

Existence theorems of the fractional Yamabe problem

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March 23, 2016

Abstract

Let X be an asymptotically hyperbolic manifold and M its conformal infinity. This paper is devoted to deduce several existence results of the fractional Yamabe problem on M under various geometric assumptions on X and M : Firstly, we handle when the boundary M has a point at which the mean curvature is negative. Secondly, we re-encounter the case when M has zero mean curvature and is either non-umbilic or umbilic but non-locally conformally flat. As a result, we replace the geometric restrictions given by González-Qing (2013) [19] and González-Wang (2015) [20] with simpler ones. Also, inspired by Marques (2007) [37] and Almaraz (2010) [1], we study lower-dimensional manifolds. Finally, the situation when X is Poincaré-Einstein, M is either locally conformally flat or 2-dimensional is covered under the validity of the positive mass theorem for the fractional conformal Laplacians.

2010 Mathematics Subject Classification. Primary: 53C21, Secondary: 35R11, 53A30.

Key words and Phrases. Fractional Yamabe problem, Conformal geometry, Existence.

1 Introduction and the Main Results

Given $n \in \mathbb{N}$, let X^{n+1} be an $(n+1)$ -dimensional smooth manifold with smooth boundary M^n . A function ρ in X is called a *defining function* of the boundary M in X if

$$\rho > 0 \quad \text{in } X, \quad \rho = 0 \quad \text{on } M \quad \text{and} \quad d\rho \neq 0 \quad \text{on } M.$$

A metric g^+ in X is *conformally compact* with *conformal infinity* $(M, [\hat{h}])$ if there exists a boundary defining function ρ so that the closure (\bar{X}, \bar{g}) of X is compact for $\bar{g} := \rho^2 g^+$ and $\bar{g}|_M \in [\hat{h}]$. A manifold (X^{n+1}, g^+) is said to be *asymptotically hyperbolic* if g^+ is conformally compact and $|\rho|_{\bar{g}} \rightarrow 1$ as $\rho \rightarrow 0$. Also if (X, g^+) is asymptotically hyperbolic and Einstein, then it is called *Poincaré-Einstein* or *conformally compact Einstein*.

Suppose that an asymptotically hyperbolic manifold (X, g^+) with the conformal infinity $(M^n, [\hat{h}])$ is given. Also, for any $\gamma \in (0, 1)$, let $P_{\hat{h}}^\gamma = P^\gamma[g^+, \hat{h}]$ be the *fractional conformal Laplacian* whose principle symbol is equal to $(-\Delta_{\hat{h}})^\gamma$ (see [22] for its precise definition). In this article, we are interested in finding a conformal metric \hat{h} on M with constant *fractional scalar curvature* $Q_{\hat{h}}^\gamma = P_{\hat{h}}^\gamma(1)$. This problem is referred to be the *fractional Yamabe problem* or the γ -*Yamabe problem*, and it was introduced and investigated by González-Qing [19] and González-Wang [20]. By imposing some restrictions on the dimension and geometric behavior of the manifold, the authors obtained the existence results when M is non-umbilic or it is umbilic but not locally conformally flat. Here we relieve the hypotheses made in [19, 20] and examine when the bubble (see (1.13) below for its precise definition) cannot be used as an appropriate test function.

As its name alludes, the fractional conformal Laplacian $P_{\hat{h}}^\gamma$ has the *conformal covariance property*: It holds that

$$P_{\hat{h}_w}^\gamma(u) = w^{-\frac{n+2\gamma}{n-2\gamma}} P_{\hat{h}}^\gamma(wu) \tag{1.1}$$

for a conformal change of the metric $\hat{h}_w = w^{4/(n-2\gamma)}\hat{h}$. Hence the fractional Yamabe problem can be formulated as looking for a positive solution of the nonlocal equation

$$P_{\hat{h}}^{\gamma}u = cu^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{on } M \quad (1.2)$$

for some $c \in \mathbb{R}$ provided $n > 2\gamma$. On the other hand, if $\gamma = 1$, $P_{\hat{h}}^{\gamma}$ and $Q_{\hat{h}}^{\gamma}$ precisely match with the classical *conformal Laplacian* $L_{\hat{h}}$ and a constant multiple of the scalar curvature $R[\hat{h}]$ on (M, \hat{h})

$$P_{\hat{h}}^1 = L_{\hat{h}} := -\Delta_{\hat{h}} + \frac{n-2}{4(n-1)}R[\hat{h}] \quad \text{and} \quad Q_{\hat{h}}^1 = \frac{n-2}{4(n-1)}R[\hat{h}], \quad (1.3)$$

respectively. If $\gamma = 2$, they coincide with the Paneitz operator [38] and Branson's Q -curvature [3]. Hence the 1 or 2-Yamabe problems are reduced to the classical Yamabe problem and the Q -curvature problem.

Thanks to the efforts of various mathematicians, a complete solution of the Yamabe problem has been known. After Yamabe [46] raised the problem and suggested an outline of the proof, Trudinger [44] first obtained a least energy solution to (1.2) under the setting that the scalar curvature of (M, \hat{h}) is nonpositive. Successively, Aubin [2] examined the case when $n \geq 6$ and M is non-locally conformally flat, and Schoen [40] gave an affirmative answer when $n = 3, 4, 5$ or M is locally conformally flat by using the positive mass theorem [41, 42, 43]. In Lee-Parker [35], the authors provided a new proof which unified the local proof of Aubin and the global proof of Schoen, introducing the notion of the conformal normal coordinates.

Also there have been lots of results on the Q -curvature problem ($\gamma = 2$) for 4-dimensional manifolds $(M^4, [\hat{h}])$. By the Chern-Gauss-Bonnet formula, the *total Q-curvature*

$$k_P := \int_{M^4} Q_{\hat{h}}^2 dv_{\hat{h}}$$

is a conformal invariant. Gursky [25] proved that if a manifold M^4 has the positive Yamabe constant $\Lambda^1(M, [\hat{h}]) > 0$ (see (1.10)) and satisfies $k_P \geq 0$, then its Paneitz operator $P_{\hat{h}}^2$ has the properties

$$\ker P_{\hat{h}}^2 = \mathbb{R} \quad \text{and} \quad P_{\hat{h}}^2 \geq 0. \quad (1.4)$$

Also Chang-Yang [11] proved that any compact 4-manifold such that (1.4) and $k_P < 8\pi^2$ hold has a solution to

$$P_{\hat{h}}^2u + 2Q_{\hat{h}}^2u = 2ce^{4u} \quad \text{on } M, \quad c \in \mathbb{R}$$

where $Q_{\hat{h}}^2$ is the Q -curvature. This result was generalized by Djadli-Malchiodi [13] where only $\ker P_{\hat{h}}^2 = \mathbb{R}$ and $k_P \neq 8m\pi^2$ for all $m \in \mathbb{N}$ are demanded. For other dimensions than 4, Gursky-Malchiodi [26] recently discovered the strong maximum principle of $P_{\hat{h}}^2$ for manifolds M^n ($n \geq 5$) with non-negative scalar curvature and semi-positive Q -curvature. Motivated by this result, Hang and Yang developed the existence theory of (1.2) for a general class of manifolds M^n including ones such that $\Lambda^1(M, [\hat{h}]) > 0$ and there exists $\hat{h}' \in [\hat{h}]$ with $Q_{\hat{h}'}^2 > 0$, provided $n \geq 5$ [28, 30] or $n = 3$ [27, 28, 29]. In [30], the positive mass theorem for the Paneitz operator [31, 26] was used to construct a test function. We also point out that a solution to (1.2) was obtained in [39] for a locally conformally flat manifold ($n \geq 5$) with positive Yamabe constant and Poincaré exponent less than $(n-4)/2$.

In addition, when $\gamma = 1/2$, the fractional Yamabe problem has a deep relationship with the boundary Yamabe problem proposed by Escobar [14], who regarded it as a generalization of the Riemann mapping theorem: It asks if a compact manifold \bar{X} with boundary is conformally equivalent to one of zero scalar curvature whose boundary M has constant mean curvature. It was solved by the series of works by Escobar himself [14, 16], Marques [36, 37] and Almaraz [1]. It is worthwhile to mention that there is another type of boundary Yamabe problem also suggested by Escobar [15]: Find a conformal metric such that the scalar curvature of X is constant and the boundary M is minimal. It was further studied by Brendle-Chen [5].

In [10] (see also [9]), Chang and González observed that the fractional conformal Laplacian, defined through the scattering theory in Graham-Zworski [22], can be described in terms of Dirichlet-Neumann operators. Especially, (1.2) has an equivalent extension problem, which is degenerate elliptic but local.

Theorem A. *Suppose that $n > 2\gamma$, $\gamma \in (0, 1)$, (X, g^+) is an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{h}])$. Assume also that ρ is a defining function associated to M such that $|d\rho|_{\bar{g}} = 1$ near M (such ρ is called geodesic), and $\bar{g} = \rho^2 g^+$ is a metric of the compact manifold \bar{X} . In addition, we let the mean curvature H on $(M, \hat{h}) \subset (\bar{X}, \bar{g})$ be 0 if $\gamma \in (1/2, 1)$, and set*

$$E(\rho) = \rho^{-1-s}(-\Delta_{g^+} - s(n-s))\rho^{n-s} \quad \text{in } X \quad (1.5)$$

where $s := n/2 + \gamma$. It can be shown that (1.5) is reduced to

$$E(\rho) = \left(\frac{n-2\gamma}{4n}\right) [R[\bar{g}] - (n(n+1) + R[g^+])\rho^{-2}] \rho^{1-2\gamma} \quad \text{near } M \quad (1.6)$$

where $R[\bar{g}]$ and $R[g^+]$ are the scalar curvature of (\bar{X}, \bar{g}) and (X, g^+) , respectively.

(1) If a positive function U satisfies

$$\begin{cases} -\operatorname{div}_{\bar{g}}(\rho^{1-2\gamma}\nabla U) + E(\rho)U = 0 & \text{in } (X, \bar{g}), \\ U = u & \text{on } M \end{cases} \quad (1.7)$$

and

$$\partial_\nu^\gamma U := -\kappa_\gamma \left(\lim_{\rho \rightarrow 0^+} \rho^{1-2\gamma} \frac{\partial U}{\partial \rho} \right) = \begin{cases} cu^{\frac{n+2\gamma}{n-2\gamma}} & \text{for } \gamma \in (0, 1) \setminus \{1/2\}, \\ cu^{\frac{n+2\gamma}{n-2\gamma}} - \left(\frac{n-1}{2}\right)Hu & \text{for } \gamma = \{1/2\} \end{cases} \quad (1.8)$$

on M , then u solves (1.2). Here $\kappa_\gamma > 0$ is the constant whose explicit value is given in (1.23) below and ν stands for the outward unit normal vector with respect to the boundary M .

(2) Assume further that the first L^2 -eigenvalue $\lambda_1(-\Delta_{g^+})$ of the Laplace-Beltrami operator $-\Delta_{g^+}$ satisfies

$$\lambda_1(-\Delta_{g^+}) > \frac{(n-1)^2}{4} - \gamma^2. \quad (1.9)$$

Then there is a special defining function ρ^* such that $E(\rho^*) = 0$ in X and $\rho^*(\rho) = \rho(1 + O(\rho^{2\gamma}))$ near M . Furthermore the function $\tilde{U} := (\rho/\rho^*)^{(n-2\gamma)/2}U$ solves a degenerate elliptic equation of pure divergent form

$$\begin{cases} -\operatorname{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma}\nabla \tilde{U}) = 0 & \text{in } (X, \bar{g}^*), \\ \partial_\nu^\gamma \tilde{U} = -\kappa_\gamma \left(\lim_{\rho^* \rightarrow 0^+} (\rho^*)^{1-2\gamma} \frac{\partial \tilde{U}}{\partial \rho^*} \right) = P_h^\gamma u - Q_h^\gamma u = cu^{\frac{n+2\gamma}{n-2\gamma}} - Q_h^\gamma u & \text{on } M \end{cases}$$

where $\bar{g}^* := (\rho^*)^2 g^+$ and Q_h^γ is the fractional scalar curvature.

Notice that in order to seek a solution of (1.2), it is natural to introduce the γ -Yamabe functional

$$I_h^\gamma[u] = \frac{\int_M u P_h^\gamma u \, dv_{\hat{h}}}{\left(\int_M u^{\frac{2n}{n-2\gamma}} \, dv_{\hat{h}}\right)^{\frac{n-2\gamma}{n}}} \quad \text{for } u \in C_c^\infty(M), \ u > 0 \text{ on } M \quad (1.10)$$

and its infimum $\Lambda^\gamma(M, [\hat{h}])$, called the γ -Yamabe constant. By the previous theorem and the energy inequality due to Case [8, Theorem 1.1], it follows under the assumption (1.9) that if one defines the functionals

$$\tilde{I}_h^\gamma[U] = \frac{\kappa_\gamma \int_X (\rho^{1-2\gamma} |\nabla U|_{\bar{g}}^2 + E(\rho)U^2) \, dv_{\bar{g}}}{\left(\int_M |U|^{\frac{2n}{n-2\gamma}} \, dv_{\hat{h}}\right)^{\frac{n-2\gamma}{n}}}, \quad (1.11)$$

$$\tilde{T}_h^\gamma[U] = \frac{\kappa_\gamma \int_X (\rho^*)^{1-2\gamma} |\nabla U|_{\frac{2}{g}}^2 dv_{\bar{g}} + \int_M Q_h^\gamma U^2 dv_{\hat{h}}}{\left(\int_M |U|^{\frac{2n}{n-2\gamma}} dv_{\hat{h}} \right)^{\frac{n-2\gamma}{n}}}$$

for $U \in W^{1,2}(X, \rho^{1-2\gamma})$ such that $U \neq 0$ on M (with a suitable modification for the $\gamma = 1/2$ case), and values

$$\begin{aligned} \bar{\Lambda}^\gamma(X, [\hat{h}]) &= \inf \left\{ \tilde{T}_h^\gamma[U] : U \in W^{1,2}(X, \rho^{1-2\gamma}), U \neq 0 \text{ on } M \right\}, \\ \tilde{\Lambda}^\gamma(X, [\hat{h}]) &= \inf \left\{ \tilde{T}_h^\gamma[U] : U \in W^{1,2}(X, \rho^{1-2\gamma}), U \neq 0 \text{ on } M \right\}, \end{aligned}$$

then

$$\Lambda^\gamma(M, [\hat{h}]) = \bar{\Lambda}^\gamma(X, [\hat{h}]) = \tilde{\Lambda}^\gamma(X, [\hat{h}]) > -\infty.$$

Besides it is shown in [19] that the sign of c in (1.2) is the same as that of $\Lambda^\gamma(M, [\hat{h}])$ as in the local case ($\gamma = 1$).

On the other hand, the Sobolev trace inequality

$$\left(\int_{\mathbb{R}^n} |U(\bar{x}, 0)|^{\frac{2n}{n-2\gamma}} d\bar{x} \right)^{\frac{n-2\gamma}{n}} \leq S_{n,\gamma} \int_0^\infty \int_{\mathbb{R}^n} x_{n+1}^{1-2\gamma} |\nabla U(\bar{x}, x_{n+1})|^2 d\bar{x} dx_{n+1} \quad (1.12)$$

is true for all functions $U \in W^{1,2}(\mathbb{R}_+^{n+1}, x_{n+1}^{1-2\gamma})$, and the equality is attained by $U = cW_{\lambda,\sigma}$ for any $c \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in \mathbb{R}^n = \partial\mathbb{R}_+^{n+1}$ where $W_{\lambda,\sigma}$ are the *bubbles* defined as

$$\begin{aligned} W_{\lambda,\sigma}(\bar{x}, x_{n+1}) &= p_{n,\gamma} \int_{\mathbb{R}^n} \frac{x_{n+1}^{2\gamma}}{(|\bar{x} - \bar{y}|^2 + x_{n+1}^2)^{\frac{n+2\gamma}{2}}} w_{\lambda,\sigma}(\bar{y}) d\bar{y} \\ &= g_{n,\gamma} \int_{\mathbb{R}^n} \frac{1}{(|\bar{x} - \bar{y}|^2 + x_{n+1}^2)^{\frac{n-2\gamma}{2}}} w_{\lambda,\sigma}^{\frac{n+2\gamma}{n-2\gamma}}(\bar{y}) d\bar{y} \end{aligned} \quad (1.13)$$

with

$$w_{\lambda,\sigma}(\bar{x}) := \alpha_{n,\gamma} \left(\frac{\lambda}{\lambda^2 + |\bar{x} - \sigma|^2} \right)^{\frac{n-2\gamma}{2}} = W_{\lambda,\sigma}(\bar{x}, 0) \quad (1.14)$$

($p_{n,\gamma}$, $g_{n,\gamma}$ and $\alpha_{n,\gamma}$ are positive numbers whose values can be found in (1.23)). Particularly, it holds that

$$\begin{cases} -\operatorname{div}(x_{n+1}^{1-2\gamma} \nabla W_{\lambda,\sigma}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \partial_\nu^\gamma W_{\lambda,\sigma} = -\kappa_\gamma \left(\lim_{x_{n+1} \rightarrow 0^+} x_{n+1}^{1-2\gamma} \frac{\partial W_{\lambda,\sigma}}{\partial x_{n+1}} \right) = (-\Delta)^\gamma w_{\lambda,\sigma} = w_{\lambda,\sigma}^{\frac{n+2\gamma}{n-2\gamma}} & \text{on } \mathbb{R}^n. \end{cases} \quad (1.15)$$

(In light of the equation that $W_{\lambda,\sigma}$ solves, we say that $W_{\lambda,\sigma}$ is γ -harmonic. Refer to [7]. For future use, let $W_\lambda = W_{\lambda,0}$ and $w_\lambda = w_{\lambda,0}$.) Moreover, if $S_{n,\gamma} > 0$ denotes the best constant one can achieve in (1.12) and $(\mathbb{S}^n, [g_c])$ is the standard unit n -dimensional sphere, then

$$\Lambda^\gamma(\mathbb{S}^n, [g_c]) = S_{n,\gamma}^{-1} \kappa_\gamma = \left(\int_{\mathbb{R}^n} w_{\lambda,\sigma}^{\frac{2n}{n-2\gamma}} dx \right)^{\frac{2\gamma}{n}}. \quad (1.16)$$

Related to this fact, we have the following compactness result.

Proposition B. *Let $n > 2\gamma$, $\gamma \in (0, 1)$ and (X^{n+1}, g^+) be an asymptotically hyperbolic manifold with the conformal infinity $(M^n, [\hat{h}])$. Also, assume that (1.9) is true. Then*

$$-\infty < \Lambda^\gamma(M, [\hat{h}]) \leq \Lambda^\gamma(\mathbb{S}^n, [g_c]), \quad (1.17)$$

and the fractional Yamabe problem (1.7)-(1.8) has a positive solution if the strict inequality holds.

Refer to [19] for its proof. Moreover since (1.17) automatically holds if the γ -Yamabe constant $\Lambda^\gamma(M, [\hat{h}])$ is negative or 0, **we assume that $\Lambda^\gamma(M, [\hat{h}]) > 0$ from now on.**

The purpose of this paper is to construct a proper nonzero test function $\Phi \in W^{1,2}(X, \rho^{1-2\gamma})$ such that $0 < \bar{I}_h^\gamma[\Phi] < \Lambda^\gamma(\mathbb{S}^n, [g_c])$ when $\gamma \in (0, 1)$, (X^{n+1}, g^+) is an asymptotically hyperbolic manifold, (1.9) holds and

- M^n has a point where the mean curvature H is negative, $n \geq 2$ and $\gamma \in (0, 1/2)$; or
- M^n is the non-umbilic boundary of X^{n+1} , $n \geq 4$ and assumption (1.18) holds; or
- M^n is the umbilic but non-locally conformally flat boundary of X^{n+1} , $n > 4 + 2\gamma$ and condition (1.19) is satisfied; or
- X^{n+1} is Poincaré-Einstein and either M^n is locally conformally flat or $n = 2$.

Then Proposition B would imply the existence of a positive solution to (1.2) automatically. The natural candidate for a positive test function is certainly the standard bubble, possibly truncated. Indeed, this is a good choice for the first case above mentioned. Nevertheless, to cover lower dimensional manifolds or locally conformally flat boundaries, it is necessary to find more accurate test functions than the truncated bubbles (cf. [19, 20]). To take into account the second and third situations, we shall add a correction term on the bubble by adapting the idea of Marques [37] and Almaraz [1]. For the fourth case, assuming the validity of the positive mass theorem for P_h^γ for $\gamma \in (0, 1)$, we will construct an appropriate test function by utilizing Green's function. In the local situation ($\gamma = 1$), such an approach was successfully applied by Schoen [40] who employed the classical positive mass theorem [41, 42, 43]. His idea was later extended by Escobar [14] in the work of the boundary Yamabe problem, which has close relationship to the fractional Yamabe problem with $\gamma = 1/2$.

Our first main result reads as follows: Let π be the second fundamental form of $(M, \hat{h}) \subset (\bar{X}, \bar{g})$. The boundary M is called *umbilic* if the tensor $T := \pi - H\bar{g}$ vanishes on M . Also M is *non-umbilic* if it possesses a point at which $T \neq 0$.

Theorem 1.1. *Suppose that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold, $(M, [\hat{h}])$ is its conformal infinity and (1.9) holds. Assume also that ρ is a geodesic defining function of (M, \hat{h}) and $\bar{g} = \rho^2 g^+ = d\rho^2 \oplus h_\rho$ near $M = \{\rho = 0\}$. If either*

- $n \geq 2$, $\gamma \in (0, 1/2)$ and M^n has a point at which the mean curvature H is negative; or
- $n \geq 4$, $\gamma \in (0, 1)$, M^n is the non-umbilic boundary of X^{n+1} and

$$R[g^+] + n(n+1) = o(\rho^2) \quad \text{as } \rho \rightarrow 0 \text{ uniformly on } M, \quad (1.18)$$

then the γ -Yamabe problem is solvable - namely, (1.2) has a positive solution.

Remark 1.2. (1) As pointed out in González-Qing [19], we are only permitted to change the metric on the conformal infinity M . Once the boundary metric \hat{h} is fixed, the geodesic boundary defining function ρ and a compact metric \bar{g} on X are automatically determined by the relations $|d\rho|_{\rho^2 g^+} = 1$ and $\bar{g} = \rho^2 g^+$. This is a huge difference between the fractional Yamabe problem (especially, with $\gamma = 1/2$) and the boundary Yamabe problem in that one has a freedom of conformal change of the metric in the whole manifold X when he/she is concerned with the boundary Yamabe problem.

Due to this reason, while it is possible to make the 'extrinsic' metric H vanish at a point by a conformal change in the boundary Yamabe problem, one cannot do the same thing in the setting of the fractional Yamabe problem. This forced us to separate the cases in the statement of Theorem 1.1.

(2) As a particular consequence of the previous discussion, the Ricci tensor $R_{\rho\rho}[\bar{g}](y)$ of (X, \bar{g}) evaluated at a point y on M is governed by \hat{h} and (1.18) (see Lemma 2.4). In the boundary Yamabe problem [14], the author could choose a metric in X such that the Ricci curvature $R_{ij}[\hat{h}](y) = 0$ of (M, \hat{h}) and $R_{\rho\rho}[\bar{g}](y) = 0$ simultaneously.

Moreover, by putting (1.6) and (1.18) together, we get

$$E(\rho) = \left(\frac{n-2\gamma}{4n} \right) R[\bar{g}] \rho^{1-2\gamma} + o(\rho^{1-2\gamma}) \quad \text{near } M.$$

Hence, on account of the energy expansion, (1.18) is the very condition that makes the boundary Yamabe problem and the 1/2-Yamabe problem identical modulo the remainder. Refer to Subsections 2.3 and 2.4.

(3) The sign of the mean curvature at a fixed point on M and (1.18) are ‘intrinsic’ curvature conditions of an asymptotically hyperbolic manifold in the sense that these properties are independent of the choice of a representative of the class $[\hat{h}]$. Refer to Lemma 2.1 below for its proof. Also Lemma 2.3 claims that (1.18) implies $H = 0$ on M .

(4) Note also that $2 + 2\gamma \in \mathbb{N}$ and $\gamma \in (0, 1)$ if and only if $\gamma = 1/2$, and the boundary Yamabe problem on non-umbilic manifolds in dimension $n = 2 + 2\gamma = 3$ was covered in [37]. We suspect that the strategy suggested in [37] can be applied for 1/2-Yamabe problem in the same setting.

(5) Suppose that $n \in \mathbb{N}$ and $\gamma \in (0, 1)$ satisfy $C'(n, \gamma) > 0$ where $C'(n, \gamma)$ is the quantity defined in (2.12) below. Moreover assume that $(M^n, [\hat{h}])$ is the conformal infinity of an asymptotic hyperbolic manifold (X, g^+) such that (1.9) and (1.18) hold, and the second fundamental form π never vanishes on M . Then the solution set of (1.2) (with $c > 0$) is compact in $C^2(M)$ as shown in [34].

We next consider the case when the boundary M is umbilic but non-locally conformally flat.

Theorem 1.3. *Suppose that $n > 4 + 2\gamma$, $\gamma \in (0, 1)$ (that is, either $n \geq 6$ and $\gamma \in (0, 1)$, or $n = 5$ and $\gamma \in (0, 1/2)$) and (X^{n+1}, g^+) is an asymptotic hyperbolic manifold such that (1.9) holds. Furthermore, assume that $(M^n, [\hat{h}])$ is the umbilic boundary of X^{n+1} and there is a point $y \in M$ such that the Weyl tensor $W[\hat{h}]$ on M is nonzero at y . If*

$$\begin{cases} R[g^+] + n(n+1) = o(\rho^4), \\ \partial_{\bar{x}}^m (R[g^+] + n(n+1)) = o(\rho^2) \quad (m = 1, 2), \\ \partial_{\rho}^m (R[g^+] + n(n+1)) = o(\rho^2) \quad (m = 1, 2) \end{cases} \quad (1.19)$$

as $\rho \rightarrow 0$ uniformly on M , then the γ -Yamabe problem is solvable. Here \bar{x} is a coordinate on M .

Remark 1.4. (1) As we will see later, the main order of the energy for the fractional Yamabe problem (1.2) is ϵ^4 on an umbilic but non-locally conformal flat boundary M , while it is ϵ^2 on a non-umbilic boundary (see (2.11), (2.14), (3.12) and (3.14)). Therefore it is natural to expect that Theorem 1.3 should require that $R[g^+] + n(n+1)$ decays ρ^2 -faster than Theorem 1.1 near M . Compare (1.18) and (1.19).

Assumption (1.19) is responsible for determining all the values of quantities which emerge in the coefficient of ϵ^4 in the energy (such as $R_{,ii}[\bar{g}](y)$ and $R_{NN,ii}[\bar{g}](y)$ - see Lemma 3.2) and controlling the term $(n(n+1) + R[g^+])\rho^{-2}$ in $E(\rho)$ to be ignorable.

(2) In light of Lemmas 2.1 and 2.3, condition (1.19) is again intrinsic and sufficient to deduce that $H = 0$ on M . Moreover every Poincaré-Einstein manifold satisfies (1.19).

(3) It is notable that $4 + 2\gamma \in \mathbb{N}$ and $\gamma \in (0, 1)$ if and only if $\gamma = 1/2$, and the boundary Yamabe problem for $n = 4 + 2\gamma = 5$ was studied in [1]. Hence it is natural to ask whether one can extend Theorem 1.3 for $\gamma = 1/2$ and $n = 5$ by following the perturbation argument given in [1].

In order to describe the last result, we first have to take into account of Green’s function under our setting.

Proposition 1.5. *Suppose that all the hypotheses of Theorem A hold true (including (1.9)) and $H = 0$ on M . In addition, assume further that $\Lambda^\gamma(M, [\hat{h}]) > 0$. Then for each $y \in M$, there exists Green’s function $G(x, y)$ on $\bar{X} \setminus \{y\}$ which satisfies*

$$\begin{cases} -\operatorname{div}_{\bar{g}}(\rho^{1-2\gamma} \nabla G(\cdot, y)) + E(\rho) G(\cdot, y) = 0 & \text{in } (X, \bar{g}), \\ \partial_y^\gamma G(\cdot, y) = \delta_y & \text{on } (M, \hat{h}) \end{cases} \quad (1.20)$$

in the distribution sense where δ_y is the Dirac measure at y . The function G is unique and positive on \bar{X} .

The proof is postponed until Subsection 4.1. The readers may compare the above result with Guillarmou-Qing [23]. Based on the previous proposition and the fact that

$$G(x, \bar{y}) = \frac{g_{n,\gamma}}{|\bar{x} - \bar{y}, x_{n+1}|^{n-2\gamma}} \quad \text{for all } (\bar{x}, x_{n+1}) \in \mathbb{R}_+^{n+1} \text{ and } \bar{y} \in \mathbb{R}^n$$

if (X, \bar{g}) is the Poincaré half-plane $(\mathbb{R}_+^{n+1}, x_{n+1}^{-2} dx)$, we conjecture the following.

Conjecture 1.6. [Positive mass theorem] Assume that $\gamma \in (0, 1)$, $n > 2\gamma$ and (X^{n+1}, g^+) is Poincaré-Einstein. Also suppose that $\Lambda^\gamma(M, [\hat{h}]) > 0$ and either $(M^n, [\hat{h}])$ is locally conformally flat or $n = 2$. Then we have an asymptotic expansion on $G(\cdot, y)$ of the form

$$G(x, y) = g_{n,\gamma} d_{\bar{g}}(x, y)^{-(n-2\gamma)} + A + \Psi(d_{\bar{g}}(x, y)) \quad \text{with } A \geq 0 \quad (1.21)$$

for any $x \in \bar{X}$ near $y \in M$, where $g_{n,\gamma} > 0$ is a constant appeared in (1.13) and Ψ is a function in a small closed neighborhood $\mathcal{N} \subset \mathbb{R}_+^{n+1}$ of 0 such that

$$\Psi(0) = 0 \quad \text{and} \quad \|\Psi\|_{C^{\vartheta_1}(\mathcal{N})} + \|\nabla_{\bar{x}} \Psi\|_{C^{\vartheta_1}(\mathcal{N})} + \left\| x_{n+1}^{1-2\gamma} \frac{\partial \Psi}{\partial x_{n+1}} \right\|_{C^{\vartheta_1}(\mathcal{N})} \leq C \quad (1.22)$$

for some $\vartheta_1 \in (0, 1)$. Furthermore, $A = 0$ if and only if (X^{n+1}, \bar{g}) is conformally diffeomorphic to the standard unit ball \mathbb{B}^{n+1} (which we denote by $(X^{n+1}, \bar{g}) \simeq \mathbb{B}^{n+1}$).

Our expectation on the regularity (1.22) of Ψ is based on the fact that Ψ is ‘approximately’ γ -harmonic near y . Now we can state our third main theorem.

Theorem 1.7. *Suppose that $\gamma \in (0, 1)$, $n > 2\gamma$ and (X^{n+1}, g^+) is a Poincaré-Einstein manifold with conformal infinity $(M^n, [\hat{h}])$. Let ρ be a geodesic defining function for (M, \hat{h}) and $\bar{g} = \rho^2 g^+$. If (1.9) holds, Conjecture 1.6 is valid, and either M^n is locally conformally flat or $n = 2$, then the fractional Yamabe problem is solvable.*

Remark 1.8. (1) Let us set a 2-tensor

$$F = \rho(\text{Ric}[g^+] + ng^+) \quad \text{in } X,$$

which is identically 0 if (X, g^+) is Poincaré-Einstein. As a matter of the fact, if M is locally conformally flat, the only property of the tensor F necessary to derive Theorem 1.7 is that $\partial_\rho^m F|_{\rho=0} = 0$ for $m = 0, \dots, n-1$ (refer to Lemma 4.3). We guess that (1.21), (1.22), and the condition on A are still valid under this assumption. Similarly, for the case $n = 2$, the assumption $\partial_\rho^m F|_{\rho=0} = 0$ for $m = 0, 1$ would suffice.

(2) Since (X^{n+1}, g^+) is Poincaré-Einstein, the second fundamental form on M is trivial. Thus the mean curvature H on M vanishes and M is umbilic.

This paper is organized as follows: In Section 2, we establish Theorem 1.1 by intensifying the ideas of Marques [37] and González-Qing [19]. Section 3 provides the proof of Theorem 1.3 which further develops the approach of Almaraz [1] and González-Wang [20]. In Section 4, Theorem 1.7 is achieved under the validity of the positive mass theorem. In particular, Subsection 4.1 is devoted to investigate the existence, uniqueness, positivity of Green’s function (i.e. Proposition 1.5). Then we are concerned with the case that M is locally conformally flat (in Subsection 4.2) and 2-dimensional (in Subsection 4.3). Finally, we examine the asymptotic behavior of the bubble $W_{1,0}$ near infinity in Appendix A, and compute some integrations regarding $W_{1,0}$ which are needed in the energy expansions in Appendix B.

Notations.

- The Einstein convention is used throughout the paper. The indices i, j, k and l always take values from 1 to n , and a and b range over values from 1 to $n+1$.

- For a tensor T , notations $T_{;a}$ and $T_{,a}$ indicate covariant differentiation and partial differentiation of T , respectively.

- For a tensor T and a number $q \in \mathbb{N}$, we use

$$\text{Sym}_{i_1 \dots i_q} T_{i_1 \dots i_q} = \frac{1}{q!} \sum_{\sigma \in S_q} T_{i_{\sigma(1)} \dots i_{\sigma(q)}}$$

where S_q is the group of all permutations of q elements.

- We denote $N = n + 1$. Also, for $x \in \mathbb{R}_+^N := \{(x_1, \dots, x_n, x_N) \in \mathbb{R}^N : x_N > 0\}$, we write $\bar{x} = (x_1, \dots, x_n, 0) \in \partial \mathbb{R}_+^N \simeq \mathbb{R}^n$ and $r = |\bar{x}|$.

- For $n > 2\gamma$, we set $p = (n + 2\gamma)/(n - 2\gamma)$.

- For any $\varrho > 0$, $B^n(0, \varrho)$ and $B_+^N(0, \varrho)$ are the n -dimensional ball and the N -dimensional upper half-ball centered at 0 whose radius is ϱ , respectively.

- $|\mathbb{S}^{n-1}|$ is the surface area of the $(n - 1)$ -dimensional unit sphere \mathbb{S}^{n-1} .

- For any $t \in \mathbb{R}$, let $t_+ = \max\{0, t\} \geq 0$ and $t_- = \max\{0, -t\} \geq 0$ so that $t = t_+ - t_-$.

- The following positive constants are given in (1.8), (1.13) and (1.14):

$$\kappa_\gamma = \frac{\Gamma(\gamma)}{2^{1-2\gamma}\Gamma(1-\gamma)}, \quad p_{n,\gamma} = \frac{\Gamma\left(\frac{n+2\gamma}{2}\right)}{\pi^{n/2}\Gamma(\gamma)}, \quad g_{n,\gamma} = \frac{\Gamma\left(\frac{n-2\gamma}{2}\right)}{\pi^{n/2}2^{2\gamma}\Gamma(\gamma)}, \quad \alpha_{n,\gamma} = 2^{\frac{n-2\gamma}{2}} \left(\frac{\Gamma\left(\frac{n+2\gamma}{2}\right)}{\Gamma\left(\frac{n-2\gamma}{2}\right)} \right)^{\frac{n-2\gamma}{4\gamma}}. \quad (1.23)$$

- $C > 0$ is a generic constant which may vary from line to line.

2 Non-minimal and Non-umbilic Conformal Infinities

2.1 Geometric Background

We initiate this section by proving that the sign of the mean curvature, (1.18) and non-umbilicity of a point on M are intrinsic conditions.

Lemma 2.1. *Suppose that (X, g^+) be an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{h}])$. Moreover, let ρ and $\tilde{\rho}$ be the geodesic boundary defining functions associated to two representatives \hat{h} and \tilde{h} of the class $[\hat{h}]$, respectively. We also define $\tilde{g} = \rho^2 g^+$ and $\tilde{\tilde{g}} := \tilde{\rho}^2 g^+$, denote by $\pi = -\tilde{g}_{,N}/2$ and $\tilde{\pi}$ the second fundamental forms of $(M, \hat{h}) \subset (\bar{X}, \tilde{g})$ and $(M, \tilde{h}) \subset (\bar{X}, \tilde{\tilde{g}})$, respectively, and set $H = \tilde{g}^{ij}\pi_{ij}/n$ and $\tilde{H} = \tilde{\tilde{g}}^{ij}\tilde{\pi}_{ij}/n$. Then we have*

$$C^{-1} \leq \frac{\tilde{\rho}}{\rho} \leq C \quad \text{in } X \quad \text{and} \quad H = \left(\frac{\tilde{\rho}}{\rho} \right) \Big|_{\rho=0} \tilde{H} \quad \text{on } M \quad (2.1)$$

for some $C > 1$. Furthermore if $H = 0$ on M , then

$$\pi = \left(\frac{\rho}{\tilde{\rho}} \right) \Big|_{\rho=0} \tilde{\pi} \quad \text{on } M. \quad (2.2)$$

Proof. The assertion on H in (2.1) is proved in [19, Lemma 2.3]. For the first inequality in (2.1), it suffices to observe that $\tilde{\rho}/\rho$ is bounded above and bounded away from 0 near M . Indeed, this follows from the fact that

$$\tilde{h} = \tilde{g}|_M = \tilde{\rho}^2 g^+|_M = \left(\frac{\tilde{\rho}}{\rho} \right)^2 \tilde{g}|_M = \left(\frac{\tilde{\rho}}{\rho} \right)^2 \hat{h} \quad \text{on } M.$$

Let us define tensors $T = \pi - H\tilde{g}$ and $\tilde{T} = \tilde{\pi} - \tilde{H}\tilde{\tilde{g}}$ on M . Then we see from [15, Proposition 1.2] that

$$\tilde{\pi} = \tilde{T} = \left(\frac{\tilde{\rho}}{\rho} \right) T = \left(\frac{\tilde{\rho}}{\rho} \right) \pi \quad \text{on } M$$

provided $H = 0$ on M , which confirms (2.2). \square

Given any fixed point $y \in M$, let $\bar{x} = (x_1, \dots, x_n)$ be normal coordinates on M at y (identified with 0) and $x_N = \rho$. In other words, let $x = (\bar{x}, x_N)$ be *Fermi coordinates*. The following lemma provides the expansion of the metric \bar{g} near $y = 0$. See [14, Lemma 3.1] for its proof.

Lemma 2.2. *Suppose that (X, g^+) is an asymptotically hyperbolic manifold and y is an arbitrary point on the conformal infinity $(M, [\hat{h}])$. Then, in terms of Fermi coordinates around y , it holds that*

$$\sqrt{|\bar{g}|}(x) = 1 - nHx_N + \frac{1}{2} \left(n^2 H^2 - \|\pi\|^2 - R_{NN}[\bar{g}] \right) x_N^2 - H_{,i} x_i x_N - \frac{1}{6} R_{ij}[\hat{h}] x_i x_j + O(|x|^3)$$

and

$$\bar{g}^{ij}(x) = \delta_{ij} + 2\pi_{ij} x_N + \frac{1}{3} R_{ikjl}[\hat{h}] x_k x_l + \bar{g}^{ij}_{,Nk} x_N x_k + (3\pi_{ik} \pi_{kj} + R_{iNjN}[\bar{g}]) x_N^2 + O(|x|^3)$$

near y (identified with a small half-ball $B_+^N(0, 2\eta_0)$ near 0 in \mathbb{R}_+^N). Here $\|\pi\|^2 = \hat{h}^{ik} \hat{h}^{jl} \pi_{ij} \pi_{kl}$ is the square of the norm of the second fundamental form π on $(M, \hat{h}) \subset (\bar{X}, \bar{g})$, $R_{ikjl}[\hat{h}]$ is a component of the Riemannian curvature tensor on M , $R_{iNjN}[\bar{g}]$ is that of the Riemannian curvature tensor in X , $R_{ij}[\hat{h}] = R_{ikjk}[\hat{h}]$ and $R_{NN}[\bar{g}] = R_{iNiN}[\bar{g}]$. Every tensor in the expansions is computed at $y = 0$.

Now notice that the transformation law of the scalar curvature (see (1.1) of [14]) implies

$$R[g^+] + n(n+1) = 2n \left(\frac{\partial_\rho \sqrt{|\bar{g}|}}{\sqrt{|\bar{g}|}} \right) \rho + R[\bar{g}] \rho^2. \quad (2.3)$$

It readily shows that (1.18) and (1.19) indicate $H = 0$ on M .

Lemma 2.3. *Suppose that (X, g^+) is an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{h}])$. If $R[g^+] + n(n+1) = o(\rho)$ as $\rho \rightarrow 0$, then $H = 0$ on M .*

Proof. Fix any $y \in M$. By (2.3), we have

$$o(1) = 2n \left(\frac{\partial_\rho \sqrt{|\bar{g}|}(y)}{\sqrt{|\bar{g}|}(y)} \right) + R[\bar{g}](y) \rho + o(1) = -2n^2 H(y) + o(1)$$

as a point tends to y . This implies $H(y) = 0$, and therefore the assertion follows. \square

We next select a good background metric on X under the validity of hypothesis (1.18).

Lemma 2.4. *Let (X, g^+) be an asymptotically hyperbolic manifold such that condition (1.18) holds. Then the conformal infinity $(M, [\hat{h}])$ admits a representative $\hat{h} \in [\hat{h}]$, the geodesic boundary defining function ρ and the metric $\bar{g} = \rho^2 g^+$ satisfying*

$$H = 0 \text{ on } M, \quad R_{ij}[\hat{h}](y) = 0 \quad \text{and} \quad R_{\rho\rho}[\bar{g}](y) = \frac{1-2n}{2(n-1)} \|\pi(y)\|^2 \quad (2.4)$$

for a fixed point $y \in M$.

Proof. According to [35, Theorem 5.2], one may choose a representative \hat{h} of the conformal class $[\hat{h}]$ such that $R_{ij}[\hat{h}](y) = 0$. Besides Lemmas 2.3 and 2.1 assure that $H = 0$ on M for any $\hat{h} \in [\hat{h}]$. Hence assumption (1.18) can be interpreted as

$$\begin{aligned} o(1) &= 2n \left(\frac{\partial_\rho \sqrt{|\bar{g}|}}{\rho \sqrt{|\bar{g}|}} \right) + R[\bar{g}] = \frac{n}{\rho} \bar{g}^{ab} \bar{g}_{ab,\rho} + R[\bar{g}] = n \left(\bar{g}^{ab}_{,\rho} \bar{g}_{ab,\rho} + \bar{g}^{ab} \bar{g}_{ab,\rho\rho} \right) + R[\bar{g}] + o(1) \\ &= -2n \left(R_{\rho\rho}[\bar{g}] + \|\pi\|^2 \right) + \left(2R_{\rho\rho}[\bar{g}] + \|\pi\|^2 + R[\hat{h}] - H^2 \right) + o(1) \end{aligned}$$

as $\rho \rightarrow 0$ where we used $H = 0$ on M for the third equality and the Gauss-Codazzi equation for the fourth equality (see the proof of Lemmas 3.1 and 3.2 of [14]). Taking the limit to $y \in M$, we get

$$0 = 2(1-n)R_{\rho\rho}[\bar{g}](y) + (1-2n)\|\pi(y)\|^2.$$

The third equality of (2.4) is its direct consequence. \square

Lastly, we recall the function E in (1.5) and (1.6). In a collar neighborhood of M where $\rho = x_N$, it can be seen that

$$E(x_N) = \left(\frac{n-2\gamma}{4n} \right) \left[R[\bar{g}] - (n(n+1) + R[g^+])x_N^{-2} \right] x_N^{1-2\gamma} = - \left(\frac{n-2\gamma}{2} \right) \left(\frac{\partial_N \sqrt{|\bar{g}|}}{\sqrt{|\bar{g}|}} \right) x_N^{-2\gamma} \quad (2.5)$$

where the second equality holds because of (2.3).

2.2 Non-minimal Conformal Infinity

Let $y \in M$ be a point identified with $0 \in \mathbb{R}^n$ such that $H(y) < 0$ and $B_+^N(0, 2\eta_0) \subset \mathbb{R}_+^N$ its neighborhood which appeared in Lemma 2.2. Also, we select any smooth radial cut-off function $\psi \in C_c^\infty(\mathbb{R}_+^N)$ such that $\psi = 1$ in $B_+^N(0, \eta_0)$ and 0 in $\mathbb{R}_+^N \setminus B_+^N(0, 2\eta_0)$. In this subsection, we shall show that $\bar{I}_h^\gamma[\psi W_\epsilon] < \Lambda^\gamma(\mathbb{S}^n, [g_c])$ for any $n \geq 2$ and $\gamma \in (0, 1/2)$ where $W_\epsilon = W_{\epsilon,0}$ as before.

Before starting the computation, let us make one useful observation: Assume that $n > m + 2\gamma$ for a certain $m \in \mathbb{N}$. Then we get from (A.3) and (A.4) that

$$\int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |x|^{m+1} |\nabla W_\epsilon|^2 dx = \eta_0^{m-\zeta} \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |x|^{m+\zeta} |\nabla W_\epsilon|^2 dx = O(\epsilon^{m+\zeta}) = o(\epsilon^m) \quad (2.6)$$

by choosing a small number $\zeta > 0$ such that $n > m + 2\gamma + \zeta$.

Proposition 2.5. *Suppose that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold with conformal infinity $(M, [\hat{h}])$ and $y \in M$ be a point such that $H(y) < 0$. Then for any $\epsilon > 0$ small, $n \geq 2$ and $\gamma \in (0, 1/2)$, we have*

$$\begin{aligned} \bar{I}_h^\gamma[\psi W_\epsilon] &\leq \Lambda^\gamma(\mathbb{S}^n, [g_c]) + \underbrace{\epsilon \left[\frac{2n^2 - 2n + 1 - 4\gamma^2}{2(1-2\gamma)} \right] \left[\frac{\kappa_\gamma \int_{\mathbb{R}_+^N} x_N^{2-2\gamma} |\nabla W_1|^2 dx}{\int_{\mathbb{R}^n} w_1^{p+1} dx} \right]}_{>0} H(y) + o(\epsilon) \\ &< \Lambda^\gamma(\mathbb{S}^n, [g_c]) \end{aligned} \quad (2.7)$$

where \bar{I}_h^γ is the γ -Yamabe functional given in (1.11), and $\Lambda^\gamma(\mathbb{S}^n, [g_c])$ and κ_γ are positive constants introduced in (1.16) and (1.23).

Proof. Since the proof is essentially the same as that of [12, Proposition 6.1], we briefly sketch it. By Lemma 2.2 and (2.6), we discover

$$\begin{aligned} &\int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|_{\bar{g}}^2 dv_{\bar{g}} \\ &= \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |\nabla W_1|^2 dx + \epsilon H \left(2 \int_{\mathbb{R}_+^N} x_N^{2-2\gamma} |\nabla_{\bar{x}} W_1|^2 dx - n \int_{\mathbb{R}_+^N} x_N^{2-2\gamma} |\nabla W_1|^2 dx \right) + o(\epsilon) \end{aligned}$$

and

$$\int_M (\psi W_\epsilon)^{p+1} dv_{\hat{h}} = \int_{B^n(0, \eta_0)} w_\epsilon^{p+1} (1 + O(|\bar{x}|^2)) d\bar{x} + O(\epsilon^n) = \int_{\mathbb{R}^n} w_1^{p+1} dx + o(\epsilon).$$

Moreover, according to Lemma 2.2 and (2.5), we have

$$\int_{B_+^N(0, \eta_0)} E(x_N) W_\epsilon^2 dv_{\bar{g}} = \left[\frac{n(n-2\gamma)}{2} \right] \epsilon H \int_{\mathbb{R}_+^N} x_N^{-2\gamma} W_1^2 dx + o(\epsilon).$$

Thus the above estimates and Lemma B.3 confirm (2.7). \square

Unlike the other existence results to be discussed later, we need to assume that $\gamma \in (0, 1/2)$ for Proposition 2.5. Such a restriction is necessary in two reasons: First of all, $\gamma \in (0, 1/2)$ is necessary for the function $x_N^{-2\gamma} W_1^2$ to be integrable in \mathbb{R}_+^N . Secondly the mean curvature H should vanish for $\gamma \in (1/2, 1)$ to guarantee the validity of the extension theorem (Theorem A).

2.3 Non-umbilic Conformal Infinity: Higher Dimensional Cases

We fix a non-umbilic point $y = 0 \in M$. Let also $B_+^N(0, 2\eta_0) \subset \mathbb{R}_+^N$ be a small neighborhood of 0 and $\psi \in C_c^\infty(B_+^N(0, 2\eta_0))$ a cut-off function chosen in the previous subsection.

Lemma 2.6. *Let J_h^γ be the energy functional defined as*

$$J_h^\gamma[U; X] = \int_X (\rho^{1-2\gamma} |\nabla U|_{\bar{g}}^2 + E(\rho)U^2) dv_{\bar{g}} \quad \text{for any } U \in W^{1,2}(X, \rho^{1-2\gamma}). \quad (2.8)$$

Assume also that (2.4) holds. Then for any $\epsilon > 0$ small, $n > 2 + 2\gamma$ and $\gamma \in (0, 1)$, it is valid that

$$J_h^\gamma[\psi W_\epsilon; B_+^N(0, \eta_0)] = \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |\nabla W_1|^2 dx + \epsilon^2 \|\pi\|^2 \left[-\left(\frac{1+b}{2}\right) \mathcal{F}_2 + \left(\frac{3+b}{n}\right) \mathcal{F}_3 + \left(\frac{n-2\gamma}{2}\right) (1+b) \mathcal{F}_1 \right] + o(\epsilon^2) \quad (2.9)$$

where $b := (1 - 2n)/(2n - 2)$, $\|\pi\|$ is the norm of the second fundamental form at $y = 0 \in M$, and the values \mathcal{F}_1 , \mathcal{F}_2 and \mathcal{F}_3 are given in Lemma B.4.

Proof. We borrow the argument presented in [19, Theorem 1.5]. According to Lemma 2.2 and (2.4), there holds that

$$\sqrt{|\bar{g}|}(\bar{x}, x_N) = 1 - \left(\frac{1+b}{2}\right) \|\pi\|^2 x_N^2 + O(|(\bar{x}, x_N)|^3) \quad \text{in } B_+^N(0, \eta_0). \quad (2.10)$$

Hence we obtain with (2.6) that

$$\int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|_{\bar{g}}^2 dv_{\bar{g}} = \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} |\nabla W_\epsilon|^2 dx + \epsilon^2 \left[(3\pi_{ik}\pi_{kj} + R_{iNjN}[\bar{g}]) \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} \partial_i W_1 \partial_j W_1 dx - \left(\frac{1+b}{2}\right) \|\pi\|^2 \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} |\nabla W_1|^2 dx \right] + o(\epsilon^2).$$

Also, in view of (2.5) and (2.10),

$$E(x_N) = \left(\frac{n-2\gamma}{2}\right) (1+b) \|\pi\|^2 x_N^{1-2\gamma} + O(|x|^2 x_N^{-2\gamma})$$

for $x_N \geq 0$ small, so

$$\int_{B_+^N(0, \eta_0)} E(x_N) W_\epsilon^2 dv_{\bar{g}} = \epsilon^2 \left(\frac{n-2\gamma}{2}\right) (1+b) \|\pi\|^2 \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} W_1^2 dx + o(\epsilon^2).$$

Collecting every calculation, we discover (2.9). \square

The previous lemma ensures the existence of a positive solution to (1.2) for non-umbilic conformal infinity M^n with $n \in \mathbb{N}$ sufficiently high.

Corollary 2.7. *Assume that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold and \hat{h} is the representative of the conformal infinity M found in Lemma 2.2. If $n > 2 + 2\gamma$ and $\gamma \in (0, 1)$, we have that*

$$\bar{T}_h^\gamma[\psi W_\epsilon] \leq \Lambda^\gamma(\mathbb{S}^n, [g_c]) - \epsilon^2 C'(n, \gamma) \Lambda^\gamma(\mathbb{S}^n, [g_c])^{-\frac{n-2\gamma}{2\gamma}} \kappa_\gamma |\mathbb{S}^{n-1}| A_3 B_2 \|\pi\|^2 + o(\epsilon^2) \quad (2.11)$$

where the positive constants $\Lambda^\gamma(\mathbb{S}^n, [g_c])$, κ_γ , A_3 and B_2 are introduced in (1.16), (1.23) and (B.3), respectively, and $C'(n, \gamma)$ is the number given by

$$C'(n, \gamma) = \frac{3n^2 + n(16\gamma^2 - 22) + 20(1 - \gamma^2)}{8n(n-1)(1 - \gamma^2)}. \quad (2.12)$$

Proof. Estimate (2.11) comes from Lemmas 2.6, B.4 and the computations made in the proof of [19, Theorem 1.5]. The details are left to the reader. \square

By (2.2), we still have that $\pi \neq 0$ at $y \in M$ even after picking a new representative of the conformal infinity. Furthermore, the number $C'(n, \gamma)$ is positive when $n \geq 4$ for $\gamma > \sqrt{5/11} \simeq 0.674$, $n \geq 5$ for $\gamma > 1/2$, $n \geq 6$ for $\gamma > \sqrt{1/19} \simeq 0.229$ and $n \geq 7$ for any $\gamma > 0$. Hence, in this regime, one is able to deduce the existence of a positive solution of (1.2) by testing the truncated standard bubble into the γ -Yamabe functional.

2.4 Non-umbilic Conformal Infinity: Lower Dimensional Cases

We remind the non-umbilic point $y \in M$ identified with the origin of \mathbb{R}_+^N , the small number $\eta_0 > 0$ and the cut-off function $\psi \in C_c^\infty(\mathbb{R}_+^N)$. Furthermore, we introduce

$$\Psi_\epsilon(\bar{x}, x_N) = M_1 \pi_{ij} x_i x_j x_N r^{-1} \partial_r W_\epsilon = \epsilon \cdot \epsilon^{-\frac{n-2\gamma}{2}} \Psi_1(\epsilon^{-1} \bar{x}, \epsilon^{-1} x_N) \quad (2.13)$$

for each $\epsilon > 0$ where $M_1 \in \mathbb{R}$ is a number to be determined later, π_{ij} 's are the coefficients of the second fundamental form at y and $r = |\bar{x}|$. Our ansatz to deal with lower dimensional cases is defined by

$$\Phi_\epsilon := \psi(W_\epsilon + \Psi_\epsilon) \quad \text{in } X.$$

The definition of Φ_ϵ is inspired by [37].

The main objective of this subsection is to prove

Proposition 2.8. *Suppose that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold. Moreover \hat{h} is the representative of the conformal infinity M satisfying (2.4). If $n > 2 + 2\gamma$ and $\gamma \in (0, 1)$, we have that*

$$\bar{I}_{\hat{h}}^\gamma[\Phi_\epsilon] \leq \Lambda^\gamma(\mathbb{S}^n, [g_c]) - \epsilon^2 C(n, \gamma) \Lambda^\gamma(\mathbb{S}^n, [g_c])^{-\frac{n-2\gamma}{2\gamma}} \kappa_\gamma |\mathbb{S}^{n-1}| |A_3 B_2| |\pi|^2 + o(\epsilon^2) \quad (2.14)$$

where $C(n, \gamma)$ is the number defined by

$$C(n, \gamma) = \frac{3n^2 + n(16\gamma^2 - 22) + 20(1 - \gamma^2)}{8n(n-1)(1 - \gamma^2)} + \frac{16(n-1)(1 - \gamma^2)}{n(3n^2 + n(2 - 8\gamma^2) + 4\gamma^2 - 4)}.$$

It can be checked that $C(n, \gamma) > 0$ whenever $n \geq 4$ and $\gamma \in (0, 1)$. Thus the above proposition justifies the statement of Theorem 1.1. While we have $C(3, \gamma) > 0$ for $\gamma > 1/2$, it also holds that $n > 2 + 2\gamma > 3$. Therefore we get no result for $n = 3$.

Proof of Proposition 2.8. The proof consists of 3 steps.

STEP 1 (ENERGY IN THE HALF-BALL $B_+^N(0, \eta_0)$). Since $\psi = 1$ in $B_+^N(0, \eta_0)$, we discover

$$\begin{aligned} & J_{\hat{h}}^\gamma[\psi(W_\epsilon + \Psi_\epsilon); B_+^N(0, \eta_0)] \\ &= J_{\hat{h}}^\gamma[\psi W_\epsilon; B_+^N(0, \eta_0)] + 2 \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} \langle \nabla W_\epsilon, \nabla \Psi_\epsilon \rangle_{\hat{g}} dv_{\hat{g}} + \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} |\nabla \Psi_\epsilon|^2 dx + o(\epsilon^2) \end{aligned} \quad (2.15)$$

where the functional $J_{\hat{h}}^\gamma$ is defined in (2.8). Moreover, we note from Lemma 2.2 that the mean curvature $H = \pi_{ii}/n$ vanishes at the origin, which yields

$$\begin{aligned} & \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} \nabla W_\epsilon \cdot \nabla \Psi_\epsilon dx \\ &= \epsilon M_1 \int_{B_+^N(0, \eta_0/\epsilon)} x_N^{2-2\gamma} \pi_{ij} x_i x_j [2r^{-2} (\partial_r W_1)^2 + r \partial_r (r^{-1} \partial_r W_1)] dx \\ &+ \epsilon M_1 \int_{B_+^N(0, \eta_0/\epsilon)} x_N^{1-2\gamma} \pi_{ij} x_i x_j r^{-1} (\partial_N W_1) [(\partial_r W_1) + x_N (\partial_{Nr} W_1)] dx \\ &= 0. \end{aligned} \quad (2.16)$$

Hence we obtain from the definition (2.13) of Ψ_ϵ and (2.16) that

$$\begin{aligned}
& 2 \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} \langle \nabla W_\epsilon, \nabla \Psi_\epsilon \rangle_{\bar{g}} dv_{\bar{g}} \\
&= 2 \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} \nabla W_\epsilon \cdot \nabla \Psi_\epsilon dx + 4\pi_{ij} \int_{\mathbb{R}_+^N} x_N^{2-2\gamma} \partial_i W_\epsilon \partial_j \Psi_\epsilon dx + o(\epsilon^2) \\
&= \epsilon^2 4M_1 \pi_{ij} \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} x_i \left[2\pi_{jk} x_k r^{-2} (\partial_r W_1)^2 + \pi_{kl} x_k x_l x_j r^{-2} (\partial_r W_1) \partial_r (r^{-1} \partial_r W_1) \right] dx + o(\epsilon^2) \quad (2.17) \\
&= \epsilon^2 4M_1 \left[\frac{2}{n} \mathcal{F}_3 + \frac{2}{n(n+2)} (-\mathcal{F}_3 + \mathcal{F}_4) \right] \|\pi\|^2 + o(\epsilon^2) \\
&= \epsilon^2 \left(\frac{4}{n} \right) M_1 |\mathbb{S}^{n-1}| A_3 B_2 \|\pi\|^2 + o(\epsilon^2)
\end{aligned}$$

where the constants $\mathcal{F}_3, \mathcal{F}_4$ as well as $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_5, \dots, \mathcal{F}_8$ are defined in Lemma B.4. In a similar fashion, it can be found that

$$\begin{aligned}
\int_{\mathbb{R}_+^N} x_N^{1-2\gamma} |\nabla \Psi_\epsilon|^2 dx &= \epsilon^2 \left[\frac{2M_1^2}{n(n+2)} \right] (\mathcal{F}_3 - 2\mathcal{F}_4 + \mathcal{F}_5 + \mathcal{F}_6 + 2\mathcal{F}_7 + \mathcal{F}_8) \|\pi\|^2 + o(\epsilon^2) \\
&= \epsilon^2 \left[\frac{3n^2 + 2n(1-4\gamma^2) - 4(1-\gamma^2)}{4n(n-1)(1-\gamma^2)} \right] M_1^2 |\mathbb{S}^{n-1}| A_3 B_2 \|\pi\|^2 + o(\epsilon^2). \quad (2.18)
\end{aligned}$$

STEP 2 (ENERGY IN THE HALF-ANNULUS $B_+^N(0, 2\eta_0) \setminus B_+^N(0, \eta_0)$). According to (A.1), (A.3) and (A.4) (cf. (2.6)), it holds

$$J_{\hat{h}}^\gamma [\psi(W_\epsilon + \Psi_\epsilon); X \setminus B_+^N(0, \eta_0)] = o(\epsilon^2). \quad (2.19)$$

Consequently, one deduces from (2.15), (2.17)-(2.19) and Lemma B.4 that

$$J_{\hat{h}}^\gamma [\psi(W_\epsilon + \Psi_\epsilon); X] \leq \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} |\nabla W_1|^2 dx - \epsilon^2 C(n, \gamma) |\mathbb{S}^{n-1}| A_3 B_2 \|\pi\|^2 + o(\epsilon^2) \quad (2.20)$$

by choosing the optimal $M_1 \in \mathbb{R}$.

STEP 3 (COMPLETION OF THE PROOF). Lemma 2.2 and the fact that $\Psi_\epsilon = 0$ on M tell us that

$$\int_M |\psi(W_\epsilon + \Psi_\epsilon)|^{p+1} dv_{\hat{h}} = \int_{B^n(0, 2\eta_0)} (\psi w_\epsilon)^{p+1} (1 + O(|\bar{x}|^3)) d\bar{x} \geq \int_{\mathbb{R}^n} w_1^{p+1} d\bar{x} + o(\epsilon^2). \quad (2.21)$$

Combining (2.20) and (2.21) gives estimate (2.14). The proof is concluded. \square

3 Umbilic Non-locally Conformally Flat Conformal Infinities

3.1 Geometric Background

For a fixed point $y \in M$ identified with $0 \in \mathbb{R}^n$, let $\bar{x} = (x_1, \dots, x_n)$ be the normal coordinate on M at y and $x_N = \rho$. The following expansion of the metric is borrowed from [36].

Lemma 3.1. *Suppose that (X, g^+) is an asymptotically hyperbolic manifold and y is a point in M such that (2.4) holds and $\pi = 0$ on M . Then, in terms of normal coordinates around y , it holds that*

$$\begin{aligned}
\sqrt{|\bar{g}|}(\bar{x}, x_N) &= 1 - \frac{1}{12} R_{ij,k}[\hat{h}] x_i x_j x_k - \frac{1}{2} R_{NN;i}[\bar{g}] x_N^2 x_i - \frac{1}{6} R_{NN;N}[\bar{g}] x_N^3 \\
&\quad - \frac{1}{20} \left(\frac{1}{2} R_{ij,kl}[\hat{h}] + \frac{1}{9} R_{miqj}[\hat{h}] R_{mkql}[\hat{h}] \right) x_i x_j x_k x_l - \frac{1}{4} R_{NN;ij}[\bar{g}] x_N^2 x_i x_j \\
&\quad - \frac{1}{6} R_{NN;Ni}[\bar{g}] x_N^3 x_i - \frac{1}{24} \left[R_{NN;NN}[\bar{g}] + 2(R_{iNjN}[\bar{g}])^2 \right] x_N^4 + O(|(\bar{x}, x_N)|^5)
\end{aligned} \quad (3.1)$$

and

$$\begin{aligned}
\bar{g}^{ij}(\bar{x}, x_N) &= \delta_{ij} + \frac{1}{3}R_{ikjl}[\hat{h}]x_k x_l + R_{iNjN}[\bar{g}]x_N^2 + \frac{1}{6}R_{ikjl;m}[\hat{h}]x_k x_l x_m + R_{iNjN;k}[\bar{g}]x_N^2 x_k \\
&+ \frac{1}{3}R_{iNjN;N}[\bar{g}]x_N^3 + \left(\frac{1}{20}R_{ikjl;mq}[\hat{h}] + \frac{1}{15}R_{iksl}[\hat{h}]R_{jmsq}[\hat{h}] \right) x_k x_l x_m x_q \\
&+ \left(\frac{1}{2}R_{iNjN;kl}[\bar{g}] + \frac{1}{3}\text{Sym}_{ij}(R_{iksl}[\hat{h}]R_{sNjN}[\bar{g}]) \right) x_N^2 x_k x_l + \frac{1}{3}R_{iNjN;kN}[\bar{g}]x_N^3 x_k \\
&+ \frac{1}{12} \left(R_{iNjN;NN}[\bar{g}] + 8R_{iNsN}[\bar{g}]R_{sNjN}[\bar{g}] \right) x_N^4 + O(|(\bar{x}, x_N)|^5)
\end{aligned} \tag{3.2}$$

near y (identified with a small half-ball $B_+^N(0, 2\eta_0)$ near 0 in \mathbb{R}_+^N). Here every tensors are computed at y and the indices m, q and s run from 1 to n as well.

To treat umbilic but non-locally conformally flat boundaries, we also need the following extension of Lemma 2.4.

Lemma 3.2. *For $n \geq 3$, let (X^{n+1}, g^+) be an asymptotically hyperbolic manifold such that the conformal infinity $(M^n, [\hat{h}])$ is umbilic and (1.19) holds. For a fixed point $y \in M$, there exist a representative \hat{h} of the class $[\hat{h}]$, the geodesic boundary defining function ρ ($= x_N$ near M) and the metric $\bar{g} = \rho^2 g^+$ such that*

- (1) $R_{ij;k}[\hat{h}](y) + R_{jk;i}[\hat{h}](y) + R_{ki;j}[\hat{h}](y) = 0$,
- (2) $\text{Sym}_{ijkl} \left(R_{ij;kl}[\hat{h}] + \frac{2}{9}R_{miqj}[\hat{h}]R_{mkql}[\hat{h}] \right) (y) = 0$,
- (3) $\pi = 0$ on M , $R_{NN;N}[\bar{g}](y) = R_{aN}[\bar{g}](y) = 0$,
- (4) $R_{;ii}[\bar{g}](y) = -\frac{n\|W\|^2}{6(n-1)}$, $R_{NN;ii}[\bar{g}](y) = -\frac{\|W\|^2}{12(n-1)}$, $R_{iNjN}[\bar{g}](y) = R_{ij}[\bar{g}](y)$,
- (5) $R_{NN;NN}[\bar{g}](y) = \frac{3}{2n}R_{;NN}[\bar{g}](y) - 2(R_{ij}[\bar{g}](y))^2$,
- (6) $R_{iNjN;ij}[\bar{g}](y) = \left(\frac{3-n}{2n} \right) R_{;NN}[\bar{g}](y) - (R_{ij}[\bar{g}](y))^2 - \frac{\|W\|^2}{12(n-1)}$

if normal coordinates around $y \in (M, \hat{h})$ is assumed. Here $\|W\|$ is the norm of the Weyl tensor of (M, \hat{h}) at y .

Note that the first partial derivatives of \hat{h} and the Christoffel symbols $\Gamma_{ij}^k[\hat{h}] = \Gamma_{ij}^k[\bar{g}]$ at y vanish. Also a simple computation utilizing $\pi = 0$ on M shows that $\Gamma_{aa}^b[\bar{g}] = \Gamma_{bN}^a[\bar{g}] = 0$ on M .

Proof of Lemma 3.2. [35, Theorem 5.2] guarantees the existence of a representative $\hat{h} \in [\hat{h}]$ on M such that (1), (2) and $R_{ij}[\hat{h}](y) = 0$ hold. Furthermore, [15, Proposition 1.2] shows that umbilicity is preserved under the conformal transformation, and so $\pi = 0$ on M . The proof of the remaining identities in (3)-(6) is presented in 2 steps.

STEP 1. By differentiating (2.3) in x_N and using the assumption that $\partial_N(R[g^+] + n(n+1)) = o(x_N^2)$ as $x_N \rightarrow 0$ (see (1.19)), we obtain

$$o(1) = n \left[\frac{\partial_N |\bar{g}|}{|\bar{g}|x_N^2} + \frac{\partial_{NN} |\bar{g}|}{|\bar{g}|x_N} - \frac{(\partial_N |\bar{g}|)^2}{|\bar{g}|^2 x_N} \right] + \frac{2R[\bar{g}]}{x_N} + R_{;N}[\bar{g}] \quad \text{as } x_N \rightarrow 0. \tag{3.3}$$

Also, since we supposed that the mean curvature H vanishes on the umbilic boundary M , we get from (2.4) that $R_{NN}[\bar{g}](y) = \pi(y) = 0$. This in turn gives that $|\bar{g}|(y) = 1$ and $\partial_N |\bar{g}|(y) = \partial_{NN} |\bar{g}|(y) = R[\bar{g}](y) = 0$. Consequently, by taking the limit to y in (3.3), we find that

$$\begin{aligned}
0 &= n \left[\frac{\partial_{NNN} |\bar{g}|(y)}{2} + \partial_{NNN} |\bar{g}|(y) - 0 \right] + 2R_{;N}[\bar{g}](y) + R_{;N}[\bar{g}](y) \\
&= n\partial_{NNN} |\bar{g}|(y) + 2R_{;N}[\bar{g}](y).
\end{aligned} \tag{3.4}$$

Now we observe from Lemma 3.1 that $\partial_{NNN}[\bar{g}](y) = -2R_{NN;N}[\bar{g}](y)$. In addition, by the second Bianchi identity, the Codazzi equation and the fact that $\pi = 0$ on M , one can achieve

$$\begin{aligned} R_{;N}[\bar{g}] &= R_{;N}[\bar{g}] = 2R_{NN;N}[\bar{g}] + R_{ijj;N}[\bar{g}] = 2R_{NN;N}[\bar{g}] + (R_{ijN;j}[\bar{g}] - R_{ijjN;i}[\bar{g}]) \\ &= 2R_{NN;N}[\bar{g}] + 2(\pi_{ii;jj} - \pi_{ij;ij}) = 2R_{NN;N}[\bar{g}] \end{aligned} \quad (3.5)$$

and

$$R_{iN}[\bar{g}] = \pi_{jj;i} - \pi_{ij;j} = 0$$

at $y \in M$. Combining (3.4) and (3.5), we get

$$0 = (2 - n)R_{NN;N}[\bar{g}](y).$$

Since $n \geq 3$, it follows that $R_{NN;N}[\bar{g}](y) = 0$ as we wanted.

STEP 2. It is well-known that $R_{;ii}[\hat{h}](y) = R_{;ii}[\hat{h}](y) = -\|W(y)\|^2/6$ in the normal coordinate around $y \in M$. Therefore the Gauss-Codazzi equation and the fact that $H = \pi = 0$ on M imply

$$R_{;ii}[\bar{g}](y) = 2R_{NN;ii}[\bar{g}](y) - \frac{\|W(y)\|^2}{6} \quad \text{and} \quad R_{iNjN}[\bar{g}](y) = R_{ij}[\bar{g}](y). \quad (3.6)$$

Moreover, since $\Delta_{\bar{x}}(R[g^+] + n(n+1)) = o(x_N^2)$ near $y \in \bar{X}$ (refer to (1.19)), by differentiating (2.3) in x_i twice, dividing the result by x_N^2 and then taking the limit to y , one obtains

$$R_{;ii}[\bar{g}](y) = 2nR_{NN;ii}[\bar{g}](y). \quad (3.7)$$

As a result, putting (3.7) into (3.6) and applying the relations at y

$$R_{;ii}[\bar{g}] = R_{;ii}[\bar{g}] \quad \text{and} \quad R_{NN;ii}[\bar{g}] = R_{NN;ii}[\bar{g}] - 2(\partial_i \Gamma_{iN}^a[\bar{g}])R_{aN}[\bar{g}] \stackrel{\text{by (3)}}{=} R_{NN;ii}[\bar{g}]$$

allow one to find (4).

On the other hand, arguing as before but using the hypothesis that $\partial_{NN}(R[g^+] + n(n+1)) = o(x_N^2)$ near $y \in \bar{X}$ at this time, one derives equalities

$$3R_{;NN}[\bar{g}](y) = -n \partial_{NNNN}[\bar{g}](y) = 2n [R_{NN;NN}[\bar{g}](y) + 2(R_{iNjN}[\bar{g}](y))^2].$$

Because $R_{;NN}[\bar{g}](y) = R_{;NN}[\bar{g}](y)$ and $R_{NN;NN}[\bar{g}](y) = R_{NN;NN}[\bar{g}](y)$, it is identical to (5). Hence the contracted second Bianchi identity, the Ricci identity and (3)-(5) give

$$\begin{aligned} R_{;NN}[\bar{g}] &= 2R_{iN;iN}[\bar{g}] + 2R_{NN;NN}[\bar{g}] = 2 [R_{iN;Ni}[\bar{g}] + (R_{ij}[\bar{g}])^2 - (R_{aN}[\bar{g}])^2] + 2R_{NN;NN}[\bar{g}] \\ &= 2 (R_{iN;Ni}[\bar{g}] + (R_{ij}[\bar{g}])^2) + \left(\frac{3}{n} R_{;NN}[\bar{g}] - 4(R_{ij}[\bar{g}])^2 \right). \end{aligned}$$

at y . Now assertion (6) directly follows from the above equality and

$$R_{iN;Ni}[\bar{g}](y) = R_{Nj;Nj}[\bar{g}](y) = -R_{iNjN;j}[\bar{g}](y) + R_{NN;ii}[\bar{g}](y) = -R_{iNjN;j}[\bar{g}](y) - \frac{\|W(y)\|^2}{12(n-1)}.$$

This finishes the proof. \square

3.2 Computation of the Energy

Like the previous section, we fix a smooth radial cut-off function $\psi \in C_c^\infty(\mathbb{R}_+^N)$ such that $\psi = 1$ in $B_+^N(0, \eta_0)$ and 0 in $\mathbb{R}_+^N \setminus B_+^N(0, 2\eta_0)$. Also, assume that $W_\epsilon = W_{\epsilon,0}$ denotes the bubble defined in (1.13).

Lemma 3.3. *Let $y = 0 \in M$ be any fixed point and J_h^y the functional given in (2.8). If (2.4) and (1)-(6) in Lemma 3.2 are valid, then*

$$\begin{aligned}
& J_h^y[\psi W_\epsilon; B_+^N(0, \eta_0)] \\
&= \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |\nabla W_1|^2 dx + \epsilon^4 \left[\frac{\|W\|^2}{4n} \left(\frac{\mathcal{F}'_5}{12(n-1)} - \frac{\mathcal{F}'_6}{2(n-1)(n+2)} - \frac{(n-2\gamma)\mathcal{F}'_4}{12n} \right) \right. \\
&\quad \left. + \frac{R_{;NN}[\bar{g}]}{2} \left(-\frac{\mathcal{F}'_2}{8n} + \frac{\mathcal{F}'_3}{4n^2} - \frac{(n-3)\mathcal{F}'_6}{n^2(n+2)} + \frac{(n-2\gamma)\mathcal{F}'_1}{4n} \right) + \frac{(R_{ij}[\bar{g}])^2}{n} \left(\frac{\mathcal{F}'_3}{2} - \frac{\mathcal{F}'_6}{n+2} \right) \right] \\
&\quad + o(\epsilon^4)
\end{aligned} \tag{3.8}$$

for any $\epsilon > 0$ small, $n > 4 + 2\gamma$ and $\gamma \in (0, 1)$. Here the tensors are computed at y and the values $\mathcal{F}'_1, \dots, \mathcal{F}'_6$ are given in Lemma B.5.

Proof. STEP 1 (ESTIMATE ON THE SECOND AND THIRD ORDER TERMS). To begin with, we ascertain that

$$J_h^y[\psi W_\epsilon; B_+^N(0, \eta_0)] = \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |\nabla W_1|^2 dx + O(\epsilon^4). \tag{3.9}$$

In fact, since $H = R_{NN}[\bar{g}] = 0$ at y and the bubbles W_ϵ depends only on the variables $|\bar{x}|$ and x_N , we have

$$\begin{aligned}
\int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|^2_{\bar{g}} dv_{\bar{g}} &= \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |\nabla W_1|^2 dx \\
&\quad + \epsilon^3 R_{NN;N}[\bar{g}](y) \left(\frac{1}{3n} \int_{\mathbb{R}_+^N} x_N^{4-2\gamma} |\nabla_{\bar{x}} W_1|^2 dx - \frac{1}{6} \int_{\mathbb{R}_+^N} x_N^{4-2\gamma} |\nabla W_1|^2 dx \right) + O(\epsilon^4).
\end{aligned} \tag{3.10}$$

Moreover, thanks to (1.19), (2.5) and $R[\bar{g}](y) = R_{,N}[\bar{g}](y) = 0$, it holds that

$$\begin{aligned}
& \int_{B_+^N(0, \eta_0)} E(x_N) W_\epsilon^2 dv_{\bar{g}} \\
&= \int_{B_+^N(0, \eta_0)} E(x_N) W_\epsilon^2 dx + O\left(\epsilon^{4+\zeta} \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} W_1^2 |x|^{4+\zeta} dx\right) \\
&= \epsilon^2 \left(\frac{n-2\gamma}{4n} \right) \int_{B_+^N(0, \eta_0/\epsilon)} x_N^{1-2\gamma} \left(R[\bar{g}](y) + \epsilon R_{,a}[\bar{g}](y) x_a + \frac{\epsilon^2}{2} R_{,ab}[\bar{g}](y) x_a x_b \right) W_1^2 dx + o(\epsilon^4) \\
&= \epsilon^4 \left(\frac{n-2\gamma}{4n} \right) \cdot \left[\frac{1}{2n} R_{;ii}[\bar{g}](y) \mathcal{F}'_4 + \frac{1}{2} R_{;NN}[\bar{g}](y) \mathcal{F}'_1 \right] + o(\epsilon^4)
\end{aligned} \tag{3.11}$$

where $\zeta > 0$ is a sufficiently small number. Because $R_{NN;N}[\bar{g}](y) = 0$ by Lemma 3.2 (3), we see from (3.10) and (3.11) that estimate (3.9) is true.

STEP 2 (ESTIMATE ON THE FOURTH ORDER TERMS). Let $\sqrt{|\bar{g}|}^{(4)}$ and $(\bar{g}^{ij})^{(4)}$ be the fourth order terms in the expansions (3.1) and (3.2) of $\sqrt{|\bar{g}|}$ and \bar{g}^{ij} . In view of (2.6), Lemma 3.2 (2) and [4, Corollary 29], one can show that

$$\begin{aligned}
& \int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|^2 \sqrt{|\bar{g}|}^{(4)} dx \\
&= -\epsilon^4 \left[\frac{1}{4n} R_{NN;ii}[\bar{g}](y) \mathcal{F}'_5 + \frac{1}{24} \left(R_{NN;NN}[\bar{g}](y) + 2(R_{iNjN}[\bar{g}](y))^2 \right) \mathcal{F}'_2 \right] + o(\epsilon^4)
\end{aligned}$$

and

$$\int_{B_+^N(0, \eta_0)} x_N^{1-2\gamma} (\bar{g}^{ij})^{(4)} \partial_i W_\epsilon \partial_j W_\epsilon dx = \epsilon^4 \left[\frac{1}{2n(n+2)} \left(R_{NN;ii}[\bar{g}](y) + 2R_{iNjN;ij}[\bar{g}](y) \right) \mathcal{F}'_6 \right]$$

$$+ \frac{1}{12n} \left(R_{NN;NN}[\bar{g}](y) + 8(R_{iN;jN}[\bar{g}](y))^2 \right) \mathcal{F}'_3 \Big] + o(\epsilon^4)$$

(cf. [20, Section 4]). Therefore (2.4), (3.10) and Lemma 3.2 (4)-(6) yield

$$\begin{aligned} & \int_{B_+^N(0,\eta_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|_{\bar{g}}^2 dv_{\bar{g}} \\ &= \int_{B_+^N(0,\eta_0)} x_N^{1-2\gamma} |\nabla W_1|^2 dx + \epsilon^4 \left[\frac{\|W\|^2}{8n(n-1)} \left(\frac{\mathcal{F}'_5}{6} - \frac{\mathcal{F}'_6}{n+2} \right) \right. \\ & \quad \left. + \frac{R_{;NN}[\bar{g}]}{2n} \left(-\frac{\mathcal{F}'_2}{8} + \frac{\mathcal{F}'_3}{4n} - \frac{(n-3)\mathcal{F}'_6}{n(n+2)} \right) + \frac{(R_{ij}[\bar{g}])^2}{n} \left(\frac{\mathcal{F}'_3}{2} - \frac{\mathcal{F}'_6}{n+2} \right) \right] + o(\epsilon^4). \end{aligned}$$

Now (3.11) and the previous estimate lead us to (3.8). The proof is accomplished. \square

Corollary 3.4. *Assume that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold, \hat{h} is the representative of the conformal infinity M in Lemma 3.1 and \bar{I}_h^γ is the γ -Yamabe functional in (1.11). If $n > 4 + 2\gamma$ and $\gamma \in (0, 1)$, we have that*

$$\begin{aligned} \bar{I}_h^\gamma[\psi W_\epsilon] &\leq \Lambda^\gamma(\mathbb{S}^n, [g_c]) + \epsilon^4 \Lambda^\gamma(\mathbb{S}^n, [g_c])^{-\frac{n-2\gamma}{2}} \kappa_\gamma |\mathbb{S}^{n-1}| A_3 B_2 \\ &\quad \times \left(-\|W\|^2 \mathcal{D}'_1(n, \gamma) + R_{;NN}[\bar{g}] \mathcal{D}'_2(n, \gamma) - (R_{ij}[\bar{g}])^2 \mathcal{D}'_3(n, \gamma) \right) + o(\epsilon^4) \end{aligned} \quad (3.12)$$

where the positive constants $\Lambda^\gamma(\mathbb{S}^n, [g_c])$, κ_γ , A_3 and B_2 are introduced in (1.16), (1.23) and (B.3), respectively. Furthermore

$$\begin{aligned} \mathcal{D}'_1(n, \gamma) &= \frac{15n^4 - 135n^3 + 10n^2(43 + 3\gamma - 4\gamma^2) - 180n(3 + \gamma - \gamma^2) + 8(24 + 35 - 30\gamma^2 - 5\gamma^3 + 6\gamma^4)}{480n(n-1)(n-4)(n-4-2\gamma)(n-4+2\gamma)(1-\gamma^2)} > 0, \\ \mathcal{D}'_2(n, \gamma) &= 0 \end{aligned} \quad (3.13)$$

and

$$\mathcal{D}'_3(n, \gamma) = \frac{5n^2 - 4n(13 - 2\gamma^2) + 28(4 - \gamma^2)}{5n(n-4)(n-4-2\gamma)(n-4+2\gamma)}.$$

Proof. By Lemmas 3.1 and 3.2 (1)-(2), it holds that

$$\begin{aligned} & \int_M (\psi W_\epsilon)^{p+1} dv_{\hat{h}} \\ &= \int_{B^n(0,\eta_0)} w_\epsilon^{p+1} \left[1 - \frac{1}{40} \left(R_{i,j,kl}[\hat{h}] + \frac{2}{9} R_{miqj}[\hat{h}] R_{mkql}[\hat{h}] \right) x_i x_j x_k x_l + O(|\bar{x}|^5) \right] d\bar{x} + O(\epsilon^n) \\ &= \int_{\mathbb{R}^n} w_1^{p+1} d\bar{x} + o(\epsilon^4). \end{aligned}$$

Thus the conclusion follows from Lemmas 3.3 and B.5 at once. \square

It is interesting to observe that the quantity $R_{;NN}[\bar{g}](y)$ does not contribute to the existence of a least energy solution, since the coefficient of $R_{;NN}[\bar{g}](y)$, denoted by $\mathcal{D}'_2(n, \gamma)$, is always zero for any n and γ . Such a phenomenon has been already observed in the boundary Yamabe problem [36]. We also observe that the number $\mathcal{D}'_3(n, \gamma)$ has a nonnegative sign in some situations: when $n = 7$ and $\gamma \in [1/2, 1)$, or $n \geq 8$ and $\gamma \in (0, 1)$. In order to cover lower dimensional cases, we need a more refined test function.

Let $y \in M$ be a point such that $W[\hat{h}](y) \neq 0$. Motivated by [1], we define functions

$$\tilde{\Psi}_\epsilon = \Psi_\epsilon(\bar{x}, x_N) = M_2 R_{iN;jN}[\bar{g}] x_i x_j x_N^2 r^{-1} \partial_r W_\epsilon = \epsilon^2 \cdot \epsilon^{-\frac{n-2\gamma}{2}} \tilde{\Psi}_1(\epsilon^{-1} \bar{x}, \epsilon^{-1} x_N)$$

for some $M_2 \in \mathbb{R}$ and

$$\tilde{\Phi}_\epsilon := \psi(W_\epsilon + \tilde{\Psi}_\epsilon) \quad \text{in } X.$$

Proposition 3.5. *Suppose that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold. Moreover \hat{h} is the representative of the conformal infinity M satisfying (2.4) and Lemma 3.2 (1)-(6). If $n > 4 + 2\gamma$ and $\gamma \in (0, 1)$, we have*

$$\begin{aligned} \bar{I}_{\hat{h}}^\gamma[\tilde{\Phi}_\epsilon] &\leq \Lambda^\gamma(\mathbb{S}^n, [g_c]) + \epsilon^4 \Lambda^\gamma(\mathbb{S}^n, [g_c])^{-\frac{n-2\gamma}{2\gamma}} \kappa_\gamma |\mathbb{S}^{n-1}| A_3 B_2 \\ &\quad \times \left(-\|W\|^2 \mathcal{D}_1(n, \gamma) + R_{;NN}[\bar{g}] \mathcal{D}_2(n, \gamma) - (R_{ij}[\bar{g}])^2 \mathcal{D}_3(n, \gamma) \right) + o(\epsilon^4) \end{aligned} \quad (3.14)$$

where

$$\mathcal{D}_1(n, \gamma) = \mathcal{D}'_1(n, \gamma), \quad \mathcal{D}_2(n, \gamma) = 0$$

(see (3.13) for the definition of the positive constant $\mathcal{D}'_1(n, \gamma)$) and

$$\mathcal{D}_3(n, \gamma) = \frac{25n^3 - 20n^2(9 - \gamma^2) + 100n(4 - \gamma^2) - 16(4 - \gamma^2)^2}{5n(n - 4 - 2\gamma)(n - 4 + 2\gamma)(5n^2 - 4n(1 + \gamma^2) - 8(4 - \gamma^2))}.$$

Proof. Since $R_{NN}[\bar{g}](y) = 0$, we obtain

$$\begin{aligned} J_{\hat{h}}^\gamma[\tilde{\Phi}_\epsilon; B_+^N(0, \eta_0)] &= J_{\hat{h}}^\gamma[\psi W_\epsilon; B_+^N(0, \eta_0)] + 2 \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} (\bar{g}^{ij} - \delta^{ij}) \partial_i W_\epsilon \partial_j \tilde{\Psi}_\epsilon dx \\ &\quad + \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} |\nabla \tilde{\Psi}_\epsilon|^2 dx + o(\epsilon^4). \end{aligned} \quad (3.15)$$

Also a tedious computation with Lemmas 3.1 and 3.2 (4) reveals that the second term of the right-hand side of (3.15) is equal to

$$\begin{aligned} \frac{2}{3} R_{ikjl}[\hat{h}] \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} x_k x_l \partial_i W_\epsilon \partial_j \tilde{\Psi}_\epsilon dx + 2 R_{iNjN}[\bar{g}] \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} \partial_i W_\epsilon \partial_j \tilde{\Psi}_\epsilon dx + o(\epsilon^4) \\ = 0 + \epsilon^4 4M_2 \left[\frac{1}{n} \mathcal{F}'_3 + \frac{1}{n(n+2)} (-\mathcal{F}'_3 + \mathcal{F}'_7) \right] (R_{ij}[\bar{g}])^2 + o(\epsilon^4) \end{aligned}$$

and it holds that

$$\int_{\mathbb{R}_+^N} x_N^{1-2\gamma} |\nabla \tilde{\Psi}_\epsilon|^2 dx = \epsilon^4 \left[\frac{2M_2^2}{n(n+2)} \right] (\mathcal{F}'_3 - 2\mathcal{F}'_7 + \mathcal{F}'_8 + 4\mathcal{F}'_6 + 4\mathcal{F}'_9 + \mathcal{F}'_{10}) (R_{ij}[\bar{g}])^2 + o(\epsilon^4)$$

(cf. (2.17) and (2.18)). Here the constants $\mathcal{F}'_1, \dots, \mathcal{F}'_{10}$ are defined in Lemma B.5.

On the other hand, we have

$$J_{\hat{h}}^\gamma[\tilde{\Phi}_\epsilon; X \setminus B_+^N(0, \eta_0)] = o(\epsilon^4),$$

and since $\tilde{\Psi}_\epsilon = 0$ on M , the integral of $|\tilde{\Phi}_\epsilon|^{p+1}$ over the boundary M does not contribute to the fourth order term in the right-hand side of (3.14). By combining all information, employing Lemma B.5 and selecting the optimal $M_2 \in \mathbb{R}$, we complete the proof. \square

One can verify that $\mathcal{D}_3(n, \gamma) > 0$ whenever $n > 4 + 2\gamma$ and $\gamma \in (0, 1)$. Consequently we deduce Theorem 1.3 from the previous proposition.

4 Locally Conformally Flat or 2-dimensional Conformal Infinities

4.1 Analysis of Green's function

In this subsection, we prove Proposition 1.5. By Theorem A, solvability of problem (1.20) for each $y \in M$ is equivalent to the existence of a solution G^* to the equation

$$\begin{cases} -\operatorname{div}_{\bar{g}^*} \left((\rho^*)^{1-2\gamma} \nabla G^*(\cdot, y) \right) = 0 & \text{in } (X, \bar{g}^*), \\ \partial_\nu^\gamma G^*(\cdot, y) = \delta_y - Q_{\hat{h}}^\gamma G^*(\cdot, y) & \text{on } (M, \hat{h}), \end{cases}$$

and it holds that $|\bar{g}_{iN}^*| + |\bar{g}_{NN}^* - 1| = O(\rho^{2\gamma})$. We also recall [19, Corollary 4.3] which states that if $\Lambda^\gamma(M, [\hat{h}]) > 0$, then M admits a metric $\hat{h}_0 \in [\hat{h}]$ such that $Q_{\hat{h}_0}^\gamma > 0$ on M . Thanks to the following lemma, it suffices to show Proposition 1.5 for $\hat{h}_0 \in [\hat{h}]$.

Lemma 4.1. *Let (X, g^+) be any conformally compact Einstein manifold with conformal infinity $(M, [\hat{h}])$, ρ the geodesic defining function of M in X and $\bar{g} = \rho^2 g^+$. For any positive smooth function w on M , define a new metric $\hat{h}_w = w^{\frac{4}{n-2\gamma}} \hat{h}$, denote the corresponding geodesic boundary defining function by ρ_w and set $\bar{g}_w = \rho_w^2 g^+$. Suppose that $G = G(x, y)$ solves (1.20). Then the function*

$$G_w(x, y) := \left(\frac{\rho(x)}{\rho_w(x)} \right)^{\frac{n-2\gamma}{2}} w^{\frac{n+2\gamma}{n-2\gamma}}(y) G(x, y) \quad \text{for } (x, y) \in \bar{X} \times M, x \neq y$$

again satisfies (1.20) with (\bar{g}_w, \hat{h}_w) and ρ_w substituted for (\bar{g}, \hat{h}) and ρ , respectively.

Proof. By (1.5), the first equality in (1.20) is re-expressed as

$$L_{\bar{g}} \left(\rho^{\frac{1-2\gamma}{2}} G(\cdot, y) \right) + \left(\gamma^2 - \frac{1}{4} \right) \rho^{-\left(\frac{3+2\gamma}{2}\right)} G(\cdot, y) = 0 \quad \text{in } (X, \bar{g}) \quad (4.1)$$

where $L_{\bar{g}}$ is the conformal Laplacian in (X, \bar{g}) defined in (1.3). Therefore one observes from (1.1) that G_w is a solution of (4.1) if \bar{g} and ρ are replaced with \bar{g}_w and ρ_w , respectively. Also, since $w = (\rho_w/\rho)^{(n-2\gamma)/2}$ on M , we see

$$\begin{aligned} \partial_v^\gamma G_w(\cdot, y) &= P_{\hat{h}_w}^\gamma G_w(\cdot, y) = w^{\frac{n+2\gamma}{n-2\gamma}}(y) P_{\frac{4}{w^{\frac{n-2\gamma}{n-2\gamma}} \hat{h}}}^\gamma \left((\rho/\rho_w)^{\frac{n-2\gamma}{2}} G(\cdot, y) \right) \\ &= w^{\frac{n+2\gamma}{n-2\gamma}}(y) P_{\frac{4}{w^{\frac{n-2\gamma}{n-2\gamma}} \hat{h}}}^\gamma \left(w^{-1} G(\cdot, y) \right) = w^{\frac{n+2\gamma}{n-2\gamma}}(y) w^{-\frac{n+2\gamma}{n-2\gamma}} P_{\hat{h}}^\gamma(G(\cdot, y)) \\ &= w^{\frac{n+2\gamma}{n-2\gamma}}(y) w^{-\frac{n+2\gamma}{n-2\gamma}} \partial_v^\gamma(G(\cdot, y)) = w^{\frac{n+2\gamma}{n-2\gamma}}(y) w^{-\frac{n+2\gamma}{n-2\gamma}} \delta_y = \delta_y \quad \text{on } M \end{aligned}$$

where we have applied Theorem A and (1.1) for the first, fourth and fifth equalities. \square

For brevity, we write $\hat{h} = \hat{h}_0$, $\bar{g} = \bar{g}^*$, $\rho = \rho^*$ and $G = G^*$ here and henceforth. Further, recalling that $Q_{\hat{h}}^\gamma > 0$ on M , let us define a norm

$$\|U\|_{\mathcal{W}^{1,q}(X; \rho^{1-2\gamma})} = \left(\int_X \rho^{1-2\gamma} |\nabla U|_{\bar{g}}^q dv_{\bar{g}} + \int_M Q_{\hat{h}}^\gamma U^q dv_{\hat{h}} \right)^{1/q}$$

for any $q \geq 1$ and set a space $\mathcal{W}^{1,q}(X; \rho^{1-2\gamma})$ as the completion of $C_c^\infty(\bar{X})$ with respect to the above norm.

Given any bounded Radon measure f (such as the dirac measures), we say that a function $U \in \mathcal{W}^{1,q}(X; \rho^{1-2\gamma})$ is a *weak solution* of

$$\begin{cases} -\operatorname{div}_{\bar{g}}(\rho^{1-2\gamma} \nabla U) = 0 & \text{in } (X, \bar{g}), \\ \partial_v^\gamma U + Q_{\hat{h}}^\gamma U = f & \text{on } (M, \hat{h}), \end{cases} \quad (4.2)$$

if it is satisfied that

$$\int_X \rho^{1-2\gamma} \langle \nabla U, \nabla \Psi \rangle_{\bar{g}} dv_{\bar{g}} + \int_M Q_{\hat{h}}^\gamma U \Psi dv_{\hat{h}} = \int_M f \Psi \quad (4.3)$$

for any $\Psi \in C^1(\bar{X})$.

The $\mathcal{W}^{1,2}(X; \rho^{1-2\gamma})$ -norm is equivalent to the standard weighted Sobolev norm $\|U\|_{W^{1,2}(X; \rho^{1-2\gamma})}$ (see [12, Lemma 3.1]). Thus for any fixed $f \in (H^\gamma(M))^*$, the existence and uniqueness of a solution $U \in W^{1,2}(X; \rho^{1-2\gamma})$ to (4.2) are guaranteed by the Riesz representation theorem.

Lemma 4.2. Assume that $n > 2\gamma$, $f \in (H^\gamma(M))^*$ and $1 \leq \alpha < \min\{\frac{n}{n-2\gamma}, \frac{2n+2}{2n+1}\}$. Then there exists a constant $C = C(\bar{X}, g^+, \rho, n, \gamma, \alpha)$ such that

$$\|U\|_{\mathcal{W}^{1,\alpha}(X;\rho^{1-2\gamma})} \leq C\|f\|_{L^1(M)} \quad (4.4)$$

for a weak solution $U \in W^{1,2}(X;\rho^{1-2\gamma})$ to (4.2). As a result, if f is the dirac measure δ_y at $y \in M$, then (4.2) has a unique nonnegative weak solution $G(\cdot, y) \in \mathcal{W}^{1,\alpha}(X;\rho^{1-2\gamma})$.

Proof. STEP 1. We are going to verify estimate (4.4) by suitably modifying the argument in [6, Section 5]. To this aim, we consider the formal adjoint of (4.2): Given any $h_0 \in L^q(M)$ and $H_1, \dots, H_N \in L^q(X;\rho^{1-2\gamma})$ for some $q > \max\{\frac{n}{2\gamma}, 2(n+1)\}$, we study a function V such that

$$\int_X \rho^{1-2\gamma} \langle \nabla V, \nabla \Psi \rangle_{\bar{g}} dv_{\bar{g}} + \int_M Q_{\bar{h}}^\gamma V \Psi dv_{\bar{h}} = \int_M h_0 \Psi dv_{\bar{h}} + \sum_{a=1}^N \int_X \rho^{1-2\gamma} H_a \partial_a \Psi dv_{\bar{g}} \quad (4.5)$$

for any $\Psi \in C^1(\bar{X})$. Indeed, by the Lax-Milgram theorem, (4.5) possesses a unique solution $V \in W^{1,2}(X;\rho^{1-2\gamma})$. Moreover, as will be seen in Step 3 below, it turns out that V satisfies

$$\|V\|_{L^\infty(M)} + \|V\|_{L^\infty(X)} \leq C \left(\|h_0\|_{L^q(M)} + \sum_{a=1}^N \|H_a\|_{L^q(X;\rho^{1-2\gamma})} \right). \quad (4.6)$$

Therefore taking $\Psi = U$ in (4.3) (which is allowed to do thanks to the density argument) and employing (4.6), we find

$$\begin{aligned} \int_M U h_0 dv_{\bar{h}} + \sum_{a=1}^N \int_X \rho^{1-2\gamma} \partial_a U H_a dv_{\bar{g}} &= \int_M f V dv_{\bar{h}} \leq \|f\|_{L^1(M)} \|V\|_{L^\infty(M)} \\ &\leq C \|f\|_{L^1(M)} \left(\|h_0\|_{L^q(M)} + \sum_{a=1}^N \|H_a\|_{L^q(X;\rho^{1-2\gamma})} \right). \end{aligned}$$

This implies the validity of (4.4) with $\alpha = q'$ where q' designates the Hölder conjugate of q .

STEP 2. Assume now that $f = \delta_y$ for some $y \in M$. Then one is capable of constructing a sequence $\{f_m\}_{m \in \mathbb{N}} \subset C^1(M)$ with an approximation to the identity or a mollifier so that $f_m \geq 0$ on M ,

$$\sup_{m \in \mathbb{N}} \|f_m\|_{L^1(M)} \leq C, \quad f_m \rightarrow 0 \text{ in } C_{\text{loc}}^1(M \setminus \{y\}) \quad \text{and} \quad f_m \rightarrow \delta_y \text{ in the distributional sense.}$$

Denote by $\{U_m\}_{m \in \mathbb{N}} \subset W^{1,2}(X;\rho^{1-2\gamma})$ a sequence of the corresponding weak solutions to (4.2). By (4.4) and elliptic regularity, there exist a function $G(\cdot, y)$ and a number $\varepsilon_0 \in (0, 1)$ such that $U_m \rightarrow G(\cdot, y)$ weakly in $\mathcal{W}^{1,\alpha}(X;\rho^{1-2\gamma})$ and $U_m \rightarrow G(\cdot, y)$ in $C_{\text{loc}}^{\varepsilon_0}(\bar{X} \setminus \{y\})$. It is a simple task to confirm that $G(\cdot, y)$ satisfies (4.3).

Also, putting $(U_m)_- \in W^{1,2}(X;\rho^{1-2\gamma})$ into (4.3) yields $U_m \geq 0$ in X , which in turn gives $G(\cdot, y) \geq 0$ in X . Finally, it is easy to see that the uniqueness of $G(\cdot, y)$ comes as a consequence of (4.4). This completes the proof of the lemma except (4.6).

STEP 3 (JUSTIFICATION OF ESTIMATE (4.6)). We shall apply Moser's iteration technique so as to get (4.6). Set

$$\zeta_0 = \|h_0\|_{L^q(M)} + \sum_{a=1}^N \|H_a\|_{L^q(X;\rho^{1-2\gamma})} \quad \text{if } (h_0, H_1, \dots, H_N) \neq (0, 0, \dots, 0).$$

Otherwise let ζ_0 be any positive number which we will make $\zeta_0 \rightarrow 0$ eventually. Then we define $\bar{V} = V_+ + \zeta_0$ and

$$\bar{V}_\ell = \begin{cases} \bar{V} & \text{if } V < \ell, \\ \ell + \zeta_0 & \text{if } V \geq \ell \end{cases}$$

for each $\ell > 0$. Testing

$$\Psi = \overline{V}_\ell^{\beta-1} \overline{V} - \zeta_0^\beta \in W^{1,2}(X; \rho^{1-2\gamma})$$

in (4.5) for a fixed exponent $\beta \geq 1$ shows that

$$\begin{aligned} & \frac{1}{2\beta} \left(\int_X \rho^{1-2\gamma} |\nabla \overline{V}_\ell|_{\overline{g}}^2 dv_{\overline{g}} + \int_M Q_h^\gamma \overline{V}_\ell^2 dv_h \right) \\ & \leq 2 \int_M Q_h^\gamma \overline{V}_\ell^2 dv_h + \frac{1}{\zeta_0} \int_M |h_0| \overline{V}_\ell^2 dv_h + \int_X \rho^{1-2\gamma} \left(\sum_{a=1}^N |H_a| \right) \left| \nabla \left(\overline{V}_\ell^{\beta-1} \overline{V} \right) \right|_{\overline{g}} dv_{\overline{g}} \end{aligned} \quad (4.7)$$

where $\widetilde{V}_\ell := \overline{V}_\ell^{\frac{\beta-1}{2}} \overline{V}$. Then one sees that (4.7) is reduced to

$$\begin{aligned} & \frac{1}{4\beta} \left(\int_X \rho^{1-2\gamma} |\nabla \widetilde{V}_\ell|_{\overline{g}}^2 dv_{\overline{g}} + \int_M Q_h^\gamma \widetilde{V}_\ell^2 dv_h \right) \\ & \leq 2 \int_M Q_h^\gamma \widetilde{V}_\ell^2 dv_h + \frac{1}{\zeta_0} \int_M |h_0| \widetilde{V}_\ell^2 dv_h + \frac{4\beta^2}{\zeta_0^2} \int_X \rho^{1-2\gamma} \left(\sum_{a=1}^N |H_a|^2 \right) \widetilde{V}_\ell^2 dv_{\overline{g}}. \end{aligned} \quad (4.8)$$

Besides an application of the Sobolev inequality and the Sobolev trace inequality (see [17, 45]) yields

$$\left(\int_M \widetilde{V}_\ell^{p+1} dv_h \right)^{\frac{2}{p+1}} + \left(\int_X \rho^{1-2\gamma} \widetilde{V}_\ell^{\frac{2(n+1)}{n}} dv_{\overline{g}} \right)^{\frac{n}{n+1}} \leq C \left[\int_X \rho^{1-2\gamma} |\nabla \widetilde{V}_\ell|_{\overline{g}}^2 dv_{\overline{g}} + \int_M Q_h^\gamma \widetilde{V}_\ell^2 dv_h \right], \quad (4.9)$$

while Hölder's inequality gives

$$\begin{aligned} & \frac{1}{\zeta_0} \int_M |h_0| \widetilde{V}_\ell^2 dv_h + \frac{4\beta^2}{\zeta_0^2} \int_X \rho^{1-2\gamma} \left(\sum_{a=1}^N |H_a|^2 \right) \widetilde{V}_\ell^2 dv_{\overline{g}} \\ & \leq \left[\delta_1^{\frac{1}{\theta_1}} \left(\int_M \widetilde{V}_\ell^{p+1} dv_h \right)^{\frac{2}{p+1}} + \delta_1^{-\frac{1}{1-\theta_1}} \left(\int_M \widetilde{V}_\ell^2 dv_h \right) \right] \\ & \quad + 4\beta^2 \left[\delta_2^{\frac{1}{\theta_2}} \left(\int_X \rho^{1-2\gamma} \widetilde{V}_\ell^{\frac{2(n+1)}{n}} dv_{\overline{g}} \right)^{\frac{n}{n+1}} + \delta_2^{-\frac{1}{1-\theta_2}} \left(\int_X \rho^{1-2\gamma} \widetilde{V}_\ell^2 dv_{\overline{g}} \right) \right] \end{aligned} \quad (4.10)$$

for any small $\delta_1, \delta_2 > 0$ and some $\theta_1, \theta_2 \in (0, 1)$ satisfying

$$\frac{2\theta_1}{p+1} + (1-\theta_1) = \frac{n\theta_2}{n+1} + (1-\theta_2) = \frac{1}{q'}.$$

Note that such numbers θ_1 and θ_2 exist because of the assumption that $q > \max\{\frac{n}{2}, 2(n+1)\}$. Collecting (4.8)-(4.10) and applying Lebesgue's monotone convergence theorem, we arrive at

$$\begin{aligned} & \left(\int_M \overline{V}^{(\beta+1) \cdot (\frac{p+1}{2})} dv_h \right)^{\frac{2}{p+1}} + \left(\int_X \rho^{1-2\gamma} \overline{V}^{(\beta+1) \cdot (\frac{n+1}{n})} dv_{\overline{g}} \right)^{\frac{n}{n+1}} \\ & \leq C\beta \left[\left(2 + \beta^{\frac{\theta_1}{1-\theta_1}} \right) \left(\int_M \overline{V}^{\beta+1} dv_h \right) + \beta^{\frac{3\theta_2}{1-\theta_2}} \left(\int_X \rho^{1-2\gamma} \overline{V}^{\beta+1} dv_{\overline{g}} \right) \right] \end{aligned}$$

for a constant $C > 0$ independent of the choice of β . Consequently, the standard iteration argument (considering also the replacement of V with $-V$) reveals that there exists $C > 0$ depending only on \overline{X} , g^+ , ρ , n , γ , $\tilde{\alpha}$ and q for each $\tilde{\alpha} \geq 2$ such that

$$\|V\|_{L^\infty(M)} + \|V\|_{L^\infty(X)} \leq C \left(\|V\|_{L^{\tilde{\alpha}}(M)} + \|V\|_{L^{\tilde{\alpha}}(X; \rho^{1-2\gamma})} + \|h_0\|_{L^q(M)} + \sum_{a=1}^N \|H_a\|_{L^q(X; \rho^{1-2\gamma})} \right). \quad (4.11)$$

Now (4.6) is achieved in view of the compactness of the Sobolev embedding $W^{1,2}(X; \rho^{1-2\gamma}) \hookrightarrow L^2(X; \rho^{1-2\gamma})$ (refer to [24] and [32, Corollary A.1]), that of the trace operator $W^{1,2}(X; \rho^{1-2\gamma}) \hookrightarrow L^{p+1-\varepsilon_1}(M)$ for any small $\varepsilon_1 > 0$, the coercivity of the bilinear form in the left-hand side of (4.5) and the assumption $q > 2(n+1) \geq 2$. \square

Completion of the proof of Proposition 1.5. The existence and nonnegativity of Green's function G is deduced in the previous lemma. Owing to Hopf's lemma (cf. [19, Theorem 3.5]), G is positive on the compact manifold \bar{X} . Remind that the coercivity of (4.3) implies the uniqueness of G . The proof is finished. \square

4.2 Locally Conformally Flat Case

This subsection is devoted to provide the proof of Theorem 1.7 under the hypothesis that M is locally conformally flat. Since the explicit solutions are known when $(X^{n+1}, \bar{g}) \simeq \mathbb{B}^{n+1}$, we shall exclude such a case throughout the section.

Pick any point $y \in M$. Since it is supposed to be locally conformally flat, we can assume that y is the origin in \mathbb{R}^N and identify a neighborhood \mathcal{U} of y in M with a Euclidean ball $B^n(0, \varrho_1)$ for some $\varrho_1 > 0$ small (namely, $\hat{h}_{ij} = \delta_{ij}$ in $\mathcal{U} = B^n(0, \varrho_1)$). Write x_N to denote the geodesic defining function ρ for the boundary M near y . Then we have smooth symmetric n -tensors $h^{(1)}, \dots, h^{(n-1)}$ on $B^n(0, \varrho_1)$ such that

$$\bar{g} = h_{x_N} \oplus dx_N^2 \quad \text{where } (h_{x_N})_{ij}(\bar{x}, x_N) = \delta_{ij} + \sum_{m=1}^{n-1} h_{ij}^{(m)}(\bar{x}) x_N^m + O(x_N^n) \quad (4.12)$$

for $(\bar{x}, x_N) \in \mathcal{R}^N(\varrho_1, \varrho_2) := B^n(0, \varrho_1) \times [0, \varrho_2] \subset \bar{X}$ where $\varrho_2 > 0$ is a number small enough. In fact, as we will see shortly, the local conformal flatness on M and the assumption that X is Poincaré-Einstein together imply that all low-order tensors $h^{(m)}$ which can be locally determined should vanish.

Lemma 4.3. *If (X, g^+) is Poincaré-Einstein, we have $h^{(m)} = 0$ in (4.12) for each $m = 1, \dots, n-1$.*

Proof. We adapt the idea in [19, Lemma 7.7] and [20, Lemma 2.2]. According to (2.5) of [21], it holds that

$$\begin{aligned} x_N h_{ij,NN} + (1-n)h_{ij,N} - h^{kl} h_{kl,N} h_{ij} - x_N h^{kl} h_{ik,N} h_{jl,N} + \frac{1}{2} x_N h^{kl} h_{kl,N} h_{ij,N} - 2x_N R_{ij}[h] \\ = -2x_N (R_{ij}[g^+] + n g^+) = 0 \end{aligned} \quad (4.13)$$

for $h := h_{x_N}$. Here the first equality is true for any metric \bar{g} satisfying (4.12), whereas the second equality holds because (X, g^+) is Poincaré-Einstein. Putting $x_N = 0$ in (4.13), we get

$$(1-n) h_{ij,N} - \hat{h}^{kl} h_{kl,N} \hat{h}_{ij} = 0,$$

from which we observe

$$(1-n) \operatorname{tr}_{\hat{h}} h_{,N} - n \operatorname{tr}_{\hat{h}} h_{,N} = (1-2n) \operatorname{tr}_{\hat{h}} h_{,N} = 0.$$

It follows that the trace $\operatorname{tr}_{\hat{h}} h_{,N}$ is 0, and eventually, one finds $h_{,N} = h^{(1)} = 0$ on $\{x_N = 0\}$.

On the other hand, it holds that $R_{ij}[\hat{h}] = 0$ on $\{x_N = 0\}$, for $\hat{h}_{ij} = \delta_{ij}$. Thus, by differentiating the both sides of (4.13) in x_N and taking $x_N = 0$, we obtain

$$(2-n) h_{ij,NN} - \hat{h}^{kl} h_{kl,NN} \hat{h}_{ij} = 0 \quad \text{and} \quad (2-2n) \operatorname{tr}_{\hat{h}} h_{,NN} = 0,$$

which again gives $h_{ij,NN} = h^{(2)} = 0$ on $\{x_N = 0\}$.

Analogously, if we differentiating (4.13) $(m-1)$ -times ($m = 3, \dots, n$) and putting $x_N = 0$, then we have

$$(m-n) \partial_N^m h_{ij} - \hat{h}^{kl} (\partial_N^m h_{kl}) \hat{h}_{ij} = 0.$$

This gives $\partial_N^m h_{ij} = h^{(m)} = 0$ on $\{x_N = 0\}$, proving the lemma. \square

In particular, the second fundamental form $h^{(1)}$ on M (up to a constant factor) is 0, which indicates Remark 1.8 (2).

Therefore it reduces to

$$\bar{g}_{ij}(\bar{x}, x_N) = \delta_{ij} + O(x_N^n) \quad \text{and} \quad |\bar{g}| = 1 + O(x_N^n) \quad \text{for } (\bar{x}, x_N) \in \mathcal{R}^N(\varrho_1, \varrho_2) \subset \bar{X}. \quad (4.14)$$

Now Conjecture 1.6 implies that there is a solution $G(\cdot, 0)$ to (1.20) with $y = 0$ such that

$$G(x, 0) = g_{n,\gamma}|x|^{-(n-2\gamma)} + A + \Psi(x) \quad \text{for } x \in \mathcal{R}^N(\varrho_1/2, \varrho_2/2)$$

where $g_{n,\gamma}, A > 0$ are fixed constants and Ψ is a function having the behavior (1.22).

Choose any smooth cut-off function $\chi : [0, \infty) \rightarrow [0, 1]$ such that $\chi(t) = 1$ for $0 \leq t \leq 1$ and 0 for $t \geq 2$. Then we construct a nonnegative, continuous and piecewise smooth function $\Phi_{\epsilon, \varrho_0}$ on \bar{X} by

$$\Phi_{\epsilon, \varrho_0}(x) = \begin{cases} W_\epsilon(x) & \text{if } x \in X \cap B^N(0, \varrho_0), \\ V_{\epsilon, \varrho_0}(x) \left(G(x, 0) - \chi_{\varrho_0}(x) \Psi(x) \right) & \text{if } x \in X \cap \left(B^N(0, 2\varrho_0) \setminus B^N(0, \varrho_0) \right), \\ V_{\epsilon, \varrho_0}(x) G(x, 0) & \text{if } x \in X \setminus B^N(0, 2\varrho_0) \end{cases}, \quad (4.15)$$

where $0 < \epsilon \ll \varrho_0 \leq \min\{\varrho_1, \varrho_2\}/5$ sufficiently small, $\chi_{\varrho_0}(x) := \chi(|x|/\varrho_0)$ and

$$V_{\epsilon, \varrho_0}(x) := \left[\alpha_{n,\gamma} \left(\frac{\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0^{n-2\gamma}} \right) + \chi_{\varrho_0}(x) \left(W_\epsilon(x) - \alpha_{n,\gamma} \frac{\epsilon^{\frac{n-2\gamma}{2}}}{|x|^{n-2\gamma}} \right) \right] \cdot \left(\varrho_0^{-(n-2\gamma)} + A \right)^{-1}. \quad (4.16)$$

We remark that the main block V_{ϵ, ϱ_0} of the test function $\Phi_{\epsilon, \varrho_0}$ is different from Escobar's (the function W in (4.2) of [14]), but they share common characteristics such as decay properties as proved in the next lemma.

Lemma 4.4. *There are constants $C, \eta_1, \eta_2 > 0$ depending only on n and γ such that*

$$|V_{\epsilon, \varrho_0}(x)| \leq C\epsilon^{\frac{n-2\gamma}{2}} \quad \text{for any } x \in X \setminus B^N(0, \varrho_0) \quad (4.17)$$

and

$$|\nabla_{\bar{x}} V_{\epsilon, \varrho_0}(x)| \leq C\varrho_0^{-\eta_1} \epsilon^{\frac{n-2\gamma+2\eta_2}{2}} \quad \text{and} \quad |\partial_N V_{\epsilon, \varrho_0}(x)| \leq C\rho_0^{-\eta_1} \left(\epsilon^{\frac{n-2\gamma+2\eta_2}{2}} + x_N^{2\gamma-1} \epsilon^{\frac{n+2\gamma}{2}} \right) \quad (4.18)$$

for $x = (\bar{x}, x_N) \in X \cap \left(B^N(0, 2\varrho_0) \setminus B^N(0, \varrho_0) \right)$. Also we have $\nabla V_{\epsilon, \varrho_0} = 0$ in $X \setminus B^N(0, 2\varrho_0)$.

Proof. We observe from (A.1) and (4.16) that

$$|V_{\epsilon, \varrho_0}(x)| \leq C\varrho_0^{n-2\gamma} \left[\left(\frac{\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0^{n-2\gamma}} \right) + \left| W_\epsilon(x) - \alpha_{n,\gamma} \frac{\epsilon^{\frac{n-2\gamma}{2}}}{|x|^{n-2\gamma}} \right| \right] \leq C \left(\epsilon^{\frac{n-2\gamma}{2}} + \frac{\epsilon^{\frac{n-2\gamma+2\vartheta_2}{2}}}{\varrho_0^{\vartheta_2}} \right) \leq C\epsilon^{\frac{n-2\gamma}{2}}$$

for all $\varrho_0 \leq |x| \leq 2\varrho_0$ and some $\vartheta_2 \in (0, 1)$, so (4.17) follows. One can derive (4.18) by making the use of both (A.1), (A.3) and (A.4). We leave the details to the reader. \square

Now we assert the following proposition, which suffices to conclude that the fractional Yamabe problem is solvable in this case.

Proposition 4.5. *For $n > 2\gamma$ and $\gamma \in (0, 1)$, let (X^{n+1}, g^+) be a Poincaré-Einstein manifold with conformal infinity $(M^n, [\hat{h}])$ such that (1.9) has the validity. Assume also that M is locally conformally flat. If (\bar{X}, \bar{g}) is not conformally diffeomorphic to the standard closed unit ball \mathbb{B}^N and Conjecture 1.6 holds, then*

$$0 < \bar{I}'_{\hat{h}}[\Phi_{\epsilon, \varrho_0}] < \Lambda^\gamma(\mathbb{S}^n, [g_c]).$$

Proof. The proof is divided into 3 steps.

STEP 1: ESTIMATION IN $X \cap B^N(0, \varrho_0)$. Applying (1.15), (1.16), (4.14), (A.3), (A.4), Lemma A.2 and integrating by parts, we obtain

$$\begin{aligned}
& \kappa_\gamma \int_{X \cap B^N(0, \varrho_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|_{\bar{g}}^2 dv_{\bar{g}} \\
& \leq \Lambda^\gamma(\mathbb{S}^n, [g_c]) \left(\int_{B^N(0, \varrho_0)} w_{\epsilon, 0}^{p+1} d\bar{x} \right)^{\frac{n-2\gamma}{n}} + \kappa_\gamma \int_{X \cap \partial B^N(0, \varrho_0)} x_N^{1-2\gamma} W_\epsilon \frac{\partial W_\epsilon}{\partial \nu} dS \\
& \quad + \underbrace{O \left(\int_{B^N(0, \varrho_0)} x_N^{n+1-2\gamma} |\nabla W_\epsilon|^2 d\bar{x} \right)}_{=O(\varrho_0^{2\gamma} \epsilon^{n-2\gamma})}
\end{aligned} \tag{4.19}$$

where ν is the outward unit normal vector and dS is the Euclidean surface measure. On the other hand, if we write $g^+ = x_N^{-2}(dx_N^2 + h_{x_N})$, then

$$E(x_N) = - \left(\frac{n-2\gamma}{4} \right) x_N^{-2\gamma} \operatorname{tr}(h_{x_N}^{-1} \partial_N h_{x_N}) = O(x_N^{n-1-2\gamma}) \tag{4.20}$$

in $X \cap B^N(0, 2\varrho_0)$ (see (2.5)). Therefore

$$\kappa_\gamma \int_{X \cap B^N(0, \varrho_0)} E(x_N) W_\epsilon^2 dv_{\bar{g}} = O(\varrho_0^{2\gamma} \epsilon^{n-2\gamma}). \tag{4.21}$$

STEP 2: ESTIMATION IN $X \setminus B^N(0, \varrho_0)$. By its own definition (4.15) of the test function $\Phi_{\epsilon, \varrho_0}$, its energy on \bar{X} can be evaluated as

$$\begin{aligned}
& \int_{X \setminus B^N(0, \varrho_0)} \left(\rho^{1-2\gamma} |\nabla \Phi_{\epsilon, \varrho_0}|_{\bar{g}}^2 + E(\rho) \Phi_{\epsilon, \varrho_0}^2 \right) dv_{\bar{g}} \\
& = \int_{X \setminus B^N(0, \varrho_0)} \left(\rho^{1-2\gamma} \langle \nabla(V_{\epsilon, \varrho_0}^2 G), \nabla G \rangle_{\bar{g}} + E(\rho) V_{\epsilon, \varrho_0}^2 G^2 + \rho^{1-2\gamma} |\nabla V_{\epsilon, \varrho_0}|^2 (G - \chi_{\varrho_0} \Psi)^2 \right) dv_{\bar{g}} \\
& \quad + \int_{X \cap (B^N(0, 2\varrho_0) \setminus B^N(0, \varrho_0))} \rho^{1-2\gamma} \left(\frac{1}{2} \langle \nabla V_{\epsilon, \varrho_0}^2, \nabla(-2G\chi_{\varrho_0}\Psi + \chi_{\varrho_0}^2\Psi^2) \rangle_{\bar{g}} \right) dv_{\bar{g}} \\
& \quad + \int_{X \cap (B^N(0, 2\varrho_0) \setminus B^N(0, \varrho_0))} \rho^{1-2\gamma} V_{\epsilon, \varrho_0}^2 \left(|\nabla(\chi_{\varrho_0}\Psi)|^2 - 2 \langle \nabla G, \nabla(\chi_{\varrho_0}\Psi) \rangle_{\bar{g}} \right) dv_{\bar{g}} \\
& \quad + \int_{X \cap (B^N(0, 2\varrho_0) \setminus B^N(0, \varrho_0))} E(\rho) V_{\epsilon, \varrho_0}^2 \left(\chi_{\varrho_0}^2 \Psi^2 - 2G\chi_{\varrho_0}\Psi \right) dv_{\bar{g}}
\end{aligned}$$

where $G = G(\cdot, 0)$. From (1.20), (1.22), (4.20) and Lemma 4.4, we see that

$$\begin{aligned}
& \kappa_\gamma \int_{X \setminus B^N(0, \varrho_0)} \left(\rho^{1-2\gamma} |\nabla \Phi_{\epsilon, \varrho_0}|_{\bar{g}}^2 + E(\rho) \Phi_{\epsilon, \varrho_0}^2 \right) dv_{\bar{g}} \\
& \leq -\kappa_\gamma \int_{X \cap \partial B^N(0, \varrho_0)} x_N^{1-2\gamma} V_{\epsilon, \varrho_0}^2 G \frac{\partial G}{\partial \nu} (1 + O(x_N^n)) dS + C\epsilon^{n-2\gamma+2\eta_2} \varrho_0^{-(n-2\gamma-2+2\eta_1)} \\
& \quad + C\epsilon^{n-2\gamma+\eta_2} \varrho_0^{\min\{\vartheta_1, 2\gamma\}+1-\eta_1} + C\epsilon^{n-2\gamma} \varrho_0^{\min\{\vartheta_1, 2\gamma\}}
\end{aligned} \tag{4.22}$$

where $\vartheta_1 \in (0, 1)$ and $C > 0$ depends only on n, γ, ϱ_1 and ϱ_2 . For instance, we have

$$\begin{aligned}
& \int_{X \setminus B^N(0, \varrho_0)} \rho^{1-2\gamma} |\nabla V_{\epsilon, \varrho_0}|^2 (G - \chi_{\varrho_0} \Psi)^2 dv_{\bar{g}} \\
& \leq C\varrho_0^{-2\eta_1} \int_{B^N(0, 2\varrho_0) \setminus B^N(0, \varrho_0)} x_N^{1-2\gamma} \left(\epsilon^{n-2\gamma+2\eta_2} + x_N^{2(2\gamma-1)} \epsilon^{n+2\gamma} \right) \cdot \left(\frac{1}{|x|^{2(n-2\gamma)}} + 1 \right) dx
\end{aligned}$$

$$\leq C \left(\epsilon^{n-2\gamma+2\eta_2} \varrho_0^{-(n-2\gamma-2+2\eta_1)} + \epsilon^{n+2\gamma} \varrho_0^{-n+6\gamma} |\log \varrho_0| \right) \leq C \epsilon^{n-2\gamma+2\eta_2} \varrho_0^{-(n-2\gamma-2+2\eta_1)}$$

for $0 < \epsilon \ll \varrho_0$ small. The other terms can be managed in a similar manner.

STEP 3: CONCLUSION. By combining (4.19), (4.21) and (4.22), we deduce

$$\begin{aligned} & \kappa_\gamma \int_X \left(\rho^{1-2\gamma} |\nabla \Phi_{\epsilon, \varrho_0}|_{\bar{g}}^2 + E(\rho) \Phi_{\epsilon, \varrho_0}^2 \right) dv_{\bar{g}} \\ & \leq \Lambda^\gamma(\mathbb{S}^n, [g_c]) \left(\int_{B^n(0, \varrho_0)} w_{\epsilon, 0}^{p+1} d\bar{x} \right)^{\frac{n-2\gamma}{n}} + \kappa_\gamma \int_{X \cap \partial B^N(0, \varrho_0)} \underbrace{x_N^{1-2\gamma} \left(W_\epsilon \frac{\partial W_\epsilon}{\partial \nu} - V_{\epsilon, \varrho_0}^2 G \frac{\partial G}{\partial \nu} \right)}_{=: I} dS \\ & + C \epsilon^{n-2\gamma} \varrho_0^{\min\{\vartheta_1, 2\gamma\}}. \end{aligned} \quad (4.23)$$

Let us compute the integral of I over the boundary $X \cap \partial B^N(0, \varrho_0)$ in the right-hand side of (4.23). Because of Lemma A.1 and (1.22), one has

$$\begin{aligned} \frac{\partial W_\epsilon}{\partial \nu} - V_{\epsilon, \varrho_0} \frac{\partial G}{\partial \nu} & \leq -\frac{\alpha_{n, \gamma}(n-2\gamma)\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0^{n-2\gamma+1}} + \left(\varrho_0^{-(n-2\gamma)} + A \right)^{-1} \frac{\alpha_{n, \gamma}(n-2\gamma)\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0^{2(n-2\gamma)+1}} \\ & + C \epsilon^{\frac{n-2\gamma}{2}} \varrho_0^{\min\{0, 2\gamma-1\}} + C \epsilon^{\frac{n-2\gamma}{2}+\vartheta_2} \varrho_0^{-(n-2\gamma+1+\vartheta_2)} \\ & \leq -\alpha_{n, \gamma}(n-2\gamma)A \frac{\epsilon^{\frac{n-2\gamma}{2}}}{\varrho_0} + C \epsilon^{\frac{n-2\gamma}{2}} \varrho_0^{\min\{0, 2\gamma-1\}} + C \epsilon^{\frac{n-2\gamma}{2}+\vartheta_2} \varrho_0^{-(n-2\gamma+1+\vartheta_2)} \end{aligned}$$

on $\{|x| = \varrho_0\}$ for some $\vartheta_2 \in (0, 1)$. Therefore using the fact that $W_1(x) \geq \alpha_{n, \gamma} \epsilon^{\frac{n-2\gamma}{2}} \varrho_0^{-(n-2\gamma)}/2$ on $\{|x| = \varrho_0\}$, we discover

$$\begin{aligned} \int_{X \cap \partial B^N(0, \varrho_0)} I dS & = \int_{X \cap \partial B^N(0, \varrho_0)} x_N^{1-2\gamma} \left[W_\epsilon \left(\frac{\partial W_\epsilon}{\partial \nu} - V_{\epsilon, \varrho_0} \frac{\partial G}{\partial \nu} \right) - V_{\epsilon, \varrho_0}^2 \frac{\partial G}{\partial \nu} \Psi \right] dS \\ & \leq -\frac{\alpha_{n, \gamma}^2(n-2\gamma)}{4} \left(\int_{\partial B^N(0, 1)} |x_N|^{1-2\gamma} dS \right) A \epsilon^{n-2\gamma} + C \epsilon^{n-2\gamma} \varrho_0^{\min\{1, 2\gamma\}} \\ & + C \epsilon^{n-2\gamma+\vartheta_2} \varrho_0^{-(n-2\gamma+\vartheta_2)} + C \epsilon^{n-2\gamma} \varrho_0^{\vartheta_1+n}. \end{aligned}$$

Now the previous estimate, (4.23) and (1.16) yield that

$$\begin{aligned} \bar{I}_h^\gamma[\Phi_{\epsilon, \varrho_0}] & \leq \Lambda^\gamma(\mathbb{S}^n, [g_c]) - \frac{\alpha_{n, \gamma}^2 \kappa_\gamma (n-2\gamma)}{8S_{n, \gamma}} \cdot \frac{|\mathbb{S}^{n-1}|}{2} B\left(1 - \gamma, \frac{n}{2}\right) \cdot A \epsilon^{n-2\gamma} \\ & + C \epsilon^{n-2\gamma+\vartheta_2} \varrho_0^{-(n-2\gamma+\vartheta_2)} + C \epsilon^{n-2\gamma} \varrho_0^{\min\{\vartheta_1, 2\gamma\}} \\ & < \Lambda^\gamma(\mathbb{S}^n, [g_c]) \end{aligned}$$

where B is the Beta function. Additionally the last strict inequality holds for $0 < \epsilon \ll \varrho_0$ small enough. This completes the proof. \square

4.3 Two Dimensional Case

We are now led to treat the case when $(M, [\hat{h}])$ is a 2-dimensional closed manifold.

Fix an arbitrary point $p \in M$ and let $\bar{x} = (x_1, x_2)$ be normal coordinates at p . Since X is Poincaré-Einstein, it holds $h^{(1)} = 0$ in (4.12), whence we have

$$\bar{g}_{ij}(\bar{x}, x_N) = \delta_{ij} + O(|x|^2) \quad \text{and} \quad |\bar{g}| = 1 + O(|x|^2) \quad \text{for } (\bar{x}, x_N) \in \mathcal{R}^N(\varrho_1, \varrho_2) \subset \bar{X} \quad (4.24)$$

where the rectangle $\mathcal{R}^N(\varrho_1, \varrho_2)$ is defined in the line following (4.12).

With Proposition B in the introduction, the next result will give the validity of Theorem 1.7 if $n = 2$.

Proposition 4.6. For $\gamma \in (0, 1)$, let (X^3, g^+) be a Poincaré-Einstein manifold with conformal infinity $(M^2, [\hat{h}])$ such that (1.9) holds. If (\bar{X}, \bar{g}) is not conformally diffeomorphic to the standard unit ball \mathbb{B}^3 and Conjecture 1.6 holds, then

$$0 < \bar{I}_h^\gamma[\Phi_{\epsilon, \varrho_0}] < \Lambda^\gamma(\mathbb{S}^2, [g_c])$$

for the test function $\Phi_{\epsilon, \varrho_0}$ introduced in (4.15).

Proof. We compute the error in $X \cap B_+^N(0, \varrho_0)$ due to the metric. As in (4.19) and (4.21), one has

$$\int_{X \cap B^N(0, \varrho_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|_{\bar{g}}^2 dv_{\bar{g}} = \int_{X \cap B^N(0, \varrho_0)} x_N^{1-2\gamma} |\nabla W_\epsilon|^2 dx + \underbrace{O\left(\int_{X \cap B^N(0, \varrho_0)} x_N^{1-2\gamma} |x|^2 |\nabla W_\epsilon|^2 dx\right)}_{=O(\varrho_0^{2\gamma} \epsilon^{2-2\gamma})}$$

and

$$\int_{X \cap B^N(0, \varrho_0)} E(x_N) W_\epsilon^2 dv_{\bar{g}} = O\left(\int_{X \cap B^N(0, \varrho_0)} x_N^{1-2\gamma} W_\epsilon^2 dx\right) = O(\varrho_0^{2\gamma} \epsilon^{2-2\gamma})$$

from (4.24). Therefore the error arising from the metric is ignorable, and the same argument in proof of Proposition 4.5 works. The proof is completed. \square

Acknowledgement. S. Kim is supported by FONDECYT Grant 3140530. M. Musso is partially supported by FONDECYT Grant 1120151 and Millennium Nucleus Center for Analysis of PDE, NC130017. The research of J. Wei is partially supported by NSERC of Canada. Part of the paper was finished while S. Kim was visiting the University of British Columbia and Wuhan University in 2015. He appreciates the institutions for their hospitality and financial support.

A Expansion of the Standard Bubble $W_{1,0}$ near Infinity

This appendix is devoted to find expansions of the function $W_1 = W_{1,0}$ (defined in (1.13)) and its derivatives near infinity. Especially we improve [12, Lemma A.2] by pursuing a new approach based on conformal properties of W_1 .

For the functions W_1 and $x \cdot \nabla W_1$, we have

Lemma A.1. Suppose that $n > 2\gamma$ and $\gamma \in (0, 1)$. For any fixed large number $R_0 > 0$, we have

$$\left| W_1(x) - \frac{\alpha_{n,\gamma}}{|x|^{n-2\gamma}} \right| + \left| x \cdot \nabla W_1(x) + \frac{\alpha_{n,\gamma}(n-2\gamma)}{|x|^{n-2\gamma}} \right| \leq \frac{C}{|x|^{n-2\gamma+\vartheta_2}} \quad (\text{A.1})$$

for $|x| \geq R_0$, where numbers $\vartheta_2 \in (0, 1)$ and $C > 0$ rely only on n , γ and R_0 .

Proof. Given any function F in \mathbb{R}_+^N , let F^* be its fractional Kelvin transform defined as

$$F^*(x) = \frac{1}{|x|^{n-2\gamma}} F\left(\frac{x}{|x|^2}\right) \quad \text{for } x \in \mathbb{R}_+^N.$$

Then it is known that $W_1^* = W_1$. Let us claim that $(x \cdot \nabla W_1)^*(0) = -\alpha_{n,\gamma}(n-2\gamma)$ and $(x \cdot \nabla W_1)^*$ is C^∞ in the \bar{x} -variable and Hölder continuous in the x_N -variable. Since

$$x_N^{2-2\gamma} \partial_{NN} W_1 = -(1-2\gamma)x_N^{1-2\gamma} \partial_N W_1 - x_N^{2-2\gamma} \Delta_{\bar{x}} W_1 \quad \text{in } \mathbb{R}_+^N,$$

we have

$$\begin{cases} -\operatorname{div}\left(x_N^{1-2\gamma} \nabla(x \cdot \nabla W_1)\right) = 0 & \text{in } \mathbb{R}_+^N, \\ \partial_\nu^\gamma(x \cdot \nabla W_1) = \sum_{i=1}^n x_i \partial_{x_i} \partial_\nu^\gamma W_1 + \partial_\nu^\gamma W_1 - \lim_{x_N \rightarrow 0} x_N^{2-2\gamma} \partial_{NN} W_1 & \text{on } \mathbb{R}^n. \\ = p \sum_{i=1}^n x_i \partial_{x_i} (w_1^p) + 2\gamma w_1^p \end{cases}$$

Employing [18, Proposition 2.6], [7] and doing some computations, we obtain that

$$\begin{cases} -\operatorname{div} \left(x_N^{1-2\gamma} \nabla (x \cdot \nabla W_1)^* \right) = 0 & \text{in } \mathbb{R}_+^N, \\ \partial_\nu^\gamma (x \cdot \nabla W_1)^* = (-\Delta)^\gamma (x \cdot \nabla W_1)^* = \alpha_{n,\gamma}^p \left(\frac{2\gamma|\bar{x}|^2 - n}{(1 + |\bar{x}|^2)^{\frac{n+2\gamma+2}{2}}} \right) & \text{on } \mathbb{R}^n. \end{cases}$$

Therefore $(x \cdot \nabla W_1)^*$ has regularity stated above, and according to Green's representation formula,

$$(x \cdot \nabla W_1)^*(0) = \alpha_{n,\gamma}^p g_{n,\gamma} \int_{\mathbb{R}^n} \frac{1}{|\bar{y}|^{n-2\gamma}} \left(\frac{2\gamma|\bar{y}|^2 - n}{(1 + |\bar{y}|^2)^{\frac{n+2\gamma+2}{2}}} \right) d\bar{y} = -\alpha_{n,\gamma}(n - 2\gamma).$$

This proves the assertion.

Now we can check (A.1) with the above observations. By standard elliptic theory, there exist constants $c_1, \dots, c_N > 0$ such that

$$\left| W_1^*(x) - \alpha_{n,\gamma} \right| + \left| (x \cdot \nabla W_1)^*(x) + \alpha_{n,\gamma}(n - 2\gamma) \right| \leq \sum_{i=1}^n c_i |x_i| + c_N x_N^{\vartheta_2} \quad (\text{A.2})$$

for any $|x| \leq R_0^{-1}$ and some $\vartheta_2 \in (0, 1)$. Hence, by taking the Kelvin transform in (A.2), we see that the desired inequality (A.1) is valid for all $|x| \geq R_0$. \square

Besides we have the following decay estimate of the derivatives of W_1 .

Lemma A.2. *Assume that $n > 2\gamma$ and $\gamma \in (0, 1)$. For any fixed large number $R_0 > 0$, there exist constants $C > 0$ and $\vartheta_3 \in (0, \min\{1, 2\gamma\})$ depending only on n, γ and R_0 such that*

$$\left| \nabla_{\bar{x}} W_1(x) + \frac{\alpha_{n,\gamma}(n - 2\gamma)\bar{x}}{|x|^{n-2\gamma+2}} \right| \leq \frac{C}{|x|^{n-2\gamma+1+\vartheta_3}} \quad (\text{A.3})$$

and

$$\left| \partial_N W_1(x) + \frac{\alpha_{n,\gamma}(n - 2\gamma)x_N}{|x|^{n-2\gamma+2}} \right| \leq C \left(\frac{1}{|x|^{n-2\gamma+2}} + \frac{x_N^{2\gamma-1}}{|x|^{n+2\gamma}} \right) \quad (\text{A.4})$$

for $|x| \geq R_0$.

Proof. The precise values of the constants $p_{n,\gamma}$, $\alpha_{n,\gamma}$ and κ_γ , which will appear during the proof, are found in (1.23).

STEP 1. By (1.13), (1.14) and Taylor's theorem, it holds

$$\begin{aligned} \partial_i W_1(x) &= p_{n,\gamma} \int_{\mathbb{R}^n} \frac{1}{(|\bar{y}|^2 + 1)^{\frac{n+2\gamma}{2}}} \partial_i w_1(\bar{x} - x_N \bar{y}) d\bar{y} \\ &= p_{n,\gamma} \int_{\mathbb{R}^n} \frac{1}{(|\bar{y}|^2 + 1)^{\frac{n+2\gamma}{2}}} \left[\partial_i w_1(-x_N \bar{y}) + \partial_{ij} w_1(-x_N \bar{y}) x_j + O(|\bar{x}|^2) \right] d\bar{y} \\ &= p_{n,\gamma} \int_{\mathbb{R}^n} \frac{1}{(|\bar{y}|^2 + 1)^{\frac{n+2\gamma}{2}}} \left[\partial_{ii} w_1(0) x_i + O((x_N |\bar{y}|)^{\vartheta_3} |\bar{x}|) + O(|\bar{x}|^2) \right] d\bar{y} \\ &= -\alpha_{n,\gamma}(n - 2\gamma) x_i + O(|x|^{1+\vartheta_3}) \end{aligned}$$

for $|x| \leq R_0^{-1}$. Here we also used the facts that the $C^2(\mathbb{R}^n)$ -norm of w_1 and the $C^{\vartheta_3}(\mathbb{R}^n)$ -norm of $\partial_{ij} w_1$ are bounded for some $\vartheta_3 \in (0, \min\{1, 2\gamma\})$. On the other hand, the uniqueness of the γ -harmonic extension yields that $(\partial_i W_1)^* = \partial_i W_1$ for $i = 1, \dots, n$. Therefore

$$\left| \partial_i W_1(x) + \frac{\alpha_{n,\gamma}(n - 2\gamma)x_i}{|x|^{n-2\gamma+2}} \right| = \left| (\partial_i W_1)^*(x) + \alpha_{n,\gamma}(n - 2\gamma)x_i^* \right| \leq C(|x|^{1+\vartheta_3})^* \leq \frac{C}{|x|^{n+2\gamma+1+\vartheta_3}}$$

for $|x| \geq R_0$, which is the desired inequality (A.3).

STEP 2. If $\gamma = 1/2$, it is known that

$$W_1(\bar{x}, x_N) = \alpha_{n,1/2} \left(\frac{1}{|\bar{x}|^2 + (x_N + 1)^2} \right)^{\frac{n-1}{2}} \quad \text{for all } (\bar{x}, x_N) \in \mathbb{R}_+^N,$$

so direct computation shows

$$\left| \partial_N W_1(x) + \frac{\alpha_{n,1/2}(n-1)x_N}{|x|^{n+1}} \right| \leq \frac{C}{|x|^{n+1}},$$

thereby implying (A.4). Therefore it is sufficient to consider when $\gamma \in (0, 1) \setminus \{1/2\}$. In light of duality [7, Subsection 2.3], we have that

$$\begin{cases} -\operatorname{div} \left(x_N^{1-2(1-\gamma)} \nabla \left(x_N^{1-2\gamma} \partial_N W_1 \right) \right) = 0 & \text{in } \mathbb{R}_+^N, \\ x_N^{1-2\gamma} \partial_N W_1 = -\kappa_\gamma^{-1} w_1^p & \text{on } \mathbb{R}^n. \end{cases}$$

Hence if we define

$$F^{**}(x) = \frac{1}{|x|^{n-2(1-\gamma)}} F \left(\frac{x}{|x|^2} \right) \quad \text{for } x \in \mathbb{R}_+^N.$$

for an arbitrary function F in \mathbb{R}_+^N , then

$$\begin{cases} -\operatorname{div} \left(x_N^{1-2(1-\gamma)} \nabla \left(x_N^{1-2\gamma} \partial_N W_1 \right)^{**} \right) = 0 & \text{in } \mathbb{R}_+^N, \\ \left(x_N^{1-2\gamma} \partial_N W_1 \right)^{**} = -\alpha_{n,\gamma}^p \kappa_\gamma^{-1} \frac{|\bar{x}|^2}{(1+|\bar{x}|^2)^{\frac{n+2\gamma}{2}}} & \text{on } \mathbb{R}^n. \end{cases}$$

This implies

$$\begin{aligned} \left(x_N^{1-2\gamma} \partial_N W_1 \right)^{**}(\bar{x}, x_N) &= -\alpha_{n,\gamma}^p \kappa_\gamma^{-1} p_{n,1-\gamma} x_N^{2-2\gamma} \int_{\mathbb{R}^n} \frac{1}{|\bar{y}|^{n-2\gamma}} \frac{1}{(1+|\bar{y}|^2)^{\frac{n+2\gamma}{2}}} d\bar{y} + O(x_N^{2-2\gamma} |x| + |x|^2) \\ &= -\alpha_{n,\gamma}(n-2\gamma) x_N^{2-2\gamma} + O(x_N^{2-2\gamma} |x| + |x|^2) \end{aligned} \quad (\text{A.5})$$

for all $|x| \leq R_0^{-1}$, where estimation of the remainder term is deferred to the end of the proof. Accordingly, we have

$$\left| x_N^{1-2\gamma} \partial_N W_1(x) + \frac{\alpha_{n,\gamma}(n-2\gamma) x_N^{2-2\gamma}}{|x|^{n-2\gamma+2}} \right| \leq C \left(\frac{x_N^{2-2\gamma}}{|x|^{n-2\gamma+3}} + \frac{1}{|x|^{n+2\gamma}} \right)$$

for $|x| \geq R_0$. Dividing the both sides by $x_N^{1-2\gamma}$ finishes the proof of (A.4).

Estimation of the remainder term in (A.5). The remainder term is equal to a constant multiple of

$$\begin{aligned} & \int_{\mathbb{R}^n} \left[\frac{1}{(1+|\bar{y}|^2)^{\frac{n-2\gamma+2}{2}}} \cdot \frac{(|\bar{x}|^2 - 2x_N \bar{x} \cdot \bar{y} + |x_N \bar{y}|^2)}{(1+|x_N \bar{y} - \bar{x}|^2)^{\frac{n+2\gamma}{2}}} - \frac{1}{|\bar{y}|^{n-2\gamma+2}} \cdot \frac{|x_N \bar{y}|^2}{(1+|x_N \bar{y}|^2)^{\frac{n+2\gamma}{2}}} \right] d\bar{y} \\ &= O(|\bar{x}|^2) + O \left(x_N |\bar{x}| \int_{\mathbb{R}^n} \frac{1}{(1+|\bar{y}|^2)^{\frac{n-2\gamma+2}{2}}} \cdot \frac{|\bar{y}|}{(1+|x_N \bar{y} - \bar{x}|^2)^{\frac{n+2\gamma}{2}}} d\bar{y} \right) \\ & \quad + x_N^2 \int_{\mathbb{R}^n} \left[\frac{|\bar{y}|^2}{(1+|\bar{y}|^2)^{\frac{n-2\gamma+2}{2}}} \cdot \frac{1}{(1+|x_N \bar{y} - \bar{x}|^2)^{\frac{n+2\gamma}{2}}} - \frac{1}{|\bar{y}|^{n-2\gamma}} \cdot \frac{1}{(1+|x_N \bar{y}|^2)^{\frac{n+2\gamma}{2}}} \right] d\bar{y} \\ &= \begin{cases} O(|\bar{x}|^2) + O(x_N |\bar{x}|) + [O(x_N^2) + O(x_N^{2-2\gamma} |\bar{x}|)] & \text{for } \gamma < 1/2, \\ O(|\bar{x}|^2) + O(x_N^{2-2\gamma} |\bar{x}|) + [O(x_N^2) + O(x_N^{2-2\gamma} |\bar{x}|)] & \text{for } \gamma > 1/2. \end{cases} \end{aligned} \quad (\text{A.6})$$

The estimate for the third term in the middle side of (A.6) can be done as

$$\begin{aligned} & x_N^2 \int_{\mathbb{R}^n} \left| \frac{|\bar{y}|^2}{(1 + |\bar{y}|^2)^{\frac{n-2\gamma+2}{2}}} - \frac{1}{|\bar{y}|^{n-2\gamma}} \right| \cdot \frac{1}{(1 + |x_N \bar{y} - \bar{x}|^2)^{\frac{n+2\gamma}{2}}} d\bar{y} \\ &= O(x_N^2) + O\left(x_N^{4-2\gamma} \int_{\{|\bar{y}| > x_N\}} \frac{1}{|\bar{y}|^{n-2\gamma+2}} \cdot \frac{1}{(1 + |\bar{y} - \bar{x}|^2)^{\frac{n+2\gamma}{2}}} d\bar{y}\right) = O(x_N^2) \end{aligned} \quad (\text{A.7})$$

with the aid of Taylor's theorem and the substitution $x_N \bar{y} \rightarrow \bar{y}$, and

$$\begin{aligned} & x_N^2 \int_{\mathbb{R}^n} \frac{1}{|\bar{y}|^{n-2\gamma}} \cdot \left[\frac{1}{(1 + |x_N \bar{y} - \bar{x}|^2)^{\frac{n+2\gamma}{2}}} - \frac{1}{(1 + |x_N \bar{y}|^2)^{\frac{n+2\gamma}{2}}} \right] d\bar{y} \\ &= x_N^{2-2\gamma} \int_{\mathbb{R}^n} \frac{1}{|\bar{y}|^{n-2\gamma}} \cdot \left[\frac{1}{(1 + |\bar{y} - \bar{x}|^2)^{\frac{n+2\gamma}{2}}} - \frac{1}{(1 + |\bar{y}|^2)^{\frac{n+2\gamma}{2}}} \right] d\bar{y} = O(x_N^{2-2\gamma} |\bar{x}|). \end{aligned}$$

Also we estimated the second term in the middle side of (A.6) by decomposing \mathbb{R}^n into two regions $\{|\bar{y}| \leq 1\}$ and $\{|\bar{y}| > 1\}$ as in (A.7). This concludes the proof. \square

B Some Integrations Regarding the Standard Bubble $W_{1,0}$ on \mathbb{R}_+^N

The following lemmas are due to González-Qing [19, Section 7] and the authors [33, Subsection 4.3].

Lemma B.1. *Suppose that $n > 4\gamma - 1$. For each $x_N > 0$ fixed, let $\widehat{W}_1(\xi, x_N)$ be the Fourier transform of $W_1(\bar{x}, x_N)$ with respect to the variable $\bar{x} \in \mathbb{R}^n$. In addition, we use K_γ to signify the modified Bessel function of the second kind of order γ . Then we have that*

$$\widehat{W}_1(\xi, x_N) = \widehat{w}_1(\xi) \varphi(|\xi|x_N) \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } x_N > 0,$$

where $\varphi(t) = d_1 t^\gamma K_\gamma(t)$ is the solution to

$$\phi''(t) + \frac{1-2\gamma}{t} \phi'(t) - \phi(t) = 0, \quad \phi(0) = 1 \text{ and } \phi(\infty) = 0 \quad (\text{B.1})$$

and $\widehat{w}_1(t) := \widehat{w}_1(|\xi|) = d_2 |\xi|^{-\gamma} K_\gamma(|\xi|)$ solves

$$\phi''(t) + \frac{1+2\gamma}{t} \phi'(t) - \phi(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0} t^{2\gamma} \phi(t) + \lim_{t \rightarrow \infty} t^{\gamma+\frac{1}{2}} e^t \phi(t) \leq C \quad (\text{B.2})$$

for some $C > 0$. The numbers $d_1, d_2 > 0$ depend only on n and γ .

Lemma B.2. *Let*

$$\begin{aligned} A_\alpha &= \int_0^\infty t^{\alpha-2\gamma} \varphi^2(t) dt, & B_\alpha &= \int_0^\infty t^{-\alpha+2\gamma} \widehat{w}_1^2(t) t^{n-1} dt, \\ A'_\alpha &= \int_0^\infty t^{\alpha-2\gamma} \varphi(t) \varphi'(t) dt, & B'_\alpha &= \int_0^\infty t^{-\alpha+2\gamma} \widehat{w}_1(t) \widehat{w}_1'(t) t^{n-1} dt, \\ A''_\alpha &= \int_0^\infty t^{\alpha-2\gamma} (\varphi'(t))^2 dt, & B''_\alpha &= \int_0^\infty t^{-\alpha+2\gamma} (\widehat{w}_1'(t))^2 t^{n-1} dt \end{aligned} \quad (\text{B.3})$$

for $\alpha \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned} A_\alpha &= \left(\frac{\alpha+2}{\alpha+1} \right) \cdot \left[\left(\frac{\alpha+1}{2} \right)^2 - \gamma^2 \right]^{-1} A_{\alpha+2} = - \left(\frac{\alpha+1}{2} - \gamma \right)^{-1} A'_{\alpha+1} \\ &= \left(\frac{\alpha+1}{2} - \gamma \right) \left(\frac{\alpha-1}{2} + \gamma \right)^{-1} A''_\alpha \end{aligned}$$

for α odd, $\alpha \geq 1$ and

$$B_\alpha = \frac{4(n-\alpha+1)B_{\alpha-2}}{(n-\alpha)(n+2\gamma-\alpha)(n-2\gamma-\alpha)} = -\frac{2B'_{\alpha-1}}{n+2\gamma-\alpha}, \quad B_{\alpha-2} = \frac{(n-2\gamma-\alpha)B''_{\alpha-2}}{n+2\gamma-\alpha+2}$$

for α even, $\alpha \geq 2$.

Proof. Apply (B.1), (B.2) and the identity

$$\int_0^\infty t^{\alpha-1} u(t) u'(t) dt = -\left(\frac{\alpha-1}{2}\right) \int_0^\infty t^{\alpha-2} u(t)^2 dt$$

which holds for any $\alpha > 1$ and $u \in C^1(\mathbb{R})$ decaying sufficiently fast. \square

Utilizing the above lemmas, we compute some integrals regarding the standard bubble W_1 and its derivatives. The next identities are necessary in the energy expansion when non-minimal conformal infinities are considered. See Subsection 2.2.

Lemma B.3. *Suppose that $n \geq 2$ and $\gamma \in (0, 1/2)$. Then*

$$\int_{\mathbb{R}_+^N} x_N^{2-2\gamma} |\nabla W_1|^2 dx = \left(\frac{4}{1+2\gamma}\right) \int_{\mathbb{R}_+^N} x_N^{2-2\gamma} (\partial_r W_1)^2 dx = \left(\frac{1-2\gamma}{2}\right) \int_{\mathbb{R}_+^N} x_N^{-2\gamma} W_1^2 dx < \infty.$$

Proof. Refer to [12, Lemma 6.3]. \square

The following information is used in the energy expansion for the non-umbilic case. Refer to Subsections 2.3 and 2.4.

Lemma B.4. *For $n > 2 + 2\gamma$, it holds that*

$$\begin{aligned} \mathcal{F}_1 &:= \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} W_1^2 dx = \left[\frac{3}{2(1-\gamma^2)}\right] |\mathbb{S}^{n-1}| A_3 B_2, \\ \mathcal{F}_2 &:= \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} |\nabla W_1|^2 dx = \left(\frac{3}{1+\gamma}\right) |\mathbb{S}^{n-1}| A_3 B_2, \\ \mathcal{F}_3 &:= \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} (\partial_r W_1)^2 dx = |\mathbb{S}^{n-1}| A_3 B_2, \\ \mathcal{F}_4 &:= \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} r (\partial_r W_1) (\partial_{rr} W_1) dx = -\frac{n}{2} |\mathbb{S}^{n-1}| A_3 B_2, \\ \mathcal{F}_5 &:= \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} r^2 (\partial_{rr} W_1)^2 dx = \left[\frac{5n^3 - 4n(1+\gamma^2) + 4(1-4\gamma^2)}{20(n-1)}\right] |\mathbb{S}^{n-1}| A_3 B_2, \\ \mathcal{F}_6 &:= \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} r^2 (\partial_r W_1)^2 dx = \left[\frac{(n+2)(3n^2 - 6n + 4 - 4\gamma^2)}{8(n-1)(1-\gamma^2)}\right] |\mathbb{S}^{n-1}| A_3 B_2, \\ \mathcal{F}_7 &:= \int_{\mathbb{R}_+^N} x_N^{2-2\gamma} r^2 (\partial_r W_1) (\partial_{rx_N} W_1) dx = -\left[\frac{(n+2)(3n^2 - 6n + 4 - 4\gamma^2)}{8(n-1)(1+\gamma)}\right] |\mathbb{S}^{n-1}| A_3 B_2, \\ \mathcal{F}_8 &:= \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} r^2 (\partial_{rx_N} W_1)^2 dx = \left[\frac{(2-\gamma)(5n^3 - 4n(2-2\gamma+\gamma^2) + 8(1-\gamma-2\gamma^2))}{20(n-1)(1+\gamma)}\right] |\mathbb{S}^{n-1}| A_3 B_2. \end{aligned}$$

Here $r = |\bar{x}|$, and the positive constants A_3 and B_2 are defined by (B.3).

Proof. The values $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and \mathcal{F}_6 were computed in [19, 33], so it suffices to consider the others.

STEP 1 (CALCULATION OF \mathcal{F}_4). Integration by parts gives

$$\mathcal{F}_4 = \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} r (\partial_r W_1) (\partial_{rr} W_1) dx = |\mathbb{S}^{n-1}| \int_0^\infty x_N^{3-2\gamma} \left(\frac{1}{2} \int_0^\infty r^n \partial_r (\partial_r W_1)^2 dr\right) dx_N$$

$$= |\mathbb{S}^{n-1}| \int_0^\infty x_N^{3-2\gamma} \left(-\frac{n}{2} \int_0^\infty r^{n-1} (\partial_r W_1)^2 dr \right) dx_N = -\frac{n}{2} \mathcal{F}_3 = -\frac{n}{2} |\mathbb{S}^{n-1}| A_3 B_2.$$

STEP 2 (CALCULATION OF \mathcal{F}_5). Since $\Delta_{\bar{x}} W_1 = W_1'' + (n-1)r^{-1}W_1'$ (where $'$ stands for the differentiation in r), it holds that

$$\int_{\mathbb{R}_+^N} x_N^{3-2\gamma} r^2 (\Delta_{\bar{x}} W_1)^2 dx = \mathcal{F}_5 + 2(n-1)\mathcal{F}_4 + (n-1)^2 \mathcal{F}_3. \quad (\text{B.4})$$

By the Plancherel theorem, Lemma B.1 and the relation

$$\begin{aligned} \Delta_\xi (|\xi|^2 \hat{w}_1(|\xi|) \varphi(|\xi|_{x_N})) &= 2n \hat{w}_1 \varphi + (n+2-2\gamma) |\xi| \hat{w}_1' \varphi + (n+2+2\gamma) |\xi| \hat{w}_1 \varphi' x_N \\ &\quad + |\xi|^2 \hat{w}_1 \varphi + 2|\xi|^2 \hat{w}_1' \varphi' x_N + |\xi|^2 \hat{w}_1 \varphi x_N^2 \end{aligned}$$

where the variable of \hat{w}_1 and \hat{w}_1' is $|\xi|$, that of φ and φ' is $|\xi|_{x_N}$, and $'$ represents the differentiation with respect to the radial variable $|\xi|$, we see

$$\begin{aligned} &\int_{\mathbb{R}_+^N} x_N^{3-2\gamma} r^2 (\Delta_{\bar{x}} W_1)^2 dx \\ &= \int_0^\infty x_N^{3-2\gamma} \int_{\mathbb{R}^n} (-\Delta_\xi) (|\xi|^2 \hat{w}_1(|\xi|) \varphi(|\xi|_{x_N})) \cdot (|\xi|^2 \hat{w}_1(|\xi|) \varphi(|\xi|_{x_N})) d\xi dx_N \\ &= |\mathbb{S}^{n-1}| \left[2n A_3 B_2 + (n+2-2\gamma) A_3 B_1' + (n+2+2\gamma) A_4' B_2 + A_3 B_0 + 2A_4' B_1' + A_5 B_2 \right]. \end{aligned}$$

Therefore Lemma B.2 implies

$$\int_{\mathbb{R}_+^N} x_N^{3-2\gamma} r^2 (\Delta_{\bar{x}} W_1)^2 dx = \left[\frac{5n^3 - 20n^2 + 4n(9-\gamma^2) - 16(1+\gamma^2)}{20(n-1)} \right] |\mathbb{S}^{n-1}| A_3 B_2.$$

Now (B.4) and the information on \mathcal{F}_3 and \mathcal{F}_4 yield the desired estimate for \mathcal{F}_5 .

STEP 3 (CALCULATION OF \mathcal{F}_7 AND \mathcal{F}_8). Since the basic strategy is similar to Step 2, we will just sketch the proof. We observe

$$\begin{aligned} \mathcal{F}_7 &= \frac{1}{2} \int_0^\infty x_N^{2-2\gamma} \partial_N \left(\int_{\mathbb{R}^n} r^2 (\partial_r W_1)^2 d\bar{x} \right) dx_N = \frac{1}{2} \int_0^\infty x_N^{2-2\gamma} \partial_N \left(\sum_{i=1}^n \int_{\mathbb{R}^n} |\bar{x}|^2 (\partial_{x_i} W_1)^2 d\bar{x} \right) dx_N \\ &= \frac{1}{2} \int_0^\infty x_N^{2-2\gamma} \partial_N \underbrace{\left(\sum_{i=1}^n \int_{\mathbb{R}^n} (-\Delta_\xi) (\xi_i \hat{w}_1(|\xi|) \varphi(|\xi|_{x_N})) \cdot (\xi_i \hat{w}_1(|\xi|) \varphi(|\xi|_{x_N})) d\xi \right)}_{=(I)} dx_N. \end{aligned}$$

Owing to Lemmas B.1 and B.2, one can compute the term

$$\begin{aligned} (I) &= - \left[(n+1) \int_{\mathbb{R}^n} \partial_N (|\xi| (\hat{w}_1 \hat{w}_1')(|\xi|) \varphi^2(|\xi|_{x_N}) + |\xi| \hat{w}_1^2(|\xi|) (\varphi \varphi')(|\xi|_{x_N}) x_N) d\xi \right. \\ &\quad + \int_{\mathbb{R}^n} \partial_N (|\xi|^2 (\hat{w}_1 \hat{w}_1')(|\xi|) \varphi^2(|\xi|_{x_N}) + 2|\xi|^2 (\hat{w}_1 \hat{w}_1')(|\xi|) (\varphi \varphi')(|\xi|_{x_N}) x_N) d\xi \\ &\quad \left. + \int_{\mathbb{R}^n} \partial_N (|\xi|^2 \hat{w}_1^2(|\xi|) (\varphi \varphi'')(|\xi|_{x_N}) x_N^2) d\xi \right] \end{aligned}$$

to get the value of \mathcal{F}_7 in the statement of the lemma. Moreover,

$$\begin{aligned} \mathcal{F}_8 &= \int_0^\infty x_N^{3-2\gamma} \left(\int_{\mathbb{R}^n} |\bar{x}|^2 |\nabla_{\bar{x}} (\partial_N W_1)|^2 d\bar{x} \right) dx_N \\ &= \int_0^\infty x_N^{3-2\gamma} \left(\sum_{i=1}^n \int_{\mathbb{R}^n} (-\Delta_\xi) (\xi_i \partial_N \widehat{W}_1) \cdot (\xi_i \partial_N \widehat{W}_1) d\xi \right) dx_N. \end{aligned}$$

The rightmost term is computable with Lemmas B.1 and B.2. The proof is completed. \square

The next lemma lists the values of some integrals which are needed in the energy expansion for the umbilic case (see Subsection 3.2). Its proof is analogous to the proofs of Lemma B.4 and [33, Lemma 4.4], so we skip it.

Lemma B.5. *For $n > 4 + 2\gamma$, we have*

$$\begin{aligned}
\mathcal{F}'_1 &:= \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} W_1^2 dx = \left[\frac{4(n-3)}{(n-4)(n-4-2\gamma)(n-4+2\gamma)} \right] |\mathbb{S}^{n-1}| A_3 B_2, \\
\mathcal{F}'_2 &:= \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} |\nabla W_1|^2 dx = \left[\frac{16(n-3)(2-\gamma)}{(n-4)(n-4-2\gamma)(n-4+2\gamma)} \right] |\mathbb{S}^{n-1}| A_3 B_2, \\
\mathcal{F}'_3 &:= \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} (\partial_r W_1)^2 dx = \left[\frac{16(n-3)(4-\gamma^2)}{5(n-4)(n-4-2\gamma)(n-4+2\gamma)} \right] |\mathbb{S}^{n-1}| A_3 B_2, \\
\mathcal{F}'_4 &:= \int_{\mathbb{R}_+^N} x_N^{1-2\gamma} r^2 W_1^2 dx = \left[\frac{n(3n^2 - 18n + 28 - 4\gamma^2)}{2(n-4)(n-4-2\gamma)(n-4+2\gamma)(1-\gamma^2)} \right] |\mathbb{S}^{n-1}| A_3 B_2, \\
\mathcal{F}'_5 &:= \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} r^2 |\nabla W_1|^2 dx = \left[\frac{n(3n^2 + 2n(-7+2\gamma) - 4(-4+3\gamma+\gamma^2))}{(n-4)(n-4-2\gamma)(n-4+2\gamma)(1+\gamma)} \right] |\mathbb{S}^{n-1}| A_3 B_2, \\
\mathcal{F}'_6 &:= \int_{\mathbb{R}_+^N} x_N^{3-2\gamma} r^2 (\partial_r W_1)^2 dx = \left[\frac{(n+2)(5n^2 - 20n + 16 - 4\gamma^2)}{5(n-4)(n-4-2\gamma)(n-4+2\gamma)} \right] |\mathbb{S}^{n-1}| A_3 B_2, \\
\mathcal{F}'_7 &:= \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} r (\partial_r W_1) (\partial_{rr} W_1) dx = - \left[\frac{8n(n-3)(4-\gamma^2)}{5(n-4)(n-4-2\gamma)(n-4+2\gamma)} \right] |\mathbb{S}^{n-1}| A_3 B_2, \\
\mathcal{F}'_8 &:= \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} r^2 (\partial_{rr} W_1)^2 dx = \left[\frac{4(4-\gamma^2)(7n^3 - 14n^2 - 4n(5+\gamma^2) + 4 - 16\gamma^2)}{35(n-4)(n-4-2\gamma)(n-4+2\gamma)} \right] |\mathbb{S}^{n-1}| A_3 B_2, \\
\mathcal{F}'_9 &:= \int_{\mathbb{R}_+^N} x_N^{4-2\gamma} r^2 (\partial_r W_1) (\partial_{rx_N} W_1) dx = - \left[\frac{(n+2)(2-\gamma)(5n^2 - 20n + 16 - 4\gamma^2)}{5(n-4)(n-4-2\gamma)(n-4+2\gamma)} \right] |\mathbb{S}^{n-1}| A_3 B_2, \\
\mathcal{F}'_{10} &:= \int_{\mathbb{R}_+^N} x_N^{5-2\gamma} r^2 (\partial_{rx_N} W_1)^2 dx = \left[\frac{4(2-\gamma)(3-\gamma)(7n^3 - 14n^2 - 4n(6-2\gamma+\gamma^2) + 8(2-3\gamma-2\gamma^2))}{35(n-4)(n-4-2\gamma)(n-4+2\gamma)} \right] |\mathbb{S}^{n-1}| A_3 B_2
\end{aligned}$$

where $r = |\bar{x}|$, and the positive constants A_3 and B_2 are defined by (B.3).

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