# INFINITELY MANY SOLUTIONS FOR THE SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$ WITH CRITICAL GROWTH

WENYI CHEN, JUNCHENG WEI, AND SHUSEN YAN

Abstract. We consider the following nonlinear problem in $\mathbb{R}^N$

$$\begin{cases} -\Delta u + V(|y|)u = u^{\frac{N+2}{N-2}}, \quad u > 0, \quad \text{in } \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{0.1}$$

where $V(r)$ is a bounded non-negative function, $N \geq 5$. We show that if $r^2V(r)$ has a local maximum point, or local minimum point $r_0 > 0$ with $V(r_0) > 0$, then (0.1) has **infinitely many non-radial** solutions, whose energy can be made arbitrarily large. As an application, we show that the solution set of the following problem

$$-\Delta u = \lambda u + u^{\frac{N+2}{N-2}}, \quad u > 0 \text{ on } S^N$$

has unbounded energy, as long as $\lambda < -\frac{N(N-2)}{4}$, $N \geq 5$.

## 1. Introduction

Standing waves for the following nonlinear Schrödinger equation in $\mathbb{R}^N$:

$$-i\frac{\partial \psi}{\partial t} = \Delta \psi - \tilde{V}(y)\psi + |\psi|^{p-1}\psi, \tag{1.1}$$

are solutions of the form $\psi(t, y) = \exp(i\lambda t)u(y)$. Assuming that the amplitude $u(y)$ is positive and vanishes at infinity, we see that $\psi$ satisfies (1.1) if and only if $u$ solves the following nonlinear elliptic problem

$$-\Delta u + V(y)u = u^p, \quad u > 0, \quad \lim_{|y|\to+\infty} u(y) = 0, \tag{1.2}$$

where $V(y) = \tilde{V}(y) + \lambda$. Throughout this paper, we will assume that $V$ is bounded, and $V(y) \geq 0$.

In this paper, we consider the critical case $p = \frac{N+2}{N-2}$:

$$\begin{cases} -\Delta u + V(y)u = u^{\frac{N+2}{N-2}}, u > 0, & y \in \mathbb{R}^N, \\ u(y) \to 0, & \text{as } |y| \to +\infty. \end{cases} \tag{1.3}$$

1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J40.

It is easy to see that if $V \geq 0$ and $V \not\equiv 0$, the mountain pass value corresponding to (1.3) is not a critical value. In contrast to the subcritical case, there are very few results to (1.3). Benci and Cerami [2] first studied (1.3) and proved the existence of at least one solution if $V \geq 0$ and $\|V\|_{L^{N/2}}$ is sufficiently small. It seems that this is the only existence result available for general $V$ in the critical exponent case. It remains a question if the smallness of the norm $\|V\|_{L^{N/2}}$ is necessary. On the other hand, the assumption $V \in L^{\frac{N}{2}}(\mathbb{R}^N)$ implies that $V$ can not have a positive lower bound in $\mathbb{R}^N$. Thus, the existence result for the case $V(y) \geq V_0 > 0$ in $\mathbb{R}^N$ is completely open.

In this paper, we consider the radially symmetric potential case, i.e. $V(y) = V(|y|)$, although this assumption can be weakened. It follows from the Pohozaev identity that (1.3) has no solution if $(r^2V(r))'$ has fixed sign and is not identically zero. Therefore, we see that to obtain a solution for (1.3), it is necessary to assume that $r^2V(r)$ has either a local maximum, or a local minimum at $r_0 > 0$. The aim of this paper is to show that this condition is not only sufficient, but also guarantees the existence of **infinitely many non-radial solutions**.

Our main result in this paper can be stated as follows:

**Theorem 1.1.** *Suppose that $V(|y|) \geq 0$ is bounded and $N \geq 5$. If $r^2V(r)$ has either an isolated local maximum, or an isolated local minimum at $r_0 > 0$ with $V(r_0) > 0$, then problem (1.3) has infinitely many non-radial solutions.*

Problem (1.3) is also related to the following Brezis-Nirenberg problem in $S^N$

$$-\Delta_{S^N} u = u^{\frac{N+2}{N-2}} + \lambda u, \ u > 0 \ \text{ on } S^N. \tag{1.4}$$

In fact, by using the stereographic projection, problem (1.4) can be reduced to (1.3) with

$$V(y) = \frac{-N(N-2) - 4\lambda}{(1+|y|^2)^2}.$$

So $V(y) > 0$ if $\lambda < -\frac{N(N-2)}{4}$. Moreover, $V(y)$ is radially symmetric.

Equation (1.4) has also been studied recently by many authors. Brezis and Li [4] proved if $\lambda > -\frac{N(N-2)}{4}$, then the only solutions to (1.4) is the constant $u \equiv (-\lambda)^{\frac{N-2}{4}}$. On the other hand, when $\lambda = -\frac{N(N-2)}{4}$, this is the Yamabe problem on $S^N$: all solutions are classified ([9]). When $\lambda < -\frac{N(N-2)}{4}$, Druet [6] (see also Druet and Hebey [7], [8]) proved that the set of positive solutions to (1.4) is compact provided the energy is bounded.

On the other hand, it has been shown that there are more and more nonradial solutions as $\lambda \to -\infty$. We refer to Brezis-Peletier [5], Bandle-Wei [3] and the references therein. Theorem 1.1 implies that as long as $\lambda < -\frac{N(N-2)}{4}$ and $N \ge 5$, there are infinitely many nonradial solutions to (1.4) whose energy can be made arbitrarily large. This shows that the boundedness of energy in [6] and [7] is necessary. We notice that when $N = 3$, Druet [6] proved that the solution set of (1.4) has bounded energy. We believe that Theorem 1.1 also holds for $N = 4$.

Let us point out that in this paper, we don't assume the condition

$$V(|y|) \ge V_0 > 0, \quad \text{for } |y| \text{ large},$$

which is essential for Schrödinger equation with sub-critical growth. In [13], we considered the following problem

$$\begin{cases} -\Delta u + V(|y|)u = u^p, u > 0, & y \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.5}$$

where $1 < p < \frac{N+2}{N-2}$. We proved that if

$$V(r) = V_0 + \frac{a}{r^m} + O\big(\frac{1}{r^{m+1}}\big), \quad \text{as } r \to +\infty$$

for some $V_0 > 0$, $a > 0$ and $m > 1$, then, (1.5) has infinitely many non-radial solutions. In fact, we showed that (1.5) has solutions with large number of bumps near the infinity. Problem (1.5) is non-compact due to the unboundedness of the domain, while (1.3) is non-compact due to the unboundedness of the domain and the critical growth of the nonlinearity. We will prove Theorem 1.1 by constructing solutions with large number of bubbles near the sphere $|y| = r_0$. So, in view of the construction of bubbling solutions, we can say that the effect from the critical growth is **stronger** than the effect from the unboundedness of the domain.

Before we close this introduction, we outline the main idea in the proof of Theorem 1.1.

Let us fix a positive integer

$$k \ge k_0,$$

where $k_0$ is large, which is to be determined later.

Let $2^* = \frac{2N}{N-2}$. It is well-known that the functions

$$U_{x,\mu}(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \left(\frac{\mu}{1 + \mu^2|y-x|^2}\right)^{\frac{N-2}{2}}, \ \mu > 0, \ x \in \mathbb{R}^N$$

are the only solutions to the problem

$$-\Delta u = u^{\frac{N+2}{N-2}},\ u > 0 \text{ in } \mathbb{R}^N.$$

Let $y = (y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$. Define

$$H_s = \big\{u : u \in D^{1,2}(\mathbb{R}^N), u \text{ is even in } y_h, h = 2, \cdots, N, \\ u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta + \frac{2\pi}{k}), r\sin(\theta + \frac{2\pi}{k}), y'')\big\}.$$

Let

$$x_j = \big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and let

$$W_{r,\mu}(y) = \sum_{j=1}^{k} U_{x_j,\mu}(y)$$

Choose $\delta > 0$ small, such that

$$V(|y|) \geq V_0 > 0, \quad \forall\ |y| \in [r_0 - 2\delta, r_0 + 2\delta].$$

In this paper, we always assume that

$$r \in [r_0 - \delta, r_0 + \delta],$$

and

$$\mu \in \big[L_0 k^{\frac{N-2}{N-4}}, L_1 k^{\frac{N-2}{N-4}}\big], \qquad \text{for some constants } \ L_1 > L_0 > 0.$$

Theorem 1.1 is a direct consequence of the following result:

**Theorem 1.2.** *Suppose that $V(|y|) \geq 0$ is bounded and $N \geq 5$. If $r^2 V(r)$ has either an isolated local maximum, or an isolated local minimum at $r_0 > 0$ with $V(r_0) > 0$, then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, (1.3) has a solution $u_k$ of the form*

$$u_k = W_{r_k,\mu_k}(y) + \omega_k,$$

*where $\omega_k \in H_s$, and as $k \to +\infty$, $\|\omega_k\|_{L^{2^*}(\mathbb{R}^N)} \to 0$, $r_k \in [r_0 - \delta, r_0 + \delta]$ and $\mu_k \in \big[L_0 k^{\frac{N-2}{N-4}}, L_1 k^{\frac{N-2}{N-4}}\big]$.*

We will use a reduction argument to prove Theorem 1.2. The reduction argument is a typical technique used in the study of perturbation problems. Problem (1.3) is not a perturbation problem. We use $k$, the number of bubbles of the solutions, as the parameter in order to carry out the reduction procedure. This technique has been used successfully to study some non-compact elliptic problems. See [11, 12, 13, 14, 15, 16]. Unlike the papers [14, 15, 16], where the reduction arguments were carried out in some weighted norm spaces, we take the advantage of the term $V(|y|)u$ in (1.3), so in this paper, we carry out the reduction argument in the standard Sobolev space as in [1, 10]. This will make the estimates a bit easier.

This paper is arranged as follows. In Section 2, we carry out the reduction. Theorem 1.2 is proved in Section 3. We put the energy expansion to the appendix.

**Acknowledgment.** The second author is supported by an Earmarked Grant from RGC of Hong Kong and Joint Overseas Grant from NSFC. The third author is partially supported by ARC. We thank Professor E. Hebey for his comments and interests in this work.

## 2. Finite-dimensional Reduction

In this section, we perform a finite-dimensional reduction.

Let

$$Z_{i,\mu,1} = \frac{\partial U_{x_i,\mu}}{\partial r}, \quad Z_{i,\mu,2} = \frac{\partial U_{x_i,\mu}}{\partial \mu}.$$

The inner product in $H_s$ is defined as follows:

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \big(DuDv + V(|y|)uv\big).$$

Let

$$E_{k,r,\mu} = \Big\{\phi : \ \phi \in H_s, \ \Big\langle \sum_{i=1}^k Z_{i,\mu,j}, \phi \Big\rangle = 0, \ j = 1, 2\Big\}.$$

Let $L_{k,r,\mu}$ be the bounded linear operator from $E_{k,r,\mu}$ to $E_{k,r,\mu}$, defined by the following relation

$$\langle L_{k,r,\mu} u, v \rangle = \int_{\mathbb{R}^N} \big(DuDv + V(|y|)uv - (2^* - 1)W_{r,\mu}^{2^*-2}uv\big), \quad u, v \in E_{k,r,\mu}. \tag{2.1}$$

**Lemma 2.1.** *There are* $\rho > 0$ *and* $k_0 > 0$*, such that for* $k \ge k_0$*,*

$$\|L_{k,r,\mu}\phi\| \ge \rho\|\phi\|, \quad \forall\, \phi \in E_{k,r,\mu}.$$

*Proof.* We argue by contradiction. Suppose that there are $k \to +\infty$, $r_k \in [r_0 - \delta, r_0 + \delta]$, $\mu_k \in \big[L_0 k^{\frac{N-2}{N-4}}, L_1 k^{\frac{N-2}{N-4}}\big]$, and $\phi_k \in E_{k,r_k,\mu_k}$, satisfying

$$\|\phi_k\| = \sqrt{k}, \quad \|L\phi_k\| = o(\sqrt{k}). \tag{2.2}$$

Let

$$\Omega_j = \big\{y: \ y = (y', y'') = \mathbb{R}^2 \times \mathbb{R}^{N-2}, \ \langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\rangle \ge \cos\frac{\pi}{k}\big\}.$$

Then, by (2.2),

$$\int_{\Omega_1} \big(|D\phi_k|^2 + V(|y|)\phi_k^2\big) = 1, \tag{2.3}$$

and

$$\int_{\Omega_1} \big(D\phi_k D\omega + V(|y|)\phi_k\omega - (2^* - 1)W_{r_k,\mu_k}^{2^*-2}\phi_k\omega\big) = o(1), \quad \forall \omega \in E_{k,r_k,\mu_k}. \tag{2.4}$$

Let $\tilde\phi_k(y) = \mu_k^{-\frac{N-2}{2}}\phi_k(\mu_k^{-1}y + x_1)$, $x_1 = (r, 0, 0, \cdots, 0)$. It follows from (2.3) that $D\tilde\phi_k$ is bounded in $L^2_{loc}(\mathbb{R}^N)$. So, we may assume that there is a $\phi \in D^{1,2}(\mathbb{R}^N)$, such that

$$D\tilde\phi_k \rightharpoonup D\phi, \quad \text{weakly in } L^2_{loc}(\mathbb{R}^N),$$

and

$$\tilde\phi_k \to \phi, \quad \text{strongly in } L^2_{loc}(\mathbb{R}^N).$$

It is easy to see that $\phi$ satisfies

$$-\Delta\phi - (2^* - 1)U_{0,1}^{2^*-2}\phi = 0, \quad \text{in } \mathbb{R}^N.$$

Moreover, from $\phi_k \in E_{k,r_k,\mu_k}$, we find that $\phi$ is even in $y_j$, $j = 2, \cdots, N$, and

$$\int_{\mathbb{R}^N} U_{0,1}^{2^*-2}\frac{\partial U_{0,1}}{\partial x_1}\phi = 0, \quad \int_{\mathbb{R}^N} U_{0,1}^{2^*-2}\frac{\partial U_{0,\mu}}{\partial \mu}\Big|_{\mu=1}\phi = 0.$$

So, we obtain $\phi = 0$. Thus, for any $R > 0$,

$$\int_{B_R(0)} |\tilde{\phi}_k|^2 = o(1).$$

As a result,

$$\int_{B_{\mu_k^{-1}R}(x_1)} W_{r_k,\mu_k}^{2^*-2}\phi_k^2 \le C\int_{B_R(0)} |\tilde{\phi}_k|^2 = o(1). \tag{2.5}$$

On the other hand, it is easy to see that $W_{r_k,\mu_k} = o(1)$ in $\Omega_1 \setminus B_{\mu_k^{-1}R}(x_1)$ for $R > 0$ large. Thus, in view of

$$V(y) \ge V_0 > 0, \quad y \in B_{r_0+2\delta}(0) \setminus B_{r_0-2\delta}(0),$$

we find

$$\begin{aligned} &\int_{\Omega_1\cap\left(B_{r_0+2\delta}(0)\setminus B_{r_0-2\delta}(0)\right)\setminus B_{\mu_k^{-1}R}(x_1)\right)} W_{r_k,\mu_k}^{2^*-2}\phi_k^2 \\ =&o(1)\int_{\Omega_1\cap\left(B_{r_0+2\delta}(0)\setminus B_{r_0-2\delta}(0)\right)\setminus B_{\mu_k^{-1}R}(x_1)\right)} \phi_k^2 = o(1)\int_{\Omega_1} V(|y|)\phi_k^2. \end{aligned} \tag{2.6}$$

Moreover, from

$$\begin{aligned} &\int_{\Omega_1\setminus(B_{r_0+2\delta}(0)\setminus B_{r_0-2\delta}(0))} W_{r_k,\mu_k}^{2^*} \\ \le& k^{2^*}\int_{\Omega_1\setminus(B_{r_0+2\delta}(0)\setminus B_{r_0-2\delta}(0))} U_{x_1,\mu_k}^{2^*} \le \frac{Ck^{2^*}}{\mu_k^N}, \end{aligned}$$

we obtain

$$
\begin{aligned}
&\int_{\Omega_1\setminus(B_{r_0+2\delta}(0)\setminus B_{r_0-2\delta}(0))} W_{r_k,\mu_k}^{2^*-2}\phi_k^2\\
\le&\Big(\int_{\Omega_1\setminus(B_{r_0+2\delta}(0)\setminus B_{r_0-2\delta}(0))} W_{r_k,\mu_k}^{2^*}\Big)^{\frac{2}{N}}\Big(\int_{\Omega_1}|\phi_k|^{2^*}\Big)^{\frac{2}{2^*}}\\
\le&\frac{Ck^{\frac{4}{N-2}}}{\mu_k^2}\frac{1}{k^{\frac{2}{2^*}}}\int_{\mathbb{R}^N}|D\phi_k|^2=\frac{Ck^{\frac{4}{N-2}+\frac{2}{N}}}{\mu_k^2}\int_{\Omega_1}|D\phi_k|^2\\
=&\frac{C}{k^{2+\frac{4}{N-4}-\frac{4}{N-2}-\frac{2}{N}}}\int_{\Omega_1}|D\phi_k|^2=o(1)\int_{\Omega_1}|D\phi_k|^2.
\end{aligned}\tag{2.7}
$$

Combining (2.5), (2.6) and (2.7), we are led to

$$
\begin{aligned}
o(1)=&\int_{\Omega_1}\big(|D\phi_k|^2+V(|y|)\phi_k^2-(2^*-1)W_{r_k,\mu_k}^{2^*-2}\phi_k^2\big)\\
=&(1+o(1))\int_{\Omega_1}\big(|D\phi_k|^2+V(|y|)\phi_k^2\big)+o(1).
\end{aligned}
$$

This is a contradiction to (2.3).

□

From Lemma 2.1, using the Fredholm alternative, we can prove the following result :

**Proposition 2.2.** *There exists $k_0>0$, such that for $k\ge k_0$, $L_{k,r,\mu}$ is an isomorphism in $E_{k,r,\mu}$.*

Define the projection $Q_{k,r,\mu}$ from $H_s$ to $E_{k,r,\mu}$ as follows:

$$
Q_{k,r,\mu}u=u-\sum_{j=1}^{2}c_j\sum_{i=1}^{k}Z_{i,\mu,j},\tag{2.8}
$$

where $c_1$ and $c_2$ are chosen such that $Q_{k,r,\mu}u\in E_{k,r,\mu}$.

Now, we consider

$$
Q_{k,r,\mu}\Big(-\Delta\big(W_{r,\mu}+\phi\big)+V(|y|)\big(W_{r,\mu}+\phi\big)-\big(W_r+\phi\big)^{2^*-1}\Big)=0,\quad \phi\in E_{k,r,\mu}.\tag{2.9}
$$

We have

**Proposition 2.3.** *There is an integer $k_0>0$, such that for each $k\ge k_0$, (2.9) has a unique solution $\phi=\phi(r,\mu)\in E_{k,r,\mu}$, satisfying*

$$\|\phi\| \le Ck\Big(\frac{\ln\mu}{\mu^{\min(\frac{N-2}{2},2)}} + \frac{1}{k^{\frac{1}{2^*}}}\big(\frac{k}{\mu}\big)^{\frac{N+1}{2}}\Big).$$

Rewrite (2.9) as

$$L_{k,r,\mu}\phi = N(\phi) + l_k, \text{ in } \mathbb{R}^N, \tag{2.10}$$

where $N(\phi) \in E_{k,r,\mu}$ and $l_k \in E_{k,r,\mu}$ are defined in the following relations respectively:

$$\big\langle N(\phi),\omega\big\rangle = \int_{\mathbb{R}^N}\Big(\big(W_{r,\mu}+\phi\big)^{2^*-1} - W_{r,\mu}^{2^*-1} - (2^*-1)W_{r,\mu}^{2^*-2}\phi\Big)\omega, \quad \omega \in E_{k,r,\mu},$$

and

$$\big\langle l_k,\omega\big\rangle = \int_{\mathbb{R}^N}\Big(W_{r,\mu}^{2^*-1} - \sum_{j=1}^k U_{x_j,\Lambda}^{2^*-1} - V(|y|)W_{r,\mu}\Big)\omega, \quad \omega \in E_{k,r,\mu}.$$

In order to use the contraction mapping theorem to prove that (2.10) is uniquely solvable, we need to estimate $N(\phi)$ and $l_k$.

**Lemma 2.4.** *If $N \ge 6$, then*

$$\big|\big\langle N(\phi),\omega\big\rangle\big| \le C\|\phi\|^{2^*-1}\|\omega\|.$$

*If $N = 5$,*

$$\big|\big\langle N(\phi),\omega\big\rangle\big| \le Ck^{\frac{1}{10}}\|\phi\|^2\|\omega\|.$$

*Proof.* We have

$$N^*(\phi) =: \big(W_{r,\mu}+\phi\big)^{2^*-1} - W_{r,\mu}^{2^*-1} - (2^*-1)W_{r,\mu}^{2^*-2}\phi = \begin{cases} C|\phi|^{2^*-1}, & N \ge 6; \\ CW_{r,\mu}^{\frac{1}{3}}\phi^2, & N = 5. \end{cases}$$

Thus, if $N \ge 6$,

$$\big|\big\langle N(\phi),\omega\big\rangle\big| \le C\int_{\mathbb{R}^N}|\phi|^{2^*-1}|\omega| \le C\|\phi\|^{2^*-1}\|\omega\|.$$

If $N = 5$,

$$\Big(\int_{\mathbb{R}^N}\big(W_{r,\mu}^{\frac{1}{3}}\phi^2\big)^{\frac{10}{7}}\Big)^{\frac{7}{10}} \le C\Big(\int_{\mathbb{R}^N}\big(W_{r,\mu}^{\frac{10}{3}}\big)^{\frac{1}{10}}\|\phi\|^2 \le Ck^{\frac{1}{10}}\|\phi\|^2.$$

□

Next, we estimate $l_k$.

**Lemma 2.5.** *If $N \geq 5$, then*

$$\|l_k\| \leq Ck\Big(\frac{\ln\mu}{\mu^{\min(\frac{N-2}{2},2)}} + \frac{1}{k^{\frac{1}{2^*}}}\big(\frac{k}{\mu}\big)^{\frac{N+1}{2}}\Big).$$

*Proof.* Write

$$l_k^* = \big(W_{r,\mu}^{2^*-1} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{2^*-1}\big) - V(|y|)W_{r,\mu} = J_1 - J_2.$$

Firstly, by symmetry

$$\big\langle J_2, \omega\big\rangle = k\big\langle V(y)U_{x_1,\mu}, \omega\big\rangle = kO\big(\int_{\mathbb{R}^N} V(y)U_{x_1,\mu}|\omega|\big). \tag{2.11}$$

We have

$$\int_{\mathbb{R}^N\setminus B_1(x_1)} V(y)U_{x_1,\mu}|\omega| \leq \frac{C}{\mu^{\frac{N-2}{2}}}\int_{\mathbb{R}^N\setminus B_1(x_1)} \frac{1}{|y-x_1|^{N-2}}V(y)|\omega| \leq \frac{C}{\mu^{\frac{N-2}{2}}}\|\omega\|.$$

On the other hand,

$$\int_{B_1(x_1)} V(y)U_{x_1,\mu}|\omega| \leq \Big(\int_{B_1(x_1)} U_{x_1,\mu}^{\frac{2N}{N+2}}\Big)^{\frac{N+2}{2N}}\|\omega\|.$$

But

$$\Big(\int_{B_1(x_1)} U_{x_1,\mu}^{\frac{2N}{N+2}}\Big)^{\frac{N+2}{2N}} = \begin{cases} O\big(\frac{1}{\mu^{\frac{3}{2}}}\big), & N=5;\\ O\big(\frac{\ln\mu}{\mu^2}\big), & N=6;\\ O\big(\frac{1}{\mu^2}\big), & N\geq 7.\end{cases}$$

So, we obtain

$$\big\langle J_2, \omega\big\rangle = kO\big(\frac{\ln\mu}{\mu^{\min(\frac{N-2}{2},2)}}\big)\|\omega\|. \tag{2.12}$$

Define

$$\Omega_j = \big\{y:\ y=(y',y'')\in\mathbb{R}^2\times\mathbb{R}^{N-2}, \big\langle\frac{y'}{|y'|}, \frac{x_j}{|x_j|}\big\rangle \geq \cos\frac{\pi}{k}\big\}.$$

By the symmetry,

$$\langle J_1, \omega \rangle = k \int_{\Omega_1} J_1 \omega. \tag{2.13}$$

We have

$$|J_1| \le C U_{x_1,\mu}^{\frac{4}{N-2}} \sum_{j=2}^{k} U_{x_j,\mu} + C \Big( \sum_{j=2}^{k} U_{x_1,\mu} \Big)^{2^*-1}, \quad y \in \Omega_1. \tag{2.14}$$

Note that

$$|y - x_j| \ge |y - x_1|, \quad \forall\, y \in \Omega_1.$$

We claim that

$$|y - x_j| \ge \frac{1}{2} |x_j - x_1|, \quad \forall\, y \in \Omega_1.$$

In fact, if $|y - x_1| \le \frac{1}{2}|x_j - x_1|$, then

$$|y - x_j| \ge |x_j - x_1| - |y - x_1| \ge \frac{1}{2}|x_j - x_1|.$$

If $|y - x_1| \ge \frac{1}{2}|x_j - x_1|$, then

$$|y - x_j| \ge |y - x_1| \ge \frac{1}{2}|x_j - x_1|.$$

So, we obtain

$$U_{x_1,\mu}^{\frac{4}{N-2}} U_{x_j,\mu} \le C \frac{\mu^{\frac{N+2}{2}}}{(1 + \mu|y - x_1|)^{\frac{N+3}{2}}} \frac{1}{(\mu|x_j - x_1|)^{\frac{N+1}{2}}}, \quad j > 1. \tag{2.15}$$

Thus

$$U_{x_1,\mu}^{\frac{4}{N-2}} \sum_{j=2}^{k} U_{x_j,\mu} \le C \frac{\mu^{\frac{N+2}{2}}}{(1 + \mu|y - x_1|)^{\frac{N+3}{2}}} \big(\frac{k}{\mu}\big)^{\frac{N+1}{2}}. \tag{2.16}$$

As a result,

$$
\begin{aligned}
&\int_{\Omega_1} U_{x_1,\mu}^{\frac{4}{N-2}} \sum_{j=2}^k U_{x_j,\mu}|\omega| \\
&\le C\big(\frac{k}{\mu}\big)^{\frac{N+1}{2}} \int_{\Omega_1} \frac{\mu^{\frac{N+2}{2}}}{(1+\mu|y-x_1|)^{\frac{N+3}{2}}}|\omega| \le C\big(\frac{k}{\mu}\big)^{\frac{N+1}{2}}\Big(\int_{\Omega_1}|\omega|^{2^*}\Big)^{\frac{1}{2^*}} \\
&\le C\frac{1}{k^{\frac{1}{2^*}}}\big(\frac{k}{\mu}\big)^{\frac{N+1}{2}}\|\omega\|.
\end{aligned}
\tag{2.17}
$$

Let $\tau>0$ be small. We have

$$
U_{x_j,\mu} \le \frac{C}{(\mu|x_j-x_1|)^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau}} \frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}.
$$

Thus

$$
\sum_{j=2}^k U_{x_j,\mu} \le C\big(\frac{k}{\mu}\big)^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau} \frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}},
$$

which, gives

$$
\Big(\sum_{j=2}^k U_{x_j,\mu}\Big)^{2^*-1} \le C\big(\frac{k}{\mu}\big)^{\frac{N+2}{2}-\tau} \frac{\mu^{\frac{N+2}{2}}}{(1+\mu|y-x_1|)^{\frac{N+2}{2}+\tau}}.
$$

As a result,

$$
\int_{\Omega_1}\Big(\sum_{j=2}^k U_{x_j,\mu}\Big)^{2^*-1}|\omega| \le C\frac{1}{k^{\frac{1}{2^*}}}\big(\frac{k}{\mu}\big)^{\frac{N+2}{2}-\tau}\|\omega\|.
$$

□

Now, we are ready to prove Proposition 2.3.

*Proof of Proposition 2.3.* Let

$$
S=\big\{\omega:\omega\in E_{k,r,\mu},\|\omega\|\le \frac{k}{k^{\frac{N-2}{N-4}}}\big\}.
$$

Then, (2.10) is equivalent to

$$
\phi = A(\phi) =: L_{k,r,\mu}^{-1}(N(\phi)+l_k).
$$

We will prove that $A$ is a contraction map from $S$ to $S$.

In fact, if $N \geq 6$,

$$\begin{aligned}\|\phi\| \leq & C\|N(\phi)\| + C\|l_k\| \\ \leq & C\|\phi\|^{2^*-1} + Ck\frac{1}{k^{\frac{N-2}{N-4}+\sigma}} \\ \leq & C\Big(\frac{k}{k^{\frac{N-2}{N-4}}}\Big)^{2^*-1} + Ck\frac{1}{k^{\frac{N-2}{N-4}+\sigma}} \leq \frac{k}{k^{\frac{N-2}{N-4}}}.\end{aligned} \tag{2.18}$$

If $N = 5$, then

$$\begin{aligned}\|\phi\| \leq & C\|N(\phi)\| + C\|l_k\| \\ \leq & Ck^{\frac{1}{10}}\|\phi\|^2 + Ck\frac{1}{k^{3+\sigma}} \leq Ck^{\frac{1}{10}}\frac{1}{k^3} + \frac{C}{k^{2+\sigma}} \leq \frac{1}{k^2}.\end{aligned} \tag{2.19}$$

Thus, $A$ maps $S$ to $S$.

On the other hand,

$$\|A(\phi_1) - A_(\phi_2)\| \leq C\|N(\phi_1) - N(\phi_2)\|.$$

If $N \geq 6$, then

$$|(N^*(t))'| \leq C|t|^{2^*-2}.$$

As a result,

$$\begin{aligned}\int_{\mathbb{R}^N} |N(\phi_1) - N(\phi_2)||\omega| \leq & C\int_{\mathbb{R}^N}\big(|\phi_1|^{2^*-2} + |\phi_2|^{2^*-2}\big)|\phi_1 - \phi_2||\omega| \\ \leq & C\big(\|\phi_1\|^{2^*-2} + \|\phi_2\|^{2^*-2}\big)\|\phi_1 - \phi_2\|\|\omega\|\end{aligned}$$

So we have

$$\|N(\phi_1) - N(\phi_2)\| \leq C\big(\|\phi_1\|^{2^*-2} + \|\phi_2\|^{2^*-2}\big)\|\phi_1 - \phi_2\| \leq \frac{1}{2}\|\phi_1 - \phi_2\|.$$

If $N = 5$,

$$|(N^*(t))'| \leq CW_{r,\mu}^{\frac{1}{3}}|t|^2.$$

So,

$$\|N(\phi_1) - N(\phi_2)\| \leq Ck^{\frac{1}{10}}\|\phi_1 - \phi_2\|^2 \leq \frac{1}{2}\|\phi_1 - \phi_2\|.$$

Thus, $A$ is a contraction map.

It follows from the contraction mapping theorem that there is a unique $\phi \in S$, such that

$$\phi = A(\phi).$$

Moreover,

$$\|\phi\| \le C\|l_k\| \le Ck\Big(\frac{\ln\mu}{\mu^{\min(\frac{N-2}{2},2)}} + \frac{1}{k^{\frac{1}{2^*}}}\big(\frac{k}{\mu}\big)^{\frac{N+1}{2}}\Big).$$

□

## 3. Proof of the main result

Let

$$F(r,\mu) = I\big(W_{r,\mu} + \phi\big),$$

where $r = |x_1|$, $\phi$ is the function obtained in Proposition 2.3, and

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\big(|Du|^2 + V(|y|)u^2\big) - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}.$$

**Proposition 3.1.** *We have*

$$\begin{aligned} F(r,\mu) &= I(W_{r,\mu}) + O\Big(\frac{k}{\mu^{2+\sigma}}\Big) \\ &= k\Big(A + \frac{B_1V(r)}{\mu^2} - \sum_{i=2}^{k}\frac{B_2}{\mu^{N-2}|x_1 - x_j|^{N-2}} + O\Big(\frac{1}{\mu^{2+\sigma}}\Big)\Big), \end{aligned}$$

*where $\sigma > 0$ is a fixed constant, $B_i > 0$, $i = 1, 2$, is some constant.*

*Proof.* Since

$$\big\langle I'\big(W_{r,\mu} + \phi\big), \phi\big\rangle = 0,$$

there is $t \in (0,1)$ such that

$$
\begin{aligned}
F(r,\mu) &= I(W_{r,\mu}) + \frac{1}{2} D^2 I\big(W_{r,\mu}+t\phi\big)(\phi,\phi)\\
=& I(W_{r,\mu}) + \frac{1}{2}\int_{\mathbb{R}^N} \big(|D\phi|^2 + V(|y|)\phi^2 - (2^*-1)\big(W_{r,\mu}+t\phi\big)^{2^*-2}\phi^2\big)\\
=& I(W_{r,\mu}) + O\Big(\|\phi\|^2 + \|\phi\|^{2^*} + \Big(\int_{\mathbb{R}^N} W_{r,\mu}^{2^*}\Big)^{\frac{2}{N}}\|\phi\|^2\Big)\\
=& I(W_{r,\mu}) + O\big(k^{\frac{2}{N}}\|\phi\|^2\big)\\
=& I(W_{r,\mu}) + k^{2+\frac{2}{N}} O\Big(\frac{\ln^2\mu}{\mu^{\min(N-2,4)}} + \frac{1}{k^{\frac{N-2}{N}}}\big(\frac{k}{\mu}\big)^{N+1}\Big).
\end{aligned}
$$

Since

$$
\big(\frac{k}{\mu}\big)^{N-2} \sim \frac{1}{\mu^2},
$$

we find

$$
k^{2+\frac{2}{N}}\frac{1}{k^{\frac{N-2}{N}}}\big(\frac{k}{\mu}\big)^{N+1} = kO\big(k^{\frac{4}{N}}\frac{1}{\mu^{\frac{6}{N-2}}}\frac{1}{\mu^2}\big) = kO\big(\frac{1}{\mu^{2+\sigma}}\big)
$$

It is also easy to check that

$$
k^{2+\frac{2}{N}}\frac{\ln^2\mu}{\mu^{\min(N-2,4)}} = kO\big(\frac{1}{\mu^{2+\sigma}}\big).
$$

So, the result follows.

□

**Proposition 3.2.** *We have*

$$
\frac{\partial F(r,\mu)}{\partial \mu} = k\Big(-\frac{2B_1 V(r)}{\mu^3} + \sum_{i=2}^{k}\frac{B_2(N-2)}{\mu^{N-1}|x_1-x_j|^{N-2}} + O\Big(\frac{1}{\mu^{3+\sigma}}\Big)\Big),
$$

*where $\sigma>0$ is a fixed constant.*

*Proof.* We have

$$
\begin{aligned}
\frac{\partial F(r,\mu)}{\partial \mu} &= \big\langle I'(W_{r,\mu}+\phi), \frac{\partial W_{r,\mu}}{\partial\mu} + \frac{\partial\phi}{\partial\mu}\big\rangle\\
=&\big\langle I'(W_{r,\mu}+\phi), \frac{\partial W_{r,\mu}}{\partial\mu}\big\rangle + \sum_{l=1}^{2}\sum_{i=1}^{k} c_l\big\langle Z_{i,\mu,l}, \frac{\partial\phi}{\partial\mu}\big\rangle.
\end{aligned}
\tag{3.20}
$$

Now

$$
\begin{aligned}
&\Big\langle I'(W_{r,\mu}+\phi), \frac{\partial W_{r,\mu}}{\partial \mu}\Big\rangle \\
=&\Big\langle I'(W_{r,\mu}), \frac{\partial W_{r,\mu}}{\partial \mu}\Big\rangle + \int_{\mathbb{R}^N} \Big(D\frac{\partial W_{r,\mu}}{\partial \mu}D\phi + V(|y|)\frac{\partial W_{r,\mu}}{\partial \mu}\phi\Big) \\
&-\int_{\mathbb{R}^N} (W_{r,\mu}+\phi)^{2^*-1}\frac{\partial W_{r,\mu}}{\partial \mu} + \int_{\mathbb{R}^N} W_{r,\mu}^{2^*-1}\frac{\partial W_{r,\mu}}{\partial \mu} \\
=&\Big\langle I'(W_{r,\mu}), \frac{\partial W_{r,\mu}}{\partial \mu}\Big\rangle - \int_{\mathbb{R}^N} (W_{r,\mu}+\phi)^{2^*-1}\frac{\partial W_{r,\mu}}{\partial \mu} + \int_{\mathbb{R}^N} W_{r,\mu}^{2^*-1}\frac{\partial W_{r,\mu}}{\partial \mu}
\end{aligned}
\tag{3.21}
$$

since $\phi \in E_{k,r,\mu}$.

On the other hand,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} (W_{r,\mu}+\phi)^{2^*-1}\frac{\partial W_{r,\mu}}{\partial \mu} - \int_{\mathbb{R}^N} W_{r,\mu}^{2^*-1}\frac{\partial W_{r,\mu}}{\partial \mu} \\
=&(2^*-1)\int_{\mathbb{R}^N} W_{r,\mu}^{2^*-2}\frac{\partial W_{r,\mu}}{\partial \mu}\phi + O\Big(\int_{\mathbb{R}^N} |\phi|^{2^*}\Big).
\end{aligned}
\tag{3.22}
$$

Moreover, from $\phi \in E_{k,r,\mu}$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Big((2^*-1)U_{x_j,\mu}^{2^*-2}\frac{\partial U_{x_j,\mu}}{\partial \mu} + V(|y|)\frac{\partial U_{x_j,\mu}}{\partial \mu}\Big)\phi \\
=&\int_{\mathbb{R}^N} \Big(D\frac{\partial U_{x_j,\mu}}{\partial \mu}D\phi + V(|y|)\frac{\partial U_{x_j,\mu}}{\partial \mu}\phi\Big) = 0.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} W_{r,\mu}^{2^*-2}\frac{\partial W_{r,\mu}}{\partial \mu}\phi \\
=&\int_{\mathbb{R}^N} \Big(W_{r,\mu}^{2^*-2}\frac{\partial W_{r,\mu}}{\partial \mu} - \sum_{j=1}^k U_{x_j,\mu}^{2^*-2}\frac{\partial U_{x_j,\mu}}{\partial \mu} - \frac{1}{2^*-1}\sum_{j=1}^k V(|y|)\frac{\partial U_{x_j,\mu}}{\partial \mu}\Big)\phi.
\end{aligned}
\tag{3.23}
$$

But

$$
\Big|\int_{\mathbb{R}^N} V(|y|)\frac{\partial U_{x_j,\mu}}{\partial \mu}\phi\Big| \le \frac{C}{\mu}\int_{\mathbb{R}^N} V(|y|)U_{x_j,\mu}|\phi| \le \frac{C}{\mu^{3+\sigma}}.
\tag{3.24}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\big(W_{r,\mu}^{2^*-2}\frac{\partial W_{r,\mu}}{\partial\mu}-\sum_{j=1}^{k}U_{x_j,\mu}^{2^*-2}\frac{\partial U_{x_j,\mu}}{\partial\mu}\big)\phi\\
=&k\int_{\Omega_1}\big(W_{r,\mu}^{2^*-2}\frac{\partial W_{r,\mu}}{\partial\mu}-\sum_{j=1}^{k}U_{x_j,\mu}^{2^*-2}\frac{\partial U_{x_j,\mu}}{\partial\mu}\big)\phi\\
\le&\frac{Ck}{\mu}\int_{\Omega_1}\Big(U_{x_1,\mu}^{2^*-2}\sum_{j=2}^{k}U_{x_j,\mu}+\sum_{j=2}^{k}U_{x_j,\mu}^{2^*-1}\Big)|\phi|\le\frac{Ck}{\mu^{3+\sigma}}.
\end{aligned}\tag{3.25}
$$

Combining (3.21)–(3.25), we obtain

$$
\Big\langle I'(W_{r,\mu}+\phi),\frac{\partial W_{r,\mu}}{\partial\mu}\Big\rangle=\Big\langle I'(W_{r,\mu}),\frac{\partial W_{r,\mu}}{\partial\mu}\Big\rangle+O\Big(\frac{k}{\mu^{3+\sigma}}\Big).\tag{3.26}
$$

To estimate $c_1$ and $c_2$, we use

$$
L_{k,r,\mu}\phi-l_k-N(\phi)=\sum_{l=1}^{2}\sum_{i=1}^{k}c_lZ_{i,\mu,l}.
$$

So,

$$
c_l\Big\langle\sum_{i=1}^{k}Z_{i,\mu,l},Z_{1,\mu,l}\Big\rangle=\Big\langle L_{k,r,\mu}\phi-l_k-N(\phi),Z_{1,\mu,l}\Big\rangle.\tag{3.27}
$$

On the other hand, similar to the estimate of (3.22), we can deduce

$$
\begin{aligned}
&\Big\langle L_{k,r,\mu}\phi,Z_{1,\mu,l}\Big\rangle=\Big\langle L_{k,r,\mu}Z_{1,\mu,l},\phi\Big\rangle\\
=&-(2^*-1)\int_{\mathbb{R}^N}W_{r,\mu}^{2^*-2}Z_{1,\mu,l}\phi=\begin{cases}O\Big(\frac{k}{\mu^{3+\sigma}}\Big), & l=2,\\ O\Big(\frac{k}{\mu^{1+\sigma}}\Big), & l=1,\end{cases}
\end{aligned}
$$

which, together with (3.27), gives

$$
c_1=\frac{1}{\mu}O\big(\|l_k\|+\|N(\phi)\|+\big)+O\Big(\frac{k}{\mu^{3+\sigma}}\Big),\quad c_2=\mu O\big(\|l_k\|+\|N(\phi)\|\big)+O\Big(\frac{k}{\mu^{1+\sigma}}\Big).
$$

But

$$
\Big\langle Z_{i,\mu,l},\frac{\partial\phi}{\partial\mu}\Big\rangle=-\Big\langle\frac{\partial Z_{i,\mu,l}}{\partial\mu},\phi\Big\rangle
$$

Thus,

$$\Big|\sum_{i=1}^{k} c_l\Big\langle Z_{i,l}, \frac{\partial \phi}{\partial \mu}\Big\rangle\Big| \le \frac{1}{\mu}\big(\|l_k\| + \|N(\phi)\|\big)\|\phi\| + O\Big(\frac{k}{\mu^{3+\sigma}}\Big) \le \frac{Ck}{\mu^{3+\sigma}}. \tag{3.28}$$

Combining (3.20), (3.26) and (3.28), we have proved

$$\frac{\partial F(r,\mu)}{\partial \mu} = \frac{\partial I(W_{r,\mu})}{\partial \mu} + O\Big(\frac{k}{\mu^{3+\sigma}}\Big),$$

and the result follows from Proposition A.2. □

Since

$$|x_j - x_1| = 2|x_1| \sin\frac{(j-1)\pi}{k}, \quad j = 2, \dots, k,$$

we have

$$\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{N-2}} = \frac{1}{(2|x_1|)^{N-2}} \sum_{j=2}^{k} \frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}}$$

$$= \begin{cases} \frac{1}{(2|x_1|)^{N-2}} \sum_{j=2}^{\frac{k}{2}} \frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}} + \frac{1}{(2|x_1|)^{N-2}}, & \text{if } k \text{ is even;} \\ \frac{1}{(2|x_1|)^{N-2}} \sum_{j=2}^{[\frac{k}{2}]} \frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}}, & \text{if } k \text{ is old.} \end{cases}$$

But

$$0 < c' \le \frac{\sin\frac{(j-1)\pi}{k}}{\frac{(j-1)\pi}{k}} \le c'', \quad j = 2, \cdots, [\frac{k}{2}].$$

So, there is a constant $B_4 > 0$, such that

$$\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{N-2}} = \frac{B_4 k^{N-2}}{|x_1|^{N-2}} + O\Big(\frac{k}{|x_1|^{N-2}}\Big).$$

Thus, we obtain

$$F(r,\mu) = k\Big(A + \frac{B_1 V(r)}{\mu^2} - \frac{B_4 k^{N-2}}{\mu^{N-2} r^{N-2}} + O\Big(\frac{1}{\mu^{2+\sigma}}\Big)\Big), \tag{3.29}$$

and

$$\frac{\partial F(r,\mu)}{\partial \mu} = k\Big(-\frac{2B_1V(r)}{\mu^3}+\frac{B_4(N-2)k^{N-2}}{\mu^{N-1}r^{N-2}}+O\Big(\frac{1}{\mu^{3+\sigma}}\Big)\Big). \tag{3.30}$$

For each fixed $r\in\big[r_0-\delta, r_0+\delta\big]$, let $\Lambda_0(r)$ be the solution of

$$-V(r)\frac{2B_1}{\Lambda^3}+\frac{B_4(N-2)}{\Lambda^{N-1}r^{N-2}}=0.$$

Then

$$\Lambda_0(r)=\Big(\frac{B_4(N-2)}{2B_1V(r)r^{N-2}}\Big)^{\frac{1}{N-4}}.$$

Note that $\Lambda_0(r)$ is the unique maximum point of the function

$$V(r)\frac{B_1}{\Lambda^2}-\frac{B_4}{\Lambda^{N-2}r^{N-2}}.$$

*Proof of Theorem 1.2 if $r_0$ is a maximum of $r^2V(r)$.* Consider

$$\max_{(r,\mu)\in D} F(r,\mu), \tag{3.31}$$

where

$$D=\Big\{(r,\Lambda): r\in[r_0-\delta,r_0+\delta],\ \ \mu=\Lambda k^{\frac{N-2}{N-4}},\ \Lambda\in[\Lambda_0(r)-\frac{1}{\mu^{\frac{3}{2}\theta}},\Lambda_0(r)+\frac{1}{\mu^{\frac{3}{2}\theta}}]\Big\},$$

where $0<\theta<<\sigma$ is a small constant, and $\sigma>0$ is the constant in (3.30). Let $(\bar r_k,\bar\mu_k)\in D$ be a solution of (3.31).

If $\theta>0$ is small enough, then it follows from (3.30) that

$$\frac{\partial F(r,\mu)}{\partial\mu}>0\ \ (\text{or }\ <0)$$

if $\bar\mu_k=k^{\frac{N-2}{N-4}}\big(\Lambda_0(r)-\frac{1}{\mu^{\frac{3}{2}\theta}}\big)$, (or $\bar\mu_k=k^{\frac{N-2}{N-4}}\big(\Lambda_0(r)+\frac{1}{\mu^{\frac{3}{2}\theta}}\big)$). So $\bar\mu_k\neq k^{\frac{N-2}{N-4}}\big(\Lambda_0(r)\pm\frac{1}{\mu^{\frac{3}{2}\theta}}\big)$.

On the other hand, for any $(r,\mu)\in D$, we have

$$
\begin{aligned}
&\frac{B_1V(r)}{\mu^2}-\frac{B_4k^{N-2}}{\mu^{N-2}r^{N-2}}=\Big(\frac{B_1V(r)}{\Lambda^2}-\frac{B_4}{\Lambda^{N-2}r^{N-2}}\Big)\frac{1}{k^{\frac{2(N-2)}{N-4}}}\\
&=\Big(\frac{B_1V(r)}{\Lambda_0^2(r)}-\frac{B_4}{\Lambda_0^{N-2}(r)r^{N-2}}+O\big(|\Lambda-\Lambda_0(r)|^2\big)\Big)\frac{1}{k^{\frac{2(N-2)}{N-4}}}\\
&=\Big(\frac{N-4}{N-2}\frac{B_1V(r)}{\Lambda_0^2(r)}+O\big(\frac{1}{\mu^{3\theta}}\big)\Big)\frac{1}{k^{\frac{2(N-2)}{N-4}}}\\
&=\Big(B'(r^2V(r))^{\frac{N-2}{2(N-4)}}+O\big(\frac{1}{\mu^{3\theta}}\big)\Big)\frac{1}{k^{\frac{2(N-2)}{N-4}}},
\end{aligned}
\tag{3.32}
$$

where $B'>0$ is a constant. Since $r^2V(r)$ has a maximum at $r_0$, from (3.29), we see that $\bar r_k\ne r_0\pm\delta$ for the maximum point $(r_k,\bar\mu_k)\in D$. So, $(r_k,\bar\mu_k)$ is an interior point of $D$, and thus a critical point of $F(r,\mu)$.

□

It remains to study the case that $r_0$ is a local minimum point of $r^2V(r)$. Define

$$
\bar F(r,\mu)=-F(r,\mu),\quad (r,\mu)\in D.
$$

Let

$$
\alpha_2=k(-A+\eta),\quad ,\alpha_1=k\Big(-A-B'(r_0^2V(r_0))^{\frac{N-2}{2(N-4)}}(1-\eta)\frac{1}{k^{\frac{2(N-2)}{N-4}}}\Big),
$$

where $\eta>0$ is a small constant, and $B'>0$ is the constant in (3.32).

Let

$$
\bar F^{\alpha}=\big\{(r,\mu)\in D,\bar F(r,\mu)\le\alpha\big\}.
$$

Consider

$$
\begin{cases}
\frac{dr}{dt}=-D_r\bar F, & t>0;\\
\frac{d\mu}{dt}=-D_\mu\bar F, & t>0;\\
(r,\mu)\in F^{\alpha_2}.
\end{cases}
$$

Then

**Proposition 3.3.** *The flow $(r(t),\mu(t))$ does not leave $D$ before it reaches $F^{\alpha_1}$.*

*Proof.* If $\mu = \Big(\Lambda_0 + \frac{1}{\mu^{\frac{3}{2}\theta}}\Big)\frac{1}{k^{\frac{N-2}{N-4}}}$, we obtain from (3.30) that

$$\frac{\partial \bar F(r,\mu)}{\partial \mu} = k\Big(\frac{c'}{\mu^{\frac{3}{2}\theta}} + O\Big(\frac{1}{\mu^{\sigma}}\Big)\Big)\frac{1}{k^{\frac{3(N-2)}{N-4}}} > 0.$$

So, the flow does not leave $D$.

Similarly, if $\mu = \Big(\Lambda_0 - \frac{1}{\mu^{\frac{3}{2}\theta}}\Big)\frac{1}{k^{\frac{N-2}{N-4}}}$, then we obtain from (3.30) that

$$\frac{\partial \bar F(r,\mu)}{\partial \mu} = k\Big(-\frac{c'}{\mu^{\frac{3}{2}\theta}} + O\Big(\frac{1}{\mu^{\sigma}}\Big)\Big)\frac{1}{k^{\frac{3(N-2)}{N-4}}} < 0.$$

So, the flow does not leave $D$.

Suppose now $|r-r_0| = \delta$. Using (3.29) and (3.32), we obtain

$$\begin{aligned}
&\bar F(r,\mu)\\
=&k\Big(-A - \Big(B'\big((r_0\pm\delta)^2 V(r_0\pm\delta)\big)^{\frac{N-2}{2(N-4)}} + O\Big(\frac{1}{\mu^{3\theta}}\Big)\Big)\frac{1}{k^{\frac{2(N-2)}{N-4}}}\Big)\\
<&k\Big(-A - B'(r_0^2 V(r_0))^{\frac{N-2}{2(N-4)}}(1-\eta)\frac{1}{k^{\frac{2(N-2)}{N-4}}}\Big)\Big) = \alpha_1,
\end{aligned} \tag{3.33}$$

if $\eta > 0$ is small.

□

*Proof of Theorem 1.2 if $r_0$ is a minimum of $r^2V(r)$.* We will prove that $\bar F$, and thus $F$, has a critical point in $D$.

Define

$$\Gamma = \Big\{h : h(r,\mu) = (h_1(r,\mu), h_2(r,\mu)) \in D, (r,\mu)\in D \\ h(r,\mu) = (r,\mu),\ \text{if } |r-r_0| = \delta\Big\}.$$

Let

$$c = \inf_{h\in\Gamma}\max_{(r,\mu)\in D} \bar F(h(r,\mu)).$$

We claim that $c$ is a critical value of $\bar F$. To prove this, we need to prove

(i) $\alpha_1 < c < \alpha_2$;

(ii) $\sup_{|r-r_0|=\delta} \bar F(h(r,\mu)) < \alpha_1,\ \forall\, h\in\Gamma$.

To prove (ii), let $h \in \Gamma$. Then for any $\bar r$ with $|\bar r - r_0| = \delta$, we have $h(\bar r, \mu) = (\bar r, \mu)$. Thus, by (3.33),

$$\bar F(h(r,\mu)) = \bar F(r,\mu) < \alpha_1.$$

Now we prove (i). It is easy to see that

$$c < \alpha_2.$$

For any $h = (h_1, h_2) \in \Gamma$. Then $h_1(r,\mu) = r$, if $|r - r_0| = \delta$. Define

$$\tilde h_1(r) = h_1(r, \Lambda_0(r) k^{\frac{N-2}{N-4}}).$$

Then $\tilde h_1(r) = r$, if $|r - r_0| = \delta$. So, there is a $\bar r \in (r_0 - \delta, r_0 + \delta)$, such that

$$\tilde h_1(\bar r) = r_0.$$

Let $\bar\mu = h_2(\bar r, \Lambda_0(\bar r) k^{\frac{N-2}{N-4}})$. Then from (3.29) and(3.32),

$$\begin{aligned}
&\max_{(r,\mu)\in D} \bar F(h(r,\mu)) \ge \bar F(h(\bar r, \Lambda_0) k^{\frac{N-2}{N-4}}) = \bar F(r_0, \bar\mu)\\
=&k\Big(-A - \Big(B'(r_0^2 V(r_0))^{\frac{N-2}{2(N-4)}} + O\big(\frac{1}{k^{\frac{3\theta(N-2)}{N-4}}}\big)\Big)\frac{1}{k^{\frac{2(N-2)}{N-4}}} > \alpha_1.
\end{aligned}$$

□

## Appendix A. Energy Expansion

In the appendix, we always assume that

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\Big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and $r \in [r_0 - \delta, r_0 + \delta]$.

Let recall that

$$I(u) = \frac12 \int_{\mathbb{R}^N} \big(|Du|^2 + V(|y|)u^2\big) - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*},$$

$$U_{x_j,\mu}(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \frac{\mu^{\frac{N-2}{2}}}{(1+\mu^2|y-x_j|^2)^{\frac{N-2}{2}}},$$

and

$$W_{r,\mu}(y) = \sum_{j=1}^{k} U_{x_j,\mu}.$$

In this section, we will calculate $I(W_{r,\mu})$.

**Proposition A.1.** *If $N \geq 5$,*

$$I(W_{r,\mu}) = k\Big(A + \frac{B_1 V(r)}{\mu^2} - \sum_{i=2}^{k} \frac{B_2}{\mu^{N-2}|x_1 - x_i|^{N-2}} + O\Big(\frac{1}{\mu^{2+\sigma}}\Big)\Big),$$

*where $B_i$, $i = 1, 2$, is some positive constant, $A > 0$ is a constant, and $r = |x_1|$.*

*Proof.* By using the symmetry, we have

$$\int_{\mathbb{R}^N} |DW_{r,\mu}|^2 = \sum_{j=1}^{k}\sum_{i=1}^{k}\int_{\mathbb{R}^N} U_{x_j,\mu}^{2^*-1} U_{x_i,\mu}$$

$$=k\Big(\int_{\mathbb{R}^N} U_{0,1}^{2^*} + \sum_{i=2}^{k}\int_{\mathbb{R}^N} U_{x_1,\mu}^{2^*-1}U_{x_i,\mu}\Big)$$

$$=k\Big(\int_{\mathbb{R}^N} U_{0,1}^{2^*} + \sum_{i=2}^{k}\frac{B_0}{\mu^{N-2}|x_1 - x_j|^{N-2}} + O\Big(\sum_{i=2}^{k}\frac{1}{(\mu|x_1 - x_j|)^{N-2+\sigma}}\Big)\Big).$$

Let

$$\Omega_j = \big\{y :\ y = (y', y'') = \mathbb{R}^2 \times \mathbb{R}^{N-2},\ \big\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\big\rangle \geq \cos\frac{\pi}{k}\big\}.$$

Then,

$$\int_{\mathbb{R}^N} |W_{r,\mu}|^{2^*} = k\int_{\Omega_1} |W_{r,\mu}|^{2^*}$$

$$=k\Big(\int_{\Omega_1} U_{x_1,\Lambda}^{2^*} + 2^*\int_{\Omega_1}\sum_{i=2}^{k} U_{x_1,\mu}^{2^*-1}U_{x_i,\mu} + O\Big(\int_{\Omega_1} U_{x_1,\mu}^{2^*/2}\big(\sum_{i=2}^{k} U_{x_i,\mu}\big)^{2^*/2}\Big)\Big).$$

Note that for $y \in \Omega_1$, $|y - x_i| \geq |y - x_1|$. So, for any $\alpha > 0$ small,

$$\sum_{i=2}^{k} U_{x_i,\mu} \leq \frac{\mu^{\frac{N-2}{2}}}{(1 + \mu|y - x_1|)^{\alpha}}\sum_{i=2}^{k}\frac{1}{(\mu|x_i - x_1|)^{N-2-\alpha}}.$$

Thus,

$$\int_{\Omega_1} U_{x_1,\Lambda}^{2^*/2}\Big(\sum_{i=2}^k U_{x_i,\Lambda}\Big)^{2^*/2} = O\Big(\big(\frac{k}{\mu}\big)^{N-\frac{\alpha N}{N-2}}\Big).$$

On the other hand, it is easy to show

$$\int_{\Omega_1} \sum_{i=2}^k U_{x_1,\mu}^{2^*-1} U_{x_i,\mu} = \sum_{i=2}^k \frac{B_0}{\mu^{N-2}|x_1-x_j|^{N-2}} + O\Big(\big(\frac{k}{\mu}\big)^{N-2+\sigma}\Big).$$

Thus, we have proved

$$\int_{\mathbb{R}^N} |W_{r,\mu}|^{2^*} = k\Big(\int_{\mathbb{R}^N} |U_{0,1}|^{2^*} + 2^* \sum_{i=2}^k \frac{B_0}{\mu^{N-2}|x_1-x_i|^{N-2}} + O\Big(\frac{1}{\mu^{2+\sigma}}\Big)\Big).$$

Finally,

$$\begin{aligned}&\int_{\mathbb{R}^N} V(y)|W_{r,\mu}|^2\\ =&k\Big(\int_{\mathbb{R}^N} V(|y|)U_{x_1,\mu}^2 + O\Big(\int_{\mathbb{R}^N} \sum_{i=2}^k U_{x_1,\mu}U_{x_i,\mu}\Big)\Big).\end{aligned}$$

But

$$\int_{\mathbb{R}^N} U_{x_1,\Lambda}U_{x_i,\mu} = O\Big(\frac{1}{\mu^{N-2}|x_j-x_1|^{N-3}}\Big),$$

Moreover,

$$\int_{\mathbb{R}^N} V(|y|)U_{x_1,\mu}^2 = V(r)\frac{1}{\mu^2}\int_{\mathbb{R}^N} U^2 + O\big(\frac{1}{\mu^{2+\sigma}}\big).$$

So,

$$\int_{\mathbb{R}^N} V(y)|W_{r,\mu}|^2 = k\big(V(r)\frac{1}{\mu^2}\int_{\mathbb{R}^N} U^2 + O\big(\frac{1}{\mu^{2+\sigma}} + \frac{1}{k^{N-3}\mu^{N-2}}\big)\big)$$

□

We also need to calculate $\frac{\partial I(W_{r,\mu})}{\partial \mu}$.

**Proposition A.2.** *We have*

$$\frac{\partial I(W_{r,\mu})}{\partial \mu} = k\Big(-\frac{2B_1V(r)}{\mu^3} + \sum_{i=2}^k \frac{B_2(N-2)}{\mu^{N-1}|x_1-x_i|^{N-2}} + O\Big(\frac{1}{\mu^{3+\sigma}}\Big)\Big),$$

*where $B_i$, $i = 1, 2$, is same positive constant in Proposition A.1*

*Proof.* The proof of this proposition is similar to the proof of Proposition A.1. So we just sketch it.

We have

$$\begin{aligned}\frac{\partial I(W_{r,\mu})}{\partial \mu} =& k\Big((2^*-1)\sum_{i=2}^{k}\int_{\mathbb{R}^N} U_{x_1,\mu}^{2^*-2}\frac{\partial U_{x_1,\mu}}{\partial \mu}U_{x_i,\mu} \\ &+\int_{\Omega_1} V(|y|)W_{r,\mu}\frac{\partial W_{r,\mu}}{\partial \mu}-\int_{\Omega_1} W_{r,\mu}^{2^*-1}\frac{\partial W_{r,\mu}}{\partial \mu}\Big).\end{aligned}$$

It is easy to check that for $y \in \Omega_1$,

$$\Big|\frac{\partial}{\partial \mu}\Big(W_{r,\mu}^{2^*}-U_{x_1,\mu}^{2^*}-2^*U_{x_1,\mu}^{2^*-1}\sum_{i=2}^{k}U_{x_i,\mu}\Big)\Big| \le \frac{C}{\mu}U_{x_1,\mu}^{2^*/2}\big(\sum_{i=2}^{k}U_{x_i,\mu}\big)^{2^*/2}.$$

Thus,

$$\frac{\partial}{\partial \mu}W_{r,\mu}^{2^*}=\frac{\partial}{\partial \mu}U_{x_1,\mu}^{2^*}+2^*\frac{\partial}{\partial \mu}\big(U_{x_1,\mu}^{2^*-1}\sum_{i=2}^{k}U_{x_i,\mu}\big)+\frac{1}{\mu}O\Big(U_{x_1,\mu}^{2^*/2}\big(\sum_{i=2}^{k}U_{x_i,\mu}\big)^{2^*/2}\Big).$$

As a result,

$$\begin{aligned}&2^*\int_{\Omega_1} W_{r,\mu}^{2^*-1}\frac{\partial W_{r,\mu}}{\partial \mu}\\ =&\int_{\Omega_1}\frac{\partial}{\partial \Lambda}U_{x_1,\mu}^{2^*}+2^*\int_{\Omega_1}\frac{\partial}{\partial \mu}\big(U_{x_1,\mu}^{2^*-1}\sum_{i=2}^{k}U_{x_i,\mu}\big)+O\Big(\frac{1}{\mu^{3+\sigma}}\Big)\\ =&2^*\frac{\partial}{\partial \mu}\int_{\Omega_1}\big(U_{x_1,\mu}^{2^*-1}\sum_{i=2}^{k}U_{x_i,\mu}\big)+O\Big(\frac{1}{\mu^{3+\sigma}}\Big).\end{aligned}$$

Similarly,

$$\int_{\Omega_1} V(|y|)W_{r,\mu}\frac{\partial W_{r,\mu}}{\partial \mu}=V(r)\frac{1}{2}\int_{\Omega_1}\frac{\partial U_{x_1,\mu}^2}{\partial \mu}+O\Big(\frac{1}{\mu^{3+\sigma}}\Big).$$

The proof is thus completed.

□

## References


[1] A. Bahri, Critical points at infinity in some variational problems, Research Notes in Mathematics. Vol 82, Longman-Pitman, 1989

[2] V. Benci and G. Cerami, Existence of positive solutions of the equation $-\Delta u+a(x)u = u^{(N+2)/(N-2)}$ in $R^N$, *J. Funct. Anal.* 88(1990), no.1, 90-117.

[3] C. Bandle and J. Wei, Non-radial clustered spike solutions for semilinear elliptic problems on $S^n$, *J. Anal. Math.* 102(2007), 181-208.

[4] H. Brezis and Y. Li, Some nonlinear elliptic equations have only constant solutions, *J. Partial Differential Equations* 19(2006), 208-217.

[5] H. Brezis and L. A. Peletier, Elliptic equations with critical exponent on spherical caps of $\mathbf{S^3}$, *J. Anal. Math.* 98(2006), 279-316.

[6] O. Druet, From one bubble to several bubbles: the low-dimensional case, *J. Diff. Geom.* 63(2003), 399-473.

[7] O. Druet and E. Hebey, Elliptic equations of Yamabe type, *IMRS Int. Math. Res. Surv.* 2005, no.1, 1-113.

[8] O. Druet and E. Hebey, Stability for strongly coupled critical elliptic systems in a fully inhomogeneous medium. *Anal. PDE* 2(2009), no.3, 305-359.

[9] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* 34(1981), 525-598.

[10] O.Rey, The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent.*J. Funct. Anal.* 89(1990), 1–52.

[11] L. Wang, J. Wei, and S. Yan, A Neumann problem with critical exponent in nonconvex domains and Lin-Ni's conjecture, *Trans. Amer. Math. Soc.* 362 (2010), 4581–4615.

[12] L. Wang, J. Wei, and S. Yan, On Lin-Ni's conjecture in convex domains, *Proc. London Math. Soc.*, to appear.

[13] J. Wei, and S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger equations in $\mathbb{R}^N$, *Calc. Var. Partial Differential Equations* 37(2010), 423–439.

[14] J. Wei, and S. Yan, Infinitely many solutions for the prescribed scalar curvature problem on $S^N$, *J. Funct. Anal.* 258 (2010), 3048–3081.

[15] J. Wei, and S. Yan, On a stronger Lazer-McKenna conjecture for Ambrosetti-Prodi type problems. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 9 (2010), 423–457.

[16] J. Wei, and S. Yan, Infinitely many positive solutions for an elliptic problem with critical or supercritical growth, *J. Math. Pure Appl.*, to appear.



Department of Mathematics, Wuhan University, Wuhan, China

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
*E-mail address*: wei@math.cuhk.edu.hk

Department of Mathematics, The University of New England, Armidale, NSW 2351, Australia
*E-mail address*: syan@turing.une.edu.au