
ENTIRE SOLUTIONS OF THE ALLEN-CAHN EQUATION AND COMPLETE EMBEDDED MINIMAL SURFACES OF FINITE TOTAL CURVATURE IN \mathbb{R}^3

by

Manuel del Pino, Michal Kowalczyk & Juncheng Wei

Abstract. — We consider minimal surfaces M which are complete, embedded and have finite total curvature in \mathbb{R}^3 , and bounded, entire solutions with finite Morse index of the Allen-Cahn equation $\Delta u + f(u) = 0$ in \mathbb{R}^3 . Here $f = -W'$ with W bistable and balanced, for instance $W(u) = \frac{1}{4}(1 - u^2)^2$. We assume that M has $m \geq 2$ ends, and additionally that M is non-degenerate, in the sense that its bounded Jacobi fields are all originated from rigid motions (this is known for instance for a Catenoid and for the Costa-Hoffman-Meeks surface of any genus). We prove that for any small $\alpha > 0$, the Allen-Cahn equation has a family of bounded solutions depending on $m-1$ parameters distinct from rigid motions, whose level sets are embedded surfaces lying close to the blown-up surface $M_\alpha := \alpha^{-1}M$, with ends possibly diverging logarithmically from M_α . We prove that these solutions are L^∞ -non-degenerate up to rigid motions, and find that their Morse index coincides with the index of the minimal surface. Our construction suggests parallels of De Giorgi conjecture for general bounded solutions of finite Morse index.

Contents

1. <i>Introduction and main results</i>	2
2. <i>The Laplacian near M_α</i>	8
3. <i>Approximation of the solution and preliminary discussion</i>	11
4. <i>The proof of Theorem 1</i>	17
5. <i>The linearized operator</i>	24
6. <i>The Jacobi operator</i>	30
7. <i>Reducing the gluing system and solving the projected problem</i> ..	36
8. <i>The reduced problem: proof of Proposition 4.4</i>	41
9. <i>Conclusion of the proof of Theorem 1</i>	44
10. <i>Negative eigenvalues and their eigenfunctions for the Jacobi operator</i>	48

2000 Mathematics Subject Classification. — 35J25, 35J20, 35B33, 35B40.

Key words and phrases. — Minimal embedded surfaces of finite total curvature, Allen-Cahn equation, Jacobi operator, Jacobi fields, Morse index.

11. <i>The proof of Theorem 2</i>	52
12. <i>Further comments and open questions</i>	72
13. <i>Appendix</i>	73
References	77

1. Introduction and main results

1.1. The Allen-Cahn equation and minimal surfaces. — The Allen-Cahn equation in \mathbb{R}^N is the semilinear elliptic problem

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

where $f(s) = -W'(s)$ and W is a “double-well potential”, bi-stable and balanced, namely

$$W(s) > 0 \text{ if } s \neq 1, -1, \quad W(1) = 0 = W(-1), \quad W''(\pm 1) = f'(\pm 1) =: \sigma_{\pm}^2 > 0. \quad (1.2)$$

A typical example of such a nonlinearity is

$$f(u) = (1 - u^2)u \quad \text{for } W(u) = \frac{1}{4}(1 - u^2)^2, \quad (1.3)$$

while we will not make use of the special symmetries enjoyed by this example.

Equation (1.1) is a prototype for the continuous modeling of phase transition phenomena. Let us consider the energy in a subregion Ω of \mathbb{R}^N

$$J_{\alpha}(v) = \int_{\Omega} \frac{\alpha}{2} |\nabla v|^2 + \frac{1}{4\alpha} W(v),$$

whose Euler-Lagrange equation is a scaled version of (1.1),

$$\alpha^2 \Delta v + f(v) = 0 \quad \text{in } \Omega. \quad (1.4)$$

We observe that the constant functions $u = \pm 1$ minimize J_{α} . They are idealized as two *stable phases* of a material in Ω . It is of interest to analyze stationary configurations in which the two phases coexist. Given any subset Λ of Ω , any discontinuous function of the form

$$v_* = \chi_{\Lambda} - \chi_{\Omega \setminus \Lambda} \quad (1.5)$$

minimizes the second term in J_{α} . The introduction of the gradient term in J_{α} makes an α -regularization of u_* a test function for which the energy gets bounded and proportional to the surface area of the *interface* $M = \partial\Lambda$, so that in addition to minimizing approximately the second term, stationary configurations should also select asymptotically interfaces M that are stationary for surface area, namely (generalized) minimal surfaces. This intuition on the Allen-Cahn equation gave important impulse to the calculus of variations, motivating the development of the theory of Γ -convergence in the 1970's. Modica [27] proved that a family of local minimizers u_{α} of J_{α} with uniformly bounded energy must converge in suitable sense to a function of the form (1.5) where $\partial\Lambda$ minimizes perimeter. Thus, intuitively, for each given $\lambda \in (-1, 1)$, the level sets $[v_{\alpha} = \lambda]$, collapse as $\alpha \rightarrow 0$ onto the interface $\partial\Lambda$. Similar

result holds for critical points not necessarily minimizers, see [23]. For minimizers this convergence is known in very strong sense, see [2, 3].

If, on the other hand, we take such a critical point u_α and scale it around an interior point $0 \in \Omega$, setting $u_\alpha(x) = v_\alpha(\alpha x)$, then u_α satisfies equation (1.1) in an expanding domain,

$$\Delta u_\alpha + f(u_\alpha) = 0 \quad \text{in } \alpha^{-1}\Omega$$

so that letting formally $\alpha \rightarrow 0$ we end up with equation (1.1) in entire space. The “interface” for u_α should thus be around the (asymptotically flat) minimal surface $M_\alpha = \alpha^{-1}M$. Modica’s result is based on the intuition that if M happens to be a smooth surface, then the transition from the equilibria -1 to 1 of u_α along the normal direction should take place in the approximate form $u_\alpha(x) \approx w(z)$ where z designates the normal coordinate to M_α . Then w should solve the ODE problem

$$w'' + f(w) = 0 \quad \text{in } \mathbb{R}, \quad w(-\infty) = -1, \quad w(+\infty) = 1. \quad (1.6)$$

This solution indeed exists thanks to assumption (1.2). It is strictly increasing and unique up to constant translations. We fix in what follows the unique w for which

$$\int_{\mathbb{R}} t w'(t)^2 dt = 0. \quad (1.7)$$

For example (1.3), we have $w(t) = \tanh(t/\sqrt{2})$. In general w approaches its limits at exponential rates,

$$w(t) - \pm 1 = O(e^{-\sigma \pm |t|}) \quad \text{as } t \rightarrow \pm\infty.$$

Observe then that

$$J_\alpha(u_\alpha) \approx \text{Area}(M) \int_{\mathbb{R}} \left[\frac{1}{2} w'^2 + W(w) \right]$$

which is what makes it plausible that M is critical for area, namely a minimal surface.

The above considerations led E. De Giorgi [9] to formulate in 1978 a celebrated conjecture on the Allen-Cahn equation (1.1), parallel to Bernstein’s theorem for minimal surfaces: The level sets $[u = \lambda]$ of a bounded entire solution u to (1.1), which is also monotone in one direction, must be hyperplanes, at least for dimension $N \leq 8$. Equivalently, up to a translation and a rotation, $u = w(x_1)$. This conjecture has been proven in dimensions $N = 2$ by Ghoussoub and Gui [13], $N = 3$ by Ambrosio and Cabré [1], and under a mild additional assumption by Savin [34]. A counterexample was recently built for $N \geq 9$ in [10, 11], see also [6, 24]. See [12] for a recent survey on the state of the art of this question.

The assumption of monotonicity in one direction for the solution u in De Giorgi conjecture implies a form of stability, locally minimizing character for u when compactly supported perturbations are considered in the energy. Indeed, if $Z = \partial_{x_N} u > 0$, then the linearized operator $L = \Delta + f'(u)$, satisfies maximum principle. This implies stability of u , in the sense that its associated quadratic form, namely the second variation of the corresponding energy,

$$\mathcal{Q}(\psi, \psi) := \int |\nabla \psi|^2 - f'(u) \psi^2 \quad (1.8)$$

satisfies $\mathcal{Q}(\psi, \psi) > 0$ for all $\psi \neq 0$ smooth and compactly supported. Stability is a basic ingredient in the proof of the conjecture dimensions 2, 3 in [1, 13], based on finding a control at infinity of the growth of the Dirichlet integral. In dimension $N = 3$ it turns out that

$$\int_{B(0,R)} |\nabla u|^2 = O(R^2) \quad (1.9)$$

which intuitively means that the embedded level surfaces $[u = \lambda]$ must have a finite number of components outside a large ball, which are all “asymptotically flat”. The question whether stability alone suffices for property (1.9) remains open. More generally, it is believed that this property is equivalent to *finite Morse index* of the solution u (which means essentially that u is stable outside a bounded set). The Morse index $m(u)$ is defined as the maximal dimension of a vector space E of compactly supported functions such that

$$\mathcal{Q}(\psi, \psi) < 0 \quad \text{for all } \psi \in E \setminus \{0\}.$$

Rather surprisingly, basically no examples of finite Morse index entire solutions of the Allen-Cahn equation seem known in dimension $N = 3$. Great progress has been achieved in the last decades, both in the theory of semilinear elliptic PDE like (1.1) and in minimal surface theory in \mathbb{R}^3 . While this link traces back to the very origins of the study of (1.1) as discussed above, it has only been partially explored in producing new solutions.

In this paper we construct a new class of entire solutions to the Allen-Cahn equation in \mathbb{R}^3 which have the characteristic (1.9), and also finite Morse index, whose level sets resemble a large dilation of a given complete, embedded minimal surface M , asymptotically flat in the sense that it has *finite total curvature*, namely

$$\int_M |K| dV < +\infty$$

where K denotes Gauss curvature of the manifold, which is also *non-degenerate* in a sense that we will make precise below.

As pointed out by Dancer [7], Morse index is a natural element to attempt classification of solutions of (1.1). Beyond De Giorgi conjecture, classifying solutions with given Morse index should be a natural step towards the understanding of the structure of the bounded solutions of (1.1). Our main results show that, unlike the stable case, the structure of the set of solutions with finite Morse index is highly complex. On the other hand, we believe that our construction contains germs of generality, providing elements to extrapolate what may be true in general, in analogy with classification of embedded minimal surfaces. We elaborate on these issues in §12.

1.2. Embedded minimal surfaces of finite total curvature. — The theory of embedded, minimal surfaces of finite total curvature in \mathbb{R}^3 , has reached a notable development in the last 25 years. For more than a century, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa, [4, 5]. The *Costa surface* is a genus one minimal

surface, complete and properly embedded, which outside a large ball has exactly three components (its *ends*), two of which are asymptotically catenoids with the same axis and opposite directions, the third one asymptotic to a plane perpendicular to that axis. The complete proof of embeddedness is due to Hoffman and Meeks [18]. In [19, 21] these authors generalized notably Costa's example by exhibiting a class of three-end, embedded minimal surface, with the same look as Costa's far away, but with an array of tunnels that provides arbitrary genus $k \geq 1$. This is known as the Costa-Hoffman-Meeks surface with genus k .

Many other examples of multiple-end embedded minimal surfaces have been found since, see for instance [25, 36] and references therein. In general all these surfaces look like parallel planes, slightly perturbed at their ends by asymptotically logarithmic corrections with a certain number of catenoidal links connecting their adjacent sheets. In reality this intuitive picture is not a coincidence. Using the Eneper-Weierstrass representation, Osserman [31] established that any embedded, complete minimal surface with finite total curvature can be described by a conformal diffeomorphism of a compact surface (actually of a Riemann surface), with a finite number of its points removed. These points correspond to the ends. Moreover, after a convenient rotation, the ends are asymptotically all either catenoids or plane, all of them with parallel axes, see Schoen [35]. The topology of the surface is thus characterized by the genus of the compact surface and the number of ends, having therefore "finite topology".

1.3. Main results. — In what follows M designates a complete, embedded minimal surface in \mathbb{R}^3 with finite total curvature (to which below we will make a further nondegeneracy assumption). As pointed out in [22], M is orientable and the set $\mathbb{R}^3 \setminus M$ has exactly two components S_+ , S_- . In what follows we fix a continuous choice of unit normal field $\nu(y)$, which conventionally we take it to point towards S_+ .

For $x = (x_1, x_2, x_3) = (x', x_3) \in \mathbb{R}^3$, we denote

$$r = r(x) = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}.$$

After a suitable rotation of the coordinate axes, outside the infinite cylinder $r < R_0$ with sufficiently large radius R_0 , then M decomposes into a finite number m of unbounded components M_1, \dots, M_m , its *ends*. From a result in [35], we know that asymptotically each end of M_k either resembles a plane or a catenoid. More precisely, M_k can be represented as the graph of a function F_k of the first two variables,

$$M_k = \{ y \in \mathbb{R}^3 / r(y) > R_0, y_3 = F_k(y') \}$$

where F_k is a smooth function which can be expanded as

$$F_k(y') = a_k \log r + b_k + b_{ik} \frac{y_i}{r^2} + O(r^{-3}) \quad \text{as } r \rightarrow +\infty, \quad (1.10)$$

for certain constants a_k, b_k, b_{ik} , and this relation can also be differentiated. Here

$$a_1 \leq a_2 \leq \dots \leq a_m, \quad \sum_{k=1}^m a_k = 0. \quad (1.11)$$

The direction of the normal vector $\nu(y)$ for large $r(y)$ approaches on the ends that of the x_3 axis, with alternate signs. We use the convention that for $r(y)$ large we have

$$\nu(y) = \frac{(-1)^k}{\sqrt{1 + |\nabla F_k(y')|^2}} (\nabla F_k(y'), -1) \quad \text{if } y \in M_k. \quad (1.12)$$

Let us consider the Jacobi operator of M

$$\mathcal{J}(h) := \Delta_M h + |A|^2 h \quad (1.13)$$

where $|A|^2 = -2K$ is the Euclidean norm of the second fundamental form of M . \mathcal{J} is the linearization of the mean curvature operator with respect to perturbations of M measured along its normal direction. A smooth function $z(y)$ defined on M is called a *Jacobi field* if $\mathcal{J}(z) = 0$. Rigid motions of the surface induce naturally some bounded Jacobi fields: Associated to respectively translations along coordinates axes and rotation around the x_3 -axis, are the functions

$$\begin{aligned} z_1(y) &= \nu(y) \cdot e_i, \quad y \in M, \quad i = 1, 2, 3, \\ z_4(y) &= (-y_2, y_1, 0) \cdot \nu(y), \quad y \in M. \end{aligned} \quad (1.14)$$

We assume that M is *non-degenerate* in the sense that these functions are actually *all* the bounded Jacobi fields, namely

$$\{ z \in L^\infty(M) / \mathcal{J}(z) = 0 \} = \text{span} \{ z_1, z_2, z_3, z_4 \}. \quad (1.15)$$

We denote in what follows by J the dimension (≤ 4) of the above vector space.

This assumption, expected to be generic for this class of surfaces, is known in some important cases, most notably the catenoid and the Costa-Hoffmann-Meeks surface which is an example of a three ended M whose genus may be of any order. See Nayatani [29, 30] and Morabito [28]. Note that for a catenoid, $z_{04} = 0$ so that $J = 3$. Non-degeneracy has been used as a tool to build new minimal surfaces for instance in Hauswirth and Pacard [17], and in Pérez and Ros [33]. It is also the basic element, in a compact-manifold version, to build solutions to the small-parameter Allen-Cahn equation in Pacard and Ritoré [32].

In this paper we will construct a solution to the Allen Cahn equation whose zero level sets look like a large dilation of the surface M , with ends perturbed logarithmically. Let us consider a large dilation of M ,

$$M_\alpha := \alpha^{-1} M.$$

This dilated minimal surface has ends parameterized as

$$M_{k,\alpha} = \{ y \in \mathbb{R}^3 / r(\alpha y) > R_0, y_3 = \alpha^{-1} F_k(\alpha y') \}.$$

Let β be a vector of given m real numbers with

$$\beta = (\beta_1, \dots, \beta_m), \quad \sum_{i=1}^m \beta_i = 0. \quad (1.16)$$

Our first result asserts the existence of a solution $u = u_\alpha$ defined for all sufficiently small $\alpha > 0$ such that given $\lambda \in (-1, 1)$, its level set $[u_\alpha = \lambda]$ defines an embedded surface lying at a uniformly bounded distance in α from the surface M_α , for points

with $r(\alpha y) = O(1)$, while its k -th end, $k = 1, \dots, m$, lies at a uniformly bounded distance from the graph

$$r(\alpha y) > R_0, \quad y_3 = \alpha^{-1} F_k(\alpha y') + \beta_k \log |\alpha y'|. \quad (1.17)$$

The parameters β must satisfy an additional constraint. It is clear that if two ends are parallel, say $a_{k+1} = a_k$, we need at least that $\beta_{k+1} - \beta_k \geq 0$, for otherwise the ends would eventually intersect. Our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate. We require

$$\beta_{k+1} - \beta_k > 4 \max \{ \sigma_-^{-1}, \sigma_+^{-1} \} \quad \text{if } a_{k+1} = a_k. \quad (1.18)$$

Let us consider the smooth map

$$X(y, z) = y + z\nu(\alpha y), \quad (y, t) \in M_\alpha \times \mathbb{R}. \quad (1.19)$$

$x = X(y, z)$ defines coordinates inside the image of any region where the map is one-to-one. In particular, let us consider a function $p(y)$ with

$$p(y) = (-1)^k \beta_k \log |\alpha y'| + O(1), \quad k = 1, \dots, m,$$

and β satisfying $\beta_{k+1} - \beta_k > \gamma > 0$ for all k with $a_k = a_{k+1}$. Then the map X is one-to-one for all small α in the region of points (y, z) with

$$|z - q(y)| < \frac{\delta}{\alpha} + \gamma \log(1 + |\alpha y'|)$$

provided that $\delta > 0$ is chosen sufficiently small.

Theorem 1. — *Let $N = 3$ and M be a minimal surface embedded, complete with finite total curvature which is nondegenerate. Then, given β satisfying relations (1.16) and (1.18), there exists a bounded solution u_α of equation (1.1), defined for all sufficiently small α , such that*

$$u_\alpha(x) = w(z - q(y)) + O(\alpha) \quad \text{for all } x = y + z\nu(\alpha y), \quad |z - q(y)| < \frac{\delta}{\alpha}, \quad (1.20)$$

where the function q satisfies

$$q(y) = (-1)^k \beta_k \log |\alpha y'| + O(1) \quad y \in M_{k,\alpha}, \quad k = 1, \dots, m.$$

In particular, for each given $\lambda \in (-1, 1)$, the level set $[u_\alpha = \lambda]$ is an embedded surface that decomposes for all sufficiently small α into m disjoint components (ends) outside a bounded set. The k -th end lies at $O(1)$ distance from the graph

$$y_3 = \alpha^{-1} F_k(\alpha y) + \beta_k \log |\alpha y'|.$$

The solution predicted by this theorem depends, for fixed α , on m parameters. Taking into account the constraint $\sum_{j=1}^m \beta_j = 0$ this gives $m - 1$ independent parameters corresponding to logarithmic twisting of the ends of the level sets. Let us observe that consistently, the combination $\beta \in \text{Span} \{(a_1, \dots, a_m)\}$ can be set in correspondence with moving α itself, namely with a dilation parameter of the surface. We are thus left with $m - 2$ parameters for the solution in addition to α . Thus, besides the trivial rigid motions of the solution, translation along the coordinates axes, and rotation about the x_3 axis, this family of solutions depends exactly on $m - 1$

“independent” parameters. Part of the conclusion of our second result is that the bounded kernel of the linearization of equation (1.1) about one of these solutions is made up exactly of the generators of the rigid motions, so that in some sense the solutions found are L^∞ -isolated, and the set of bounded solutions nearby is actually $m - 1 + J$ -dimensional. A result parallel to this one, in which the *moduli space* of the minimal surface M is described by a similar number of parameters, is found in [33].

Next we discuss the connection of the Morse index of the solutions of Theorem 1 and the *index* of the minimal surface M , $i(M)$, which has a similar definition relative to the quadratic form for the Jacobi operator: The number $i(M)$ is the largest dimension for a vector space E of compactly supported smooth functions in M with

$$\int_M |\nabla k|^2 dV - \int_M |A|^2 k^2 dV < 0 \quad \text{for all } k \in E \setminus \{0\}.$$

We point out that for complete, embedded surfaces, finite index is equivalent to finite total curvature, see [16] and also §7 of [22] and references therein. Thus, for our surface M , $i(M)$ is indeed finite. Moreover, in the Costa-Hoffmann-Meeks surface it is known that $i(M) = 2l - 1$ where l is the genus of M . See [29], [30] and [28].

Our second result is that the Morse index and non-degeneracy of M are transmitted into the linearization of equation (1.1).

Theorem 2. — *Let u_α the solution of problem (1.1) given by Theorem 1. Then for all sufficiently small α we have*

$$m(u_\alpha) = i(M).$$

Besides, the solution is non-degenerate, in the sense that any bounded solution of

$$\Delta\phi + f'(u_\alpha)\phi = 0 \quad \text{in } \mathbb{R}^3$$

must be a linear combination of the functions Z_i , $i = 1, 2, 3, 4$ defined as

$$Z_i = \partial_i u_\alpha, \quad i = 1, 2, 3, \quad Z_4 = -x_2 \partial_1 u_\alpha + x_1 \partial_2 u_\alpha.$$

We will devote the rest of this paper to the proofs of Theorems 1 and 2.

2. The Laplacian near M_α

2.1. The Laplace-Beltrami Operator of M_α . — Let D be the set

$$D = \{y \in \mathbb{R}^2 \mid |y| > R_0\}.$$

We can parameterize the end M_k of M as

$$y \in D \longmapsto y := Y_k(y) = y_i e_i + F_k(y) e_3. \quad (2.1)$$

and F_k is the function in (1.10). In other words, for $y = (y', y_3) \in M_k$ the coordinate y is just defined as $y = y'$. We want to represent Δ_M —the Laplace-Beltrami operator of M —with respect to these coordinates. For the coefficients of the metric g_{ij} on M_k we have

$$\partial_{y_i} Y_k = e_i + O(r^{-1}) e_3$$

so that

$$g_{ij}(\mathbf{y}) = \langle \partial_i Y_k, \partial_j Y_k \rangle = \delta_{ij} + O(r^{-2}), \quad (2.2)$$

where $r = |\mathbf{y}|$. The above relations “can be differentiated” in the sense that differentiation makes the terms $O(r^{-j})$ gain corresponding negative powers of r . Then we find the representation

$$\Delta_M = \frac{1}{\sqrt{\det g_{ij}}} \partial_i (\sqrt{\det g_{ij}} g^{ij} \partial_j) = \Delta_{\mathbf{y}} + O(r^{-2}) \partial_{ij} + O(r^{-3}) \partial_i \quad \text{on } M_k. \quad (2.3)$$

The normal vector to M at $y \in M_k$ $k = 1, \dots, m$, corresponds to

$$\nu(y) = (-1)^k \frac{1}{\sqrt{1 + |\nabla F_k(\mathbf{y})|^2}} (\partial_i F_k(\mathbf{y}) e_i - e_3), \quad y = Y_k(\mathbf{y}) \in M_k$$

so that

$$\nu(y) = (-1)^k e_3 + \alpha_k r^{-2} \mathbf{y}_i e_i + O(r^{-2}), \quad y = Y_k(\mathbf{y}) \in M_k. \quad (2.4)$$

Let us observe for later reference that since $\partial_i \nu = O(r^{-2})$, then the principal curvatures of M , k_1, k_2 satisfy $k_i = O(r^{-2})$. In particular, we have that

$$|A(y)|^2 = k_1^2 + k_2^2 = O(r^{-4}). \quad (2.5)$$

To describe the entire manifold M we consider a finite number $N \geq m + 1$ of local parametrizations

$$\mathbf{y} \in \mathcal{U}_k \subset \mathbb{R}^2 \mapsto y = Y_k(\mathbf{y}), \quad Y_k \in C^\infty(\bar{\mathcal{U}}_k), \quad k = 1, \dots, N. \quad (2.6)$$

For $k = 1, \dots, m$ we choose them to be those in (2.1), with $\mathcal{U}_k = D$, so that $Y_k(\mathcal{U}_k) = M_k$, and $\bar{\mathcal{U}}_k$ is bounded for $k = m + 1, \dots, N$. We require then that

$$M = \bigcup_{k=1}^N Y_k(\mathcal{U}_k).$$

We remark that the Weierstrass representation of M implies that we can actually take $N = m + 1$, namely only one extra parametrization is needed to describe the bounded complement of the ends in M . We will not use this fact. In general, we represent for $y \in Y_k(\mathcal{U}_k)$,

$$\Delta_M = a_{ij}^0(y) \partial_{ij} + b_i^0(y) \partial_i, \quad y = Y_k(\mathbf{y}), \quad y \in \mathcal{U}_k, \quad (2.7)$$

where a_{ij}^0 is a uniformly elliptic matrix and the index k is not made explicit in the coefficients. For $k = 1, \dots, m$ we have

$$a_{ij}^0(y) = \delta_{ij} + O(r^{-2}), \quad b_i^0 = O(r^{-3}), \quad \text{as } r(y) = |\mathbf{y}| \rightarrow \infty. \quad (2.8)$$

The parametrizations set up above induce naturally a description of the expanded manifold $M_\alpha = \alpha^{-1} M$ as follows. Let us consider the functions

$$Y_{k\alpha} : \mathcal{U}_{k\alpha} := \alpha^{-1} \mathcal{U}_k \rightarrow M_\alpha, \quad \mathbf{y} \mapsto Y_{k\alpha}(\mathbf{y}) := \alpha^{-1} Y_k(\alpha \mathbf{y}), \quad k = 1, \dots, N. \quad (2.9)$$

Obviously we have

$$M_\alpha = \bigcup_{k=1}^N Y_{k\alpha}(\mathcal{U}_{k\alpha}).$$

The computations above lead to the following representation for the operator Δ_{M_α} :

$$\Delta_{M_\alpha} = a_{ij}^0(\alpha y) \partial_{ij} + b_i^0(\alpha y) \partial_i, \quad y = Y_{k\alpha}(y), \quad y \in \mathcal{U}_{k\alpha}, \quad (2.10)$$

where a_{ij}^0, b_i^0 are the functions in (2.7), so that for $k = 1, \dots, m$ we have

$$a_{ij}^0 = \delta_{ij} + O(r_\alpha^{-2}), \quad b_i^0 = O(r_\alpha^{-3}), \quad \text{as } r_\alpha(y) := |\alpha y| \rightarrow \infty. \quad (2.11)$$

2.2. The Euclidean Laplacian near M_α . — We will describe in coordinates relative to M_α the Euclidean Laplacian Δ_x , $x \in \mathbb{R}^3$, in a setting needed for the proof of our main results. Let us consider a smooth function $h : M \rightarrow \mathbb{R}$, and the smooth map X_h defined as

$$X_h : M_\alpha \times \mathbb{R} \rightarrow \mathbb{R}^3, \quad (y, t) \longmapsto X_h(y, t) := y + (t + h(\alpha y)) \nu(\alpha y) \quad (2.12)$$

where ν is the unit normal vector to M . Let us consider an open subset \mathcal{O} of $M_\alpha \times \mathbb{R}$ and assume that the map $X_h|_{\mathcal{O}}$ is one to one, and that it defines a diffeomorphism onto its image $\mathcal{N} = X_h(\mathcal{O})$. Then

$$x = X_h(y, t), \quad (y, t) \in \mathcal{O},$$

defines smooth coordinates to describe the open set \mathcal{N} in \mathbb{R}^3 . Moreover, the maps

$$x = X_h(Y_{k\alpha}(y), t), \quad (y, t) \in (\mathcal{U}_{k\alpha} \times \mathbb{R}) \cap \mathcal{O}, \quad k = 1, \dots, N,$$

define local coordinates (y, t) to describe the region \mathcal{N} . We shall assume in addition that for certain small number $\delta > 0$, we have

$$\mathcal{O} \subset \{(y, t) / |t + h(\alpha y)| < \frac{\delta}{\alpha} \log(2 + r_\alpha(y))\}. \quad (2.13)$$

We have the validity of the following expression for the Euclidean Laplacian operator in \mathcal{N} .

Lemma 2.1. — *For $x = X_h(y, t)$, $(y, t) \in \mathcal{O}$ with $y = Y_{k\alpha}(y)$, $y \in \mathcal{U}_{k\alpha}$, we have the validity of the identity*

$$\begin{aligned} \Delta_x = & \partial_{tt} + \Delta_{M_\alpha} - \alpha^2[(t+h)|A|^2 + \Delta_M h] \partial_t - 2\alpha a_{ij}^0 \partial_j h \partial_{it} + \\ & \alpha(t+h)[a_{ij}^1 \partial_{ij} - 2\alpha a_{ij}^1 \partial_i h \partial_{jt} + \alpha b_i^1 (\partial_i - \alpha \partial_i h \partial_t)] + \\ & \alpha^3(t+h)^2 b_3^1 \partial_t + \alpha^2[a_{ij}^0 + \alpha(t+h)a_{ij}^1] \partial_i h \partial_j h \partial_{tt}. \end{aligned} \quad (2.14)$$

Here, in agreement with (2.10), $\Delta_{M_\alpha} = a_{ij}^0(\alpha y) \partial_{ij} + b_i^0(\alpha y) \partial_i$.

The functions a_{ij}^1, b_i^1, b_3^1 in the above expressions appear evaluated at the pair $(\alpha y, \alpha(t+h(\alpha y)))$, while the functions $h, \partial_i h, \Delta_M h, |A|^2, a_{ij}^0, b_i^0$ are evaluated at αy . In addition, for $k = 1, \dots, m, l = 0, 1$,

$$a_{ij}^l = \delta_{ij} \delta_{0l} + O(r_\alpha^{-2}), \quad b_i^l = O(r_\alpha^{-3}), \quad b_3^1 = O(r_\alpha^{-6}),$$

as $r_\alpha(y) = |\alpha y| \rightarrow \infty$, uniformly in their second variables. The notation $\partial_j h$ refers to $\partial_j[h \circ Y_k]$.

We postpone proof of this fact for the appendix. The proof actually yields that the coefficients a_{ij}^1 and b_i^1 can be further expanded as follows:

$$a_{ij}^1 = a_{ij}^1(\alpha y, 0) + \alpha(t+h) a_{ij}^{(2)}(\alpha y, \alpha(t+h)) =: a_{ij}^{1,0} + \alpha(t+h) a_{ij}^2,$$

with $a_{ij}^{(2)} = O(r_\alpha^{-3})$, and similarly

$$b_j^1 = b_j^1(\alpha y, 0) + \alpha(t+h) b_j^{(2)}(\alpha y, \alpha(t+h)) =: b_j^{1,0} + \alpha(t+h) b_j^2,$$

with $b_j^{(2)} = O(r_\alpha^{-4})$. As an example of the previous formula, let us compute the Laplacian of a function that separates variables t and y , that will be useful in §3 and §11.

Lemma 2.2. — *Let $v(x) = k(y)\psi(t)$. Then the following holds.*

$$\begin{aligned} \Delta_x v &= k\psi'' + \psi \Delta_{M_\alpha} k - \alpha^2[(t+h)|A|^2 + \Delta_M h] k\psi' - 2\alpha a_{ij}^0 \partial_j h \partial_i k \psi' + \\ &\alpha(t+h) [a_{ij}^{1,0} \partial_{ij} k \psi - 2\alpha a_{ij}^{1,0} \partial_j h \partial_i k \psi' + \alpha(b_i^{1,0} \partial_i k \psi - \alpha b_i^{1,0} \partial_i h k \psi')] + \\ &\alpha^2(t+h)^2 [a_{ij}^2 \partial_{ij} k \psi - 2\alpha a_{ij}^2 \partial_j h \partial_i k \psi' + \alpha(b_i^2 \partial_i k \psi - \alpha b_i^2 \partial_i h k \psi')] + \\ &\alpha^3(t+h)^2 b_3^1 k \psi' + \alpha^2 [a_{ij}^0 + \alpha(t+h) a_{ij}^1] \partial_i h \partial_j h k \psi'' . \end{aligned} \quad (2.15)$$

3. Approximation of the solution and preliminary discussion

3.1. Approximation of order zero and its projection. — Let us consider a function h and sets \mathcal{O} and \mathcal{N} as in §2.2. Let $x = X_h(y, t)$ be the coordinates introduced in (2.12). At this point we shall make a more precise assumption about the function h . We need the following preliminary result whose proof we postpone for §6.2.

We consider a fixed m -tuple of real numbers $\beta = (\beta_1, \dots, \beta_m)$ such that

$$\sum_{i=1}^m \beta_i = 0. \quad (3.1)$$

Lemma 3.1. — *Given any real numbers β_1, \dots, β_m satisfying (3.1), there exists a smooth function $h_0(y)$ defined on M such that*

$$\mathcal{J}(h_0) = \Delta_M h_0 + |A|^2 h_0 = 0 \quad \text{in } M,$$

$$h_0(y) = (-1)^j \beta_j \log r + \theta \quad \text{as } r \rightarrow \infty \quad \text{in } M_j \quad \text{for all } y \in M_j,$$

where θ satisfies

$$\|\theta\|_\infty + \|r^2 D\theta\|_\infty < +\infty. \quad (3.2)$$

We fix a function h_0 as in the above lemma and consider a function h in the form

$$h = h_0 + h_1.$$

We allow h_1 to be a parameter which we will adjust. For now we will assume that for a certain constant \mathcal{K} we have

$$\|h_1\|_{L^\infty(M)} + \|(1+r^2)Dh_1\|_{L^\infty(M)} \leq \mathcal{K}\alpha. \quad (3.3)$$

We want to find a solution to

$$S(u) := \Delta_x u + f(u) = 0.$$

We consider in the region \mathcal{N} the approximation

$$u_0(x) := w(t) = w(z - h_0(\alpha y) - h_1(\alpha y))$$

where z designates the normal coordinate to M_α . Thus, whenever $\beta_j \neq 0$, the level sets $[u_0 = \lambda]$ for a fixed $\lambda \in (-1, 1)$ departs logarithmically from the end $\alpha^{-1}M_j$ being still asymptotically catenoidal, more precisely it is described as the graph

$$y_3 = (\alpha^{-1}a_j + \beta_j) \log r + O(1) \text{ as } r \rightarrow \infty.$$

Note that, just as in the minimal surface case, the coefficients of the ends are balanced in the sense that they add up to zero.

It is clear that if two ends are parallel, say $a_{j+1} = a_j$, we need at least that $\beta_{j+1} - \beta_j \geq 0$, for otherwise the ends of this zero level set would eventually intersect. We recall that our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate:

$$\beta_{j+1} - \beta_j > 4 \max\{\sigma_-^{-1}, \sigma_+^{-1}\} \quad \text{if } a_{j+1} = a_j. \quad (3.4)$$

We will explain later the role of this condition. Let us evaluate the error of approximation $S(u_0)$. Using Lemma 2.2 and the fact that $w'' + f(w) = 0$, we find

$$\begin{aligned} S(u_0) &:= \Delta_x u_0 + f(u_0) = \\ &-\alpha^2[|A|^2 h_1 + \Delta_M h_1] w' + \\ &-\alpha^2 |A|^2 t w' + 2 \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0 w'' + \\ &\alpha^2 a_{ij}^0 (2 \partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_1) w'' + \\ &2 \alpha^3 (t + h_0 + h_1) a_{ij}^1 \partial_i (h_0 + h_1) \partial_j (h_0 + h_1) w'' + \\ &\alpha^3 (t + h_0 + h_1) b_i^1 \partial_i (h_0 + h_1) w' + \alpha^3 (t + h_0 + h_1)^3 b_3^1 w' \end{aligned} \quad (3.5)$$

where the formula above has been broken into ‘‘sizes’’, keeping in mind that h_0 is fixed while $h_1 = O(\alpha)$. Since we want that u_0 be as close as possible to be a solution of (1.1), then we would like to choose h_1 in such a way that the quantity (3.5) be as small as possible. Examining the above expression, it does not look like we can do that in absolute terms. However part of the error could be made smaller by adjusting

h_1 . Let us consider the “ L^2 -projection” onto $w'(t)$ of the error for each fixed y , given by

$$\Pi(y) := \int_{-\infty}^{\infty} S(u_0)(y, t) w'(t) dt$$

where for now, and for simplicity we assume the coordinates are defined for all t , the difference with the integration is taken in all the actual domain for t produces only exponentially small terms in α^{-1} . Then we find

$$\begin{aligned} \Pi(y) = & \alpha^2(\Delta_M h_1 + h_1|A|^2) \int_{-\infty}^{\infty} w'^2 dt + \alpha^3 \partial_i(h_0 + h_1) \int_{-\infty}^{\infty} b_i^1(t + h_0 + h_1) w'^2 dt + \\ & \alpha^3 \partial_i(h_0 + h_1) \partial_j(h_0 + h_1) \int_{-\infty}^{\infty} (t + h_0 + h_1) a_{ij}^1 w'' w' dt + \alpha^3 \int_{-\infty}^{\infty} (t + h_0 + h_1)^3 b_3^1 w'^2 dt \end{aligned} \quad (3.6)$$

where we have used $\int_{-\infty}^{\infty} t w'^2 dt = \int_{-\infty}^{\infty} w'' w' dt = 0$ to get rid in particular of the terms of order α^2 .

Making all these “projections” equal to zero amounts to a nonlinear differential equation for h of the form

$$\mathcal{J}(h_1) = \Delta_M h_1 + h_1|A(y)|^2 = G_0(h_1) \quad y \in M \quad (3.7)$$

where G_0 is easily checked to be a contraction mapping of small constant in h_1 , in the ball radius $O(\alpha)$ with the C^1 norm defined by the expression in the left hand side of inequality (3.3). This is where the nondegeneracy assumption on the Jacobi operator \mathcal{J} enters, since we would like to invert it, in such a way to set up equation (3.7) as a fixed point problem for a contraction mapping of a ball of the form (3.3).

3.2. Improvement of approximation. — The previous considerations are not sufficient since even after adjusting optimally h , the error in absolute value does not necessarily decrease. As we observed, the “large” term in the error,

$$-\alpha^2|A|^2 t w' + \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0 w''$$

did not contribute to the projection. In order to eliminate, or reduce the size of this remaining part $O(\alpha^2)$ of the error, we improve the approximation through the following argument. Let us consider the differential equation

$$\psi_0''(t) + f'(w(t))\psi_0(t) = t w'(t),$$

which has a unique bounded solution with $\psi_0(0) = 0$, given explicitly by the formula

$$\psi_0(t) = w'(t) \int_0^t w'(s)^{-2} \int_{-\infty}^s s w'(s)^2 ds.$$

Observe that this function is well defined and it is bounded since $\int_{-\infty}^{\infty} s w'(s)^2 ds = 0$ and $w'(t) \sim e^{-\sigma_{\pm}|t|}$ as $t \rightarrow \pm\infty$, with $\sigma_{\pm} > 0$. Note also that $\psi_1(t) = \frac{1}{2} t w'(t)$ solves

$$\psi_1''(t) + f'(w(t))\psi_1(t) = w''(t).$$

We consider as a second approximation

$$u_1 = u_0 + \phi_1, \quad \phi_1(y, t) := \alpha^2|A(\alpha y)|^2 \psi_0(t) - \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0(\alpha y) \psi_1(t). \quad (3.8)$$

Let us observe that

$$S(u_0 + \phi) = S(u_0) + \Delta_x \phi + f'(u_0)\phi + N_0(\phi), \quad N_0(\phi) = f(u_0 + \phi) - f(u_0) - f'(u_0)\phi.$$

We have that

$$\partial_{tt}\phi_1 + f'(u_0)\phi_1 = \alpha^2 |A(\alpha y)|^2 t w' - \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0(\alpha y) w''.$$

Hence we get that the largest remaining term in the error is canceled. Indeed, we have

$$S(u_1) = S(u_0) - (2\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0 w'' - \alpha^2 |A(\alpha y)|^2 t w') + [\Delta_x - \partial_{tt}]\phi_1 + N_0(\phi_1).$$

Since ϕ_1 has size of order α^2 , a smooth dependence in αy and it is of size $O(r_\alpha^{-2} e^{-\sigma|t|})$ using Lemma 2.2, we readily check that the “error created”

$$[\Delta_x - \partial_{tt}]\phi_1 + N_0(\phi_1) := -\alpha^4 (|A|^2 t \psi'_0 - a_{ij}^0 \partial_i h_0 \partial_j h_0 t \psi'_1) \Delta h_1 + R_0$$

satisfies

$$|R_0(y, t)| \leq C \alpha^3 (1 + r_\alpha(y))^{-4} e^{-\sigma|t|}.$$

Hence we have eliminated the h_1 -independent term $O(\alpha^2)$ that did not contribute to the projection $\Pi(y)$, and replaced it by one smaller and with faster decay. Let us be slightly more explicit for later reference. We have

$$S(u_1) := \Delta u_1 + f(u_1) =$$

$$-\alpha^2 [|A|^2 h_1 + \Delta_M h_1] w' + \alpha^2 a_{ij}^0 (\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_0 + \partial_i h_1 \partial_j h_1) w''$$

$$-\alpha^4 (|A|^2 t \psi'_0 - a_{ij}^0 \partial_i h_0 \partial_j h_0 t \psi'_1) \Delta_M h_1 + 2\alpha^3 (t + h) a_{ij}^1 \partial_i h \partial_j h w'' + R_1 \quad (3.9)$$

where

$$R_1 = R_1(y, t, h_1(\alpha y), \nabla_M h_1(\alpha y))$$

with

$$|D_i R_1(y, t, \iota, j)| + |D_j R_1(y, t, \iota, j)| + |R_1(y, t, \iota, j)| \leq C \alpha^3 (1 + r_\alpha(y))^{-4} e^{-\sigma|t|}$$

and the constant C above possibly depends on the number \mathcal{K} of condition (3.3).

The above arguments are in reality the way we will actually solve the problem: two separate, but coupled steps are involved: (1) Eliminate the parts of the error that do not contribute to the projection Π and (2) Adjust h_1 so that the projection Π becomes identically zero.

3.3. The condition of diverging ends. — Let us explain the reason to introduce condition (3.4) in the parameters β_j . To fix ideas, let us assume that we have two consecutive planar ends of M , M_j and M_{j+1} , namely with $a_j = a_{j+1}$ and with $d = b_{j+1} - b_j > 0$. Assuming that the normal in M_j points upwards, the coordinate t reads approximately as

$$t = x_3 - \alpha^{-1}b_j - h \quad \text{near } M_{j\alpha}, \quad t = \alpha^{-1}b_{j+1} - x_3 - h \quad \text{near } M_{j+1\alpha}.$$

If we let $h_0 \equiv 0$ both on $M_{j\alpha}$ and $M_{j+1\alpha}$ which are separated at distance d/α , then a good approximation in the entire region between $M_{j\alpha}$ and $M_{j+1\alpha}$ that matches the parts of $w(t)$ coming both from M_j and M_{j+1} should read near M_j approximately as

$$w(t) + w(\alpha^{-1}d - t) - 1.$$

When computing the error of approximation, we observe that the following additional term arises near $M_{j\alpha}$:

$$\begin{aligned} E &:= f(w(t) + w(\alpha^{-1}d - t) - 1) - f(w(t)) - f(w(\alpha^{-1}d - t)) \sim \\ &\sim [f'(w(t)) - f'(1)](w(\alpha^{-1}d - t) - 1). \end{aligned}$$

Now in the computation of the projection of the error this would give rise to

$$\int_{-\infty}^{\infty} [f'(w(t)) - f'(1)](w(\alpha^{-1}d - t) - 1)w'(t) dt \sim c_* e^{-\sigma + \frac{d}{\alpha}}.$$

where $c_* \neq 0$ is a constant. Thus equation (3.7) for h_1 gets modified with a term which even though very tiny, it has no decay as $|y| \rightarrow +\infty$ on M_j , unlike the others involved in the operator G_0 in (3.7). That term eventually dominates and the equation for h_1 for very large r would read in M_j as

$$\Delta_M h_1 \sim e^{-\frac{\sigma}{\alpha}} \neq 0,$$

which is inconsistent with the assumption that h is bounded. Worse yet, its solution would be quadratic thus eventually intersecting another end. This nuisance is fixed with the introduction of h_0 satisfying condition (3.4). In that case the term E created above will now read near $M_{j\alpha}$ as

$$E \sim C e^{-\sigma + \frac{d}{\alpha}} e^{-(\beta_{j+1} - \beta_j) \log r_\alpha} e^{-\sigma|t|} = O(e^{-\frac{\sigma}{\alpha}} r_\alpha^{-4} e^{-\sigma|t|})$$

which is qualitatively of the same type of the other terms involved in the computation of the error.

3.4. The global first approximation. — The approximation $u_1(x)$ in (3.2) will be sufficient for our purposes, however it is so far defined only in a region of the type \mathcal{N} which we have not made precise yet. Since we are assuming that M_α is connected, the fact that M_α is properly embedded implies that $\mathbb{R}^3 \setminus M_\alpha$ consists of precisely two components S_- and S_+ . Let us use the convention that ν points in the direction of S_+ . Let us consider the function \mathbb{H} defined in $\mathbb{R}^3 \setminus M_\alpha$ as

$$\mathbb{H}(x) := \begin{cases} 1 & \text{if } x \in S_+ \\ -1 & \text{if } x \in S_- \end{cases}. \quad (3.10)$$

Then our approximation $u_1(x)$ approaches $\mathbb{H}(x)$ at an exponential rate $O(e^{-\sigma_\pm|t|})$ as $|t|$ increases. The global approximation we will use consists simply of interpolating u_1 with \mathbb{H} sufficiently well-inside $\mathbb{R}^3 \setminus M_\alpha$ through a cut-off in $|t|$. In order to avoid the problem described in §3.3 and having the coordinates (y, t) well-defined, we consider this cut-off to be supported in a region y -dependent that expands logarithmically in r_α . Thus we will actually consider a region \mathcal{N}_δ expanding at the ends, thus becoming wider as $r_\alpha \rightarrow \infty$ than the set $\mathcal{N}_\delta^\alpha$ previously considered, where the coordinates are still well-defined.

We consider the open set \mathcal{O} in $M_\alpha \times \mathbb{R}$ defined as

$$\mathcal{O} = \{ (y, t) \in M_\alpha \times \mathbb{R}, \quad |t + h_1(\alpha y)| < \frac{\delta}{\alpha} + 4 \max\{\sigma_-^{-1}, \sigma_+^{-1}\} \log(1 + r_\alpha(y)) =: \rho_\alpha(y) \} \quad (3.11)$$

where δ is small positive number. We consider the the region $\mathcal{N} =: \mathcal{N}_\delta$ of points x of the form

$$x = X_h(y, t) = y + (t + h_0(\alpha y) + h_1(\alpha y))\nu(\alpha y), \quad (y, t) \in \mathcal{O},$$

namely $\mathcal{N}_\delta = X_h(\mathcal{O})$. The coordinates (y, t) are well-defined in \mathcal{N}_δ for any sufficiently small δ : indeed the map X_h is one to one in \mathcal{O} thanks to assumption (3.4) and the fact that $h_1 = O(\alpha)$. Moreover, Lemma 2.1 applies in \mathcal{N}_δ .

Let $\eta(s)$ be a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$. and define

$$\eta_\delta(x) := \begin{cases} \eta(|t + h_1(\alpha y)| - \rho_\alpha(y) - 3) & \text{if } x \in \mathcal{N}_\delta, \\ 0 & \text{if } x \notin \mathcal{N}_\delta \end{cases} \quad (3.12)$$

where ρ_α is defined in (3.11). Then we let our global approximation $\mathbf{w}(x)$ be simply defined as

$$\mathbf{w} := \eta_\delta u_1 + (1 - \eta_\delta)\mathbb{H} \quad (3.13)$$

where \mathbb{H} is given by (3.10) and $u_1(x)$ is just understood to be $\mathbb{H}(x)$ outside \mathcal{N}_δ .

Since \mathbb{H} is an exact solution in $\mathbb{R}^3 \setminus M_\delta$, the global error of approximation is simply computed as

$$S(\mathbf{w}) = \Delta \mathbf{w} + f(\mathbf{w}) = \eta_\delta S(u_1) + E \quad (3.14)$$

where

$$E = 2\nabla \eta_\delta \nabla u_1 + \Delta \eta_\delta (u_1 - \mathbb{H}) + f(\eta_\delta u_1 + (1 - \eta_\delta)\mathbb{H}) - \eta_\delta f(u_1).$$

The new error terms created are of exponentially small size $O(e^{-\frac{\sigma}{\alpha}})$ but have in addition decay with r_α . In fact we have

$$|E| \leq C e^{-\frac{\sigma}{\alpha}} r_\alpha^{-4}.$$

Let us observe that $|t + h_1(\alpha y)| = |z - h_0(\alpha y)|$ where z is the normal coordinate to M_α , hence η_δ does not depend on h_1 , in particular the term $\Delta \eta_\delta$ does involves second derivatives of h_1 on which we have not made assumptions yet.

4. The proof of Theorem 1

The proof of Theorem 1 involves various ingredients whose detailed proofs are fairly technical. In order to keep the presentation as clear as possible, in this section we carry out the proof, skimming it from several (important) steps, which we state as lemmas or propositions, with complete proofs postponed for the subsequent sections.

We look for a solution u of the Allen Cahn equation (1.1) in the form

$$u = \mathbf{w} + \varphi \quad (4.1)$$

where \mathbf{w} is the global approximation defined in (3.13) and φ is in some suitable sense small. Thus we need to solve the following problem

$$\Delta\varphi + f'(\mathbf{w})\varphi = -S(\mathbf{w}) - N(\varphi) \quad (4.2)$$

where

$$N(\varphi) = f(\mathbf{w} + \varphi) - f(\mathbf{w}) - f'(\mathbf{w})\varphi.$$

Next we introduce various norms that we will use to set up a suitable functional analytic scheme for solving problem (4.2). For a function $g(x)$ defined in \mathbb{R}^3 , $1 < p \leq +\infty$, $\mu > 0$, and $\alpha > 0$ we write

$$\|g\|_{p,\mu,*} := \sup_{x \in \mathbb{R}^3} (1 + r(\alpha x))^\mu \|g\|_{L^p(B(x,1))}, \quad r(x', x_3) = |x'|.$$

On the other hand, given numbers $\mu \geq 0$, $0 < \sigma < \min\{\sigma_+, \sigma_-\}$, $p > 3$, and functions $g(y, t)$ and $\phi(y, t)$ defined in $M_\alpha \times \mathbb{R}$ we consider the norms

$$\|g\|_{p,\mu,\sigma} := \sup_{(y,t) \in M_\alpha \times \mathbb{R}} r_\alpha(y)^\mu e^{\sigma|t|} \left(\int_{B((y,t),1)} |f|^p dV_\alpha \right)^{\frac{1}{p}}. \quad (4.3)$$

Consistently we set

$$\|g\|_{\infty,\mu,\sigma} := \sup_{(y,t) \in M_\alpha \times \mathbb{R}} r_\alpha(y)^\mu e^{\sigma|t|} \|f\|_{L^\infty(B((y,t),1))} \quad (4.4)$$

and let

$$\|\phi\|_{2,p,\mu,\sigma} := \|D^2\phi\|_{p,\mu,\sigma} + \|D\phi\|_{\infty,\mu,\sigma} + \|\phi\|_{\infty,\mu,\sigma}. \quad (4.5)$$

We consider also for a function $g(y)$ defined in M the L^p -weighted norm

$$\|f\|_{p,\beta} := \left(\int_M |f(y)|^p (1 + |y|^\beta)^p dV(y) \right)^{1/p} = \| (1 + |y|^\beta) f \|_{L^p(M)} \quad (4.6)$$

where $p > 1$ and $\beta > 0$.

We assume in what follows, that for a certain constant $\mathcal{K} > 0$ and $p > 3$ we have that the parameter function $h_1(y)$ satisfies

$$\|h_1\|_* := \|h_1\|_{L^\infty(M)} + \|(1 + r^2)Dh_1\|_{L^\infty(M)} + \|D^2h_1\|_{p,4-\frac{4}{p}} \leq \mathcal{K}\alpha. \quad (4.7)$$

Next we reduce problem (4.2) to solving one qualitatively similar (equation (4.20) below) for a function $\phi(y, t)$ defined in the whole space $M_\alpha \times \mathbb{R}$.

4.1. Step 1: the gluing reduction. — We will follow the following procedure. Let us consider again $\eta(s)$, a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$, and define

$$\zeta_n(x) := \begin{cases} \eta(|t + h_1(\alpha y)| - \frac{\delta}{\alpha} + n) & \text{if } x \in \mathcal{N}_\delta \\ 0 & \text{if } x \notin \mathcal{N}_\delta \end{cases}. \quad (4.8)$$

We look for a solution $\varphi(x)$ of problem (4.2) of the following form

$$\varphi(x) = \zeta_2(x)\phi(y, t) + \psi(x) \quad (4.9)$$

where ϕ is defined in entire $M_\alpha \times \mathbb{R}$, $\psi(x)$ is defined in \mathbb{R}^3 and $\zeta_2(x)\phi(y, t)$ is understood as zero outside \mathcal{N}_δ .

We compute, using that $\zeta_2 \cdot \zeta_1 = \zeta_1$,

$$S(\mathbf{w} + \varphi) = \Delta\varphi + f'(\mathbf{w})\varphi + N(\varphi) + S(\mathbf{w}) =$$

$$\zeta_2 [\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) + H(t))\psi + \zeta_1 N(\psi + \phi) + S(u_1)] +$$

$$\Delta\psi - [(1 - \zeta_1)f'(u_1) + \zeta_1 H(t)]\psi +$$

$$(1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 \quad (4.10)$$

where $H(t)$ is any smooth, strictly negative function satisfying

$$H(t) = \begin{cases} f'(1) & \text{if } t > 1, \\ f'(-1) & \text{if } t < -1 \end{cases}.$$

Thus, we will have constructed a solution $\varphi = \zeta_2\phi + \psi$ to problem (4.2) if we require that the pair (ϕ, ψ) satisfies the following coupled system

$$\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi + \phi) + S(u_1) = 0 \text{ for } |t| < \frac{\delta}{\alpha} + 3 \quad (4.11)$$

$$\Delta\psi + [(1 - \zeta_1)f'(u_1) + \zeta_1 H(t)]\psi +$$

$$(1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 = 0 \quad \text{in } \mathbb{R}^3. \quad (4.12)$$

In order to find a solution to this system we will first extend equation (4.11) to entire $M_\alpha \times \mathbb{R}$ in the following manner. Let us set

$$\mathbf{B}(\phi) = \zeta_4[\Delta_x - \partial_{tt} - \Delta_{y, M_\alpha}] \phi \quad (4.13)$$

where Δ_x is expressed in (y, t) coordinates using expression (2.14) and $\mathbf{B}(\phi)$ is understood to be zero for $|t + h_1| > \frac{\delta}{\alpha} + 5$. The other terms in equation (4.11) are simply extended as zero beyond the support of ζ_1 . Thus we consider the extension of equation (4.11) given by

$$\partial_{tt}\phi + \Delta_{y, M_\alpha}\phi + \mathbf{B}(\phi) + f'(w(t))\phi = -\tilde{S}(u_1)$$

$$- \{ [f'(u_1) - f'(w)]\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi + \phi) \} \text{ in } M_\alpha \times \mathbb{R}, \quad (4.14)$$

where we set, with reference to expression (3.9),

$$\begin{aligned} \tilde{S}(u_1) &= -\alpha^2 [|A|^2 h_1 + \Delta_M h_1] w' + \alpha^2 a_{ij}^0 (2\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_0) w'' \\ &- \alpha^4 (|A|^2 t \psi'_0 - a_{ij}^0 \partial_i h_0 \partial_j h_0 t \psi'_1) \Delta h_1 + \zeta_4 [\alpha^3 (t+h) a_{ij}^1 \partial_i h \partial_j h w'' + R_1(y, t)] \end{aligned} \quad (4.15)$$

and, we recall

$$R_1 = R_1(y, t, h_1(\alpha y), \nabla_M h_1(\alpha y))$$

with

$$|D_i R_1(y, t, \iota, j)| + |D_j R_1(y, t, \iota, j)| + |R_1(y, t, \iota, j)| \leq C \alpha^3 (1 + r_\alpha(y))^{-4} e^{-\sigma|t|}. \quad (4.16)$$

In summary $\tilde{S}(u_1)$ coincides with $S(u_1)$ if $\zeta_4 = 1$ while outside the support of ζ_4 , their parts that are not defined for all t are cut-off.

To solve the resulting system (4.12)-(4.14), we first solve equation (4.12) in ψ for a given ϕ a small function in absolute value. Noticing that the potential $[(1 - \zeta_1)f'(u_1) + \zeta_1 H(t)]$ is uniformly negative, so that the linear operator is qualitatively like $\Delta - 1$ and using contraction mapping principle, a solution $\psi = \Psi(\phi)$ is found according to the following lemma, whose detailed proof we carry out in §7.1.2.

Lemma 4.1. — *For all sufficiently small α the following holds. Given ϕ with $\|\phi\|_{2,p,\mu,\sigma} \leq 1$, there exists a unique solution $\psi = \Psi(\phi)$ of problem (4.12) such that*

$$\|\psi\|_X := \|D^2 \psi\|_{p,\mu,*} + \|\psi\|_{p,\mu,*} \leq C e^{-\frac{\sigma\delta}{\alpha}}. \quad (4.17)$$

Besides, Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_X \leq C e^{-\frac{\sigma\delta}{\alpha}} \|\phi_1 - \phi_2\|_{2,p,\mu,\sigma}. \quad (4.18)$$

Thus we replace $\psi = \Psi(\phi)$ in the first equation (4.11) so that by setting

$$\mathbf{N}(\phi) := \mathbf{B}(\phi) + [f'(u_1) - f'(w)]\phi + \zeta_1(f'(u_1) - H(t))\Psi(\phi) + \zeta_1 N(\Psi(\phi) + \phi), \quad (4.19)$$

our problem is reduced to finding a solution ϕ to the following nonlinear, nonlocal problem in $M_\alpha \times \mathbb{R}$.

$$\partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathbf{N}(\phi) \quad \text{in } M_\alpha \times \mathbb{R}. \quad (4.20)$$

Thus, we concentrate in the remaining of the proof in solving equation (4.20). As we hinted in §3.2, we will find a solution of problem (4.20) by considering two steps: (1) “Improving the approximation”, roughly solving for ϕ that eliminates the part of the error that does not contribute to the “projections” $\int [\tilde{S}(U_1) + \mathbf{N}(\phi)] w'(t) dt$, which amounts to a nonlinear problem in ϕ , and (2) Adjust h_1 in such a way that the resulting projection is actually zero. Let us set up the scheme for step (1) in a precise form.

4.2. Step 2: Eliminating terms not contributing to projections. — Let us consider the problem of finding a function $\phi(y, t)$ such that for a certain function $c(y)$ defined in M_α , we have

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y, M_\alpha}\phi &= -\tilde{S}(u_1) - \mathbb{N}(\phi) + c(y)w'(t) \quad \text{in } M_\alpha \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi(y, t) w'(t) dt &= 0, \quad \text{for all } y \in M_\alpha. \end{aligned} \quad (4.21)$$

Solving this problem for ϕ amounts to “eliminating the part of the error that does not contribute to the projection” in problem (4.20). To justify this phrase let us consider the associated linear problem in $M_\alpha \times \mathbb{R}$

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y, M_\alpha}\phi + f'(w(t))\phi &= g(y, t) + c(y)w'(t), \quad \text{for all } (y, t) \in M_\alpha \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y, t) w'(t) dt &= 0, \quad \text{for all } y \in M_\alpha. \end{aligned} \quad (4.22)$$

Assuming that the corresponding operations can be carried out, let us multiply the equation by $w'(t)$ and integrate in t for fixed y . We find that

$$\Delta_{y, M_\alpha} \int_{\mathbb{R}} \phi(y, t) w'(t) dt + \int_{\mathbb{R}} \phi(y, t) [w''' + f'(w)w'] dt = \int_{\mathbb{R}} g w' + c(y) \int_{\mathbb{R}} w'^2.$$

The left hand side of the above identity is zero and then we find that

$$c(y) = -\frac{\int_{\mathbb{R}} g(y, t) w'(t) dt}{\int_{\mathbb{R}} w'^2 dt}, \quad (4.23)$$

hence a ϕ solving problem (4.22). ϕ *precisely* solves or *eliminates* the part of g which does not contribute to the projections in the equation $\Delta\phi + f'(w)\phi = g$, namely the same equation with g replaced by \tilde{g} given by

$$\tilde{g}(y, t) = g(y, t) - \frac{\int_{\mathbb{R}} f(y, \cdot) w'}{\int_{\mathbb{R}} w'^2} w'(t). \quad (4.24)$$

The term $c(y)$ in problem (4.21) has a similar role, except that we cannot find it so explicitly.

In order to solve problem (4.21) we need to devise a theory to solve problem (4.22) where we consider a class of right hand sides g with a qualitative behavior similar to that of the error $S(u_1)$. As we have seen in (4.15), typical elements in this error are of the type $O((1 + r_\alpha(y))^{-\mu} e^{-\sigma|t|})$, so this is the type of functions $g(y, t)$ that we want to consider. This is actually the motivation to introduce the norms (4.3), (4.4) and (4.5). We will prove that problem (4.22) has a unique solution ϕ which respects the size of g in norm (4.3) up to its second derivatives, namely in the norm (4.5). The following fact holds.

Proposition 4.1. — *Given $p > 3$, $\mu \geq 0$ and $0 < \sigma < \min\{\sigma_-, \sigma_+\}$, there exists a constant $C > 0$ such that for all sufficiently small $\alpha > 0$ the following holds. Given*

f with $\|g\|_{p,\mu,\sigma} < +\infty$, then Problem (4.22) with $c(y)$ given by (4.23), has a unique solution ϕ with $\|\phi\|_{\infty,\mu,\sigma} < +\infty$. This solution satisfies in addition that

$$\|\phi\|_{2,p,\mu,\sigma} \leq C\|g\|_{p,\mu,\sigma}. \quad (4.25)$$

We will prove this result in §5. After Proposition 4.1, solving Problem (4.21) for a small ϕ is easy using the small Lipschitz character of the terms involved in the operator $\mathbb{N}(\phi)$ in (4.19) and contraction mapping principle. The error term $\tilde{S}(u_1)$ satisfies

$$\|\tilde{S}(u_1) + \alpha^2 \Delta h_1 w'\|_{p,4,\sigma} \leq C\alpha^3. \quad (4.26)$$

Using this, and the fact that $\mathbb{N}(\phi)$ defines a contraction mapping in a ball center zero and radius $O(\alpha^3)$ in $\|\cdot\|_{2,p,4,\sigma}$, we conclude the existence of a unique small solution ϕ to problem (4.21) whose size is $O(\alpha^3)$ for this norm. This solution ϕ turns out to define an operator in h_1 $\phi = \Phi(h_1)$ which is Lipschitz in the norms $\|\cdot\|_*$ appearing in condition (4.7). In precise terms, we have the validity of the following result, whose detailed proof we postpone for §7.2.

Proposition 4.2. — Assume $p > 3$, $0 \leq \mu \leq 3$, $0 < \sigma < \min\{\sigma_+, \sigma_-\}$. There exists a $K > 0$ such that problem (7.8) has a unique solution $\phi = \Phi(h_1)$ such that

$$\|\phi\|_{2,p,\mu,\sigma} \leq K\alpha^3.$$

Besides, Φ has a Lipschitz dependence on h_1 satisfying (4.7) in the sense that

$$\|\Phi(h_1) - \Phi(h_2)\|_{2,p,\mu,\sigma} \leq C\alpha^2 \|h_1 - h_2\|_*. \quad (4.27)$$

4.3. Step 3: Adjusting h_1 to make the projection zero. — In order to conclude the proof of the theorem, we have to carry out the second step, namely adjusting h_1 , within a region of the form (4.7) for suitable \mathcal{K} in such a way that the “projections” are identically zero, namely making zero the function $c(y)$ found for the solution $\phi = \Phi(h_1)$ of problem (4.21). Using expression (4.23) for $c(y)$ we find that

$$c(y) \int_{\mathbb{R}} w'^2 = \int_{\mathbb{R}} \tilde{S}(u_1) w' dt + \int_{\mathbb{R}} \mathbb{N}(\Phi(h_1)) w' dt. \quad (4.28)$$

Now, setting $c_* := \int_{\mathbb{R}} w'^2 dt$ and using same computation employed to derive formula (3.6), we find from expression (4.15) that

$$\int_{\mathbb{R}} \tilde{S}(u_1)(y,t) w'(t) dt = -c_* \alpha^2 (\Delta_M h_1 + h_1 |A|^2) + c_* \alpha^2 G_1(h_1)$$

where

$$\begin{aligned} c_* G_1(h_1) = & -\alpha^2 \Delta h_1 (|A|^2 \int_{\mathbb{R}} t \psi'_0 w' dt - a_{ij}^0 \partial_i h_0 \partial_j h_0 \int_{\mathbb{R}} t \psi'_1 w' dt) + \\ & \alpha \partial_i (h_0 + h_1) \partial_j (h_0 + h_1) \int_{\mathbb{R}} \zeta_4(t+h) a_{ij}^1 w'' w' dt + \alpha^{-2} \int_{\mathbb{R}} \zeta_4 R_1(y,t,h_1, \nabla_M h_1) w' dt \end{aligned} \quad (4.29)$$

and we recall that R_1 is of size $O(\alpha^3)$ in the sense (4.16). Thus, setting

$$c_* G_2(h_1) := \alpha^{-2} \int_{\mathbb{R}} \mathbb{N}(\Phi(h_1)) w' dt, \quad G(h_1) := G_1(h_1) + G_2(h_1), \quad (4.30)$$

we find that the equation $c(y) = 0$ is equivalent to the problem

$$\mathcal{J}(h_1) = \Delta_M h_1 + |A|^2 h_1 = G(h_1) \quad \text{in } M. \quad (4.31)$$

Therefore, we will have proven Theorem 1 if we find a function h_1 defined on M satisfying constraint (4.7) for a suitable \mathcal{K} that solves equation (4.31). Again, this is not so direct since the operator \mathcal{J} has a nontrivial bounded kernel. Rather than solving directly (4.31), we consider first a projected version of this problem, namely that of finding h_1 such that for certain scalars c_1, \dots, c_J we have

$$\mathcal{J}(h_1) = G(h_1) + \sum_{i=1}^J \frac{c_i}{1+r^4} \hat{z}_i \quad \text{in } M,$$

$$\int_M \frac{\hat{z}_i h}{1+r^4} dV = 0, \quad i = 1, \dots, J. \quad (4.32)$$

Here $\hat{z}_1, \dots, \hat{z}_J$ is a basis of the vector space of bounded Jacobi fields.

In order to solve problem (4.32) we need a corresponding linear invertibility theory. This leads us to consider the linear problem

$$\mathcal{J}(h) = f + \sum_{i=1}^J \frac{c_i}{1+r^4} \hat{z}_i \quad \text{in } M,$$

$$\int_M \frac{\hat{z}_i h}{1+r^4} dV = 0, \quad i = 1, \dots, J. \quad (4.33)$$

Here $\hat{z}_1, \dots, \hat{z}_J$ are bounded, linearly independent Jacobi fields, and J is the dimension of the vector space of bounded Jacobi fields.

We will prove in §6.1 the following result.

Proposition 4.3. — *Given $p > 2$ and f with $\|f\|_{p, 4-\frac{4}{p}} < +\infty$, there exists a unique bounded solution h of problem (4.33). Moreover, there exists a positive number $C = C(p, M)$ such that*

$$\|h\|_* := \|h\|_\infty + \|(1+|y|^2) Dh\|_\infty + \|D^2 h\|_{p, 4-\frac{4}{p}} \leq C \|f\|_{p, 4-\frac{4}{p}}. \quad (4.34)$$

Using the fact that G is a small operator of size $O(\alpha)$ uniformly on functions h_1 satisfying (4.7), Proposition 4.3 and contraction mapping principle yield the following result, whose detailed proof we carry out in §8.

Proposition 4.4. — *Given $p > 3$, there exists a number $\mathcal{K} > 0$ such that for all sufficiently small $\alpha > 0$ there is a unique solution h_1 of problem (4.32) that satisfies constraint (4.7).*

4.4. Step 3: Conclusion. — At the last step we prove that the constants c_i found in equation (4.32) are in reality all zero, without the need of adjusting any further parameters but rather as a consequence of the natural invariances of the of the full equation. The key point is to realize what equation has been solved so far.

First we observe the following. For each h_1 satisfying (4.7), the pair (ϕ, ψ) with $\phi = \Phi(h_1)$, $\psi = \Psi(\phi)$, solves the system

$$\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi + \phi) + S(u_1) = c(y)w'(t) \text{ for } |t| < \frac{\delta}{\alpha} + 3$$

$$\Delta\psi + [(1 - \zeta_1)f'(u_1) + \zeta_1 H(t)]\psi +$$

$$(1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 = 0 \quad \text{in } \mathbb{R}^3.$$

Thus setting

$$\varphi(x) = \zeta_2(x)\phi(y, t) + \psi(x), \quad u = \mathbf{w} + \varphi,$$

we find from formula (4.10) that

$$\Delta u + f(u) = S(\mathbf{w} + \varphi) = \zeta_2 c(y) w'(t).$$

On the other hand choosing h_1 as that given in Proposition 4.4 which solves problem (4.32), amounts precisely to making

$$c(y) = c_* \alpha^2 \sum_{i=1}^J c_i \frac{\hat{z}_i(\alpha y)}{1 + r_\alpha(y)^4}$$

for certain scalars c_i . In summary, we have found h_1 satisfying constraint (4.7) such that

$$u = \mathbf{w} + \zeta_2(x)\Phi(h_1) + \Psi(\Phi(h_1)) \quad (4.35)$$

solves the equation

$$\Delta u + f(u) = \sum_{j=1}^J \frac{\tilde{c}_j}{1 + r_\alpha^4} \hat{z}_j(\alpha y) w'(t) \quad (4.36)$$

where $\tilde{c}_i = c_* \alpha^2 c_i$. Testing equation (4.36) against the generators of the rigid motions $\partial_i u$ $i = 1, 2, 3$, $-x_2 \partial_1 u + x_1 \partial_2 u$, and using the balancing formula for the minimal surface and the zero average of the numbers β_j in the definition of h_0 , we find a system of equations that leads us to $c_i = 0$ for all i , thus conclude the proof. We will carry out the details in §9.

In sections §5-9 we will complete the proofs of the intermediate steps of the program designed in this section.

5. The linearized operator

In this section we will prove Proposition 4.1. At the core of the proof of the stated a priori estimates is the fact that the one-variable solution w of (1.1) is *nondegenerate* in $L^\infty(\mathbb{R}^3)$ in the sense that the linearized operator

$$L(\phi) = \Delta_y \phi + \partial_{tt} \phi + f'(w(t))\phi, \quad (y, t) \in \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R},$$

is such that the following property holds.

Lemma 5.1. — *Let ϕ be a bounded, smooth solution of the problem*

$$L(\phi) = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}. \quad (5.1)$$

Then $\phi(y, t) = Cw'(t)$ for some $C \in \mathbb{R}$.

Proof. — We begin by reviewing some known facts about the one-dimensional operator $L_0(\psi) = \psi'' + f'(w)\psi$. Assuming that $\psi(t)$ and its derivative decay sufficiently fast as $|t| \rightarrow +\infty$ and defining $\psi(t) = w'(t)\rho(t)$, we get that

$$\int_{\mathbb{R}} [|\psi'|^2 - f'(w)\psi^2] dt = \int_{\mathbb{R}} L_0(\psi)\psi dt = \int_{\mathbb{R}} w'^2 |\rho'|^2 dt,$$

therefore this quadratic form is positive unless ψ is a constant multiple of w' . Using this and a standard compactness argument we get that there is a constant $\gamma > 0$ such that whenever $\int_{\mathbb{R}} \psi w' = 0$ with $\psi \in H^1(\mathbb{R})$ we have that

$$\int_{\mathbb{R}} (|\psi'|^2 - f'(w)\psi^2) dt \geq \gamma \int_{\mathbb{R}} (|\psi'|^2 + |\psi|^2) dt. \quad (5.2)$$

Now, let ϕ be a bounded solution of equation (5.1). We claim that ϕ has exponential decay in t , uniform in y . Let us consider a small number $\sigma > 0$ so that for a certain $t_0 > 0$ and all $|t| > t_0$ we have that

$$f'(w) < -2\sigma^2.$$

Let us consider for $\varepsilon > 0$ the function

$$g_\varepsilon(t, y) = e^{-\sigma(|t|-t_0)} + \varepsilon \sum_{i=1}^2 \cosh(\sigma y_i)$$

Then for $|t| > t_0$ we get that

$$L(g_\varepsilon) < 0 \quad \text{if } |t| > t_0.$$

As a conclusion, using maximum principle, we get

$$|\phi| \leq \|\phi\|_\infty g_\varepsilon \quad \text{if } |t| > t_0,$$

and letting $\varepsilon \rightarrow 0$ we then get

$$|\phi(y, t)| \leq C \|\phi\|_\infty e^{-\sigma|t|} \quad \text{if } |t| > t_0.$$

Let us observe the following fact: the function

$$\tilde{\phi}(y, t) = \phi(y, t) - \left(\int_{\mathbb{R}} w'(\zeta) \phi(y, \zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} w'^2}$$

also satisfies $L(\tilde{\phi}) = 0$ and, in addition,

$$\int_{\mathbb{R}} w'(t) \tilde{\phi}(y, t) dt = 0 \quad \text{for all } y \in \mathbb{R}^2. \quad (5.3)$$

In view of the above discussion, it turns out that the function

$$\varphi(y) := \int_{\mathbb{R}} \tilde{\phi}^2(y, t) dt$$

is well defined. In fact so are its first and second derivatives by elliptic regularity of ϕ , and differentiation under the integral sign is thus justified. Now, let us observe that

$$\Delta_y \varphi(y) = 2 \int_{\mathbb{R}} \Delta_y \tilde{\phi} \cdot \tilde{\phi} dt + 2 \int_{\mathbb{R}} |\nabla_y \tilde{\phi}|^2$$

and hence

$$\begin{aligned} 0 &= \int_{\mathbb{R}} (L(\tilde{\phi}) \cdot \tilde{\phi}) \\ &= \frac{1}{2} \Delta_y \varphi - \int_{\mathbb{R}} |\nabla_y \tilde{\phi}|^2 dz - \int_{\mathbb{R}} (|\tilde{\phi}_t|^2 - f'(w) \tilde{\phi}^2) dt. \end{aligned} \quad (5.4)$$

Let us observe that because of relations (5.3) and (5.2), we have that

$$\int_{\mathbb{R}} (|\tilde{\phi}_t|^2 - f'(w) \tilde{\phi}^2) dt \geq \gamma \varphi.$$

It follows then that

$$\frac{1}{2} \Delta_y \varphi - \gamma \varphi \geq 0.$$

Since φ is bounded, from maximum principle we find that φ must be identically equal to zero. But this means

$$\phi(y, t) = \left(\int_{\mathbb{R}} w'(\zeta) \phi(y, \zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} w'^2}. \quad (5.5)$$

Then the bounded function

$$g(y) = \int_{\mathbb{R}} w_\zeta(\zeta) \phi(y, \zeta) d\zeta$$

satisfies the equation

$$\Delta_y g = 0, \quad \text{in } \mathbb{R}^2. \quad (5.6)$$

Liouville's theorem implies that $g \equiv \text{constant}$ and relation (5.5) yields $\phi(y, t) = C w'(t)$ for some C . This concludes the proof. \square

5.1. A priori estimates. — We shall consider problem (4.22) in a slightly more general form, also in a domain finite in y -direction. For a large number $R > 0$ let us set

$$M_\alpha^R := \{y \in M_\alpha / r(\alpha y) < R\}$$

and consider the variation of Problem (4.22) given by

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y, M_\alpha}\phi + f'(w(t))\phi &= g(y, t) + c(y)w'(t) \quad \text{in } M_\alpha^R \times \mathbb{R}, \\ \phi &= 0, \quad \text{on } \partial M_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y, t) w'(t) dt &= 0 \quad \text{for all } y \in M_\alpha^R, \end{aligned} \quad (5.7)$$

where we allow $R = +\infty$ and

$$c(y) \int_{\mathbb{R}} w'^2 dt = - \int_{\mathbb{R}} g(y, t) w' dt .$$

We begin by proving a priori estimates.

Lemma 5.2. — *Let us assume that $0 < \sigma < \min\{\sigma_-, \sigma_+\}$ and $\mu \geq 0$. Then there exists a constant $C > 0$ such that for all small α and all large R , and every solution ϕ to Problem (5.13) with $\|\phi\|_{\infty, \mu, \sigma} < +\infty$ and right hand side g satisfying $\|g\|_{p, \mu, \sigma} < +\infty$ we have*

$$\|D^2\phi\|_{p, \mu, \sigma} + \|D\phi\|_{\infty, \mu, \sigma} + \|\phi\|_{\infty, \mu, \sigma} \leq C\|g\|_{p, \mu, \sigma}. \quad (5.8)$$

Proof. — For the purpose of the a priori estimate, it clearly suffices to consider the case $c(y) \equiv 0$. By local elliptic estimates, it is enough to show that

$$\|\phi\|_{\infty, \mu, \sigma} \leq C\|g\|_{p, \mu, \sigma}. \quad (5.9)$$

Let us assume by contradiction that (5.9) does not hold. Then we have sequences $\alpha = \alpha_n \rightarrow 0$, $R = R_n \rightarrow \infty$, g_n with $\|g_n\|_{p, \mu, \sigma} \rightarrow 0$, ϕ_n with $\|\phi_n\|_{\infty, \mu, \sigma} = 1$ such that

$$\begin{aligned} \partial_{tt}\phi_n + \Delta_{y, M_\alpha}\phi_n + f'(w(t))\phi_n &= g_n \quad \text{in } M_\alpha^R \times \mathbb{R}, \\ \phi_n &= 0 \quad \text{on } \partial M_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi_n(y, t) w'(t) dt &= 0 \quad \text{for all } y \in M_\alpha^R. \end{aligned} \quad (5.10)$$

Then we can find points $(y_n, t_n) \in M_\alpha^R \times \mathbb{R}$ such that

$$e^{-\sigma|t_n|} (1 + r(\alpha_n y_n))^\mu |\phi_n(y_n, t_n)| \geq \frac{1}{2}.$$

We will consider different possibilities. We may assume that either $r_\alpha(y_n) = O(1)$ or $r_\alpha(y_n) \rightarrow +\infty$.

5.1.1. *Case $r(\alpha_n y_n)$ bounded.* — We have $\alpha_n y_n$ lies within a bounded subregion of M , so we may assume that

$$\alpha_n y_n \rightarrow \tilde{y}_0 \in M.$$

Assume that $\tilde{y}_0 \in Y_k(\mathcal{U}_k)$ for one of the local parametrization of M . We consider $\tilde{y}_n, \tilde{y}_0 \in \mathcal{U}_k$ with $Y_k(\tilde{y}_n) = \alpha_n y_n, Y_k(\tilde{y}_0) = \tilde{y}_0$.

On $\alpha_n^{-1} Y_k(\mathcal{U}_k)$, M_α is parameterized by $Y_{k, \alpha_n}(\mathbf{y}) = \alpha_n^{-1} Y_k(\alpha_n \mathbf{y}), \mathbf{y} \in \alpha_n^{-1} \mathcal{U}_k$. Let us consider the local change of variable,

$$\mathbf{y} = \alpha^{-1} \tilde{\mathbf{y}}_n + \mathbf{y}.$$

5.1.2. *Subcase t_n bounded.* — Let us assume first that $|t_n| \leq C$. Then, setting

$$\tilde{\phi}_n(\mathbf{y}, t) := \tilde{\phi}_n(\alpha^{-1} \tilde{\mathbf{y}}_n + \mathbf{y}, t),$$

the local equation becomes

$$a_{ij}^0(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}) \partial_{ij} \tilde{\phi}_n + \alpha_n b_j^0(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}) \partial_j \tilde{\phi}_n + \partial_{tt} \tilde{\phi}_n + f'(w(t)) \tilde{\phi}_n = \tilde{g}_n(\mathbf{y}, t)$$

where $\tilde{g}_n(\mathbf{y}, t) := g_n(\tilde{\mathbf{y}}_n + \alpha \mathbf{y}, t)$. We observe that this expression is valid for \mathbf{y} well-inside the domain $\alpha^{-1} \mathcal{U}_k$ which is expanding to entire \mathbb{R}^2 . Since $\tilde{\phi}_n$ is bounded, and $\tilde{g}_n \rightarrow 0$ in $L_{loc}^p(\mathbb{R}^2)$, we obtain local uniform $W^{2,p}$ -bound. Hence we may assume, passing to a subsequence, that $\tilde{\phi}_n$ converges uniformly in compact subsets of \mathbb{R}^3 to a function $\tilde{\phi}(\mathbf{y}, t)$ that satisfies

$$a_{ij}^0(\tilde{\mathbf{y}}) \partial_{ij} \tilde{\phi} + \partial_{tt} \tilde{\phi} + f'(w(t)) \tilde{\phi} = 0.$$

Thus $\tilde{\phi}$ is non-zero and bounded. After a rotation and stretching of coordinates, the constant coefficient operator $a_{ij}^0(\tilde{\mathbf{y}}) \partial_{ij}$ becomes $\Delta_{\mathbf{y}}$. Hence Lemma 5.1 implies that, necessarily, $\tilde{\phi}(\mathbf{y}, t) = Cw'(t)$. On the other hand, we have

$$0 = \int_{\mathbb{R}} \tilde{\phi}_n(\mathbf{y}, t) w'(t) dt \longrightarrow \int_{\mathbb{R}} \tilde{\phi}(\mathbf{y}, t) w'(t) dt \quad \text{as } n \rightarrow \infty.$$

Hence, necessarily $\tilde{\phi} \equiv 0$. But we have $(1 + r(\alpha_n y_n))^\mu |\tilde{\phi}_n(0, t_n)| \geq \frac{1}{2}$, and since t_n and $r(\alpha_n y_n)$ were bounded, the local uniform convergence implies $\tilde{\phi} \neq 0$. We have reached a contradiction.

5.1.3. *Subcase t_n unbounded.* — If y_n is in the same range as above, but, say, $t_n \rightarrow +\infty$, the situation is similar. The variation is that we define now

$$\tilde{\phi}_n(\mathbf{y}, t) = e^{\sigma(t_n+t)} \phi_n(\alpha_n^{-1} \mathbf{y}_n + \mathbf{y}, t_n + t), \quad \tilde{g}_n(\mathbf{y}, t) = e^{\sigma(t_n+t)} g_n(\alpha_n^{-1} \mathbf{y}_n + \mathbf{y}, t_n + t).$$

Then $\tilde{\phi}_n$ is uniformly bounded, and $\tilde{g}_n \rightarrow 0$ in $L_{loc}^p(\mathbb{R}^3)$. Now $\tilde{\phi}_n$ satisfies

$$\begin{aligned} a_{ij}^0(\mathbf{y}_n + \alpha_n \mathbf{y}) \partial_{ij} \tilde{\phi}_n + \partial_{tt} \tilde{\phi}_n + \alpha_n b_j(\mathbf{y}_n + \alpha_n \mathbf{y}) \partial_j \tilde{\phi}_n \\ - 2\sigma \partial_t \tilde{\phi}_n + (f'(w(t+t_n)) + \sigma^2) \tilde{\phi}_n = \tilde{g}_n. \end{aligned}$$

We fall into the limiting situation

$$a_{ij}^* \partial_{ij} \tilde{\phi} + \partial_{tt} \tilde{\phi} - 2\sigma \partial_t \tilde{\phi} - (\sigma_+^2 - \sigma^2) \tilde{\phi} = 0 \quad \text{in } \mathbb{R}^3 \quad (5.11)$$

where a_{ij}^* is a positive definite, constant matrix and $\tilde{\phi} \neq 0$. But since, by hypothesis $\sigma_+^2 - \sigma^2 > 0$, maximum principle implies that $\tilde{\phi} \equiv 0$. We obtain a contradiction.

5.1.4. Case $r(\alpha_n y_n) \rightarrow +\infty$.— In this case we may assume that the sequence $\alpha_n y_n$ diverges along one of the ends, say M_k . Considering now the parametrization associated to the end, $y = \psi_k(\mathbf{y})$, given by (2.1), which inherits that for $M_{\alpha_n, k}$, $y = \alpha_n^{-1} \psi_k(\alpha_n \mathbf{y})$. Thus in this case $a_{ij}^0(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}) \rightarrow \delta_{ij}$, uniformly in compact subsets of \mathbb{R}^2 .

5.1.5. Subcase t_n bounded.— Let us assume first that the sequence t_n is bounded and set

$$\tilde{\phi}_n(\mathbf{y}, t) = (1 + r(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}))^\mu \phi_n(\alpha_n^{-1} \tilde{\mathbf{y}}_n + \mathbf{y}, t_n + t).$$

Then

$$\begin{aligned} \partial_j(r_{\alpha_n}^{-\mu} \tilde{\phi}_n) &= -\mu \alpha r^{-\mu-1} \partial_j r \tilde{\phi} + r^{-\mu} \partial_j \tilde{\phi} \\ \partial_{ij}(r_{\alpha_n}^{-\mu} \tilde{\phi}_n) &= \mu(\mu+1) \alpha^2 r^{-\mu-2} \partial_i r \partial_j r \tilde{\phi} - \mu \alpha^2 r^{-\mu-1} \partial_{ij} r \tilde{\phi} - \mu \alpha r^{-\mu-1} \partial_j r \partial_i \tilde{\phi} \\ &\quad + r^{-\mu} \partial_{ij} \tilde{\phi} - \mu \alpha r^{-\mu-1} \partial_i r \partial_j \tilde{\phi}. \end{aligned}$$

Now $\partial_i r = O(1)$, $\partial_{ij} r = O(r^{-1})$, hence we have

$$\partial_j(r_{\alpha_n}^{-\mu} \tilde{\phi}_n) = r^{-\mu} \left[\partial_j \tilde{\phi} + O(\alpha r_{\alpha}^{-1}) \tilde{\phi} \right],$$

$$\partial_{ij}(r_{\alpha_n}^{-\mu} \tilde{\phi}_n) = r_{\alpha_n}^{-\mu} \left[\partial_{ij} \tilde{\phi} + O(\alpha r_{\alpha}^{-1}) \partial_i \tilde{\phi} + O(\alpha^2 r_{\alpha}^{-2}) \tilde{\phi} \right],$$

and the equation satisfied by $\tilde{\phi}_n$ has therefore the form

$$\Delta_{\mathbf{y}} \tilde{\phi}_n + \partial_{tt} \tilde{\phi}_n + o(1) \partial_{ij} \tilde{\phi}_n + o(1) \partial_j \tilde{\phi}_n + o(1) \tilde{\phi}_n + f'(w(t)) \tilde{\phi}_n = \tilde{g}_n.$$

where $\tilde{\phi}_n$ is bounded, $\tilde{g}_n \rightarrow 0$ in $L_{loc}^p(\mathbb{R}^3)$. From elliptic estimates, we also get uniform bounds for $\|\partial_j \tilde{\phi}_n\|_{\infty}$ and $\|\partial_{ij} \tilde{\phi}_n\|_{p,0,0}$. In the limit we obtain a $\tilde{\phi} \neq 0$ bounded, solution of

$$\Delta_{\mathbf{y}} \tilde{\phi} + \partial_{tt} \tilde{\phi} + f'(w(t)) \tilde{\phi} = 0, \quad \int_{\mathbb{R}} \tilde{\phi}(\mathbf{y}, t) w'(t) dt = 0, \quad (5.12)$$

a situation which is discarded in the same way as before if $\tilde{\phi}$ is defined in \mathbb{R}^3 . There is however, one more possibility which is that $r(\alpha_n y_n) - R_n = O(1)$. In such a case we would see in the limit equation (5.12) satisfied in a half-space, which after a rotation in the \mathbf{y} -plane can be assumed to be

$$H = \{(\mathbf{y}, t) \in \mathbb{R}^2 \times \mathbb{R} / y_2 < 0\}, \quad \text{with } \phi(\mathbf{y}_1, 0, t) = 0 \quad \text{for all } (\mathbf{y}_1, t) \in \mathbb{R}^2.$$

By Schwarz's reflection, the odd extension of $\tilde{\phi}$, which achieves for $y_2 > 0$, $\tilde{\phi}(y_1, y_2, t) = -\tilde{\phi}(y_1, -y_2, t)$, satisfies the same equation, and thus we fall into one of the previous cases, again finding a contradiction.

5.1.6. *Subcase t_n unbounded.* — Let us assume now $|t_n| \rightarrow +\infty$. If $t_n \rightarrow +\infty$ we define

$$\tilde{\phi}_n(\mathbf{y}, t) = (1 + r(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}))^\mu e^{t_n+t} \phi_n(\alpha_n^{-1} \tilde{\mathbf{y}}_n + \mathbf{y}, t_n + t).$$

In this case we end up in the limit with a $\tilde{\phi} \neq 0$ bounded and satisfying the equation

$$\Delta_{\mathbf{y}} \tilde{\phi} + \partial_{tt} \tilde{\phi} - 2\sigma \partial_t \tilde{\phi} - (\sigma_+^2 - \sigma^2) \tilde{\phi} = 0$$

either in entire space or in a Half-space under zero boundary condition. This implies again $\tilde{\phi} = 0$, and a contradiction has been reached that finishes the proof of the a priori estimates. \square

5.2. Existence: conclusion of proof of Proposition 4.1. — Let us prove now existence. We assume first that g has compact support in $M_\alpha \times \mathbb{R}$.

$$\begin{aligned} \partial_{tt} \phi + \Delta_{y, M_\alpha} \phi + f'(w(t)) \phi &= g(y, t) + c(y) w'(t) \quad \text{in } M_\alpha^R \times \mathbb{R}, \\ \phi &= 0, \quad \text{on } \partial M_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y, t) w'(t) dt &= 0 \quad \text{for all } y \in M_\alpha^R, \end{aligned} \tag{5.13}$$

where we allow $R = +\infty$ and

$$c(y) \int_{\mathbb{R}} w'^2 dt = - \int_{\mathbb{R}} g(y, t) w' dt .$$

Problem (5.13) has a weak formulation which is the following. Let

$$H = \{ \phi \in H_0^1(M_\alpha^R \times \mathbb{R}) / \int_{\mathbb{R}} \phi(y, t) w'(t) dt = 0 \quad \text{for all } y \in M_\alpha^R \} .$$

H is a closed subspace of $H_0^1(M_\alpha^R \times \mathbb{R})$, hence a Hilbert space when endowed with its natural norm,

$$\|\phi\|_H^2 = \int_{M_\alpha^R} \int_{\mathbb{R}} (|\partial_t \phi|^2 + |\nabla_{M_\alpha} \phi|^2 - f'(w(t)) \phi^2) dV_\alpha dt .$$

ϕ is then a weak solution of Problem (5.13) if $\phi \in H$ and satisfies

$$\begin{aligned} a(\phi, \psi) &:= \int_{M_\alpha^R \times \mathbb{R}} (\nabla_{M_\alpha} \phi \cdot \nabla_{M_\alpha} \psi - f'(w(t)) \phi \psi) dV_\alpha dt = \\ &\quad - \int_{M_\alpha^R \times \mathbb{R}} g \psi dV_\alpha dt \quad \text{for all } \psi \in H. \end{aligned}$$

It is standard to check that a weak solution of problem (5.13) is also classical provided that g is regular enough. Let us observe that because of the orthogonality condition defining H we have that

$$\gamma \int_{M_\alpha^R \times \mathbb{R}} \psi^2 dV_\alpha dt \leq a(\psi, \psi) \quad \text{for all } \psi \in H.$$

Hence the bilinear form a is coercive in H , and existence of a unique weak solution follows from Riesz's theorem. If g is regular and compactly supported, ψ is also

regular. Local elliptic regularity implies in particular that ϕ is bounded. Since for some $t_0 > 0$, the equation satisfied by ϕ is

$$\Delta\phi + f'(w(t))\phi = c(y)w'(t), \quad |t| > t_0, \quad y \in M_\alpha^R, \quad (5.14)$$

and $c(y)$ is bounded, then enlarging t_0 if necessary, we see that for $\sigma < \min\{\sigma_+, \sigma_-\}$, the function $v(y, t) := Ce^{-\sigma|t|} + \varepsilon e^{\sigma|t|}$ is a positive supersolution of equation (5.14), for a large enough choice of C and arbitrary $\varepsilon > 0$. Hence $|\phi| \leq Ce^{-\sigma|t|}$, from maximum principle. Since M_α^R is bounded, we conclude that $\|\phi\|_{p,\mu,\sigma} < +\infty$. From Lemma 5.2 we obtain that if R is large enough then

$$\|D^2\phi\|_{p,\mu,\sigma} + \|D\phi\|_{\infty,\mu,\sigma} + \|\phi\|_{\infty,\mu,\sigma} \leq C\|g\|_{p,\mu,\sigma} \quad (5.15)$$

Now let us consider Problem (5.13) for $R = +\infty$, allowed above, and for $\|g\|_{p,\mu,\sigma} < +\infty$. Then solving the equation for finite R and suitable compactly supported g_R , we generate a sequence of approximations ϕ_R which is uniformly controlled in R by the above estimate. If g_R is chosen so that $g_R \rightarrow g$ in $L^p_{loc}(M_\alpha \times \mathbb{R})$ and $\|g_R\|_{p,\mu,\sigma} \leq C\|g\|_{p,\mu,\sigma}$, We obtain that ϕ_R is locally uniformly bounded, and by extracting a subsequence, it converges uniformly locally over compacts to a solution ϕ to the full problem which respects the estimate (4.25). This concludes the proof of existence, and hence that of the proposition. \square

6. The Jacobi operator

We consider this section the problem of finding a function h such that for certain constants c_1, \dots, c_J ,

$$\mathcal{J}(h) = \Delta_M h + |A|^2 h = f + \sum_{j=1}^J \frac{c_j}{1+r^4} \hat{z}_j \quad \text{in } M, \quad (6.1)$$

$$\int_M \frac{\hat{z}_i h}{1+r^4} = 0, \quad i = 1, \dots, J \quad (6.2)$$

and prove the result of Proposition 4.3. We will also deduce the existence of Jacobi fields of logarithmic growth as in Lemma 3.1. We recall the definition of the norms $\|\cdot\|_{p,\beta}$ in (4.6).

Outside of a ball of sufficiently large radius R_0 , it is natural to parameterize each end of M , $y_3 = F_k(y_1, y_2)$ using the Euclidean coordinates $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. The requirement in f on each end amounts to $\tilde{f} \in L^p(B(0, 1/R_0))$ where

$$\tilde{f}(\mathbf{y}) := |\mathbf{y}|^{-4} f(|\mathbf{y}|^{-2} \mathbf{y}). \quad (6.3)$$

Indeed, observe that

$$\|\tilde{f}\|_{L^p(B(0, 1/R_0))}^p = \int_{B(0, 1/R_0)} |\mathbf{y}|^{-4p} |f(|\mathbf{y}|^{-2} \mathbf{y})|^p d\mathbf{y} = \int_{\mathbb{R}^2 \setminus B(0, R_0)} |\mathbf{y}|^{4(p-1)} |f(\mathbf{y})|^p d\mathbf{y}.$$

In order to prove the proposition we need some a priori estimates.

Lemma 6.1. — *Let $p > 2$. For each $R_0 > 0$ sufficiently large there exists a constant $C > 0$ such that if*

$$\|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(M)} < +\infty$$

and h solves

$$\Delta_M h + |A|^2 h = f, \quad y \in M, \quad |y| > R_0,$$

then

$$\begin{aligned} \|h\|_{L^\infty(|y|>2R_0)} + \| |y|^2 Dh \|_{L^\infty(|y|>2R_0)} + \| |y|^{4-\frac{4}{p}} D^2 h \|_{L^p(|y|>2R_0)} \leq \\ C [\|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(R_0<|y|<3R_0)}]. \end{aligned}$$

Proof. — Along each end M_k of M , Δ_M can be expanded in the coordinate y as

$$\Delta_M = \Delta + O(|y|^{-2})D^2 + O(|y|^{-3})D.$$

A solution of h of equation (6.1) satisfies

$$\Delta_M h + |A|^2 h = f, \quad |y| > R_0$$

for a sufficiently large R_0 . Let us consider a Kelvin's transform

$$h(y) = \tilde{h}(y/|y|^2).$$

Then we get

$$\Delta h(y) = |y|^{-4}(\Delta \tilde{h})(y/|y|^2).$$

Besides

$$O(|y|^{-2})D^2 h(y) + O(|y|^{-3})Dh(y) = O(|y|^{-6})D^2 \tilde{h}(y/|y|^2) + O(|y|^{-5})D\tilde{h}(y/|y|^2).$$

Hence

$$(\Delta_M h)(y/|y|^2) = |y|^4 \left[\Delta \tilde{h}(y) + O(|y|^2)D^2 \tilde{h}(y) + O(|y|)D\tilde{h}(y) \right].$$

Then \tilde{h} satisfies the equation

$$\Delta \tilde{h} + O(|y|^2)D^2 \tilde{h} + O(|y|)D\tilde{h} + O(1)h = \tilde{f}(y), \quad 0 < |y| < \frac{1}{R_0}$$

where \tilde{f} is given by (6.3). The operator above satisfies maximum principle in $B(0, \frac{1}{R_0})$ if R_0 is fixed large enough. This, the fact that \tilde{h} is bounded, and L^p -elliptic regularity for $p > 2$ in two dimensional space imply that

$$\|\tilde{h}\|_{L^\infty(B(0,1/2R_0))} + \|D\tilde{h}\|_{L^\infty(B(0,1/2R_0))} + \|D^2 \tilde{h}\|_{L^p(B(0,1/2R_0))} \leq$$

$$C[\|\tilde{f}\|_{L^p(B(0,1/R_0))} + \|\tilde{h}\|_{L^\infty(1/3R_0<|y|<1/R_0)}] \leq$$

$$C[\|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(B(R_0<|y|<3R_0))}].$$

Let us observe that

$$\|\tilde{h}\|_{L^\infty(B(0,1/2R_0))} = \|h\|_{L^\infty(|y|>2R_0)},$$

$$\|D\tilde{h}\|_{L^\infty(B(0,1/2R_0))} = \| |y|^2 Dh \|_{L^\infty(|y|>2R_0)}.$$

Since

$$|D^2 h(\mathbf{y})| \leq C(|\mathbf{y}|^{-4} |D^2 \tilde{h}(|\mathbf{y}|^{-2} \mathbf{y})| + |\mathbf{y}|^{-3} |D \tilde{h}(|\mathbf{y}|^{-2} \mathbf{y})|)$$

then

$$|\mathbf{y}|^{4-\frac{4}{p}} |D^2 h(\mathbf{y})| \leq C(|\mathbf{y}|^{-4/p} |D^2 \tilde{h}(|\mathbf{y}|^{-2} \mathbf{y})| + |\mathbf{y}|^{-\frac{4}{p}-1} |D \tilde{h}(|\mathbf{y}|^{-2} \mathbf{y})|).$$

Hence

$$\begin{aligned} & \int_{|\mathbf{y}| > 2R_0} |\mathbf{y}|^{4p-4} |D^2 h|^p d\mathbf{y} \leq \\ & C \left(\int_{B(0,1/2R_0)} |D^2 \tilde{h}(\mathbf{y})|^p d\mathbf{y} + \|D \tilde{h}\|_{L^\infty(B(0,1/2R_0))}^p \int_{|\mathbf{y}| > 2R_0} |\mathbf{y}|^{-4-p} d\mathbf{y} \right). \end{aligned}$$

It follows that

$$\|h\|_{L^\infty(|\mathbf{y}| > 2R_0)} + \|\mathbf{y}|^2 D h\|_{L^\infty(|\mathbf{y}| > 2R_0)} + \|\mathbf{y}|^{4-\frac{4}{p}} D^2 h\|_{L^p(|\mathbf{y}| > 2R_0)} \leq$$

$$C \left[\|f\|_{p, 4-\frac{4}{p}} + \|h\|_{L^\infty(B(R_0 < |\mathbf{y}| < 3R_0))} \right].$$

Since this estimate holds at each end, the result of the lemma follows, after possibly changing slightly the value R_0 . \square

Lemma 6.2. — *Under the conditions of Lemma 6.1, assume that h is a bounded solution of Problem (6.1)-(6.2). Then the a priori estimate (4.34) holds.*

Proof. — Let us observe that this a priori estimate in Lemma 6.1 implies in particular that the Jacobi fields \hat{z}_i satisfy

$$\nabla \hat{z}_i(\mathbf{y}) = O(|\mathbf{y}|^{-2}) \quad \text{as } |\mathbf{y}| \rightarrow +\infty.$$

Using \hat{z}_i as a test function in a ball $B(0, \rho)$ in M we obtain

$$\begin{aligned} & \int_{\partial B(0, \rho)} (h \partial_\nu \hat{z}_i - \hat{z}_i \partial_\nu h) + \int_{|\mathbf{y}| < \rho} (\Delta_M \hat{z}_i + |A|^2 \hat{z}_i) h = \\ & \int_{|\mathbf{y}| < \rho} f \hat{z}_i + \sum_{j=1}^J c_j \int_M \frac{\hat{z}_i \hat{z}_j}{1+r^4}. \end{aligned}$$

Since the boundary integral in the above identity is of size $O(\rho^{-1})$ we get

$$\int_M f \hat{z}_i + \sum_{j=1}^J c_j \int_M \frac{\hat{z}_i \hat{z}_j}{1+r^4} = 0 \quad (6.4)$$

so that in particular

$$|c_j| \leq C \|f\|_{p, 4-\frac{4}{p}} \quad \text{for all } j = 1, \dots, J. \quad (6.5)$$

In order to prove the desired estimate, we assume by contradiction that there are sequences h_n, f_n with $\|h_n\|_\infty = 1$ and $\|f_n\|_{p, 4-\frac{4}{p}} \rightarrow 0$, such that

$$\Delta_M h_n + |A|^2 h_n = f_n + \sum_{j=1}^J \frac{c_j^n \hat{z}_j}{1+r^4}$$

$$\int_M \frac{h_n \hat{z}_i}{1+r^4} = 0 \quad \text{for all } i = 1, \dots, J.$$

Thus according estimate (6.5), we have that $c_i^n \rightarrow 0$. From Lemma 6.1 we find

$$\|h_n\|_{L^\infty(|y|>2R_0)} \leq C[o(1) + \|h_n\|_{L^\infty(B(0,3R_0))}].$$

The latter inequality implies that

$$\|h_n\|_{L^\infty(B(0,3R_0))} \geq \gamma > 0.$$

Local elliptic estimates imply a C^1 bound for h_n on bounded sets. This implies the presence of a subsequence h_n which we denote the same way such that $h_n \rightarrow h$ uniformly on compact subsets of M , where h satisfies

$$\Delta_M h + |A|^2 h = 0.$$

h is bounded hence, by the nondegeneracy assumption, it is a linear combination of the functions \hat{z}_i . Besides $h \neq 0$ and satisfies

$$\int_M \frac{h \hat{z}_i}{1+r^4} = 0 \quad \text{for all } i = 1, \dots, J.$$

The latter relations imply $h = 0$, hence a contradiction that proves the validity of the a priori estimate. \square

6.1. Proof of Proposition 4.3. — Thanks to Lemma 6.2 it only remains to prove existence of a bounded solution to problem (6.1)-(6.2). Let f be as in the statement of the proposition. Let us consider the Hilbert space H of functions $h \in H_{loc}^1(M)$ with

$$\|h\|_H^2 := \int_M |\nabla h|^2 + \frac{1}{1+r^4} |h|^2 < +\infty,$$

$$\int_M \frac{1}{1+r^4} h \hat{z}_i = 0 \quad \text{for all } i = 1, \dots, J.$$

Problem (6.1)-(6.2) can be formulated in weak form as that of finding $h \in H$ with

$$\int_M \nabla h \nabla \psi - |A|^2 h \psi = - \int_M f \psi \quad \text{for all } \psi \in H.$$

In fact, a weak solution $h \in H$ of this problem must be bounded thanks to elliptic regularity, with the use of Kelvin's transform in each end for the control at infinity. Using that $|A|^2 \leq Cr^{-4}$, Riesz representation theorem and the fact that H is compactly embedded in $L^2((1+r^4)^{-1}dV)$ (which follows for instance by inversion at each end), we see that this weak problem can be written as an equation of the form

$$h - T(h) = \tilde{f}$$

where T is a compact operator in H and $\tilde{f} \in H$ depends linearly on f . When $f = 0$, the a priori estimates found yield that necessarily $h = 0$. Existence of a solution then follows from Fredholm's alternative. The proof is complete. \square

6.2. Jacobi fields of logarithmic growth. The proof of Lemma 3.1. — We will use the theory developed above to construct Jacobi fields with logarithmic growth as $r \rightarrow +\infty$, whose existence we stated and use to set up the initial approximation in Lemma 3.1. One of these Jacobi fields is the generator of dilations of the surface, $z_0(y) = y \cdot \nu(y)$. We will prove next that there are another $m - 2$ linearly independent logarithmically growing Jacobi fields.

Let us consider an m -tuple of numbers β_1, \dots, β_m with $\sum_j \beta_j = 0$, and any smooth function $p(y)$ in M such that on each end M_j we have that for sufficiently large $r = r(y)$,

$$p(y) = (-1)^j \beta_j \log r(y), \quad y \in M_j$$

for certain numbers β_1, \dots, β_m that we will choose later. To prove the result of Lemma 3.1 we need to find a solution h_0 of the equation $\mathcal{J}(h_0) = 0$ of the form $h_0 = p + h$ where h is bounded. This amounts to solving

$$\mathcal{J}(h) = -\mathcal{J}(p). \quad (6.6)$$

Let us consider the cylinder $C_R = \{x \in \mathbb{R}^3 / r(x) < R\}$ for a large R . Then

$$\int_{M \cap C_R} \mathcal{J}(p) z_3 dV = \int_{M \cap C_R} \mathcal{J}(z_3) z_3 dV + \int_{\partial C_R \cap M} (z_3 \partial_n p - p \partial_n z_3) d\sigma(y).$$

Hence

$$\int_{M \cap C_R} \mathcal{J}(p) z_3 dV = \sum_{j=1}^m \int_{\partial C_R \cap M_j} (z_3 \partial_n p - p \partial_n z_3) d\sigma(y).$$

Thus using the graph coordinates on each end, we find

$$\begin{aligned} & \int_{M \cap C_R} \mathcal{J}(p) z_3 dV = \\ & \sum_{j=1}^m (-1)^j \left[\frac{\beta_j}{R} \int_{|y|=R} \nu_3 d\sigma(y) - \beta_j \log R \int_{|y|=R} \partial_r \nu_3 d\sigma(y) \right] + O(R^{-1}). \end{aligned}$$

We have that, on each end M_j ,

$$\nu_3(y) = \frac{(-1)^j}{\sqrt{1 + |\nabla F_k(y)|^2}} = (-1)^j + O(r^{-2}), \quad \partial_r \nu_3(y) = O(r^{-3}).$$

Hence we get

$$\int_{M \cap C_R} \mathcal{J}(p) z_3 dV = 2\pi \sum_{j=1}^m \beta_j + O(R^{-1}).$$

It is easy to see, using the graph coordinates that $\mathcal{J}(p) = O(r^{-4})$ and it is hence integrable. We pass to the limit $R \rightarrow +\infty$ and get

$$\int_M \mathcal{J}(p) z_3 dV = 2\pi \sum_{j=1}^m \beta_j = 0. \quad (6.7)$$

We make a similar integration for the remaining bounded Jacobi fields. For $z_i = \nu_i(y)$ $i = 1, 2$ we find

$$\int_{M \cap C_R} \mathcal{J}(p) z_i dV = \sum_{j=1}^m (-1)^j \left[\frac{\beta_j}{R} \int_{|y|=R} \nu_i d\sigma(y) - \beta_j \log R \int_{|y|=R} \partial_r \nu_i d\sigma(y) \right] + O(R^{-1}).$$

Now, on M_j ,

$$\nu_2(\mathbf{y}) = \frac{(-1)^j}{\sqrt{1 + |\nabla F_k(\mathbf{y})|^2}} = (-1)^j a_j \frac{x_i}{r^2} + O(r^{-3}), \quad \partial_r \nu_2(\mathbf{y}) = O(r^{-2}).$$

Hence

$$\int_M \mathcal{J}(p) z_i dV = 0 \quad i = 1, 2.$$

Finally, for $z_4(y) = (-y_2, y_1, 0) \cdot \nu(y)$ we find on M_j ,

$$(-1)^j z_4(\mathbf{y}) = -\mathbf{y}_2 \partial_2 F_j + \mathbf{y}_1 \partial_1 F_j = b_{j1} \frac{y_2}{r^2} - b_{j2} \frac{y_1}{r^2} + O(r^{-2}), \quad \partial_r z_4 = O(r^{-2})$$

and hence again

$$\int_M \mathcal{J}(p) z_4 dV = 0.$$

From the solvability theory developed, we can then find a bounded solution to the problem

$$\mathcal{J}(h) = -\mathcal{J}(p) + \sum_{j=1}^J q c_j \hat{z}_j.$$

Since $\int_M \mathcal{J}(p) z_i dV = 0$ and hence $\int_M \mathcal{J}(p) \hat{z}_i dV = 0$, relations (6.4) imply that $c_i = 0$ for all i .

We have thus found a bounded solution to equation (6.6) and the proof is concluded. \square

Remark 6.1. — Observe that, in particular, the explicit Jacobi field $z_0(y) = y \cdot \nu(y)$ satisfies that

$$z(y) = (-1)^j a_j \log r + O(1) \quad \text{for all } y \in M_j$$

and we have indeed $\sum_j a_j = 0$. Besides this one, we thus have the presence of another $m - 2$ linearly independent Jacobi fields with $|z(y)| \sim \log r$ as $r \rightarrow +\infty$, where m is the number of ends.

These are in reality *all* Jacobi fields with exact logarithmic growth. In fact if $\mathcal{J}(z) = 0$ and

$$|z(y)| \leq C \log r, \tag{6.8}$$

then the argument in the proof of Lemma 6.1 shows that the Kelvin's inversion $\tilde{z}(\mathbf{y})$ as in the proof of Lemma 6.2 satisfies near the origin $\Delta \tilde{z} = \tilde{f}$ where \tilde{f} belongs to any L^p near the origin, so it must equal a multiple of $\log |\mathbf{y}|$ plus a regular function. It follows that on M_j there is a number β_j with

$$z(\mathbf{y}) = (-1)^j \beta_j \log |\mathbf{y}| + h$$

where h is smooth and bounded. The computations above force $\sum_j \beta_j = 0$. It follows from Lemma 3.1 that then z must be equal to one of the elements there predicted plus a bounded Jacobi field. We conclude in particular that the dimension of the space of Jacobi fields satisfying (6.8) must be at most $m - 1 + J$, thus recovering a fact stated in Lemma 5.2 of [33].

7. Reducing the gluing system and solving the projected problem

In this section we prove Lemma 4.1, which reduces the gluing system (4.12)-(4.14) to solving the nonlocal equation (4.20) and prove Proposition 4.2 on solving the nonlinear projected problem (4.21), in which the basic element is linear theory stated in Proposition 4.1. In what follows we refer to notation and objects introduced in §4.1, §4.2.

7.1. Reducing the gluing system. — Let us consider equation (4.12) in the gluing system (4.12)-(4.14),

$$\Delta\psi - W_\alpha(x)\psi + (1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 = 0 \quad \text{in } \mathbb{R}^3 \quad (7.1)$$

where

$$W_\alpha(x) := [(1 - \zeta_1)f'(u_1) + \zeta_1H(t)] .$$

7.1.1. Solving the linear outer problem. — We consider first the linear problem

$$\Delta\psi - W_\alpha(x)\psi + g(x) = 0 \quad \text{in } \mathbb{R}^3 \quad (7.2)$$

We observe that globally we have $0 < a < W_\alpha(x) < b$ for certain constants a and b . In fact we can take $a = \min\{\sigma_-^2, \sigma_+^2\} - \tau$ for arbitrarily small $\tau > 0$.

We consider for the purpose the norms for $1 < p \leq +\infty$,

$$\|g\|_{p,\mu} := \sup_{x \in \mathbb{R}^3} (1 + r(\alpha x))^\mu \|g\|_{L^p(B(x,1))}, \quad r(x', x_3) = |x'| .$$

Lemma 7.1. — *Given $p > 3$, $\mu \geq 0$, there is a $C > 0$ such that for all sufficiently small α and any g with $\|g\|_{p,\mu} < +\infty$ there exists a unique ψ solution to Problem (7.2) with $\|\psi\|_{\infty,\mu} < +\infty$. This solution satisfies in addition,*

$$\|D^2\psi\|_{p,\mu} + \|\psi\|_{\infty,\mu} \leq C\|g\|_{p,\mu}. \quad (7.3)$$

Proof. — We claim that the a priori estimate

$$\|\psi\|_{\infty,\mu} \leq C\|g\|_{p,\mu} \quad (7.4)$$

holds for solutions ψ with $\|\psi\|_{\infty,\mu} < +\infty$ to problem (7.2) with $\|g\|_{p,\mu} < +\infty$ provided that α is small enough. This and local elliptic estimates in turn implies the validity of (7.3). To see this, let us assume the opposite, namely the existence $\alpha_n \rightarrow 0$, and solutions ψ_n to equation (7.2) with $\|\psi_n\|_{\infty,\mu} = 1$, $\|g_n\|_{p,\mu} \rightarrow 0$. Let us consider a point x_n with

$$(1 + r(\alpha_n x_n))^\mu \psi_n(x_n) \geq \frac{1}{2}$$

and define

$$\begin{aligned}\tilde{\psi}_n(x) &= (1 + r(\alpha_n(x_n + x)))^\mu \psi_n(x_n + x), & \tilde{g}_n(x) &= (1 + r(\alpha_n(x_n + x)))^\mu g_n(x_n + x), \\ \tilde{W}_n(x) &= W_{\alpha_n}(x_n + x).\end{aligned}$$

Then, similarly to what was done in the previous section, we check that the equation satisfied by $\tilde{\psi}_n$ has the form

$$\Delta \tilde{\psi}_n - \tilde{W}_n(x) \tilde{\psi}_n + o(1) \nabla \tilde{\psi}_n + o(1) \tilde{\psi}_n = \tilde{g}_n.$$

$\tilde{\psi}_n$ is uniformly bounded. Then elliptic estimates imply L^∞ -bounds for the gradient and the existence of a subsequence uniformly convergent over compact subsets of \mathbb{R}^3 to a bounded solution $\tilde{\psi} \neq 0$ to an equation of the form

$$\Delta \tilde{\psi} - W_*(x) \tilde{\psi} = 0 \quad \text{in } \mathbb{R}^3$$

where $0 < a \leq W_*(x) \leq b$. But maximum principle makes this situation impossible, hence estimate (7.4) holds.

Now, for existence, let us consider g with $\|g\|_{p,\mu} < +\infty$ and a collection of approximations g_n to g with $\|g_n\|_{\infty,\mu} < +\infty$, $g_n \rightarrow g$ in $L^p_{loc}(\mathbb{R}^3)$ and $\|g_n\|_{p,\mu} \leq C\|g\|_{p,\mu}$. The problem

$$\Delta \psi_n - W_n(x) \psi_n = g_n \quad \text{in } \mathbb{R}^3$$

can be solved since this equation has a positive supersolution of the form $C_n(1 + r(\alpha x))^{-\mu}$, provided that α is sufficiently small, but independently of n . Let us call ψ_n the solution thus found, which satisfies $\|\psi_n\|_{\infty,\mu} < +\infty$. The a priori estimate shows that

$$\|D^2 \psi_n\|_{p,\mu} + \|\psi_n\|_{\infty,\mu} \leq C\|g\|_{p,\mu}.$$

and passing to the local uniform limit up to a subsequence, we get a solution ψ to problem (7.2), with $\|\psi\|_{\infty,\mu} < +\infty$. The proof is complete. \square

7.1.2. The proof of Lemma 4.1. — Let us call $\psi := \Upsilon(g)$ the solution of Problem (7.2) predicted by Lemma 7.1. Let us write Problem (7.1) as fixed point problem in the space X of $W_{loc}^{2,p}$ -functions ψ with $\|\psi\|_X < +\infty$,

$$\psi = \Upsilon(g_1 + K(\psi)) \tag{7.5}$$

where

$$g_1 = (1 - \zeta_2)S(\mathbf{w}) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1, \quad K(\psi) = (1 - \zeta_1)N(\psi + \zeta_2\phi).$$

Let us consider a function ϕ defined in $M_\alpha \times \mathbb{R}$ such that $\|\phi\|_{2,p,\mu,\sigma} \leq 1$. Then,

$$|2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1| \leq Ce^{-\sigma\frac{\delta}{\alpha}} (1 + r(\alpha x))^{-\mu} \|\phi\|_{2,p,\mu,\sigma}.$$

We also have that $\|S(\mathbf{w})\|_{p,\mu,\sigma} \leq C\alpha^3$, hence

$$|(1 - \zeta_2)S(\mathbf{w})| \leq Ce^{-\sigma\frac{\delta}{\alpha}} (1 + r(\alpha x))^{-\mu}$$

and

$$\|g_1\|_{p,\mu} \leq Ce^{-\sigma\frac{\delta}{\alpha}}.$$

Let consider the set

$$\Lambda = \{\psi \in X \mid \|\psi\|_X \leq Ae^{-\sigma\frac{\delta}{\alpha}}\},$$

for a large number $A > 0$. Since

$$|K(\psi_1) - K(\psi_2)| \leq C(1 - \zeta_1) \sup_{t \in (0,1)} |t\psi_1 + (1-t)\psi_2 + \zeta_2\phi| |\psi_1 - \psi_2|,$$

we find that

$$\|K(\psi_1) - K(\psi_2)\|_{\infty, \mu} \leq C e^{-\sigma \frac{\delta}{\alpha}} \|\psi_1 - \psi_2\|_{\infty, \mu}$$

while $\|K(0)\|_{\infty, \mu} \leq C e^{-\sigma \frac{\delta}{\alpha}}$. It follows that the right hand side of equation (7.5) defines a contraction mapping of Λ , and hence a unique solution $\psi = \Psi(\phi) \in \Lambda$ exists, provided that the number A in the definition of Λ is taken sufficiently large and $\|\phi\|_{2,p,\mu,\sigma} \leq 1$. In addition, it is direct to check the Lipschitz dependence of Ψ (4.18) on $\|\phi\|_{2,p,\mu,\sigma} \leq 1$. \square

Thus, we replace $\psi = \Psi(\phi)$ into the equation (4.14) of the gluing system (4.12)-(4.14) and get the (nonlocal) problem,

$$\partial_{tt}\phi + \Delta_{y, M_\alpha}\phi = -\tilde{S}(u_1) - \mathbf{N}(\phi) \quad \text{in } M_\alpha \times \mathbb{R} \quad (7.6)$$

where

$$\mathbf{N}(\phi) := \underbrace{\mathbf{B}(\phi) + [f'(u_1) - f'(w)]\phi}_{\mathbf{N}_1(\phi)} + \underbrace{\zeta_1(f'(u_1) - H(t))\Psi(\phi)}_{\mathbf{N}_2(\phi)} + \underbrace{\zeta_1 N(\Psi(\phi) + \phi)}_{\mathbf{N}_3(\phi)}, \quad (7.7)$$

which is what we concentrate in solving next.

7.2. Proof of Proposition 4.2. — We recall from §4.2 that Proposition 4.2 refers to solving the projected problem

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y, M_\alpha}\phi &= -\tilde{S}(u_1) - \mathbf{N}(\phi) + c(y)w'(t) \quad \text{in } M_\alpha \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi(y, t) w'(t) dt &= 0, \quad \text{for all } y \in M_\alpha, \end{aligned} \quad (7.8)$$

and then adjust h_1 so that $c(y) \equiv 0$. Let $\phi = T(g)$ be the linear operator providing the solution in Proposition 4.1. Then Problem (7.8) can be reformulated as the fixed point problem

$$\phi = T(-\tilde{S}(u_1) - \mathbf{N}(\phi)) =: \mathcal{T}(\phi), \quad \|\phi\|_{2,p,\mu,\sigma} \leq 1 \quad (7.9)$$

which is equivalent to

$$\phi = T(-\tilde{S}(u_1) + \alpha^2 \Delta h_1 w' - \mathbf{N}(\phi)), \quad \|\phi\|_{2,p,\mu,\sigma} \leq 1, \quad (7.10)$$

since the term added has the form $\rho(y)w'$ which thus adds up to $c(y)w'$. The reason to absorb this term is that because of assumption (4.7), $\|\alpha^2 \Delta h_1 w'\|_{p,4,\sigma} = O(\alpha^{3-\frac{2}{p}})$ while the remainder has a priori size slightly smaller, $O(\alpha^3)$.

7.2.1. *Lipschitz character of \mathbf{N} .* — We will solve Problem (7.10) using contraction mapping principle, so that we need to give account of a suitable Lipschitz property for the operator \mathcal{T} . We claim the following.

Claim. *We have that for a certain constant $C > 0$ possibly depending on \mathcal{K} in (4.7) but independent of $\alpha > 0$, such that for any ϕ_1, ϕ_2 with*

$$\|\phi_l\|_{2,p,\mu,\sigma} \leq K\alpha^3,$$

$$\|\mathbf{N}(\phi_1) - \mathbf{N}(\phi_2)\|_{p,\mu+1,\sigma} \leq C\alpha \|\phi_1 - \phi_2\|_{2,p,\mu,\sigma} \quad (7.11)$$

where the operator \mathbf{N} is defined in (7.7).

We study the Lipschitz character of the operator \mathbf{N} through analyzing each of its components. Let us start with N_1 . This is a second order linear operator with coefficients of order α plus a decay of order at least $O(r_\alpha^{-1})$. We recall that $B = \zeta_2 B$ where in coordinates

$$\begin{aligned} B = & (f'(u_1) - f'(w)) - \alpha^2[(t+h_1)|A|^2 + \Delta_M h_1] \partial_t - 2\alpha a_{ij}^0 \partial_j h \partial_{it} + \\ & \alpha(t+h) [a_{ij}^1 \partial_{ij} - \alpha a_{ij}^1 (\partial_j h \partial_{it} + \partial_i h \partial_{jt}) + \alpha(b_i^1 \partial_i - \alpha b_i^1 \partial_i h \partial_t)] + \\ & \alpha^3(t+h)^2 b_3^1 \partial_t + \alpha^2 [a_{ij}^0 + \alpha(t+h)a_{ij}^1] \partial_i h \partial_j h \partial_{tt} \end{aligned} \quad (7.12)$$

where, we recall,

$$\begin{aligned} a_{ij}^1 = O(r_\alpha^{-2}), \quad a_{ij}^1 = O(r_\alpha^{-2}), \quad b_i^1 = O(r_\alpha^{-3}), \quad b_i^3 = O(r_\alpha^{-6}), \\ f'(u_1) - f'(w) = O(\alpha^2 r_\alpha^{-2} e^{-\sigma|t|}) \quad \partial_j h = O(r_\alpha^{-1}), \quad |A|^2 = O(r_\alpha^{-4}). \end{aligned}$$

We claim that

$$\|\mathbf{N}_1(\phi)\|_{p,\mu+1,\sigma} \leq C\alpha \|\phi\|_{2,p,\mu,\sigma}. \quad (7.13)$$

The only term of $N_1(\phi)$ that requires a bit more attention is $\alpha^2(\Delta h_1)(\alpha y) \partial_t \phi$. We have

$$\begin{aligned} & \int_{B((y,t),1)} |\alpha^2(\Delta h_1)(\alpha z) \partial_t \phi|^p dV_\alpha(z) d\tau \leq \\ & C\alpha^{2p} \|\partial_t \phi\|_{L^\infty(B((y,t),1))} (1+r_\alpha(y))^{-4p+4} \int_{B((y,t),1)} |(1+r_\alpha(z))^{4-\frac{4}{p}} (\Delta h_1)(\alpha z)|^p |dV_\alpha(z)| \leq \\ & C\alpha^{2p-2} \|\Delta h_1\|_{L^p(M)}^p e^{-p\sigma|t|} (1+r_\alpha(y))^{-p\mu-4p+4} \|\nabla \phi\|_{\infty,\mu,\sigma}, \end{aligned}$$

and hence in particular for $p \geq 3$,

$$\|\alpha^2(\Delta h_1)(\alpha y) \partial_t \phi\|_{p,\mu+2,\sigma} \leq C\alpha^{2-\frac{2}{p}} \|h_1\|_* \|\phi\|_{2,p,\mu,\sigma} \leq C\alpha^{3-\frac{2}{p}} \|\phi\|_{2,p,\mu,\sigma}.$$

Let us consider now functions ϕ_l with

$$\|\phi_l\|_{2,p,\mu,\sigma} \leq 1, \quad l = 1, 2.$$

Now, according to Lemma 4.1, we get that

$$\|\mathbf{N}_2(\phi_1) - \mathbf{N}_2(\phi_2)\|_{p,\mu,\sigma} \leq C e^{-\sigma \frac{\delta}{\alpha}} \|\phi_1 - \phi_2\|_{p,\mu,\sigma}. \quad (7.14)$$

Finally, we also have that

$$|\mathbf{N}_3(\phi_1) - \mathbf{N}_3(\phi_2)| \leq$$

$$C\zeta_1 \sup_{t \in (0,1)} |t(\Psi(\phi_1) + \phi_1) + (1-t)(\Psi(\phi_2) + \phi_2)| [|\phi_1 - \phi_2| + |\Psi(\phi_1) - \Psi(\phi_2)|],$$

hence

$$\|\mathbb{N}_3(\phi_1) - \mathbb{N}_3(\phi_2)\|_{p,2\mu,\sigma} \leq C (\|\phi_1\|_{\infty,\mu,\sigma} + \|\phi_2\|_{\infty,\mu,\sigma} + e^{-\sigma \frac{\delta}{\alpha}}) \|\phi_1 - \phi_2\|_{\infty,\mu,\sigma}. \quad (7.15)$$

From (7.13), (7.14) and (7.15), inequality (7.11) follows. The proof of the claim is concluded.

7.2.2. *Conclusion of the proof of Proposition 4.2.* — The first observation is that choosing $\mu \leq 3$, we get

$$\|\tilde{S}(u_1) + \alpha^2 \Delta h_1 w'\|_{p,\mu,\sigma} \leq C\alpha^3. \quad (7.16)$$

Let us assume now that $\phi_1, \phi_2 \in B_\alpha$ where

$$B_\alpha = \{\phi / \|\phi\|_{2,p,\mu,\sigma} \leq K\alpha^3\}$$

where K is a constant to be chosen. Then we observe that for small α

$$\|\mathbb{N}(\phi)\|_{p,\mu+1,\sigma} \leq C\alpha^4, \quad \text{for all } \phi \in B_\alpha,$$

where C is independent of K . Then, from relations (7.16)-(7.15) we see that if K is fixed large enough independent of α , then the right hand side of equation (7.5) defines an operator that applies B_α into itself, which is also a contraction mapping of B_α endowed with the norm $\|\cdot\|_{p,\mu,\sigma}$, provided that $\mu \leq 3$. We conclude, from contraction mapping principle, the existence of ϕ as required.

The Lipschitz dependence (4.27) is a consequence of series of lengthy but straightforward considerations of the Lipschitz character in h_1 of the operator in the right hand side of equation (7.5) for the norm $\|\cdot\|_*$ defined in (4.34). Let us recall expression (7.12) for the operator B , and consider as an example, two terms that depend linearly on h_1 :

$$A(h_1, \phi) := \alpha a_{ij}^0 \partial_j h_1 \partial_{it} \phi.$$

Then

$$|A(h_1, \phi)| \leq C\alpha |\partial_j h_1| |\partial_{it} \phi|.$$

Hence

$$\|A(h_1, \phi)\|_{p,\mu+2,\sigma} \leq C\alpha \|(1+r_\alpha^2) \partial_j h_1\|_\infty \|\partial_{it} \phi\|_{p,\mu,\sigma} \leq C\alpha^4 \|h_1\|_* \|\phi\|_{2,p,\mu,\sigma}.$$

Similarly, for $A(\phi, h_1) = \alpha^2 \Delta_M h_1 \partial_t \phi$ we have

$$|A(\phi, h_1)| \leq C\alpha^2 |\Delta_M h_1(\alpha y)| (1+r_\alpha)^{-\mu} e^{-\sigma|t|} \|\phi\|_{2,p,\mu,\sigma}.$$

Hence

$$\|\alpha^2 \Delta_M h_1 \partial_t \phi\|_{p,\mu+2,\sigma} \leq C\alpha^{5-\frac{2}{p}} \|h_1\|_* \|\phi\|_{2,p,\mu,\sigma}.$$

We should take into account that some terms involve nonlinear, however mild dependence, in h_1 . We recall for instance that $a_{ij}^1 = a_{ij}^1(\alpha y, \alpha(t+h_0+h_1))$. Examining the rest of the terms involved we find that the whole operator \mathbb{N} produces a dependence on h_1 which is Lipschitz with small constant, and gaining decay in r_α ,

$$\|\mathbb{N}(h_1, \phi) - \mathbb{N}(h_2, \phi)\|_{p,\mu+1,\sigma} \leq C\alpha^2 \|h_1 - h_2\|_* \|\phi\|_{2,p,\mu,\sigma}. \quad (7.17)$$

Now, in the error term

$$\mathcal{R} = -\tilde{S}(u_1) + \alpha^2 \Delta h_1 w',$$

we have that

$$\|\mathcal{R}(h_1) - \mathcal{R}(h_2)\|_{p,3,\sigma} \leq C \alpha^2 \|h_1 - h_2\|_* . \quad (7.18)$$

To see this, again we go term by term in expansion (4.15). For instance the linear term $\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_1 w''$. We have

$$|\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_1| \leq C \alpha^2 (1 + r_\alpha)^{-3} e^{-\sigma|t|} \|h_1\|_*$$

so that

$$\|\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_1\|_{p,3,\sigma} \leq C \alpha^2 \|h_1\|_*,$$

the remaining terms are checked similarly.

Combining estimates (7.17), (7.18) and the fixed point characterization (7.5) we obtain the desired Lipschitz dependence (4.27) of Φ . This concludes the proof. \square

8. The reduced problem: proof of Proposition 4.4

In this section we prove Proposition 4.4 based on the linear theory provided by Proposition 4.3 Thus, we want to solve the problem

$$\mathcal{J}(h_1) = \Delta_M h_1 + h_1 |A|^2 = G(h_1) + \sum_{i=1}^J \frac{c_i}{1+r^4} \hat{z}_i \quad \text{in } M, \quad (8.1)$$

$$\int_M \frac{h_1 \hat{z}_i}{1+r^4} dV = 0 \quad \text{for all } i = 1, \dots, J,$$

where the linearly independent Jacobi fields \hat{z}_i will be chosen in (9.1) and (9.2) of §8, and $G = G_1 + G_2$ was defined in (4.29), (4.30). We will use contraction mapping principle to determine the existence of a unique solution h_1 for which constraint (4.7), namely

$$\|h_1\|_* := \|h_1\|_{L^\infty(M)} + \|(1+r^2)Dh_1\|_{L^\infty(M)} + \|D^2 h_1\|_{p,4-\frac{4}{p}} \leq \mathcal{K} \alpha, \quad (8.2)$$

is satisfied after fixing \mathcal{K} sufficiently large.

We need to analyze the size of the operator G , for which the crucial step is the following estimate.

Lemma 8.1. — *Let $\psi(y, t)$ be a function defined in $M_\alpha \times \mathbb{R}$ such that*

$$\|\psi\|_{p,\mu,\sigma} := \sup_{(y,t) \in M_\alpha \times \mathbb{R}} e^{\sigma|t|} (1+r_\alpha^\mu) \|\psi\|_{L^p(B((y,t),1))} < +\infty$$

for $\sigma, \mu \geq 0$. The function defined in M as

$$q(y) := \int_{\mathbb{R}} \psi(y/\alpha, t) w'(t) dt$$

satisfies

$$\|q\|_{p,a} \leq C \|\psi\|_{p,\mu,\sigma} \quad (8.3)$$

provided that

$$\mu > \frac{2}{p} + a.$$

In particular, for any $\tau > 0$,

$$\|q\|_{p, 2-\frac{2}{p}-\tau} \leq C \|\psi\|_{p, 2, \sigma} \quad (8.4)$$

and

$$\|q\|_{p, 4-\frac{4}{p}} \leq C \|\psi\|_{p, 4, \sigma}. \quad (8.5)$$

Proof. — We have that for $|y| > R_0$

$$\int_{|y|>R_0} |y|^{ap} \left| \int_{\mathbb{R}} \psi(y/\alpha, t) w'(t) dt \right|^p dV \leq C \int_{\mathbb{R}} w'(t) dt \int_{|y|>R_0} |y|^{ap} |\psi(y/\alpha, t)|^p dV.$$

Now

$$\int_{|y|>R_0} |y|^{ap} |\psi(y/\alpha, t)|^p dV = \alpha^{ap+2} \int_{|y|>R_0/\alpha} |y|^{ap} |\psi(y, t)|^p dV_\alpha$$

and

$$\int_{|y|>R_0/\alpha} |y|^{ap} |\psi(y, t)|^p dV_\alpha \leq C \sum_{i \geq [R_0/\alpha]} i^{ap} \int_{i < |y| < i+1} |\psi(y, t)|^p dV_\alpha.$$

Now, $i < |y| < i+1$ is contained in $O(i)$ balls with radius one centered at points of the annulus, hence

$$\begin{aligned} \int_{i < |y| < i+1} |\psi(y, t)|^p dV_\alpha &\leq C e^{-\sigma p |t|} i^{1-\mu p} \|\psi\|_{p, \mu}^p \\ &\leq C e^{-\sigma p |t|} \|\psi\|_{p, \mu}^p \int_{i < |y| < i+1} (1+r_\alpha)^{-\mu p} dV_\alpha \\ &\leq C e^{-\sigma p |t|} \|\psi\|_{p, \mu}^p \int_{i < |y| < i+1} |\alpha y|^{-\mu p} dV_\alpha \\ &\leq C e^{-\sigma p |t|} \|\psi\|_{p, \mu}^p \alpha^{-\mu p} i^{1-\mu p}. \end{aligned}$$

Then we find

$$\| |y|^a q \|_{L^p(|y|>R_0)}^p \leq C \alpha^{ap-\mu p+2} \|\psi\|_{p, \mu}^p \sum_{i \geq [R_0/\alpha]} i^{ap-\mu p+1}.$$

The sum converges if $\mu > \frac{2}{p} + a$ and in this case

$$\| |y|^a q \|_{L^p(|y|>R_0)}^p \leq C \alpha^{ap-\mu p+2} \alpha^{-ap+\mu p-2} \|\psi\|_{p, \mu}^p = C \|\psi\|_{p, \mu}^p$$

so that

$$\| |y|^a q \|_{L^p(|y|>R_0)} \leq C \|\psi\|_{p, \mu}.$$

Now, for the inner part $|y| < R_0$ in M , the weights play no role. We have

$$\begin{aligned} \int_{|y|<R_0} |\psi(y/\alpha, t)|^p dV &= \alpha^2 \int_{|y|<R_0/\alpha} |\psi(y, t)|^p dV_\alpha \leq \\ C \alpha^2 \sum_{i \leq R_0/\alpha} \int_{i < |y| < i+1} |\psi(y, t)|^p dV_\alpha &\leq C \alpha^2 \|\psi\|_{p, \mu}^p e^{-\sigma p |t|} \sum_{i \leq R_0/\alpha} i \end{aligned}$$

$$\leq C \|\psi\|_{p,\mu}^p e^{-\sigma p|t|}.$$

Hence if $\mu > \frac{2}{p} + a$ we finally get

$$\|q\|_{p,a} \leq C \|\psi\|_{p,\mu}$$

and the proof of (8.3) is concluded. Letting $(\mu, a) = (2, 2 - \frac{2}{p} - \tau)$, $(\mu, a) = (4, 4 - \frac{4}{p})$ respectively in (8.3), we obtain (8.4) and (8.5). \square

Let us apply this result to $\psi(y, t) = \mathbf{N}(\Phi(h_1))$ to estimate the size of the operator G_2 in (4.30). For $\phi = \Phi(h_1)$ we have that

$$G_2(h_1)(y) := c_*^{-1} \alpha^{-2} \int_{\mathbb{R}} \mathbf{N}(\phi)(y/\alpha, t) w' dt$$

satisfies

$$\|G_2(h_1)\|_{p,4-\frac{4}{p}} \leq C \alpha^{-2} \|\mathbf{N}(\phi)\|_{p,4,\sigma} \leq C \alpha^2.$$

On the other hand, we have that, similarly, for $\phi_l = \Phi(h_l)$, $l = 1, 2$,

$$\|G_2(h_1) - G_2(h_2)\|_{p,4-\frac{4}{p}} \leq C \alpha^{-2} \|\mathbf{N}(\phi_1, h_1) - \mathbf{N}(\phi_2, h_2)\|_{p,4,\sigma}.$$

Now,

$$\|\mathbf{N}(\phi_1, h_1) - \mathbf{N}(\phi_1, h_2)\|_{p,4,\sigma} \leq C \alpha^2 \|h_1 - h_2\|_* \|\phi_1\|_{2,p,3,\sigma} \leq C \alpha^5 \|h_1 - h_2\|_*,$$

according to inequality (7.17), and

$$\|\mathbf{N}(\phi_1, h_1) - \mathbf{N}(\phi_2, h_1)\|_{p,4,\sigma} \leq C \alpha^2 \|\phi_1 - \phi_2\|_{p,3,\sigma} \leq C \alpha^4 \|h_1 - h_2\|_*.$$

We conclude then that

$$\|G_2(h_1) - G_2(h_2)\|_{p,4-\frac{4}{p}} \leq C \alpha^2 \|h_1 - h_2\|_*.$$

In addition, we also have that

$$\|G_2(0)\|_{p,4-\frac{4}{p}} \leq C \alpha^2.$$

for some $C > 0$ possibly dependent of \mathcal{K} . On the other hand, it is similarly checked that the remaining small operator $G_1(h_1)$ in (4.29) satisfies

$$\|G_1(h_1) - G_1(h_2)\|_{p,4-\frac{4}{p}} \leq C_1 \alpha \|h_1 - h_2\|_*.$$

A simple but crucial observation we make is that

$$c_* G_1(0) = \alpha \partial_i h_0 \partial_j h_0 \int_{\mathbb{R}} \zeta_4(t + h_0) a_{ij}^1 w'' w' dt + \alpha^{-2} \int_{\mathbb{R}} \zeta_4 R_1(y, t, 0, 0) w' dt$$

so that for a constant C_2 independent of \mathcal{K} in (8.2) we have

$$\|G_1(0)\|_{p,4-\frac{4}{p}} \leq C_2 \alpha.$$

In all we have that the operator $G(h_1)$ has an $O(\alpha)$ Lipschitz constant, and in addition satisfies

$$\|G(0)\|_{p,4-\frac{4}{p}} \leq 2C_2 \alpha.$$

Let $h = T(g)$ be the linear operator defined by Proposition 4.3. Then we consider the problem (8.1) written as the fixed point problem

$$h_1 = T(G(h_1)), \quad \|h\|_* \leq \mathcal{K} \alpha. \quad (8.6)$$

We have

$$\|T(G(h_1))\|_* \leq \|T\| \|G(0)\|_{p,4-\frac{4}{p}} + C\alpha \|h_1\|_*.$$

Hence fixing $\mathcal{K} > 2C_2\|T\|$, we find that for all α sufficiently small, the operator TG is a contraction mapping of the ball $\|h\|_* \leq \mathcal{K}\alpha$ into itself. We thus have the existence of a unique solution of the fixed problem (8.6), namely a unique solution h_1 to problem (8.1) satisfying (8.2) and the proof of Proposition 4.4 is concluded. \square

9. Conclusion of the proof of Theorem 1

We denote in what follows

$$r(x) = \sqrt{x_1^2 + x_2^2}, \quad \hat{r} = \frac{1}{r}(x_1, x_2, 0), \quad \hat{\theta} = \frac{1}{r}(-x_2, x_1, 0).$$

We consider the four Jacobi fields associated to rigid motions, z_1, \dots, z_4 introduced in (1.14). Let J be the number of bounded, linearly independent Jacobi fields of \mathcal{J} . By our assumption and the asymptotic expansion of the ends (1.12), $3 \leq J \leq 4$. (Note that when M is a catenoid, $z_4 = 0$ and $J = 3$.) Let us choose

$$\hat{z}_j = \sum_{l=1}^4 d_{jl} z_{0l}, \quad j = 1, \dots, J \quad (9.1)$$

be normalized such that

$$\int_M q(y) \hat{z}_i \hat{z}_j = 0, \quad \text{for } i \neq j, \quad \int_M q(y) \hat{z}_i^2 = 1, \quad i, j = 1, \dots, J. \quad (9.2)$$

In what follows we fix the function q as

$$q(y) := \frac{1}{1 + r(y)^4}. \quad (9.3)$$

So far we have built, for certain constants \tilde{c}_i a solution u of equation (4.36), namely

$$\Delta u + f(u) = \sum_{j=1}^J \tilde{c}_j \hat{z}_j(\alpha y) w'(t) q(\alpha y) \zeta_2$$

where u , defined in (4.35) satisfies the following properties

$$u(x) = w(t) + \phi(y, t) \quad (9.4)$$

near the manifold, meaning this $x = y + (t + h(\alpha y)) \nu(\alpha y)$ with

$$y \in M_\alpha, \quad |t| \leq \frac{\delta}{\alpha} + \gamma \log(2 + r(\alpha y)).$$

The function ϕ satisfies in this region the estimate

$$|\phi| + |\nabla \phi| \leq C\alpha^2 \frac{1}{1 + r^2(\alpha y)} e^{-\sigma|t|}. \quad (9.5)$$

Moreover, we have the validity of the global estimate

$$|\nabla u(x)| \leq \frac{C}{1 + r^3(\alpha x)} e^{-\sigma \frac{\delta}{\alpha}}.$$

We introduce the functions

$$Z_i(x) = \partial_{x_i} u(x), \quad i = 1, 2, 3, \quad Z_4(x) = -\alpha x_2 \partial_{x_2} u + \alpha x_1 \partial_{x_1} u.$$

From the expansion (9.4) we see that

$$\nabla u(x) = w'(t) \nabla t + \nabla \phi.$$

Now, $t = z - h(\alpha y)$ where z designates normal coordinate to M_α . Since $\nabla z = \nu = \nu(\alpha y)$ we then get

$$\nabla t = \nu(\alpha y) - \alpha \nabla h(\alpha y).$$

Let us recall that h satisfies $h = (-1)^k \beta_k \log r + O(1)$ along the k -th end, and

$$\nabla h = (-1)^k \frac{\beta_k}{r} \hat{r} + O(r^{-2}).$$

From estimate (9.5) we we find that

$$\nabla u(x) = w'(t) (\nu - \alpha (-1)^k \frac{\beta_k}{r_\alpha} \hat{r}) + O(\alpha r_\alpha^{-2} e^{-\sigma|t|}). \quad (9.6)$$

From here we get that near the manifold,

$$Z_i(x) = w'(t) (z_i(\alpha y) - \alpha (-1)^k \frac{\beta_k}{r_\alpha} \hat{r} e_i) + O(\alpha r_\alpha^{-2} e^{-\sigma|t|}), \quad i = 1, 2, 3, \quad (9.7)$$

$$Z_4(x) = w'(t) z_{04}(\alpha y) + O(\alpha r_\alpha^{-1} e^{-\sigma|t|}). \quad (9.8)$$

Using the characterization (4.36) of the solution u and barriers (in exactly the same way as in Lemma 11.4 below which estimates eigenfunctions of the linearized operator), we find the following estimate for $r_\alpha(x) > R_0$:

$$|\nabla u(x)| \leq C \sum_{k=1}^m e^{-\sigma|x_3 - \alpha^{-1}(F_k(\alpha x') + \beta_j \alpha \log |\alpha x'|)|}. \quad (9.9)$$

We claim that

$$\int_{\mathbb{R}^3} (\Delta u + f(u)) Z_i(x) dx = 0 \quad \text{for all } i = 1, \dots, 4 \quad (9.10)$$

so that

$$\sum_{j=1}^J \tilde{c}_j \int_{\mathbb{R}^3} q(\alpha x) \hat{z}_j(\alpha y) w'(t) Z_i(x) \zeta_2 dx = 0 \quad \text{for all } i = 1, \dots, 4. \quad (9.11)$$

Let us accept this fact for the moment. Let us observe that from estimates (9.7) and (9.8),

$$\alpha^2 \int_{\mathbb{R}^3} q(\alpha x) \hat{z}_j(\alpha y) w'(t) \sum_{l=1}^4 d_{il} Z_l(x) \zeta_2 dx = \int_{-\infty}^{\infty} w'(t)^2 dt \int_M q \hat{z}_j \hat{z}_i dV + o(1)$$

with $o(1)$ is small with α . Since the functions \hat{z}_i are linearly independent on any open set because they solve an homogeneous elliptic PDE, we conclude that the matrix with the above coefficients is invertible. Hence from (9.11) and (9.2), all \tilde{c}_i 's are

necessarily zero. We have thus found a solution to the Allen Cahn equation (1.1) with the properties required in Theorem 1.

It remains to prove identities (9.10). The idea is to use the invariance of $\Delta + f(u)$ under rigid translations and rotations. This type of Pohozaev identity argument has been used in a number of places, see for instance [15].

In order to prove that the identity (9.10) holds for $i = 3$, we consider a large number $R \gg \frac{1}{\alpha}$ and the infinite cylinder

$$C_R = \{x / x_1^2 + x_2^2 < R^2\}.$$

Since in C_R the quantities involved in the integration approach zero at exponential rate as $|x_3| \rightarrow +\infty$ uniformly in (x_1, x_2) , we have that

$$\int_{C_R} (\Delta u + f(u)) \partial_{x_3} u - \int_{\partial C_R} \nabla u \cdot \hat{r} \partial_{x_3} u = \int_{C_R} \partial_{x_3} (F(u) - \frac{1}{2} |\nabla u|^2) = 0.$$

We claim that

$$\lim_{R \rightarrow +\infty} \int_{\partial C_R} \nabla u \cdot \hat{r} \partial_{x_3} u = 0.$$

Using estimate (9.6) we have that near the manifold,

$$\partial_{x_3} u \nabla u(x) \cdot \hat{r} = w'(t)^2 ((\nu - \alpha(-1)^k \frac{\beta_k}{r_\alpha} \hat{r}) \cdot \hat{r}) \nu_3 + O(\alpha e^{-\sigma|t|} \frac{1}{r^2}).$$

Let us consider the k -th end, which for large r is expanded as

$$x_3 = F_{k,\alpha}(x_1, x_2) = \alpha^{-1}(a_k \log \alpha r + b_k + O(r^{-1}))$$

so that

$$(-1)^k \nu = \frac{1}{\sqrt{1 + |\nabla F_{k,\alpha}|^2}} (\nabla F_{k,\alpha}, -1) = \frac{a_k}{\alpha} \frac{\hat{r}}{r} - e_3 + O(r^{-2}). \quad (9.12)$$

Then on the portion of C_R near this end we have that

$$(\nu - \alpha(-1)^k \frac{\beta_k}{r_\alpha} \hat{r}) \cdot \hat{r} \nu_3 = -\alpha^{-1} \frac{a_k + \alpha \beta_k}{R} + O(R^{-2}). \quad (9.13)$$

In addition, also, for $x_1^2 + x_2^2 = R^2$ we have the expansion

$$t = (x_3 - F_{k,\alpha}(x_1, x_2) - \beta_k \log \alpha r + O(1))(1 + O(R^{-2}))$$

with the same order valid after differentiation in x_3 , uniformly in such (x_1, x_2) . Let us choose $\rho = \gamma \log R$ for a large, fixed γ . Observe that on ∂C_R the distance between ends is greater than 2ρ whenever α is sufficiently small. We get,

$$\int_{F_{k,\alpha}(x_1, x_2) + \beta_k \log \alpha r - \rho}^{F_{k,\alpha}(x_1, x_2) + \beta_k \log \alpha r + \rho} w'(t)^2 dx_3 = \int_{-\infty}^{\infty} w'(t)^2 dt + O(R^{-2}).$$

Because of estimate (9.9) we conclude, fixing appropriately γ , that

$$\int_{\bigcap_k \{|x_3 - F_{k,\alpha}| > \rho\}} \partial_{x_3} u \nabla u(x) \cdot \hat{r} dx_3 = O(R^{-2}).$$

As a conclusion

$$\int_{-\infty}^{\infty} \partial_{x_3} u \nabla u \cdot \hat{r} \, dx_3 = -\frac{1}{\alpha R} \sum_{k=1}^m (a_k + \alpha \beta_k) \int_{-\infty}^{\infty} w'(t)^2 \, dt + O(R^{-2})$$

and hence

$$\int_{\partial C_R} \partial_{x_3} u \nabla u(x) \cdot \hat{r} = -\frac{2\pi}{\alpha} \sum_{k=1}^m (a_k + \alpha \beta_k) + O(R^{-1}).$$

But $\sum_{k=1}^m a_k = \sum_{k=1}^m \beta_k = 0$ and hence (9.10) for $i = 3$ follows after letting $R \rightarrow \infty$.

Let us prove the identity for $i = 2$. We need to carry out now the integration against $\partial_{x_2} u$. In this case we get

$$\int_{C_R} (\Delta u + f(u)) \partial_{x_2} u = \int_{\partial C_R} \nabla u \cdot \hat{r} \partial_{x_2} u + \int_{C_R} \partial_{x_2} (F(u) - \frac{1}{2} |\nabla u|^2).$$

We have that

$$\int_{C_R} \partial_{x_2} (F(u) - \frac{1}{2} |\nabla u|^2) = \int_{\partial C_R} (F(u) - \frac{1}{2} |\nabla u|^2) n_2$$

where $n_2 = x_2/r$. Now, near the ends estimate (9.6) yields

$$|\nabla u|^2 = |w'(t)|^2 + O(e^{-\sigma|t|} \frac{1}{r^2})$$

and arguing as before, we get

$$\int_{-\infty}^{\infty} |\nabla u|^2 \, dx_3 = m \int_{-\infty}^{\infty} |w'(t)|^2 \, dt + O(R^{-2}).$$

Hence

$$\int_{\partial C_R} |\nabla u|^2 n_2 = m \int_{-\infty}^{\infty} |w'(t)|^2 \, dt \int_{[r=R]} n_2 + O(R^{-1}).$$

Since $\int_{[r=R]} n_2 = 0$ we conclude that

$$\lim_{R \rightarrow +\infty} \int_{\partial C_R} |\nabla u|^2 n_2 = 0.$$

In a similar way we get

$$\lim_{R \rightarrow +\infty} \int_{\partial C_R} F(u) n_2 = 0.$$

Since near the ends we have

$$\partial_{x_2} u = w'(t) (\nu_2 - \alpha (-1)^k \frac{\beta_k}{r_\alpha} \hat{r} e_2) + O(\alpha r^{-2} e^{-\sigma|t|})$$

and from (9.12) $\nu_2 = O(R^{-1})$, completing the computation as previously done yields

$$\int_{\partial C_R} \nabla u \cdot \hat{r} \partial_{x_2} u = O(R^{-1}).$$

As a conclusion of the previous estimates, letting $R \rightarrow +\infty$ we finally find the validity of (9.10) for $i = 2$. Of course the same argument holds for $i = 1$.

Finally, for $i = 4$ it is convenient to compute the integral over C_R using cylindrical coordinates. Let us write $u = u(r, \theta, z)$. Then

$$\begin{aligned} & \int_{C_R} (\Delta u + f(u)) (x_2 \partial_{x_1} u - x_1 \partial_{x_2} u) = \\ & \int_0^{2\pi} \int_0^R \int_{-\infty}^{\infty} [u_{zz} + r^{-1}(ru_r)_r + f(u)] u_\theta r \, d\theta \, dr \, dz = \\ & -\frac{1}{2} \int_0^{2\pi} \int_0^R \int_{-\infty}^{\infty} \partial_\theta [u_z^2 + u_r^2 - 2F(u)] r \, d\theta \, dr \, dz + R \int_{-\infty}^{\infty} \int_0^{2\pi} u_r u_\theta(R, \theta, z) \, d\theta \, dz = \\ & 0 + \int_{\partial C_R} u_r u_\theta . \end{aligned}$$

On the other hand, on the portion of ∂C_R near the ends we have

$$u_r u_\theta = w'(t)^2 R(\nu \cdot \hat{r})(\nu \cdot \hat{\theta}) + O(R^{-2}e^{-\sigma|t|}).$$

From (9.12) we find

$$(\nu \cdot \hat{r})(\nu \cdot \hat{\theta}) = O(R^{-3}),$$

hence

$$u_r u_\theta = w'(t)^2 O(R^{-2}) + O(R^{-2}e^{-\sigma|t|})$$

and finally

$$\int_{\partial C_R} u_r u_\theta = O(R^{-1}).$$

Letting $R \rightarrow +\infty$ we obtain relation (9.10) for $i = 4$. The proof is concluded. \square

10. Negative eigenvalues and their eigenfunctions for the Jacobi operator

For the proof of Theorem 2 we need to translate the information on the index of the minimal surface M into spectral features of the Jacobi operator. Since M has finite total curvature, the index $i(M)$ of the minimal surface M is finite. We will translate this information into an eigenvalue problem for the operator \mathcal{J} . Let

$$\mathbf{Q}(k, k) := \int_M |\nabla k|^2 \, dV - \int_M |A|^2 k^2 \, dV .$$

The number $i(M)$ is, by definition, the largest dimension for a vector space E of compactly supported smooth functions in M such that

$$\mathbf{Q}(z, z) < 0 \quad \text{for all } z \in E \setminus \{0\}.$$

The number $i(M)$ when finite has the following convenient characterization, whose proof is straightforward. In what follows we fix the function q as

$$p(y) := \frac{1}{1 + r(y)^4} . \tag{10.14}$$

Let us consider for a large number R , the region

$$M^R = \{y \in M / r(y) < R\}$$

and the eigenvalue problem

$$\begin{aligned} \Delta_M k + |A|^2 k + \lambda p(y) k &= 0 \quad \text{in } M^R, \\ k &= 0 \quad \text{on } \partial M^R. \end{aligned} \quad (10.15)$$

Let $m_R(p)$ denote the number of negative eigenvalues (counting multiplicities) for this problem. Then we have

$$i(M) = \sup_{R>0} m_R(p). \quad (10.16)$$

Let us also consider the eigenvalue problem in entire space

$$\Delta_M k + |A|^2 k + \lambda p(y) k = 0 \quad \text{in } M, \quad k \in L^\infty(M). \quad (10.17)$$

We will prove the following result.

Lemma 10.1. — *Problem (10.17) has exactly $i(M)$ negative eigenvalues, counting multiplicities.*

10.0.3. A priori estimates in M^R . — For the proof of Lemma 10.1, and for later purposes, it is useful to have a priori estimates uniform in large $R > 0$ for the linear problem

$$\begin{aligned} \Delta_M k + |A|^2 k - \gamma p(y) k &= f \quad \text{in } M^R, \\ k &= 0 \quad \text{on } \partial M^R. \end{aligned} \quad (10.18)$$

We have the following result.

Lemma 10.2. — *Let $p > 1$, $\sigma > 0$. Then for $R_0 > 0$ large enough and fixed and $\gamma_0 > 0$, there exist a $C > 0$ such that for all $R > R_0 + 1$, $0 \leq \gamma < \gamma_0$, any f , and any solution k of problem (10.18), we have that*

(a) *If $\|f\|_{p, 4-\frac{4}{p}} < +\infty$ then*

$$\|k\|_\infty \leq C [\|f\|_{p, 4-\frac{4}{p}} + \|k\|_{L^\infty(|y| < 3R_0)}]. \quad (10.19)$$

(b) *If $\|f\|_{p, 2-\frac{2}{p}-\sigma} < +\infty$, then ,*

$$\|D^2 k\|_{p, 2-\frac{2}{p}-\sigma} + \|Dk\|_{p, 1-\frac{2}{p}-\sigma} \leq C [\|f\|_{p, 2-\frac{2}{p}-\sigma} + \|k\|_\infty]. \quad (10.20)$$

If $p > 2$, we have in addition

$$\|(1 + |y|)^{1-\sigma} Dk\|_\infty \leq C [\|f\|_{p, 2-\frac{2}{p}-\sigma} + \|k\|_\infty]. \quad (10.21)$$

Proof. — Let us consider the equation in M

$$\Delta_M \psi + |A|^2 \psi = -|f| \chi_{|y| < R}, \quad |y| > R_0, \quad (10.22)$$

$$\psi(y) = 0, \quad |y| = R_0. \quad (10.23)$$

For a large and fixed R_0 , solving this problem amounts to doing it on each separate end. As in Lemma 6.1, after a Kelvin's transform the problem reduces in each end to solving in a ball in \mathbb{R}^2 an equation of the form

$$\Delta \tilde{\psi} + O(|\mathbf{y}|^2)D^2\tilde{\psi} + O(|\mathbf{y}|)D\tilde{\psi} + O(1)\tilde{\psi} = -|\tilde{f}|\chi_{|\mathbf{y}|>\frac{1}{R}}, \quad |\mathbf{y}| < \frac{1}{R_0},$$

$$\tilde{\psi}(\mathbf{y}) = 0, \quad |\mathbf{y}| = \frac{1}{R_0}.$$

Enlarging R_0 if necessary, this problem has a unique solution, which is also positive. This produces a bounded, positive solution ψ of (10.22)-(10.23) with

$$\|\psi\|_\infty \leq C\|f\|_{p,4-\frac{4}{p}}.$$

On the other hand, on this end the Jacobi field $z_3 = \nu \cdot e_3$ can be taken positive with $z_3 \geq 1$ on $|\mathbf{y}| > R_0$. Thus the function $\psi + \|k\|_{L^\infty(|\mathbf{y}|=R_0)}z_3$ is a positive, bounded supersolution for the problem (10.18) in this end, where $|\mathbf{y}| > R_0$, and estimate (10.19) then readily follows.

Let us prove now estimate (10.20). Fix a large number $R_0 > 0$ and another number $R \gg R_0$. Consider also a large $\rho > 0$ with $3\rho < R$. On a given end we parameterize with Euclidean coordinates $\mathbf{y} \in \mathbb{R}^2$ and get that the equation satisfied by $k = k(\mathbf{y})$ reads

$$\Delta k + O(|\mathbf{y}|^{-2})D^2k + O(|\mathbf{y}|^{-3})Dk + O(|\mathbf{y}|^{-4})k = f, \quad R_0 < |\mathbf{y}| < R.$$

Consider the function $k_\rho(z) = k(\rho z)$ wherever it is defined. Then

$$\Delta k_\rho + O(\rho^{-2}|z|^{-2})D^2k_\rho + O(\rho^{-2}|z|^{-3})Dk_\rho + O(\rho^{-2}|z|^{-4})k_\rho = f_\rho$$

where $f_\rho(z) = \rho^2 f(\rho z)$. Then interior elliptic estimates (see Theorem 9.11 of [14]) yield the existence of a constant $C = C(p)$ such that for any sufficiently large ρ

$$\|Dk_\rho\|_{L^p(1 < |z| < 2)} + \|D^2k_\rho\|_{L^p(1 < |z| < 2)} \leq C(\|k_\rho\|_{L^\infty(\frac{1}{2} < |z| < 3)} + \|f_\rho\|_{L^p(\frac{1}{2} < |z| < 3)}). \quad (10.24)$$

Now,

$$\|f_\rho\|_{L^p(\frac{1}{2} < |z| < 3)}^p = \rho^{2p} \int_{(\frac{1}{2} < |z| < 3)} |f(\rho z)|^p dz \leq$$

$$C\rho^{p\sigma} \int_{(\frac{1}{2} < |z| < 3)} |\rho z|^{2p-2-p\sigma} |f(\rho z)|^p \rho^2 dz = C\rho^{p\sigma} \int_{(\frac{\rho}{2} < |\mathbf{y}| < 3\rho)} |\mathbf{y}|^{2p-2-p\sigma} |f(\mathbf{y})|^p d\mathbf{y}.$$

Similarly

$$\|D^2k_\rho\|_{L^p(1 < |z| < 2)}^p \geq C\rho^{p\sigma} \int_{(\rho < |\mathbf{y}| < 2\rho)} |\mathbf{y}|^{2p-2-p\sigma} |D^2k(\mathbf{y})|^p d\mathbf{y}.$$

Thus

$$\int_{(\rho < |\mathbf{y}| < 2\rho)} |\mathbf{y}|^{2p-2-p\sigma} |D^2k(\mathbf{y})|^p d\mathbf{y} \leq C \int_{(\frac{\rho}{2} < |\mathbf{y}| < 4\rho)} |\mathbf{y}|^{2p-2-p\sigma} |f(\mathbf{y})|^p d\mathbf{y} + \rho^{-p\sigma} \|k\|_\infty^p.$$

Take $\rho = \rho_j = 2^j$. Then

$$\int_{(\rho_j < |\mathbf{y}| < \rho_{j+1})} |\mathbf{y}|^{2p-2-p\sigma} |D^2k(\mathbf{y})|^p d\mathbf{y} \leq$$

$$C \int_{(\rho_{j-1} < |y| < \rho_{j+2})} |y|^{2p-2-p\sigma} |f(y)|^p dy + 2^{-j p \sigma} \|k\|_\infty^p.$$

Then, adding up these relations wherever they are defined, taking in addition into account boundary elliptic estimates which give that for $\rho = \frac{R}{2}$,

$$\|D^2 k_\rho\|_{L^p(1 < |z| < 2)} \leq C \left(\|k_\rho\|_{L^\infty(\frac{1}{2} < |z| < 2)} + \|f_\rho\|_{L^p(\frac{1}{2} < |z| < 2)} \right),$$

plus a local elliptic estimate in a bounded region, we obtain that for some $C > 0$ independent of R ,

$$\|D^2 k\|_{p, 2 - \frac{2}{p} - \sigma} \leq C (\|k\|_\infty + \|f\|_{p, 2 - \frac{2}{p} - \sigma}).$$

The corresponding estimate for the gradient follows immediately from (10.24). We have proven (10.20). If $p > 2$ we can use Sobolev's embedding to include $\|Dk_\rho\|_{L^\infty(1 < |z| < 2)}$ on the left hand side of (10.24), and estimate (10.21) follows. The proof is complete. \square

10.0.4. Proof of Lemma 10.1. — We will prove first that problem (10.17) has at least $i(M)$ linearly independent eigenfunctions associated to negative eigenvalues in $L^\infty(M)$. For all $R > 0$ sufficiently large, problem (10.15) has $n = i(M)$ linearly independent eigenfunctions $k_{1,R}, \dots, k_{n,R}$ associated to negative eigenvalues

$$\lambda_{1,R} \leq \lambda_{2,R} \leq \dots \leq \lambda_{n,R} < 0.$$

Through the min-max characterization of these eigenvalues, we see that they can be chosen to define decreasing functions of R . On the other hand, $\lambda_{1,R}$ must be bounded below. Indeed, for a sufficiently large $\gamma > 0$ we have that

$$|A|^2 - \gamma p < 0 \quad \text{in } M$$

and by maximum principle we must have $\lambda_{1,R} > -\gamma$. The eigenfunctions can be chosen orthogonal in the sense that

$$\int_{M^R} p k_{i,R} k_{j,R} dV = 0 \quad \text{for all } i \neq j. \quad (10.25)$$

Let us assume that $\|k_{i,R}\|_\infty = 1$. Then the a priori estimate in Lemma 10.2 imply that, passing to a subsequence in $R \rightarrow +\infty$, we may assume that

$$\lambda_{i,R} \downarrow \lambda_i < 0, \quad k_{i,R}(y) \rightarrow k_i(y),$$

uniformly on compact subsets of M , where $k_i \neq 0$ is a bounded eigenfunction of (10.17) associated to the negative eigenvalue λ_i . Moreover, relations (10.25) pass to the limit and yield

$$\int_M p k_i k_j dV = 0 \quad \text{for all } i \neq j. \quad (10.26)$$

Thus, problem (10.17) has at least $n = i(M)$ negative eigenvalues. Let us assume there is a further bounded eigenfunction k_{n+1} , linearly independent of k_1, \dots, k_n , say with

$$\int_M p k_i k_{n+1} dV = 0 \quad \text{for all } i = 1, \dots, n, \quad (10.27)$$

associated to a negative eigenvalue λ_{n+1} . Then the a priori estimate of Lemma 6.1 implies that

$$\|(1+r^2)\nabla k_{n+1}\| < +\infty.$$

The same of course holds for the remaining k_i 's. It follows that

$$\mathcal{Q}(k, k) < 0 \quad \text{for all } k \in \text{span}\{k_1, \dots, k_{n+1}\} \setminus \{0\}.$$

However, again since ∇k_j decays fast, the same relation above will hold true for the k_i 's replaced by suitable smooth truncations far away from the origin. This implies, by definition, $i(M) \geq n+1$ and we have reached a contradiction. The proof is concluded. \square

11. The proof of Theorem 2

In this section we will prove that the Morse index $m(u_\alpha)$ of the solution we have built in Theorem 1 coincides with the index of the surface M , as stated in Theorem 2. We recall that this number is defined as the supremum of all dimensions of vector spaces E of compactly supported smooth functions for which

$$\mathcal{Q}(\psi, \psi) = \int_{\mathbb{R}^3} |\nabla \psi|^2 - f'(u_\alpha)\psi^2 < 0 \quad \text{for all } \psi \in E \setminus \{0\}.$$

We provide next a more convenient characterization of this number, analogous to that for the Jacobi operator of §10. Let us consider a smooth function $p(x)$ defined in \mathbb{R}^3 such that

$$p(\alpha x) = \frac{1}{1+r_\alpha(y)^4} \quad \text{if } x = y + (t+h(\alpha y))\nu(\alpha y) \in \mathcal{N}_\delta,$$

and such that for positive numbers a, b ,

$$\frac{a}{1+|\alpha x'|^4} \leq p(\alpha x) \leq \frac{b}{1+|\alpha x'|^4} \quad \text{for all } x = (x', x_3) \in \mathbb{R}^3.$$

For each $R > 0$, we consider the eigenvalue problem in the cylinder

$$\mathcal{C}_R = \{(x', x_3) / |x'| < R\alpha^{-1}, |x_3| < R\alpha^{-1}\},$$

$$\Delta \phi + f'(u_\alpha)\phi + \lambda p(\alpha x)\phi = 0 \quad \text{in } \mathcal{C}_R, \tag{11.1}$$

$$\phi = 0 \quad \text{on } \partial \mathcal{C}_R.$$

We also consider the problem in entire space

$$\Delta \phi + f'(u_\alpha)\phi + \lambda p(\alpha x)\phi = 0 \quad \text{in } \mathbb{R}^3, \quad \phi \in L^\infty(\mathbb{R}^3). \tag{11.2}$$

Let $m_R(u_\alpha)$ be the number of negative eigenvalues λ (counting multiplicities) of this Problem (11.1). Then we readily check that

$$m(u_\alpha) = \sup_{R>0} m_R(u_\alpha).$$

On the other hand, we have seen in §10 that the index $i(M)$ of the minimal surface can be characterized as the number of linearly independent eigenfunctions associated to negative eigenvalues of the problem

$$\Delta z + |A|^2 z + \lambda p(y)z = 0 \quad \text{in } M, \quad z \in L^\infty(M), \quad (11.3)$$

which corresponds to the maximal dimension of the negative subspace in $L^\infty(M)$ for the quadratic form

$$\mathcal{Q}(z, z) = \int_M |\nabla_M z|^2 - |A|^2 z^2 dV.$$

We shall prove in this section that $m(u_\alpha) = i(M)$ for any sufficiently small α .

The idea of the proof is to put in correspondence eigenfunctions for negative eigenvalues of problem (11.1) for large R with those of problem (11.3). This correspondence comes roughly as follows. If z is such an eigenfunction for problem (11.3) then the function defined near M_α as

$$k(y)w'(t), \quad k(y) = z(\alpha y) \quad (11.4)$$

defines after truncation a negative direction for the quadratic form \mathcal{Q} on any large ball. Reciprocally, an eigenfunction for negative eigenvalue of problem (11.1) will look for any sufficiently small α and all large R like a function of the form (11.4). In the following two lemmas we clarify the action of the operator L on functions of this type, and the corresponding connection at the level of the quadratic forms \mathcal{Q} and \mathcal{Q} .

Lemma 11.1. — *Let $k(y)$ be a function of class C^2 defined in some open subset \mathcal{V} of M_α . Let us consider the function $v(x)$ defined for $x \in \mathcal{N}_\delta$, $y \in \mathcal{V}$ as*

$$v(x) = v(y, t) := k(y)w'(t), \quad y \in \mathcal{V}, \quad |t + h_1(\alpha y)| < \rho_\alpha(y)$$

where ρ_α is the function in the definition of \mathcal{N}_α , (3.11). Then $L(v) := \Delta_x v + f'(u_\alpha)v$ can be expanded as in (11.8) below. Besides we have

$$\begin{aligned} \int_{|t+h_1|<\rho_\alpha} L(v)w' dt &= (\Delta_{M_\alpha} k + \alpha^2 |A|^2 k + \alpha h a_{ij}^{1,0} \partial_{ij} k) \int_{\mathbb{R}} w'^2 dt \\ &+ O(\alpha^2 r_\alpha^{-2}) \partial_{ij} k + O(\alpha^2 r_\alpha^{-3}) \partial_i k + O(\alpha^3 r_\alpha^{-4}) k. \end{aligned} \quad (11.5)$$

Here

$$a_{ij}^{1,0} = a_{ij}^{1,0}(\alpha y) = O(r_\alpha^{-2}).$$

The same conclusions hold for the function

$$v(x) = v(y, t) := k(y)w'(t)\eta_\delta(y, t), \quad y \in \mathcal{V}, \quad |t + h_1(\alpha y)| < \rho_\alpha(y)$$

where the cut-off function η_δ is defined in (3.12).

Proof. — Let us recall that

$$\begin{aligned} \Delta_x &= \partial_{tt} + \Delta_{M_\alpha} - \alpha^2 [(t+h)|A|^2 + \Delta_M h] \partial_t - 2\alpha a_{ij}^0 \partial_j h \partial_{it} + \\ &\alpha(t+h) [a_{ij}^1 \partial_{ij} - 2\alpha a_{ij}^1 \partial_j h \partial_{it} + \alpha(b_i^1 \partial_i - \alpha b_i^1 \partial_i h \partial_t)] + \\ &\alpha^3 (t+h)^2 b_3^1 \partial_t + \alpha^2 [a_{ij}^0 + \alpha(t+h)a_{ij}^1] \partial_i h \partial_j h \partial_{it}. \end{aligned}$$

Hence, using Lemma 2.2 in the appendix we get

$$\begin{aligned}
\Delta_x v + f'(u_\alpha)v &= k(w''' + f'(w)w') + [f'(u_\alpha) - f'(w)]kw' + \\
&w' \Delta_{M_\alpha} k - \alpha^2[(t+h_1)|A|^2 + \Delta_M h_1]kw'' - 2\alpha a_{ij}^0 \partial_j h \partial_i k w'' + \\
&\alpha(t+h) [a_{ij}^1 \partial_{ij} k w' - \alpha a_{ij}^1 (\partial_j h \partial_i k + \partial_i h \partial_j k) w'' + \alpha(b_i^1 \partial_i k w' - \alpha b_i^1 \partial_i h w'')]] + \\
&\alpha^3(t+h)^2 b_3^1 k w'' + \alpha^2 [a_{ij}^0 + \alpha(t+h)a_{ij}^1] \partial_i h \partial_j h k w'''. \tag{11.6}
\end{aligned}$$

We can expand

$$a_{ij}^1 = a_{ij}^1(\alpha y, 0) + \alpha(t+h) a_{ij}^2(\alpha y, \alpha(t+h)) =: a_{ij}^{1,0} + \alpha(t+h) a_{ij}^2,$$

with $a_{ij}^2 = O(r_\alpha^{-2})$, and similarly

$$b_j^1 = b_j^1(\alpha y, 0) + \alpha(t+h) b_j^2(\alpha y, \alpha(t+h)) =: b_j^{1,0} + \alpha(t+h) b_j^2,$$

with $b_j^2 = O(r_\alpha^{-3})$. On the other hand, let us recall that

$$u_\alpha - w = \phi_1 + O(\alpha^3 r_\alpha^{-4} e^{-\sigma|t|})$$

where ϕ_1 is given by (3.2),

$$\phi_1(y, t) = \alpha^2 |A(\alpha y)|^2 \psi_0(t) - \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0(\alpha y) \psi_1(t) \tag{11.7}$$

and ψ_0, ψ_1 decay exponentially as $|t| \rightarrow +\infty$. Hence

$$[f'(u_\alpha) - f'(w)]w' = f''(w)\phi_1 w' + O(\alpha^3 e^{-\sigma|t|} r_\alpha^{-4}).$$

Using these considerations and expression (11.6), we can write,

$$\begin{aligned}
Q &:= \Delta_x v + f'(u_\alpha)v = \\
&\underbrace{\Delta_{M_\alpha} k w' - \alpha^2 |A|^2 k t w'' + \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0 k w'''}_{Q_1} + \underbrace{f''(w)\phi_1 k w'}_{Q_2} \\
&\underbrace{-w'' \left[\alpha a_{ij}^0 (\partial_j h \partial_i k + \partial_i h \partial_j k) + \alpha^2 k \Delta_M h_1 + \alpha^2 h a_{ij}^{1,0} (\partial_j h \partial_i k + \partial_i h \partial_j k) \right]}_{Q_3} \\
&\quad + \underbrace{\alpha t w' \left[a_{ij}^{1,0} \partial_{ij} k + \alpha b_i^{1,0} \partial_i k \right]}_{Q_4} \\
&\quad + \underbrace{\alpha^2 (t+h)^2 a_{ij}^2 \partial_{ij} k w' + \alpha^2 (t+h) a_{ij}^2 (\partial_j h \partial_i k + \partial_i h \partial_j k) w''}_{Q_5} + \underbrace{O(\alpha^3 e^{-\sigma|t|} r_\alpha^{-2})}_{Q_6}. \tag{11.8}
\end{aligned}$$

The precise meaning of the remainder Q_6 is

$$Q_6 = O(\alpha^3 e^{-\sigma|t|} r_\alpha^{-2}) \partial_{ij} k + O(\alpha^3 e^{-\sigma|t|} r_\alpha^{-3}) \partial_j k.$$

We will integrate the above relation against $w'(t)$ in the region $|t + h_1(\alpha y)| < \rho_\alpha(y)$.

Let us observe that the terms Q_i for $i = 1, \dots, 4$ are in reality defined for all t and that

$$\int_{|t+h_1|<\rho_\alpha} Q_i w' dt = \int_{\mathbb{R}} Q_i w' dt + O(\alpha^3 r_\alpha^{-4}) \quad (11.9)$$

where the remainder means

$$O(\alpha^3 r_\alpha^{-4}) := O(\alpha^3 r_\alpha^{-4}) \partial_{ij} k + O(\alpha^3 r_\alpha^{-4}) \partial_i k + O(\alpha^3 r_\alpha^{-4}) k .$$

Let us observe that

$$\int_{\mathbb{R}} (Q_3 + Q_4) w' dt = 0 . \quad (11.10)$$

On the other hand, since

$$\int_{\mathbb{R}} t w'' w' dt = -\frac{1}{2} \int_{\mathbb{R}} w'^2 dt ,$$

we get that

$$\int_{\mathbb{R}} Q_1 w' dt = (\Delta_{M_\alpha} k + \frac{1}{2} |A|^2 k + \alpha h a_{ij}^{1;0} \partial_{ij} k) \int_{\mathbb{R}} w'^2 dt + a_{ij}^0 \partial_i h_0 \partial_j h_0 \int_{\mathbb{R}} w''' w' dt . \quad (11.11)$$

Next we will compute $\int_{\mathbb{R}} Q_2 w' dt$. We recall that, setting $L_0(\psi) = \psi'' + f'(w)\psi$, the functions ψ_0 and ψ_1 in (11.7) satisfy

$$L_0(\psi_0) = t w'(t), \quad L_0(\psi_1) = w'' .$$

Differentiating these equations we get

$$L_0(\psi_0') + f''(w) w' \psi_0 = (t w')', \quad L_0(\psi_1') + f''(w) w' \psi_1 = w''' .$$

Integrating by parts against w' , using $L_0(w') = 0$ we obtain

$$\int_{\mathbb{R}} f''(w) w'^2 \psi_0 = - \int_{\mathbb{R}} t w'' w' = \frac{1}{2} \int_{\mathbb{R}} w'^2, \quad \int_{\mathbb{R}} f''(w) w'^2 \psi_1 = \int_{\mathbb{R}} w''' w' .$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}} Q_1 w' dt &= \int_{\mathbb{R}} f''(w) \phi_1 k w'^2 dt = \\ &= \alpha^2 k |A|^2 \int_{\mathbb{R}} f''(w) \psi_0 w'^2 dt - \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0 k \int_{\mathbb{R}} f''(w) \psi_1 w'^2 dt = \\ &= \alpha^2 k |A|^2 \frac{1}{2} \int_{\mathbb{R}} w'^2 - \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0 k \int_{\mathbb{R}} w''' w' . \end{aligned} \quad (11.12)$$

Thus, combining relations (11.10)-(11.12) we get

$$\int_{\mathbb{R}} (Q_1 + \dots + Q_4) w' dt = (\Delta_{M_\alpha} k + |A|^2 k + \alpha h a_{ij}^{1;0} \partial_{ij} k) \int_{\mathbb{R}} w'^2 dt . \quad (11.13)$$

On the other hand, we observe that

$$\int_{|t+h_1|<\rho_\alpha} (Q_5 + Q_6) w' dt = O(\alpha^2 r_\alpha^{-2}) \partial_{ij} k + O(\alpha^2 r_\alpha^{-3}) \partial_i k . \quad (11.14)$$

Combining relations (11.13), (11.14) and (11.9), expansion (11.5) follows. Finally, for v replaced by $\eta_\delta k w'$ we have that

$$\int L(kw\eta_\delta)\eta_\delta kw' dt = \int \eta_\delta^2 L(kw)kw dt + \int \eta_\delta(\Delta\eta_\delta kw' + 2\nabla\eta_\delta\nabla(kw')) kw dt .$$

The arguments above apply to obtain the desired expansion for the first integral in the right hand side of the above decomposition. The second integral produces only smaller order operators in k since $\Delta\eta_\delta, \nabla\eta_\delta$ are both of order $O(r_\alpha^{-4}\alpha^4)$ inside their supports. The proof is concluded. \square

Let us consider now the region

$$\mathcal{W} := \{x \in \mathcal{N}_\delta / r_\alpha(y) < R\},$$

where R is a given large number.

Lemma 11.2. — *Let $k(y)$ be a smooth function in M_α that vanishes when $r_\alpha(y) = R$, and set $v(y, t) := \eta_\delta(y, t) k(y) w'(t)$. Then the following estimate holds.*

$$\begin{aligned} \mathcal{Q}(v, v) &= \int_{\mathcal{W}} |\nabla v|^2 - f'(u_\alpha) v^2 dx = \\ &= \int_{r_\alpha(y) < R} [|\nabla_{M_\alpha} k|^2 - \alpha^2 |A(\alpha y)|^2 k^2] dV_\alpha \int_{\mathbb{R}} w'^2 dt \\ &+ O\left(\alpha \int_{r_\alpha(y) < R} [|\nabla k|^2 + \alpha^2 (1 + r_\alpha^4)^{-1} k^2] dV_\alpha\right). \end{aligned} \quad (11.15)$$

Proof. — Let us estimate first the quantity

$$\int_{\mathcal{W}} L(kw') kw' dx .$$

Let us express the Euclidean element of volume dx in the coordinates (y, t) . Consider one of the charts $Y_l(y)$, $y \in \mathcal{U}_l$ of M , $l = 1, \dots, N$ introduced in (2.6), which induce corresponding charts in M_α as in (2.9). Then, dropping the index l , we compute the element of volume through the change of coordinates

$$x = X(y, t) = \alpha^{-1} Y(\alpha y) + (t + h)\nu(\alpha y) . \quad (11.16)$$

Below, as in subsequent computations, the computation of the integral in the entire region is performed by localization through smooth partition of unity ξ_1, \dots, ξ_m subordinated to the covering $Y_l(\mathcal{U}_k)$, $l = 1, \dots, N$ of M , namely with the support of ξ_l is contained in $Y_l(\mathcal{U}_k)$ and $\sum_l \xi_l \equiv 1$. We perform typically a computation of an integral of a function $g(x)$ defined in \mathcal{N}_δ as

$$\int_{\mathcal{N}_\delta} g(x) dx = \sum_{l=1}^N \int_{\mathcal{N}_\delta} \xi_l(\alpha y) g(y, t) dx(y, t) . \quad (11.17)$$

Let us keep this in mind particularly for estimates obtained by integration by parts relative to local variables for y .

Let us consider the coordinates X in (11.16). Then we have that
 $DX(y, t) = [\partial_1 Y + \alpha(t+h)\partial_1 \nu + \alpha\partial_1 h \nu \mid \partial_2 Y + \alpha(t+h)\partial_2 \nu + \alpha\partial_2 h \nu \mid \nu](\alpha y)$,
hence

$$\begin{aligned} \det DX(y, t) &= \det [\partial_1 Y + \alpha(t+h)\partial_1 \nu \mid \partial_2 Y + \alpha(t+h)h\partial_2 \nu \mid \nu](\alpha y) = \\ &= \det [\partial_1 Y \mid \partial_2 Y \mid \nu] + \alpha^2(t+h)^2 \det [\partial_1 \nu \mid \partial_2 \nu \mid \nu] + \\ &= \alpha(t+h) \{ \det [\partial_1 Y \mid \partial_2 \nu \mid \nu] + \det [\partial_1 \nu \mid \partial_2 Y \mid \nu] \}. \end{aligned}$$

Since mean curvature of M vanishes and the Gauss curvature equals $|A|^2$, we obtain

$$\begin{aligned} dx &= |\det DX(y, t)| dy dt = \\ &= (1 + \alpha^2(t+h)^2 |A|^2) |\det [\partial_1 Y \mid \partial_2 Y \mid \nu]|(\alpha y) dy dt = \\ &= (1 + \alpha^2(t+h)^2 |A(\alpha y)|^2) dV_\alpha(y) dt. \end{aligned}$$

Using this, we estimate

$$\begin{aligned} I &= \int_{\mathcal{W}} L(kw') kw' dx = \\ &= \int_{r_\alpha(y) < R} \int_{|t+h_1| < \rho_\alpha} [L(kw') kw'] (1 + \alpha^2(t+h)^2 |A(\alpha y)|^2) dV_\alpha(y) dt. \end{aligned}$$

According to Lemma 11.1,

$$\begin{aligned} I &= \underbrace{\int (\Delta_{M_\alpha} k + \alpha^2 |A(\alpha y)|^2 k) k dV_\alpha(y)}_{I_1} + \\ &= \underbrace{\int_{r_\alpha(y) < R} [O(\alpha r_\alpha^{-2} \log r_\alpha) k \partial_{ij} k + O(\alpha^2 r_\alpha^{-3}) k \partial_j k + O(\alpha^3 r_\alpha^{-4}) k^2] dV_\alpha(y)}_{I_2} \\ &= \underbrace{\alpha^2 \int_{\mathcal{W}} L(kw') kw' (t+h)^2 |A(\alpha y)|^2 dV_\alpha(y) dt}_{I_3}. \end{aligned}$$

Integrating by parts, we see that

$$-I_1 = \int_{r_\alpha(y) < R} [|\nabla_{M_\alpha} k|^2 - \alpha^2 |A(\alpha y)|^2] dV_\alpha(y).$$

The quantity I_2 involves some abuse of notation since it is expressed in local coordinates for y associated to each chart, and the total should be understood in the sense (11.17). Integrating by parts in those coordinates, we get

$$I_2 = \int_{r_\alpha(y) < R} [O(\alpha r_\alpha^{-2} \log r_\alpha) \partial_i k \partial_j k + O(\alpha^2 r_\alpha^{-3} \log r_\alpha) \partial_i k k + O(\alpha^3 r_\alpha^{-4}) k^2] dV_\alpha(y).$$

Now,

$$|\alpha^2 O(r_\alpha^{-3} \log r_\alpha) \partial_i k k| \leq C [\alpha |\nabla k|^2 + \alpha^3 (1 + r_\alpha^4)^{-1} k^2],$$

and hence we have

$$|I_2| \leq C \alpha \int_{r_\alpha(y) < R} [|\nabla k|^2 + \alpha^2 (1 + r_\alpha^4)^{-1} k^2] dV_\alpha(y) \quad (11.18)$$

where C is independent of ν and all small α . Finally, to deal with the term I_3 , we consider the expression (11.8) for $L(kw')$ and integrate by parts once the terms involving second derivatives of k . Using that $|A|^2 = O(r_\alpha^{-4})$ we then get that

$$|I_3| \leq C\alpha \int_{r_\alpha(y) < R} [|\nabla k|^2 + \alpha^2(1+r_\alpha^4)^{-1}k^2] dV_\alpha(y). \quad (11.19)$$

The same considerations above hold for kw' replaced by $\eta_\delta kw'$, at this point we observe that since $\eta_\delta kw'$ satisfies Dirichlet boundary conditions, we have

$$\begin{aligned} - \int_{\mathcal{W}} L(kw'\eta_\delta) \eta_\delta kw' dx &= \int_{\mathcal{W}} |\nabla(kw'\eta_\delta)|^2 - \alpha^2 |\eta_\delta kw'|^2 dx = \\ &= \int_{r_\alpha(y) < R} \{ |\nabla_{M_\alpha} k|^2 - \alpha^2 |A(\alpha y)|^2 k^2 \} dV_\alpha \int_{\mathbb{R}} w'^2 dt \\ &\quad + O\left(\alpha \int_{r_\alpha(y) < R} [|\nabla k|^2 + \alpha^2(1+r_\alpha^4)^{-1}k^2] dV_\alpha\right) \end{aligned}$$

and estimate (11.15) has been established. This concludes the proof. \square

After Lemma 11.2, the inequality

$$m(u_\alpha) \geq i(M) \quad (11.20)$$

for small α follows at once. Indeed, we showed in §10 that the Jacobi operator has exactly $i(M)$ linearly independent bounded eigenfunctions \hat{z}_i associated to negative eigenvalues λ_i of the weighted problem in entire space M . According to the theory developed in §6, we also find that $\nabla \hat{z}_i = O(r^{-2})$, hence we may assume

$$\mathbf{Q}(\hat{z}_i, \hat{z}_j) = \lambda_i \int_M q \hat{z}_i \hat{z}_j dV. \quad (11.21)$$

Let us set $k_i(y) := \hat{z}_i(\alpha y)$. According to Lemma 11.2, setting $v_i(x) = k_i(y) w'(t) \eta_\delta$ and changing variables we get

$$\mathcal{Q}(v_i, v_j) = \alpha^2 \mathbf{Q}(\hat{z}_i, \hat{z}_j) \int_{\mathbb{R}} w'^2 + O(\alpha^3) \sum_{l=i,j} \int_M |\nabla \hat{z}_l|^2 + (1+r^4)^{-1} \hat{z}_l^2 dV. \quad (11.22)$$

From here and relations (11.21), we find that the quadratic form \mathcal{Q} is negative on the space spanned by the functions $v_1, \dots, v_{i(M)}$. The same remains true for the functions v_i smoothly truncated around $r_\alpha(y) = R$, for very large R . We have proven then inequality (11.20).

In what remains this section we will carry out the proof of the inequality

$$m(u_\alpha) \leq i(M). \quad (11.23)$$

Relation (11.22) suggests that associated to a negative eigenvalue λ_i of problem (11.3), there is an eigenvalue of (11.1) approximated by $\sim \lambda_i \alpha^2$. We will show next that negative eigenvalues of problem (11.1) cannot exceed a size $O(\alpha^2)$.

Lemma 11.3. — *There exists a $\mu > 0$ independent of $R > 0$ and all small α such that if λ is an eigenvalue of problem (11.1) then*

$$\lambda \geq -\mu \alpha^2 .$$

Proof. — Let us denote

$$\mathcal{Q}_\Omega(\psi, \psi) := \int_\Omega |\nabla \psi|^2 - f'(u_\alpha) \psi^2 .$$

Then if $\psi(x)$ is any function that vanishes for $|x'| > R\alpha^{-1}$ then we have

$$\mathcal{Q}(\psi, \psi) \geq \mathcal{Q}_{\mathcal{N}_\delta \cap \{r_\alpha(y) < R\}}(\psi, \psi) + \gamma \int_{\mathbb{R}^3 \setminus \mathcal{N}_\delta} \psi^2$$

where $\gamma > 0$ is independent of α and R . We want to prove that for some $\mu > 0$ we have in $\Omega = \mathcal{N}_\delta \cap \{r_\alpha(y) < R\}$ that

$$\mathcal{Q}_\Omega(\psi, \psi) \geq -\mu \alpha^2 \int_\Omega \frac{\psi^2}{1 + r_\alpha^4} dx . \quad (11.24)$$

Equivalently, let us consider the eigenvalue problem

$$L(\psi) + \lambda p(\alpha x) \psi = 0 \quad \text{in } \Omega, \quad (11.25)$$

$$\psi = 0 \quad \text{on } r_\alpha = R, \quad \partial_n \psi = 0 \quad \text{on } |t + h_1| = \rho_\alpha .$$

Then we need to show that for any eigenfunction ψ associated to a negative eigenvalue, inequality (11.24) holds. Here ∂_n denotes normal derivative. Let us express this boundary operator in terms of the coordinates (t, y) . Let us consider the portion of $\partial \mathcal{N}_\delta$ where

$$t + h_1(\alpha y) = \rho_\alpha(y). \quad (11.26)$$

We recall that for some $\gamma > 0$, $\rho_\alpha(y) = \rho(\alpha y) = \gamma \log(1 + r_\alpha(y))$. Relation (11.26) is equivalent to

$$z - h_0(\alpha y) - \rho_\alpha(y) = 0 \quad (11.27)$$

where z denotes the normal coordinate to M_α . Then, for $\nabla = \nabla_x$, we have that a normal vector to the boundary at a point satisfying (11.27) is

$$n = \nabla z - \nabla_{M_\alpha}(h_0 + \rho) = \nu(\alpha y) - \alpha \nabla_M(h_0 + \rho)(\alpha y).$$

Now, we have that $\partial_t \psi = \nabla_x \psi \cdot \nu(\alpha y)$. Hence, on points (11.26), condition $\partial_n \psi = 0$ is equivalent to

$$\partial_t \psi - \alpha \nabla_M(h_0 + \rho) \cdot \nabla_{M_\alpha} \psi = 0, \quad (11.28)$$

and similarly, for

$$t + h_1(\alpha y) = \rho_\alpha(y). \quad (11.29)$$

it corresponds to

$$\partial_t \psi - \alpha \nabla_M(h_0 - \rho) \cdot \nabla_{M_\alpha} \psi = 0 . \quad (11.30)$$

Let us consider a solution ψ of problem (11.25). We decompose

$$\psi = k(y)w'(t)\eta_\delta + \psi^\perp$$

where η_δ is the cut-off function (3.12) and

$$\int_{|\tau+h_1(\alpha y)|<\rho_\alpha(y)} \psi^\perp(y, \tau) w'(\tau) d\tau = 0 \quad \text{for all } y \in M_\alpha \cap \{r_\alpha(y) < R\},$$

namely

$$k(y) = \frac{\int_{|\tau+h_1(\alpha y)|<\rho_\alpha(y)} \psi(y, \tau) w'(\tau) d\tau}{\int_{\mathbb{R}} w'(t)^2 \eta_\delta dt}. \quad (11.31)$$

Then we have

$$\mathcal{Q}_\Omega(\psi, \psi) = \mathcal{Q}_\Omega(\psi^\perp, \psi^\perp) + \mathcal{Q}_\Omega(kw'\eta_\delta, kw'\eta_\delta) + 2\mathcal{Q}_\Omega(kw'\eta_\delta, \psi^\perp).$$

Since ψ^\perp satisfies the same boundary conditions as ψ we have that

$$\mathcal{Q}_\Omega(\psi^\perp, \psi^\perp) = - \int_\Omega (\psi^\perp \Delta_x \psi^\perp + f'(u_\alpha) \psi^{\perp 2}) dx.$$

Thus,

$$\mathcal{Q}_\Omega(\psi^\perp, \psi^\perp) = - \int_{r_\alpha < R} \int_{|t+h_1|<\rho_\alpha} [\psi^\perp \Delta_x \psi^\perp + f'(u_\alpha) \psi^{\perp 2}] (1 + \alpha^2(t+h)^2 |A|^2) dV_\alpha dt.$$

Let us fix a smooth function $H(t)$ with $H(t) = +1$ if $t > 1$, $H(t) = -1$ for $t < -1$. Let us write

$$\begin{aligned} -\Delta_x \psi^\perp - f'(u_\alpha) \psi^\perp &= -\partial_{tt} \psi^\perp - f'(w) \psi^\perp + \alpha \partial_t [\nabla_M(h_0 + H(t)\rho) \cdot \nabla_{M_\alpha} \psi^\perp] \\ &\quad - \Delta_{M_\alpha} \psi^\perp + B(\psi^\perp). \end{aligned}$$

Then, integrating by parts in t , using the Neumann boundary condition, we get that the integral

$$\begin{aligned} I &:= \\ &- \int_{|t+h_1|<\rho_\alpha} [\partial_{tt} \psi^\perp + f'(w) \psi^\perp - \alpha \partial_t (\nabla_M(h_0 + H(t)\rho) \cdot \nabla_{M_\alpha} \psi^\perp)] \psi^\perp (1 + \alpha^2(t+h)^2 |A|^2) dt \\ &= \int_{|t+h_1|<\rho_\alpha} [\partial_t \psi^\perp - \alpha (\nabla_M(h_0 + H(t)\rho) \cdot \nabla_{M_\alpha} \psi^\perp)] \partial_t \psi^\perp (1 + \alpha^2(t+h)^2 |A|^2) dt \\ &\quad - \int_{|t+h_1|<\rho_\alpha} f'(w) \psi^{\perp 2} (1 + \alpha^2(t+h)^2 |A|^2) dt \\ &\quad + \int_{|t+h_1|<\rho_\alpha} [\partial_t \psi^\perp - \alpha (\nabla_M(h_0 + H(t)\rho) \cdot \nabla_{M_\alpha} \psi^\perp)] \psi^\perp 2\alpha^2(t+h)^2 |A|^2 dt \\ &= \int_{|t+h_1|<\rho_\alpha} [|\partial_t \psi^\perp|^2 - f'(w) |\psi^\perp|^2] (1 + o(1)) + \alpha O(r_\alpha^{-1}) \nabla_{M_\alpha} \psi^\perp \partial_t \psi^\perp + o(1) \partial_t \psi^\perp \psi^\perp dt. \end{aligned}$$

Now we need to make use of the following fact: there is a $\gamma > 0$ such that if $a > 0$ is a sufficiently large number, then for any smooth function $\xi(t)$ with $\int_{-a}^a \xi w' dt = 0$ we have that

$$\int_{-a}^a \xi'^2 - f'(w) \xi^2 dt \geq \gamma \int_{-a}^a \xi'^2 + \xi^2 dt. \quad (11.32)$$

Inequality (11.32) is just a perturbation of the inequality (5.2). We leave the details to the reader.

Hence

$$I \geq \frac{\gamma}{2} \int_{|t+h_1| < \rho_\alpha} [|\partial_t \psi^\perp|^2 + |\psi^\perp|^2] dt + \int_{|t+h_1| < \rho_\alpha} \alpha O(r_\alpha^{-1}) \nabla_M \psi^\perp \partial_t \psi^\perp dt. \quad (11.33)$$

On the other hand, for the remaining part, integrating by parts in the y variable the terms that involve two derivatives of ψ^\perp we get that

$$\begin{aligned} II := & - \int_{|t+h_1| < \rho_\alpha} dt \int_{r_\alpha(y) < R} (\Delta_{M_\alpha} \psi^\perp + B\psi^\perp) \psi^\perp (1 + \alpha^2(t+h)^2|A|^2) dV_\alpha(y) \geq \\ & \int_{|t+h_1| < \rho_\alpha} dt \int_{r_\alpha(y) < R} |\nabla_{M_\alpha} \psi^\perp|^2 + o(1) (\psi^{\perp 2} + |\partial_t \psi^\perp|^2 |\nabla_{M_\alpha} \psi^\perp|^2). \end{aligned} \quad (11.34)$$

Using estimates (11.33), (11.34), we finally get

$$\mathcal{Q}_\Omega(\psi^\perp, \psi^\perp) \geq 3\mu \int_\Omega (|\partial_t \psi^\perp|^2 + |\nabla_{M_\alpha} \psi^\perp|^2 + \psi^{\perp 2}) dx, \quad (11.35)$$

for some $\mu > 0$.

Now, we estimate the crossed term. We have

$$-\mathcal{Q}_\Omega(\psi^\perp, kw' \eta_\delta) = \int_\Omega L(kw' \eta_\delta) \psi^\perp (1 + \alpha^2(t+h)^2|A|^2) dV_\alpha dt.$$

Let us consider expression (11.8) for $L(kw')$, and let us also consider the fact that

$$L(\eta_\delta kw') = \eta_\delta L(kw') + 2\nabla \eta_\delta \nabla(kw') + \Delta \eta_\delta kw',$$

with the last two terms producing a first order operator in k with exponentially small size, at the same time with decay $O(r_\alpha^{-4})$. Thus all main contributions come from the integral

$$I = \int_\Omega \eta_\delta L(kw') \psi^\perp (1 + \alpha^2(t+h)^2|A|^2) dV_\alpha dt.$$

Examining the expression (11.8), integrating by parts once in y variable those terms involving two derivatives in k , we see that most of the terms obtained produce straightforwardly quantities of the type

$$\theta := o(1) \int_{M_\alpha} (|\nabla k|^2 + \alpha^2|A|^2 k^2) dV_\alpha + o(1) \int_\Omega (|\psi^\perp|^2 + |\nabla \psi^\perp|^2).$$

In fact we have

$$\begin{aligned} I = & \underbrace{\int_\Omega \Delta_{M_\alpha} k w' \eta_\delta \psi^\perp dV_\alpha dt}_{I_1} + \underbrace{\int_\Omega \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0 k w''' \psi^\perp dV_\alpha dt}_{I_2} + \\ & \underbrace{\int_\Omega f''(w) \phi_1 k w' \psi^\perp dV_\alpha dt}_{I_3} + \theta. \end{aligned}$$

On the other hand, the orthogonality definition of ψ^\perp essentially eliminates I_1 . Indeed,

$$I_1 = - \int_\Omega \Delta_{M_\alpha} k w' (1 - \eta_\delta) \psi^\perp dV_\alpha dt = \int_\Omega \nabla_{M_\alpha} k w' [(1 - \eta_\delta) \nabla_{M_\alpha} \psi^\perp - \nabla \eta_\delta \psi^\perp] dV_\alpha dt = \theta.$$

On the other hand, for a small, fixed number $\nu > 0$ we have

$$|I_2| \leq C\alpha^2 \int_{\Omega} \frac{1}{1+r_{\alpha}^2} |k| |w''| |\psi^{\perp}| dV_{\alpha} dt \leq C\nu^{-1}\alpha^2 \int_{M_{\alpha}} \frac{1}{1+r_{\alpha}^4} k^2 dV_{\alpha} + \nu \int_{\Omega} |\psi|^2 dx.$$

A similar control is valid for I_3 since $\phi_1 = O(\alpha^2 r_{\alpha}^{-2})$. We then get

$$I \geq -C\nu^{-1}\alpha^2 \int_{M_{\alpha}} \frac{1}{1+r_{\alpha}^4} k^2 dV_{\alpha} - \nu \int_{\Omega} |\psi^{\perp}|^2. \quad (11.36)$$

Finally, we recall that from Lemma 11.2,

$$\mathcal{Q}_{\Omega}(kw'\eta_{\delta}, kw'\eta_{\delta}) = \int_{r_{\alpha}(y) < R} [|\nabla_{M_{\alpha}} k|^2 - \alpha^2 |A(\alpha y)|^2 k^2] dV_{\alpha} \int_{\mathbb{R}} w'^2 dt + \theta. \quad (11.37)$$

From estimates (11.35), (11.36), (11.37), we obtain that if ν is chosen sufficiently small, then

$$\mathcal{Q}_{\Omega}(\psi, \psi) \geq -C\alpha^2 \int_{M_{\alpha}} \frac{1}{1+r_{\alpha}^4} k^2 dV_{\alpha} \geq -\mu\alpha^2 \int_{\Omega} \frac{1}{1+r_{\alpha}^4} |\psi|^2 dx,$$

for some $\mu > 0$ and inequality (11.24) follows. \square

In the next result, we show that an eigenfunction with negative eigenvalue of problem (11.1) or (11.2) decays exponentially, away from the interface of u_{α} .

Lemma 11.4. — *let ϕ be a solution of either (11.1) or (11.2) with $\lambda \leq 0$. Then ϕ satisfies in the subregion of \mathcal{N}_{α} where it is defined that*

$$|\phi(y, t)| \leq C \|\phi\|_{\infty} e^{-\sigma|t|} \quad (11.38)$$

where $\sigma > 0$ can be taken arbitrarily close to $\min\{\sigma_+, \sigma_-\}$. The number C depends on σ but it is independent of small α and large R . We have, moreover, that for $|\alpha x'| > R_0$,

$$|\phi(x)| \leq C \sum_{j=1}^m e^{-\sigma|x_3 - \alpha^{-1}(F_k(\alpha x') + \beta_j \alpha \log |\alpha x'|)|}. \quad (11.39)$$

where R_0 is independent of α . Finally, we have that

$$|\phi(x)| \leq C e^{-\sigma \frac{\delta}{\alpha}} \quad \text{for } \text{dist}(x, M_{\alpha}) > \frac{\delta}{\alpha}. \quad (11.40)$$

Proof. — Let ϕ solve problem (11.1) for a large R . Let us consider the region between two consecutive ends $M_{j,\alpha}$ and $M_{j+1,\alpha}$. For definiteness, we assume that this region lies inside S_+ so that $f'(u_{\alpha})$ approaches σ_+^2 inside it. So, let us consider the region S of points $x = (x', x_3)$ such that $r_{\alpha}(x) > R_0$ for a sufficiently large but fixed $R_0 > 0$ and

$$(a_j + \alpha\beta_j) \log \alpha|x'| + b_j + \alpha\gamma < \alpha x_3 < (a_{j+1} + \alpha\beta_{j+1}) \log \alpha|x'| + b_{j+1} - \alpha\gamma.$$

In terms of the coordinate t near $M_{j,\alpha}$, saying that

$$\alpha x_3 \sim (a_j + \alpha\beta_j) \log \alpha|x'| + b_j + \alpha\gamma$$

is up to lower order terms, the same as saying $t \sim \gamma$, similarly near $M_{j+1,\alpha}$. Thus given any small number $\tau > 0$ we can choose γ sufficiently large but fixed, independently of all R_0 sufficiently large and any small α , such that

$$f'(u_\alpha) < -(\sigma_+ - \tau)^2 \quad \text{in } S.$$

Let us consider, for $x \in S$ and $\sigma = \sigma_+ - 2\tau$ the function

$$v_1(x) := e^{-\sigma[x_3 - \alpha^{-1}(a_j + \alpha\beta_j) \log \alpha |x'| + b_j]} + e^{-\sigma(\alpha^{-1}[a_{j+1} + \alpha\beta_{j+1}] \log \alpha |x'| + b_{j+1}) - x_3}.$$

Then v has the form

$$v_1 = A_1 e^{-\sigma x_3} r^{A_2} + B_1 e^{\sigma x_3} r^{-B_2}, \quad r = |x'|,$$

so that

$$\begin{aligned} \Delta v_1 &= A_2^2 r^{-2} r^{A_2} A_1 e^{-\sigma x_3} + B_2^2 r^{-2} B_1 r^{-B_2} e^{\sigma x_3} + \sigma^2 v_1 < \\ &[\alpha^2 A_2^2 R_0^{-2} + \alpha^2 B_2^2 R_0^{-2} + \sigma^2] v_1. \end{aligned}$$

Here

$$A_2 = \sigma \alpha^{-1}(a_j + \alpha\beta_j), \quad B_2 = \sigma \alpha^{-1}(a_{j+1} + \alpha\beta_{j+1}).$$

Hence, enlarging R_0 if necessary, we achieve

$$\Delta v_1 + f'(u_\alpha) v_1 < 0 \quad \text{in } S.$$

Therefore v so chosen is a positive supersolution of

$$\Delta v + f'(u_\alpha) v + \lambda p(\alpha x) v \leq 0 \quad \text{in } S. \quad (11.41)$$

Observe that the definition of v also achieves that

$$\inf_{\partial S \setminus \{r_\alpha = R_0\}} \geq \gamma > 0$$

where γ is independent of α . Now, let us observe that the function $v_2 = e^{-\sigma(|x'| - \frac{R_0}{\alpha})}$ also satisfies, for small α , inequality (11.41). As a conclusion, for ϕ , solution of (11.1), we have that

$$|\phi(x)| \leq C \|\phi\|_\infty [v_1(x) + v_2(x)] \quad \text{for all } x \in S, r_\alpha(x) < R. \quad (11.42)$$

Using the form of this barrier, we then obtain the validity of estimate (11.39), in particular that of (11.38), in the subregion of \mathcal{N}_δ in the positive t direction of $M_{j,\alpha}$ and $M_{j+1,\alpha}$ when $r_\alpha(y) > R_0$. The remaining subregions of $\mathcal{N}_\delta \cap \{r_\alpha(y) > R_0\}$ are dealt with in a similar manner. Finally, to prove the desired estimate for $r_\alpha(y) < r_0$ we consider the region where $|t| < \frac{2\delta}{\alpha}$ assuming that the local coordinates are well defined there. In this case we use, for instance in the region

$$\nu < t < \frac{2\delta}{\alpha}$$

for $\nu > 0$ large and fixed, a barrier of the form

$$v(y, t) = e^{-\sigma t} + e^{-\sigma(\frac{2\delta}{\alpha} - t)}.$$

It is easily seen that for small α this function indeed satisfies

$$\Delta_x v - f'(u_\alpha) v < 0$$

where σ can be taken arbitrarily close to σ_+ . We conclude that

$$|\phi(y, t)| \leq C \|\phi\|_\infty e^{-\sigma t} \quad \text{for } \nu < t < \frac{\delta}{\alpha}.$$

Thus estimate (11.38) holds true. Inequality (11.40) follows from maximum principle.

Finally, for a solution of problem (11.2) the same procedure works, with only minor difference introduced. Estimate (11.42) can be obtained after adding a growing barrier. Indeed, we obtain

$$|\phi(x)| \leq C \|\phi\|_\infty [v_1(x) + v_2(x) + \varepsilon v_3(x)] \quad \text{for all } x \in S$$

with $v_3(x) = \varepsilon e^{\sigma|x'|}$, and then we let $\varepsilon \rightarrow 0$. We should also use $\varepsilon e^{\sigma x_3}$ to deal with the region above the last end M_m and similarly below M_1 . We then use the controls far away to deal with the comparisons at the second step. The proof is concluded. \square

11.1. The proof of inequality (10.23). — Let us assume by contradiction that there is a sequence $\alpha = \alpha_n \rightarrow 0$ along which

$$m(u_\alpha) > i(M) =: N.$$

This implies that for some sequence $R_n \rightarrow +\infty$ we have that, for all $R > R_n$, Problem (11.1) has at least $N + 1$ linearly independent eigenfunctions

$$\phi_{1,\alpha,R}, \dots, \phi_{N+1,\alpha,R}$$

associated to negative eigenvalues

$$\lambda_{1,\alpha,R} \leq \lambda_{2,\alpha,R} \leq \dots \leq \lambda_{N+1,\alpha,R} < 0.$$

We may assume that $\|\phi_{i,\alpha,R}\|_\infty = 1$ and that

$$\int_{\mathbb{R}^3} p(\alpha x) \phi_{i,\alpha,R} \phi_{j,\alpha,R} dx = 0 \quad \text{for all } i, j = 1, \dots, N+1, \quad i \neq j.$$

Let us observe that then the estimates in Lemma 11.4 imply that the contribution to the above integrals of the region outside \mathcal{N}_δ is small. We have at most

$$\int_{\mathcal{N}_\delta} p(\alpha y) \phi_{i,\alpha,R} \phi_{j,\alpha,R} dx = O(\alpha^3) \quad \text{for all } i, j = 1, \dots, N+1, \quad i \neq j. \quad (11.43)$$

From the variational characterization of the eigenvalues, we may also assume that $\lambda_{i,\alpha,R}$ defines a decreasing function of R . On the other hand, from Lemma 11.3 we know that $\lambda_{i,\alpha,R} = O(\alpha^2)$, uniformly in R , so that we write for convenience

$$\lambda_{i,\alpha,R} = \mu_{i,\alpha,R} \alpha^2, \quad \mu_{i,\alpha,R} < 0.$$

We may assume $\mu_{i,\alpha,R} \rightarrow \mu_{i,\alpha} < 0$ as $R \rightarrow +\infty$. We will prove that $\phi_{i,\alpha,R}$ converges, up to subsequences, uniformly over compacts to a nonzero bounded limit $\phi_{i,\alpha}$ which is an eigenfunction with eigenvalue $\mu_{i,\alpha} \alpha^2$ of Problem (11.2). We will then take limits when $\alpha \rightarrow 0$ and find a contradiction with the fact that \mathcal{J} has at $i(M)$ negative eigenvalues.

We fix an index i and consider the corresponding pair $\phi_{i,\alpha,R}$, $\mu_{i,\alpha,R}$, to which temporarily we drop the subscripts i, α, R .

Note that by maximum principle, $|\phi|$ can have values that stay away from zero only inside \mathcal{N}_δ . Besides, from Lemma 11.4, $\phi = O(e^{-\sigma|t|})$ in \mathcal{N}_δ . We observe then that since λ remains bounded, local elliptic estimates imply the stronger assertion

$$|D^2\phi| + |D\phi| + |\phi| \leq C e^{-\sigma|t|} \quad \text{in } \mathcal{N}_\delta. \quad (11.44)$$

In particular, considering its dependence in R , ϕ approaches up to subsequences, locally uniformly in \mathbb{R}^3 a limit. We will prove by suitable estimates that that limit is nonzero. Moreover, we will show that $\phi \approx z(\alpha y)w'(t)$ in \mathcal{N}_δ where z is an eigenfunction with negative eigenvalue $\approx \mu$ of the Jacobi operator \mathcal{J} .

First, let us localize ϕ inside \mathcal{N}_δ . Let us consider the cut-off function η_δ in (3.12), and the function

$$\tilde{\phi} = \eta_\delta \phi.$$

Then $\tilde{\phi}$ satisfies

$$L(\tilde{\phi}) + \mu\alpha^2 q(\alpha x)\tilde{\phi} = E_\alpha := -2\nabla\eta_\delta\nabla\phi - \Delta\eta_\delta\phi \quad (11.45)$$

with $L(\tilde{\phi}) = \Delta\tilde{\phi} + f'(u_\alpha)\tilde{\phi}$. Then from (11.44) we have that for some $\sigma > 0$,

$$|E_\alpha| \leq C\alpha^3 e^{-\sigma|t|} (1 + r_\alpha^4)^{-1}.$$

Inside \mathcal{N}_δ we write in (y, t) coordinates equation (11.45) as

$$L_*(\tilde{\phi}) + B(\tilde{\phi}) + \lambda p(\alpha y)\tilde{\phi} = E_\alpha \quad (11.46)$$

where

$$L_*(\tilde{\phi}) = \partial_{tt}\tilde{\phi} + \Delta_{M_\alpha}\tilde{\phi} + f'(w(t))\tilde{\phi}.$$

Extending $\tilde{\phi}$ and E_α as zero, we can regard equation (11.46) as the solution of a problem in entire $M_\alpha \times \mathbb{R}$ for an operator L that interpolates L inside \mathcal{N}_δ with L_* outside. More precisely $\tilde{\phi}$ satisfies

$$L(\tilde{\phi}) := L_*(\tilde{\phi}) + B(\tilde{\phi}) + \lambda p(\alpha y)\tilde{\phi} = E_\alpha \quad \text{in } M_\alpha^R \times \mathbb{R}, \quad (11.47)$$

where for a function $\psi(y, t)$ we denote

$$B(\psi) := \begin{cases} \chi B(\psi) & \text{if } |t + h_1(\alpha y)| < \rho_\alpha(y) + 3 \\ 0 & \text{otherwise} \end{cases} \quad (11.48)$$

and

$$\chi(y, t) = \zeta_1(y + (t + h)\nu_\alpha(y))$$

with ζ_1 the cut off function defined by (4.8) for $n = 1$. In particular, $L = L_*$ in \mathcal{N}_δ .

Now, we decompose

$$\tilde{\phi}(y, t) = \varphi(y, t) + k(y)\eta_\delta w'(t) \quad (11.49)$$

where

$$k(y) = -w'(t) \frac{\int_{\mathbb{R}} \tilde{\phi}(y, \cdot) w' d\tau}{\int_{\mathbb{R}} \eta_\delta w'^2 d\tau}$$

so that

$$\int_{\mathbb{R}} \varphi(y, t) w'(t) dt = 0 \quad \text{for all } y \in M_\alpha^R.$$

From (11.44), k is a bounded function, of class C^2 defined on M_α^R with first and second derivatives uniformly bounded independently of large R . A posteriori we expect that k has also bounded smoothness as a function of αy , which means in particular that $Dk = O(\alpha)$. We will see that this is indeed the case.

The function φ satisfies the equation

$$\mathbf{L}(\varphi) + \mu\alpha^2 p(\alpha y) \varphi = -\mathbf{L}(kw') + E_\alpha - \mu\alpha^2 p k w' \quad \text{in } M_\alpha^R \times \mathbb{R}. \quad (11.50)$$

We observe that the expansion (11.8) holds true globally in $M_\alpha^R \times \mathbb{R}$ for $\mathbf{L}(kw')$ replacing $L(kw')$. We also have the validity of expansion (11.5) for the corresponding projection, namely

$$\begin{aligned} \int_{\mathbb{R}} \mathbf{L}(kw') w' dt &= (\Delta_{M_\alpha} k + \alpha^2 |A|^2 k) \int_{\mathbb{R}} w'^2 dt \\ &+ O(\alpha r_\alpha^{-2}) \partial_{ij} k + O(\alpha^2 r_\alpha^{-3}) \partial_i k + O(\alpha^3 r_\alpha^{-4}) k. \end{aligned} \quad (11.51)$$

Thus, integrating equation (11.50) against w' we find that k satisfies

$$\begin{aligned} \Delta_{M_\alpha} k + \alpha^2 |A|^2 k + \mu\alpha^2 p(\alpha y) k + \\ O(\alpha r_\alpha^{-2}) \partial_{ij} k + O(\alpha^2 r_\alpha^{-3}) \partial_i k + O(\alpha^3 r_\alpha^{-4}) k = \\ O(\alpha^3 r_\alpha^{-4}) - \frac{1}{\int_{\mathbb{R}} w'^2} \int_{\mathbb{R}} \mathbf{B}(\varphi) w' dt, \quad y \in M_\alpha^R. \end{aligned} \quad (11.52)$$

Let us consider the function $z(y)$ defined in M by the relation $k(y) = z(\alpha y)$. Then (11.52) translates in terms of z as

$$\begin{aligned} \Delta_M z + |A(y)|^2 z + \mu q(y) z = \\ \alpha [O(r^{-2}) \partial_{ij} z + O(r^{-3}) \partial_i z + O(r^{-4}) z + O(r^{-4})] + \mathcal{B} \quad y \in M^R. \end{aligned} \quad (11.53)$$

where

$$\mathcal{B}(y) := \frac{1}{\int_{\mathbb{R}} w'^2} \alpha^{-2} \int_{\mathbb{R}} \mathbf{B}(\varphi)(\alpha^{-1} y, t) w' dt, \quad y \in M^R. \quad (11.54)$$

In other words we have that $k(y) = z(\alpha y)$, where z solves “a perturbation” of the eigenvalue equation for the Jacobi operator that we treated in §10. We need to make this assertion precise, the basic element being to prove that the operator $\mathcal{B}[z]$ is “small”. For this we will derive estimates for φ from equation (11.50).

We shall refer to the decomposition $Q_1 + \dots + Q_6$ in (11.8) to identify different terms in $\mathbf{L}(kw')$. Let us consider the decomposition

$$\varphi = \varphi_1 + \varphi_2,$$

where φ_1 solves the linear problem for the operator L_* and the part of $\mathbf{L}(kw')$ that “does not contribute to projections”, namely

$$\begin{aligned} Q_3 + Q_4 = -w'' \left[\alpha a_{ij}^0 (\partial_j h \partial_i k + \partial_i h \partial_j k) + \alpha^2 k \Delta_M h_1 + \alpha^2 h a_{ij}^{1,0} (\partial_j h \partial_i k + \partial_i h \partial_j k) \right] \\ + \alpha t w' \left[a_{ij}^{1,0} \partial_{ij} k + \alpha b_i^{1,0} \partial_i k \right]. \end{aligned} \quad (11.55)$$

More precisely φ_1 solves the equation

$$L_*(\varphi_1) + \alpha^2 \mu p \varphi_1 = Q_3 + Q_4 \quad \text{in } M_\alpha^R \times \mathbb{R}. \quad (11.56)$$

This problem can indeed be solved: according to the linear theory developed, there exists a unique solution to the problem

$$L_*(\varphi_1) + \mu\alpha^2 p \varphi_1 = Q_3 + Q_4 + c(y)w'(t) \quad \text{in } M_\alpha^R \times \mathbb{R},$$

such that

$$\int_{\mathbb{R}} \varphi_1 w' dt = 0 \quad \text{for all } y \in M_\alpha^R$$

and

$$\|D^2\varphi_1\|_{p,1,\sigma} + \|D\varphi_1\|_{\infty,1,\sigma} + \|\varphi_1\|_{\infty,1,\sigma} \leq \|Q_3 + Q_4\|_{p,1,\sigma} \leq C\alpha. \quad (11.57)$$

But since

$$\int_{\mathbb{R}} (Q_3 + Q_4) w' dt = 0 \quad \text{for all } y \in M_\alpha^R$$

it follows that actually $c(y) \equiv 0$, namely φ_1 solves equation (11.56).

We claim that φ_2 has actually a smaller size than φ_1 . Indeed φ_2 solves the equation

$$L_*(\varphi_2) + \mathbf{B}(\varphi_2) + \mu\alpha^2 p \varphi_2 = E_\alpha - \mathbf{B}(\varphi_1) - (Q_1 + Q_2 + Q_5 + Q_6) - \mu\alpha^2 q k w' \quad \text{in } M_\alpha^R \times \mathbb{R}. \quad (11.58)$$

Now, we have that

$$\begin{aligned} & Q_1 + Q_2 + Q_5 + Q_6 = \\ & \left[\Delta_{M_\alpha} k + \alpha h a_{ij}^{1,0} \partial_{ij} k \right] w' + \alpha^2 \left[-|A|^2 k t w'' + a_{ij}^0 \partial_i h_0 \partial_j h_0 k w''' + \right. \\ & \left. \alpha^{-2} f''(w) \phi_1 k w' + (t+h)^2 a_{ij}^2 \partial_{ij} k w' + 2(t+h) a_{ij}^2 \partial_i h \partial_j k w'' \right] + \\ & \alpha^3 \left[O(e^{-\sigma|t|} r_\alpha^{-2}) \partial_{ij} k + O(e^{-\sigma|t|} r_\alpha^{-3}) \partial_j k \right] = \\ & O(\alpha^2 r_\alpha^{-2} \log^2 r_\alpha e^{-\sigma|t|}) + \rho(y) w'(t), \end{aligned} \quad (11.59)$$

for a certain function $\rho(y)$. On the other hand, let us recall that

$$\begin{aligned} B &= (f'(u_\alpha) - f'(w)) - \alpha^2 [(t+h_1)|A|^2 + \Delta_M h_1] \partial_t - \alpha a_{ij}^0 (\partial_j h \partial_{it} + \partial_i h \partial_{jt}) + \\ & \alpha(t+h) [a_{ij}^1 \partial_{ij} - 2\alpha a_{ij}^1 \partial_i h \partial_{jt} + \alpha(b_i^1 \partial_i - \alpha b_i^1 \partial_i h \partial_t)] + \\ & \alpha^3 (t+h)^2 b_3^1 \partial_t + \alpha^2 [a_{ij}^0 + \alpha(t+h) a_{ij}^1] \partial_i h \partial_j h \partial_{tt} \end{aligned} \quad (11.60)$$

Thus the order of $\mathbf{B}(\varphi_1)$ carries both an extra α and an extra r_α^{-1} over those of φ_1 , in the sense that

$$\|\mathbf{B}(\varphi_1)\|_{p,2,\sigma} \leq C\alpha^2. \quad (11.61)$$

From relations (11.59) and (11.61) we find that φ_2 satisfies an equation of the form

$$L_*(\varphi_2) + \mathbf{B}(\varphi_2) + \mu\alpha^2 q \varphi_2 = g + c(y) w' \quad \text{in } M_\alpha^R \times \mathbb{R} \quad (11.62)$$

where for arbitrarily small $\sigma' > 0$ we have

$$\|g\|_{p,2-\sigma',\sigma} \leq C\alpha^2.$$

Since φ_2 satisfies $\int_{\mathbb{R}} \varphi_2 w' dt \equiv 0$, the linear theory for the operator L_* yields then that

$$\|D^2\varphi_2\|_{p,2-\sigma',\sigma} + \|D\varphi_2\|_{\infty,2-\sigma',\sigma} + \|\varphi_2\|_{\infty,2-\sigma',\sigma} \leq C\alpha^2, \quad (11.63)$$

which compared with (11.57) gives us the claimed extra smallness:

$$\|\mathbf{B}(\varphi_2)\|_{p,3-\sigma',\sigma} \leq C\alpha^3. \quad (11.64)$$

Let us decompose in (11.54)

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$$

where

$$\mathcal{B}_l := \frac{1}{\int_{\mathbb{R}} w'^2} \alpha^{-2} \int_{\mathbb{R}} \mathbf{B}(\varphi_l)(\alpha^{-1}y, t) w' dt, \quad l = 1, 2. \quad (11.65)$$

From Lemma 8.1 we get that

$$\|\mathcal{B}_1\|_{p,2-\frac{2}{p}-\sigma'} \leq C\alpha^{-2} \|\mathbf{B}(\varphi_1)\|_{p,2,\sigma} \leq C \quad (11.66)$$

and

$$\|\mathcal{B}_2\|_{p,3-\frac{2}{p}-2\sigma'} \leq C\alpha^{-2} \|\mathbf{B}(\varphi_2)\|_{p,3-\sigma',\sigma} \leq C\alpha \quad (11.67)$$

Now, we apply the estimate in part (b) of Lemma 10.2 to equation (11.53) and then get for $z(y) = k(\frac{y}{\alpha})$ the estimate

$$\|D^2z\|_{p,2-\frac{2}{p}-2\sigma'} + \|(1+|x|)^{1-2\sigma'} Dz\|_{\infty} \leq C[\|f\|_{p,2-\frac{2}{p}-2\sigma'} + \|z\|_{\infty}] \quad (11.68)$$

where

$$f = \alpha [O(r^{-2}) \partial_{ij}z + O(r^{-3}) \partial_i z + O(r^{-4}) z + O(r^{-4})] + \mathcal{B}.$$

Then from estimate (11.68) it follows that for small α ,

$$\|D^2z\|_{p,2-\frac{2}{p}-2\sigma'} + \|(1+|x|)^{1-2\sigma'} Dz\|_{\infty} \leq C\alpha. \quad (11.69)$$

Using this new information, let us go back to equation (11.56) and to the expression (11.55) for $Q_3 + Q_4$. The terms contributing the largest sizes in this function can be bounded by

$$C\alpha e^{-\sigma|t|} \left[\frac{|Dk|}{1+r_{\alpha}} + \frac{|D^2k|}{1+r_{\alpha}^2} \right].$$

Now, we compute

$$\begin{aligned} & (1+r_{\alpha}(y)^2)^p \int_{B(y,1)} \frac{|D^2k|^p}{(1+r_{\alpha}^2)^p} dV_{\alpha} \leq \\ & C\alpha^{2p-2} \int_{B(y,\alpha)} |D^2z|^p dV \leq C\alpha^{2p-2} \|D^2z\|_{p,2-\frac{2}{p}-2\sigma'}^p \leq C\alpha^{2p-2}, \end{aligned}$$

and

$$\begin{aligned} & (1+r_{\alpha}(y)^{2-2\sigma'})^p \int_{B(y,1)} \frac{|Dk|^p}{(1+r_{\alpha})^p} dV_{\alpha} \leq \\ & C\| |Dk| (1+r_{\alpha})^{1-2\sigma'} \|_{\infty}^p = C\alpha^p \| |Dz| (1+r)^{1-2\sigma'} \|_{\infty}^p \leq C\alpha^p. \end{aligned}$$

As a conclusion, from expression (11.55) we obtain that

$$\|Q_3 + Q_4\|_{p,2-2\sigma',\sigma} \leq C\alpha^2,$$

and therefore a substantial reduction of the size of φ_1 , compared with (11.57), we have

$$\|D^2\varphi_1\|_{p,2-2\sigma',\sigma} + \|D\varphi_1\|_{\infty,2-2\sigma',\sigma} \leq C\alpha^2, \quad (11.70)$$

hence, using again Lemma 8.1 we get

$$\|\mathcal{B}_1\|_{p,3-\frac{2}{p}-3\sigma'} \leq C\alpha^{-2}\|\mathbf{B}(\varphi_1)\|_{p,3-2\sigma',\sigma} \leq C\alpha \quad (11.71)$$

which matches the size we initially found for \mathcal{B}_2 in (11.67).

We recall that $\phi = \phi_{i,\alpha,R}$ has a uniform C^1 bound (11.44). Thus, passing to a subsequence if necessary, we may assume that

$$\phi_{i,\alpha,R} \rightarrow \phi_{i,\alpha} \quad \text{as } R \rightarrow +\infty,$$

locally uniformly, where $\phi_{i,\alpha}$ is bounded and solves

$$\Delta\phi_{i,\alpha} + f'(u_\alpha)\phi_{i,\alpha} + \mu_{i,\alpha}\alpha^2 p(\alpha x)\phi_{i,\alpha} = 0 \quad \text{in } \mathbb{R}^3. \quad (11.72)$$

Let us return to equation (11.53) including the omitted subscripts. Thus $k = k_{i,\alpha,R}$ satisfies the local uniform convergence in M^α ,

$$k_{i,\alpha,R}(y) = c \int_{|t+h_1|<\rho_\alpha} \phi_{i,\alpha,R} w' dt \rightarrow c \int_{|t+h_1|<\rho_\alpha} \phi_{i,\alpha} w' dt =: k_{i,\alpha}(y).$$

We have that $z = z_{i,\alpha,R}$ satisfies

$$\begin{aligned} & \Delta_M z_{i,\alpha,R} + |A(y)|^2 z_{i,\alpha,R} + \mu_{i,\alpha,R} q(y) z_{i,\alpha,R} = \\ & \alpha [O(r^{-2}) \partial_{ij} z_{i,\alpha,R} + O(r^{-3}) \partial_i z_{i,\alpha,R} + O(r^{-4}) z_{i,\alpha,R} + O(r^{-4})] + \\ & \mathcal{B}_{i,\alpha,R}, \quad y \in M^R, \end{aligned}$$

where

$$\|\mathcal{B}_{i,\alpha,R}\|_{p,3-\frac{2}{p}-3\sigma'} \leq C\alpha \quad (11.73)$$

with arbitrarily small $\sigma' > 0$ and C independent of R . We apply now the estimates in Lemma 10.2 for some $1 < p < 2$ and find that for C independent of R we have

$$\|z_{i,\alpha,R}\|_{L^\infty(M^R)} \leq C [\|z_{i,\alpha,R}\|_{L^\infty(r < R_0)} + O(\alpha)]$$

or equivalently

$$\|k_{i,\alpha,R}\|_{L^\infty(M_\alpha^R)} \leq C [\|k_{i,\alpha,R}\|_{L^\infty(r_\alpha < R_0)} + O(\alpha)]. \quad (11.74)$$

Since from (11.49) we have that

$$\phi_{i,R,\alpha}(y, t) = \varphi_{i,R,\alpha}(y, t) + k_{i,R,\alpha}(y) w'(t) \quad \text{in } \mathcal{N}_\delta, \quad r_\alpha(y) \leq R, \quad (11.75)$$

where we have uniformly in R

$$|\varphi_{i,R,\alpha}(y, t)| = O(\alpha e^{-\sigma|t|} r_\alpha^{-2}),$$

while $\phi_{i,R,\alpha} = O(e^{-\frac{\alpha}{R}})$ outside \mathcal{N}_δ , and

$$\|\phi_{i,R,\alpha}\|_\infty = 1,$$

then

$$\|k_{i,\alpha,R}\|_{L^\infty(M_\alpha^R)} \geq \gamma > 0$$

uniformly in R . Thus from (11.74), the limit $k_{i,\alpha}$ as $R \rightarrow +\infty$ cannot be zero. We have thus found that $\phi_{i,\alpha}$ is non-zero. Moreover, we observe the following: Since the functions

$$Z_i := \partial_i u_\alpha, \quad i = 1, 2, 3, \quad Z_4 := -x_2 \partial_1 u_\alpha + x_1 \partial_2 u_\alpha$$

are bounded solutions of (11.2) for $\lambda = 0$, we necessarily have that

$$\int_{\mathbb{R}^3} p(\alpha x) Z_j \phi_{i,\alpha} dx = 0, \quad j = 1, 2, 3, 4. \quad (11.76)$$

Let

$$\hat{Z}_i = \sum_{l=1}^4 d_{il} Z_l, \quad i = 1, \dots, J.$$

Then we also have

$$\int_{\mathbb{R}^3} p(\alpha x) \hat{Z}_i \phi_{i,\alpha} dx = 0, \quad i = 1, \dots, J. \quad (11.77)$$

Now we want to let $\alpha \rightarrow 0$. $z_{i,\alpha}$ satisfies

$$\begin{aligned} \Delta_M z_{i,\alpha} + |A(y)|^2 z_{i,\alpha} + \mu_{i,\alpha} p(y) z_{i,\alpha} = \\ \alpha [O(r^{-2}) \partial_{ij} z_{i,\alpha} + O(r^{-3}) \partial_i z_{i,\alpha} + O(r^{-4}) z_{i,\alpha} + O(r^{-4})] + \mathcal{B}_{i,\alpha}, \quad y \in M, \end{aligned}$$

with

$$\|\mathcal{B}_{i,\alpha}\|_{p, 3-\frac{2}{p}-3\sigma'} \leq C\alpha. \quad (11.78)$$

Moreover,

$$\|z_{i,\alpha}\|_{L^\infty(M)} \leq C [\|z_{i,\alpha,R}\|_{L^\infty(r < R_0)} + O(\alpha)].$$

Since we also have that

$$\|D^2 z_{i,\alpha}\|_{L^p(M)} \leq C,$$

Sobolev's embedding implies that passing to a subsequence in α , $z_{i,\alpha}$ converges as $\alpha \rightarrow 0$, uniformly over compact subsets of M to a non-zero bounded solution \bar{z}_i of the equation

$$\Delta_M \bar{z}_i + |A(y)|^2 \bar{z}_i + \mu_i q(y) \bar{z}_i = 0 \quad \text{in } M,$$

with $\mu_i \leq 0$.

Now, we have that

$$\phi_{i,\alpha} = z_{i,\alpha}(\alpha y) w'(t) + \varphi_i(y, t) \quad \text{in } \mathcal{N}_\delta$$

where

$$|\varphi_i(y, t)| \leq C\alpha e^{-\sigma|t|}.$$

We recall that

$$\int_{\mathcal{N}_\alpha} q(\alpha y) \phi_{i,\alpha} \phi_{j,\alpha} dx = O(\alpha) \quad \text{for all } i \neq j.$$

Since on \mathcal{N}_δ ,

$$dx = (1 + \alpha^2 |A|^2(t+h)) dV_\alpha dt,$$

we get then that

$$\int_{M_\alpha} q(\alpha y) z_{i,\alpha}(\alpha y) z_{j,\alpha}(\alpha y) dV_\alpha = O(\alpha)$$

or

$$\int_M q(y) z_{i,\alpha}(y) z_{j,\alpha}(y) dV = O(\alpha^3) \quad \text{for all } i \neq j.$$

We conclude, passing to the limit, that the z_i 's $i = 1, \dots, N+1$ satisfy

$$\int_M q \bar{z}_i \bar{z}_j dV = 0 \quad \text{for all } i \neq j.$$

Since, as we have seen in §10, this problem has exactly $N = i(M)$ negative eigenvalues, it follows that $\mu_{N+1} = 0$, so that that z_{N+1} is a bounded Jacobi field.

But we recall that, also

$$Z_i = z_i(\alpha y) w'(t) + O(\alpha e^{-\sigma|t|}) \quad \text{for all } i = 1, \dots, J,$$

hence the orthogonality relations (11.77) pass to the limit to yield

$$\int_M q \hat{z}_i \cdot \bar{z}_{N+1} dV = 0, \quad i = 1, \dots, J.$$

where \hat{z}_i 's are the J linearly independent Jacobi fields. We have thus reached a contradiction with the non-degeneracy assumption for M and the proof of $m(u_\alpha) = i(M)$ is concluded.

Finally, the proof of the non-degeneracy of u_α for all small α goes along the same lines. Indeed, the above arguments are also valid for a bounded eigenfunction in entire space, in particular for $\mu = 0$. If we assume that a bounded solution Z_5 of equation (11.2) is present, linearly independent from Z_1, \dots, Z_4 , then we assume that

$$\int_{\mathbb{R}^3} p(\alpha x) Z_5 \hat{Z}_i dx = 0 \quad i = 1, \dots, J. \quad (11.79)$$

Thus, if in the same way as before, we have that in \mathcal{N}_δ ,

$$Z_5 = z_5(\alpha y) w'(t) + \varphi$$

with φ orthogonal to $w'(t)$ for all y and φ small with size α and uniform exponential decay in t . The function z_α solves an equation of the form (11.78), now for $\mu = 0$. In the same way as we did before, it converges uniformly on compacts to a non-zero limit which is a bounded Jacobi field. But the orthogonality (11.79) passes to the limit, thus implying the existence of at least $J+1$ linearly independent Jacobi field. We have reached a contradiction that finishes the proof of Theorem 2. \square

12. Further comments and open questions

12.1. Symmetries. — As it is natural, the invariances of the surface are at the same time inherited from the construction. If M is a catenoid, revolved around the x_3 axis, the solution in Theorem 1 is radial in the first two variables,

$$u_\alpha(x) = u_\alpha(|x'|, x_3).$$

This is a consequence of the construction. The invariance of the Laplacian under rotations and the autonomous character of the nonlinearity imply that the entire proof can be carried out in spaces of functions with this radial symmetry. More generally, if M is invariant a group of linear isometries, so will be the solution found, at least in the case that $f(u)$ is odd. This assumption allows for odd reflections. The Costa-Hoffmann-Meeks surface is invariant under a discrete group constituted of combination of dihedral symmetries and reflections to which this remark apply.

12.2. Towards a classification of finite Morse index solutions. —

Understanding bounded, entire solutions of nonlinear elliptic equations in \mathbb{R}^N is a problem that has always been at the center of PDE research. This is the context of various classical results in PDE literature like the Gidas-Ni-Nirenberg theorems on radial symmetry of one-signed solutions, Liouville type theorems, or the achievements around De Giorgi conjecture. In those results, the geometry of level sets of the solutions turns out to be a posteriori very simple (planes or spheres). More challenging seems the problem of classifying solutions with finite Morse index, in a model as simple as the Allen-Cahn equation. While the solutions predicted by Theorem 1 are generated in an asymptotic setting, it seems plausible that they contain germs of generality, in view of parallel facts in the theory of minimal surfaces. In particular we believe that the following two statements hold true for a bounded solution u to equation (1.1) in \mathbb{R}^3 .

(1) *If u has finite Morse index and $\nabla u(x) \neq 0$ outside a bounded set, then each level set of u must have outside a large ball a finite number of components, each of them asymptotic to either a plane or to a catenoid. After a rotation of the coordinate system, all these components are graphs of functions of the same two variables.*

(2) *If u has Morse index equal to one. Then u must be axially symmetric, namely after a rotation and a translation, u is radially symmetric in two of its variables. Its level sets have two ends, both of them catenoidal.*

It is worth mentioning that a *balancing formula* for the “ends” of level sets to the Allen-Cahn equation is available in \mathbb{R}^2 , see [15]. An extension of such a formula to \mathbb{R}^3 should involve the configuration (1) as its basis. The condition of finite Morse index can probably be replaced by the energy growth (1.9).

On the other hand, (1) should not hold if the condition $\nabla u \neq 0$ outside a large ball is violated. For instance, let us consider the octant $\{x_1, x_2, x_3 \geq 0\}$ and the odd nonlinearity $f(u) = (1 - u^2)u$. Problem (1.1) in the octant with zero boundary data can be solved by a super-subsolution scheme (similar to that in [8]) yielding a positive solution. Extending by successive odd reflections to the remaining octants,

one generates an entire solution (likely to have finite Morse index), whose zero level set does not have the characteristics above: the condition $\nabla u \neq 0$ far away corresponds to *embeddedness of the ends*.

Various rather general conditions on a minimal surface imply that it is a catenoid. For example, R. Schoen [35] proved that a complete embedded minimal surface in \mathbb{R}^3 with two ends must be catenoid (and hence it has index one). One may wonder if a bounded solution to (1.1) whose zero level set has only two ends is radially symmetric in two variables. On the other hand a one-end minimal surface is forced to be a plane [20]. We may wonder whether or not the zero level set lies on a half space implies that the solution depends on only one variable.

These questions seem rather natural generalizations of that by De Giorgi, now on the classification finite Morse index entire solutions of (1.1). The case in which the minimal surfaces have finite topology but infinite total curvature, like the helicoid, are natural objects to be considered. While results parallel to that in Theorem 1 may be expected possible, they may have rather different nature. The condition of diverging ends in β is not just technical. If it fails a solution may still be associated to the manifold but interactions between neighboring interfaces, which are inherent to the Allen-Cahn equation but not to the minimal surface problem, will come into play. The case of infinite topology may also give rise to very complicated patterns, we refer to Pacard and Hauswirth [17] and references therein for recent result on construction of minimal surfaces in this scenario.

13. Appendix

In this appendix we carry out the computations that lead to Lemma 2.1.

13.1. Coordinates near M and the Euclidean Laplacian. — Let us consider the smooth map

$$(y, z) \in M \times \mathbb{R} \longmapsto x = \tilde{X}(y, z) = y + z\nu(y) \in \mathbb{R}^3. \quad (13.1)$$

Let \mathcal{O} be a set as in the statement of Lemma 2.1, and consider the subset of $M \times \mathbb{R}$ defined as

$$\tilde{\mathcal{O}} = \{(\alpha y, \alpha(t + h(y))) \in M \times \mathbb{R} / (t, y) \in \mathcal{O}\}.$$

Then $\tilde{X}|_{\tilde{\mathcal{O}}}$ is one to one, and

$$\tilde{\mathcal{O}} \subset \{(y, z) \in M \times \mathbb{R} / |z| < \delta \log(1 + r(y))\}.$$

Since along ends $\partial_i \nu = O(r^{-2})$ so that $z\partial_i \nu$ is uniformly small in $\tilde{\mathcal{O}}$, it follows that \tilde{X} is actually a diffeomorphism onto its image, $\tilde{\mathcal{N}} = \tilde{X}(\tilde{\mathcal{O}}) = \alpha\mathcal{N}$.

The Euclidean Laplacian Δ_x can be computed in such a region by the well-known formula in terms of the coordinates $(y, z) \in \tilde{\mathcal{O}}$ as

$$\Delta_x = \partial_{zz} + \Delta_{M_z} - H_{M_z} \partial_z, \quad x = \tilde{X}(y, z), \quad (y, z) \in \mathcal{O} \quad (13.2)$$

where M_z is the manifold

$$M_z = \{y + z\nu(y) / y \in M\}.$$

Local coordinates $y = Y_k(\mathbf{y})$, $\mathbf{y} \in \mathbb{R}^2$ as in (2.1) induce natural local coordinates in M_z . The metric $g_{ij}(z)$ in M_z can then be computed as

$$g_{ij}(z) = \langle \partial_i Y, \partial_j Y \rangle + z(\langle \partial_i Y, \partial_j \nu \rangle + \langle \partial_j Y, \partial_i \nu \rangle) + z^2 \langle \partial_i \nu, \partial_j \nu \rangle \quad (13.3)$$

or

$$g_{ij}(z) = g_{ij} + zO(r^{-2}) + z^2O(r^{-4}).$$

where these relations can be differentiated. Thus we find from the expression of Δ_{M_z} in local coordinates that

$$\Delta_{M_z} = \Delta_M + za_{ij}^1(y, z)\partial_{ij} + zb_i^1(y, z)\partial_i, \quad y = Y(\mathbf{y}) \quad (13.4)$$

where a_{ij}^1, b_i^1 are smooth functions of their arguments. Let us examine this expansion closer around the ends of M_k where $y = Y_k(\mathbf{y})$ is chosen as in (2.1). In this case, from (13.3) and (2.2) we find

$$g^{ij}(z) = g^{ij} + zO(r^{-2}) + z^2O(r^{-4}) + \dots$$

Then we find that for large r ,

$$\Delta_{M_z} = \Delta_M + zO(r^{-2})\partial_{ij} + zO(r^{-3})\partial_i. \quad (13.5)$$

Let us consider the remaining term in the expression for the Laplacian, the mean curvature H_{M_z} . We have the validity of the formula

$$H_{M_z} = \sum_{i=1}^2 \frac{k_i}{1 - k_i z} = \sum_{i=1}^2 k_i + k_i^2 z + k_i^3 z^2 + \dots$$

where k_i , $i = 1, 2$ are the principal curvatures. Since M is a minimal surface, we have that $k_1 + k_2 = 0$. Thus

$$|A|^2 = k_1^2 + k_2^2 = -2k_1 k_2 = -2K$$

where $|A|$ is the Euclidean norm of the second fundamental form, and K the Gauss curvature. As $r \rightarrow +\infty$ we have seen that $k_i = O(r^{-2})$ and hence $|A|^2 = O(r^{-4})$. More precisely, we find for large r ,

$$H_{M_z} = |A|^2 z + z^2 O(r^{-6}).$$

Thus we have found the following expansion for the Euclidean Laplacian,

$$\Delta_x = \partial_{zz} + \Delta_M - z|A|^2 \partial_z + B \quad (13.6)$$

where expressed in local coordinates in M the operator B has the form

$$B = za_{ij}^1(y, z)\partial_{ij} + zb_i^1(y, z)\partial_i + z^2 b_3^1(y, z)\partial_z \quad (13.7)$$

with a_{ij}^1, b_i^1, b_3^1 smooth functions. Besides, we find that

$$a_{ij}^1(y, z) = O(r^{-2}), \quad b_i^1(y, z) = O(r^{-3}), \quad b_3^1(y, z) = O(r^{-6}), \quad (13.8)$$

uniformly in z for $(y, z) \in \tilde{\mathcal{O}}$. Moreover, the way these coefficients are produced from the metric yields for instance that

$$\begin{aligned} a_{i,j}^1(y, z) &= a_{i,j}^1(y, 0) + z a_{i,j}^{(2)}(y, z), & a_{i,j}^2(y, z) &= O(r^{-3}), \\ b_i^1(y, z) &= b_i^1(y, 0) + z b_i^{(2)}(y, z), & b_i^{(2)}(y, z) &= O(r^{-4}). \end{aligned}$$

We summarize the discussion above. Let us consider the parameterization in (13.1) of the region $\tilde{\mathcal{N}}$.

Lemma 13.1. — *The Euclidean Laplacian can be expanded in $\tilde{\mathcal{N}}$ as*

$$\begin{aligned} \Delta_x &= \partial_{zz} + \Delta_{M_z} - H_{M_z} \partial_z = \\ &= \partial_{zz} + \Delta_M - z |A|^2 \partial_z + z [a_{i,j}^1(y, z) \partial_{ij} + b_i^1(y, z) \partial_i] + z^2 b_3^1(y, z) \partial_z, \\ \Delta_M &= a_{i,j}^0 \partial_{ij} + b_i^0 \partial_i, \quad x = \tilde{X}(y, z), \quad (y, z) \in \tilde{\mathcal{O}}, \end{aligned}$$

where $a_{i,j}^l, b_j^l$ are smooth, bounded functions, with the index k omitted. In addition, for $k = 1, \dots, m$,

$$a_{i,j}^l = \delta_{ij} \delta_{0l} + O(r^{-2}), \quad b_i^l = O(r^{-3}), \quad b_3^l = O(r^{-6}),$$

as $r = |y| \rightarrow \infty$, uniformly in z variable.

13.2. Laplacian in expanded variables. — Now we consider the expanded minimal surface $M_\alpha = \alpha^{-1}M$ for a small number α . We have that $\mathcal{N} = \alpha^{-1}\tilde{\mathcal{N}}$. We describe \mathcal{N} via the coordinates

$$x = X(y, z) := y + z\nu_\alpha(y), \quad (y, z) \in \alpha^{-1}\tilde{\mathcal{O}}. \quad (13.9)$$

Let us observe that

$$X(y, z) = \alpha^{-1}\tilde{X}(\alpha y, \alpha z)$$

where $\tilde{x} = \tilde{X}(\tilde{y}, \tilde{z}) = \tilde{y} + \tilde{z}\nu(\tilde{y})$, where the coordinates in \mathcal{N}_δ previously dealt with. We want to compute the Euclidean Laplacian in these coordinates associated to M_α . Observe that

$$\Delta_x[u(x)]|_{x=X(y,z)} = \alpha^2 \Delta_{\tilde{x}}[u(\alpha^{-1}\tilde{x})]|_{\tilde{x}=\tilde{X}(\alpha y, \alpha z)}$$

and that the term in the right hand side is the one we have already computed. In fact setting $v(y, z) := u(y + z\nu_\alpha(y))$, we get

$$\Delta_x u|_{x=X(y,z)} = \alpha^2 (\Delta_{\tilde{y}, M_{\tilde{z}}} + \partial_{\tilde{z}\tilde{z}} - H_{M_{\tilde{z}}} \partial_{\tilde{z}}) [v(\alpha^{-1}\tilde{y}, \alpha^{-1}\tilde{z})]|_{(\tilde{y}, \tilde{z})=(\alpha y, \alpha z)}. \quad (13.10)$$

We can then use the discussion summarized in Lemma 13.1 to obtain a representation of Δ_x in \mathcal{N} via the coordinates $X(y, t)$ in (13.9). Let us consider the local coordinates $Y_{k\alpha}$ of M_α in (2.9).

Lemma 13.2. — *In \mathcal{N} we have*

$$\Delta_x = \partial_{zz} + \Delta_{M_{\alpha,z}} - H_{M_{\alpha,z}} \partial_z =$$

$$\partial_{zz} + \Delta_{M_\alpha} - \alpha^2 z |A(\alpha y)|^2 \partial_z + \alpha z [a_{ij}^1(\alpha y, \alpha z) \partial_{ij} + \alpha b_i^1(\alpha y, \alpha z) \partial_i] + \alpha^3 z^2 b_3^1(\alpha y, \alpha z) \partial_z,$$

$$\Delta_{M_\alpha} = a_{ij}^0(\alpha y) \partial_{ij} + b_i^1(\alpha y) \partial_i, \quad (y, z) \in \alpha^{-1} \tilde{\mathcal{O}}, \quad y = Y_{k\alpha}(\mathbf{y})$$

where a_{ij}^l, b_j^l are smooth, bounded functions. In addition, for $k = 1, \dots, m$,

$$a_{ij}^l = \delta_{ij} \delta_{0l} + O(r_\alpha^{-2}), \quad b_i^l = O(r_\alpha^{-3}), \quad b_3^l = O(r_\alpha^{-6}),$$

as $r_\alpha(y) = |\alpha y| \rightarrow \infty$, uniformly in z variable.

13.3. The proof of Lemma 2.1. — Let us consider a function u defined in \mathcal{N} , expressed in coordinates $x = X(y, z)$, and consider the expression of u in the coordinates $x = X_h(y, t)$, namely the function $v(\mathbf{y}, t)$ defined by the relation in local coordinates $y = Y_k(\mathbf{y})$,

$$v(\mathbf{y}, z - h(\alpha y)) = u(\mathbf{y}, z),$$

(by slight abuse of notation we are denoting just by h the function $h \circ Y_k$). Then we compute

$$\partial_i u = \partial_i v - \alpha \partial_t v \partial_i h, \quad \partial_z u = \partial_t v,$$

$$\partial_{ij} u = \partial_{ij} v - \alpha \partial_{it} v \partial_j h - \alpha \partial_{jt} v \partial_i h + \alpha^2 \partial_{tt} v \partial_i h \partial_j h - \alpha^2 \partial_t v \partial_{ij} h.$$

Observe that, in the notation for coefficients in Lemma 13.2,

$$a_{ij}^0 \partial_{ij} h + b_i^0 \partial_i h = \Delta_M h, \quad a_{ij}^0 \partial_{ij} v + \alpha b_i^0 \partial_i v = \Delta_{M_\alpha} v.$$

We find then

$$\Delta_x = \partial_{tt} + \Delta_{M_\alpha} - \alpha^2 [(t+h)|A|^2 + \Delta_M h] \partial_t - 2\alpha a_{ij}^0 \partial_j h \partial_{it} +$$

$$\alpha(t+h) [a_{ij}^1 \partial_{ij} - 2\alpha a_{ij}^1 \partial_i h \partial_{jt} + \alpha(b_i^1 \partial_i - \alpha b_i^1 \partial_i h \partial_t)] +$$

$$\alpha^3 (t+h)^2 b_3^1 \partial_t + \alpha^2 [a_{ij}^0 + \alpha(t+h)a_{ij}^1] \partial_i h \partial_j h \partial_{tt} \quad (13.11)$$

where all the coefficients are understood to be evaluated at αy or $(\alpha y, \alpha(t+h(\alpha y)))$. The desired properties of the coefficients have already been established. The proof of Lemma 2.1 is concluded. \square

Acknowledgments: We are indebted to N. Dancer for a useful discussion, and to F. Pacard for pointing us out reference [36]. The first author has been partly supported by research grants Fondecyt 1070389 and FONDAF, Chile. The second author has been supported by Fondecyt grant 1050311, and FONDAF, Chile. He thanks the Department of Mathematics of Politecnico di Torino for its hospitality while part of the work was completed. The research of the third author is partially supported by a General Research Fund from RGC of Hong Kong and Focused Research Scheme from CUHK.

References

- [1] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in \mathbb{R}^3 and a conjecture of De Giorgi*, Journal Amer. Math. Soc. 13 (2000), 725–739.
- [2] L. Caffarelli, A. Córdoba, *Uniform convergence of a singular perturbation problem*, Comm. Pure Appl. Math. XLVII (1995), 1–12.
- [3] L. Caffarelli, A. Córdoba, *Phase transitions: uniform regularity of the intermediate layers*. J. Reine Angew. Math. 593 (2006), 209–235.
- [4] C.J. Costa, *Imersões mínimas em \mathbb{R}^3 de gênero um e curvatura total finita*. PhD thesis, IMPA, Rio de Janeiro, Brasil (1982).
- [5] C.J. Costa, *Example of a complete minimal immersions in \mathbb{R}^3 of genus one and three embedded ends*, Bol. Soc. Bras. Mat. 15(1-2)(1984), 47–54.
- [6] X. Cabré, J. Terra *Saddle-shaped solutions of bistable diffusion equations in all of \mathbb{R}^{2m}* . Preprint 2008.
- [7] E. N. Dancer, *Stable and finite Morse index solutions on \mathbf{R}^n or on bounded domains with small diffusion*. Trans. Amer. Math. Soc. 357 (2005), no. 3, 1225–1243
- [8] H. Dang, P.C. Fife, L.A. Peletier, *Saddle solutions of the bistable diffusion equation*. Z. Angew. Math. Phys. 43 (1992), no. 6, 984–998.
- [9] E. De Giorgi, *Convergence problems for functionals and operators*, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131–188, Pitagora, Bologna(1979).
- [10] M. del Pino, M. Kowalczyk, J. Wei, *On De Giorgi’s Conjecture in Dimensions $N \geq 9$* , preprint 2008, arXiv.org/0806.3141.
- [11] M. del Pino, M. Kowalczyk, J. Wei, *A counterexample to a conjecture by De Giorgi in large dimensions*, Comp. Rend. Mathématique 346 (2008), 23–24, 1261–1266.
- [12] A. Farina and E. Valdinoci, *The state of art for a conjecture of De Giorgi and related questions*, to appear in “Reaction-Diffusion Systems and Viscosity Solutions”, World Scientific, 2008.
- [13] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. 311 (1998), 481–491.
- [14] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second order*, 2nd edition, Springer-Verlag, 1983.
- [15] C. Gui, *Hamiltonian identities for elliptic partial differential equations*, J. Funct. Anal. 254 (2008), no. 4, 904–933
- [16] R. Gulliver. *Index and total curvature of complete minimal surfaces*. Proc. Symp. Pure Math., 44, 207212, (1986).
- [17] L. Hauswirth and F. Pacard, *Higher genus Riemann minimal surfaces*, Invent. Math. 169(2007), 569–620.
- [18] D. Hoffman and W.H. Meeks III, *A complete embedded minimal surface in \mathbb{R}^3 with genus one and three ends*, J. Diff. Geom. 21 (1985), 109–127.
- [19] D. Hoffman and W.H. Meeks III, *The asymptotic behavior of properly embedded minimal surfaces of finite topology*, J. Am. Math. Soc. 4(2) (1989), 667–681.
- [20] D. Hoffman and W. H. Meeks III. *The strong halfspace theorem for minimal surfaces*. Inventiones Math., 101, 373–377, 1990.
- [21] D. Hoffman and W.H. Meeks III, *Embedded minimal surfaces of finite topology*, Ann. Math. 131 (1990), 1–34.

- [22] D. Hoffman and H. Karcher, *Complete embedded minimal surfaces of finite total curvature*. In Geometry V, Encyclopaedia Math. Sci., vol. 90, pp. 5-93, 262-272. Springer Berlin (1997).
- [23] J.E. Hutchinson, Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*. Calc. Var. Partial Differential Equations 10 (2000), no. 1, 49–84.
- [24] D. Jerison and R. Monneau, *Towards a counter-example to a conjecture of De Giorgi in high dimensions*, Ann. Mat. Pura Appl. 183 (2004), 439–467.
- [25] N. Kapuleas, *Complete embedded minimal surfaces of finite total curvature*, J. Differential Geom. 45 (1997), 95–169.
- [26] F. J. López, A. Ros, *On embedded complete minimal surfaces of genus zero*, J. Differential Geom. 33 (1991), no. 1, 293–300.
- [27] L. Modica, *Convergence to minimal surfaces problem and global solutions of $\Delta u = 2(u^3 - u)$* . Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 223–244, Pitagora, Bologna, (1979).
- [28] F. Morabito, *Index and nullity of the Gauss map of the Costa-Hoffman-Meeks surfaces*, preprint 2008.
- [29] S. Nayatani, *Morse index and Gauss maps of complete minimal surfaces in Euclidean 3-space*, Comm. Math. Helv. 68(4)(1993), 511–537.
- [30] S. Nayatani, *Morse index of complete minimal surfaces*. In: Rassias, T.M. (ed.) *The Problem of Plateau*, pp. 181–189(1992)
- [31] R. Osserman, *A survey of minimal surfaces*, Math. Studies 25, Van Nostrand, New York 1969.
- [32] F. Pacard and M. Ritoré, *From the constant mean curvature hypersurfaces to the gradient theory of phase transitions*, J. Differential Geom. 64 (2003), no. 3, 359–423.
- [33] J. Pérez, A. Ros, *The space of properly embedded minimal surfaces with finite total curvature*. Indiana Univ. Math. J. 45 (1996), no. 1, 177–204.
- [34] O. Savin, *Regularity of flat level sets in phase transitions*, Annals of Mathematics, 169 (2009), 41–78.
- [35] R. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. 18 (1983), 791–809.
- [36] M. Traizet, *An embedded minimal surface with no symmetries*, J. Differential Geom. 60 (2002) 103–153.

MANUEL DEL PINO, Departamento de Ingeniería Matemática and CMM, Universidad de Chile,
Casilla 170 Correo 3, Santiago, Chile. • *E-mail* : delpino@dim.uchile.cl

MICHAL KOWALCZYK, Departamento de Ingeniería Matemática and CMM, Universidad de Chile,
Casilla 170 Correo 3, Santiago, Chile. • *E-mail* : kowalczy@dim.uchile.cl

JUNCHENG WEI, Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong
E-mail : wei@math.cuhk.edu.hk