Nonradial Solutions for a Conformally Invariant Fourth Order Equation in $\mathbb{R}^4$

Juncheng Wei[1], Dong Ye[2]

November 27, 2006

[1]Department of Mathematics,
The Chinese University of Hong Kong, Shatin, Hong Kong.
E-mail address: wei@math.cuhk.edu.hk
[2]Département de Mathématiques et UMR 8088,
Université de Cergy-Pontoise,
95302 Cergy-Pontoise, France.
E-mail address: dong.ye@u-cergy.fr

1 Introduction

Recently, there have been lots of work on the study of higher order conformally invariant operators. A notable example is the so-called Paneitz operator (see [10])

$$Pu = \Delta^2 u + \delta\left(\frac{2}{3}K_0 I - 2Ric\right)du \tag{1}$$

and the associated $Q-$curvature:

$$Q = \frac{1}{12}\Big(K_0^2 - \Delta K_0 - 3|Ric|^2\Big), \tag{2}$$

where δ denotes the divergence, d the differential, Ric is the Ricci curvature of $(\mathcal{M}, g_0)$, a four dimensional manifold and K_0 is the scalar curvature. In [2]-[3], Chang and Yang studied the existence of extreme functions for the associated variational problem:

$$II[u] = \langle Pu, u\rangle + \int Q_0 w dV_{g_0} - \left(\int Q_0 dV_{g_0}\right)\log\int e^{4w}dV_{g_0}.$$

For background material and other related problems, we refer to [2], [3], [4], [6] and the references therein. The extreme function u of $II[u]$ satisfies a conformal invariant elliptic equation of fourth order:

$$Pu + 2Q_0 = 2Qe^{4w}. \tag{3}$$

where Q is a constant. To study the qualitative behavior (such as blow up, *a priori* estimates) of solutions of (3), it is important to classify all solutions to the following reduced fourth order equation

$$\Delta^2 u = 6e^{4u} \ \text{ in } \ \mathbb{R}^4, \quad \int_{\mathbb{R}^4} e^{4u}dx < \infty. \tag{4}$$

In [7] (see also [12] for higher order cases), Lin classified the solutions to (4) and proved the following theorem.

Theorem 1.1 *(Theorem 1.1 and 1.2 of [7]) Suppose u is a solution to (4). Then the following statements hold true.*

(i) After an orthogonal transformation, $u(x)$ can be represented by

$$
\begin{aligned}
u(x) &= \frac{3}{4\pi^2}\int_{\mathbb{R}^4} e^{4u(y)}\log\frac{|y|}{|x-y|}dy - \sum_{j=1}^{4} a_j(x_j - x_j^0)^2 + c_0 \\
&= -\sum_{j=1}^{4} a_j(x_j - x_j^0)^2 - \alpha\log|x| + c_0 + o(1),
\end{aligned} \tag{5}
$$

as $|x|$ tends to ∞. Here $a_j \ge 0$, c_0 are constants and $x^0 = (x_1^0 \dots, x_4^0) \in \mathbb{R}^4$. Moreover, if $a_j \neq 0$ for all j, then u is symmetric with respect to the hyperplane $\{x \mid x_j = x_j^0\}$. If $a_1 = a_2 = a_3 = a_4$, then u is radially symmetric with respect to x^0.
(ii) The total integration

$$\alpha = \frac{3}{4\pi^2}\int_{\mathbb{R}^4} e^{4u(y)}dy \le 2.$$

If $\alpha = 2$, then all a_j are zero and u has the following form:

$$u(x) = \log\frac{2\lambda}{1+\lambda^2|x-x^0|^2}, \quad \textit{with} \quad \lambda > 0. \tag{6}$$

(iii) If $u(x) = o(|x|^2)$ at ∞, then $\alpha = 2$.

Lin's theorem shows a striking difference between (4) and its second order analogue:

$$\Delta u + e^{2u} = 0 \ \text{ in } \ \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{2u}dx < \infty. \tag{7}$$

It is known (see [5]) that all solutions to (7) are radially symmetric (with respect to one point) and have the form (6).

An interesting question in Lin's theorem is: given any $a_j \ge 0$, $1 \le j \le 4$ and $\alpha > 0$, are there solutions to (4) satisfying (5)? In the case $a_1 = a_2 = a_3 = a_4$, by Lin's theorem, solutions are radially symmetric up to translation. Chang and Chen [1] proved the existence of radially symmetric solutions to (4) for any $\alpha \in (0, 2)$. It remains to study the nonradially symmetric case, that is exactly the purpose of this paper.

Here, we show that the converse of Lin's theorem is also true.

Theorem 1.2 *Let $x^0 \in \mathbb{R}^4$, $1 \le k \le 4$, $\alpha \in \left(1 - \frac{k}{4}, 2\right)$ and $a_j > 0$ for $1 \le j \le k$. Then there exists $c_0 \in \mathbb{R}$ and a solution of (4) such that*

$$u(x) = -\sum_{j=1}^{k} a_j(x_j - x_j^0)^2 - \alpha\log|x| + c_0 + o(1) \tag{8}$$

for $|x|$ tends to ∞. Moreover,

$$\int_{\mathbb{R}^4} e^{4u(x)}dx = \frac{4\pi^2\alpha}{3}.$$

Remark 1.3 *Lin remarked that the condition $\alpha > 1 - \frac{k}{4}$ is* **necessary** *for the existence of solution if we have just $a_j > 0$ for $1 \le j \le k$, see the comments under (3.8) in page 224 of [7]. Our result means that this condition is also* **sufficient**. *Note that when $k = 4$, this condition becomes just $\alpha \in (0, 2)$, so we recover the result in [1].*

Theorem 1.2 shows that there are abundant nonradially symmetric solutions to the conformally invariant equation (4). More precisely, for any $\alpha \in (0, 2)$, even up to translation and the gauge transformation $u(\lambda x) + 4\log\lambda$, there exist *infinitely many* nonradial solutions. This is quite surprising.

In the next section, we shall prove Theorem 1.2. We make use an idea of McOwen [8], where he constructed solutions to

$$\Delta u + k(x)e^{2u} = 0 \ \text{ in } \ \mathbb{R}^2, \tag{9}$$

with prescribed asymptotic behavior. Our difficulty is to show a *priori* estimates, our main arguments are blow-up analysis and Pohozaev's identity.

2 Proof of Theorem 1.2

Fix $1 \leq k \leq 4$, $a_j > 0$ for $1 \leq j \leq k$ and $1 - k/4 < \alpha < 2$. Using translation, we can assume that $x^0 = 0$. First, we fix a radially symmetric function $u_0 \in C^\infty(\mathbb{R}^4)$ such that $u_0(x) = -\log|x|$ for any $|x| \geq 1$. Clearly, $\Delta^2 u_0$ is compactly supported and

$$\int_{\mathbb{R}^4} \Delta^2 u_0(x) dx = 8\pi^2. \tag{10}$$

Define

$$v = u + \sum_{j=1}^{k} a_j x_j^2 - \alpha u_0 \stackrel{\text{def}}{=} u + u_1 - \alpha u_0. \tag{11}$$

Then u is a solution of (4) if and only if $\Delta^2 v = Ke^{4v} - \alpha\Delta^2 u_0$ where $K(x) = 6e^{-4u_1+4\alpha u_0}$. For constructing v, we shall use some ideas of McOwen. Let $\mathcal{M}^p_{s,\delta}$ be the weighted Sobolev spaces, the completion of $C_0^\infty(\mathbb{R}^4)$ with the norm

$$\sum_{|\ell|\leq s} \left\|(1+|x|^2)^{(\delta+|\ell|)/2} D^\ell \phi\right\|_{L^p(\mathbb{R}^4)}$$

where $p \in (1,\infty)$, $s \in \mathbb{N}$ and $\delta \in \mathbb{R}$. Let $L^p_\delta = \mathcal{M}^p_{0,\delta}$. The following are some useful properties of $\mathcal{M}^p_{s,\delta}$ (cf. [9]).

Lemma 2.1 *Let $p > 1$ and $\delta \in \left(-\frac{4}{p}, -\frac{4}{p}+1\right)$. Then the operator Δ^2 is an isomorphism from $\mathcal{M}^p_{4,\delta}$ into*

$$\Lambda = \left\{ f \in L^p_{4+\delta}, \quad \int_{\mathbb{R}^4} f dx = 0 \right\}.$$

On the other hand, if $p > 1$, $\delta > -4/p$ and $s > 4/p$, $\mathcal{M}^p_{s,\delta}$ is compactly embedded in $C_0(\mathbb{R}^4)$.

Here $C_0(\mathbb{R}^4)$ denotes the space of continuous functions which tend to zero at ∞, endowed with the norm $\|\cdot\|_\infty$.

Remark 2.2 *For $1 \leq k \leq 4$, if $a_j > 0$ for $1 \leq j \leq k$ and $\alpha > 1 - k/4$, we can always choose $p > 1$ and $\delta \in \left(-\frac{4}{p}, -\frac{4}{p}+1\right)$ such that $K \in L^p_{4+\delta} \cap L^1(\mathbb{R}^4)$. In fact, a sufficient condition is just $p(4\alpha - \delta - 4) > 4 - k$.*

For any $v \in C_0(\mathbb{R}^4)$, define

$$c_v = \frac{\log(8\pi^2\alpha)}{4} - \frac{1}{4}\log\left(\int_{\mathbb{R}^4} Ke^{4v} dx\right). \tag{12}$$

Thanks to remark 2.2, c_v is well defined and it is easy to see that $Ke^{4(v+c_v)} - \alpha\Delta^2 u_0$ belongs to Λ for suitable $p > 1$ and $\delta \in \left(-\frac{4}{p}, -\frac{4}{p}+1\right)$. By lemma 2.1, there exists unique $\overline{v} \in \mathcal{M}^p_{4,\delta}$ such that $\Delta^2 \overline{v} = Ke^{4(v+c_v)} - \alpha\Delta^2 u_0$, we define then $\overline{v} = \mathbb{T}v$. Applying again lemma 2.1, $\mathbb{T}$ is a continuous and compact mapping from $C_0(\mathbb{R}^4)$ into itself. Now we will try to find a fixed point for $\mathbb{T}$, which enables us to get a solution of (4).

In our analysis, a crucial argument is the following result.

Lemma 2.3 *Let u be a smooth function satisfying $\Delta^2 u = Ke^{4u}$ in $\overline{B}_1 \subset \mathbb{R}^4$, such that K is continuous et positive in $\overline{B}_1$,*

$$\int_{B_1} Ke^{4u} dy \leq \beta < 16\pi^2 \tag{13}$$

and there exists $C_0 > 0$ verifying

$$\forall\, B(x,r) \subset B_1, \quad \|\Delta u\|_{L^1(B(x,r))} \leq C_0 r^2. \tag{$*$}$$

Then there exists $C \in \mathbb{R}$ (depending on β and C_0) such that

$$\max_{B_{1/4}} u \leq C. \tag{14}$$

The proof of this lemma is given by contradiction and blow-up analysis. One of the key points is that the condition $(*)$ remains stable under the gauge transformation $u_\lambda(y) = u(x+\lambda y) + 4\log\lambda$. Moreover, this condition prevents to have some $a_j > 0$ for the solution after the blow-up, which will force the total integration to be just $16\pi^2$ and contradicts then $\beta < 16\pi^2$.

Proof of lemma 2.3. Suppose that the constant C does not exist, so we have a family of smooth functions u_n such that $\Delta^2 u_n = Ke^{4u_n}$ in B_1, verifies $(*)$ and

$$\int_{B_1} Ke^{4u_n}dy \leq \beta, \quad \lim_{n\to\infty}\max_{B_{1/4}} u_n \geq n.$$

Consider

$$h_n(x) = u_n(x) + 4\log\left(\frac{1}{2} - |x|\right) \quad \text{in } \; B_{1/2}.$$

Then $\max_{B_{1/2}} h_n \geq n - 4\log 4 \to \infty$. Define

$$\mu_n = h_n(x_n) = \max_{B_{1/2}} h_n, \quad \sigma_n = \frac{1}{2} - |x_n| \quad \text{and} \quad \lambda_n = \sigma_n e^{-\mu_n/4}.$$

Clearly $\sigma_n/\lambda_n \to \infty$. Define also $w_n(y) = u_n(x_n + \lambda_n y) + 4\log\lambda_n$. For $|y| \leq \sigma_n/(2\lambda_n)$, we have

$$\frac{1}{2} - |x_n + \lambda_n y| \geq \frac{1}{2} - |x_n| - \lambda_n|y| \geq \sigma_n - \frac{\sigma_n}{2} = \frac{\sigma_n}{2},$$

hence

$$u_n(x_n + \lambda_n y) \leq \mu_n - 4\log\left(\frac{\sigma_n}{2}\right) = -4\log\lambda_n + \log 16.$$

In other words, $w_n(y) \leq \log 16$ for $|y| \leq \sigma_n/(2\lambda_n)$. Therefore, we obtain

$$\begin{cases} \Delta^2 w_n &= K(x_n+\lambda_n y)e^{4w_n} \quad \text{in } \; B_{\sigma_n/(2\lambda_n)} \\ \quad w_n &\leq \log 16 \qquad\qquad\quad\;\; \text{in } \; B_{\sigma_n/(2\lambda_n)} \\ w_n(0) &= 1. \end{cases}$$

Moreover, for any $R > 0$, $y_0 \in \mathbb{R}^4$ such that $B(y_0, R) \subset B_{\sigma_n/(2\lambda_n)}$, the condition $(*)$ implies

$$\int_{B(y_0,R)} |\Delta w_n|dy = \frac{1}{\lambda_n^2}\int_{B(x_n+\lambda_n y_0, \lambda_n R)} |\Delta u_n|dx \leq C_0 R^2. \tag{15}$$

Using standard elliptic theory, it is not difficult to prove

Lemma 2.4 *Let $R > 0$ and w be a family of functions satisfying $w(0) = 1$ and*

$$\|\Delta^2 w\|_{L^\infty(B_R)} + \|\Delta w\|_{L^1(B_R)} + \sup_{B_R} w \leq A,$$

then there exists $C_R > 0$ depending on R and A such that $w \geq -C_R$ in $B_{R/2}$.

Applying this result on w_n. Up to a subsequence, we can assume that $x_n \to x_*$, $w_n \to w$ in $C^\infty_{loc}(\mathbb{R}^4)$, solution of

$$\Delta^2 w = K(x_*)e^{4w} \quad \text{in } \; \mathbb{R}^4$$

and

$$\int_{\mathbb{R}^4} K(x_*)e^{4w}dy \leq \liminf_{n\to\infty}\int_{B_{\sigma_n/(2\lambda_n)}} K(x_n+\lambda_n y)e^{4w_n}dy \leq \beta < 16\pi^2.$$

Noting that $K(x_*)$ is a constant, w must be a solution given by (5), then

$$\Delta w(x) = -\frac{K(x_*)}{4\pi^2}\int_{\mathbb{R}^4}\frac{e^{4w(y)}}{|x-y|^2}dy - 2\sum_{j=1}^4 a_j = O(|x|^{-2}) - 2\sum_{j=1}^4 a_j.$$

Otherwise, if we take limit in (15), we get $\|\Delta w\|_{L^1(B_R)} \leq C_0 R^2$ for any $R > 0$. Since $a_j \geq 0$, all the coefficients a_j must be equal to zero. By Lin's result, we have $w(y) = o(|y|^2)$, hence

$$\int_{\mathbb{R}^4} K(x_*)e^{4w}dy = 16\pi^2,$$

which is a contradiction. Our proof is completed. □

Remark 2.5 *We can prove similar results for a family of equicontinuous functions K_n which verifies $0 < a \leq K_n \leq b < \infty$. The gauge transformation yields also that the result is true in any ball B_R.*

Proof of Theorem 1.2 completed. Suppose that v is a fixed point for the operator $t\mathbb{T}$ in $C_0(\mathbb{R}^4)$ with $t \in (0, 1]$, that is $v = t\mathbb{T}v$ and $v \in C_0(\mathbb{R}^4)$. We claim then

$$(16) \qquad v(x) = -\frac{t}{8\pi^2}\int_{\mathbb{R}^4} \log|x-y| K e^{4(v+c_v)} dy - t\alpha u_0(x) + C_1 \stackrel{\text{def}}{=} \widetilde{v}(x) + C_1.$$

Indeed, as $e^{4(v+c_v)} \in L^\infty(\mathbb{R}^4)$, under the assumption on α and k, $\widetilde{v}$ is well defined. It is clear that $\Delta^2(\widetilde{v} - v) = 0$ in $\mathbb{R}^4$. Moreover, since for $|x| > 1$,

$$(17) \qquad \widetilde{v}(x) = \frac{t}{8\pi^2}\int_{\mathbb{R}^4} K e^{4(v+c_v)} \log\frac{|x|}{|x-y|} dy$$

and $Ke^{4(v+c_v)} \in C_0 \cap L^1(\mathbb{R}^4)$, we get $\widetilde{v}(x) = o(\log|x|)$ at ∞. Liouville's theorem yields then $v - \widetilde{v} \equiv$ constant.

Take

$$w = v + c_v + t\alpha u_0 + \frac{\log t}{4},$$

then $\Delta^2 w = Qe^{4w}$ in $\mathbb{R}^4$ with

$$Q = Ke^{-4t\alpha u_0} = 6e^{-\sum_{j\leq k} a_j x_j^2 + 4(1-t)\alpha u_0}.$$

Clearly,

$$\int_{\mathbb{R}^4} Qe^{4w} dx = t\int_{\mathbb{R}^4} Ke^{4(v+c_v)} dx \leq 8\pi^2\alpha < 16\pi^2,$$

since $t \in (0, 1]$ and $\alpha < 2$. On the other hand, thanks to (16),

$$\Delta w(x) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{Qe^{4w}}{|x-y|^2} dy.$$

Since $-\Delta w \geq 0$ and

$$\begin{aligned}
-\int_{B(x_0,r)} \Delta w(x) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^4} Qe^{4w(y)} \int_{B(x_0,r)} \frac{1}{|x-y|^2} dx dy \\
&\leq \frac{1}{4\pi^2}\int_{\mathbb{R}^4} Qe^{4w(y)} \int_{B(y,r)} \frac{1}{|x-y|^2} dx dy \\
&\leq \frac{r^2}{4}\int_{\mathbb{R}^4} Qe^{4w(y)} dy \\
&\leq 2\pi^2\alpha r^2.
\end{aligned}$$

Thus w satisfies the condition $(*)$. By lemma 2.3 and remark 2.5, we obtain then w is locally uniformly upper bounded. Using the representation formula, we get also $|\nabla v|$ and Δv are locally bounded, as e^{4w} is locally bounded. For example, fix $R > 0$, for any $x \in B_R$,

$$\begin{aligned}
|\Delta v(x)| = |\Delta w(x)| &= \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{Qe^{4w}}{|x-y|^2} dy \\
&= \frac{1}{4\pi^2}\int_{B_{2R}} \frac{Qe^{4w}}{|x-y|^2} dy + \frac{1}{4\pi^2}\int_{\mathbb{R}^4\setminus B_{2R}} \frac{Qe^{4w}}{|x-y|^2} dy \\
&\leq C_R\int_{B_{2R}} \frac{1}{|x-y|^2} dy + \frac{1}{4\pi^2 R^2}\int_{\mathbb{R}^4\setminus B_{2R}} Qe^{4w} dy \\
&\leq C_R + \frac{2\alpha}{R^2}.
\end{aligned}$$

From the uniform upper bound of w, it follows then $\overline{w} = v + c_v + \log t/4$ is locally uniformly upper bounded. So we conclude

Lemma 2.6 *Let k, a_j and α be as in Theorem 1.2. For any $R > 0$, there exists $C_R > 0$ such that if $v = t\mathbb{T}v$ with $t \in (0, 1]$,*

$$\sup_{B_R} v + c_v + \log t + \|\nabla v\|_{L^\infty(B_R)} + \|\Delta v\|_{L^\infty(B_R)} \leq C_R.$$

It remains to study the exterior domain. For that, we apply the Pohozaev's identity (see [11]). For any $R \geq 1$, as $\mathrm{supp}(\Delta^2 u_0) \subset \overline{B}_1$,

$$\begin{aligned} &\int_{B_R} K e^{4\overline{w}} dx + \frac{1}{4}\int_{B_R} (x \cdot \nabla K) e^{4\overline{w}} dx \\ = & -\int_{B_1} t\alpha (x \cdot \nabla v)\Delta^2 u_0 dx + \frac{1}{4}\int_{\partial B_R} K(x)|x| e^{4\overline{w}} dx \\ & - \int_{\partial B_R} |x| \frac{(\Delta v)^2}{2} d\sigma + \int_{\partial B_R} \frac{\partial v}{\partial r}\frac{\partial(\Delta v)}{\partial r} d\sigma + \int_{\partial B_R} \Delta v \frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) d\sigma. \end{aligned} \tag{18}$$

By lemma 2.6, we know that the first term in the right-hand side is uniformly bounded. The following lemma shows the behavior of last three terms, its proof is technical and delayed to the next section. There we will use intensively the assumption $\alpha > 1 - k/4$.

Lemma 2.7 *Let k, a_j and α be as in Theorem 1.2. For each* **fixed** *v satisfying $v = t\mathbb{T}v$ with $t \in (0, 1]$, the last three terms in (18) tend to zero as $R \to \infty$.*

Since

$$\forall\, x \in \mathbb{R}^4 \setminus B_1, \quad x \cdot \nabla K = -\Big(\sum_{j \leq k} 8a_j x_j^2 + 4\alpha|x|\Big) K \leq 0,$$

passing to the limit $R \to \infty$ in (18), we obtain

$$-\frac{1}{4}\int_{\mathbb{R}^4} (x \cdot \nabla K) e^{4\overline{w}} dx \leq \int_{B_1} t\alpha (x \cdot \nabla v)\Delta^2 u_0 dx + \int_{\mathbb{R}^4} K e^{4\overline{w}} dx.$$

Using again lemma 2.6,

$$\alpha R \int_{\mathbb{R}^4 \setminus B_R} K e^{4\overline{w}} dx \leq C + 8\pi^2 t\alpha + \frac{1}{4}\int_{B_1} (x \cdot \nabla K) e^{4\overline{w}} dx \leq C', \quad \text{if } R > 1.$$

For any $\varepsilon > 0$, there exists $R_0 > 1$ (depending only on ε) such that

$$\int_{\mathbb{R}^4 \setminus B_{R_0}} K e^{4\overline{w}} dx \leq \varepsilon. \tag{19}$$

As $\alpha > 1 - k/4$, we can verify that $|x|^{-8} K(x|x|^{-2}) \in L^p(B_1)$ for some $p > 1$. Choose $\varepsilon = 16\pi^2/q$ where

$$q = \frac{p(p+1)}{p-1},$$

and R_0 such that (19) holds. Consider the Kelvin's transformation

$$\zeta = \overline{w} \circ \varphi \quad \text{with} \quad \varphi(x) = \frac{R_0 x}{|x|^2} \quad \text{for} \ |x| \leq 1.$$

Therefore $\Delta^2 \zeta = R_0^4 |x|^{-8} K \circ \varphi(x) e^{4\zeta}$ in B_1, since $\mathrm{supp}(\Delta^2 u_0) \subset \overline{B}_1$. As

$$\int_{B_1} R_0^4 |x|^{-8} K \circ \varphi(x) e^{4\zeta} dx = \int_{\mathbb{R}^4 \setminus B_{R_0}} K e^{4\overline{w}} dx \leq \varepsilon,$$

by Moser-Trudinger's inequality (see [7]), the upper bound for ζ and $|\Delta\zeta|$ on ∂B_1, we can prove $\|e^{4\zeta}\|_{L^q(B_1)} \leq C$ so that $\||x|^{-8} K \circ \varphi(x) e^{4\zeta}\|_{L^{(p+1)/2}(B_1)} \leq C$. Thus ζ is uniformly upper bounded in B_1 by elliptic theory.

Finally, $\overline{w}$ is uniformly upper bounded in $\mathbb{R}^4 \setminus B_{R_0}$, so $\overline{w}$ is uniformly upper bounded in $\mathbb{R}^4$. Furthermore, as $\Delta^2 v = Ke^{4\overline{w}} - t\alpha\Delta^2 u_0$, we get easily that v is uniformly bounded by lemma 2.1, that is

$$\text{If } \; v \in C_0(\mathbb{R}^4), \quad v = t\mathbb{T}v \; \text{ with } \; t \in (0,1], \quad \text{then } \|v\|_\infty \leq C.$$

In conclusion, as $\mathbb{T}$ is compact, the Leray-Schauder's theory ensures the existence of a fixed point v for $\mathbb{T}$, so we get the desired solution as $u = v - u_1 + \alpha u_0$. □

3 Proof of Lemma 2.7

For **fixed** v, we recall the Pohozaev identity (18)

$$\begin{aligned}
&\int_{B_R} Ke^{4\overline{w}}dx + \frac{1}{4}\int_{B_R}(x\cdot\nabla K)e^{4\overline{w}}dx \\
&= -\int_{B_1} t\alpha(x\cdot\nabla v)\Delta^2 u_0 dx + \frac{1}{4}\int_{\partial B_R} K(x)|x|e^{4\overline{w}}dx \\
&\quad -\int_{\partial B_R}|x|\frac{(\Delta v)^2}{2}d\sigma - \int_{\partial B_R}|x|\frac{\partial v}{\partial r}\frac{\partial(\Delta v)}{\partial r}d\sigma + \int_{\partial B_R}\Delta v\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)d\sigma \\
&= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

We claim then, under the condition of Theorem 1.2 for α and a_j,

$$\lim_{R\to\infty} J_3 = \lim_{R\to\infty} J_4 = \lim_{R\to\infty} J_5 = 0. \tag{20}$$

Remark 3.1 *By similar arguments, we can also show that* $\lim_{R\to\infty} J_2 = 0$, *but it is not necessary for the proof of Theorem 1.2.*

In fact, for $|x| > 1$,

$$\begin{aligned}
|x|^2\Delta v(x) = -\frac{|x|^2}{4\pi^2}\int_{\mathbb{R}^4}\frac{Ke^{4\overline{w}}}{|x-y|^2}dy - |x|^2 t\alpha\Delta u_0 &= -\frac{|x|^2}{4\pi^2}\int_{\mathbb{R}^4}\frac{Ke^{4\overline{w}}}{|x-y|^2}dy + 2t\alpha \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^4}Ke^{4\overline{w}}\left[1 - \frac{|x|^2}{|x-y|^2}\right]dy \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^4}\frac{|y|^2 - 2x\cdot y}{|x-y|^2}Ke^{4\overline{w}}dy
\end{aligned}$$

We decompose the integral over three sub domains, $\Omega_1 = \{|y| \leq R_1\}$; $\Omega_2 = B(x, |x|/2)$ and $\Omega_3 = \mathbb{R}^4 \setminus (\Omega_1 \cup \Omega_2)$, assuming that $|x| = R > 2R_1 > 2$.

On Ω_3, since $|x-y| \geq |x|/2$ implies $|x-y| \geq |y|/4$ (we can discuss the cases $|y| \leq 2|x|$ and $|y| \geq 2|x|$), by taking R_1 big enough (depending on v),

$$\left|\int_{\Omega_3}\frac{|y|^2 - 2x\cdot y}{|x-y|^2}Ke^{4\overline{w}}dy\right| \leq C\int_{\mathbb{R}^4\setminus B_{R_1}}Ke^{4\overline{w}}dy \leq \varepsilon. \tag{21}$$

Fix R_1, for $|x| > 2R_1$,

$$\left|\int_{\Omega_1}\frac{|y|^2 - 2x\cdot y}{|x-y|^2}Ke^{4\overline{w}}dy\right| \leq C|x|^{-1}. \tag{22}$$

It remains to consider Ω_2 where $|y| \leq 3|x|/2$. Denote $\overline{y} = (y_j)_{1\leq j\leq k} \in \mathbb{R}^k$ for any $y \in \mathbb{R}^4$,

$$\begin{aligned}
\left|\int_{\Omega_2}\frac{|y|^2 - 2x\cdot y}{|x-y|^2}Ke^{4\overline{w}}dy\right| &\leq C\int_{\Omega_2}\frac{|x|^2 e^{-a|\overline{y}|^2}}{|x-y|^2}|y|^{-4\alpha}dy \\
&= CR^{4-4\alpha}\int_{B(\xi,1/2)}\frac{e^{-aR^2|\overline{\eta}|^2}}{|\xi-\eta|^2}|\eta|^{-4\alpha}d\eta
\end{aligned}$$

where $a = \min_{1\le j\le k} a_j$ is positive and we use the change of variables $x = R\xi$ and $y = R\eta$. Since $|\eta| \ge 1/2$, we have

$$|x|^2 \Delta v(x) = O(\varepsilon) + O(R^{-1}) + O(A_2) \tag{23}$$

where

$$A_2 \stackrel{\text{def}}{=} R^{4-4\alpha} \int_{B(\xi,1/2)} \frac{e^{-aR^2|\overline{\eta}|^2}}{|\xi-\eta|^2} d\eta.$$

with $|\xi| = 1$. Similarly, if $|x| = R > 2R_1$, by decomposing $\mathbb{R}^4$ as above,

$$\begin{aligned} |x|\frac{\partial v}{\partial r}(x) &= -\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \frac{x\cdot(x-y)}{|x-y|^2} Ke^{4\overline{w}} dy + t\alpha = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \frac{-y\cdot(x-y)}{|x-y|^2} Ke^{4\overline{w}} dy \\ &= O(\varepsilon) + O(R^{-1}) + O(A_1), \end{aligned} \tag{24}$$

$$\begin{aligned} |x|\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)(x) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{\left[|x|^2\cdot y - (x\cdot y)x\right]\cdot(x-y)}{|x-y|^4} Ke^{4\overline{w}} dy \\ &\quad + \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \frac{x\cdot y}{|x-y|^2} Ke^{4\overline{w}} dy \\ &= O(\varepsilon) + O(R^{-1}) + O(A_2) + O(A_3), \end{aligned} \tag{25}$$

and

$$\begin{aligned} |x|^3\frac{\partial(\Delta v)}{\partial r}(x) &= \frac{1}{2\pi^2}\int_{\mathbb{R}^4} \frac{|x|^2(y-x)\cdot x}{|x-y|^4} Ke^{4\overline{w}} dy + 4t\alpha \\ &= \frac{1}{2\pi^2}\int_{\mathbb{R}^4} \left[\frac{|y|^2 - 2x\cdot y}{|x-y|^2} - \frac{|x|^2(x-y)\cdot y}{|x-y|^4}\right] Ke^{4\overline{w}} dy^{\cdot} \\ &= O(\varepsilon) + O(R^{-1}) + O(A_2) + O(A_3) \end{aligned} \tag{26}$$

Here

$$A_j(x) \stackrel{\text{def}}{=} R^{4-4\alpha} \int_{B(\xi,1/2)} \frac{e^{-aR^2|\overline{\eta}|^2}}{|\xi-\eta|^j} d\eta, \quad \forall\, 1 \le j \le 3$$

with $|\xi| = 1$. Of course, $A_1 \le 2A_2$ and $A_2 \le 2A_3$, so it suffices to estimate A_3.

If $k = 4$, it is easy to see that $e^{-aR^2|\overline{\eta}|^2} \le e^{-aR^2/4}$ in $B(\xi, 1/2)$ since $|\overline{\eta}| = |\eta| > 1/2$, so $\|A_3\|_{L^\infty(B_R)} = o(1)$ as R tends to infinity. Finally, (20) follows easily from

$$\lim_{|x|\to\infty} |x|^2\Delta v = \lim_{|x|\to\infty} |x|\frac{\partial v}{\partial r} = \lim_{|x|\to\infty} |x|^3\frac{\partial(\Delta v)}{\partial r} = \lim_{|x|\to\infty} |x|\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = 0. \tag{27}$$

3.1 Case $k = 1$

Consider now $k = 1$. By rearrangement argument, the integral of fg is less than that of their Schwarz-symmetrizations f^*g^*. Applying that to each hyperplane $(\eta_i)_{2\le i\le 4}$ = constant, we obtain

$$\int_{B(\xi,1/2)} \frac{e^{-aR^2\eta_1^2}}{|\xi-\eta|^3} d\eta = \int_{B(0,1/2)} \frac{e^{-aR^2(\xi_1+\eta_1)^2}}{|\eta|^3} d\eta \le \int_{B(0,1/2)} \frac{e^{-aR^2\eta_1^2}}{|\eta|^3} d\eta.$$

Using the sphere coordinates, $\eta_1 = r\cos\theta$, $\eta_2 = r\sin\theta\cos\varphi$ etc, we get

$$\begin{aligned} R^{4\alpha-4}A_3 &\le C\int_0^{1/2}\int_0^{\pi/2} e^{-aR^2r^2\cos^2\theta}\sin^2\theta\, dr\, d\theta = C\int_0^{1/2}\int_0^1 e^{-aR^2s^2t^2}\sqrt{1-t^2}\, ds\, dt \\ &\le C\,(I_1 + I_2) \end{aligned}$$

where

$$I_1 = \int_0^{1/2}\int_{1/2}^{1} e^{-aR^2s^2t^2}\sqrt{1-t^2}dsdt \le \frac{1}{2}\int_0^{1/2} e^{-aR^2s^2/4}ds \le \frac{1}{R}\int_0^{\infty} e^{-as^2}ds = \frac{C}{R}$$

and

$$I_2 = \int_0^{1/2}\int_0^{1/2} e^{-aR^2s^2t^2}\sqrt{1-t^2}dsdt \le \frac{1}{R}\int_0^{\sqrt{R}/2}\int_0^{\sqrt{R}/2} e^{-as^2t^2}dsdt \le \frac{C\log R}{R},$$

thanks to the following lemma.

Lemma 3.2 *Let*

$$\ell(M) = \int_0^M\int_0^M e^{-as^2t^2}dsdt,$$

then $\ell(M) = O(\log M)$ *as* M *tends to* ∞.

Therefore $A_3 = O\left(R^{3-4\alpha}\log R\right)$, then $\lim_{R\to\infty}\|A_3\|_{L^\infty(B_R)} = 0$ as $\alpha > 3/4$. Since ε is arbitrary, we obtain easily (27). We finish by the proof of lemma 3.2. Indeed (for $M \ge 1$),

$$\begin{aligned}\ell(M) = 2\int_0^M\int_0^s e^{-as^2t^2}dsdt = 2\int_0^M\int_0^1 e^{-as^4t^2}sdsdt &= \int_0^{M^2}\int_0^1 e^{-as^2t^2}dsdt \\ &\le C + \int_1^{M^2}\left(\int_0^\infty e^{-as^2t^2}dt\right)ds \\ &= C + C'\int_1^{M^2}\frac{ds}{s}\end{aligned}$$

which yields $\ell(M) \le C\log M$ for all $M \ge 2$.

3.2 Case $k = 2$

In this case, we take the change of variables $\eta_1 + i\eta_2 = r\cos\theta e^{i\varphi}$ and $\eta_3 + i\eta_4 = r\sin\theta e^{i\psi}$.

$$\begin{aligned}R^{4\alpha-4}A_2 \le \int_{B(0,1/2)}\frac{e^{-aR^2(\eta_1^2+\eta_2^2)}}{|\eta|^2}d\eta &\le C\int_0^{1/2}\int_0^{\pi/2} e^{-aR^2r^2\cos^2\theta}r\sin\theta\cos\theta drd\theta \\ &= \frac{C}{R^2}\int_0^{1/2}\frac{1}{r}\left(1-e^{-aR^2r^2}\right)dr \\ &= \frac{C}{R^2}\int_0^{R/2}\frac{1}{s}\left(1-e^{-as^2}\right)ds \\ &\le \frac{C\log R}{R^2}.\end{aligned}$$

Therefore, $A_2 = O(R^{2-4\alpha}\log R)$ and tends uniformly to zero as $\alpha > 1/2$. Consequently,

$$\lim_{|x|\to\infty}|x|^2\Delta v = \lim_{|x|\to\infty}|x|\frac{\partial v}{\partial r} = 0.$$

For $A_3(x)$, we cannot prove a uniform estimate tending to zero at ∞ as in previous case. However, we show that $\lim_{R\to\infty}\|A_3\|_{L^1(\partial B_R)} = o(R^3)$ by suitable pointwise estimate. In fact, denote $\overline{y} = (y_1, y_2)$ and $y' = (y_3, y_4)$ for any $y \in \mathbb{R}^4$, we have

$$\begin{aligned}R^{4\alpha-4}A_3(x) &\le C\int_{\mathbb{R}^2} e^{-aR^2|\overline{\eta}|^2}\left(\int_{B_{\mathbb{R}^2}(0,R)}\frac{d\eta'}{|\overline{\xi}-\overline{\eta}|^3+|\eta'|^3}\right)d\overline{\eta} \\ &\le C\int_{\mathbb{R}^2} e^{-aR^2|\overline{\eta}|^2}\left(\int_0^R\frac{rdr}{|\overline{\xi}-\overline{\eta}|^3+r^3}\right)d\overline{\eta},\end{aligned}$$

which implies

$$R^{4\alpha-4}A_3(x) \le C\int_{\mathbb{R}^2}\frac{e^{-aR^2|\overline{\eta}|^2}}{|\xi-\overline{\eta}|}d\overline{\eta} = \frac{C}{R}\int_{\mathbb{R}^2}\frac{e^{-a|\overline{y}|^2}}{|\overline{x}-\overline{y}|}d\overline{y}$$
$$\le \frac{C|\overline{x}|}{R}e^{-a|\overline{x}|^2/4} + \frac{C}{R|\overline{x}|}\int_{\mathbb{R}^2\setminus B(\overline{x},|\overline{x}|/2)} e^{-a|\overline{y}|^2}d\overline{y}.$$

Hence

$$R^{4\alpha-4}A_3(x) \le \frac{C}{R(1+|\overline{x}|)}.$$

Since $\alpha > 1/2$,

$$\int_{\partial B_R} A_3(x)d\sigma \le CR^{3-4\alpha}\int_0^{\pi/2}\frac{R^3\sin\theta\cos\theta}{1+R\cos\theta}d\theta \le CR^{5-4\alpha} = o(R^3),$$

Now we can claim (20). For example,

$$J_5 = \int_{\partial B_R} o\left(R^{-2}\right)\times\left[o\left(R^{-1}\right)+O\left(R^{-1}A_3\right)\right]d\sigma = o(1)+O\left(R^{-3}\int_{\partial B_R}A_3(x)d\sigma\right) = o(1).$$

3.3 Case $k = 3$

Here, we prove the following pointwise estimates for A_1, A_2 and A_3.

Lemma 3.3 *If $k = 3$, for any $R > 2$ and $x \in \partial B_R$, we have*

(28) $$A_2(x) \le CR^{1-4\alpha}\log R, \quad A_2(x) \le \frac{CR^{2-4\alpha}}{1+|\overline{x}|} \quad \text{and} \quad A_3(x) \le \frac{CR^{3-4\alpha}}{1+|\overline{x}|^2},$$

where $\overline{x} = (x_1, x_2, x_3)$.

With these estimates, we obtain again (20). As the proof is very similar to that for $A_3(x)$ in the case $k = 2$, we just show how to handle J_3 using (28) and leave other details for interested readers.

As $\Delta v(x) = o(R^{-2}) + O(R^{-2}A_2)$ for $|x| = R$,

$$2|J_3| = \int_{\partial B_R}|x|(\Delta v)^2 d\sigma \le CR\int_{\partial B_R}o(R^{-4})d\sigma + CR^{-3}\int_{\partial B_R}\left(\frac{R^{2-4\alpha}}{1+|\overline{x}|}\right)^2 d\sigma$$
$$\le o(1) + CR^{1-8\alpha}\int_{\partial B_R}\frac{d\sigma}{1+|\overline{x}|^2}.$$

Taking the sphere coordinates $x_4 = r\cos\theta$, $x_3 = r\sin\theta\cos\varphi$ etc,

$$\int_{\partial B_R}\frac{d\sigma}{1+|\overline{x}|^2} = C\int_0^\pi\frac{R^3\sin^2\theta}{1+R^2\sin^2\theta}d\theta \le CR,$$

so we have $|J_3| \le o(1) + CR^{2-8\alpha}$, which yields $\lim_{R\to\infty} J_3 = 0$ when $\alpha > 1/4$. The proof is completed. □

Acknowledgments *The research of the first author is supported by an Earmarked Grant from RGC of Hong Kong. This work was initiated when the second author was visiting the Chinese University of Hong Kong, he would like to thank the department of Mathematics for its kind hospitality.*

References

[1] S-Y A. Chang and W. Chen, A note on a class of higher order conformally covariant equations, *Discrete Contin. Dynam. Systems* **7(2)** (2001), 275–281.

[2] S-Y A. Chang and P.C. Yang, Extremal metrics of zeta function determinants on 4-manifolds, *Ann. Math.* **142** (1995), 171-212.

[3] S-Y A. Chang and P. C. Yang, On uniqueness of solutions of n-th order differential equations in conformal geometry, *Math. Research Letters* **4** (1997), 91-102.

[4] S-Y A. Chang and J. Qing, The zeta functional determinants on manifolds with boundary. I. The formula. *J. Funct. Anal.* **147(2)** (1997), 327–362.

[5] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.* **63** (1991), 615-622.

[6] C.R. Graham, R. Jenne, L. Mason and G. Sparling, Conformally invariant powers of the Laplacian, I: Existence, *J. London Math. Soc.* **46(2)** (1992), 557-565.

[7] C.S. Lin, A classification of solutions of a conformally invariant fourth order equation in R^n, *Comment. Math. Helv.* **73** (1998), 206-231.

[8] R.C. McOwen, Conformal metrics in R^2 with prescribed Gaussian curvature and positive total curvature, *Indiana Univ. Math. J.* **34(1)** (1985), 97–104.

[9] R.C. McOwen, The behavior of the Laplacian on weighted Sobolev spaces, *Comm. Pure Appl. Math.* **32(6)** (1979), 783–795.

[10] S. Paneitz, A quadratic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds, preprint (1983).

[11] P. Pucci and J. Serrin, Critical exponents and critical dimensions for polyharmonic operators, *J. Math. Pures Appl.* **69** (1990), 55-83.

[12] J. Wei and X. Xu, Classification of solutions of higher order conformally invariant equations, *Math. Ann.* **313** (1999), 207-228.