linear system, which yields $\alpha_i = 0, \quad 0 \leq i \leq n$. So $\quad \phi_\varepsilon(y-\xi) \to 0 \quad$ in $C^1_{loc}(\Omega_\lambda)$.

If $|y-\xi| \leq \frac{1}{2} d_a \lambda$, using Lemma 3.2 we can obtain

$$\begin{aligned} \|\phi_\varepsilon\|_* & \leq C\|W^{\frac{4}{n-2}}\phi_\varepsilon\|_{**} + C\|h_\varepsilon\|_{**} + C\|\sum_i c_i Z_i\|_{**}, \\ (1+|y-\xi|)^{\frac{n+2+\sigma}{2}} |W^{\frac{4}{n-2}}\phi_\varepsilon| & \leq C(1+|y-\xi|)^{\frac{n+2+\sigma}{2}} (|W_1|^{\frac{4}{n-2}} + |W_2|^{\frac{4}{n-2}})|\phi_\varepsilon| \\ & \leq C(1+|y-\xi|)^{\frac{n+2+\sigma}{2}} \frac{|\phi_\varepsilon|}{(1+|y-\xi|)^4} \\ & \quad + C(1+|y-\xi|)^2 (\lambda\varepsilon)^{-2} e^{-\frac{(1-\sigma)}{2\varepsilon} d_a} \|\phi_\varepsilon\|_*. \end{aligned}$$

Since the first term on the right hand is dominated by $(1+|y-\xi|)^{-2}\|\phi_\varepsilon\|_*$ if $|y-\xi| \geq \sigma d_a \lambda$ and goes uniformly to zero in any ball $B_R(\xi)$ which, through the choice of R, can be made as small as desired.

If $|y-\xi| \geq \frac{1}{2} d_a \lambda$, let ψ_ε be such that

$$\begin{cases} \Delta\psi_\varepsilon - \varepsilon^{-2}\psi_\varepsilon & = 0 \qquad \text{in} \quad R^n \backslash \bar{B}_{a,\frac{1}{4}d_a}, \\ \psi_\varepsilon & = 1 \qquad \text{on} \quad \partial B_{a,\frac{1}{4}d_a}. \end{cases}$$

By the following transformation $\psi_\varepsilon = e^{-\frac{h_\varepsilon}{\varepsilon}}$, we see that h_ε satisfies

$$\begin{cases} \varepsilon\Delta h_\varepsilon - |\nabla h_\varepsilon|^2 + 1 & = 0 \qquad \text{in} \quad R^n \backslash \bar{B}_{a,\frac{1}{4}d_a}, \\ h_\varepsilon & = 0 \qquad \text{on} \quad \partial B_{a,\frac{1}{4}d_a}. \end{cases}$$

The solutions to the limit problem $|\nabla h|^2 = 1$ are $C_1 + C_2(-\frac{1}{4}d_a + r)$. By the method of viscosity solutions, we conclude that $h_\varepsilon \to r - \frac{1}{4}d_a$ as $\varepsilon \to 0$.

Consider the function $\tilde{\phi}_\varepsilon = \phi_\varepsilon - d\psi_\varepsilon(\frac{y-\xi}{\lambda})$ with $d = \phi_\varepsilon(\frac{1}{4}d_a\lambda) \sim \lambda^{-\frac{n-2+\sigma}{2}}$. Then in the x coordinate we have

$$\varepsilon^2\Delta\tilde{\phi}_\varepsilon - \tilde{\phi}_\varepsilon + n(n+2)W_\epsilon^{\frac{4}{n-2}}\tilde{\phi}_\varepsilon = g$$

where $g = (\lambda\varepsilon)^2 h(\lambda x) - n(n+2)\Big((\varepsilon^{\frac{n-2}{2}}V_{a,\lambda} + W_\epsilon)^{\frac{4}{n-2}} - W_\varepsilon^{\frac{4}{n-2}}\Big)\tilde{\phi}_\varepsilon - n(n+2)dW_\varepsilon^{\frac{4}{n-2}}\psi_\varepsilon$. Now we estimate term by term.

$$(\lambda\varepsilon)^2|h(\lambda x)| \le (\lambda\varepsilon)^2(1+|y-\xi|)^{-\frac{n+2+\sigma}{2}}\|h\|_{**} = o(\varepsilon^\alpha)(1+|y-\xi|)^{-\frac{n-2+\sigma}{2}},$$

$$\begin{aligned} n(n+2)|(\varepsilon^{\frac{n-2}{2}}V_{a,\lambda} + W_\varepsilon)^{\frac{4}{n-2}} - W_\varepsilon^{\frac{4}{n-2}}||\tilde{\phi}_\varepsilon| &\le C(\varepsilon^2|V_{a,\lambda}|^{\frac{4}{n-2}} + \varepsilon^{\frac{6-n}{2}}|V_{a,\lambda}|^{\frac{6-n}{n-2}}W_\varepsilon)(|\phi_\varepsilon + d\psi_\varepsilon|) \\ &\le C(\varepsilon^2\lambda^{-\frac{4}{n-2}} + \varepsilon^{\frac{6-n}{2}}\lambda^{-\frac{6-n}{2}}) \\ &\quad \Big((1+|y-\xi|)^{-\frac{n-2+\sigma}{2}}\|\phi_\varepsilon\|_* + \lambda^{-\frac{n-2+\sigma}{2}}\Big) \\ &= o(\varepsilon^\alpha)\lambda^{-\frac{n-2+\sigma}{2}}, \\ n(n+2)d|W_\varepsilon^{\frac{4}{n-2}}\psi_\varepsilon| \le C\lambda^{-\frac{n-2+\sigma}{2}}\psi_\varepsilon|W_\varepsilon^{\frac{4}{n-2}}| &\le Ce^{-\frac{\sigma d_a}{\varepsilon}}\lambda^{-\frac{n-2+\sigma}{2}} = o(\varepsilon^\alpha)\lambda^{-\frac{n-2+\sigma}{2}}. \end{aligned}$$

Using Corollary 2.1 we have

$$\|\tilde{\phi}_\varepsilon\|_{L^\infty(\Omega_\lambda\setminus\bar{B}_{\xi,\frac{1}{2}\lambda d_a})} \le o(1)(1+|y-\xi|)^{-\frac{n-2+\sigma}{2}}.$$

Thus

$$\sup_{y\in\Omega_\lambda\setminus\bar{B}_{\xi,\frac{1}{2}\lambda d_a}}(1+|y-\xi|)^{\frac{n-2+\sigma}{2}}|\phi| \le C\lambda^{-\frac{n-2+\sigma}{2}}\sup_{y\in\Omega_\lambda\setminus\bar{B}_{\xi,\frac{1}{2}\lambda d_a}}(|\tilde{\phi}_\varepsilon| + d\psi_\varepsilon) = o(1),$$

i.e. $\|\phi_\varepsilon\|_* = o(1)$, contradiction.

Now we set

$$H = \{\phi \in H^1(\Omega_\lambda), \langle Z_i, \phi\rangle = 0, \quad 0 \le i \le n\}$$

equipped with the scalar product $(\cdot,\cdot)_\lambda$. Problem(3.1) is equivalent to finding $\phi \in H$ such that

$$(\phi,\theta)_\lambda = \langle n(n+2)W^{\frac{4}{n-2}}\phi + h, \theta\rangle \quad \forall\theta \in H$$

that is

$$\phi = T_\lambda(\phi) + \tilde{h}$$

$\tilde{h}$ depending linearly on h, and T_λ being a compact operator in H. Fredholm's alternative ensures the existence of a unique solution, provided that the kernel of $Id - T_\lambda$ is reduced to 0. We notice that $\phi \in Ker(Id - T_\lambda)$ solves (3.1) with $h = 0$. Thus by the first part estimate we know that $\|\phi\|_* = o(1)$ as ε goes to zero. As $Ker(Id - T_\lambda)$ is a vector space, $Ker(Id - T_\lambda) = \{0\}$. This completes the proof of the first part of Proposition 3.1.

The smoothness of L_λ with respect to λ is a consequence of the smoothness of T_λ and $\tilde{h}$. Inequalities (3.2) are obtained differentiating (3.1), writing the derivatives of ϕ with respect to λ as a linear combination of the Z_i' and an orthogonal part, and estimating each term using the first part of the Proposition 3.1– see[11] for detailed computations. □

4. Finite-dimensional reduction: a nonlinear problem

In this section, we turn our attention to the nonlinear problem which we solve in the finite-codimensional subspace orthogonal to the Z_i's with $n = 3, 4, 5$. Let $S_\lambda[u]$ be as defined at (1.6). Then (1.5) is equivalent to

$$S_\lambda[u] = 0 \quad \text{in} \quad \Omega_\lambda, \quad u_+ \not\equiv 0, \quad \frac{\partial u}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega_\lambda. \tag{4.1}$$

Indeed, if u satisfies (4.1) maximum principle ensures that $u > 0$ in Ω_λ and (1.5) is satisfied. Observe that

$$S_\lambda[W+\phi] = \Delta(W+\phi) - (\lambda\varepsilon)^{-2}(W+\phi) + n(n-2)(W+\phi)_+^{\frac{n+2}{n-2}}$$

may be written as

$$S_\lambda[W+\phi] = \Delta\phi - (\lambda\varepsilon)^{-2}\phi + n(n+2)W^{\frac{4}{n-2}}\phi + R^\lambda + n(n-2)N_\lambda(\phi)$$

with

$$N_\lambda(\phi) = (W+\phi)_+^{\frac{n+2}{n-2}} - W^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}W^{\frac{4}{n-2}}\phi \tag{4.2}$$

and

$$R^\lambda = S_\lambda[W] = \Delta W - (\lambda\varepsilon)^{-2}W + n(n-2)W^{\frac{n+2}{n-2}}.$$

We now consider the following nonlinear problem: find ϕ such that for some numbers c_i's,

(4.3)
$$\begin{cases} \Delta\phi - (\lambda\varepsilon)^{-2}\phi + n(n+2)W^{\frac{4}{n-2}}\phi = -R^\lambda - n(n-2)N_\lambda(\phi) + \sum_{i=0}^{n} c_i Z_i \quad \text{in} \quad \Omega_\lambda, \\ \frac{\partial\phi}{\partial\nu} = 0 \qquad \text{on} \quad \partial\Omega_\lambda, \\ \langle Z_i, \phi\rangle = 0 \qquad 0 \le i \le n. \end{cases}$$

Lemma 4.1. *There exists a constant C independent of $\varepsilon, \lambda, \xi$ such that if $\|\phi\|_* \le \lambda^{-\frac{\beta_n + \frac{\sigma}{2}}{2}}$, then*

$$\|N_\lambda(\phi)\|_{**} \le C\lambda^{-\frac{\beta_n}{2}}\|\phi\|_* \tag{4.4}$$

and if $\|\phi_j\|_* \le \lambda^{-\frac{\beta_n + \frac{\sigma}{2}}{2}}, j = 1, 2,$ then

$$\|N_\lambda(\phi_1) - N_\lambda(\phi_2)\|_{**} \le C\lambda^{-\frac{\beta_n}{2}}\|\phi_1 - \phi_2\|_*. \tag{4.5}$$

Proof.

$$\begin{aligned} |N_\lambda(\phi)| &\le C(|W|^{\frac{6-n}{n-2}}|\phi|^2 + |\phi|^{\frac{n+2}{n-2}}) \\ &\le C(|W_1|^{\frac{6-n}{n-2}}|\phi|^2 + |W_2|^{\frac{6-n}{n-2}}|\phi|^2 + |\phi|^{\frac{n+2}{n-2}}) \\ &:= I_1 + I_2 + I_3 \end{aligned} \tag{4.6}$$

where I_1, I_2, I_3 are defined as the last equality. Then

$$\begin{aligned} (1+|y-\xi|)^{\frac{n+2+\sigma}{2}} I_1 &\le C\|\phi\|_*^2 (1+|y-\xi|)^{\frac{6-n-\sigma}{2}}(1+|y-\xi|)^{-6+n} \\ &\le C\|\phi\|_*^2, \\ (1+|y-\xi|)^{\frac{n+2+\sigma}{2}} I_2 &\le C\|\phi\|_*^2 (1+|y-\xi|)^{\frac{6-n-\sigma}{2}}(\lambda\varepsilon)^{\frac{-6+n}{2}} \\ &\le C\|\phi\|_*^2, \\ (1+|y-\xi|)^{\frac{n+2+\sigma}{2}} I_3 &\le C\|\phi\|_*^{\frac{n+2}{n-2}} (1+|y-\xi|)^{\frac{n+2+\sigma}{2}}(1+|y-\xi|)^{-\frac{n-2+\sigma}{2}\cdot\frac{n+2}{n-2}} \\ &\le C\|\phi\|_*^{\frac{n+2}{n-2}}. \end{aligned} \tag{4.7}$$

Using (4.6), (4.7) we get (4.4).

Using (4.2) we get

$$N_\lambda(\phi_1) - N_\lambda(\phi_2) = \partial_\eta N_\lambda(\eta)(\phi_1 - \phi_2)$$

where $\eta = t\phi_1 + (1-t)\phi_2, 0 \le t \le 1$, and

$$|\partial_\eta N_\lambda(\eta)| = |\frac{n+2}{n-2}(W+\eta)_+^{\frac{4}{n-2}} - \frac{n+2}{n-2}W^{\frac{4}{n-2}}| = O(|W^{\frac{6-n}{n-2}}\eta| + |\eta|^{\frac{4}{n-2}}).$$

Hence

$$\begin{aligned}(1+|y-\xi|)^{\frac{n+2+\sigma}{2}}|N_\lambda(\phi_1) - N_\lambda(\phi_2)| &\le C\|\phi_1-\phi_2\|_*(1+|y-\xi|)^2(W_1^{\frac{6-n}{n-2}}|\eta| \\ &+ W_2^{\frac{6-n}{n-2}}|\eta| + |\eta|^{\frac{4}{n-2}}) := J_1 + J_2 + J_3\end{aligned} \tag{4.8}$$

where J_1, J_2, J_3 are defined as the last equality. Then

$$\begin{aligned} J_1 &\le C\|\eta\|_*(1+|y-\xi|)^{\frac{6-n-\sigma}{2}}(1+|y-\xi|)^{-6+n}\|\phi_1-\phi_2\|_* \\ &\le C\lambda^{-\frac{\beta_n}{2}}\|\phi_1-\phi_2\|_*, \\ J_2 &\le C\|\eta\|_*(1+|y-\xi|)^{\frac{6-n-\sigma}{2}}(\lambda\varepsilon)^{-\frac{6-n}{2}}\|\phi_1-\phi_2\|_* \\ &\le C\lambda^{-\frac{\beta_n}{2}}\|\phi_1-\phi_2\|_*, \\ J_3 &\le C(1+|y-\xi|)^2(|\phi_1|^{\frac{4}{n-2}} + |\phi_2|^{\frac{4}{n-2}})\|\phi_1-\phi_2\|_* \\ &\le C(\|\phi_1\|_*^{\frac{4}{n-2}} + \|\phi_2\|_*^{\frac{4}{n-2}})\|\phi_1-\phi_2\|_* \\ &\le C\lambda^{-\frac{\beta_n}{2}}\|\phi_1-\phi_2\|_*. \end{aligned} \tag{4.9}$$

Using (4.8), (4.9) we get (4.5). □

By Proposition 3.1, Lemma 2.3, Lemma 4.1 and contraction mapping , we drive the following main result:

Proposition 4.1. *There exists a constant C, independent of $\varepsilon, \lambda, \xi$ such that problem (4.3) has a unique solution $\phi_{\lambda,\xi} = \phi(\varepsilon, \lambda, \xi)$ with*

$$\|\phi_{\lambda,\xi}\|_* \le C\lambda^{-\frac{\beta_n + \frac{\sigma}{2}}{2}}.$$

Moreover, $\lambda \mapsto \phi_{\lambda,\xi}$ is C^2 with respect to the L^∞_*-norm and

$$\|\partial_\lambda \phi_{\lambda,\xi}\|_* \le C\lambda^{-\frac{\beta_n+\frac{\sigma}{2}+2}{2}}, \quad \|\partial_\lambda^2 \phi_{\lambda,\xi}\|_* \le C\lambda^{-\frac{\beta_n+\frac{\sigma}{2}+4}{2}}. \tag{4.10}$$

Proof. We consider the map $\mathcal{A}_\lambda$ from

$$\mathcal{F} = \{\phi \in H^1(\Omega_\lambda) \mid \|\phi\|_* \le \bar{C}\lambda^{-\frac{\beta_n+\frac{\sigma}{2}}{2}}\}$$

to $H^1(\Omega_\lambda)$, defined as

$$\mathcal{A}_\lambda(\phi) = -L_\lambda(n(n-2)N_\lambda(\phi) + R^\lambda)$$

where $\bar{C}$ is a large number, to be determined later and L_λ is given by Proposition 3.1. We note that finding a solution ϕ to problem(4.3) is equivalent to finding a fixed point of $\mathcal{A}_\lambda$.

$\forall \phi \in \mathcal{F}$, Proposition 3.1 gives us

$$\begin{aligned}\|A_\lambda(\phi)\|_* &\le \|L_\lambda(N_\lambda(\phi)\|_* + \|L_\lambda(R^\lambda\|_* \le C\varepsilon^{-\alpha}(\|N_\lambda(\phi)\|_{**} + \|R^\lambda\|_{**}) \\ &\le C\varepsilon^{-\alpha}\lambda^{-\frac{\beta_n}{2}}\|\phi\|_* + C\varepsilon^{-\alpha}\lambda^{-\frac{\beta_n+\sigma}{2}} \le C\lambda^{-\frac{\beta_n}{4}}\|\phi\|_* + C\lambda^{-\frac{\beta_n+\frac{\sigma}{2}}{2}}.\end{aligned}$$

Let $\bar{C} = 2C$ and ε small enough, then $\mathcal{A}_\lambda$ sends $\mathcal{F}$ into itself.

On the other hand, $\mathcal{A}_\lambda$ is a contraction map. Indeed, for ϕ_1 and ϕ_2 in $\mathcal{F}$, we have

$$\begin{aligned}\|\mathcal{A}_\lambda(\phi_1)-\mathcal{A}_\lambda(\phi_2)\|_* &\le C\varepsilon^{-\alpha}\|N_\lambda(\phi_1)-N_\lambda(\phi_2)\|_{**}\\ &\le C\varepsilon^{-\alpha}\lambda^{-\frac{\beta_n}{2}}\|\phi_1-\phi_2\|_*\le \tfrac{1}{2}\|\phi_1-\phi_2\|_*,\end{aligned}$$

which implies that $\mathcal{A}_\lambda$ has a unique fixed point in $\mathcal{F}$.

In order to prove that $\lambda\mapsto\phi_{\lambda,\xi}$ is C^2, we remark that if we set for $\eta\in\mathcal{F}$,

$$B(\lambda,\xi,\eta)=\eta+L_\lambda(n(n-2)N_\lambda(\phi)+R^\lambda), \tag{4.11}$$

then $\phi_{\lambda,\xi}$ is defined as $B(\lambda,\xi,\phi_{\lambda,\xi})=0$.

We have

$$\partial_\eta B(\lambda,\xi,\eta)[\theta]=\theta+n(n-2)L_\lambda(\theta\partial_\eta N_\lambda(\eta)).$$

Using Proposition 3.1, we write

$$\begin{aligned}\|L_\lambda(\theta\partial_\eta N_\lambda(\eta))\|_* &\le C\varepsilon^{-\alpha}\|\theta\partial_\eta N_\lambda(\eta)\|_{**}\le C\varepsilon^{-\alpha}\|\theta\|_*\|(1+|y-\xi|)^{-\frac{n-2+\sigma}{2}}\partial_\eta N_\lambda(\eta)\|_{**}\\ &\le C\lambda^{-\frac{\beta_n}{4}}\|\theta\|_*.\end{aligned}$$

Consequently, $\partial_\eta B(\lambda,\xi,\eta)$ is invertible with uniformly bounded inverse. Then the fact that $\lambda\mapsto\phi_{\lambda,\xi}$ is C^2 follows from the fact that $(\lambda,\eta)\mapsto L_\lambda(N_\varepsilon(\eta))$ is C^2 and implicit function theorem.

Finally, let us show how estimate (4.10) may be obtained. Differentiating (4.11) with respect to λ, we find

$$\partial_\lambda\phi_{\lambda,\xi}=-\big(\partial_\eta B(\lambda,\xi,\phi_{\lambda,\xi})\big)^{-1}\Big((\partial_\lambda L_\lambda)(N_\lambda(\phi_{\lambda,\xi})+R^\lambda)+L_\lambda((\partial_\lambda N_\lambda)(\phi_{\lambda,\xi}))+L_\lambda(\partial_\lambda R^\lambda)\Big).$$

Using Proposition 3.1, we have

$$\begin{aligned}\|(\partial_\lambda L_\lambda)(N_\lambda(\phi_{\lambda,\xi})+R^\lambda)\|_* &\le C\varepsilon^{-\alpha}\lambda^{-1}(\|N_\lambda(\phi_{\lambda,\xi})\|_{**}+\|R^\lambda\|_{**})\le C\lambda^{-\frac{\beta_n+\frac{\sigma}{2}+2}{2}},\\ |(\partial_\lambda N_\lambda)(\phi_{\lambda,\xi})| &\le C|(W+\phi_{\lambda,\xi})_+^{\frac{4}{n-2}}-W^{\frac{4}{n-2}}-\tfrac{4}{n-2}W^{\frac{6-n}{n-2}}\phi_{\lambda,\xi}||\partial_\lambda W|\\ &\le CW^{\frac{6-n}{n-2}}|\phi_{\lambda,\xi}||\partial_\lambda W|+|\phi_{\lambda,\xi}|^{\frac{4}{n-2}}|\partial_\lambda W|:=H_1+H_2\end{aligned}$$

where H_1,H_2 are defined as the last equality.

$$\begin{aligned}\varepsilon^{-\alpha}(1+|y-\xi|)^{\frac{n+2+\sigma}{2}}H_1 &\le C\|\phi_{\lambda,\xi}\|_*\varepsilon^{-\alpha}(1+|y-\xi|)^2\big((1+|y-\xi|)^{-(6-n)}\\ &\quad+(\lambda\varepsilon)^{-\frac{6-n}{2}}\big)(\lambda\varepsilon)^{-\frac{n+2}{2}}\le C\lambda^{-\frac{\beta_n+\frac{\sigma}{2}+2}{2}}.\end{aligned}$$

Just as the above, we get

$$\varepsilon^{-\alpha}\|H_2\|_{**}\le C\lambda^{-\frac{\beta_n+\frac{\sigma}{2}+2}{2}},\qquad \|L_\lambda(\partial_\lambda R^\lambda)\|_*\le C\varepsilon^{-\alpha}\|\partial_\lambda R^\lambda\|_{**}\le C\lambda^{-\frac{\beta_n+\frac{\sigma}{2}+2}{2}}$$

which implies the first part of (4.10).

The second derivatives of $\phi_{\lambda,\xi}$ with respect to λ may be estimated in the same way. This concludes the proof of Proposition 4.1. □

5. Proof of Theorem 1.1

Let us define a reduced energy functional as

$$I_\varepsilon(\lambda,a)\equiv J_\lambda[W+\phi_{\lambda,\xi}]. \tag{5.1}$$

Then we state:

Proposition 5.1. *The function $u=W+\phi_{\lambda,\xi}$ is a solution to problem (1.2) if and only if (λ,a) is a critical point of $I_\varepsilon(\lambda,a)$.*

The proof is similar to those of Proposition 5.1 in [16], [17]. We omit it.

In view of Proposition 5.1, to prove Theorem 1.1, we have to find a critical point of $I_\varepsilon(\lambda,a)$. First we establish a C^2 expansion of $I_\varepsilon(\lambda,a)$.

Proposition 5.2. *For ε sufficiently small, we have*

$$I_\varepsilon(\lambda,a)=\begin{cases} J_\lambda[W]+\varepsilon^{-1}\lambda^{-\frac{3}{2}}D_1 & n=3,\\ J_\lambda[W]+\varepsilon^{-1}\lambda^{-2}D_2 & n=4,5.\end{cases}\tag{5.2}$$

with $D_1=O(1), D_2=o(1)$ and

$$\partial_\lambda D_1=O(\lambda^{-1}),\quad \partial_\lambda^2 D_1=O(\lambda^{-2}),\quad \partial_\lambda D_2=o(\lambda^{-1}),\quad \partial_\lambda^2 D_2=o(\lambda^{-2}).$$

Proof. Actually, in view of (5.1) and the fact that $J_\lambda'[W+\phi_{\lambda,\xi}][\phi_{\lambda,\xi}]=0$ yields

$$\begin{aligned}I_\varepsilon(\lambda,a)-J_\lambda[W]&=J_\lambda[W+\phi_{\lambda,\xi}]-J_\lambda[W]=-\int_0^1 J_\lambda''[W+t\phi_{\lambda,\xi}][\phi_{\lambda,\xi},\phi_{\lambda,\xi}]tdt\\ &=-\int_0^1\int_{\Omega_\lambda}\big(|\nabla\phi_{\lambda,\xi}|^2+(\lambda\varepsilon)^{-2}\phi_{\lambda,\xi}^2-n(n+2)(W+t\phi_{\lambda,\xi})_+^{\frac{4}{n-2}}\phi_{\lambda,\xi}^2\big)tdt\\ &=-\int_0^1\int_{\Omega_\lambda}\Big(n(n+2)\big(W^{\frac{4}{n-2}}-(W+t\phi_{\lambda,\xi})_+^{\frac{4}{n-2}}\big)\phi_{\lambda,\xi}^2+R^\lambda\phi_{\lambda,\xi}\\ &\quad+n(n-2)N_\lambda(\phi_{\lambda,\xi})\phi_{\lambda,\xi}\Big)tdt.\end{aligned}$$

If $n=4,5$,

$$\begin{aligned}\int_{\Omega_\lambda}|R^\lambda\phi_{\lambda,\xi}|&\le C\|R^\lambda\|_{**}\|\phi_{\lambda,\xi}\|_*\int_{\Omega_\lambda}(1+|y-\xi|)^{-(n+\sigma)}\\ &\le C\lambda^{-\frac{2+\sigma}{2}}\lambda^{-\frac{2+\frac{\sigma}{2}}{2}}=o(\varepsilon^{-1}\lambda^{-2}),\end{aligned}\tag{5.3}$$

(5.4)
$$\begin{aligned}\int_{\Omega_\lambda}|N_\lambda(\phi_{\lambda,\xi})\phi_{\lambda,\xi}|&\le C\int_{\Omega_\lambda}(|W|^{\frac{6-n}{n-2}}|\phi_{\lambda,\xi}|^3+|\phi_{\lambda,\xi}|^{\frac{2n}{n-2}})\\ &\le C\int_{\Omega_\lambda}(|W_1|^{\frac{6-n}{n-2}}+(\lambda\varepsilon)^{-\frac{6-n}{2}})(1+|y-\xi|)^{-\frac{3(n-2+\sigma)}{2}}\|\phi_{\lambda,\xi}\|_*^3\\ &\quad+C\|\phi_{\lambda,\xi}\|_*^{\frac{2n}{n-2}}\int_{\Omega_\lambda}(1+|y-\xi|)^{-n(1+\frac{\sigma}{n-2})}\\ &\le C\lambda^{-\frac{2+\frac{\sigma}{2}}{2}\min\{3,\frac{2n}{n-2}\}}=o(\varepsilon^{-1}\lambda^{-2}),\end{aligned}$$

$$\begin{aligned}n(n+2)\int_{\Omega_\lambda}|W^{\frac{4}{n-2}}-(W+t\phi_{\lambda,\xi})_+^{\frac{4}{n-2}}|\phi^2&\le C\int_{\Omega_\lambda}(|W|^{\frac{6-n}{n-2}}|\phi|^3+|\phi_{\lambda,\xi}|^{\frac{2n}{n-2}})\\ &=o(\varepsilon^{-1}\lambda^{-2}).\end{aligned}\tag{5.5}$$

Using (5.3) – (5.5) we get the second part of (5.2).

If $n=3$, since $|R^\lambda|\le C\big((\lambda\varepsilon)^{-1}(1+|y-\xi|)^{-4}+(\lambda\varepsilon)^{-4}(1+|y-\xi|)^{-1}\big)$, then

$$\begin{aligned}\int_{\Omega_\lambda}|R^\lambda\phi_{\lambda,\xi}|&\le C\|\phi_{\lambda,\xi}\|_*\int_{\Omega_\lambda}\Big((\lambda\varepsilon)^{-1}(1+|y-\xi|)^{-4-\frac{1+\sigma}{2}}+(\lambda\varepsilon)^{-4}(1+|y-\xi|)^{-1-\frac{1+\sigma}{2}}\Big)\\ &\le C\lambda^{-\frac{1+\frac{\sigma}{2}}{2}}(\lambda\varepsilon)^{-1}=o(\varepsilon^{-1}\lambda^{-\frac{3}{2}}).\end{aligned}$$

The other two terms is the same as (5.4) and (5.5). Hence we get (5.2).

Differentiating (5.3) with respect to λ, by the similar computation the estimates of $D_i, i=1,2$ hold for the first and second derivatives with respect to λ. This concludes the proof of Proposition 5.2. □

since

$$J_\lambda[W_2]=\frac{n(n-2)}{2}\int_\Omega W_\varepsilon^{\frac{2n}{n-2}}-\frac{(n-2)^2}{2}\int_\Omega W_\varepsilon^{\frac{2n}{n-2}}=\varepsilon^{-n}(n-2)\int_\Omega W_\varepsilon^{\frac{2n}{n-2}}$$

which has no relation with λ or a. According to Lemma 2.3 and Proposition 5.2 we have the following corollary:

Corollary 5.1. *When $n=3$, noticing $\int_{R^3}U_{0,1}^5=\frac{4\pi}{3}$,*

$$I_\varepsilon(\lambda,a)=J_\lambda[W_2]+\int_{R^3}U_{0,1}^6+\frac{2\pi}{\lambda\varepsilon}-(B_3+o(1))(\lambda\varepsilon)^{-\frac{1}{2}}e^{-\frac{\beta d_a}{\varepsilon}}+O(\varepsilon^{-1}\lambda^{-1}e^{-\frac{d_a}{\varepsilon}}+\varepsilon^{-1}\lambda^{-\frac{3}{2}}),$$

$$\partial_\lambda I_\varepsilon=-\frac{2\pi}{\lambda^2\varepsilon}+\frac{(B_3+o(1))}{2}\lambda^{-\frac{3}{2}}\varepsilon^{-\frac{1}{2}}e^{-\frac{\beta d_a}{\varepsilon}}+O(\varepsilon^{-1}\lambda^{-2}e^{-\frac{d_a}{\varepsilon}}+\varepsilon^{-1}\lambda^{-\frac{5}{2}}),$$

$$\partial_\lambda^2 I_\varepsilon=\frac{1}{\lambda^3\varepsilon}(4\pi-\frac{3(B_3+o(1))}{4}\lambda^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}e^{-\frac{\beta d_a}{\varepsilon}})+O(\varepsilon^{-1}\lambda^{-3}e^{-\frac{d_a}{\varepsilon}}+\varepsilon^{-1}\lambda^{-\frac{7}{2}}).$$

When $n=4,5$,

$$\begin{aligned}I_\varepsilon(\lambda,a)=\;&J_\lambda[W_2]+(n-2)\textstyle\int_{R^n}U_{0,1}^{\frac{2n}{n-2}}+\frac{n-2}{2}(A_n+o(1))(\lambda\varepsilon)^{-2}(\ln\lambda)^m\\&-(B_n+o(1))(\lambda\varepsilon)^{-\frac{n-2}{2}}e^{-\frac{\beta d_a}{\varepsilon}}+O(\varepsilon^{-1}\lambda^{-2}),\end{aligned}$$

$$\begin{aligned}\partial_\lambda I_\varepsilon&=-(n-2)(A_n+o(1))\lambda^{-3}\varepsilon^{-2}\big((\ln\lambda)^m-\frac{m}{2}\big)\\&+\frac{(n-2)(B_n+o(1))}{2}\lambda^{-\frac{n}{2}}\varepsilon^{-\frac{n-2}{2}}e^{-\frac{\beta d_a}{\varepsilon}}+O(\varepsilon^{-1}\lambda^{-3}),\end{aligned}$$

$$\begin{aligned}\partial_\lambda^2 I_\varepsilon&=3(n-2)(A_n+o(1))\lambda^{-4}\varepsilon^{-2}\big((\ln\lambda)^m-\frac{5m}{6}\big)\\&-\frac{n(n-2)(B_n+o(1))}{4}\lambda^{-\frac{n+2}{2}}\varepsilon^{-\frac{n-2}{2}}e^{-\frac{\beta d_a}{\varepsilon}}+O(\varepsilon^{-1}\lambda^{-4})\end{aligned}$$

where $m=1$ if $n=4$ and $m=0$ if $n=5$.

Proof of Theorem 1.1: When $n=3$, let $\lambda_0=\big(\frac{4\pi+o(1)}{B_3}\big)^2\varepsilon^{-1}e^{\frac{2\beta d_a}{\varepsilon}}\in\Lambda$, then

$$\partial_\lambda^2 I_\varepsilon\mid_{\lambda=\lambda_0}=\frac{\pi+o(1)}{\lambda_0^3\varepsilon}\neq 0.$$

The implicit functions theorem provides us, for ε small enough, with a C^1–map $a\in M_\gamma\to\lambda(a)$, such that

$$\partial_\lambda I_\varepsilon(\lambda(a),a)=0,\quad \lambda(a)=\Big(\frac{4\pi+o(1)}{B_3}\Big)^2\varepsilon^{-1}e^{\frac{2\beta d_a}{\varepsilon}}\in\Lambda.$$

Then by Corollary 5.1

$$I_\varepsilon(\lambda(a),a)=J_\lambda[W_2]+\int_{R^3}U_{0,1}^6-\frac{B_3^2}{8\pi}e^{-\frac{2\beta d_a}{\varepsilon}}+o(e^{-\frac{2\beta d_a}{\varepsilon}}).$$

Obviously, there exists a maximum point a_ε of $I_\varepsilon(\lambda(a),a)$ and $a_\varepsilon\to a_0$ as $\varepsilon\to 0$ where a_0 satisfies $d_{a_0}=\max_{a\in M_\gamma}d_a$, i.e. $(\lambda(a_\varepsilon),a_\varepsilon)$ is a critical point of $I_\varepsilon(\lambda,a)$.

When $n=4,5$, using the same discussion as $n=3$, we find $I_\varepsilon(\lambda,a)$ has a critical point $(\lambda(a_\varepsilon),a_\varepsilon)$ where $\lambda(a_\varepsilon)\sim e^{\frac{2+o(1)}{(6-n)\varepsilon}\beta d_{a_\varepsilon}}$ and $a_\varepsilon\to a_0$ as $\varepsilon\to 0$.

Let $u_\varepsilon(x)=(\lambda(a_\varepsilon)\varepsilon)^{\frac{n-2}{2}}\Big(W_1(\lambda(a_\varepsilon)x)+W_2(\lambda(a_\varepsilon)x)+\phi_{\lambda(a_\varepsilon),\lambda a_\varepsilon}(\lambda(a_\varepsilon)x)\Big)$, then $u_\varepsilon(x)$ has all properties in Theorem 1.1. □

6. Appendix

Proof of Lemma 2.3: Using (1.5), we have

$$|S_\lambda[W]|\le C|W_1|^{\frac{4}{n-2}}W_2+C|W_2|^{\frac{4}{n-2}}W_1+C|U_{\xi,1}|^{\frac{4}{n-2}}|W_1-U_{\xi,1}|. \tag{6.1}$$

When $n=3$, using (2.8) and (2.11),

$$|U_{\xi,1}|^4|W_1-U_{\xi,1}|\le C(1+|y-\xi|^2)^{-2}(\lambda\varepsilon)^{-1}\le C(1+|y-\xi|)^{-\frac{5+\sigma}{2}}\lambda^{-\frac{1+\sigma}{2}}.$$

If $|y-\xi|\ge\sigma\lambda d_a$, then

$$\begin{aligned}|W_1^4W_2| &\le C(\lambda^{-2}+\tfrac{1}{\lambda\varepsilon}e^{-\frac{|y-\xi|}{\lambda\varepsilon}})^4(\lambda\varepsilon)^{-\frac12}\le C(1+|y-\xi|)^{-\frac{5+\sigma}{2}}\lambda^{-\frac{1+\sigma}{2}},\\ |W_2^4W_1| &\le C(\lambda^{-2}+\tfrac{1}{\lambda\varepsilon}e^{-\frac{|y-\xi|}{\lambda\varepsilon}})(\lambda\varepsilon)^{-2}\le C(1+|y-\xi|)^{-\frac{5+\sigma}{2}}\lambda^{-\frac{1+\sigma}{2}}.\end{aligned}$$

If $|y-\xi|\le\sigma\lambda d_a$, then

$$\begin{aligned}|W_1^4W_2| &\le C(1+|y-\xi|^2)^{-2}(\lambda\varepsilon)^{-\frac12}e^{-\frac{(1-\sigma)^2d_a}{\varepsilon}}\\ &\le C(1+|y-\xi|)^{-\frac{5+\sigma}{2}}\lambda^{-\frac{1+\sigma}{2}},\\ |W_2^4W_1| &\le C(1+|y-\xi|^2)^{-\frac12}(\lambda\varepsilon)^{-2}e^{-\frac{4(1-\sigma)^2d_a}{\varepsilon}}\\ &\le C(1+|y-\xi|)^{-\frac{5+\sigma}{2}}\lambda^{-\frac{1+\sigma}{2}}.\end{aligned} \tag{6.2}$$

Using (6.1), (6.2), the first part of (2.15) holds for $n=3$.

When $n=4,5$,

$$\begin{aligned}|U_{\xi,1}|^{\frac{4}{n-2}}|W_1-U_{\xi,1}| &\le C(1+|y-\xi|^2)^{-2}\lambda^{-2}\varepsilon^{-3}(1+|y-\xi|)^{-n+4}\\ &\le C(1+|y-\xi|)^{-\frac{n+2+\sigma}{2}}\lambda^{-\frac{2+\sigma}{2}}.\end{aligned} \tag{6.3}$$

If $|y-\xi|\ge\sigma\lambda d_a$, then

$$\begin{aligned}|W_1^{\frac{4}{n-2}}W_2| &\le C(\lambda\varepsilon)^{-\frac{n-2}{2}}(1+|y-\xi|)^{-2}\le C(1+|y-\xi|)^{-\frac{n+2+\sigma}{2}}\lambda^{-\frac{2+\sigma}{2}},\\ |W_2^{\frac{4}{n-2}}W_1| &\le C(\lambda\varepsilon)^{-2-n}\le C(1+|y-\xi|)^{-\frac{n+2+\sigma}{2}}\lambda^{-\frac{2+\sigma}{2}}.\end{aligned}$$

If $|y-\xi|\le\sigma\lambda d_a$, then

$$\begin{aligned}|W_1^{\frac{4}{n-2}}W_2| &\le C(1+|y-\xi|^2)^{-2}(\lambda\varepsilon)^{-\frac{n-2}{2}}e^{-\frac{(1-\sigma)^2d_a}{\varepsilon}}\\ &\le C(1+|y-\xi|)^{-\frac{n+2+\sigma}{2}}\lambda^{-\frac{2+\sigma}{2}},\\ |W_2^{\frac{4}{n-2}}W_1| &\le C(1+|y-\xi|^2)^{-\frac{n-2}{2}}(\lambda\varepsilon)^{-2}e^{-\frac{4(1-\sigma)^2d_a}{(n-2)\varepsilon}}\\ &\le C(1+|y-\xi|)^{-\frac{n+2+\sigma}{2}}\lambda^{-\frac{2+\sigma}{2}}\end{aligned} \tag{6.4}$$

since $\frac{4}{n-2}\ge\frac43$.

Using (6.1), (6.3), (6.4), the first part of (2.15) holds for $n=4,5$.

Differentiating (1.6) with respect to λ and by the similar computation , (2.15) holds for $n=3,4,5$.

Now we compute $J_\lambda[W]$. By the definition of $J_\lambda[W]$, we have

$$(6.5)\quad \begin{aligned} J_\lambda[W] = J_\lambda[W_1+W_2] = J_\lambda[W_1] + J_\lambda[W_2] + n(n-2)\int_{\Omega_\lambda} W_2^{\frac{n+2}{n-2}} W_1 \\ -\frac{(n-2)^2}{2}\int_{\Omega_\lambda}\left((W_1+W_2)^{\frac{2n}{n-2}} - W_1^{\frac{2n}{n-2}} - W_2^{\frac{2n}{n-2}}\right) \\ := J_\lambda[W_1] + J_\lambda[W_2] - I_\lambda, \end{aligned}$$

where

$$\begin{aligned} I_\lambda &= \tfrac{(n-2)^2}{2}\int_{\Omega_\lambda}\Big((W_1+W_2)^{\frac{2n}{n-2}} - W_1^{\frac{2n}{n-2}} - W_2^{\frac{2n}{n-2}} - \tfrac{2n}{n-2}W_2^{\frac{n+2}{n-2}}W_1 - \tfrac{2n}{n-2}W_1^{\frac{n+2}{n-2}}W_2\Big) \\ &\quad + n(n-2)\int_{\Omega_\lambda}\tfrac{2n}{n-2}W_1^{\frac{n+2}{n-2}}W_2 \quad := I_{\lambda,1} + I_{\lambda,2}. \end{aligned}$$

Since W_1 satisfies

$$\begin{cases} \Delta W_1 - (\lambda\varepsilon)^{-2}W_1 + n(n-2)U_{\xi,1}^{\frac{n+2}{n-2}} = 0 & \text{in } \Omega_\lambda, \\ \frac{\partial W_1}{\partial \nu} = 0 & \text{on } \partial\Omega_\lambda, \end{cases}$$

then

$$J_\lambda[W_1] = \frac{n(n-2)}{2}\int_{\Omega_\lambda} U_{\xi,1}^{\frac{n+2}{n-2}} W_1 - \frac{(n-2)^2}{2}\int_{\Omega_\lambda} W_1^{\frac{2n}{n-2}}.$$

First we consider the case $n=3$.

$$\begin{aligned} J_\lambda[W_1] &= \tfrac{3}{2}\int_{\Omega_\lambda} U_{\xi,1}^5 W_1 - \tfrac{1}{2}\int_{\Omega_\lambda} W_1^6 \\ &= \tfrac{3}{2}\int_{\Omega_\lambda} U_{\xi,1}^5 (U_{\xi,1} + W_1 - U_{\xi,1}) - \tfrac{1}{2}\int_{\Omega_\lambda}(U_{\xi,1} + W_1 - U_{\xi,1})^6 \\ &= \tfrac{3}{2}\int_{\Omega_\lambda} U_{\xi,1}^6 + \tfrac{3}{2}\int_{\Omega_\lambda} U_{\xi,1}^5(W_1 - U_{\xi,1}) - \tfrac{1}{2}\int_{\Omega_\lambda} U_{\xi,1}^6 - 3\int_{\Omega_\lambda} U_{\xi,1}^5(W_1 - U_{\xi,1}) \\ &\quad + O\Big(\int_{\Omega_\lambda} U_{\xi,1}^4 (W_1 - U_{\xi,1})^2 + \int_{\Omega_\lambda}(W_1 - U_{\xi,1})^6\Big) \\ &= \int_{\Omega_\lambda} U_{\xi,1}^6 - \tfrac{3}{2}\int_{\Omega_\lambda} U_{\xi,1}^5(W_1 - U_{\xi,1}) + O\Big(\int_{\Omega_\lambda} U_{\xi,1}^4(W_1 - U_{\xi,1})^2 + \int_{\Omega_\lambda}(W_1 - U_{\xi,1})^6\Big). \end{aligned}$$

Compute directly,

$$\begin{aligned} \int_{\Omega_\lambda} U_{\xi,1}^6 &= \int_{R^3} U_{0,1}^6 + O(\lambda^{-3}), \\ -\tfrac{3}{2}\int_{\Omega_\lambda} U_{\xi,1}^5 (W_1 - U_{\xi,1}) &= \tfrac{3}{2}(\lambda\varepsilon)^{-1}\int_{R^3} U_{0,1}^5 + O(\varepsilon^{-1}\lambda^{-1}e^{-\frac{d_a}{\varepsilon}}), \\ O\Big(\int_{\Omega_\lambda} U_{\xi,1}^4 (W_1 - U_{\xi,1})^2 + \int_{\Omega_\lambda}(W_1 - U_{\xi,1})^6\Big) &= o(\varepsilon^{-1}\lambda^{-1}e^{-\frac{d_a}{\varepsilon}}). \end{aligned}$$

Hence

$$(6.6)\quad J_\lambda[W_1] = \int_{R^3} U_{0,1}^6 + \frac{3}{2}(\lambda\varepsilon)^{-1}\int_{R^3} U_{0,1}^5 + O(\varepsilon^{-1}\lambda^{-1}e^{-\frac{d_a}{\varepsilon}}).$$

By direct computation, we get

$$I_{\lambda,1} = \frac{15}{2}\varepsilon^{-2}\int_{\Omega_\lambda} W_\varepsilon^4 V_{a,\lambda}^2 + O(\varepsilon^{-\frac{3}{2}}\int_{\Omega_\lambda} W_\varepsilon^3 V_{a,\lambda}^3 + \varepsilon^{-1}\int_{\Omega_\lambda} W_\varepsilon^2 V_{a,\lambda}^4) = O(\varepsilon^{-1}\lambda^{-1}e^{-\frac{d_a}{\varepsilon}}).$$

On the other hand,

$$(6.7)\quad \begin{aligned} I_{\lambda,2} &= 3\int_{\Omega_\lambda} W_2 W_1^5 = 3\varepsilon^{-\frac{1}{2}}\int_{\Omega} W_\varepsilon(x) V_{a,\lambda}^5 dx \\ &= 3\varepsilon^{-\frac{1}{2}}\int_{|x-a|\le\sigma d_a} W_\varepsilon(x)V_{a,\lambda}^5 dx + 3\varepsilon^{-\frac{1}{2}}\int_{|x-a|>\sigma d_a} W_\varepsilon(x) V_{a,\lambda}^5 dx \\ &= (B_3 + o(1))(\lambda\varepsilon)^{-\frac{1}{2}} e^{-\frac{\beta d_a}{\varepsilon}} + O(\varepsilon^{-1}\lambda^{-1}e^{-\frac{d_a}{\varepsilon}}) \end{aligned}$$

where β is fixed and $1-\sigma < \beta < 1+\sigma$.

Using (6.5) – (6.7) , (2.16) holds for $n=3$.

When $n=4,5$, using (6.5), we have

$$
\begin{aligned}
J_\lambda[W_1] &= (n-2)\int_{\Omega_\lambda} U_{\xi,1}^{\frac{2n}{n-2}} - \frac{n(n-2)}{2}\int_{\Omega_\lambda} U_{\xi,1}^{\frac{n+2}{n-2}}(W_1-U_{\xi,1}) \\
&\quad + O\Big(\int_{\Omega_\lambda} U_{\xi,1}^{\frac{4}{n-2}}(W_1-U_{\xi,1})^2 + \int_{\Omega_\lambda}|W_1-U_{\xi,1}|^{\frac{2n}{n-2}}\Big).
\end{aligned} \tag{6.8}
$$

Now compute term by term.

$$
(n-2)\int_{\Omega_\lambda} U_{\xi,1}^{\frac{2n}{n-2}} = (n-2)\int_{R^n} U_{0,1}^{\frac{2n}{n-2}} + O(\lambda^{-n}),
$$

$$
\begin{aligned}
-\frac{n(n-2)}{2}\int_{\Omega_\lambda} U_{\xi,1}^{\frac{n+2}{n-2}}(W_1-U_{\xi,1}) &= -\frac{n(n-2)}{2}\int_{\Omega} U_{a,\lambda}^{\frac{n+2}{n-2}}(V_{a,\lambda}-U_{a,\lambda}) \\
&= \frac{1}{2}\int_\Omega \Delta U_{a,\lambda}(V_{a,\lambda}-U_{a,\lambda}) \\
&= \frac{1}{2}\int_\Omega U_{a,\lambda}(\Delta V_{a,\lambda} + n(n-2)U_{a,\lambda}^{\frac{n+2}{n-2}}) \\
&= \frac{1}{2}\varepsilon^{-2}\int_\Omega U_{a,\lambda}^2 + O(\varepsilon^{-1}\lambda^{-2}) \\
&= (A_n+o(1))(\lambda\varepsilon)^{-2}(\ln\lambda)^m + O(\varepsilon^{-1}\lambda^{-2})
\end{aligned}
$$

where A_n is positive, $m=1$ if $n=4$ and $m=0$ if $n=5$.

$$
\begin{aligned}
\int_{\Omega_\lambda} U_{\xi,1}^{\frac{4}{n-2}}(W_1-U_{\xi,1})^2 &= O\Big(\varepsilon^{-6}\lambda^{-4}\int_{\Omega_\lambda}(1+|y-\xi|)^{-2n+4}\Big) \\
&= O(\varepsilon^{-1}\lambda^{-n+2}), \\
\int_{\Omega_\lambda}|W_1-U_{\xi,1}|^{\frac{2n}{n-2}} &= O(\varepsilon^{-1}\lambda^{-n+2}).
\end{aligned} \tag{6.9}
$$

Using (6.8), (6.9) we get

$$
J_\lambda[W_1] = (n-2)\int_{R^n} U_{0,1}^{\frac{2n}{n-2}} + (A_n+o(1))(\lambda\varepsilon)^{-2}(\ln\lambda)^m + O(\varepsilon^{-1}\lambda^{-2}). \tag{6.10}
$$

Similarly, we have

$$
\begin{aligned}
I_{\lambda,1} &= O\Big(\int_{\Omega_\lambda}|W_1|^{\frac{4}{n-2}}W_2^2 + |W_2|^{\frac{4}{n-2}}W_1^2\Big) = O(\varepsilon^{-1}\lambda^{-2}), \\
I_{\lambda,2} &= n(n-2)\int_{\Omega_\lambda}(\lambda\varepsilon)^{-\frac{n-2}{2}}W_\varepsilon\frac{y}{\lambda}V_{a,\lambda}^{\frac{n+2}{n-2}}\lambda^{-\frac{n+2}{2}} \\
&= n(n-2)\int_\Omega \varepsilon^{-\frac{n-2}{2}}W_\varepsilon(x)V_{a,\lambda}^{\frac{n+2}{n-2}}dx \\
&= n(n-2)\varepsilon^{-\frac{n-2}{2}}\int_\Omega W_\varepsilon(x)V_{a,\lambda}^{\frac{n+2}{n-2}} + O(\varepsilon^{-1}\lambda^{-2}) \\
&= (B_n+o(1))(\lambda\varepsilon)^{-\frac{n-2}{2}}e^{-\frac{\beta d_a}{\varepsilon}} + O(\varepsilon^{-1}\lambda^{-2})
\end{aligned} \tag{6.11}
$$

where β is defined as before and B_n is positive. According to (6.5), (6.8)–(6.11), (2.16) holds for $n=4,5$.

Differentiating (6.5) with respect to λ and by the similar computation, the estimates of $E_i, i=1,2$ hold for the first and second derivatives with respect to λ. This concludes the proof of Lemma 2.3. □

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