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BIFURCATIONS OF SOME ELLIPTIC PROBLEMS WITH A SINGULAR NONLINEARITY VIA MORSE INDEX

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Abstract. We study the boundary value problem

$$\Delta u = \lambda |x|^\alpha f(u) \text{ in } \Omega, \quad u = 1 \text{ on } \partial\Omega \tag{1}$$

where $\lambda > 0$, $\alpha \geq 0$, Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 2$) containing 0 and f is a C^1 function satisfying $\lim_{s \to 0^+} s^p f(s) = 1$. We show that for each $\alpha \geq 0$, there is a critical power $p_c(\alpha) > 0$, which is decreasing in α, such that the branch of positive solutions possesses infinitely many bifurcation points provided $p > p_c(\alpha)$ or $p > p_c(0)$, and this relies on the shape of the domain Ω. We get some important estimates of the Morse index of the regular and singular solutions. Moreover, we also study the radial solution branch of the related problems in the unit ball. We find that the branch possesses infinitely many turning points provided that $p > p_c(\alpha)$ and the Morse index of any radial solution (regular or singular) in this branch is finite provided that $0 < p \leq p_c(\alpha)$. This implies that the structure of the radial solution branch of (1) changes for $0 < p \leq p_c(\alpha)$ and $p > p_c(\alpha)$.

1. **Introduction.** We study the structure of positive solutions to the following problem

$$(T_\lambda) \qquad \begin{cases} \Delta u = \lambda |x|^\alpha f(u) & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega \end{cases}$$

where $\lambda > 0$, $\alpha \geq 0$, $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded smooth domain which contains 0. The nonlinearity f satisfies the following assumptions, either

(F_1) $f \in C^1(0, \infty)$, $f(0) = \infty$, $f'(s) < 0$ for s near 0,

(F_2) $\lim_{s \to 0^+} s^p f(s) = 1$ with $p > 0$ and $f(s) > 0$ for $s > 0$;

or

(G_1) $f \in C^1(0, 1]$, $f(0) = \infty$, $f'(s) < 0$ for s near 0, $f(1) = 0$, $f'(1) < 0$,

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(G_2) $\lim_{s\to 0^+} s^p f(s) = 1$ with $p > 0$ and $f(s) > 0$ for $s \in (0,1)$.

The typical example of f satisfying (F_1) and (F_2) is $f(s) = s^{-p}$ for $p > 0$. For f satisfying (G_1) and (G_2), we take the examples as $f(s) = s^{-p} - 1$ or

$$f(s) = s^{-p} - s^{-q}, \quad 0 < q < p.$$

By a positive solution u of (T_λ) we mean that $u \in H^1(\Omega) \cap C^0(\Omega)$, $u > 0$ in Ω and satisfying

$$\int_\Omega (\nabla u \cdot \nabla \phi + \lambda |x|^\alpha f(u)\phi) = 0, \qquad \forall \phi \in H_0^1(\Omega).$$

Observe that if u is a positive solution of (T_λ), then by standard elliptic regularity $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and hence is a classical solution of (T_λ) in Ω.

Equation (T_λ) arises in the study of steady states of thin films. Equations as

$$u_t = -\nabla \cdot (h(u)\nabla \Delta u) - \nabla \cdot (g(u)\nabla u) \tag{2}$$

have been used to model the dynamics of thin films of viscous liquids, where $u(x,t)$ presents the height of the air/liquid interface. The zero set $\Sigma = \{x \in \Omega : \ u(x,t) = 0\}$ is the liquid/solid interface and is sometimes called set of **ruptures**. It plays a very important role in the study of thin films. The coefficient $h(u)$ in (2) reflects surface tension effects - a typical choice is $h(u) = u^3$. The coefficient of the second-order term can reflect additional forces such as gravity $g(u) = u^3$, van der Waals interactions $g(u) = u^m - \gamma u^l$ with $\gamma \geq 0$ and $m < 0$, $l \leq 0$ and $|l| < |m|$. For background on (2), we refer to [2], [3], [4], [22], [23], [24], [25], [26] and the references therein.

Taking for example $h(u) = u^3$, $g(u) = u^m - \gamma u^l$ with $\gamma \geq 0$, $m < 0$, $l \leq 0$ and $|l| < |m|$. Then if we consider the steady-state of (2), we see that u satisfying

$$h(u)\nabla\Delta u + g(u)\nabla u = \mathcal{C} \ \text{ in } \Omega$$

is a steady state of (2), where $\mathcal{C} = (C_1, C_2, \ldots, C_n)$ is a constant vector. Assuming $\mathcal{C} = \mathbf{0}$, we get that

$$\Delta u + \frac{1}{m-2} u^{m-2} - \frac{\gamma}{l-2} u^{l-2} = C \ \text{ in } \Omega$$

where C is a constant. For $C = 0$ and $v = \tau^{\frac{1}{l-m}} (2-m)^{1/(3-m)} u$ where $\tau = \frac{\gamma(2-m)^{(3-l)/(3-m)}}{(2-l)}$, then v satisfies the equation

$$\Delta v = \tau^{(3-m)/(l-m)} \left[v^{m-2} - v^{l-2} \right] \tag{3}$$

which is the form of (T_λ).

The problem

$$(P_\lambda) \qquad -\Delta v = \frac{\lambda g(x)}{(1-v)^2}, \quad 0 < v < 1 \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega$$

models a simple electrostatic MEMS(Micro-Electromechanical System) device, consisting of a thin dielectric elastic membrane with boundary supported at level 0 below a rigid plate located at $+1$ immersed in an external electric field, where v is the (normalized) deflection of the elastic membrane and g represents the permittivity of profile. When a voltage, represented here by λ, is applied, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates a so called "pull-in instability" which greatly affects the design of many devices (see [10] and [28], [27] for a detailed discussion on MEMS devices).

In recent papers [11]-[13], [8], [9] and [21], the authors studied the problem (P_λ) where $g \in C(\overline{\Omega})$ is a nonnegative function. They gave a detailed study on the minimal solutions of the problem (P_λ) with different forms of $g(x)$. Similar problems with singular nonlinearities to (P_λ) have also been studied by the authors in [16]-[18] and the references therein. In [29], a general family of nonautonomous elliptic and parabolic equations related to MEMS modeling has been considered.

For $N = 2$ or Ω is the unit ball, and some special nonlinearities, problem (T_λ) has been studied in [16], [17], [15]. For $p = 2$ and $2 \le N \le 7$, (T_λ) was studied in [8], the author obtained some results similar to the case $f(s) = \frac{1}{s^p}$ and $p > p_c(0)$ (see the definition below) in the present paper. The purpose of our work is to provide a rather unified approach to the general problem (T_λ), which in particular reveals the underlying relationship of the results in [11], [8]. We present here a sharp condition on p, N and α, which reveals the change of structure of positive solutions of (T_λ).

On the other hand, it was considered in [7] the equation

$$\Delta u = |x|^\alpha u^{-p} \quad \text{in } \mathbb{R}^N. \tag{4}$$

It was proved that for $\alpha > -2$, $p > 0$ and $N \ge 2$, there is a critical number

$$p_c(\alpha) = \begin{cases} 0 & \text{if } N = 2 \\ P(N,\alpha) & \text{if } 3 \le N < 10 + 4\alpha \\ +\infty & \text{if } N \ge 10 + 4\alpha, \end{cases}$$

with

$$P(N,\alpha) := \frac{(N-2)^2 - 2(N-2)(\alpha+2) - 2(\alpha+2)^2 + 2\sqrt{(\alpha+2)^3(2N-2+\alpha)}}{(N-2)(10+4\alpha-N)}$$

such that for $p > p_c(\alpha)$, (4) has no positive stable solution. Moreover, for $\alpha \in (-2, 0]$ and $p > p_c(\alpha)$, (4) has no finite Morse index solution in $\mathbb{R}^N$. As observed in [7], $p_c(\alpha)$ is strictly decreasing in α when $3 \le N < 10 + 4\alpha$.

In this paper, we first obtain the results similar to those in [7] for $\alpha > 0$ in Section 2. We show that (4) has no lower bounded positive finite Morse index solution u in $\mathbb{R}^N$ when $p > p_c(\alpha)$. Here we say that u is lower bounded if there is some $c > 0$ such that $u \ge c$ in $\mathbb{R}^N \backslash B_R$ for $R >> 1$. Then using these results, we study the structure of positive solutions of (T_λ) in Section 3 and Section 4, where f satisfies (F_1) - (F_2) and (G_1) - (G_2) respectively. In the sequel of the paper, the symbol C, D denote generic positive constants independent of λ, it could be changed from one line to another.

2. **Infinite Morse index solutions of (4) for $\alpha > 0$.**

In this section we will obtain the results similar to those in [7]. More precisely we will prove

Theorem 2.1. *Equation (4) has no lower bounded positive solution that has finite Morse index provided $\alpha > 0$ and $p > p_c(\alpha)$.*

Proof. The proof is similar to that of Theorem 3.3 of [7]. The main difficulty is that we can not obtain the Harnack inequality as in [7] directly for all $\alpha > 0$. For convenience of the readers, we present a sketch of the proof here.

Arguing indirectly we assume that (4) has a lower bounded positive solution u with finite Morse index. Then there exists $R^* > 0$ sufficiently large such that u

is stable in $\mathbb{R}^N \backslash B_{R^*}$. We show that this leads a contradiction. The proof can be divided into several steps. We denote $\gamma(p) := -1 - 2p - 2\sqrt{p(p+1)}$.

Step 1. There exists $R_0 > R^*$ such that for every $\gamma \in (\gamma(p), -1]$ and every $r > 2R_0$, we have

$$\int_{R_0+2<|x|<r} \left(|\nabla(u^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u^{\gamma-p} \right) \leq C + Dr^{N+\frac{(\alpha+2)\gamma+\alpha-2p}{p+1}}, \tag{5}$$

where C and D are positive constants independent of r and u.

Step 2. For every $\gamma \in (\gamma(p), -1]$ and every open ball $B_R(y)$ with $|y| > \frac{6}{5}R^*$ and $R = |y|/4$, we have

$$\int_{B_R(y)} \left(|\nabla(u^{\frac{\gamma+1}{2}})|^2 + |x|^\alpha u^{\gamma-p} \right) \leq CR^{N+\frac{(\alpha+2)\gamma+\alpha-2p}{p+1}}, \tag{6}$$

where C is a positive constant independent of y.

Step 3. There exists a small $\epsilon_0 = \epsilon_0(p, N) > 0$ such that for every $\epsilon \in (0, \epsilon_0]$ and every open ball $B_{2R}(y)$ with $|y| \geq \frac{4}{3}R^*$ and $R = |y|/8$, we have

$$\int_{B_{2R}(y)} \left(|x|^\alpha u^{-(p+1)} \right)^{\frac{N}{2-\epsilon}} \leq CR^{N-\frac{2N}{2-\epsilon}}, \tag{7}$$

where C is a positive constant independent of y and ϵ.

Step 4. Harnack inequality: Under the conditions of Step 3, there exists a positive constant K such that

$$\max_{|x|=r} u(x) \leq K \min_{|x|=r} u(x), \qquad \forall r \geq R^*. \tag{8}$$

Step 5. Under the conditions of Step 3, there exist positive constants C_1 and C_2 such that

$$C_1|x|^{\frac{\alpha+2}{p+1}} \leq u(x) \leq C_2|x|^{\frac{\alpha+2}{p+1}}, \qquad \forall x \in \mathbb{R}^N \backslash B_{R^*}. \tag{9}$$

Step 6. Reaching a contradiction.

Except Step 3, all of other steps can be obtained by arguments exactly the same as those in the proof of Theorem 3.3 of [7]. To prove Step 3, we consider the function

$$\begin{aligned} \Delta(p, \gamma, \alpha) : &= N(p+1) + (\alpha+2)\gamma + \alpha - 2p \\ &= N(p+1) + 2\gamma - 2p + \alpha(\gamma+1). \end{aligned}$$

We find that

$$\Delta(p, \gamma(p), \alpha) = 0 \text{ for } p = p_c(\alpha); \quad \Delta(p, \gamma(p), \alpha) < 0 \text{ for } p > p_c(\alpha).$$

Hence $\Delta(p, \gamma(p), 0) < 0$ if $p > p_c(0)$. This implies that

$$\frac{N(p+1)}{2} < p - \gamma(p) \quad \text{for } p > p_c(0). \tag{10}$$

This inequality holds for $p > p_c(\alpha)$ and $\alpha \in (-2, 0]$, since $p_c(\alpha) \geq p_c(0)$. But (10) does not hold for $p > p_c(\alpha)$ and $\alpha > 0$, since $p_c(\alpha) < p_c(0)$ if $\alpha > 0$. It is easily known that

$$\frac{N(p+1)}{2} \geq p - \gamma(p) \quad \text{for } p_c(\alpha) < p \leq p_c(0). \tag{11}$$

Then we can choose $0 < \rho := \rho(p, N) < p$ such that

$$\frac{\rho N}{2} < p - \gamma(p) \quad \text{for } p_c(\alpha) < p \leq p_c(0). \tag{12}$$

We now show that (7) still holds. It follows from (12) that we can fix $\gamma_* \in (\gamma(p), -1)$ such that $\frac{p-\gamma_*}{\rho N/2} > 1$. Therefore we can find $\epsilon_0 > 0$ sufficiently small so that

$$\frac{p-\gamma_*}{\rho\theta} > 1, \qquad \forall \theta \in \Big[\frac{N}{2}, \frac{N}{2-\epsilon_0}\Big].$$

Fix such θ and set

$$\xi = \frac{p-\gamma_*}{\rho\theta} \quad \text{and} \quad \tau = p-\rho.$$

Note that $B_{2R}(y) \subset \mathbb{R}^N \backslash B_{R^*}$, then

$$\begin{aligned}
\int_{B_{2R}(y)} (|x|^\alpha u^{-(p+1)})^\theta &= \int_{B_{2R}(y)} u^{-(\tau+1)\theta} (|x|^\alpha u^{-\rho})^\theta \\
&\leq c^{-(\tau+1)\theta} \int_{B_{2R}(y)} (|x|^\alpha u^{-\rho})^\theta \\
&\leq C\Big(\int_{B_{2R}(y)} |x|^\alpha u^{\gamma_*-p}\Big)^{1/\xi} \Big(\int_{B_{2R}(y)} |x|^{\frac{\alpha(\theta\xi-1)}{\xi-1}}\Big)^{(\xi-1)/\xi} \\
&\leq CR^{(N+\frac{(\alpha+2)\gamma_*+\alpha-2p}{p+1})\frac{1}{\xi}} R^{(N+\frac{\alpha(\theta\xi-1)}{\xi-1})\frac{\xi-1}{\xi}} \\
&= CR^{N-2\theta},
\end{aligned}$$

where in the first inequality, we have used the lower bounded assumption. This gives (7) if we take $\theta = \frac{N}{2-\epsilon}$ which completes the proof. □

3. **Structure of positive solutions of (T_λ) when f satisfies (F_1) - (F_2).** In this section we study the structure of positive solutions of (T_λ) for f satisfying (F_1) and (F_2). Clearly, with $v := 1-u$, (T_λ) is transformed to the following Dirichlet problem:

$$(S_\lambda) \qquad \begin{cases} -\Delta v = \lambda |x|^\alpha f(1-v) & \text{in } \Omega \\ 0 < v < 1 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 1. *There exists $0 < \lambda^* < \infty$ such that for $\lambda \in (0, \lambda^*)$, (S_λ) admits a minimal positive solution $\underline{v}_\lambda$ and has no positive solution if $\lambda > \lambda^*$.*

Proof. We first show the second statement. Let (σ_1, ϕ_1) be the first eigenvalue and eigenfunction pair of the Dirichlet problem

$$-\Delta\phi = \sigma|x|^\alpha \phi \ \text{ in } \Omega, \quad \phi = 0 \ \text{ on } \partial\Omega.$$

It is clear that such (σ_1, ϕ_1) exists. In fact, let $H^* \subset H_0^1(\Omega)$ be the space with the norm

$$\|\phi\|_{H^*} = \Big(\int_\Omega [|\nabla\phi|^2 + |x|^\alpha \phi^2] dx\Big)^{1/2}.$$

Let us consider

$$\sigma_1 = \inf_{\phi \in H^* \cap C^0(\Omega)} \frac{\int_\Omega |\nabla\phi|^2 dx}{\int_\Omega |x|^\alpha \phi^2 dx}.$$

By the standard theory, we see that σ_1 can be attained by some $\phi_1 \in H^* \cap C^0(\Omega)$. Since

$$\int_\Omega |\nabla|\phi_1||^2 dx \leq \int_\Omega |\nabla\phi_1|^2 dx$$

we can assume that $\phi_1 \geq 0$ in Ω. On the other hand, it is easily seen that $\phi_1 \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

If (S_λ) has a positive solution v_λ, multiplying ϕ_1 on both the sides of (S_λ) and integrating over Ω, we have that

$$\sigma_1 \int_\Omega |x|^\alpha \phi_1 dx > \sigma_1 \int_\Omega |x|^\alpha v_\lambda \phi_1 dx = \lambda \int_\Omega |x|^\alpha f(1-v_\lambda)\phi_1 dx \geq C\lambda \int_\Omega |x|^\alpha \phi_1 dx. \tag{13}$$

Note that $f(1-v_\lambda) \geq C > 0$. (13) implies that λ is bounded.

Now we show that for $\lambda > 0$ sufficiently small, (S_λ) has a minimal positive solution $\underline{v}_\lambda$. It is clear that 0 is a subsolution of (S_λ) since $f(1) > 0$. Let $\Omega_1 \subset \mathbb{R}^N$ be a bounded smooth domain such that $\Omega \subset\subset \Omega_1$ and (σ_1^*, ψ_1) be the first eigenvalue and eigenfunction pair of the problem

$$-\Delta\psi = \sigma|x|^\alpha\psi \ \text{ in } \Omega_1, \quad \psi = 0 \ \text{ on } \partial\Omega_1.$$

We see that for any $\Omega \subset K \subset\subset \Omega_1$, $\psi_1 \geq \delta > 0$ on $\overline{K}$. Let $M = \sup_K \psi_1$ and $\tilde{\psi} = \psi_1/M$. Then $\sup_K \tilde{\psi} = 1$. For any $\epsilon > 0$, there exists $\tilde{\tau} := \tilde{\tau}(\epsilon, \delta) > 0$ such that

$$\frac{f(1-\epsilon s)}{\epsilon s} \leq \tilde{\tau} \quad \text{for } s \in [\tfrac{\delta}{M}, 1].$$

Therefore

$$f(1-\epsilon\tilde{\psi}) \leq \tilde{\tau}\epsilon\tilde{\psi} \ \text{ in } \overline{K}.$$

Thus,

$$-\Delta(\epsilon\tilde{\psi}) = \sigma_1^*\epsilon|x|^\alpha\tilde{\psi} \geq \frac{\sigma_1^*}{\tilde{\tau}}|x|^\alpha f(1-\epsilon\tilde{\psi}) \geq \lambda|x|^\alpha f(1-\epsilon\tilde{\psi})$$

for $0 < \lambda \leq \frac{\sigma_1^*}{\tilde{\tau}}$. It is clear that $\epsilon\tilde{\psi}$ is a supersolution of (S_λ). The sub- and supersolution argument implies that there exists a minimal positive solution $\underline{v}_\lambda$ of (S_λ) for $0 < \lambda \leq \frac{\sigma_1^*}{\tilde{\tau}}$. Let

$$\lambda^* = \sup\{\lambda \in (0,\infty) : \ (S_\lambda) \text{ possesses a minimal positive solution}\}.$$

Then λ^* is bounded.

We now show that the mapping: $\lambda \mapsto \underline{v}_\lambda$ is increasing. Indeed, for any $\lambda_1 > \lambda_2$, we see that $\underline{v}_{\lambda_1}$ is a supersolution to the problem of S_{λ_2}. The sub- and supersolution argument implies that there is a positive solution $\underline{v}_{\lambda_2}$ of (S_{λ_2}) between 0 and $\underline{v}_{\lambda_1}$. Since $\underline{v}_{\lambda_2}$ is a minimal solution, we see that $\underline{v}_{\lambda_2} \leq \underline{v}_{\lambda_1}$ in Ω. The monotonicity of $\underline{v}_\lambda$ about λ implies that $V_{\lambda^*}(x) := \lim_{\lambda\nearrow\lambda^*} \underline{v}_\lambda(x)$ exists and satisfies (S_{λ^*}) almost everywhere. This completes the proof. □

Theorem 3.1. *Let f satisfy (F_1) and (F_2). Then there is an unlimited branch $\Gamma := \{(\lambda, v_\lambda) : \ 0 < v_\lambda < 1$ satisfies $(S_\lambda)\}$ starting from $(0,0)$. If $p > p_c(\alpha)$ or $p > p_c(0)$ (which relies on the domain Ω), then Γ has infinitely many bifurcation points.*

By an unlimited branch, we mean a solution branch along which the solutions approach a singular state, i.e., there exists a sequence $\{(\lambda_n, v_{\lambda_n})\}$ such that $\lambda_n \to \hat{\lambda} \geq 0$ and $\max_\Omega v_{\lambda_n} \to 1$ as $n \to \infty$.

Note that the conclusions hold provided $p > p_c(0)$, since $p_c(0) \geq p_c(\alpha)$.

Proof. By Proposition 1, there is a minimal positive solution branch starting from (0,0). Let $\mathbb{D}$ denote the component of $\{(\lambda, v) \in (0,\lambda^*) \times C(\overline{\Omega}) : \text{v is solution of } (S_\lambda)\}$ containing in its closure $(0,0)$. Note that we can talk about the component since we know from Proposition 1 that it is a simple curve near the starting point (0,0). According to Theorem 2.2 of [5], there exists an analytic curve $(\lambda(s), v(s)) \in \mathbb{D}$

for $s \geq 0$ such that $\max_\Omega v(s) \to 1$ as $s \to \infty$. Note that we allow the curve $(\lambda(s), v(s))$ to have isolated intersections and that for each $s > 0$, $v(s) \in C_0^2(\Omega)$. By usual argument of finding a minimal continuum in $\{(\lambda(s), v(s)) : s \geq 0\}$ joining $(\lambda(0), v(0)) = (0, 0)$ to "infinity", we obtain a curve with no self intersections, but it is only piecewise analytic and continuous. It is easy to see that the minimal continuum is an unlimited solution branch.

We study the behavior of the branch near "infinity". For any sequence $\{s_n\}$ and $(\lambda_n, v_n) \equiv (\lambda(s_n), v(s_n))$ such that $\lambda_n \to \hat{\lambda} \geq 0$, $\max_\Omega v_n \to 1$ as $n \to \infty$, let $\{x_n\} \subset \Omega$ be a sequence of points such that $v_n(x_n) = \max_\Omega v_n$. Suppose that $x_n \to a \in \overline{\Omega}$ (by a subsequences if necessary and it is the same in the sequel).

Set $\epsilon_n = 1 - v_n(x_n)$ so that $\epsilon_n \to 0$ as $n \to \infty$. Define

$$w_n(y) := \frac{1 - v_n(x)}{\epsilon_n}$$

for $y \in \tilde{\Omega}_n := \{\tau_n(x - x_n) : \ \tau_n = \lambda_n^{1/(2+\alpha)} \epsilon_n^{-(p+1)/(2+\alpha)}, x \in \Omega\}$. We see that w_n satisfies $w_n \geq 1$, $w_n(0) = 1$ and

$$\Delta w_n = |y + \tau_n x_n|^\alpha w_n^{-p}[(\epsilon_n w_n)^p f(\epsilon_n w_n)] \ \text{ in } \tilde{\Omega}_n, \quad w_n|_{\partial \tilde{\Omega}_n} = \epsilon_n^{-1}. \tag{14}$$

We claim that,

$$\tau_n \to \infty \ \text{ as } n \to \infty \tag{15}$$

We need to prove that the following two cases do not occur.

(i) $\tau_n \to 0$ as $n \to \infty$,

(ii) $\tau_n \to R > 0$ as $n \to \infty$.

Assume that the first case occurs. Since v_n satisfies

$$-\Delta v_n = \lambda_n |x|^\alpha \epsilon_n^{-p} \frac{f(1 - v_n)}{\epsilon_n^{-p}}$$

and f is decreasing near 0, $f(s) \leq C s^{-p} + C$ for $s > 0$, we see that $\frac{f(1-v_n)}{\epsilon_n^{-p}} \leq M$ and

$$\epsilon_n^{-p} \lambda_n \frac{f(1 - v_n)}{\epsilon_n^{-p}} \to 0 \ \text{ in } C^0(\Omega) \text{ as } n \to \infty.$$

The standard regularity theory implies that $v_n \to v$ in $C^1(\overline{\Omega})$ as $n \to \infty$, where v satisfies $\|v\|_{L^\infty(\Omega)} = 1$ and $-\Delta v = 0$ in Ω, $v = 0$ on $\partial\Omega$. This is impossible.

For the second case, we see from (14) that $w_n \to w$ in $C^1_{loc}(\tilde{\Omega})$ as $n \to \infty$ and w satisfies the problem

$$\Delta w = |y + y_0|^\alpha w^{-p}, \ \ w \geq 1 \ \text{ in } \tilde{\Omega}, \quad w(0) = 1, \ \ w = \infty \ \text{ on } \partial\tilde{\Omega}, \tag{16}$$

where $\tilde{\Omega} = \{R(x - a) : \ x \in \Omega\}$ and $y_0 = \lim_{n\to\infty} \tau_n x_n = Ra$. We show such w can not exist. Let η be the solution of the problem

$$-\Delta \eta = 1 \ \text{ in } \tilde{\Omega}, \quad \eta = 0 \ \text{ on } \partial\tilde{\Omega}.$$

We see $\eta > 0$ in $\tilde{\Omega}$. Thus,

$$\Delta(w + C\eta) = |y + y_0|^\alpha w^{-p} - C \leq 0$$

if $C > 0$ is sufficiently large. But $w + C\eta = \infty$ on $\partial\tilde{\Omega}$. This contradicts the maximum principle. Therefore, our claim (15) holds.

Now we consider three cases for y_0:

(i) $|y_0| = 0$,

(ii) $|y_0| \neq 0, |y_0| < \infty$,

(iii) $|y_0| = \infty$.

We first complete the proof of this theorem for the first two cases. The proof of the third case postpone at the end of this section. We recall that in the sequel, the convergence maybe considered for a subsenquence. We see that $x_n \to 0$ as $n \to \infty$. By a standard blow-up argument, we get from (14) and (15) that $w_n \to W$ in $C^1_{loc}(\mathbb{R}^N)$ as $n \to \infty$, where W satisfies $W(0) = 1$, $W \geq 1$ in $\mathbb{R}^N$ and

$$\Delta W = |y|^\alpha W^{-p} \text{ in } \mathbb{R}^N \tag{17}$$

or

$$\Delta W = |y + y_0|^\alpha W^{-p} \text{ in } \mathbb{R}^N. \tag{18}$$

Note that W is a lower bounded positive solution of (17) or (18). It is known from Theorem 2.1 that W is an infinite Morse index solution of (17) or (18) if $p > p_c(\alpha)$. (We can change (18) to (17) by a simple transformation $z = y + y_0$.) By Dancer's argument in [6], we see that for any $M >> 1$, there is $n^* := n(M)$ such that for $n > n^*$, the Morse index of v_n is bigger than M. This also implies that the branch Γ has infinitely many bifurcation points.

Now we show that $\hat{\lambda} > 0$ for the first two cases. Otherwise $\hat{\lambda} = 0$ and we should have a contradiction. Recall that $r_n := \tau_n^{-1} \to 0$ as $n \to \infty$ by (15). On the other hand, we have

$$\lambda_n^{-1/(p+1)} |r_n y + x_n|^{-(2+\alpha)/(p+1)} (1 - v_n(r_n y + x_n)) \to |y + y_0|^{-(2+\alpha)/(p+1)} W(y) \tag{19}$$

uniformly for $0 < |y + y_0| < R$. This implies (note $W \geq 1$)

$$1 - v_n(x) \geq C\lambda_n^{1/(p+1)} |y + y_0|^{-(2+\alpha)/(p+1)} |x|^{(2+\alpha)/(p+1)} \tag{20}$$

for $|x|$ sufficiently small. Thus, by the conditions of f,

$$\lambda_n |x|^\alpha f(1 - v_n) \leq C\lambda_n^{1/(p+1)} |y|^{p(2+\alpha)/(p+1)} |x|^{(\alpha-2p)/(p+1)} \leq C\lambda_n^{1/(p+1)} |x|^{(\alpha-2p)/(p+1)}, \tag{21}$$

where we have used $s^p f(s) \to 1$ as $s \to 0$ and $f'(s) < 0$ for s near 0. If $\alpha \geq 2p$, we see that

$$\|\lambda_n |x|^\alpha f(1 - v_n)\|_{L^\gamma(\Omega)} \to 0 \text{ as } n \to \infty \tag{22}$$

for any $\gamma > N/2$; if $\alpha - 2p < 0$, then (22) still holds provided $N/2 < \gamma < N(p + 1)/(2p - \alpha)$. Therefore we have that $v_n \in C^0(\overline{\Omega})$ and $\|v_n\|_{C^0(\overline{\Omega})} \to 0$ as $n \to \infty$. This contradicts $v_n(0) \to 1$ as $n \to \infty$. This completes the proof for cases (i) and (ii). □

Remark 1. Note that whether the first two cases occur or not relies on the shape of Ω. If Ω is a ball and we consider the radial branch of (S_λ), then the first case occurs. We see that the radial branch Γ has infinitely many turning points (see [18]) provided $p > p_c(\alpha)$. When Ω is a good behaved domain as in [6], [17], since the moving plane method can not be used for $\alpha > 0$, we can not obtain the symmetry properties for the positive solutions as those of Ω. If all the solutions in the branch Γ have the symmetry properties as those of Ω, then Γ possesses infinitely many bifurcation points when $p > p_c(\alpha)$. But, if $\alpha = 0$, the first case occurs and the branch Γ has infinitely many bifurcation points provided $p > p_c(0)$.

The conditions on p, α and N of Theorem 3.1 are sharp for the infinitely many bifurcation points. In fact, if Ω is the unit ball of $\mathbb{R}^N$ and $f(s) = s^{-p}$, $p > 0$, from Theorem 3.1, there is an unlimited positive solution branch Γ of (S_λ), which has infinitely many bifurcation points, provided $p > p_c(\alpha)$. Arguments similar to those in Lemma 4.1 of [18] imply that all the bifurcation points are turning points.

We can obtain the exact structure of the radial branch Γ of the problem

$$\Delta u = \lambda |x|^{\alpha} u^{-p} \ \text{ in } B, \quad u = 1 \ \text{ on } \partial B. \tag{23}$$

Lemma 3.2. *There is a unique singular radial solution u_* of (23) attained at $\lambda = \lambda_*$.*

Proof. Let u_λ be a radial solution of (23). Setting

$$\rho = \lambda^{1/(2+\alpha)} r, \quad v(\rho) = u_\lambda(r), \tag{24}$$

we see that v satisfies the problem

$$v'' + \frac{N-1}{\rho} v' = \rho^{\alpha} v^{-p} \ \ \rho \in (0, \lambda^{1/(2+\alpha)}), \quad v(\lambda^{1/(2+\alpha)}) = 1. \tag{25}$$

It follows from Theorem 2.3 of [7] that if $v(0) = 0$, then

$$v(\rho) = \Lambda \rho^{\frac{2+\alpha}{p+1}}, \quad \Lambda^{-(p+1)} = \frac{2+\alpha}{p+1}\Big[N - 2 + \frac{2+\alpha}{p+1}\Big].$$

Therefore,

$$\lambda_* = \frac{2+\alpha}{p+1}\Big[N - 2 + \frac{2+\alpha}{p+1}\Big]$$

and the unique singular solution corresponded is $u_*(r) = r^{\frac{2+\alpha}{p+1}}$. □

Now we treat the case where $p > p_c(\alpha)$.

Theorem 3.3. *Assume that $N \geq 2$, $p > p_c(\alpha)$. Then there is a unique number λ_* as in Lemma 3.2 such that, for any integer $k \geq 1$, there exist at least k positive radial solutions of (23) for any λ sufficiently close to λ_*. In particular, there are infinitely many classical solutions of (23) for $\lambda = \lambda_*$.*

Proof. We know that the solution branch has infinitely many bifurcation points. It is also known from [18] that the radial solution branch has infinitely many turning points. What we need to show is that for any solution sequence $\{(\lambda_n, u_n)\} \equiv \{(\lambda_n, u_{\lambda_n})\}$ satisfying $\lambda_n \to \lambda_*$, $u_n(0) \to 0$ as $n \to \infty$ and any integer $k >> 1$, there exists an $N^* := N^*(k)$ such that for $n > N^*$, the graph of $u_n(r)$ intersects that of $u_*(r)$ at least k times in the interval $(0,1)$. Indeed, if this is proved, then $u_* - u_n$ has at least k zeros in $(0,1)$. By the changes of type (24),

$$\rho = \lambda_*^{1/(2+\alpha)} r, \quad v_*(\rho) = u_*(r)$$

and

$$\rho = \lambda_n^{1/(2+\alpha)} r, \quad v_n(\rho) = u_n(r)$$

then v_* and v_n satisfy the problems

$$\Delta v_* = v_*^{-p} \ \text{ in } (0, \lambda_*^{1/(2+\alpha)}), \quad v_*(\lambda_*^{1/(2+\alpha)}) = 1$$

and

$$\Delta v_n = v_n^{-p} \ \text{ in } (0, \lambda_n^{1/(2+\alpha)}), \quad v_n(\lambda_n^{1/(2+\alpha)}) = 1$$

respectively. Moreover, $v_* - v_n$ has at least k zeros in $(0, \min\{\lambda_n^{1/(2+\alpha)}, \lambda_*^{1/(2+\alpha)}\})$. Thus there are at least $\left[\frac{k}{2}\right] - 1$ intervals I_i $(i = 1, 2, \ldots, [\frac{k}{2}] - 1)$ on which $v_* - v_n < 0$. Setting $\Omega_i = \{y : |y| \in I_i\}$, we see that $h_n^i := v_* - v_n$ satisfies satisfies

$$\int_{\Omega_i} [|\nabla h_n^i|^2 - p v_*^{-(p+1)} (h_n^i)^2] < 0.$$

Since each $h_n^i \in H_0^1(B_*)$, where $B_* = \{y : |y| < \lambda_*^{1/(2+\alpha)}\}$, and $\int_{B_*} h_n^i h_n^j dy = 0$ for $i \neq j$, the arbitrariness of k implies that the Morse index of v_* is ∞. Arguments similar to those in [6] imply that the radial solution branch Γ of (23) turns infinitely many times around $\lambda = \lambda_*$.

We use contradiction argument to prove our claim. On the contrary, we see that there is $r_* \in (0,1)$ independent of n such that, for any solution sequence $\{(\lambda_n, u_n)\}$ with $\lambda_n \to \lambda_*$ and $u_n(0) \to 0$ as $n \to \infty$, we have $u_n > u_*$ in $[0, r_*]$ provided that n is large. Note that $u_*(0) = 0$. This also implies that there is $\rho_* > 0$ independent of n such that

$$v_n(\rho) > v_*(\rho) \qquad \text{for } \rho \in (0, \rho_*). \tag{26}$$

Let $w_n(t) = v_n/v_*$, $t = \ln\rho$. Then w_n satisfies

$$w_n''(t) + \Big(\frac{2\rho v_*'(\rho)}{v_*(\rho)} + N - 2\Big)w_n' + \rho^{2+\alpha} v_*^{-(p+1)}(\rho)(w_n - w_n^{-p})(t) = 0, \tag{27}$$

for $t \leq \ln\rho_*$ and $\lim_{t\to-\infty} w_n(t) = \infty$. A direct calculation shows that

$$\frac{2\rho v_*'(\rho)}{v_*(\rho)} = \frac{2(2+\alpha)}{p+1}, \qquad \rho^{2+\alpha} v_*^{-(p+1)}(\rho) = \Lambda^{-(p+1)}.$$

Thus w_n satisfies

$$w_n''(t) + \Big(N - 2 + \frac{2(2+\alpha)}{p+1}\Big)w_n'(t) + \Lambda^{-(p+1)}(w_n - w_n^{-p})(t) = 0, \;\; t \leq \ln\rho_*. \tag{28}$$

Since $w_n(t) > 1$ for $t \in (-\infty, \ln\rho_*)$, we have

$$w_n''(t) + \beta w_n'(t) \leq 0 \qquad \text{for } t \in (-\infty, \ln\rho_*),$$

where $\beta = \Big(N - 2 + \frac{2(2+\alpha)}{p+1}\Big)$. Thus,

$$e^{\beta\tau} w_n'(\tau) \leq e^{\beta t} w_n'(t) \qquad \text{for } t < \tau.$$

This implies

$$w_n'(t) \leq 0 \qquad \text{for } t \in (-\infty, \ln\rho_*). \tag{29}$$

We know that $w_n'(t) \not\equiv 0$ since $v_n \not\equiv v_*$.

On the other hand, it clear that (28) is equivalent to

$$z_n''(t) + \beta z_n' + \Lambda^{-(p+1)}\frac{(w_n - w_n^{-p})}{(w_n - 1)} z_n = 0, \;\; z_n > 0 \;\text{ on } (-\infty, \ln\rho_*). \tag{30}$$

where $z_n = w_n - 1$. Note that $w_n \to 1$ in $C^1_{loc}(-\infty, \ln\lambda_*^{1/(2+\alpha)})$ as $n \to \infty$. Then

$$\frac{(w_n - w_n^{-p})}{(w_n - 1)} \to (p+1) \qquad \text{as } n \to \infty.$$

Since

$$\beta^2 - 4(p+1)\Lambda^{-(p+1)} < 0 \qquad \text{for } p > p_c(\alpha),$$

we have for any interval $[a,b] \subset (-\infty, \ln\rho_*)$ such that for all n sufficiently large

$$g_n(t) := \frac{(w_n - w_n^{-p})}{(w_n - 1)} \geq \kappa \qquad \text{in } [a,b]$$

with

$$\beta^2 - 4\kappa\Lambda^{-(p+1)} < 0$$

and

$$z_n'' + \beta z_n' + \Lambda^{-(p+1)}\kappa z_n \leq 0, \quad z_n > 0 \qquad \text{on } [a,b]. \tag{31}$$

Observe that any solution of

$$Z''(t) + \beta Z' + \Lambda^{-(p+1)}\kappa Z = 0 \tag{32}$$

is oscillatory. In particular, there exist $[c, d] \subset [a, b]$ such that $Z(c) = Z(d) = 0$, $Z > 0$ in (c, d) (and hence $Z'(c) > 0 > Z'(d)$). Multiplying (31) by Z and (32) by z_n, we have

$$z_n'' Z + \beta z_n' Z + \Lambda^{-(p+1)}\kappa z_n Z \le 0 \quad \text{ on } [c, d] \tag{33}$$

$$Z'' z_n + \beta Z' z_n + \Lambda^{-(p+1)}\kappa Z z_n = 0 \quad \text{ on } [c, d]. \tag{34}$$

Subtracting (34) from (33) yields $(Zz_n' - Z' z_n)' \le 0$ on $[c, d]$. In particular, we have

$$-Z'(d)z_n(d) \le -Z'(c)z_n(c).$$

This is clearly impossible. $\square$

Now we consider the case of $0 < p \le p_c(\alpha)$.

Theorem 3.4. *Assume that $N \ge 3$, $0 < p \le p_c(\alpha)$. Then the radial solution branch of (23) is*

$$\Gamma = \{(\lambda, \overline{u}_\lambda) : 0 < \lambda \le \lambda_*, \ \overline{u}_\lambda \textit{ is the maximal positive radial solution of (23)}\}.$$

Proof. Note that $p_c(\alpha) = 0$ for $N = 2$. An elementary calculation shows that

$$p\frac{2+\alpha}{p+1}\Big(N - 2 + \frac{2+\alpha}{p+1}\Big) \le \frac{(N-2)^2}{4}, \qquad \text{if } 0 < p \le p_c(\alpha).$$

Arguments similar to those in the proof of Theorem 3.1 imply that there is a maximal positive solution branch of (23) starting from $(0, 1)$. It is clear that each maximal solution $\overline{u}_\lambda$ is a radially symmetric. Moreover, the mapping $\lambda \mapsto \overline{u}_\lambda$ is decreasing. We know that $(\lambda_*, |x|^{(2+\alpha)/(p+1)})$ is the unique singular solution of (23).

Now we show that $u_*(x) = |x|^{(2+\alpha)/(p+1)}$ is linearized semistable in B. For any $\varphi \in C_0^1(B)$, we see that

$$\begin{aligned}\int_B [|\nabla\varphi|^2 - p\lambda^*|x|^\alpha u_*^{-(p+1)}\varphi^2] &= \int_B [|\nabla\varphi|^2 - p\lambda^*|x|^{-2}\varphi^2] \\ &\ge \int_B [|\nabla\varphi|^2 - \frac{(N-2)^2}{4}|x|^{-2}\varphi^2] \ge 0.\end{aligned}$$

The last inequality is by the well-known Hardy inequality. Therefore u_* is linearized semistable.

Let $v(\rho)$ and $v_*(\rho)$ be as defined by the change (24) from u_λ and u_* respectively, they can be uniquely extended to be radial solutions of the equation

$$\Delta v = |x|^\alpha v^{-p} \qquad \text{in } \mathbb{R}^N.$$

Arguments similar to those in the proof of (3.3) in [7] imply that $v(\rho) > v_*(\rho)$ for $\rho \in (0, \infty)$ provided $0 < p \le p_c(\alpha)$. Since $v(\lambda^{1/(2+\alpha)}) = 1$ and $v_*(\lambda_*^{1/(2+\alpha)}) = 1$, we see that $\lambda < \lambda^*$. Therefore, $u_\lambda > u_*$ in $(0, 1)$. Moreover, for any $\varphi \in C_0^1(B)$, we see that

$$\int_B [|\nabla\varphi|^2 - p\lambda|x|^\alpha u_\lambda^{-(p+1)}\varphi^2] \quad \ge \quad \int_B [|\nabla\varphi|^2 - p\lambda_*|x|^\alpha u_*^{-(p+1)}\varphi^2] \ge 0.$$

Now we show that $u_\lambda = \overline{u}_\lambda$. Under the change as in (24), we see that v and $\overline{v}$ satisfies the same equation and

$$v(\lambda^{1/(2+\alpha)}) = \overline{v}(\lambda^{1/(2+\alpha)}) = 1.$$

For $0 < p \leq p_c(\alpha)$, we easily show that $v(0) = \overline{v}(0)$. Otherwise, we can show by arguments similar to those in the proof of (3.3) of [7] that $v < \overline{v}$ in $[0, \lambda^{1/(2+\alpha)}]$. This is a contradiction. □

Proof of Theorem 3.1 for the case $|y_0| = \infty$. We distinguish the following two cases: (a) $a = 0$ and (b) $a \neq 0$.

For the first case, if we set $\tilde{w}_n(z) = w_n(y)$ where $\rho_n = \tau_n|x_n|, z = \rho_n^{\alpha/2} y$, we see from (14) that $\tilde{w}_n(0) = 1, \tilde{w}_n \geq 1$ satisfies the equation

$$\Delta \tilde{w}_n = |1 + \frac{z}{\rho_n^{(2+\alpha)/2}}|^\alpha \tilde{w}_n^{-p}[(\epsilon_n \tilde{w}_n)^p f(\epsilon_n \tilde{w}_n)] \text{ in } \hat{\Omega}_n, \ \ \tilde{w}_n|_{\partial \hat{\Omega}_n} = \epsilon_n^{-1}, \tag{35}$$

where $\hat{\Omega}_n = \{\rho_n^{\alpha/2} y : \ \ y \in \tilde{\Omega}_n\}$. The standard blow-up argument implies that $\tilde{w} \to \tilde{W}$ in $C^1_{loc}(\mathbb{R}^N)$ as $n \to \infty$ and $\tilde{W}$ satisfies the equation

$$\Delta \tilde{W} = \tilde{W}^{-p}, \ \ \tilde{W} \geq 1 \ \text{ in } \mathbb{R}^N, \ \ \tilde{W}(0) = 1. \tag{36}$$

It is known from [7] that the Morse index of $\tilde{W}$ is infinity for $p > p_c(0)$. Thus the branch Γ has infinitely many bifurcation points in this case. This completes the proof of case (a).

Now we consider case (b). We first see that $\lambda_n^{1/2}\epsilon_n^{-(p+1)/2} d_n \to +\infty$ as $n \to \infty$ where $d_n := \text{dist}(x_n, \partial\Omega)$. Note that it follows from (15)

$$t_n := \lambda_n^{1/2}\epsilon_n^{-(p+1)/2} \to \infty \ \text{ as } n \to \infty.$$

So it can be done by arguments similar to those in the proof (3.16) in [8].

By arguments as above, we see that

$$\frac{1 - u_n(t_n^{-1} y + x_n)}{\epsilon_n} \to U \qquad \text{in } C^1_{loc}(\mathbb{R}^N) \text{ as } n \to \infty$$

where U satisfies

$$\Delta U = |a|^\alpha U^{-p}, \ \ U \geq 1 \ \text{ in } \mathbb{R}^N, \ \ U(0) = 1. \tag{37}$$

Thus the same conclusion holds as for case (a). This completes the proof of Theorem 3.1 for $|y_0| = \infty$. □

4. **Structure of positive solutions of (T_λ) when f satisfies (G_1) - (G_2).** In this section we study the structure of positive solutions of (T_λ) for f satisfying (G_1) and (G_2). As in Section 3, we study its equivalent form (S_λ). Recall that the existence of positive entire radial solutions of equation

$$\Delta u = f(u) \qquad \text{in } \mathbb{R}^N$$

with f satisfying (G_1) and (G_2) was established in [23] and [20]. Moreover, arguments similar to those in [6] and [5] imply that there exists a solution branch which starts from $(\lambda^*, 0)$, where $\lambda^* = -\sigma_1/f'(0)$. Similar arguments to those in the proof of Theorem 3.1 proves also

Theorem 4.1. *There exists an unlimited solution branch*

$$\Gamma := \{(\lambda, v_\lambda) : \ 0 < v_\lambda < 1 \textit{ satisfies } (S_\lambda)\}$$

starting from $(\lambda^*, 0)$. Moreover, if $p > p_c(\alpha)$ or $p > p_c(0)$ (which relies on the shape of Ω), then Γ has infinitely many bifurcation points.

Now we study the exact structure of positive radial solutions of the problem

$$\Delta u = \lambda |x|^\alpha [u^{-p} - 1], \quad 0 < u < 1 \ \text{ in } B, \quad u = 1 \ \text{ on } \partial B. \tag{38}$$

Lemma 4.2. *There is a unique singular radial solution u_* of (38) attained at $\lambda = \lambda_*$.*

Proof. Let u_λ be a radial solution of (38). Let again ρ, $v(\rho)$ define by (24), then v satisfies the equation

$$v'' + \frac{N-1}{\rho} v' = \rho^\alpha [v^{-p} - 1], \qquad \rho \in (0, \lambda^{1/(2+\alpha)}), \quad v(\lambda^{1/(2+\alpha)}) = 1. \tag{39}$$

Arguments similar to those in Lemmas 4.1 and 4.2 in [23] imply that if $v(0) = 0$, then

$$\rho^{-(2+\alpha)/(p+1)} v(\rho) \to \Lambda \ \text{ as } \rho \to 0, \tag{40}$$

$$\rho^{1-(2+\alpha)/(p+1)} v'(\rho) \to \frac{(2+\alpha)}{(p+1)} \Lambda \ \text{ as } \rho \to 0 \tag{41}$$

where Λ is defined as in Lemma 3.2. Moreover, v can be uniquely extended to be a singular radial solution of the equation

$$\Delta v = |y|^\alpha [v^{-p} - 1], \qquad y \in \mathbb{R}^N. \tag{42}$$

As in Section 4 of [23], it can be prove that (42) has a unique singular radial solution. Let λ_* satisfy $v(\lambda_*^{1/(2+\alpha)}) = 1$. Then $\lambda_* > 0$. It is clear that $u_*(r) = v(\lambda_*^{1/(2+\alpha)} r)$ is the unique singular solution to (38). $\square$

Theorem 4.3. *Assume that $N \geq 2$, $p > p_c(\alpha)$. Then there is a unique $\lambda_* > 0$ as in Lemma 4.2 such that, for any integer $k \geq 1$, there exist at least k positive radial solutions of (38) for any λ sufficiently close to λ_*. In particular, there are infinitely many classical solutions of (38) for $\lambda = \lambda_*$.*

Proof. Similar to the proof of Theorem 3.3, we only need to show that for any sequence $\{(\lambda_n, u_n)\} \equiv \{(\lambda_n, u_{\lambda_n})\}$ satisfying $\lambda_n \to \lambda_*$, $u_n(0) \to 0$ as $n \to \infty$ and any integer $k >> 1$, there exists an $N^* := N^*(k)$ such that for $n > N^*$, the graph of $u_n(r)$ intersects that of $u_*(r)$ at least k times in the interval $(0, 1)$.

We keep the same notations as in the proof of Theorem 3.3 and also use a contradiction argument to prove this. Assume that there exist $r_* \in (0, 1)$ independent of n and a solution sequence $\{(\lambda_n, u_n)\}$ of (38) such that $u_n > u_*$ in $[0, r_*]$ provided that n is large.

Observe that both functions $v_*(\rho)$ and $v_n(\rho)$ satisfy the same equation

$$v'' + \frac{N-1}{\rho} v' = \rho^\alpha [v^{-p} - 1].$$

We also have that there is $\rho_* > 0$ such that $v_n(\rho) > v_*(\rho)$ for $\rho \in (0, \rho_*)$.

Let $w_n(t) = v_n / v_*$, $t = \ln \rho$. Then w_n satisfies

$$w_n''(t) + \Big(\frac{2\rho v_*'(\rho)}{v_*(\rho)} + N - 2 \Big) w_n'(t) + \rho^{2+\alpha} v_*^{-(p+1)}(\rho) [(w_n - w_n^{-p})(t) + v_*^p(\rho)(1 - w_n)(t)] = 0, \tag{43}$$

for $t \leq \ln \rho_*$ and $\lim_{t \to -\infty} w_n(t) = \infty$. From (40) and (41), we obtain

$$\frac{2\rho v_*'(\rho)}{v_*(\rho)} \to \frac{2(2+\alpha)}{p+1}, \quad \rho^{2+\alpha} v_*^{-(p+1)} \to \Lambda^{-(p+1)} \ \text{ as } \rho \to 0. \tag{44}$$

Since $w_n(t) > 1$ for $t \in (-\infty, \ln\rho_*)$ and $v_* \to 0$ as $\rho \to 0$, there exists $-\infty < T < \ln\rho_*$ such that

$$w_n''(t) + g_1(t) w_n'(t) \leq 0 \qquad \text{for } t \in (-\infty, T) \tag{45}$$

where $g_1(t) = \frac{2e^t v_*'(e^t)}{v_*(e^t)} + N - 2$. Thus,

$$\exp\Big(\int_{-\infty}^{t} g_1(s)ds\Big) w_n'(t) \le \exp\Big(\int_{-\infty}^{\tau} g_1(s)ds\Big) w_n'(\tau) \quad \text{ if } t \ge \tau > -\infty. \tag{46}$$

We know $g_1(t) \to (N - 2 + \frac{2(2+\alpha)}{p+1})$ as $t \to -\infty$. Since $w_n(t) \to \infty$ as $t \to -\infty$, (46) implies that

$$w_n'(t) \le 0 \ \text{ for } -\infty < t < T. \tag{47}$$

The strict inequality in (47) must be true since $v' \equiv 0$ is impossible (note $w_n \not\equiv 1$).

Let $z_n = w_n - 1$. Then by (43), we get

$$z_n''(t) + g_1(t) z_n' + g_2^n(t) z_n = 0, \ \ z_n > 0 \ \text{ on } (-\infty, T) \tag{48}$$

where

$$g_{2,n}(t) = \rho^{2+\alpha} v_*^{-(p+1)}(\rho)\Big[\frac{(w_n - w_n^{-p})}{(w_n - 1)} - v_*^p(\rho)\Big].$$

Since $w_n \to 1$ in $C^0_{loc}(-\infty, T)$ as $n \to \infty$, we have

$$g_{2,n} \to \rho^{2+\alpha} v_*^{-(p+1)}(\rho)[(p+1) - v_*^p] \quad \text{ in } C^0_{loc}(-\infty, T) \text{ as } n \to \infty.$$

Noticing that

$$\rho^{2+\alpha} v_*^{-(p+1)}(\rho) \to \Lambda^{-(p+1)}, \ \ v_*(\rho) \to 0 \quad \text{ as } \rho \to 0$$

and

$$\Big(N - 2 + \frac{2(2+\alpha)}{p+1}\Big)^2 - 4(p+1)\Lambda^{-(p+1)} < 0 \quad \text{ for } p > p_c(\alpha),$$

we deduce that for any closed interval $[T_2, T_1] \subset (-\infty, T)$ and n sufficiently large

$$g_1^2 - 4 g_{2,n} < 0 \quad \text{ in } [T_2, T_1].$$

Thus there exist $b_1 > 0$ and $c_1 > 0$ such that $b_1^2 - 4c_1 < 0$, and $g_1(t) < b_1$, $g_{2,n}(t) > c_1$ if $t \in [T_2, T_1]$. Observe that any solution of

$$Z''(t) + b_1 Z' + c_1 Z = 0 \tag{49}$$

is oscillatory. In particular, there exist $T_2 < a_2 < b_2 < T_1$ such that $Z(a_2) = Z(b_2) = 0$, $Z > 0$ in (a_2, b_2) (and hence $Z'(a_2) > 0 > Z'(b_2)$). Then as before, we have

$$e^{b_1 b_2}(Z z_n' - Z' z_n)(b_2) < e^{b_1 a_2}(Z z_n' - Z' z_n)(a_2).$$

This is impossible since $Z'(a_2) > 0 > Z'(b_2)$. This completes the proof. □

We now study the structure of radial solution branch Γ of (38) for $0 < p \le p_c(\alpha)$. Note that for that case, we have $\lambda^* = \sigma_1/p$. We first give the asymptotic behavior of $u_*(r)$ as $r \to 0$.

Lemma 4.4.

$$u_*(r) = \lambda_*^{1/(p+1)} r^{\delta}\Big(\Lambda - B r^{p\delta} + o(r^{p\delta})\Big) \quad \textit{as } r \to 0,$$

where Λ is as in Lemma 3.4 and

$$\delta = \frac{2+\alpha}{p+1}, \ \ B = \frac{1}{(p\delta)^2 + (N - 2 + 2\delta)p\delta + (p+1)\Lambda^{-(p+1)}}.$$

Proof. Introducing the Emden-Fowler transformation for u_*

$$v_*(\rho) = (\lambda_*^{\frac{1}{2+\alpha}} r)^{-\delta} u_*(r), \quad \rho = \frac{1}{2+\alpha}\ln\lambda + \ln r,$$

we see that v_* satisfies, for $\rho \in (-\infty, \frac{\ln\lambda_*}{2+\alpha})$

$$v_*''(\rho) + (N-2+2\delta)v_*'(\rho) + \Lambda^{-(p+1)}v_*(\rho) - v_*^{-p}(\rho) + e^{p\delta\rho} = 0, \tag{50}$$

with $v_* > 0$, $\lim_{\rho\to-\infty} v_*(\rho) = \Lambda$ and $v(\frac{\ln\lambda_*}{2+\alpha}) = \lambda_*^{-\delta/(2+\alpha)}$. Therefore if we assume that $v_*(\rho) = \Lambda + Ae^{p\delta\rho} + o(e^{p\delta\rho})$ as $\rho \to -\infty$, we easily obtain from (50) that $A = -B$. Here we are using the fact that

$$(\Lambda + Ae^{p\delta\rho} + o(e^{p\delta\rho}))^{-p} = \Lambda^{-p} - p\Lambda^{-(p+1)}Ae^{p\delta\rho} + o(e^{p\delta\rho}).$$

This completes the proof. $\square$

Next we show that for any solution (regular or singular) $(\lambda, u_\lambda) \in \Gamma$, the Morse index $m(u_\lambda)$ of u_λ is bounded. More precisely, we have

Theorem 4.5. *There exists an integer $C \geq 1$, independent of λ, such that*

$$1 \leq m(u_\lambda) \leq C. \tag{51}$$

Proof. We first consider the singular solution case. For any solution consequence $\{(\lambda_n, u_n)\} \equiv \{(\lambda_n, u_{\lambda_n})\}$ with $\lambda_n \to \lambda_*$ and $u_n(0) \to 0$ as $n \to \infty$, we consider the eigenvalue problem

$$-\Delta h - p\lambda_n|x|^\alpha u_n^{-(p+1)}h = \mu h \ \text{ in } B, \quad h = 0 \ \text{ on } \partial B. \tag{52}$$

Firstly, by multiplying $k_n := 1 - u_n \in H_0^1(B)$ on both sides of the equation of u_n and integrate it over B, we obtain

$$\int_B |\nabla k_n|^2 = \int_B \lambda_n|x|^\alpha(u_n^{-p} - 1)k_n < \int_B p\lambda_n|x|^\alpha u_n^{-(p+1)}k_n^2$$

This implies that for any n, the first eigenvalue $\mu_{1,n}$ of (52) is negative. Therefore $m(u_n) \geq 1$.

To obtain the conclusion of Theorem 4.5, we only need to show that $|\mu_{1,n}| \leq C$. First, for near $r = 0$, it follows from Lemma 4.4 that

$$\begin{aligned} u_*^{-(p+1)}(r) &= \lambda_*^{-1}r^{-(2+\alpha)}\Big(\Lambda - B\lambda_*^{\frac{p}{p+1}}r^{p\delta} + o(r^{p\delta})\Big)^{-(p+1)} \\ &= \lambda_*^{-1}r^{-(2+\alpha)}\Big(\Lambda^{-(p+1)} + (p+1)B\lambda_*^{\frac{p}{p+1}}\Lambda^{-(p+2)}r^{p\delta} + o(r^{p\delta})\Big). \end{aligned}$$

That is, for r near 0

$$\lambda_* r^\alpha u_*^{-(p+1)} = r^{-2}\Big(\Lambda^{-(p+1)} + (p+1)B\lambda_*^{\frac{p}{p+1}}\Lambda^{-(p+2)}r^{p\delta} + o(r^{p\delta})\Big). \tag{53}$$

Since u_* is increasing in $(0,1)$, there exists a small $\epsilon > 0$ such that

$$\lambda_* r^\alpha u_*^{-(p+1)}(r) \leq \begin{cases} r^{-2}\Big(\Lambda^{-(p+1)} + (p+2)B\lambda_*^{\frac{p}{p+1}}\Lambda^{-(p+2)}r^{p\delta}\Big) & \text{for } r \in (0,\epsilon) \\ C_* & \text{for } r \in [\epsilon, 1) \end{cases} \tag{54}$$

where $C_* > 0$ depends on ϵ and λ_*. Therefore for $r \in (0,1)$,

$$\lambda_* r^\alpha u_*^{-(p+1)} \leq r^{-2}\Big(\Lambda^{-(p+1)} + (p+2)B\lambda_*^{\frac{p}{p+1}}\Lambda^{-(p+2)}r^{p\delta}\Big) + C_*. \tag{55}$$

On the other hand, we have that

$$p\Lambda^{-(p+1)} \le \frac{(N-2)^2}{4} \quad \text{for } 0 < p \le p_c(\alpha)$$

and

$$\int_B |\nabla h|^2 - \frac{(N-2)^2}{4} \int_B \frac{h^2}{|x|^2} \ge C \int_B \frac{h^2}{|x|^2(\ln R/|x|)^2} \tag{56}$$

for every $h \in H_0^1(B)$ (see [1]), where $C > 0$ is a constant and $R \ge e$. Therefore for any $h \in H_0^1(B)$,

$$\begin{aligned}
&\int_B [|\nabla h|^2 - p\lambda_* r^\alpha u_*^{-(p+1)} h^2] \\
&\ge \int_B \Big[|\nabla h|^2 - p\Lambda^{-(p+1)} \frac{h^2}{r^2} - r^{-2}\Big((p+2)B\lambda_*^{\frac{p}{p+1}} \Lambda^{-(p+2)} r^{p\delta}\Big) h^2 - C_{**} h^2\Big] \\
&\ge \int_B \Big[|\nabla h|^2 - \frac{(N-2)^2}{4} \frac{h^2}{r^2} - r^{-2}\Big((p+2)B\lambda_*^{\frac{p}{p+1}} \Lambda^{-(p+2)} r^{p\delta}\Big) h^2 - C_{**} h^2\Big] \\
&\ge \int_B \Big[\frac{Ch^2}{r^2(\ln R/r)^2} - r^{-2}\Big((p+2)B\lambda_*^{\frac{p}{p+1}} \Lambda^{-(p+2)} r^{p\delta}\Big) h^2\Big] - C_{**} \int_B h^2 \\
&= \int_B \Big[\frac{C}{(\ln R/r)^2} - \Big((p+2)B\lambda_*^{\frac{p}{p+1}} \Lambda^{-(p+2)} r^{p\delta}\Big)\Big] \frac{h^2}{r^2} - C_{**} \int_B h^2 \\
&= \Big\{\int_{B_\epsilon} + \int_{B \setminus B_\epsilon}\Big\} \Big[\frac{C}{(\ln R/r)^2} - \Big((p+2)B\lambda_*^{\frac{p}{p+1}} \Lambda^{-(p+2)} r^{p\delta}\Big)\Big] \frac{h^2}{r^2} - C_{**} \int_B h^2 \\
&\ge C^* \int_{B_\epsilon} h^2 + \int_{B \setminus B_\epsilon} \Big[\frac{C}{(\ln R/r)^2} - \Big((p+2)B\lambda_*^{\frac{p}{p+1}} \Lambda^{-(p+2)} r^{p\delta}\Big)\Big] \frac{h^2}{r^2} - C_{**} \int_B h^2 \\
&\ge -C^{**} \int_B h^2
\end{aligned}$$

where $C^*, C^{**} > 0$ depend on C_*, ϵ and λ_* but are independent of h. Thus the first eigenvalue μ_* of the problem

$$-\Delta h - p\lambda_* |x|^\alpha u_*^{-(p+1)} h = \mu h \ \text{ in } B, \quad h = 0 \ \text{ on } \partial B. \tag{57}$$

satisfies $\mu_* \ge -C^{**}$. Since $u_n \to u_*$ in $B\setminus\{0\}$ and $\lambda_n \to \lambda_*$, we see that $\mu_{1,n} \to \mu_*$ as $n \to \infty$. Therefore, $|\mu_{1,n}| \le 2C^{**}$ (note that $\mu_{1,n} < 0$). This implies that there exists an integer $C \ge 1$ such that $1 \le m(u_*) \le C$.

The proof for a regular solution (λ, u_λ) case is similar. Note that for $r \in (0, \epsilon)$,

$$u_\lambda(r) \ge \lambda^{1/(p+1)} \Lambda r^\delta. \tag{58}$$

Since u_λ is increasing in $(0,1)$, we see that for $r \in (0,1)$,

$$p\lambda r^\alpha u_\lambda^{-(p+1)} \le p\Lambda^{-(p+1)} r^{-2} + C^{***} \tag{59}$$

where $C^{***} > 0$ depends on σ_1 and ϵ but is independent of λ. We deduce with the same conclusion. □

Corollary 1. *For $0 < p \le p_c(\alpha)$, the graph of any regular solution u_λ intersects with that of u_* finitely many times in $(0,1)$. Moreover, the graphs of any two different regular solutions intersect finitely many times in $(0,1)$.*

Proof. We only prove the first conclusion, the second one is obtained similarly.

By contradiction, suppose that the graph of u_λ intersects with that of u_* infinitely many times. Then we can find infinitely many intervals J_i $(i = 1, 2, \dots)$ such that

$u_* < u_\lambda$ in J_i. We first consider the case $\lambda \geq \lambda_*$. Setting $h_i = u_* - u_\lambda$, we see that $h_i < 0$ in J_i and $h_i = 0$ on ∂J_i. Moreover,

$$\begin{aligned} \Delta h_i &= \lambda_* r^\alpha [u_*^{-p} - 1] - \lambda r^\alpha [u_\lambda^{-p} - 1] \\ &\leq \lambda_* r^\alpha [u_*^{-p} - u_\lambda^{-p}] \\ &= -p\lambda_* r^\alpha \xi^{-(p+1)} h_i \\ &< -p\lambda_* r^\alpha u_*^{-(p+1)} h_i. \end{aligned}$$

Therefore we have

$$\int_{B_i} [|\nabla h_i|^2 - \lambda_* |x|^\alpha u_*^{-(p+1)} h_i^2] < 0$$

where $B_i = \{x : \ |x| \in J_i\}$. This implies that $m(u_*) = \infty$, a contradiction.

For the case $\lambda < \lambda_*$, we can find infinitely many intervals K_i ($i = 1, 2, \ldots$) such that $u_* > u_\lambda$ in K_i. Thus $h_i > 0$ in K_i and $h_i = 0$ on ∂K_i. Moreover, one has

$$\Delta h_i \geq \lambda r^\alpha [u_*^{-p} - u_\lambda^{-p}] > -p\lambda r^\alpha u_\lambda^{-(p+1)} h_i,$$

which implies

$$\int_{\tilde{B}_i} [|\nabla h_i|^2 - p\lambda |x|^\alpha u_\lambda^{-(p+1)} h_i^2] < 0$$

where $\tilde{B}_i = \{x : \ |x| \in K_i\}$. This means that $m(u_\lambda) = \infty$, a contradiction again. $\square$

Remark 2. We can also consider the problem

$$(T_\lambda^\kappa) \qquad \begin{cases} \Delta u = \lambda |x|^\alpha f(u) & \text{in } \Omega, \\ 0 < u < \kappa & \text{in } \Omega, \\ u = \kappa & \text{on } \partial\Omega \end{cases}$$

with $0 < \kappa < 1$. Under the assumptions (G_1) and (G_2), we see that $f(\kappa) > 0$. Arguments similar to those in the proof of Theorem 3.1 imply that the corresponding problem (S_λ^κ) has an unlimited positive solution branch starting from $(0, 0)$, that has infinitely many bifurcation points, provided $p > p_c(\alpha)$ or $p > p_c(0)$. When $\Omega = B$ and $g(s) = (\kappa - s)^{-p} - 1$ with $0 < p \leq p_c(\alpha)$, the behavior of the solution branch depends on κ. If κ is close to 1, the solution branch is close to the branch of the case $\kappa = 1$. This implies that the Morse index of any solution in this branch is finite.

Remark 3. The main conclusions of Theorems 4.3, 4.4 and 4.5 still hold for the radial solution branch of the problem

$$\Delta u = \lambda |x|^\alpha [u^{-p} - u^{-q}], \quad 0 < u < 1 \ \text{ in } B, \quad u = 1 \ \text{ on } \partial B \tag{60}$$

where $\alpha \geq 0$ and $0 < q < p$. We know that (60) has an unlimited radial solution branch Γ starting from $(\sigma_1/(p - q), 1)$ and that $u \equiv 1$ is a trivial solution to (60). The conclusions as in Theorems 4.3, 4.4 and 4.5 can be obtained by variants of the proofs as above. When Ω is an annulus, the structure of radial solution branch of (60) can be found in [19].

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