Bubbling Solutions for an Anisotropic Emden-Fowler Equation

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Abstract

We consider the following anisotropic Emden-Fowler equation

$$\nabla(a(x)\nabla u) + \varepsilon^2 a(x)e^u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a smooth and bounded domain and $a(x)$ is a positive and smooth function. We investigate the effect of anisotropic coefficient $a(x)$ on the existence of bubbling solutions. We show that at given local maximum points of $a(x)$, there exists arbitrarily many bubbles. As a consequence, the quantity

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega a(x)e^u dx$$

can approach to $+\infty$ as $\varepsilon \to 0$. These results show a striking difference with the isotropic case ($a(x) \equiv Constant$).

1 Introduction

We consider the following generalized Emden-Fowler equation

$$\nabla(a(x)\nabla u) + \varepsilon^2 a(x)e^u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, $\varepsilon > 0$ and $a(x)$ is a smooth function over Ω satisfying

$$0 < a_1 \leq a(x) \leq a_2 < +\infty. \tag{H}$$

Equation (1) was motivated by the study of the following Emden-Fowler equation, or Gelfand's equation

$$\Delta u + \varepsilon^2 e^u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain. When $N = 2$, (2) relates to the geometric problem of Riemann surfaces with constant Gaussian curvature. When $N \geq 3$, it also arises in the theory of thermionic emission, isothermal gas sphere, gas combustion and many other physical applications. (see [1], [14], [18], [22] and the references therein.) It is well-known that a positive solution to (2) exists only when $\varepsilon < \lambda_1(\Omega)$, the first Dirichlet eigenvalue. Therefore it is reasonable to assume that ε is small.

The asymptotic behavior of solutions to (2), or to a more general equation

$$\Delta u + \varepsilon^2 k(x) e^u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{3}$$

where $k(x)$ is a positive smooth function has been studied in [5], [20], [19], [21], [24] and [27]. In particular, we mention the results of [21] (see e.g. [27]): let

$$\mathcal{T}_\varepsilon = \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon}\, dx$$

Suppose $\max_{x\in\Omega} u_\varepsilon \to +\infty$. Then along a subsequence of solutions u_ε, $\mathcal{T}_\varepsilon$ tends to l as $\varepsilon \to 0$, there holds either $l = \infty$, $u_\varepsilon \to +\infty$ for all $x \in \Omega$; or $l = 8\pi m$ and u_ε makes m points (simple) blow-up on $\mathcal{S} = \{x_1, ..., x_m\} \subset \Omega$ such that $\varepsilon^2 k(x) e^{u_\varepsilon} \to 8\pi \sum_j \delta_{x_j}$ and $(x_1, ..., x_m)$ is a critical point of

$$\Psi(x) = \sum_{j=1}^m H_D(x_j, x_j) + \sum_{i\neq j} G_D(x_i, x_j) + 2\sum_{j=1}^m \log k(x_j),$$

where G_D denotes the standard Green's function of $-\Delta$ with Dirichlet boundary condition and H denotes the regular part of G_D, i.e.

$$H_D(x, y) = G_D(x, y) + \frac{1}{2\pi} \log |x - y|. \tag{4}$$

Conversely, many authors have tried to construct blow-up solutions. See [2, 12, 13, 3]. In particular in [12], the following result was established: If the domain is **not simply connected**, then at least one solution with m bubbles exists.

Our motivation in (1) are two-folds. First, equation (1) is a natural generalization of (3). One may expect similar results hold. (In fact, this is **not** true, as we will show below.) Secondly, equation (1) is a special case of equation (2) in higher-dimension ($N \geq 3$). In fact, when we work with the cross-section of a 3-dimensional torus having axial symmetry, we can find that problem (2) is reduced to (1): let the torus be $T = \{(x, y, z) : (\sqrt{x^2 + y^2} - 1)^2 + z^2 \leq R^2\}$ with $R < 1$. If we look for solutions in the form $u(x, y, x) = u(r, z)$ with $r = \sqrt{x^2 + y^2}$ for equation (2), a direct calculus shows that the problem is transformed to

$$\nabla(r\nabla u) + \varepsilon^2 r e^u = 0 \text{ in } \Omega = \{(r, z) : (r - 1)^2 + z^2 < R^2\}, \quad u = 0 \text{ on } \partial\Omega$$

which is just the equation (1) with $a(r, z) = r$.

In [6], Chanillo and Li studied (1) (and more general uniformly elliptic type problems) and generalized the Brezis-Merle results to (1). In [28], Ye and Zhou studied the asymptotic behavior of bubbling solutions to (1). They proved that if $\mathcal{T}_\varepsilon = O(1)$, then either $u_\varepsilon \to 0$ uniformly on any compact subset of Ω, or there exists a finite set $\mathcal{S} = \{x_1, ..., x_m\} \subset \Omega$ such that $u_\varepsilon \to u^*$ weakly in $W^{1,p}$ for any $p \in (1, 2)$, where u^* satisfies

$$\nabla(a\nabla u^*) + 8\pi \sum_i m_i a(x_i) \delta_{x_i} = 0 \text{ in } \Omega, \ u^* = 0 \text{ on } \partial\Omega.$$

Moreover, they proved that in the last case each x_i must be a critical point of the function a. In the case of $\Omega = B_1(0)$, $a = a(|x|)$, they also constructed a single blowing up radially symmetric solution.

Several important questions have left open:

Q1. Can $m_i > 1$? Are all bubbles **simple**?

Q2. How do we construct bubbling solution for the non-radial and general $a(x)$?

In this paper, we answer these two questions affirmatively. To state our results, we need to introduce some notations. It is well known that the solutions to the following Liouville-type equation

$$\Delta u + e^u = \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u dx < +\infty$$

can be all written in the following form

$$U_{\mu,\xi} = \log \frac{8\mu^2}{(\mu^2 + |x-\xi|^2)^2}, \quad \mu > 0, \ \xi \in \mathbb{R}^2.$$

Let

$$\Delta_a u = \frac{1}{a(x)} \nabla(a(x)\nabla u) = \Delta u + \nabla \log a \nabla u$$

and $G(x,y)$ be the Green's function satisfying

$$\Delta_a G(x,y) + 8\pi\delta_y = 0 \ \text{ in } \Omega, \quad G(x,y) = 0 \ \text{ on } \partial\Omega. \tag{5}$$

We decompose $G(x,y)$ as

$$G(x,y) = -4\log|x-y| + H(x,y). \tag{6}$$

Our first result is the following

Theorem 1.1 *Let $\bar{x} \in \Omega$ be a strict local maximum point of $a(x)$, i.e. there exists an neighborhood $B_\delta(\bar{x})$ such that*

$$a(x) < a(\bar{x}), \quad \forall\, x \in B_\delta(\bar{x}) \backslash \{\bar{x}\}.$$

Then for any $m \in \mathbb{N}^*$, problem (1) has a family of solutions u_ε such that as $\varepsilon \to 0^+$,

$$\varepsilon^2 \int_\Omega a(x) e^{u_\varepsilon} \to 8\pi m a(\bar{x}), \quad u_\varepsilon \to u^* \quad \text{in } C^2_{loc}(\overline{\Omega}\backslash\{\bar{x}\})$$

where $u^* = mG(x,\bar{x})$ or equivalently,

$$-\nabla(a(x)\nabla u^*) = 8\pi m a(\bar{x})\delta_{\bar{x}} \quad \text{in } \Omega, \quad u^* = 0 \quad \text{on } \partial\Omega.$$

More precisely, we have

$$u_\varepsilon(x) = \sum_{j=1}^m \left[\log \frac{1}{(\varepsilon^2\mu_j^2 + |x - \xi_j^\varepsilon|^2)^2} + H(\xi_j^\varepsilon, x) \right] + o(1) \tag{7}$$

where μ_j satisfies

$$\frac{1}{C} \le \mu_j \le |\log\varepsilon|^C$$

and $(\xi_1^\varepsilon, ..., \xi_m^\varepsilon)$ satisfies

$$\xi_j^\varepsilon \to \bar{x} \text{ for all } j \quad \text{and} \quad |\xi_i^\varepsilon - \xi_j^\varepsilon| > |\log\varepsilon|^{-\frac{m^2+1}{2}}, \ \forall\, i \ne j. \tag{8}$$

Theorem 1.1 answers **Q1**.

Remark 1.2 *It is quite surprising that accumulation of bubbles can occur for anisotropic Emden-Fowler equation. The only known result for such phenomena is due to [8]. In [28], it is shown that if $\bar{x}$ is a nondegenerate local minimum point of $a(x)$, then $m = 1$. Here we show that if $\bar{x}$ is a (strict) local maximum point, then we can allow arbitrary $m > 1$. We believe that if $\bar{x}$ is a saddle point, similar result should hold.*

Remark 1.3 *A consequence of Theorem 1.1 is that if $a(x)$ has a local maximum in Ω, then there exists solutions u_ε such that $\mathcal{T}_\varepsilon \to +\infty$. This is* **new** *and* **unexpected**. *When $a \equiv C$ and the domain is simply-connected, Mizoguchi-Suzuki [23] have shown that $\mathcal{T}_\varepsilon$ is uniformly bounded. In [28], it is shown that if the domain is simply connected, and if there exists a point $x_0 \in \Omega$ such that $\nabla a \cdot (x - x_0) \geq 0$ in Ω and $(x - x_0) \cdot \nu > 0$ on $\partial\Omega$, then $\mathcal{T}_\varepsilon$ is uniformly bounded.*

Remark 1.4 *In Theorem 1.1, if we have the following expansion of a at $\bar{x}$:*

$$a(x) \geq a(\bar{x}) - c|x - \bar{x}|^\alpha + o(|x - \bar{x}|^\alpha)$$

in a neighborhood of $\bar{x}$, then the distance between bubbles satisfies

$$|\xi_i^\varepsilon - \xi_j^\varepsilon| \geq |\log \varepsilon|^{-\frac{m^2+1}{\alpha}}, \quad \forall\, i \neq j. \tag{9}$$

This implies that the flatter the anisotropic coefficient is, the larger are the distances between the bubbles. A sketch of proof is given at the end of section 7.

To answer **Q2**, our proof gives also

Theorem 1.5 *Let $\bar{x} \in \Omega$ be a* topologically nontrivial critical point *of $a(x)$. Then for $\varepsilon > 0$ sufficiently small, problem (1) has a solution u_ε such that*

$$\varepsilon^2 \int_\Omega a(x) e^{u_\varepsilon} \to 8\pi a(\bar{x}), \quad u_\varepsilon \to G(x, \bar{x}) \ \ in\ C^2_{loc}(\bar{\Omega} \backslash \{\bar{x}\}).$$

For the notion of "topologically nontrivial critical point", we refer to [10]. Theorem 1.5 answers Q2.

Remark 1.6 *Theorems 1.1 and 1.5 can be extended trivially to the equation $\nabla(a(x)\nabla u) + \varepsilon^2 k(x) e^u = 0$ with any positive function $k \in C^2(\overline{\Omega})$. More generally, these results hold also for the following uniformly elliptic problem*

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \varepsilon^2 k(x) e^u = 0 \ \ in\ \Omega, \quad u = 0 \ \ on\ \partial\Omega,$$

where a_{ij}, $k \in C^2(\overline{\Omega})$, $k > 0$ and the symmetric matrix (a_{ij}) is positive definite uniformly. In this case, the role of $a(x)$ is replaced by $\lambda(x) = \sqrt{det(a_{ij}(x))}$.

Theorems 1.1 and 1.5 are proved via the so-called "localized energy method"- a combination of Liapunov-Schmidt reduction method and variational techniques. Namely, we first use Liapunov-Schmidt reduction method to reduce the problem to a finite dimensional one, with some *reduced energy*. Then, the solutions in Theorems 1.1 and 1.5 turn out to be generated by critical points of the reduced energy functionals. Such an idea has been used in many other papers. See for instance [4], [11], [9], [15], [16], [17], [13], [25], and references therein. Here we follow those of [12]. However, a new functional setting has to be introduced, since the distance between the bubbles is **small** (see Theorem 1.1), and an appropriate variational argument to be developed to make the approach successful.

Throughout the paper, the symbol C denotes always a positive constant independent of ε, it could be changed from one line to another.

In the rest of the sections, we only prove Theorem 1.1. The proof of the Theorem 1.5 is simpler and follows from the proof of Theorem 2 in [12]. We omit the details.

2 Ansatz for the solution

The purpose of this section is to provide an ansatz for solutions of problem (1) and give some basic estimates for the "error term" in the corresponding scaling problem.

The function $H(x,y)$ defined by (6) plays an essential role in our construction, so we need to well understand its behavior, which is characterized by

Lemma 2.1 *Let $H_y(x) = H(y,x)$, $\forall\ y \in \Omega$. Then $y \mapsto H_y \in C(\Omega, C^\gamma(\overline{\Omega}))$ for any $\gamma \in (0,1)$. Let H_D be the regular part of Green's function defined by (4), then we have*

$$H(x,y) = 8\pi H_D(x,y) + \nabla \log a(y) \cdot \nabla(|x-y|^2 \log|x-y|) + H_1(x,y) \tag{10}$$

where $x \mapsto H_1(x,y) \in C^{1,\gamma}(\overline{\Omega})$ for all $\gamma \in (0,1)$. Furthermore, the function $(x,y) \mapsto H_1(x,y) \in C^1(\Omega \times \Omega)$, in particular $x \mapsto H(x,x) \in C^1(\Omega)$.

Proof. For any $y \in \Omega$, it is easy to see that H_y satisfies

$$\Delta_a H_y - 4\nabla \log a(x) \cdot \frac{(x-y)}{|x-y|^2} = 0 \ \text{ in } \Omega, \quad H_y(x) = 4\log|x-y| \ \text{ on } \partial\Omega.$$

Since $|x-y|^{-1} \in L^p(\Omega)$ for any $p < 2$, by elliptic regularity, $H(x,y) \in W^{2,p}(\Omega)$. Then by Sobolev embedding, $H_y \in C^\gamma(\overline{\Omega})$ for any $0 < \gamma < 1$. Notice the continuity of $x \mapsto |x-y|^{-1}$ in $L^p(\Omega)$ with respect to y and the regularity of the boundary condition, it is easy to get $y \mapsto H_y$ belongs to $C(\Omega, C^\gamma(\overline{\Omega}))$. To prove (10), we observe that the function $x \mapsto S(x,y) = |x-y|^2 \log|x-y|$ satisfies

$$\Delta(|x-y|^2 \log|x-y|) = 4\log|x-y| + 4$$

and hence

$$\Delta\left[\nabla \log a(y) \cdot \nabla(|x-y|^2 \log|x-y|)\right] = 4\nabla \log a(y) \cdot \frac{x-y}{|x-y|^2}$$

Then for all $y \in \Omega$, the equation for $x \mapsto H_1(x,y)$ becomes

$$\begin{aligned}
&\Delta_{a(x)} H_1(x,y) \\
&= 4\big[\nabla \log a(x) - \nabla \log a(y)\big] \cdot \frac{x-y}{|x-y|^2} - 8\pi \nabla \log a(x) \nabla_x H_D(x,y) \\
&\quad - \nabla_x^2(|x-y|^2 \log|x-y|) \cdot (\nabla \log a(x), \nabla \log a(y))\,.
\end{aligned}$$

in Ω and $H_1(x,y) = -\nabla \log a(y) \cdot \nabla_x(|x-y|^2 \log|x-y|)$ if $x \in \partial\Omega$. We can verify that the function

$$A(x,y) = 4\big[\nabla \log a(x) - \nabla \log a(y)\big] \cdot \frac{x-y}{|x-y|^2} \in L^\infty(\Omega \times \Omega).$$

On the other hand, $x \mapsto \nabla_x^2(|x-y|^2 \log|x-y|) \in L^p(\Omega)$ for any $p > 1$. Applying elliptic theory, we obtain that $x \mapsto H_1(x,y)$ is in $C^{1,\gamma}(\overline{\Omega})$, $\forall\ \gamma \in (0,1)$. Furthermore, by the continuity of the second member and the boundary condition with respect to y in $L^p(\Omega)$ and $C^2(\partial\Omega)$ respectively, we can get $\nabla_x H_1 \in C(\Omega \times \Omega)$.

Similarly, if we check carefully the equation for $\nabla_y H_1$, using the regularity of $H_D(x,y)$, we can prove that $\nabla_y H_1 \in C(\Omega \times \Omega)$, then H_1 is a C^1 function over $\Omega \times \Omega$. Finally, $H(x,x) = 8\pi H_D(x,x) + H_1(x,x)$ is clearly in $C^1(\Omega)$. □

Remark 2.2 *By the same proof, we can get that $y \mapsto H_1(\cdot, y)$ belongs to $C(\Omega, C^{1,\gamma}(\overline{\Omega}))$ for all $\gamma \in (0,1)$. Using the symmetry of function H, we get also that $(x,y) \mapsto H(x,y) \in C^\gamma_{loc}(\Omega \times \Omega)$ for any $\gamma \in (0,1)$.*

Remark 2.3 *From the expansion* (10), *we observe easily that the function* $x \mapsto H(x,y)$ *is never in* $C^1(\Omega)$ *for any* $y \in \Omega$, *but it is clearly in* $C^2(\Omega \setminus \{y\})$ *by the equation and elliptic theory. We can also estimate easily* $|\nabla_x H|$ *in function of* $|x-y|$ *using* (10). *The argument that the function* $x \mapsto H(x,x) \in C^1(\Omega)$ *is crucial for us to get the* C^1 *dependance of solutions for the linearized problem, for the nonlinear problem and also for the variational reduction.*

Given now $\xi_j \in \Omega$, $\mu_j > 0$, we define

$$u_j(x) = \log \frac{8\mu_j^2}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2}.$$

The configuration space for $(\xi_1, ..., \xi_m)$ we choose is the following

$$\Lambda := \left\{ \xi = (\xi_1, ..., \xi_m) \in B_\delta(\bar{x}) \times ... \times B_\delta(\bar{x}) \,\Big|\, \min_{i \neq j} |\xi_i - \xi_j| > |\log \varepsilon|^{-M} \right\} \tag{11}$$

where M is given by

$$M = \frac{m^2+1}{2}. \tag{12}$$

Note that by the choice of ξ_j, we have

$$\left| \log \frac{1}{|\xi_i - \xi_j|} \right| \le C \log|\log\varepsilon|, \quad \forall\, i \neq j, \quad (\xi_1, ..., \xi_m) \in \Lambda. \tag{13}$$

The choice of μ_j will be made later on.

The ansatz is

$$U(x) = \sum_{j=1}^{m} \left[u_j(x) + H_j^\varepsilon(x) \right] \tag{14}$$

where H_j^ε is a correction term defined as the solution of

$$\begin{cases} \Delta_a H_j^\varepsilon + \nabla \log a(x) \nabla u_j = 0 & \text{in } \Omega \\ \qquad\qquad\qquad\quad H_j^\varepsilon = -u_j & \text{on } \partial\Omega. \end{cases} \tag{15}$$

We will need the following lemma which proof is given at the end of this section.

Lemma 2.4 *For any* $0 < \alpha < 1$,

$$H_j^\varepsilon(x) = H(x, \xi_j) - \log\left(8\mu_j^2\right) + O(\varepsilon^\alpha) \tag{16}$$

uniformly in $\overline{\Omega}$, *where* H *is the regular part of Green's function defined by* (6).

It will be convenient to work with the scaling of u given by

$$v(y) = u(\varepsilon y) + 4\log\varepsilon.$$

If u is a solution of (1) then v satisfies

$$\Delta_{a(\varepsilon y)} v + e^v = 0 \ \text{ in } \Omega_\varepsilon\,, \quad v = 4\log\varepsilon \ \text{ on } \partial\Omega_\varepsilon, \tag{17}$$

where $\Omega_\varepsilon = \Omega/\varepsilon$. With this scaling u_j becomes

$$v_j(y) = \log \frac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}$$

where $\xi_j' = \xi_j/\varepsilon$. Note that $u_j + H_j^\varepsilon$ with the scaling εy, satisfies

$$\Delta_{a(\varepsilon y)}(u_j + H_j^\varepsilon) + e^{v_j} = 0 \;\text{ in } \Omega_\varepsilon\ , \quad u_j + H_j^\varepsilon = 0 \;\text{ on } \partial\Omega_\varepsilon. \tag{18}$$

We will seek a solution v of (17) of the form

$$v = V + \phi$$

where

$$V(y) = U(\varepsilon y) + 4\log\varepsilon \tag{19}$$

and U is defined by (14). Problem (17) can be stated as to find ϕ a solution to

$$\Delta_{a(\varepsilon y)}\phi + e^V\phi + N(\phi) + R = 0 \;\text{ in } \Omega_\varepsilon, \quad \phi = 0 \;\text{ on } \partial\Omega_\varepsilon$$

where the "nonlinear term" is

$$N(\phi) = e^V(e^\phi - 1 - \phi) \tag{20}$$

and the "error term" is given by

$$R = \Delta_{a(\varepsilon y)}V + e^V. \tag{21}$$

At this point it is convenient to make a choice of the parameters μ_j, the objective being to make the error term small. We choose μ_j such that

$$\log\left(8\mu_j^2\right) = H(\xi_j, \xi_j) + \sum_{i\neq j} G(\xi_i, \xi_j). \tag{22}$$

Observe that μ_j is not $O(1)$ since $\xi_j \to \bar{x}$ implies that $\lim G(\xi_i, \xi_j) = \infty$. But using $0 < G(\xi_i, \xi_j) \le -4\log|\xi_i - \xi_j| + O(1) \le C\log|\log\varepsilon|$, we derive that

$$\frac{1}{C} \le \mu_j \le |\log\varepsilon|^C \tag{23}$$

for some fixed positive number C.

We claim the following behavior for R holds: for any $0 < \alpha < 1$, there exists C independent of ε such that

$$|R(y)| \le C\varepsilon^\alpha \sum_{j=1}^m \frac{1}{\mu_j^2\left(1 + \mu_j^{-3}|y - \xi_j'|^3\right)}, \quad \forall\, y \in \Omega_\varepsilon \tag{24}$$

and for $W = e^V$

$$W(y) = \begin{cases} \dfrac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}(1 + \theta_j(y)) & \text{if } |y - \xi_j'| \le \dfrac{1}{\varepsilon|\log\varepsilon|^{2M}} \\ O(\varepsilon^{4-\alpha}) & \text{if } |y - \xi_j'| \ge \dfrac{1}{\varepsilon|\log\varepsilon|^{2M}},\ \forall\, j \end{cases} \tag{25}$$

with θ_j satisfying the following estimate

$$|\theta_j(y)| \le C_\alpha\left(\varepsilon^\alpha|y - \xi_j'|^\alpha + \varepsilon^\alpha\right).$$

Proof of (25). We write W as

$$\begin{aligned}
W(y) &= \varepsilon^4 \exp\Big\{ \sum_{j=1}^{m} \big[u_j(\varepsilon y) + H_j^\varepsilon(\varepsilon y)\big] \Big\} \\
&= \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2} \\
&\quad \times \exp\Big\{ H_j^\varepsilon(\varepsilon y) + \sum_{i\neq j}^{m} \Big[\log \frac{8\mu_i^2}{(\varepsilon^2\mu_i^2 + \varepsilon^2|y-\xi_i'|^2)^2} + H_i^\varepsilon(\varepsilon y)\Big] \Big\}.
\end{aligned}$$

Using (16) and the fact that H is $C^\alpha(\Omega^2)$, we have

$$\begin{aligned}
H_i^\varepsilon(\varepsilon y) &= H(\varepsilon y, \xi_i) - \log(8\mu_i^2) + O(\varepsilon^\alpha) \\
&= H(\xi_j, \xi_i) - \log(8\mu_i^2) + O(\varepsilon^\alpha) + O(\varepsilon^\alpha |y-\xi_j'|^\alpha), \quad \forall\, y \in \Omega_\varepsilon.
\end{aligned}$$

Hence for $|y-\xi_j'| \le \frac{1}{\varepsilon|\log\varepsilon|^{2M}}$

$$\begin{aligned}
& H_j^\varepsilon(\varepsilon y) + \sum_{i\neq j}^{m} \Big[\log \frac{8\mu_i^2}{(\varepsilon^2\mu_i^2 + \varepsilon^2|y-\xi_i'|^2)^2} + H_i^\varepsilon(\varepsilon y)\Big] \\
&= H(\xi_j,\xi_j) - \log(8\mu_j^2) + \sum_{i\neq j}^{m} \Big[\log \frac{8\mu_i^2}{|\xi_j - \xi_i|^4} + H(\xi_j,\xi_i) - \log(8\mu_i^2)\Big] \\
&\quad + O(\varepsilon^\alpha) + O(\varepsilon^\alpha|y-\xi_j'|^\alpha) \\
&= H(\xi_j,\xi_j) - \log(8\mu_j^2) + \sum_{i\neq j}^{m} G(\xi_j,\xi_i) + O(\varepsilon^\alpha) + O(\varepsilon^\alpha|y-\xi_j'|^\alpha) \\
&= O(\varepsilon^\alpha) + O(\varepsilon^\alpha|y-\xi_j'|^\alpha).
\end{aligned}$$

The last equality is due to the choice of μ_j, c.f. (22). The first equality comes from the expansion of $H_i^\varepsilon(\varepsilon y)$, the choice of ξ_j, (23) and

$$\begin{aligned}
\varepsilon^2\mu_i^2 + \varepsilon^2|y-\xi_i'|^2 &= [|\xi_i - \xi_j| + O(|\varepsilon y - \xi_j|)]^2 + \varepsilon^2\mu_i^2 \\
&= |\xi_i-\xi_j|^2 \times \Big[1 + O\Big(\frac{|\varepsilon y - \xi_j|^2}{|\xi_i-\xi_j|^2}\Big) + \frac{\varepsilon^2\mu_i^2}{|\xi_i-\xi_j|^2}\Big] \\
&= |\xi_i-\xi_j|^2 \times \big[1 + O(\varepsilon^2|\log\varepsilon|^{2M}|y-\xi_j'|^2) + O(\varepsilon^{2-\alpha})\big] \\
&= |\xi_i-\xi_j|^2 \times \big[1 + O(\varepsilon^\alpha) + O(\varepsilon^\alpha|y-\xi_j'|^\alpha)\big]\,.
\end{aligned}$$

Therefore when $|y-\xi_j'| \le \frac{1}{\varepsilon|\log\varepsilon|^{2M}}$,

$$W(y) = \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2}\Big[1 + O(\varepsilon^\alpha) + O(\varepsilon^\alpha|y-\xi_j'|^\alpha)\Big]. \tag{26}$$

If $|y-\xi_j'| \ge \frac{1}{\varepsilon|\log\varepsilon|^{2M}}$ for all $j = 1,\ldots,m$, we have easily $W = O(\varepsilon^{4-\alpha})$, and this together with (26) implies (25). □

Proof of (24). We defined $R = \Delta_{a(\varepsilon r)}V + e^V$ with V given by (19). By our definition and (18),

$$R = \varepsilon^4 e^{\sum_{j=1}^m (u_j + H_j^\varepsilon)(\varepsilon y)} - \sum_{j=1}^{m} e^{v_j}.$$

It is clear that $R = O(\varepsilon^{4-\beta})$ for any $\beta \in (0,1)$ if $|y - \xi_j'| \geq \frac{1}{\varepsilon|\log\varepsilon|^{2M}}$ for all j. If $|y-\xi_j'| \leq \frac{1}{\varepsilon|\log\varepsilon|^{2M}}$ for some j, we have, according to (26)

$$\begin{aligned} R &= e^{v_j}\left[1 + O(\varepsilon^\gamma) + O(\varepsilon^\gamma|y-\xi_j'|^\gamma)\right] - e^{v_j} + O(\varepsilon^{4-\gamma}) \\ &= O\left(e^{v_j}\left[\varepsilon^\gamma + \varepsilon^\gamma|y-\xi_j'|^\gamma\right]\right) + O(\varepsilon^{4-\gamma}) \end{aligned}$$

where γ is any constant in $(0,1)$. By choosing suitable β and γ, we can prove (24) for any fixed $\alpha \in (0,1)$. □

Proof of lemma 2.4. The boundary condition satisfied by H_j^ε is

$$H_j^\varepsilon = -u_j = 4\log|x-\xi_j| - \log(8\mu_j^2) + O(\varepsilon^2\mu_j^2). \tag{27}$$

The regular part of Green's function $H(x,y)$ satisfies

$$\begin{cases} -\Delta_a H(x,y) = 4\nabla\log a(x)\nabla\log\dfrac{1}{|x-y|} & \text{in } \Omega \\ \qquad H(x,y) = 4\log|x-y| & \text{on } \partial\Omega. \end{cases}$$

For the difference $z_\varepsilon(x) = H_j^\varepsilon(x) + \log(8\mu_j^2) - H(x,\xi_j)$ we have

$$\begin{cases} -\Delta_a z_\varepsilon = \nabla\log a(x)\nabla\left[\log\dfrac{|x-\xi_j|^4}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2}\right] & \text{in } \Omega \\ \qquad z_\varepsilon = O(\varepsilon^2\mu_j^2) & \text{on } \partial\Omega. \end{cases}$$

For any $1 < p < 2$, let us estimate now

$$I_p = \left\|\nabla\left[\log\frac{1}{|x-\xi_j|^2} - \log\frac{1}{\varepsilon^2\mu_j^2 + |x-\xi_j|^2}\right]\right\|_{L^p(\Omega)}^p.$$

Note that

$$\left|\nabla\left[\log\frac{1}{|x-\xi_j|^2} - \log\frac{1}{\varepsilon^2\mu_j^2 + |x-\xi_j|^2}\right]\right| = \frac{2\varepsilon^2\mu_j^2}{|x-\xi_j|(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)},$$

applying polar coordinates with center ξ_j, i.e. $r = |x-\xi_j|$

$$\begin{aligned} I_p \leq 2^{p+1}\pi\int_0^\infty\left[\frac{\varepsilon^2\mu_j^2}{r(\varepsilon^2\mu_j^2 + r^2)}\right]^p r\,dr &= 2^{p+1}\pi(\varepsilon\mu_j)^{2-p}\int_0^\infty\frac{s}{(1+s^2)^p s^p}ds \\ &= C(\varepsilon\mu_j)^{2-p}. \end{aligned}$$

In conclusion, for any $1 < p < 2$ we have

$$\left\|\nabla a(x)\nabla\left[\log\frac{1}{|x-\xi_j|^2} - \log\frac{1}{\varepsilon^2\mu_j^2 + |x-\xi_j|^2}\right]\right\|_{L^p(\Omega)} \leq C(\varepsilon\mu_j)^{\frac{2-p}{p}}.$$

Applying L^p theory,

$$\|z_\varepsilon\|_{W^{2,p}(\Omega)} \leq C\Big(\|\Delta z_\varepsilon\|_{L^p(\Omega)} + \|z_\varepsilon\|_{C^2(\partial\Omega)}\Big) \leq C(\varepsilon\mu_j)^{\frac{2-p}{p}}.$$

By the Morrey embedding, we obtain

$$\|z_\varepsilon\|_{C^\gamma(\overline{\Omega})} \leq C(\varepsilon\mu_j)^{\frac{2-p}{p}}$$

for any $0 < \gamma < 2 - \frac{2}{p}$. This proves the result with $\alpha < \frac{2-p}{p}$. □

3 Solvability of a linear equation

The main result of this section is the solvability of the following linear problem: Given $h \in L^\infty(\Omega_\varepsilon)$, find ϕ, $c_{11}, \dots, c_{2m}$ such that

$$
\left\{
\begin{array}{ll}
-\Delta_{a(\varepsilon y)}\phi = W\phi + h + \dfrac{1}{a(\varepsilon y)} \displaystyle\sum_{j=1}^{m}\sum_{i=1}^{2} c_{ij}\chi_j Z_{ij} & \text{in } \Omega_\varepsilon \\
\phi = 0 & \text{on } \partial\Omega_\varepsilon \\
\displaystyle\int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0, \quad \forall\, 1 \le j \le m,\ i = 1,2 &
\end{array}
\right. \tag{28}
$$

where W is a function satisfying (25), and Z_{ij}, χ_j are defined as follows. Let z_{ij} be

$$
z_{0j} = \frac{1}{\mu_j} - 2\frac{\mu_j}{\mu_j^2 + |y|^2}, \quad z_{ij} = \frac{y_i}{\mu_j^2 + |y|^2}, \quad 1 \le i \le 2, \quad 1 \le j \le m.
$$

It is well-known that any solution to

$$
\Delta\phi + \frac{8}{(1+|z|^2)^2}\phi = 0, \quad |\phi| \le C
$$

is a linear combination of $Z_i, i = 0,1,2$ where $Z_0 = \frac{1-|z|^2}{1+|z|^2}$ and $Z_i = \frac{z_i}{1+|z|^2}$ for $i = 1,2$ (See lemma 2.1 of [7]).

Next we choose a large but fixed number R_0 and nonnegative smooth function $\chi : \mathbb{R} \to \mathbb{R}$ so that $\chi(r) = 1$ for $r \le R_0$ and $\chi(r) = 0$ for $r \ge R_0 + 1$, $0 \le \chi \le 1$. For $j = 1, \dots, m$, we define

$$
\chi_j(y) = \chi\left(\frac{|y - \xi_j'|}{\mu_j}\right), \quad Z_{ij}(y) = z_{ij}(y - \xi_j'), \quad i = 0,1,2,\ j = 1,\dots,m. \tag{29}
$$

It is important to note that

$$
\begin{aligned}
\Delta_{a(\varepsilon y)} Z_{0j} + e^{v_j} Z_{0j} &= \varepsilon \nabla \log a(\varepsilon y) \nabla Z_{0j} \\
&= O\left(\varepsilon\mu_j^{-2}\left[1 + \frac{|y - \xi_j'|}{\mu_j}\right]^{-3}\right).
\end{aligned} \tag{30}
$$

All functions above depend on ε but we omit this dependence in the notation.

Equation (28) will be solved for $h \in L^\infty(\Omega_\varepsilon)$, but we will be able to estimate the size of the solution in terms of the following norm

$$
\|h\|_* = \sup_{y \in \Omega_\varepsilon} \frac{|h(y)|}{\varepsilon^2 + \sum_{j=1}^m \mu_j^{-2}\left(1 + \frac{|y - \xi_j'|}{\mu_j}\right)^{-3}}. \tag{31}
$$

Proposition 3.1 *Let m be a positive integer. Then there exist $\varepsilon_0 > 0$, $C > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, any family of points $\xi = (\xi_1, \dots, \xi_m) \in \Lambda$ and any $h \in L^\infty(\Omega_\varepsilon)$, there is a unique solution $\phi \in L^\infty(\Omega_\varepsilon)$, $c_{ij} \in \mathbb{R}$ to (28). Moreover*

$$
\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C|\log\varepsilon|\|h\|_*.
$$

We begin by stating an *a priori* estimate for solutions of linearized problem satisfying orthogonality conditions with respect to all Z_{ij}, $0 \le i \le 2$, $1 \le j \le m$.

Lemma 3.2 *There are $R_0 > 0$ and $\varepsilon_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$ and any solution ϕ of*

$$
\left\{\begin{array}{ll} -\Delta_{a(\varepsilon y)}\phi = W\phi + h & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \displaystyle\int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0 & \forall\, j = 1, \dots, m;\ i = 0, 1, 2, \end{array}\right. \tag{32}
$$

we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C\|h\|_*$$

where C is independent of $\varepsilon \in (0, \varepsilon_0)$.

For the proof of this lemma we need to construct a suitable barrier.

Lemma 3.3 *There exist constants $R_1 > 0$, $C > 0$ such that for any $\varepsilon > 0$ small enough and $\xi \in \Lambda$, there exists*

$$\psi : \Omega_\varepsilon \setminus \cup_{j=1}^m B_{\mu_j R_1}(\xi_j') \to [1, \infty)$$

smooth and positive verifying

$$-\Delta_{a(\varepsilon y)}\psi - W\psi \ge \sum_{j=1}^m \frac{\mu_j}{|y - \xi_j'|^3} + \varepsilon^2 \quad \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{\mu_j R_1}(\xi_j').$$

Moreover ψ is bounded uniformly,

$$1 < \psi \le C \quad \text{in } \Omega_\varepsilon \setminus \cup_{j=1}^m B_{\mu_j R_1}(\xi_j').$$

Proof of lemma 3.2. Since $\xi \in \Lambda$, for ε small enough, we have $B_{\mu_j R_1}(\xi_j')$ disjointed and included in Ω_ε. We take $R_0 = 2R_1$, R_1 being the constant of lemma 3.3. Thanks to the barrier ψ of that lemma, we deduce that the following maximum principle holds in $\Omega_\varepsilon \setminus \cup_{j=1}^m B_{\mu_j R_1}(\xi_j')$: if $\phi \in H^1(\Omega_\varepsilon \setminus \cup_{j=1}^m B_{\mu_j R_1}(\xi_j'))$ satisfies

$$
\left\{\begin{array}{ll} -\Delta_{a(\varepsilon y)}\phi \ge W\phi & \text{in } \Omega_\varepsilon \setminus \bigcup_{j=1}^m B_{\mu_j R_1}(\xi_j') \\ \phi \ge 0 & \text{on } \bigcup_{j=1}^m \partial B_{\mu_j R_1}(\xi_j') \cup \partial\Omega_\varepsilon, \end{array}\right.
$$

then $\phi \ge 0$ in $\Omega_\varepsilon \setminus \cup_{j=1}^m B_{\mu_j R_1}(\xi_j')$.

Let h be bounded and ϕ a solution to (32). Following [12] we first claim that $\|\phi\|_{L^\infty(\Omega_\varepsilon)}$ can be controlled in terms of $\|h\|_*$ and the following inner norm of ϕ:

$$\|\phi\|_i = \sup_{\cup_{j=1}^m B_{\mu_j R_1}(\xi_j')} |\phi|.$$

Indeed, set

$$\widetilde{\phi} = (\|\phi\|_i + \|h\|_*)\,\psi.$$

By the above maximum principle, we have $|\phi| \le \widetilde{\phi}$ in $\Omega_\varepsilon \setminus \cup_{j=1}^m B_{\mu_j R_1}(\xi_j')$. Since ψ is uniformly bounded, we get

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C\,(\|\phi\|_i + \|h\|_*)\,, \tag{33}$$

for some constant C independent of ϕ and ε.

We prove the lemma by contradiction. Assume that there exist a sequence $\varepsilon_n \to 0$, points $\xi^n = (\xi_1^n, \dots \xi_m^n) \in \Lambda$ and functions ϕ_n, h_n with $\|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} = 1$ and $\|h_n\|_* \to 0$ so that for each n, ϕ_n solves (32). By (33) we see that $\|\phi_n\|_i$ stays away from zero. Up to a subsequence, for one of the indices, say j, we can assume that $\sup_{B_{\mu_j^n R_1}((\xi_j^n)')} |\phi_n| \geq c > 0$ for all n. Consider $\widehat{\phi}_n(z) = \phi_n\left(\mu_j^n z + (\xi_j^n)'\right)$, note that

$$\Delta_{a(\varepsilon y)}\phi_n + W_n\phi_n\big|_{\mu_j^n z+(\xi_j^n)'} = (\mu_j^n)^{-2}\left(\Delta_{\widetilde{a}_n}\widehat{\phi}_n + (\mu_j^n)^2\widetilde{W}_n\widehat{\phi}_n\right)$$

where

$$\widetilde{a}_n(y) = a\left(\varepsilon_n\mu_j^n y + \xi_j^n\right) \quad \text{and} \quad \widetilde{W}_n(y) = W_n\left(\mu_j^n y + (\xi_j^n)'\right).$$

Then by elliptic estimates, $\widehat{\phi}_n$ converges uniformly on compact sets to a nontrivial solution $\widehat{\phi}$ of

$$\Delta\phi + \frac{8}{(1+|z|^2)^2}\phi = 0, \quad |\phi| \leq 1 \quad \text{in } \mathbb{R}^2.$$

Thus $\widehat{\phi}$ is a linear combination of Z_i, $i = 0, 1, 2$. On the other hand, we can take the limit in the orthogonality relations (32), observing that limits of the functions Z_{ij} are just rotations and translations of Z_i, so $\widehat{\phi}$ is orthogonal to Z_i for $i = 0, 1, 2$. This contradicts the fact that $\widehat{\phi} \not\equiv 0$. □

Proof of lemma 3.3. Fix any $\alpha \in (0,1)$, define

$$\psi_{1j}(r_j) = \Psi_0(\varepsilon y) - \frac{\mu_j^\alpha}{r_j^\alpha} \quad \text{where} \quad r_j = |y - \xi_j'|,$$

and Ψ_0 satisfies $-\Delta_a\Psi_0 = 1$ in Ω, $\Psi_0 = 2$ on $\partial\Omega$. Clearly $\Psi_0 \geq 2$ in Ω and

$$-\Delta_{a(\varepsilon y)}\psi_{1j} = \varepsilon^2 + \alpha^2\frac{\mu_j^\alpha}{r_j^{2+\alpha}} - \frac{\varepsilon\alpha\mu_j^\alpha}{r_j^{\alpha+1}}\nabla\log a(\varepsilon y)\nabla r_j.$$

Consider the function $\Psi = \sum_j \psi_{1j}(r_j)$. First we take $R_1 \geq 1$ and $\mu_j R_1 \leq |y - \xi_j'| \leq \frac{1}{\varepsilon|\log\varepsilon|^{2M}}$, for sufficiently small ε, we have $\Psi \geq 1$ and

$$\begin{aligned} -\Delta_{a(\varepsilon y)}\psi_{1j} - W\psi &\geq \varepsilon^2 + \frac{\alpha\mu_j^\alpha}{r_j^{2+\alpha}} \times (\alpha - \varepsilon\|\nabla\log a\|_\infty r_j) - W \times m\|\Psi_0\|_\infty \\ &\geq \varepsilon^2 + \frac{\alpha^2\mu_j^\alpha}{2r_j^{2+\alpha}} - \frac{2mC\mu_j^2}{r_j^4} \\ &\geq \varepsilon^2 + \frac{\alpha^2\mu_j^\alpha}{4r_j^{2+\alpha}}. \end{aligned}$$

The last inequality holds when we choose R_1 great enough. Otherwise, for $i \neq j$, if $|y - \xi_j'| \leq \frac{1}{\varepsilon|\log\varepsilon|^{2M}}$, we have (for sufficiently small ε)

$$-\Delta_{a(\varepsilon y)}\psi_{1i} \geq \varepsilon^2 - \frac{\alpha\varepsilon\mu_i^\alpha\|\nabla\log a\|_\infty}{r_i^{\alpha+1}} \geq \varepsilon^2 - C\varepsilon^{2+\alpha}|\log\varepsilon|^{2(\alpha+1)M+\alpha C} \geq \frac{\mu_i^\alpha}{2r_i^{2+\alpha}}.$$

Then

$$-\Delta_{a(\varepsilon y)}\psi - W\psi \geq \varepsilon^2 + \sum_{1\leq i\leq m}\frac{\alpha^2\mu_i^\alpha}{4r_i^{2+\alpha}}, \quad \text{if} \quad \mu_j R_1 \leq |y - \xi_j'| \leq \frac{1}{\varepsilon|\log\varepsilon|^{2M}}.$$

If $|y - \xi_i'| \geq \frac{1}{\varepsilon|\log\varepsilon|^{2M}}$ for all i,

$$\begin{aligned} -\Delta_{a(\varepsilon y)}\psi - W\psi &\geq m\varepsilon^2 - C\varepsilon^{2+\alpha}|\log\varepsilon|^{(M+1)(\alpha+1)+\alpha C} - O(\varepsilon^{4-\alpha})\psi \\ &\geq \frac{\varepsilon^2}{2} + \sum_{1\leq i\leq m}\frac{\mu_i^\alpha}{2r_i^{2+\alpha}}. \end{aligned} \tag{34}$$

Since $\mu_j/r_j < R_1^{-1} < 1$, it is easy to see that $4\psi/\alpha^2$ satisfies all the properties of the lemma. $\square$

We will establish next an *a priori* estimate for solutions to problem (32) that satisfy orthogonality conditions with respect to Z_{ij}, $i = 1, 2$ only. More precisely,

Lemma 3.4 *For $\varepsilon > 0$ sufficiently small, if ϕ solves*

$$\begin{cases} -\Delta_{a(\varepsilon y)}\phi - W\phi = h & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{35}$$

and satisfies

$$\int_{\Omega_\varepsilon} Z_{ij}\chi_j\,\phi = 0 \qquad \forall\, j = 1, \dots, m,\ i = 1, 2 \tag{36}$$

then

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C|\log\varepsilon|\|h\|_* \tag{37}$$

where C is independent of ε.

Proof. Let ϕ satisfy (35) and (36). We will modify ϕ to satisfy all orthogonality relations in (32) and for this purpose we consider modifications with compact support of the functions Z_{0j}. Let $R > R_0 + 1$ be large and fixed. Let

$$a_{0j} = \frac{1}{\mu_j\left[-4\log(\varepsilon\mu_j R) + H(\xi_j, \xi_j)\right]}. \tag{38}$$

Note that by (23),

$$\frac{|\log\varepsilon|}{2} \le -\log(\varepsilon\mu_j R) \le 2|\log\varepsilon|. \tag{39}$$

Let η_1 be radial smooth cut-off function on $\mathbb{R}^2$ so that

$$0 \le \eta_1 \le 1, \quad |\nabla\eta_1| \le C \text{ in } \mathbb{R}^2; \quad \eta_1 \equiv 1 \text{ in } B_R(0), \quad \eta_1 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{R+1}(0)$$

and η_2 be radial smooth cut-off function on $\mathbb{R}^2$ so that

$$0 \le \eta_2 \le 1, \quad |\nabla\eta_2| \le C \text{ in } \mathbb{R}^2; \quad \eta_2 \equiv 1 \text{ in } B_1(0), \quad \eta_2 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_2(0).$$

Without loss of generality, we assume that $\delta < 1$ and $B(\bar{x}, 2+\delta) \subset \Omega$. Let

$$\eta_{1j}(y) = \eta_1\left(\frac{|y - \xi_j'|}{\mu_j}\right), \quad \eta_{2j}(y) = \eta_2\left(4\varepsilon|y - \xi_j'|\right) \tag{40}$$

and also

$$\widehat{Z}_{0j}(y) = Z_{0j}(y) - \frac{1}{\mu_j} + a_{0j}G(\varepsilon y, \xi_j). \tag{41}$$

Now define

$$\widetilde{Z}_{0j} = \eta_{1j}Z_{0j} + (1 - \eta_{1j})\eta_{2j}\widehat{Z}_{0j}.$$

Given ϕ satisfying (35) and (36), let

$$\tilde{\phi} = \phi + \sum_{j=1}^m d_j\widetilde{Z}_{0j} + \sum_{i=1}^2\sum_{j=1}^m e_{ij}Z_{ij}\chi_j. \tag{42}$$

We adjust $\widetilde{\phi}$ to satisfy the orthogonality condition:

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij} \widetilde{\phi} = 0, \quad \text{for all } 0 \le i \le 2,\ 1 \le j \le m.$$

Estimate (37) is a direct consequence of the following two claims:

Claim 1. Let $L = -\Delta_{a(\varepsilon y)} - W$, then

$$\|L(Z_{ij}\chi_j)\|_* \le \frac{C}{\mu_j}, \quad \forall\, i = 1, 2;\ j = 1, \dots, m. \tag{43}$$

and

$$\|L\widetilde{Z}_{0j}\|_* \le C \frac{\log|\log\varepsilon|}{\mu_j |\log\varepsilon|} \quad \forall\, j = 1, \dots, m. \tag{44}$$

Claim 2.

$$|d_j| \le C\mu_j |\log\varepsilon| \|h\|_*, \quad |e_{ij}| \le C\mu_j |\log\varepsilon| \|h\|_* \quad \forall\, i = 1, 2;\ j = 1, \dots, m. \tag{45}$$

In fact, we have that

$$L(\widetilde{\phi}) = h + \sum_{j=1}^{m} d_j L(\widetilde{Z}_{0j}) + \sum_{i=1}^{2}\sum_{j=1}^{m} e_{ij} L(Z_{ij}\chi_j) \quad \text{in } \Omega_\varepsilon. \tag{46}$$

Thus by lemma 3.2, we have

$$\begin{aligned} \|\widetilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} &\le C \sum_{j=1}^{m} |d_j| \|L(\widetilde{Z}_{0j})\|_* + C \sum_{i=1}^{2}\sum_{j=1}^{m} |e_{ij}| \|L(Z_{ij}\chi_j)\|_* + C\|h\|_* \\ &\le C|\log\varepsilon| \|h\|_*. \end{aligned} \tag{47}$$

Using (42), the estimate (37) then follows from (47), Claims 1 and 2.

First, we prove the existence of d_j and e_{ij}. Remark that $\widetilde{Z}_{0j}$ coincide with Z_{0j} in $B_{\mu_j R}(\xi'_j)$, they are still orthogonal to $Z_{ij}\chi_j$ for $i = 1, 2$, so

$$e_{ij} = -\int_{\Omega_\varepsilon} \sum_{l \ne j} d_l \widetilde{Z}_{0l} Z_{ij}\chi_j \Big/ \int_{\Omega_\varepsilon} Z_{ij}^2 \chi_j^2. \tag{48}$$

We need just to consider d_j. Multiplying (42) by $Z_{0k}\chi_k$, we get a system of d_j

$$\sum_j d_j \int_{\Omega_\varepsilon} \widetilde{Z}_{0j} Z_{0k}\chi_k = -\int_{\Omega_\varepsilon} \phi Z_{0k}\chi_k, \quad \text{for any } k = 1, \dots, m.$$

Note that

$$\int_{\Omega_\varepsilon} \widetilde{Z}_{0k} Z_{0k}\chi_k = \int_{\Omega_\varepsilon} Z_{0k}^2 \chi_k = C, \quad \forall\, k$$

and

$$\int_{\Omega_\varepsilon} \widetilde{Z}_{0j} Z_{0k}\chi_k = O\left(\frac{\mu_k \log|\log\varepsilon|}{\mu_j |\log\varepsilon|}\right), \quad \forall\, j \ne k$$

where we have used (13) for the last estimate.

We can present the system as $\mathcal{M}D = S$, where $D = (d_j)$, S is the second member and $\mathcal{M} = (m_{ij})$ is the coefficient matrix. By above estimates, $\mathcal{M}' = (m_{ij}\mu_j\mu_i^{-1})$ is diagonally dominated, so invertible. As $\mathcal{M} = P\mathcal{M}'P^{-1}$ where $P = \mathrm{Diag}(\mu_1, \dots, \mu_m)$, $\mathcal{M}$ is also invertible, hence D is well defined.

We now prove Claim 1. We begin with (44). Consider four regions: $\Omega_1 = \{|y-\xi_j^{'}| \le \mu_j R\}$, $\Omega_2 = \{\mu_j R < |y-\xi_j^{'}| \le \mu_j(R+1)\}$,

$$\Omega_3 = \left\{\mu_j(R+1) < |y-\xi_j^{'}| \le \frac{1}{4\varepsilon}\right\} \quad \text{and} \quad \Omega_4 = \left\{\frac{1}{4\varepsilon} < |y-\xi_j'| \le \frac{1}{2\varepsilon}\right\}.$$

On Ω_1, using (30), we have

$$L\widetilde{Z}_{0j} = LZ_{0j} = O\left(\varepsilon\mu_j^{-2}\left[1+\frac{|y-\xi_j^{'}|}{\mu_j}\right]^{-3}\right) + (e^{v_j}-W)Z_{0j}. \tag{49}$$

According to (26),

$$(e^{v_j}-W)Z_{0j} = O\left(\varepsilon^{\alpha}\mu_j^{-2}\left[1+\frac{|y-\xi_j^{'}|}{\mu_j}\right]^{-4}\right)\times O(\mu_j^{-1}),$$

hence for any fixed $\beta \in (0,1)$

$$|L\widetilde{Z}_{0j}(y)| = O\left(\varepsilon^{\beta}\mu_j^{-2}\left[1+\frac{|y-\xi_j^{'}|}{\mu_j}\right]^{-3}\right), \quad \forall\, y \in \Omega_1. \tag{50}$$

On Ω_2 where $\mu_j R < |y-\xi_j^{'}| \le \mu_j(R+1)$, we have

$$\begin{aligned} L\widetilde{Z}_{0j} &= \eta_{1j}LZ_{0j} + (1-\eta_{1j})L\widehat{Z}_{0j} + 2\nabla\eta_{1j}\nabla(\widehat{Z}_{0j}-Z_{0j}) + (\widehat{Z}_{0j}-Z_{0j})\Delta_{a(\varepsilon y)}\eta_{1j} \\ &= LZ_{0j} - (1-\eta_{1j})W(\widehat{Z}_{0j}-Z_{0j}) + 2\nabla\eta_{1j}\nabla(\widehat{Z}_{0j}-Z_{0j}) \\ &\quad + (\widehat{Z}_{0j}-Z_{0j})\Delta_{a(\varepsilon y)}\eta_{1j}. \end{aligned}$$

Note that on Ω_2,

$$\widehat{Z}_{0j} - Z_{0j} = -\frac{1}{\mu_j} + a_{0j}G(\varepsilon y, \xi_j) = a_{0j}\left[4\log\frac{\mu_j R}{|y-\xi_j'|} + H(\varepsilon y, \xi_j) - H(\xi_j, \xi_j)\right],$$

Using (10), we derive that for $r = |y-\xi_j^{'}|/\mu_j \in (R, R+1)$,

$$\widehat{Z}_{0j} - Z_{0j} = O\left(\frac{1}{\mu_j|\log\varepsilon|}\right), \quad \nabla(\widehat{Z}_{0j}-Z_{0j}) = O\left(\frac{1}{\mu_j^2|\log\varepsilon|}\right). \tag{51}$$

Moreover $|\nabla\eta_{1j}| = O(\mu_j^{-1})$, $\Delta_{a(\varepsilon y)}\eta_{1j} = O(\mu_j^{-2})$. From (51), we conclude that

$$\|L\widetilde{Z}_{0j}\|_{L^\infty(\Omega_2)} = O\left(\frac{1}{\mu_j^3|\log\varepsilon|}\right). \tag{52}$$

For Ω_3, we get

$$\begin{aligned} L\widetilde{Z}_{0j} &= -\Delta_{a(\varepsilon y)}Z_{0j} - W\widehat{Z}_{0j} \\ &= -\Delta_{a(\varepsilon y)}Z_{0j} - e^{v_j}Z_{0j} + (e^{v_j}-W)Z_{0j} + W\left[\frac{1}{\mu_j} - a_{0j}G(\varepsilon y, \xi_j)\right]. \end{aligned}$$

We have always

$$-\Delta_{a(\varepsilon y)} Z_{0j} - e^{v_j} Z_{0j} = O\left(\varepsilon \mu_j^{-2}\left[1+\frac{|y-\xi_j'|}{\mu_j}\right]^{-3}\right).$$

For the estimation of the last two terms, we decompose Ω_3 to some subregions: $\Omega_{3j} = \{\mu_j(R+1) < |y-\xi_j'| \le \frac{1}{\varepsilon|\log\varepsilon|^{2M}}\}$, $\Omega_{3k} = \{y \in \Omega_3, |y-\xi_k'| \le \frac{1}{\varepsilon|\log\varepsilon|^{2M}}\}$ if $k \neq j$, $\widetilde{\Omega}_3 = \{y \in \Omega_3, |y-\xi_l'| \ge \frac{1}{\varepsilon|\log\varepsilon|^{2M}}, \ \forall\, l\}$. We have then

$$(e^{v_j} - W)Z_{0j} = \begin{cases} \dfrac{8\mu_j}{(\mu_j^2+|y-\xi_j'|^2)^2} \times O(\varepsilon^\alpha + \varepsilon^\alpha |y-\xi_j'|^\alpha) & \text{if } y \in \Omega_{3j} \\ O(\varepsilon^{4-\alpha}\mu_j^{-1}) & \text{if } y \in \widetilde{\Omega}_3. \end{cases}$$

Moreover, $W[\mu_j^{-1} - a_{0j}G(\varepsilon y, \xi_j)] = O(\varepsilon^{4-\alpha})$ in $\widetilde{\Omega}_3$. If $y \in \Omega_{3j}$,

$$\begin{aligned} & W\left[\frac{1}{\mu_j} - a_{0j}G(\varepsilon y,\xi_j)\right] \\ &= O\left(\frac{8\mu_j^2}{(\mu_j^2+|y-\xi_j'|^2)^2}\right) \times a_{0j} \times O\left(\log\frac{|y-\xi_j'|}{\mu_j R} + \varepsilon^\alpha |y-\xi_j'|^\alpha\right) \\ &= O\left(\mu_j^{-2}\left[1+\frac{|y-\xi_j'|}{\mu_j}\right]^{-3} \frac{1}{\mu_j|\log\varepsilon|}\right). \end{aligned}$$

Therefore we get

$$|L\widetilde{Z}_{0j}(y)| = O\left(\mu_j^{-2}\left[1+\frac{|y-\xi_j'|}{\mu_j}\right]^{-3} \frac{1}{\mu_j|\log\varepsilon|}\right), \quad \forall\, y \in \widetilde{\Omega}_3 \cup \Omega_{3j}. \tag{53}$$

In Ω_{3k} with $k \neq j$, $e^{v_j}Z_{0j} = O(\varepsilon^{4-\alpha})$, but

$$W = O\left(\mu_k^{-2}\left[1+\frac{|y-\xi_k'|}{\mu_k}\right]^{-4}\right) \quad \text{and} \quad |\widehat{Z}_{0j}| = O\left(\frac{\log|\log\varepsilon|}{\mu_j|\log\varepsilon|}\right).$$

Hence if $y \in \Omega_{3k}$ for some $k \neq j$, we obtain

$$e^{v_j}Z_{0j} - W\widehat{Z}_{0j} = O\left(\mu_k^{-2}\left[1+\frac{|y-\xi_k'|}{\mu_k}\right]^{-3} \frac{\log|\log\varepsilon|}{\mu_j|\log\varepsilon|}\right). \tag{54}$$

Finally,

$$|L\widetilde{Z}_{0j}(y)| = O\left(\frac{\log|\log\varepsilon|}{\mu_j|\log\varepsilon|}\right) \times \sum_{k=1}^m \mu_k^{-2}\left[1+\frac{|y-\xi_k'|}{\mu_k}\right]^{-3}, \quad \forall\, y \in \Omega_3. \tag{55}$$

On Ω_4, we have

$$\begin{aligned} L\widetilde{Z}_{0j} &= \eta_{2j}[-\Delta_{a(\varepsilon y)}Z_{0j} - e^{v_j}Z_{0j}] - 2\nabla\widehat{Z}_{0j}\nabla\eta_{2j} - \widehat{Z}_{0j}\Delta_{a(\varepsilon y)}\eta_{2j} \\ &\quad - \eta_{2j}(W\widehat{Z}_{0j} - e^{v_j}Z_{0j}). \end{aligned}$$

Since $|\nabla\eta_{2j}| = O(\varepsilon)$, $|\Delta_{a(\varepsilon y)}\eta_{2j}| = O(\varepsilon^2)$,

$$|\nabla Z_{0j}| = O\left(\mu_j^{-2}\left[1+\frac{|y-\xi_j'|}{\mu_j}\right]^{-3}\right) \quad \text{and} \quad |\widehat{Z}_{0j}| = O\left(\frac{1}{\mu_j|\log\varepsilon|}\right),$$

hence

$$\|L\widetilde{Z}_{0j}\|_{L^\infty(\Omega_4)} = O\left(\frac{\varepsilon^2}{\mu_j|\log\varepsilon|}\right). \tag{56}$$

Combining (50), (52), (55) and (56), we arrive at

$$\|L\widetilde{Z}_{0j}\|_* \leq \frac{C\log|\log\varepsilon|}{\mu_j|\log\varepsilon|}.$$

The estimate (43) is easy to get as it is very similar to the consideration of (30) and that of $\widetilde{Z}_{0j}$ over Ω_1 and Ω_2. We leave the details to readers.

Now we prove Claim 2. Multiplying equation (46) by $a(\varepsilon y)\widetilde{Z}_{0j}$, integrating by parts, using (47) and (43), we find

$$\begin{aligned}
&\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_{0j}\sum_k d_k L\widetilde{Z}_{0k}\\
&= -\int_{\Omega_\varepsilon} a(\varepsilon y)h\widetilde{Z}_{0j} + \int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{\phi}L\widetilde{Z}_{0j} - \sum_{k=1}^{2}\sum_{l=1}^{m} e_{kl}\int_{\Omega_\varepsilon} a(\varepsilon y)Z_{kl}\chi_l L(\widetilde{Z}_{0j})\\
&\leq C\|h\|_*\left[\frac{1}{\mu_j} + \|L\widetilde{Z}_{0j}\|_*\right] + C\sum_{k=1}^{m}|d_k|\|L\widetilde{Z}_{0k}\|_*\|L\widetilde{Z}_{0j}\|_*\\
&\quad + C\sum_{k=1}^{2}\sum_{l=1}^{m}|e_{kl}|\frac{\|L\widetilde{Z}_{0j}\|_*}{\mu_l}.
\end{aligned}$$

Remark that

$$\int_{\Omega_\varepsilon} Z_{ij}^2\chi_j^2 = C, \quad \forall\, i = 1,2;\ j = 1\ldots m$$

and

$$\int_{\Omega_\varepsilon} \widetilde{Z}_{0l}Z_{ij}\chi_j = O\left(\frac{\mu_j\log|\log\varepsilon|}{\mu_l|\log\varepsilon|}\right), \quad \forall\, l \neq j.$$

Using (48), we get

$$|e_{ij}| \leq C\sum_{l\neq j}|d_l|\frac{\mu_j\log|\log\varepsilon|}{\mu_l|\log\varepsilon|}, \quad \forall\, i = 1,2;\ j = 1,\ldots,m. \tag{57}$$

So

$$\begin{aligned}
&|d_j|\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_{0j}L\widetilde{Z}_{0j}\\
\leq\ &\frac{C}{\mu_j}\|h\|_* + \frac{C}{\mu_j}\sum_{k=1}^{m}\frac{|d_k|(\log|\log\varepsilon|)^2}{\mu_k(\log\varepsilon)^2} + \sum_{k\neq j}\left|d_k\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_{0k}L\widetilde{Z}_{0j}\right|.
\end{aligned} \tag{58}$$

We decompose

$$\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_{0j}L\widetilde{Z}_{0j} = \sum_{1\leq l\leq 4}\int_{\Omega_l} a(\varepsilon y)\widetilde{Z}_{0j}L\widetilde{Z}_{0j} = \sum_{1\leq l\leq 4} I_l. \tag{59}$$

From (50), we have

$$I_1 = O(\varepsilon^\beta), \quad \forall\, \beta \in (0,1).$$

From (53), we derive that

$$\int_{\Omega_{3j}\cup\widetilde{\Omega}_3} a(\varepsilon y)\widetilde{Z}_{0j}L\widetilde{Z}_{0j} \le \frac{C}{\mu_j^2|\log\varepsilon|}\int_{R+1}^{\infty}\frac{rdr}{1+r^3} = O\left(\frac{1}{R\mu_j^2|\log\varepsilon|}\right).$$

From (54), we get also for $j\neq k$,

$$\int_{\Omega_{3k}} a(\varepsilon y)\widetilde{Z}_{0j}L\widetilde{Z}_{0j} \le \frac{C(\log|\log\varepsilon|)^2}{\mu_j^2|\log\varepsilon|^2}\int_{0}^{\infty}\frac{rdr}{1+r^4} = O\left(\frac{(\log|\log\varepsilon|)^2}{\mu_j^2|\log\varepsilon|^2}\right).$$

For I_4, we have

$$\begin{aligned}
I_4 &= \int_{\Omega_4} a(\varepsilon y)\widetilde{Z}_{0j}L\widetilde{Z}_{0j} = \int_{\Omega_4} a(\varepsilon y)\eta_{2j}\widehat{Z}_{0j}L(\eta_{2j}\widehat{Z}_{0j})\\
&= \int_{\Omega_4} a(\varepsilon y)\eta_{2j}^2\widehat{Z}_{0j}L\widehat{Z}_{0j} - 2a(\varepsilon y)\eta_{2j}\widehat{Z}_{0j}\nabla\eta_{2j}\nabla\widehat{Z}_{0j} - \eta_{2j}\widehat{Z}_{0j}^2\nabla\cdot[a(\varepsilon y)\nabla\eta_{2j}]\\
&= \int_{\Omega_4} a(\varepsilon y)\eta_{2j}^2\widehat{Z}_{0j}L\widehat{Z}_{0j} - \eta_{2j}\nabla\cdot\left[a(\varepsilon y)\widehat{Z}_{0j}^2\nabla\eta_{2j}\right]\\
&= \int_{\Omega_4} a(\varepsilon y)\eta_{2j}^2\widehat{Z}_{0j}L\widehat{Z}_{0j} + a(\varepsilon y)\widehat{Z}_{0j}^2|\nabla\eta_{2j}|^2,
\end{aligned}$$

Thus

$$I_4 = O\left(\frac{1}{\mu_j^2|\log\varepsilon|^2}\right).$$

It remains to estimate I_2:

$$\begin{aligned}
I_2 = &\int_{\Omega_2} a(\varepsilon y)\widetilde{Z}_{0j}\left[\eta_{1j}LZ_{0j} + (1-\eta_{1j})L\widehat{Z}_{0j}\right]\\
&+\int_{\Omega_2} 2a(\varepsilon y)\widetilde{Z}_{0j}\nabla\eta_{1j}\nabla(\widehat{Z}_{0j}-Z_{0j}) + \widetilde{Z}_{0j}(\widehat{Z}_{0j}-Z_{0j})\nabla\cdot[a(\varepsilon y)\nabla\eta_{1j}].
\end{aligned}$$

Thus integrating by parts for the last term implies that

$$\begin{aligned}
I_2 = &\int_{\Omega_2} a(\varepsilon y)\widetilde{Z}_{0j}\left[\eta_{1j}LZ_{0j} + (1-\eta_{1j})L\widehat{Z}_{0j}\right] + a(\varepsilon y)(\widehat{Z}_{0j}-Z_{0j})^2|\nabla\eta_{1j}|^2\\
&+\int_{\Omega_2} a(\varepsilon y)\widehat{Z}_{0j}\nabla\eta_{1j}\nabla(\widehat{Z}_{0j}-Z_{0j}) - a(\varepsilon y)(\widehat{Z}_{0j}-Z_{0j})\nabla\eta_{1j}\nabla\widehat{Z}_{0j}\\
= &\, I_{21}+I_{22}+I_{23}+I_{24}.
\end{aligned}$$

We observe that in the considered region, we have (51), $\widetilde{Z}_{0j}=O(\mu_j^{-1})$ and

$$\eta_{1j}LZ_{0j} + (1-\eta_{1j})L\widehat{Z}_{0j} = O\left(\frac{1}{\mu_j^3|\log\varepsilon|(1+r^4)}\right),\quad |\nabla\widehat{Z}_{0j}(y)| = O\left(\frac{1}{\mu_j^2r^3}\right)$$

where $r=|y-\xi_j'|/\mu_j$. We get then

$$I_{21} = O\left(\frac{1}{\mu_j^2|\log\varepsilon|R^3}\right),\quad I_{22} = O\left(\frac{R}{\mu_j^2|\log\varepsilon|^2}\right),\quad I_{24} = O\left(\frac{1}{\mu_j^2|\log\varepsilon|R^2}\right).$$

Now $\widehat{Z}_{0j} - Z_{0j} = o(Z_{0j})$ gives

$$\begin{aligned} I_{23} &= \int_{\Omega_2} a(\varepsilon y)\widehat{Z}_{0j}\nabla\eta_{1j}\nabla(\widehat{Z}_{0j} - Z_{0j}) \\ &= \mu_j^{-1} a_{0j} \int_R^{R+1} a(\xi_j)\eta_1'(r)\frac{1-r^2}{1+r^2}\big[4+o(1)\big]dr \\ &= \frac{a(\xi_j)}{\mu_j^2|\log\varepsilon|}\big[1+o(1)\big]. \end{aligned}$$

Combining all these estimates, we conclude that for R large enough and ε small enough,

$$\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_{0j}L\widetilde{Z}_{0j} \geq \frac{a(\xi_j)}{\mu_j^2|\log\varepsilon|}\left[\frac{1}{2}+o(1)\right]. \tag{60}$$

For the integral of $a(\varepsilon y)\widetilde{Z}_{0k}L\widetilde{Z}_{0j}$ when $k \neq j$, first by the estimate of $L\widetilde{Z}_{0j}$ and $\widetilde{Z}_{0k}$, we have readily

$$\int_{\widetilde{\Omega}_3\cup\Omega_1} a(\varepsilon y)\widetilde{Z}_{0k}L\widetilde{Z}_{0j} = O(\varepsilon^\alpha), \quad \int_{\Omega_2\cup\Omega_4} a(\varepsilon y)\widetilde{Z}_{0k}L\widetilde{Z}_{0j} = O\left(\frac{\log|\log\varepsilon|}{\mu_j\mu_k|\log\varepsilon|^2}\right)$$

and

$$\int_{\Omega_{3l}} a(\varepsilon y)\widetilde{Z}_{0k}L\widetilde{Z}_{0j} = O\left(\frac{(\log|\log\varepsilon|)^2}{\mu_j\mu_k|\log\varepsilon|^2}\right), \quad \forall\, l \neq k.$$

It remains to consider the integral over Ω_{3k}, we make an integration by parts,

$$\begin{aligned} &\int_{\Omega_{3k}} a(\varepsilon y)\widetilde{Z}_{0k}L\widetilde{Z}_{0j} \\ &= \int_{\Omega_{3k}} a(\varepsilon y)\widetilde{Z}_{0j}L\widetilde{Z}_{0k} - \int_{\partial\Omega_{3k}} a(\varepsilon y)\widetilde{Z}_{0k}\frac{\partial\widetilde{Z}_{0j}}{\partial\nu} + \int_{\partial\Omega_{3k}} a(\varepsilon y)\widetilde{Z}_{0j}\frac{\partial\widetilde{Z}_{0k}}{\partial\nu} \\ &= Q_1 + Q_2 + Q_3. \end{aligned}$$

As above, we have now

$$Q_1 = O\left(\frac{(\log|\log\varepsilon|)^2}{\mu_j\mu_k|\log\varepsilon|^2}\right).$$

On $\partial\Omega_{3k}$, we have

$$\widetilde{Z}_{0k} = O\left(\frac{\log|\log\varepsilon|}{\mu_k|\log\varepsilon|}\right), \quad |\nabla\widetilde{Z}_{0j}| = O\left(\frac{\varepsilon|\log\varepsilon|^{2M}}{\mu_j|\log\varepsilon|}\right).$$

It is easy to get

$$Q_2 = O\left(\frac{\log|\log\varepsilon|}{\mu_j\mu_k|\log\varepsilon|^2}\right).$$

Similar arguments hold for Q_3, so we conclude that

$$\int_{\Omega_{3k}} a(\varepsilon y)\widetilde{Z}_{0k}L\widetilde{Z}_{0j} = O\left(\frac{(\log|\log\varepsilon|)^2}{\mu_j\mu_k|\log\varepsilon|^2}\right), \quad \text{if } \ k \neq j.$$

This, combined with (58), gives

$$|d_j|\int_{\Omega_\varepsilon} a(\varepsilon y)Z_{0j}L\widetilde{Z}_{0j} \leq \frac{C}{\mu_j}\|h\|_* + \frac{C}{\mu_j}\sum_{k=1}^m \frac{|d_k|(\log|\log\varepsilon|)^2}{\mu_k|\log\varepsilon|^2}.$$

Using (60) and linear algebra arguments, we prove the Claim 2 for d_j and conclude the proof by (57). □

Proof of proposition 3.1. First we prove that for any ϕ, c_{ij} solution to (28) the bound

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C|\log\varepsilon|\|h\|_* \tag{61}$$

holds. In fact, the previous lemma yields

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C|\log\varepsilon|\left(\|h\|_* + \sum_{j=1}^m\sum_{i=1}^2 |c_{ij}|\mu_j\right). \tag{62}$$

So it suffices to estimate the values of the constants c_{ij}. To this end, we multiply (28) by $a(\varepsilon y)\widetilde{Z}_{ij}$ with $\widetilde{Z}_{ij} = Z_{ij}\eta_{2j}$, η_{2j} given by (40), and integrate to find

$$\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_{ij}L\phi = \int_{\Omega_\varepsilon} a(\varepsilon y)h\widetilde{Z}_{ij} + \sum_{k=1}^2\sum_{l=1}^m c_{kl}\int_{\Omega_\varepsilon}\widetilde{Z}_{ij}Z_{kl}\chi_l. \tag{63}$$

We claim that $\|L\widetilde{Z}_{ij}\|_* \le C\varepsilon^{1/3}$ for any $i = 1, 2$ and $j = 1, \dots, m$. For proving that, we decompose the domain into several regions: Fix j, let $\Omega_{1k} = \{|y - \xi_k'| \le \frac{1}{\varepsilon|\log\varepsilon|^{2M}}\}$ for any $1 \le k \le m$,

$$\Omega_2 = \left\{|y-\xi_j'| \le \frac{2}{\varepsilon},\ |y-\xi_k'| \ge \frac{1}{\varepsilon|\log\varepsilon|^{2M}},\ \forall\, k\right\}, \quad \Omega_3 = \left\{\frac{2}{\varepsilon} < |y-\xi_j'| \le \frac{4}{\varepsilon}\right\}.$$

For any k, $\widetilde{Z}_{ij} = Z_{ij}$ in Ω_{1k} since $\eta_{2j} \equiv 1$, so

$$L\widetilde{Z}_{ij} = -\Delta_{a(\varepsilon y)}Z_{ij} - WZ_{ij} = -\varepsilon\nabla\log a(\varepsilon y)\nabla Z_{ij} - (W - e^{v_j})Z_{ij}. \tag{64}$$

We have clearly, by Young's inequality

$$\begin{aligned} -\varepsilon\nabla\log a(\varepsilon y)\nabla Z_{ij} &= O\left(\varepsilon\mu_j^{-2}\left[1+\frac{|y-\xi_j'|}{\mu_j}\right]^{-2}\right)\\ &= O(\varepsilon^{7/3}) + O\left(\varepsilon^{1/3}\mu_j^{-3}\left[1+\frac{|y-\xi_j'|}{\mu_j}\right]^{-3}\right). \end{aligned} \tag{65}$$

In Ω_{1j}, using (25), we can obtain

$$(W - e^{v_j})Z_{ij} = O\left(\varepsilon^{1/3}\mu_j^{-2}\left[1+\frac{|y-\xi_j'|}{\mu_j}\right]^{-3}\right) \times O(\mu_j^{-1}),$$

so

$$|L\widetilde{Z}_{ij}(y)| = O(\varepsilon^{7/3}) + O\left(\varepsilon^{1/3}\mu_j^{-2}\left[1+\frac{|y-\xi_j'|}{\mu_j}\right]^{-3}\right), \quad \forall\, y \in \Omega_{1j}. \tag{66}$$

In Ω_{1k} with $k \ne j$, we get as for (54)

$$(e^{v_j} - W)Z_{ij} = O\left(\varepsilon|\log\varepsilon|^M\mu_k^{-2}\left[1+\frac{|y-\xi_k'|}{\mu_k}\right]^{-3}\right), \tag{67}$$

thus

$$|L\widetilde{Z}_{ij}(y)| = O(\varepsilon^{1/3}) \times \left(\varepsilon^2 + \sum_{l=1}^m \mu_l^{-2}\left[1+\frac{|y-\xi_l'|}{\mu_l}\right]^{-3}\right).$$

In $\Omega_2 \cup \Omega_3$, we have $e^{v_j} = O(\varepsilon^{4-\alpha})$, $W = O(\varepsilon^{4-\alpha})$. As $Z_{ij} = O(|y-\xi_j'|^{-1})$, $\nabla Z_{ij} = O(|y-\xi_j'|^{-2})$ and

$$L\widetilde{Z}_{ij} = -\left[\varepsilon\nabla \log a(\varepsilon y)\nabla Z_{ij} + (W - e^{v_j})Z_{ij}\right]\eta_{2j} - 2\nabla\eta_{2j}\nabla Z_{ij} - Z_{ij}\Delta_{a(\varepsilon y)}\eta_{2j}. \tag{68}$$

Using the estimates for η_{2j}, we obtain easily $\|L\widetilde{Z}_{ij}\|_{L^\infty(\Omega_2\cup\Omega_3)} = O(\varepsilon^{2+\alpha})$ for any $\alpha \in (0,1)$. Combining all these estimates, we prove the claim. Hence

$$\int_{\Omega_\varepsilon} a(\varepsilon y)\widetilde{Z}_{ij}L\phi = \int_{\Omega_\varepsilon} a(\varepsilon y)\phi L\widetilde{Z}_{ij} = O\left(\varepsilon^{1/3}\|\phi\|_\infty\right). \tag{69}$$

Moreover

$$\int_{\Omega_\varepsilon} a(\varepsilon y)h\widetilde{Z}_{ij} = O(\mu_j^{-1}\|h\|_*). \tag{70}$$

Now we need just to estimate the coefficient matrix for c_{ij}. By definition,

$$\int_{\Omega_\varepsilon} \widetilde{Z}_{il}Z_{kl}\chi_l = \int_{\Omega_\varepsilon} Z_{il}Z_{kl}\chi_l = C\delta_{ik}$$

and if $j \neq l$,

$$\int_{\Omega_\varepsilon} \widetilde{Z}_{ij}Z_{kl}\chi_l = \int_{\Omega_\varepsilon} Z_{ij}Z_{kl}\chi_l = O\left(\varepsilon|\log\varepsilon|^M\mu_l\right),$$

it is clear that the coefficient matrix is diagonally dominated and the inverse is uniformly bounded for small ε. Substituting (70) and (69) into (63), we obtain the estimate $\sup|c_{ij}| \leq C(\mu_j^{-1}\|h\|_* + \varepsilon^{1/3}\|\phi\|_\infty)$ and thus (61).

Now consider the Hilbert space

$$H = \left\{\phi \in H_0^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0 \quad \forall\, j = 1,\dots,m,\ i = 1,2\right\}$$

with the norm $\|\phi\|_{H_0^1} = \|\nabla\phi\|_{L^2(\Omega_\varepsilon)}$. Equation (28) is equivalent to find $\phi \in H$ such that

$$\int_{\Omega_\varepsilon} a(\varepsilon y)\nabla\phi\nabla\psi - a(\varepsilon y)W\phi\psi = \int_{\Omega_\varepsilon} a(\varepsilon y)h\psi, \qquad \forall\, \psi \in H.$$

By Fredholm's alternative this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by (61). □

The result of Proposition 3.1 implies that the unique solution $\phi = T(h)$ of (28) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$, into L^∞. We need to understand the differentiability of the operator T with respect to the variable ξ'. Fix $h \in \mathcal{C}_*$ and let $\phi = T(h)$. We can compute the derivatives of ϕ with respect to ξ' and obtain their estimates as follows

$$\|\partial_{\xi'}T(h)\|_\infty \leq C|\log\varepsilon|^3\|h\|_*, \quad \text{for all } k = 1,...,m;\ l = 1,2. \tag{71}$$

Sketch of proof: The proof is similar to that in [12], the most delicate point is to estimate $\|\partial_{\xi'}W\|_*$. Since $W = e^V$, we need just to estimate $\|\partial_{\xi'}V\|_\infty$.

For that, we consider first μ_j. Thanks to (22) and lemma 2.1, we prove easily that $|\partial_\xi\mu_j| = O(\mu_j|\log\varepsilon|^M)$ over Λ. This will lead to $\|\partial_\xi u_j\|_\infty = O(\varepsilon^{-1})$. Now using the equation for the ansatz U and by maximum principle, we get

$$\|\partial_\xi U\|_\infty \leq \|\partial_\xi u_j\|_\infty\|U\|_\infty = O\left(\frac{|\log\varepsilon|}{\varepsilon}\right).$$

After the scaling, we obtain finally $\|\partial_{\xi'}V\|_\infty = O(|\log\varepsilon|)$. □

4 The nonlinear problem

Consider the nonlinear equation

$$
(72) \qquad \begin{cases} -\Delta_{a(\varepsilon y)}\phi - W\phi = R + N(\phi) + \dfrac{1}{a(\varepsilon y)}\displaystyle\sum_{i,j} c_{ij}\chi_j Z_{ij} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \displaystyle\int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0, \quad \forall\, j = 1,\dots,m,\ i = 1,2 \end{cases}
$$

where W is as in (25) and N, R are defined in (20) and (21) respectively. We have the following result.

Lemma 4.1 *Let $m \in \mathbb{N}$ and $\alpha \in (0,1)$. Then there exist $\varepsilon_0 > 0$, $C > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $\xi = (\xi_1, \dots, \xi_m) \in \Lambda$ the problem (72) admits a unique solution ϕ, c_{ij} such that*

$$
(73) \qquad \|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon^\alpha |\log\varepsilon|.
$$

Furthermore, the function $\xi' \mapsto \phi(\xi')$ is C^1 and

$$
(74) \qquad \|\nabla_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)} \le C\,\varepsilon^\alpha |\log\varepsilon|^3.
$$

Proof. The proof of this lemma can be done along the lines of those of lemma 4.1 of [12]. We omit the details. □

5 Variational reduction

In view of lemma 4.1, given $\xi = (\xi_1, \dots, \xi_m) \in \Lambda$, we define $\phi(\xi')$ and $c_{ij}(\xi')$ to be the unique solution to (72) satisfying (73). Given $\xi = (\xi_1, \dots, \xi_m) \in \Omega^m$ we recall

$$
U(\xi) = \sum_{j=1}^m \left(u_j(x) + H_j^\varepsilon(x)\right)
$$

the ansatz defined in (14). Set

$$
(75) \qquad \mathcal{F}_\varepsilon(\xi) = J_\varepsilon(U(\xi) + \tilde{\phi}(\xi)),
$$

where J_ε is the functional defined by

$$
J_\varepsilon(v) = \frac{1}{2}\int_\Omega a(x)|\nabla v|^2 - \varepsilon^2 \int_\Omega a(x)e^v
$$

and

$$
(76) \qquad \tilde{\phi}(\xi)(x) = \phi\left(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon}\right), \quad x \in \Omega.
$$

Lemma 5.1 *If $\xi = (\xi_1, \dots, \xi_m) \in \Lambda$ is a critical point of $\mathcal{F}_\varepsilon$ then $u = U(\xi) + \tilde{\phi}(\xi)$ is a critical point of J_ε, that is, a solution to (1).*

Proof. The proof is similar to that of lemma 5.1 of [12]. □

In order to solve for critical points of the function $\mathcal{F}_\varepsilon$, a key step is its expected closeness to the function $J_\varepsilon(U)$, which we will analyze in the next section.

Lemma 5.2 *The following expansion holds*

$$\mathcal{F}_\varepsilon(\xi) = J_\varepsilon(U(\xi)) + \theta_\varepsilon(\xi),$$

where

$$|\theta_\varepsilon| + \|\nabla\theta_\varepsilon\| \to 0,$$

uniformly on Λ.

Proof. Let $\tilde\theta(\xi') = I_\varepsilon(V+\phi) - I_\varepsilon(V)$ where

$$I_\varepsilon(V) = \frac{1}{2}\int_{\Omega_\varepsilon} a(\varepsilon y)|\nabla V|^2 - \int_{\Omega_\varepsilon} a(\varepsilon y)e^V.$$

In order to get the proof of this lemma, we need to show that

$$|\tilde\theta| + \varepsilon^{-1}\|\nabla_{\xi'}\tilde\theta_\varepsilon\| = o(1).$$

Using $DI_\varepsilon(V+\phi)\cdot\phi = 0$, a Taylor expansion and an integration by parts give

$$\begin{aligned} & I_\varepsilon(V+\phi) - I_\varepsilon(V) \\ &= -\int_0^1 tD^2 I_\varepsilon(V+t\phi)(\phi,\phi)dt \\ &= -\int_0^1 \left(\int_{\Omega_\varepsilon}[N(\phi)+R]\,\phi + \int_{\Omega_\varepsilon} e^V[1-e^{t\phi}]\phi^2\right) tdt, \end{aligned} \tag{77}$$

so we get

$$\tilde\theta_\varepsilon = I_\varepsilon(V+\phi) - I_\varepsilon(V) = O(\varepsilon^{2\alpha}|\log\varepsilon|),$$

taking into account that $\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C\varepsilon^\alpha|\log\varepsilon|$. Let us differentiate with respect to ξ',

$$\begin{aligned} & \partial_{\xi'}[I_\varepsilon(V+\phi) - I_\varepsilon(V)] \\ &= -\int_0^1 \left(\int_{\Omega_\varepsilon}\partial_{\xi'}[(N(\phi)+R)\,\phi] + \int_{\Omega_\varepsilon}\partial_{\xi'}[e^V(1-e^{t\phi})\phi^2]\right) tdt. \end{aligned}$$

Using the fact that $\|\nabla_{\xi'}\phi\|_\infty \le C\,\varepsilon^\alpha|\log\varepsilon|^3$ and the estimates of the previous sections we get

$$\partial_{\xi'}[I_\varepsilon(V+\phi) - I_\varepsilon(V)] = \nabla_{\xi'}\tilde\theta_\varepsilon = O(\varepsilon^{2\alpha}|\log\varepsilon|).$$

The continuity in ξ of all these expressions is inherited from that of ϕ and its derivatives in ξ in the L^∞ norm. The proof is complete. $\square$

6 Expansion of the energy

In this section we will give an asymptotic estimate of $J_\varepsilon(U)$ where U is the approximate solution defined in (14) and J_ε is given in the previous section. We prove now the following result.

Lemma 6.1 *Let* U *be given by* (14). *Then*

$$\begin{aligned} J_\varepsilon(U) =\; & -16\pi\sum_{j=1}^m a(\xi_j)\log\varepsilon - 4\pi\sum_{i\neq j} a(\xi_j)G(\xi_i,\xi_j) \\ & - 4\pi\sum_j a(\xi_j)H(\xi_j,\xi_j) + O(1) \end{aligned} \tag{78}$$

where the term $O(1)$ *is uniform for* $\xi = (\xi_1, \ldots\xi_m) \in \Lambda$.

Proof. Define

$$U_j(x) = u_j(x) + H_j^\varepsilon(x)$$

so we may rewrite (14) in equivalent form $U = \sum_{j=1}^m U_j$. Then

$$\begin{aligned} J_\varepsilon(U) &= \frac{1}{2}\int_\Omega a(x)\Big|\sum_{j=1}^m \nabla U_j\Big|^2 - \varepsilon^2 \int_\Omega a(x)\exp\Big(\sum_{j=1}^m U_j\Big) \\ &= \frac{1}{2}\sum_{j=1}^m \int_\Omega a(x)|\nabla U_j|^2 + \sum_{i<j}\int_\Omega a(x)\nabla U_i \nabla U_j - \varepsilon^2\int_\Omega a(x)\exp\Big(\sum_{j=1}^m U_j\Big) \\ &= J_A + J_B + J_C. \end{aligned}$$

Let us analyze the behavior of J_A. Note that U_j satisfies

$$\Delta_a U_j + \varepsilon^2 e^{u_j} = 0 \text{ in } \Omega, \quad U_j = 0 \text{ on } \partial\Omega$$

which gives

$$\int_\Omega a(x)|\nabla U_j|^2 = \varepsilon^2\int_\Omega a(x)e^{u_j}(u_j + H_j^\varepsilon). \tag{79}$$

Let us find the asymptotic behavior of the expression:

$$\begin{aligned} &\int_\Omega a(x)|\nabla U_j|^2 \\ &= \varepsilon^2\int_\Omega a(x)\frac{8\mu_j^2}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2}\left[\log\frac{1}{(\varepsilon^2\mu_j^2 + |x-\xi_j|^2)^2} + H(x,\xi_j) + O(\varepsilon^\alpha)\right]. \end{aligned}$$

Changing variables $\varepsilon\mu_j y = x - \xi_j$, denote by $\omega_j = \{y \in \mathbb{R}^2, \xi_j + \varepsilon\mu_j y \in \Omega\}$,

$$\begin{aligned} &\int_\Omega a(x)|\nabla U_j|^2 \\ &= \int_{\omega_j}\frac{8a(\xi_j + \varepsilon\mu_j y)}{(1+|y|^2)^2}\left[\log\frac{1}{(1+|y|^2)^2} + H(\xi_j + \varepsilon\mu_j y, \xi_j) - 4\log(\varepsilon\mu_j)\right] + O(\varepsilon^\alpha). \end{aligned}$$

But

$$\int_{\omega_j} a(\xi_j + \varepsilon\mu_j y)\frac{8}{(1+|y|^2)^2} = 8\pi a(\xi_j) + O(\varepsilon\mu_j)$$

and

$$\int_{\omega_j}\frac{8}{(1+|y|^2)^2}|H(\xi_j + \varepsilon\mu_j y, \xi_j) - H(\xi_j,\xi_j)| = O(\varepsilon^\alpha).$$

Therefore we get

$$\int_\Omega a(x)|\nabla U_j|^2 = 8\pi a(\xi_j)H(\xi_j,\xi_j) - 32\pi a(\xi_j)\log(\varepsilon\mu_j) + O(1) \tag{80}$$

which derives

$$J_A = 4\pi\sum_{j=1}^m a(\xi_j)H(\xi_j,\xi_j) - 16\pi\sum_{j=1}^m a(\xi_j)\log(\varepsilon\mu_j) + O(1). \tag{81}$$

Regarding the expression J_C, we have

$$\begin{aligned} J_C &= -\varepsilon^2 \int_\Omega a(x) e^{\sum_{k=1}^m U_k} \\ &= -\varepsilon^2 \sum_{j=1}^m \int_{B_{\varepsilon^\alpha}(\xi_j)} a(x) e^{\sum_{k=1}^m (u_k + H_k^\varepsilon)} + O(\varepsilon^{2-2\alpha}). \end{aligned}$$

Fix $0 < \alpha < 1$. Using the definition of u_j and (16) for each term we have

$$\begin{aligned} &\varepsilon^2 \int_{B_{\varepsilon^\alpha}(\xi_j)} a(x) e^{\sum_{k=1}^m (u_k + H_k^\varepsilon)} \\ &= \varepsilon^2 \int_{B_{\varepsilon^\alpha}(\xi_j)} a(x) e^{u_j} e^{H(x,\xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha)} E_j(x) \end{aligned}$$

where

$$E_j(x) = \exp\Big\{ \sum_{i \neq j} \Big[\log \frac{1}{(\varepsilon^2 \mu_i^2 + |x - \xi_i|^2)^2} + H(x, \xi_i) + O(\varepsilon^\alpha) \Big] \Big\}.$$

Take the change of variable $x = \xi_j + \varepsilon \mu_j y$ in $B_{\varepsilon^\alpha}(\xi_j)$,

$$e^{H(x,\xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha)} = e^{H(\xi_j,\xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha) + O(\varepsilon^\alpha |y|^\alpha)}.$$

But $\varepsilon^\alpha |y|^\alpha = O(\varepsilon^{\alpha^2})$ for $x \in B_{\varepsilon^\alpha}(\xi_j)$. Similarly, since $|\xi_i - \xi_j| \geq |\log \varepsilon|^{-M}$,

$$E_j(x) = \exp\Big[\sum_{i \neq j} G(\xi_j, \xi_i) + O(\varepsilon^\alpha |\log \varepsilon|^M) + O(\varepsilon^{\alpha^2}) \Big] \quad \text{in} \quad B_{\varepsilon^\alpha}(\xi_j).$$

Therefore, by the definition of μ_j in (22)

$$\varepsilon^2 \int_{B_{\varepsilon^\alpha}(\xi_j)} a(x) e^{\sum_{k=1}^m (u_k + H_k^\varepsilon)} = \varepsilon^2 \int_{B_{\varepsilon^\alpha}(\xi_j)} a(x) e^{u_j + O(\varepsilon^{\alpha^2})} = 8\pi a(\xi_j) + O(\varepsilon^{\alpha^2}).$$

Hence

$$J_C = -8\pi \sum_j a(\xi_j) + O(\varepsilon^{\alpha^2}). \tag{82}$$

Consider now

$$J_B = \sum_{i<j} \int_\Omega a(x) \nabla U_i \nabla U_j = \varepsilon^2 \sum_{i<j} \int_\Omega a(\xi_j) e^{u_i} (u_j + H_j^\varepsilon),$$

similar arguments as for J_A and J_C show that

$$J_B = 8\pi \sum_{i<j} a(\xi_i) G(\xi_i, \xi_j) + O(1). \tag{83}$$

Thanks to (81)-(83) and employing (22), we obtain (78). □

7 Proof of Theorem 1.1

We assume that $\bar{x} \in \Omega$ is a strict local maximum point of $a(x)$, i.e., there exists an open subset $B_\delta(\bar{x}) \subset \Omega$ such that $\forall\, y \in B_\delta(\bar{x}) \setminus \{\bar{x}\}$, $a(y) < a(\bar{x})$. Then we have

Lemma 7.1 *For $\varepsilon > 0$ small enough, the following maximization problem*

$$\max_{(\xi_1,\ldots,\xi_m)\in\bar{\Lambda}} \mathcal{F}_\varepsilon(\xi_1,...,\xi_m)$$

has a solution in the interior of Λ.

Proof: Let $(\xi_1^\varepsilon,...,\xi_m^\varepsilon)\in\bar{\Lambda}$ be the maximizer of $\mathcal{F}_\varepsilon$. We need to prove that $(\xi_1^\varepsilon,...,\xi_m^\varepsilon)\in\Lambda^\circ$, the interior of Λ. First, we obtain a lower bound. Let

$$\xi_j^0 = \bar{x} + \frac{1}{\sqrt{|\log\varepsilon|}}\widehat{\xi}_j^0$$

where $\widehat{\xi}_j^0, j = 1,...,m$, form a m-regular polygon in $\mathbb{R}^2$. Then it is easy to see that $(\xi_1^0,...,\xi_m^0)\in\Lambda$ since $M\geq 1$. From (78) and lemma 5.2, using that $\bar{x}$ is a strict local maximum point of a, we obtain

$$\begin{aligned}(84)\qquad &\max\mathcal{F}_\varepsilon(\xi_1,...,\xi_m)\\ &\geq 4\pi m a(\bar{x})\log\frac{1}{\varepsilon^4} - 16\pi\sum_{i\neq j} a(\xi_j^0)\log\frac{1}{|\xi_j^0-\xi_i^0|} + O(1)\\ &\geq -16\pi m a(\bar{x})\log\varepsilon - 8(m-1)m\pi a(\bar{x})\log|\log\varepsilon| + O(1).\end{aligned}$$

Now suppose $(\xi_1^\varepsilon,...,\xi_m^\varepsilon)\in\partial\Lambda$. There are two possibilities: either there exists a j_0 such that $\xi_{j_0}^\varepsilon\in\partial B_\delta(\bar{x})$, in which case, $a(\xi_{j_0}^\varepsilon)\leq a(\bar{x})-\delta_0$ for some $\delta_0>0$; or there exists $i_0\neq j_0$ such that $|\xi_{i_0}^\varepsilon-\xi_{j_0}^\varepsilon| = |\log\varepsilon|^{-M}$.

In the first case, we have

$$\begin{aligned}(85)\qquad \max\mathcal{F}_\varepsilon(\xi_1,...,\xi_m) &\leq 4\pi\left[(m-1)a(\bar{x})+a(\bar{x})-\delta_0\right]\log\frac{1}{\varepsilon^4} + O(\log|\log\varepsilon|)\\ &\leq -16\pi\left[m a(\bar{x})-\delta_0\right]\log\varepsilon + O(\log|\log\varepsilon|)\end{aligned}$$

which contradicts to (84). This also shows that $a(\xi_j^\varepsilon)\to a(\bar{x})$. By the condition over a, we get $\xi_j^\varepsilon\to\bar{x}$.

In the second case, we have

$$\begin{aligned}(86)\qquad \max\mathcal{F}_\varepsilon(\xi_1,...,\xi_m) &\leq 4\pi m a(\bar{x})\log\frac{1}{\varepsilon^4} - 16\pi a(\xi_{j_0}^\varepsilon)\log\frac{1}{|\xi_{i_0}^\varepsilon-\xi_{j_0}^\varepsilon|} + O(1)\\ &\leq -16\pi m a(\bar{x})\log\varepsilon - 16M\pi a(\xi_{j_0}^\varepsilon)\log|\log\varepsilon| + O(1).\end{aligned}$$

Comparing with (84), we have

$$(87)\qquad 16M\pi a(\xi_{j_0}^\varepsilon)\log|\log\varepsilon| + O(1) \leq 8m(m-1)\pi a(\bar{x})\log|\log\varepsilon| + O(1)$$

which is impossible by the choice of M at (12). □

Proof of Theorem 1.1 (completed): According to lemma 5.1, the function $U(\xi)+\tilde{\phi}(\xi)$, where U and $\tilde{\phi}$ are defined respectively by (14) and (76), is a solution of problem (1), if we adjust ξ so that it is a critical point of $\mathcal{F}_\varepsilon(\xi) = J_\varepsilon(U(\xi)+\tilde{\phi}(\xi))$ defined by (75). Lemma 7.1 then guarantees the existence of a such critical point and thus a solution u_ε for (1). Furthermore, from the ansatz (14), we get that, as $\varepsilon\to 0$, u_ε remains uniformly bounded on $\Omega\setminus\cup_{j=1}^m B_{\varepsilon^\alpha}(\xi_i^\varepsilon)$, and

$$\sup_{B_{\varepsilon^\alpha}(\xi_i^\varepsilon)} u_\varepsilon \to +\infty.$$

The rest of the properties of u_ε can be easily seen from the decomposition of u_ε. □

Proof of Remark 1.4: We choose now $M = -(m^2+1)/\alpha$. We need just to change the lower bound estimate in the proof of lemma 7.1. Take

$$\xi_j^0 = \bar{x} + |\log\varepsilon|^{-\frac{1}{\alpha}}\hat{\xi}_j^0$$

where $\hat{\xi}_j^0$ $(1 \le j \le m)$ form a m-regular polygon in $\mathbb{R}^2$, we get then

$$\begin{aligned} &\max \mathcal{F}_\varepsilon(\xi_1, ..., \xi_m) \\ \ge &- 16\pi m a(\bar{x})\log\varepsilon - \frac{16(m-1)m\pi}{\alpha} a(\bar{x})\log|\log\varepsilon| + O(1). \end{aligned} \tag{88}$$

Using (85) and (86), we can proof again that $\mathcal{F}_\varepsilon$ reaches its maximum in the interior of Λ. □

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