

ON SOLUTIONS WITH POINT RUPTURES FOR A SEMILINEAR ELLIPTIC PROBLEM WITH SINGULARITY

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ABSTRACT. We consider the following semilinear elliptic equation with singular nonlinearity:

$$\Delta u - \frac{\lambda}{u^\nu} = 0 \text{ in } B, \quad u = \psi \text{ on } \partial B$$

where $\lambda > 0, \nu > 0$ and $\psi \in C^{2,\alpha}(\partial B)$ and B is the unit ball in \mathbb{R}^N . Under various conditions on λ, ν and ψ , we construct solutions with one isolated zero in B .

1. INTRODUCTION

Let B be the unit ball of \mathbb{R}^N ($N \geq 2$). The main purpose of this paper is to construct nonnegative solutions with one isolated zero point of the semilinear elliptic Dirichlet problem

$$(1.1) \quad \Delta u - \lambda u^{-\nu} = 0 \text{ in } B, \quad u = \psi \text{ on } \partial B,$$

where $\lambda, \nu > 0$, $\psi \in C^{2,\alpha}(\partial B)$ with $\psi(\theta) > 0$ for $\theta \in S^{N-1} = \partial B$.

Problem (1.1) appears in several applications in mechanics and physics, and in particular can be used to model the electrostatic Micro-Electromechanic System (MEMS) devices. See [FMP], [GG1], [GG2], [GG3], [GPW] and the references therein. In particular, in [GG1], [GG2] and [GG3], Ghoussoub and Guo have given a thorough study on the following problem

$$(1.2) \quad \begin{cases} u_t = \Delta u - \frac{\lambda f(x)}{u^2}, & x \in \Omega, t > 0, \\ u(x, 0) = 1 \text{ for } x \in \Omega, \quad u(x, t) = 1 \text{ for } x \in \partial\Omega \end{cases}$$

where $\lambda > 0$, $f(x)$ is a positive function and Ω is a bounded smooth domain in \mathbb{R}^N . (1.1) is just the steady state of (1.2) with $f(x) \equiv 1$ and $\nu = 2$. The set $\{x | u(x) = 0\}$ is called **touch town** set and plays an important role in MEMS.

Problem (1.1) can also be considered as steady state problem of thin films problems. Equations of the type

$$(1.3) \quad u_t = -\nabla \cdot (f(u)\nabla \Delta u) - \nabla \cdot (g(u)\nabla u)$$

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x, t)$ is the height of the air/liquid interface. The zero set $\Sigma_u = \{u = 0\}$ is the liquid/solid interface and is sometimes called set of **ruptures**. Ruptures play a very important role in the study of thin films. The coefficient $f(u)$ reflects surface tension effects- a typical choice is $f(u) = u^3$. The coefficient of the second-order term

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can reflect additional forces such as gravity $g(u) = u^3$, van der Waals interactions $g(u) = u^m, m < 0$. For more background on thin films, we refer to [BBD, BP1, BP2, LP1, LP2, LP3, WB, YD, YH] and the references therein. By choosing $f(u) = u^p, g(u) = u^{-m}$, (1.3) is equivalent to a fourth order equation

$$(1.4) \quad u_t = -\nabla \cdot (u^p \nabla (\Delta u - u^{-\nu}))$$

with $\nu = p + m - 1$. Again, solutions to (1.1) are steady-states of (1.4).

In [GW], we computed the Hausdorff dimension of rupture sets for

$$(1.5) \quad \Delta u - \frac{\lambda}{u^\nu} + h(x) = 0 \text{ in } \Omega$$

We showed that if u is a nonnegative **stationary** solution of (1.5) such that $u \in H^1(\Omega)$ and $\int_\Omega u^{1-\nu} dx < \infty$, then the zero set of u has locally finite Hausdorff $[(N-2)\nu + (N+2)]/(\nu+1)$ -dimensional measure. However, it is a difficult question to construct solutions to (1.1) exhibiting point ruptures. If $\nu > 0$, it is easy to see that there exists a radial solution $u_0(x) = |x|^{2/(\nu+1)}$ of the problem

$$(1.6) \quad \Delta u - \lambda_0 u^{-\nu} = 0 \text{ in } B, \quad u = 1 \text{ on } \partial B,$$

where $\lambda_0 = \frac{2(N+(N-2)\nu)}{(\nu+1)^2} > 0$. It seems that this is the only known example.

The purpose of this paper is to construct nonnegative solutions of (1.1) with one isolated zero point, under various conditions on ψ and ν . Our main idea is to study the surjectivity properties of the linearized operator associated with the known rupture solution $|x|^{\frac{2}{\nu+1}}$ in some weighted Hölder spaces. The weighted Hölder space has been introduced and used by Mazzeo and Pacard [MP], Mazzeo-Pacard-Uhlenbeck [MPU] in constructing singular solutions to Yamabe type problems ([MP1], [MP2]). It is also used by Rebai [R1], [R2] to construct solutions singular on submanifolds.

The corresponding Neumann problem

$$(1.7) \quad \Delta u - \frac{1}{u^\nu} + h(|x|) = 0 \text{ in } B, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial B$$

has been studied by del Pino and Hernandez [DH] for $\nu > 1$. They showed that (1.7) has at least one nonnegative radial solution $u = u(r)$ satisfying $a_1 r^{2/(\nu+1)} \leq u(r) \leq a_2$, $a_1, a_2 > 0$.

A different kind of problem

$$(1.8) \quad \Delta u + k(x) \frac{1}{u^\alpha} = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

was studied in [CR, De, GHW, Go, GL] and the references therein, where $k(x) > 0$. The regularity of ∇u is obtained. Problem (1.8) is fundamentally different from (1.1): the sign of nonlinearity makes the Maximum Principle applicable to (1.8) which allow the use of e.g. a super-sub solutions scheme. In fact the following problem

$$\Delta u + \frac{1}{u^\alpha} - h(x) = 0 \text{ in } \Omega, \quad u = \psi \text{ on } \partial\Omega,$$

possesses a (unique) positive solution in case that h is, for example, positive.

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2. PRELIMINARY COMPUTATION

Let (λ_0, u_0) be the radial solution of (1.6). We define the linearized operator $\mathcal{L} : w \mapsto \Delta w + \lambda_0 \nu u_0^{-(\nu+1)} w$. Clearly we have

$$(2.1) \quad \lambda_0 \nu u_0^{-(\nu+1)} = \frac{c_0}{r^2},$$

where c_0 is a positive constant. Precisely,

$$c_0 = \frac{2\nu(N + (N - 2)\nu)}{(\nu + 1)^2}.$$

It is known that the eigenvalues of the problem

$$(2.2) \quad -\Delta_{\theta} v = \sigma v, \quad \theta \in S^{N-1}$$

are $\sigma_k = k(N + k - 2)$, $k \geq 0$ with multiplicity $m_k = \frac{(N-3+k)!(N-2+2k)}{k!(N-2)!}$. In particular, we denote that $\sigma_0 = 0$, $\sigma_1 = N - 1$, $\sigma_2 = N - 1$, \dots , $\sigma_N = N - 1$, $\sigma_{N+1} = 2N$ and $\varphi_j(\theta)$ ($j = 0, 1, \dots$) the eigenfunction corresponding to σ_j which is normalized in such a way that

$$\int_{S^{N-1}} \varphi_j^2(\theta) d\theta = 1.$$

Note that $\varphi_0(\theta) \equiv \text{Const}$.

We define the indicial roots of \mathcal{L} by

$$(2.3) \quad \gamma_j^{\pm} = \frac{2 - N}{2} \pm \left(\left(\frac{N - 2}{2} \right)^2 + \sigma_j - c_0 \right)^{1/2}, \quad j \geq 0.$$

We deduce the following proposition by simple computations.

Proposition 2.1. *The following inequalities hold:*

- (1) If $N = 2$, then $\Re(\gamma_0^{\pm}) = 0$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > 0$.
- (2) If $N = 3$, then $\gamma_0^- \leq (2 - N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (2^{1/2}8 - 11)/7$; $\Re(\gamma_0^{\pm}) = (2 - N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (2^{1/2}8 - 11)/7$.
- (3) If $N = 4$, then $\gamma_0^- \leq (2 - N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (3^{1/2}2 - 3)/3$; $\Re(\gamma_0^{\pm}) = (2 - N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (3^{1/2}2 - 3)/3$.
- (4) If $N = 5$, then $\gamma_0^- \leq (2 - N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq 1/3$; $\Re(\gamma_0^{\pm}) = (2 - N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > 1/3$.
- (5) If $N = 6$, then $\gamma_0^- \leq (2 - N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (5^{1/2} - 1)/2$, $\Re(\gamma_0^{\pm}) = (2 - N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (5^{1/2} - 1)/2$.
- (6) If $N = 7$, then $\gamma_0^- \leq (2 - N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (6^{1/2}8 - 3)/15$, $\Re(\gamma_0^{\pm}) = (2 - N)/2$, $\Im(\gamma_0^{\pm}) \neq 0$ for $\nu > (6^{1/2}8 - 3)/15$.

(7) If $N = 8$, then $\gamma_0^- \leq (2 - N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (7^{1/2}2 + 1)/3$, $\Re(\gamma_0^\pm) = (2 - N)/2$, $\Im(\gamma_0^\pm) \neq 0$ for $\nu > (7^{1/2}2 + 1)/3$.

(8) If $N = 9$, then $\gamma_0^- \leq (2 - N)/2 \leq \gamma_0^+ < 0$ are real numbers for $0 < \nu \leq (8^{1/2}8 + 13)/7$, $\Re(\gamma_0^\pm) = (2 - N)/2$, $\Im(\gamma_0^\pm) \neq 0$ for $\nu > (8^{1/2}8 + 13)/7$.

(9) If $N \geq 10$, then $\gamma_0^- \leq (2 - N)/2 \leq \gamma_0^+ < 0$ are real numbers for $\nu > 0$.

(10)

$$\gamma_1^\pm = \gamma_2^\pm = \dots \gamma_N^\pm = \frac{(2 - N)}{2} \pm \left| \frac{(N + (N - 4)\nu)}{2(\nu + 1)} \right|.$$

Thus, for $N = 2$,

$$\gamma_1^\pm = \gamma_2^\pm = \begin{cases} \pm \frac{(1-\nu)}{(1+\nu)} & \text{if } 0 < \nu \leq 1 \\ \pm \frac{(\nu-1)}{(1+\nu)} & \text{if } \nu > 1, \end{cases}$$

which implies that

$$\gamma_1^+ = \gamma_2^+ = \begin{cases} \frac{(1-\nu)}{(1+\nu)} & \text{if } 0 < \nu \leq 1 \\ \frac{(\nu-1)}{(1+\nu)} & \text{if } \nu > 1, \end{cases}$$

$$\gamma_1^- = \gamma_2^- = \begin{cases} \frac{(\nu-1)}{(1+\nu)} & \text{if } 0 < \nu \leq 1 \\ \frac{(1-\nu)}{(1+\nu)} & \text{if } \nu > 1. \end{cases}$$

For $N = 3$,

$$\gamma_1^\pm = \gamma_2^\pm = \gamma_3^\pm = \begin{cases} -\frac{1}{2} \pm \frac{(3-\nu)}{2(1+\nu)} & \text{if } 0 < \nu \leq 3 \\ -\frac{1}{2} \pm \frac{(\nu-3)}{2(1+\nu)} & \text{if } \nu > 3, \end{cases}$$

which implies that

$$\gamma_1^+ = \gamma_2^+ = \gamma_3^+ = \begin{cases} \frac{(1-\nu)}{(1+\nu)} & \text{if } 0 < \nu \leq 3 \\ -\frac{2}{(1+\nu)} & \text{if } \nu > 3, \end{cases}$$

$$\gamma_1^- = \gamma_2^- = \gamma_3^- = \begin{cases} -\frac{2}{(1+\nu)} & \text{if } 0 < \nu \leq 3 \\ \frac{(1-\nu)}{(1+\nu)} & \text{if } \nu > 3. \end{cases}$$

For $N \geq 4$,

$$\gamma_1^+ = \gamma_2^+ = \dots = \gamma_N^+ = \frac{(1 - \nu)}{(1 + \nu)},$$

$$\gamma_1^- = \gamma_2^- = \dots = \gamma_N^- = \frac{[(3 - N)\nu + (1 - N)]}{(1 + \nu)}.$$

(11)

$$\begin{aligned}
\gamma_{N+1}^+ &= \frac{(2-N)}{2} + \left(\frac{(N-2)^2}{4} + 2N - \frac{2\nu(N+(N-2)\nu)}{(\nu+1)^2} \right)^{1/2} \\
&= \frac{(2-N)}{2} + \left(\frac{(N^2-4N+20)\nu^2 + 2(N^2+4)\nu + (N+2)^2}{2(\nu+1)} \right)^{1/2} \\
&> \frac{(2-N)}{2} + \left(\frac{(N-2)^2\nu^2 + 2(N-2)(N+2)\nu + (N+2)^2}{2(\nu+1)} \right)^{1/2} \\
&= \frac{(2-N)}{2} + \frac{(N-2)\nu + (N+2)}{2(\nu+1)} \\
&= \frac{2}{(\nu+1)}.
\end{aligned}$$

3. A RIGHT INVERSE FOR \mathcal{L} .

We introduce the weighted Hölder spaces as in [MS, MP, MPU, Re1, Re2]. For any $k \geq 0$, $\alpha \in (0, 1)$ and $\mu \in \mathbb{R}$, we define some weighted Hölder spaces $C_\mu^{k,\alpha}$ as follows

$$C_\mu^{k,\alpha} = \{u \in C_{loc}^{k,\alpha}(B \setminus \{0\}) : \|u\|_{C_\mu^{k,\alpha}} = \sup_{r \leq 1/2} (r^{-\mu} |u|_{k,\alpha,[r,2r]}) < +\infty\},$$

where, by definition

$$|u|_{k,\alpha,[r,2r]} = \sup_{r \leq |x| \leq 2r} (\sum_{j=0}^k r^j |\nabla^j u|) + r^{k+\alpha} \sup_{r \leq |x|, |y| \leq 2r; x \neq y} \frac{|\nabla^k u(y) - \nabla^k u(x)|}{|y-x|^\alpha}.$$

In addition, for all $j \geq 0$, we define

$$(3.1) \quad C_{\mu,j}^{2,\alpha} = \{v \in C_\mu^{2,\alpha} : v|_{\partial B} \in \text{span}(\varphi_0(\theta), \dots, \varphi_j(\theta))\}.$$

It follows from (2.1) that the linear operator \mathcal{L} is well defined from $C_\mu^{2,\alpha}$ into $C_{\mu-2}^{0,\alpha}$. The proof of the following proposition is a little variant of the proof of Proposition 3 of [Re2].

Proposition 3.1. *Assume that $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, and $0 < 2/(\nu+1) < \mu < \gamma_{N+1}^+$. Then for any $g \in C_{\mu-2}^{0,\alpha}$ there exists a unique solution of $\mathcal{L}w = g$ in $B \setminus \{0\}$ which belongs to the space $C_{\mu,N}^{2,\alpha}$. In addition, the mapping $g \in C_{\mu-2}^{0,\alpha} \rightarrow w \in C_{\mu,N}^{2,\alpha}$ is bounded.*

Proof. By our assumptions, we know from Proposition 2.1 that for $N \geq 3$ and $\nu > 0$,

$$\begin{aligned}
\Re(\gamma_0^+) &< \mu \text{ and } \Re(\gamma_0^+) + \mu + N - 3 > -1, \\
\gamma_1^+ &= \dots = \gamma_N^+ < \mu < \gamma_{N+1}^+ \text{ and } \mu + \gamma_1^+ > -1.
\end{aligned}$$

For $N = 2$ and $0 < \nu \leq 3$,

$$\begin{aligned}
\Re(\gamma_0^+) &< \mu \text{ and } \Re(\gamma_0^+) + \mu + N - 3 > -1, \\
\gamma_1^+ &= \gamma_2^+ < \mu < \gamma_3^+ \text{ and } \mu + \gamma_1^+ > 0.
\end{aligned}$$

Choosing

$$w_i(r) = -r^{\gamma_i^+} \int_r^1 \kappa^{1-N-2\gamma_i^+} \int_0^\kappa s^{N-1+\gamma_i^+} g_i(s) ds d\kappa$$

for $i = N + 1, N + 2, \dots$, and

$$w_i(r) = \Re(r^{\gamma_i^+} \int_0^r \kappa^{1-N-2\gamma_i^+} \int_0^\kappa s^{N-1+\gamma_i^+} g_i(s) ds d\kappa)$$

for $i = 0, 1, \dots, N$ as those in the proof of Proposition 3 of [Re2], we easily know that $w_i(r)$ exists for each i and that there exists some constant $c_i > 0$ such that for all $r \in (0, 1]$, $r^{-\mu} |w_i(r)| \leq c_i \|g\|_{C_{\mu-2}^{2,\alpha}}$. Thus, this proposition can be easily obtained from Proposition 3 of [Re2] by choosing $j = N + 1$. \square

Let us define

$$\begin{aligned} & (C_\mu^{2,\alpha} \oplus r^{\gamma_1^+} \text{span}\{\varphi_1(\theta), \varphi_2(\theta), \dots, \varphi_N(\theta)\})_0 \\ &= \left\{ w \in C_\mu^{2,\alpha} \oplus r^{\gamma_1^+} \text{span}\{\varphi_1(\theta), \varphi_2(\theta), \dots, \varphi_N(\theta)\} : w|_{\partial B} \in \text{span}\{1\} \right\}. \end{aligned}$$

We easily obtain the following corollaries from the previous propositions.

Corollary 3.2. *Assume that $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, and $0 < 2/(\nu + 1) < \mu < \gamma_{N+1}^+$. Then for any $g \in C_{\mu-2}^{0,\alpha}$ there exists a unique solution of $\mathcal{L}w = g$ in $B \setminus \{0\}$ which belongs to the space $(C_\mu^{2,\alpha} \oplus r^{\gamma_1^+} \text{span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\})_0$. In addition, the mapping $g \in C_{\mu-2}^{0,\alpha} \rightarrow w \in (C_\mu^{2,\alpha} \oplus r^{\gamma_1^+} \text{span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\})_0$ is bounded.*

The same results hold for $N = 2$ and $1 < \nu \leq 3$; $N = 3$ and $\nu > 3$, if the space

$$(C_\mu^{2,\alpha} \oplus r^{\gamma_1^+} \text{span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\})_0$$

is replaced by $(C_\mu^{2,\alpha} \oplus r^{\gamma_1^-} \text{span}\{\varphi_1(\theta), \dots, \varphi_N(\theta)\})_0$.

Proof. Choosing

$$w_i(r) = -r^{\gamma_i^+} \int_r^1 \kappa^{1-N-2\gamma_i^+} \int_0^\kappa s^{N-1+\gamma_i^+} g_i(s) ds d\kappa$$

for $i = 1, 2, \dots$, and

$$w_0(r) = \Re(r^{\gamma_0^+} \int_0^r \kappa^{1-N-2\gamma_0^+} \int_0^\kappa s^{N-1+\gamma_0^+} g_0(s) ds d\kappa)$$

as those in the proof of Proposition 3 of [Re2], we easily know that $w_i(r)$ exists for each i . We know that for $i = 0, N + 1, N + 2, \dots$, there exist constants $c_i > 0$ such that for all $r \in (0, 1]$, $r^{-\mu} |w_i(r)| \leq c_i \|g\|_{C_{\mu-2}^{2,\alpha}}$. Thus, the first part of this corollary can be easily obtained from Corollary 2 of [Re2].

To show the second part, we choose

$$w_i(r) = -r^{\gamma_i^+} \int_r^1 \kappa^{1-N-2\gamma_i^+} \int_0^\kappa s^{N-1+\gamma_i^+} g_i(s) ds d\kappa$$

for $i = N + 1, N + 2, \dots$;

$$w_i(r) = -r^{\gamma_i^-} \int_r^1 \kappa^{1-N-2\gamma_i^-} \int_0^\kappa s^{N-1+\gamma_i^-} g_i(s) ds d\kappa$$

for $i = 1, 2, \dots, N$; and

$$w_0(r) = \Re(r^{\gamma_0^+} \int_0^r \kappa^{1-N-2\gamma_0^+} \int_0^\kappa s^{N-1+\gamma_0^+} g_0(s) ds d\kappa)$$

as those in the proof of Proposition 3 of [Re2]. It is easily known that, for $N = 2$ and $1 < \nu \leq 3$; $N = 3$ and $\nu > 3$, $w_i(r)$ exists for each i and, for $i = 0, N + 1, N + 2, \dots$, there exist constants $c_i > 0$ such that for all $r \in (0, 1]$, $r^{-\mu}|w_i(r)| \leq c_i \|g\|_{C_{\mu-2}^{2,\alpha}}$. Moreover, if $0 < 2/(\nu + 1) < \mu < \gamma_{N+1}^+$, then $\gamma_1^- < \mu$. Thus, the second part of this corollary can also be easily obtained from Corollary 2 of [Re2]. \square

Corollary 3.3. *Assume that $N = 2$ and $\nu > 3$, and $0 < 2/(\nu + 1) < \mu < \min\{\gamma_1^+, \gamma_3^+\}$. Then for any $g \in C_{\mu-2}^{0,\alpha}$ there exists a unique solution of $\mathcal{L}w = g$ in $B \setminus \{0\}$ which belongs to the space $C_{\mu,0}^{2,\alpha}$. In addition, the mapping $g \in C_{\mu-2}^{0,\alpha} \rightarrow w \in C_{\mu,0}^{2,\alpha}$ is bounded.*

Proof. It is easily known from Proposition 2.1 that $\gamma_1^+ = \gamma_2^+ > 2/(1 + \nu)$ for $N = 2$ and $\nu > 3$. Therefore, choosing

$$w_i(r) = -r^{\gamma_i^+} \int_r^1 \kappa^{1-N-2\gamma_i^+} \int_0^\kappa s^{N-1+\gamma_i^+} g_i(s) ds d\kappa$$

for $i = 1, 2, \dots$, and

$$w_0(r) = \Re(r^{\gamma_0^+} \int_0^r \kappa^{1-N-2\gamma_0^+} \int_0^\kappa s^{N-1+\gamma_0^+} g_0(s) ds d\kappa)$$

as those in the proof of Proposition 3 of [Re2], we easily know that $w_i(r)$ exists for each i and that there exists some constant $c_i > 0$ such that for all $r \in (0, 1]$, $r^{-\mu}|w_i(r)| \leq c_i \|g\|_{C_{\mu-2}^{2,\alpha}}$. Thus, this corollary can be easily obtained from Proposition 3 of [Re2]. \square

4. THE CASE OF $\psi(\theta) = 1 + \zeta(\theta)$

In this section we will find nonnegative solutions u of (1.1) with $\psi(\theta) = 1 + \zeta(\theta)$ and $\|\zeta\|_{C^{2,\alpha}(S^{N-1})}$ being sufficiently small. Moreover, u has a nonremovable zero point. We first obtain the following theorem.

Theorem 4.1. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$, such that, for any $\eta \in C^{2,\alpha}(S^{N-1})$, if $\|\eta\|_{C^{2,\alpha}(S^{N-1})} < \epsilon$, there exist $\zeta_\eta \in C^{2,\alpha}(S^{N-1})$ satisfying $\|\zeta_\eta\|_{C^{2,\alpha}(S^{N-1})} \leq \Lambda\epsilon < 1/4$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_η of the problem*

$$(4.1) \quad \Delta u = \lambda_0 u^{-\nu} \text{ in } B \setminus \{0\}, \quad u = 1 + \eta + \zeta_\eta \text{ on } S^{N-1}$$

with a nonremovable zero at 0.

Proof. Choosing $0 < 2/(\nu + 1) < \mu < \gamma_{N+1}^+$, we have from Proposition 2.1 that for $N \geq 3$ and $\nu > 0$ or $N = 2$ and $0 < \nu \leq 3$,

$$\gamma_1^+ = \dots = \gamma_N^+ < \mu < \gamma_{N+1}^+.$$

For any $\eta \in C^{2,\alpha}(S^{N-1})$ we define $w_\eta(x) = \chi(r)\eta(\theta)$ where χ is some fixed regular function which equals to 0 in $B_{1/2}$ and equals to 1 outside $B_{3/4}$.

We are going to find a solution $v \in C_{\mu,N}^{2,\alpha}$ of the equation

$$(4.2) \quad \Delta(u_0 + v + w_\eta) = \lambda_0|u_0 + v + w_\eta|^{-(\nu+1)}(u_0 + v + w_\eta) \quad \text{in } B \setminus \{0\}.$$

To this end, we define, for all $(v, \eta) \in C_{\mu,N}^{2,\alpha} \times C^{2,\alpha}(S^{N-1})$

$$\mathcal{N}(v, \eta) \equiv \Delta(u_0 + v + w_\eta) - \lambda_0|u_0 + v + w_\eta|^{-(\nu+1)}(u_0 + v + w_\eta).$$

It is easy to see that \mathcal{N} is well defined from $C_{\mu,N}^{2,\alpha} \times C^{2,\alpha}(S^{N-1})$ into $C_{\mu-2}^{0,\alpha}$. In addition, $\mathcal{N}(0, 0) = 0$ and $D\mathcal{N}|_{(0,0)}(v, 0) = \mathcal{L}v$. It follows easily from the implicit function theorem and Proposition 3.1 that all solutions of the equation (4.2) near $(0, 0)$ are of the form (v_η, η) where $\eta \in C^{2,\alpha}(S^{N-1}) \rightarrow v_\eta \in C_{\mu,N}^{2,\alpha}$ is a regular mapping. Thus, we can choose $\epsilon > 0$ sufficiently small, which satisfies that for any η satisfying $\|\eta\|_{C^{2,\alpha}(S^{N-1})} < \epsilon$, there is $v_\eta \in C_{\mu,N}^{2,\alpha}$ satisfying $\|v_\eta\|_{C_{\mu,N}^{2,\alpha}} \leq \Lambda\epsilon < 1/4$, where $\Lambda > 0$ is independent of ϵ , such that $u_\eta := u_0 + v_\eta + w_\eta$ is a solution of (4.2). It is easy to see that $v_\eta(0) + u_0(0) + w_\eta(0) = 0$. Since $u_0(x) = |x|^{2/(\nu+1)}$ and $v_\eta(x) \leq |x|^\mu/4$ with $2/(\nu + 1) < \mu$, we know that $u_0(x) + v_\eta(x) + w_\eta(x) > 0$ for $x \in B_\delta \setminus \{0\}$, where $\delta > 0$ is a sufficiently small number. Note that $u_0(x) \geq \delta^{2/(\nu+1)}$ for $x \in B \setminus B_\delta$. By choosing $\epsilon > 0$ small enough, we obtain that

$$(4.3) \quad u_0 + v_\eta + w_\eta > 0 \quad \text{in } B \setminus \{0\}.$$

This implies that $u_\eta = u_0 + v_\eta + w_\eta$ is a nonnegative solution of the equation in (4.1) with $u_\eta(0) = 0$. Moreover,

$$(4.4) \quad u_\eta(\theta) = 1 + v_\eta(\theta) + \eta(\theta) \quad \text{for } \theta \in S^{N-1}.$$

Defining $\zeta(\theta) = v_\eta(\theta)$, we easily see that ζ is the required function. This completes the proof of Theorem 4.1. \square

From Theorem 4.1 and Corollary 3.3, we easily obtain the following corollary.

Corollary 4.2. *Given $N = 2$ and $\nu > 3$, there exists $\epsilon > 0$ sufficiently small, such that, for any $\eta \in C^{2,\alpha}(S^1)$, if $\|\eta\|_{C^{2,\alpha}(S^1)} < \epsilon$, there exist a constant c_η satisfying $|c_\eta| \leq (\Lambda + 1)\epsilon < 1/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_η of the problem*

$$(4.5) \quad \Delta u = \lambda_0 u^{-\nu} \quad \text{in } B \setminus \{0\}, \quad u = 1 + c_\eta + \eta \quad \text{on } S^1$$

with a nonremovable zero at 0.

Theorem 4.3. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $y \in B_\epsilon \subset B$, there exist $\zeta_y \in C^{2,\alpha}(S^{N-1})$ satisfying $\|\zeta_y\|_{C^{2,\alpha}(S^{N-1})} \leq (\Lambda + 1)\epsilon < 1/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative*

solution u_y of the problem

$$(4.6) \quad \Delta u = \lambda_0 u^{-\nu} \text{ in } B \setminus \{y\}, \quad u = 1 + \zeta_y \text{ on } S^{N-1}.$$

with a nonremovable zero at y .

Proof. Let $T : B \times B_{1/4} \rightarrow B$ be a $C^{2,\alpha}$ map which satisfies that, for all $y \in B_{1/4}$, $T(\cdot, y)$ is a $C^{2,\alpha}$ diffeomorphism from the unit ball into itself. Moreover, T satisfies that

$$T(x, y) = \begin{cases} x - y & \text{for all } x, y \in B_{1/4}, \\ x & \text{for all } x \in B \setminus B_{3/4} \text{ and all } y \in B_{1/4} \end{cases}$$

and

$$T(x, 0) = x.$$

For $0 < 2/(\nu + 1) < \mu < \gamma_{N+1}^+$, we define the nonlinear mapping

$$\mathcal{N}(v, y) = \Delta((u_0 + v) \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) - \lambda_0 |u_0 + v|^{-(\nu+1)}(u_0 + v).$$

It is easy to see that \mathcal{N} is well defined from $C_{\mu,N}^{2,\alpha} \times B_{1/4}$ into $C_{\mu-2}^{0,\alpha}$. In addition, $\mathcal{N}(0, 0) = 0$ and

$$D\mathcal{N}|_{(0,0)}(v, 0) = \mathcal{L}v.$$

It follows easily from the implicit function theorem and Proposition 3.1 that all solutions of the equation $\mathcal{N}(v, y) = 0$ near $(0, 0)$ are of the form (v_y, y) where

$$y \in B_{1/4} \rightarrow v_y \in C_{\mu,N}^{2,\alpha}$$

is some regular mapping. That is, we can choose $\epsilon > 0$ which satisfies that, for any $y \in B_\epsilon$ there is $v_y \in C_{\mu,N}^{2,\alpha}$ satisfying $\|v_y\|_{C_{\mu,N}^{2,\alpha}} \leq \Lambda\epsilon < 1/4$, where $\Lambda > 0$ is independent of ϵ , such that $u_y := u_0 + v_y$ is a solution of the equation

$$(4.7) \quad \Delta(u \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) = \lambda_0 |u|^{-(\nu+1)} u \text{ in } B \setminus \{0\}.$$

Arguments similar to those in the proof of Theorem 4.1 imply that $u_y = u_0 + v_y$ satisfies $u_y(y) = 0$ and $u_y > 0$ in $B \setminus \{y\}$. Define $\zeta_y(\theta) = v_y(\theta)$ for $\theta \in S^{N-1}$. Then $\|\zeta_y\|_{C^{2,\alpha}(S^{N-1})} \leq (\Lambda + 1)\epsilon$. This completes the proof. \square

From Theorem 4.3 and Corollary 3.3, we easily obtain the following corollary.

Corollary 4.4. *Given $N = 2$ and $\nu > 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $y \in B_\epsilon \subset B$, there exists a nonnegative solution (λ, u_y) of the problem*

$$(4.8) \quad \Delta u = \lambda u^{-\nu} \text{ in } B \setminus \{y\}, \quad u = 1 \text{ on } S^1.$$

with a nonremovable zero at y .

Proof. Using the same idea as in the proof of Theorem 4.3 and Corollary 3.3, we see that there exists $\epsilon > 0$ such that, for any $y \in B_\epsilon$, there exist a constant c_y satisfying $|c_y| \leq 1/2$ and a nonnegative solution \tilde{u}_y of the problem

$$\Delta u = \lambda_0 u^{-\nu} \text{ in } B \setminus \{y\}, \quad u = 1 + c_y \text{ on } S^1.$$

with a nonremovable zero at y . Setting $u_y = \tilde{u}_y/(1 + c_y)$, we easily see that u_y satisfies (4.8) with $\lambda = \lambda_0(1 + c_y)^{-(\nu+1)}$. Moreover, y is a nonremovable zero point of u_y . \square

Theorem 4.5. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $\eta \in C^{2,\alpha}(S^{N-1})$, if $\|\eta\|_{C^{2,\alpha}(S^{N-1})} < \epsilon$, there exist $x_\eta \in B$; a constant c_η satisfying $|c_\eta| \leq (\Lambda + 1)\epsilon < 1/2$ ($\Lambda > 0$ independent of ϵ) and u_η a nonnegative solution of the problem*

$$(4.9) \quad \Delta u = \lambda_0 u^{-\mu} \text{ in } B \setminus \{x_\eta\}, \quad u = 1 + c_\eta + \eta \text{ on } S^{N-1}$$

with a nonremovable zero at x_η .

Proof. It is known from Proposition 2.1 that for $N \geq 4$ and $\nu > 0$,

$$\gamma_1^+ = \gamma_2^+ = \dots = \gamma_N^+ = \frac{(1 - \nu)}{(1 + \nu)};$$

for $N = 2$ and $0 < \nu \leq 1$,

$$\gamma_1^+ = \gamma_2^+ = \frac{(1 - \nu)}{(1 + \nu)};$$

for $N = 2$ and $\nu > 1$,

$$\gamma_1^- = \gamma_2^- = \frac{(1 - \nu)}{(1 + \nu)};$$

for $N = 3$ and $0 < \nu \leq 3$,

$$\gamma_1^+ = \gamma_2^+ = \gamma_3^+ = \frac{(1 - \nu)}{(1 + \nu)};$$

for $N = 3$ and $\nu > 3$,

$$\gamma_1^- = \gamma_2^- = \gamma_3^- = \frac{(1 - \nu)}{(1 + \nu)}.$$

We choose μ such that $0 < 2/(\nu + 1) < \mu < \gamma_{N+1}^+$ and define the space \mathbb{M} as follows:

$$\mathbb{M} = \text{span}\{\varphi_1(\theta), \varphi_2(\theta), \dots, \varphi_N(\theta)\}.$$

Thanks to Corollary 3.2, for all $g \in C_{\mu-2}^{0,\alpha}$, the problem

$$\mathcal{L}w = g \text{ in } B \setminus \{0\}$$

has a solution in the space $(C_\mu^{2,\alpha} \oplus r^{(1-\nu)/(1+\nu)}\mathbb{M})_0$. Note that for $N \geq 4$ and $\nu > 0$; $N = 2$ and $0 < \nu \leq 1$; $N = 3$ and $0 < \nu \leq 3$, we use γ_1^+ in Corollary 3.2. For $N = 2$ and $1 < \nu \leq 3$; $N = 3$ and $\nu > 3$, we use γ_1^- in Corollary 3.2. It is clear that

$$\nabla|x|^{2/(\nu+1)} = |x|^{(1-\nu)/(1+\nu)}\nabla|x|.$$

Given a function $\eta \in C^{2,\alpha}(S^{N-1})$ we have to find a solution $(v, y) \in C_{\mu,0}^{2,\alpha} \times \mathbb{R}^N$ of the equation

$$\Delta((u_0 + w_\eta + v) \circ T(\cdot, y)) \circ T^{-1}(\cdot, y) - \lambda_0 f(u_0 + w_\eta + v) = 0 \text{ in } B \setminus \{0\}$$

where $f(s) = |s|^{-(\nu+1)}s$. We define the nonlinear mapping

$$\mathcal{N}(v, y, \eta) = [\Delta((u_0 + w_\eta + v) \circ T(\cdot, y)) - \lambda_0 f((u_0 + w_\eta + v) \circ T(\cdot, y))] \circ T^{-1}(\cdot, y).$$

Obviously, \mathcal{N} is well defined from $C_{\mu,0}^{2,\alpha} \times \mathbb{R}^N \times C^{2,\alpha}(S^{N-1})$ into the space $C_{\mu-2}^{0,\alpha}$. We notice that $\mathcal{N}(0, 0, 0) = 0$. Furthermore

$$D\mathcal{N}|_{(0,0,0)}(v, 0, 0) = \mathcal{L}v$$

and since $\Delta u_0 = \lambda_0 u_0^{-\nu}$ in B ,

$$\begin{aligned} D\mathcal{N}|_{(0,0,0)}(0, z, 0) &= \Delta(Du_0|_x \circ D_y T|_{(x,0)}(z)) \\ &\quad + \lambda_0 \nu u_0^{-(\nu+1)}(Du_0|_x \circ D_y T|_{(x,0)}(z)) \\ &= \mathcal{L}(Du_0|_x \circ D_y T|_{(x,0)}(z)). \end{aligned}$$

Therefore,

$$D\mathcal{N}|_{(0,0,0)}(w, z, 0) = \mathcal{L}(w + Du_0|_x \circ D_y T|_{(x,0)}(z)).$$

Since $D_y T|_{(x,0)}(z) = 0$ if $x \in B \setminus B_{3/4}$ and since $D_y T|_{(x,0)}(z) = -z$ if $x \in B_{1/4}$ we see from [Re2] that $(C_{\mu}^{2,\alpha} \oplus r^{(1-\nu)/(1+\nu)} \mathbb{M})_0 = C_{\mu,0}^{2,\alpha} \oplus \text{span}\{Du_0|_x \circ D_y T|_{(x,0)}(z) : z \in \mathbb{R}^N\}$.

We can use the implicit function theorem to prove that all solutions $\mathcal{N}(v, y, \eta) = 0$ near $(0, 0, 0)$ are given by (v_η, y_η, η) where

$$\eta \in C^{2,\alpha} \rightarrow (v_\eta, y_\eta) \in C_{\mu,0}^{2,\alpha} \times \mathbb{R}^N$$

is a regular mapping. Therefore, arguments similar to those in the proof of Theorem 4.1 imply that $u_\eta := u_0 + w_\eta + v_\eta$ is a nonnegative solution of the equation in (4.9) which satisfies that $u_\eta = u_0 + w_\eta + v_\eta > 0$ in $B \setminus \{y_\eta\}$ and $u_\eta(y_\eta) = 0$. Moreover, $u_\eta(\theta) = 1 + c_\eta + \eta(\theta)$ for $\theta \in S^{N-1}$, where $c_\eta = v_\eta|_{S^{N-1}}$ is a constant. This completes the proof. \square

The following corollary is an easy consequence of Theorem 4.5.

Corollary 4.6. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ sufficiently small such that, for any constant ρ , if $|\rho| < \epsilon$, there exist $x_\rho \in B$ and (λ, u_ρ) a nonnegative solution of the problem*

$$(4.10) \quad \Delta u = \lambda u^{-\nu} \text{ in } B \setminus \{x_\rho\}, \quad u = 1 + \rho \text{ on } S^{N-1}$$

with a nonremovable zero at x_ρ .

Proof. It follows from Theorem 4.5 that for any constant ρ (since $\rho \in C^{2,\alpha}(S^{N-1})$) satisfying $|\rho| < \epsilon$, there exist $x_\rho \in B$; a constant c_ρ satisfying $|c_\rho| \leq (\Lambda + 1)\epsilon < 1/2$ and \tilde{u}_ρ a nonnegative solution of the problem

$$(4.11) \quad \Delta u = \lambda_0 u^{-\nu} \text{ in } B \setminus \{x_\rho\}, \quad u = 1 + c_\rho + \rho \text{ on } S^{N-1}$$

with a nonremovable zero at x_ρ . Defining $u_\rho := \tilde{u}_\rho / (1 + c_\rho + \rho)$, we have that u_ρ satisfies the problem

$$(4.12) \quad \Delta u = \lambda u^{-\nu} \text{ in } B \setminus \{x_\rho\}, \quad u = 1 \text{ on } S^{N-1}$$

where $\lambda = \lambda_0(1 + c_\rho + \rho)^{-(\nu+1)}$. It is clear that x_ρ is a removable zero of u_ρ . \square

5. THE CASE OF $\psi(\theta) = C + \zeta(\theta)$

In this section we use the results obtained in Section 4 to consider the case that $\psi(\theta) = C + \zeta(\theta)$, where $C > 1$ or $0 < C < 1$, for $\theta \in S^{N-1}$ and $\zeta \in C^{2,\alpha}(S^{N-1})$ satisfying that $\|\zeta\|_{C^{2,\alpha}(S^{N-1})}$ is sufficiently small.

By simple calculations, we easily know that $u_C(x) = C|x|^{2/(\nu+1)}$ satisfies the problem

$$(5.1) \quad \Delta u = \frac{2C^{\nu+1}(N + (N-2)\nu)}{(\nu+1)^2} u^{-\nu} \quad \text{in } B \setminus \{0\}, \quad u = C \quad \text{on } \partial B.$$

Define $\lambda_C = \frac{2C^{\nu+1}(N+(N-2)\nu)}{(\nu+1)^2}$ and the linear operator

$$\mathcal{L} : w \mapsto \Delta w + \lambda_C \nu u_C^{-(\nu+1)} w.$$

Clearly we have

$$(5.2) \quad \lambda_C \nu u_C^{-(\nu+1)} = \frac{c_0}{r^2},$$

where c_0 is same as in (2.1). Thus, \mathcal{L} is exactly same as that we defined in Section 2. Thus, the indicial roots of \mathcal{L} are defined in (2.3). By arguments similar to those in the proofs of Theorems 4.1, 4.3, 4.5, we easily obtain the following results.

Theorem 5.1. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ such that, for any $\eta \in C^{2,\alpha}(S^{N-1})$, if $\|\eta\|_{C^{2,\alpha}(S^{N-1})} < \epsilon$, there exist $\zeta_\eta \in C^{2,\alpha}(S^{N-1})$ satisfying $\|\zeta_\eta\|_{C^{2,\alpha}(S^{N-1})} \leq (\Lambda + 1)\epsilon < C/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_η of the problem*

$$(5.3) \quad \Delta u = \lambda_C u^{-\nu} \quad \text{in } B \setminus \{0\}, \quad u = C + \zeta_\eta \quad \text{on } S^{N-1}$$

with a nonremovable zero at 0.

Corollary 5.2. *Given $N = 2$ and $\nu > 3$, there exists $\epsilon > 0$ sufficiently small, such that, for any $\eta \in C^{2,\alpha}(S^1)$, if $\|\eta\|_{C^{2,\alpha}(S^1)} < \epsilon$, there exist a constant c_η satisfying $|c_\eta| \leq (\Lambda + 1)\epsilon < C/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_η of the problem*

$$(5.4) \quad \Delta u = \lambda_C u^{-\nu} \quad \text{in } B \setminus \{0\}, \quad u = C + c_\eta + \eta \quad \text{on } S^1$$

with a nonremovable zero at 0.

Theorem 5.3. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $y \in B_\epsilon \subset B$, there exist $\zeta_y \in C^{2,\alpha}(S^{N-1})$ satisfying $\|\zeta_y\|_{C^{2,\alpha}(S^{N-1})} \leq (\Lambda + 1)\epsilon < C/2$ ($\Lambda > 0$ independent of ϵ) and a nonnegative solution u_y of the problem*

$$(5.5) \quad \Delta u = \lambda_C u^{-\nu} \quad \text{in } B \setminus \{y\}, \quad u = C + \zeta_y \quad \text{on } S^{N-1}.$$

with a nonremovable zero at y .

Corollary 5.4. *Given $N = 2$ and $\nu > 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $y \in B_\epsilon \subset B$, there exists a nonnegative solution (λ, u_y) of the problem*

$$(5.6) \quad \Delta u = \lambda u^{-\nu} \text{ in } B \setminus \{y\}, \quad u = C \text{ on } S^1.$$

with a nonremovable zero at y .

Theorem 5.5. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ sufficiently small such that, for any $\eta \in C^{2,\alpha}(S^{N-1})$, if $\|\eta\|_{C^{2,\alpha}(S^{N-1})} < \epsilon$, there exist $x_\eta \in B$; a constant c_η satisfying $|c_\eta| \leq (\Lambda + 1)\epsilon < C/2$ ($\Lambda > 0$ independent of ϵ) and u_η a nonnegative solution of the problem*

$$(5.7) \quad \Delta u = \lambda_C u^{-\nu} \text{ in } B \setminus \{x_\eta\}, \quad u = C + c_\eta + \eta \text{ on } S^{N-1}$$

with a nonremovable zero at x_η .

We can also obtain the existence for a class of Dirichlet problems with constant boundary values.

Corollary 5.6. *Given $N \geq 3$ and $\nu > 0$, or $N = 2$ and $0 < \nu \leq 3$, there exists $\epsilon > 0$ such that, for any constant ρ , if $|\rho| < \epsilon$, there exist $x_\rho \in B$ and (λ_ρ, u_ρ) a nonnegative solution of the problem*

$$(5.8) \quad \Delta u = \lambda u^{-\nu} \text{ in } B \setminus \{x_\rho\}, \quad u = C + \rho \text{ on } S^{N-1}$$

with a nonremovable zero at x_ρ .

Remark. It is easily seen from Theorems 5.1, 5.3 and 5.5 that the parameter λ depends upon the boundary value C . We can also obtain the existence for any $\lambda > 0$, but the boundary value changes. Indeed, for any fixed $\lambda > 0$, we easily know that $(\lambda, u_\lambda(r))$ is a nonnegative solution, with one isolated zero at 0, of the problem

$$(5.9) \quad \Delta u = \lambda u^{-\nu} \text{ in } B \setminus \{0\}$$

where

$$u_\lambda(r) = \left[\frac{\lambda(\nu + 1)^2}{2((N - 2)\nu + N)} \right]^{1/(\nu+1)} r^{2/(\nu+1)}.$$

It is clear that the boundary value of u_λ is the constant in the expression of $u_\lambda(r)$. Now we define

$$\mathcal{L} : w \longmapsto \Delta w + \lambda \nu u_\lambda^{-(\nu+1)} w.$$

It is clear that $\lambda \nu u_\lambda^{(-\nu+1)} = c_0/r^{-2}$ and \mathcal{L} is exactly same as that we defined in Section 2. Thus, we can derive results similar to Theorems 5.1, 5.3, 5.5, but with different boundary conditions.

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