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# NONLOCAL $s$-MINIMAL SURFACES AND LAWSON CONES

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Abstract. The nonlocal $s$-fractional minimal surface equation for $\Sigma = \partial E$ where $E$ is an open set in $\mathbb{R}^N$ is given by

$$H^s_\Sigma(p) := \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x-p|^{N+s}}\, dx \;=\; 0 \quad \text{for all} \quad p \in \Sigma.$$

Here $0 < s < 1$, $\chi$ designates characteristic function, and the integral is understood in the principal value sense. The classical notion of minimal surface is recovered by letting $s \to 1$. In this paper we exhibit the first concrete examples (beyond the plane) of nonlocal $s-$minimal surfaces. When $s$ is close to 1, we first construct a connected embedded $s$-minimal surface of revolution in $\mathbb{R}^3$, the **nonlocal catenoid**, an analog of the standard catenoid $|x_3| = \log(r + \sqrt{r^2-1})$. Rather than eventual logarithmic growth, this surface becomes asymptotic to the cone $|x_3| = r\sqrt{1-s}$. We also find a two-sheet embedded $s$-minimal surface asymptotic to the same cone, an analog to the simple union of two parallel planes.

On the other hand, for any $0 < s < 1$, $n, m \geq 1$, $s-$minimal Lawson cones $|v| = \alpha|u|$, $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$, are found to exist. In sharp contrast with the classical case, we prove their stability for small $s$ and $n+m = 7$, which suggests that unlike the classical theory (or the case $s$ close to 1), the regularity of $s$-area minimizing surfaces may not hold true in dimension 7.

## 1. Introduction

1.1. **Fractional minimal surfaces.** Phase transition models where the motion of the interface region is driven by curvature type flows arise in many applications. The standard flow by mean curvature of surfaces $\Sigma(t)$ in $\mathbb{R}^N$ is that in which the normal speed of each point $x \in \Sigma(t)$ is proportional to its mean curvature $H_{\Sigma(t)}(x) = \sum_{i=1}^{N-1} k_i(x)$ where the $k_i$'s designate the principal curvatures, namely the eigenvalues of the second fundamental form. Evans [13] showed that standard mean curvature flow for level surfaces of a function can be recovered as the limit of a discretization scheme in time where heat flow $u_t - \Delta u = 0$ of suitable initial data is used to connect consecutive time steps, which was introduced in [19]. When standard diffusion is replaced by that of the fractional Laplacian $u_t + (-\Delta)^{\frac{s}{2}} u = 0$ in order to describe long range, nonlocal interactions between points in the two distinct phases by a Levy process, Caffarelli and Souganidis [6], see also Imbert [16], found that for $1 \leq s < 2$ flow by mean curvature is still recovered, while for $0 < s < 1$, the stronger nonlocal effect makes the surfaces evolve in normal velocity according to their *fractional mean curvature*, defined for a surface $\Sigma = \partial E$ where $E$ is an open subset of $\mathbb{R}^N$ as

$$H^s_\Sigma(p) := \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x-p|^{N+s}}\, dx \quad \text{for } p \in \Sigma. \tag{1.1}$$

Here $\chi$ denotes characteristic function, $E^c = \mathbb{R}^N \setminus E$ and the integral is understood in the principal value sense,

$$H^s_\Sigma(p) = \lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus B_\delta(p)} \frac{\chi_E(x) - \chi_{E^c}(x)}{|x-p|^{N+s}}\, dx.$$

This quantity is well-defined provided that $\Sigma$ is regular near $p$. It agrees with usual mean curvature in the limit $s \to 1$ by the relation

$$\lim_{s \to 1} (1-s)\, H^s_\Sigma(p) = c_N H_\Sigma(p), \tag{1.2}$$

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see [16]. Stationary surfaces for the fractional mean curvature flow are naturally called fractional minimal surfaces. We say that $\Sigma$ is an *s-minimal surface* in an open set $\Omega$, if the surface $\Sigma\cap\Omega$ is sufficiently regular, and it satisfies the *nonlocal minimal surface equation*

$$H^s_\Sigma(p)=0 \quad \text{for all } p\in\Sigma\cap\Omega. \tag{1.3}$$

For instance, it is clear by symmetry and definition (1.1) that a hyperplane is a $s$-minimal surface in $\mathbb{R}^N$ for all $0<s<1$. Similarly, the *Simons cone*

$$C^m_m=\{(u,v)\in\mathbb{R}^m\times\mathbb{R}^m \ / \ |v|=|u|\}$$

is a $s$-minimal surface in $\mathbb{R}^{2m}\setminus\{0\}$. As far as we know, no other explicit minimal surfaces in $\mathbb{R}^N$ have been found in the literature. The purpose of this paper is to exhibit a new class of non-trivial examples. The hyperplane is not just a minimal surface but also established in [5] to be *locally area minimizing* in a sense that we describe next.

Caffarelli, Roquejoffre and Savin introduced in [5] a nonlocal notion of surface area of $\Sigma=\partial E$ whose Euler-Lagrange equation corresponds to equation (1.3). For $0<s<1$, the $s$-perimeter of a measurable set $E\subset\mathbb{R}^N$ is defined as

$$\mathcal{I}_s(E)=\int_E\int_{E^c}\frac{dx\,dy}{|x-y|^{N+s}}.$$

The above quantity corresponds to a total interaction between points of $E$ and $E^c$, where the interaction density $1/|x-y|^{N+s}$ is largest possible when the points $x\in E$ and $y\in E^c$ are both close to a given point of the boundary. $\mathcal{I}_s(E)$ has a simple representation in terms of the usual semi-norm in the fractional Sobolev space $H^{\frac{s}{2}}(\mathbb{R}^N)$. In fact,

$$\mathcal{I}_s(E)=[\chi_E]_{H^{\frac{s}{2}}(\mathbb{R}^N)}:=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\chi_E(x)-\chi_E(y))^2}{|x-y|^{N+s}}dx\,dy. \tag{1.4}$$

Alternatively, we can also write

$$\mathcal{I}_s(E)=[\chi_E]_{W^{s,1}(\mathbb{R}^N)}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{N+s}}dx\,dy.$$

If $E$ is an open set and $\Sigma=\partial E$ is a smooth bounded surface we have that

$$(1-s)\mathcal{I}_s(E)\to c_N\mathcal{H}^{N-1}(\Sigma)=\int_{\mathbb{R}^N}|\nabla\chi_E|$$

where the latter equality is classically understood in the sense of functions of bounded variation. $\mathcal{I}_s$ can also be achieved as the $\Gamma$-limit as $\varepsilon\to0$ of the nonlocal Allen-Cahn phase transition functional $\int\frac{\varepsilon}{2}|\nabla^{\frac{s}{2}}u|^2+\frac{1}{4\varepsilon}(1-u^2)^2$ along functions that $\varepsilon$-regularize $\chi_E-\chi_{E^c}$. See [22, 25].

This nonlocal notion of perimeter is localized to a bounded open set $\Omega$ by taking away the contribution of points of $E$ and $E^c$ outside $\Omega$, formally setting

$$\mathcal{I}_s(E,\Omega)=\int_E\int_{E^c}\frac{dx\,dy}{|x-y|^{N+s}}-\int_{E\cap\Omega^c}\int_{E^c\cap\Omega^c}\frac{dx\,dy}{|x-y|^{N+s}}.$$

This quantity makes sense, even if the last two terms above are infinite, by rewriting it in the form

$$\mathcal{I}_s(E,\Omega)=\int_{E\cap\Omega}\int_{E^c}\frac{dx\,dy}{|x-y|^{N+s}}+\int_{E\cap\Omega^c}\int_{E^c\cap\Omega}\frac{dx\,dy}{|x-y|^{N+s}}.$$

Again, if $E$ is an open set with $\Sigma\cap\Omega$ smooth, $\Sigma=\partial E$. The usual notion of perimeter is recovered by the relation

$$\lim_{s\to1}(1-s)\mathcal{I}_s(E,\Omega)=c_N\mathcal{H}^{N-1}(\Sigma\cap\Omega),$$

see [21]. Let $h$ be a smooth function on $\Sigma$ supported in $\Omega$, and $\nu$ a normal vector field to $\Sigma$ exterior to $E$. For a sufficiently small number $t$ we let $E_{th}$ be the set whose boundary $\partial E_{th}$ is parametrized as

$$\partial E_{th}=\{x+th(x)\nu(x) \ / \ x\in\partial E\}.$$

The first variation of the perimeter along these normal perturbations yields precisely

$$\frac{d}{dt}\mathcal{I}_s(E_{th},\Omega)\Big|_{t=0} = -\int_\Sigma H^s_\Sigma h$$

and this quantity vanishes for all such $h$ if and only if (1.3) holds. Thus $\Sigma = \partial E$ is an $s$-minimal surface in $\Omega$ if the first variation of perimeter for normal perturbations of $E$ inside $\Omega$ is identically equal to zero.

If $\Sigma = \partial E$ is a nonlocal minimal surface the *second variation* of the $s$-perimeter in $\Omega$ can be computed as

$$\frac{d^2}{dt^2}Per_s(E_{th},\Omega)\Big|_{t=0} \;=\; -2\int_\Sigma \mathcal{J}^s_\Sigma[h]\,h. \tag{1.5}$$

We call $\mathcal{J}^s_\Sigma[h]$ the *fractional Jacobi operator*. It is explicitly computed as

$$\mathcal{J}^s_\Sigma[h](p) = \int_\Sigma \frac{h(x)-h(p)}{|p-x|^{N+s}}dx + h(p)\int_\Sigma \frac{\langle \nu(p)-\nu(x),\nu(p)\rangle}{|p-x|^{N+s}}dx, \quad p\in\Sigma, \tag{1.6}$$

where the first integral is understood in a pricipal value sense. In agreement with formula (1.5), we say that an $s$-minimal surface $\Sigma$ is *stable* in $\Omega$ if

$$-\int_\Sigma \mathcal{J}^s_\Sigma[h]\,h \,\geq\, 0 \quad \text{for all} \quad h\in C^\infty_0(\Sigma\cap\Omega).$$

Naturally we get the correspondence between this nonlocal operator and the usual Jacobi operator

$$\lim_{s\to 1}(1-s)\mathcal{J}^s_\Sigma[h] = c_N\mathcal{J}_\Sigma[h], \quad \mathcal{J}_\Sigma[h] = \Delta_\Sigma h + |A_\Sigma|^2 h \tag{1.7}$$

where $\Delta_\Sigma$ is the Laplace-Beltrami operator and $|A_\Sigma|^2 = \sum_{i=1}^{N-1} k_i^2$ where the $k_i$ are the principal curvatures.

A basic example of a stable fractional minimal surface $\Sigma = \partial E$ is a *fractional minimizing surface*. In [5] the existence of fractional perimeter-minimizing sets is proven in the following sense: let $\Omega$ be a bounded domain with Lipschitz boundary, and $E_0\subset\Omega^c$ a given set. Let $\mathcal{F}$ be the class of all sets $F$ with $F\cap\Omega^c = E_0$. Then there exists a set $E\in\mathcal{F}$ with

$$\mathcal{I}_s(E,\Omega) \;=\; \inf_{F\in\mathcal{F}}\mathcal{I}_s(F,\Omega).$$

Moreover, $\partial E\cap\Omega$ is a $(N-1)$-dimensional set, which is a surface of class $C^{1,\alpha}$ except possibly on a singular set of Hausdorff dimension at most $N-2$. Minimizers $E$ are proven to satisfy in a viscosity sense the fractional minimal surface equation (1.3). In fact, a hyperplane is minimizing in the above sense inside any bounded set. No other example of embedded smooth fractional minimal surface in $\mathbb{R}^N$ (minimizing or not) is known.

1.2. **Axially symmetric $s$-minimal surfaces.** After a plane, next in complexity in $\mathbb{R}^3$ is the *axially symmetric case*, namely the case of a surface of revolution around the $x_3$-axis. In the classical case, the minimal surface equation reduces to a simple ODE from which the catenoid $C_1$ is obtained:

$$C_1 = \{(x_1,x_2,x_3)\ /\ |x_3| = \log(r+\sqrt{r^2-1}), \quad r = \sqrt{x_1^2+x_2^2} > 1\}.$$

A main purpose of this paper is the construction of an axially symmetric $s$-minimal surface $C_s$ for $s$close to 1 in such a way that $C_s\to C_1$ as $s\to 1$ on bounded sets. We call this surface the *fractional catenoid*. A striking feature of the surface of revolution $C_s$ is that it becomes at main order as $r\to\infty$ a cone with small slope rather than having logarithmic growth. It is precisely in this feature where the strength of the nonlocal effect is felt.

The usual catenoid $C_1$ cannot be obtained by an area minimization scheme in expanding domains since it is linearly unstable, hence non-minimizing, inside any sufficiently large domain. It is unlikely that $C_s$ can be captured with a scheme based on the results in [5]. In fact, even worse, this is a highly unstable object compared with the classical case: there are elements in an approximate kernel of its $s$-Jacobi operator that change sign infinitely many times. The Morse index of $C_s$ is infinite in any reasonable sense (unlike the standard catenoid, whose Morse index is one).

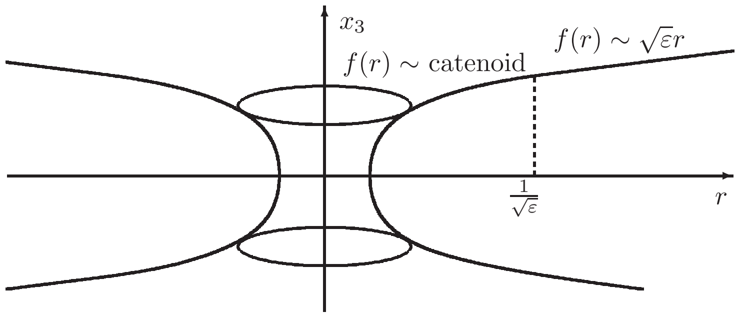


FIGURE 1. Fractional catenoid

**Theorem 1. (The fractional catenoid)** *For all $0 < s < 1$ sufficiently close to 1 there exists a connected surface of revolution $C_s$ such that if we set $\varepsilon = (1-s)$ then*

$$\sup_{x \in C_s \cap B(0,2)} \operatorname{dist}(x, C_1) \le c \frac{\sqrt{\varepsilon}}{|\log \varepsilon|},$$

*and, for $r = \sqrt{x_1^2 + x_2^2} > 2$ the set $C_s$ can be described as $|x_3| = f(r)$, where*

$$f(r) = \begin{cases} \log(r + \sqrt{r^2 - 1}) + O\left(\frac{r\sqrt{\varepsilon}}{|\log \varepsilon|}\right) & \text{if} \quad r < \frac{1}{\sqrt{\varepsilon}} \\ r\sqrt{\varepsilon} + O(|\log \varepsilon|) + O\left(\frac{r\sqrt{\varepsilon}}{|\log \varepsilon|}\right) & \text{if} \quad r > \frac{1}{\sqrt{\varepsilon}}. \end{cases}$$

As we have mentioned, a plane is an $s$-minimal surface for any $0 < s < 1$. In the classical scenario, so is the union of two parallel planes, say $x_3 = 1$ and $x_3 = -1$. This is no longer the case when $0 < s < 1$ since the nonlocal interaction between the two components deforms them and in fact equilibria is reached when the two components diverge becoming cones. Our second results states the existence of a *two-sheet* nontrivial $s$-minimal surface $D_s$ for $s$ close to 1 where the components eventually become at main order the cone $x_3 = \pm r\sqrt{\varepsilon}$. As in the $s$-catenoid, this is a highly unstable object.

**Theorem 2. (The two-sheet $s$-minimal surface)** *For all $0 < s < 1$ sufficiently close to 1 there exists a two-component surface of revolution $D_s = D_s^+ \cup D_s^-$ such that if we set $\varepsilon = (1-s)$ then $D_s^\pm$ is the graph of the radial functions $x_3 = \pm f(r)$ where $f$ is a positive function of class $C^2$ with $f(0) = 1$, $f'(0) = 0$, and*

$$f(r) = \begin{cases} 1 + \frac{\varepsilon}{4} r^2 + O(\varepsilon r) & \text{if} \quad r < \frac{1}{\sqrt{\varepsilon}} \\ r\sqrt{\varepsilon} + O(1) + O(\varepsilon r) & \text{if} \quad r > \frac{1}{\sqrt{\varepsilon}}. \end{cases}$$

As we shall discuss later, Theorem 2 can be generalized to the existence of a $k$-sheet axially symmetric $s$-minimal surface constituted by the union of the graphs of $k$ radial functions $x_3 = f_j(r)$, $j = 1, \dots, k$, with

$$f_1 > f_2 > \cdots > f_k$$

where asymptotically we have

$$f_j(r) = a_j r\sqrt{\varepsilon} + O(\varepsilon r) \quad \text{as } r \to +\infty. \tag{1.8}$$

Here the constants $a_i$ are required to satisfy the constraints

$$a_1 > a_2 > \cdots > a_k, \quad \sum_{i=1}^k a_i = 0 \tag{1.9}$$

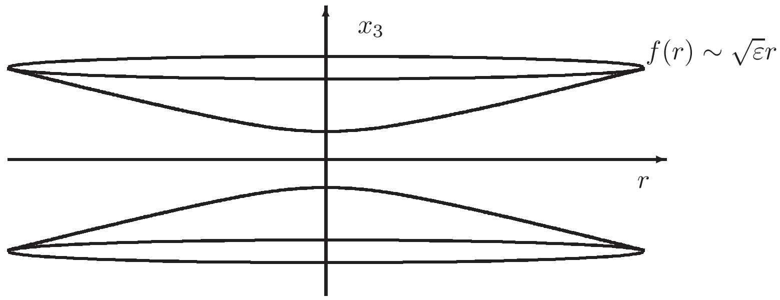


Figure 2. Two-sheet $s$-minimal surface

and the balancing conditions

$$a_i = 2 \sum_{j \neq i} \frac{(-1)^{i+j+1}}{a_i - a_j}, \qquad \text{for all} \quad i = 1, \ldots, k. \tag{1.10}$$

A solution of the system (1.10) can be obtained by minimization of

$$E(a_1, \ldots, a_k) = \frac{1}{2} \sum_{i=1}^{k} a_i^2 + \sum_{i \neq j} (-1)^{i+j} \log(|a_i - a_j|)$$

in the set of $k$-tuples $a = (a_1, \ldots, a_k)$ that satisfy (1.9). If this minimizer or, more generally, a critical point $a$ of $E$ constrained to (1.9) is non-degenerate, in the sense that $D^2 E(a)$ is non-singular, then an $s$-minimal surface with the required properties (1.8) can indeed be found. This condition is evidently satisfied by $a = (1, -1)$ when $k = 2$.

The method for the proofs of the above results relies in a simple idea of obtaining a good initial approximation $\Sigma_0$ to a solution of the equation $H_\Sigma = 0$ Then we consider the surface perturbed normally by a small function $h$, $\Sigma_h$. As we will see, regardless that $\Sigma_0$ is a minimal surface or not, we can expand

$$H_{\Sigma_h} = H_{\Sigma_0} + J_s[h] + N(h)$$

where $N(h)$ is at main order quadratic in $h$. In the classical case, $N(h)$ depends on first and second derivatives of $h$ with various terms that can be qualitatively described (see [17]). We shall see that if the approximation $\Sigma_0$ is properly chosen, in particular so that the error $H_{\Sigma_0}$ is small in $\varepsilon = 1 - s$ and has suitable decay along the manifold, then this equation can be solved by a fixed point argument. To do so, we need to identify the functional spaces to set up the problem, that take into account the delicate issues of non-compactness and strong long range interactions. These spaces should be such that a left inverse of $J_s$ can be found with good transformation properties, and $N(h)$ has a small Lipschitz dependence for the corresponding norms. The latter issue is especially delicate, for $N(h)$ is made out of various pieces, all strongly singular integral nonlinear operators involving fractional derivatives up to the nearly second order. The transformation properties of these nonlinear terms have suitable analogs with to those found by Kapouleas [17], but the proofs in the current situation are considerably harder.

The procedure we set up in this paper, and the associated computations, apply in large generality, not just to the axially symmetric case. For instance most of the calculations actually apply to a general setting of finding as $s \to 1$ a connected surface with multiple ends that are eventually conic and satisfy relations (1.9), where the starting point is a multiple-logarithmic-end minimal surface. This paper sets the basis of the gluing arguments for the construction of fractional minimal surfaces, in a way similar that the paper [17] did for the construction by gluing methods of classical minimal and CMC surfaces. The fractional scenario makes the analysis considerably harder.

1.3. **Fractional Lawson cones.** The pictures associated to Theorems 1 and 2 resemble that of "one-sheet" and "two-sheet" revolution hyperboloids, asymptotic to a cone $|x_3| = r\sqrt{1-s}$. It is reasonable to believe that a cone of this form, with aperture close to $\sqrt{1-s}$ is a fractional minimal surface with a singularity at the origin. We consider, more in general, for given $n, m \ge 1$, and $0 < s < 1$ the problem of finding a value $\alpha > 0$ such that the *Lawson cone*

$$C_\alpha = \{(u, v) \in \mathbb{R}^m \times \mathbb{R}^n \ / \ |v| = \alpha |u|\} \tag{1.11}$$

is a $s$-minimal surface in $\mathbb{R}^{m+n} \setminus \{0\}$. For the classical case $s = 1$ this is easy: since $\Sigma = C_\alpha$ is the zero level set of the function $g(u, v) = |v| - \alpha|u|$ then $(u, v) \in C_\alpha$ we have

$$H_\Sigma(u, v) = \operatorname{div}\left(\frac{\nabla g}{|\nabla g|}\right) = \frac{1}{\sqrt{1+\alpha^2}}\left[\frac{n-1}{|v|} - \alpha\frac{m-1}{|u|}\right],$$

and the latter quantity is equal to zero on $\Sigma$ if and only if $n = m = 1$ and $\alpha = 1$ or

$$n \ge 2,\ m \ge 2, \quad \alpha = \sqrt{\frac{n-1}{m-1}}.$$

Following [18], we call this one the minimal Lawson cone $C_m^n$. For the fractional situation we have the following result.

**Theorem 3.** (Existence of $s$-Lawson cones) *For any given $m \ge 1$, $n \ge 1$, $0 < s < 1$, there is a unique $\alpha = \alpha(s, m, n) > 0$ such that the cone $C_\alpha$ given by (1.11) is an $s$-fractional minimal surface. We call this $C_m^n(s)$ the $s$-Lawson cone.*

A notable different between classical and nonlocal cases is that in the latter, a nontrivial minimal cone in $\mathbb{R}^n$

$$C_1^{n-1}(s) \ = \ \{(x', x_n) \in \mathbb{R}^n \ / \ |x_n| = \alpha_n(s)|x'|\,\},$$

with $n \ge 3$ does exist. This is not true in the classical case. The bottomline is that when aperture becomes very large ($\alpha$ small), in the standard case mean curvature approaches 0, while the nonlocal interaction between the two pieces of the cone makes its fractional mean curvature go to $-\infty$. For $n = 2$, $C_1^2(s)$ is precisely the $s$-minimal cone that represents at main order the asymptotic behavior of the revolution $s$-minimal surfaces of Theorems 1 and 2. Letting $\varepsilon = 1 - s \to 0$, we have, as suspected

$$\alpha_2(s) = \sqrt{\varepsilon} + O(\varepsilon),$$

so that the two halves of the minimal cone become planes. In the opposite limit, $s \to 0$, there is no collapsing. In fact, if $n \le m$ we have

$$\lim_{s\to 0} \alpha(s, m, n) = \alpha_0$$

where $\alpha_0 > 0$ is the unique number $\alpha$ such that

$$\int_\alpha^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{m+n}{2}}}dt - \int_0^\alpha \frac{t^{n-1}}{(1+t^2)^{\frac{m+n}{2}}}dt = 0.$$

An interesting analysis of asymptotics for the fractional perimeter $\mathcal{I}_s$ and assoaciated $s$-minimizing surfaces as $s \to 0$ is contained in [11].

Minimal cones are important objects in the regularity theory of classical minimal surfaces and Bernstein type results for minimal graphs. Simons [24] proved that *no stable minimal cone exists in dimension $N \le 7$, except for hyperplanes*. This result implies that locally area minimizing surfaces must be smooth outside a closed set of Hausdorff dimension at most $N - 8$. He also proved that the cone $C_4^4$ (Simons' cone) was stable, and conjectured its minimizing character. This was proved in a deep work by Bombieri, De Giorgi and Giusti [4].

Savin and Valdinoci [21] proved the nonexistence of fractional minimizing cones in $\mathbb{R}^2$, which implies regularity of fractional minimizing surfaces except for a set of Hausdorff dimension at most $N - 3$, thus improving the original result in [5]. Figalli and Valdinoci [14] prove that, in every dimension, Lipschitz nonlocal minimal surfaces are smooth, see also [1]. Also, They extend to the nonlocal setting a famous theorem of De Giorgi stating that the validity of Bernstein's theorem as a consequence of the nonexistence of singular minimal cones in one dimension less.

In [8], Caffarelli and Valdinoci proved that regularity of non-local minimizers holds up to a $(N-8)$-dimensional set, whenever $s$ is sufficiently close to 1. Thus, there remains a conspicuous gap between the best general regularity result found so far and the case $s$ close to 1. Our second results concerns this issue. Its most interesting feature is that, in strong contrast with the classical case, when $s$ is sufficiently close to zero, Lawson cones **are all stable** in dimension $N = 7$, which suggests that a regularity theory up to a $(N-7)$-dimensonal set should be the best possible for general $s$.

**Theorem 4.** (Stability of $s$-Lawson cones) *There is a $s_0 > 0$ such that for each $s \in (0, s_0)$, all minimal cones $C_m^n(s)$ are unstable if $N = m + n \le 6$ and stable if $N = 7$.*

Besides the reults in [24, 4], we remark that for $N > 8$ the cones $C_m^n$ are all area minimizing. For $N = 8$ they are area minimizing if and only if $|m - n| \le 2$. These facts were established by Lawson [18] and Simoes [23], see also [20, 9, 2, 10].

The rest of this paper will be devoted to the proofs of Theorems 1–4. The proof of Theorem 2 is actuallly a simpler variation of that of Theorem 1. We will just concentrate in the proof of Theorem 1, whose scheme we explain in Section 2. There we shall isolate the main steps in the form of intermediate results which we prove in the subsequent sections. The proofs of Theorems 3 and 4 rely on explicit computations of singular integral quantities, and are carried out in Sections 10 and 11.

We leave for the Appendix self contained proofs of asymptotic formulas (1.2), (1.7) in Section A, and the computation of first and second variations of the $s$-perimeter in Section B.

## 2. Scheme of the proof of Theorem 1

In this section we shall outline the proof of Theorem 1, isolating the main steps whose proofs are delayed to later sections. We look for a set $E \subseteq \mathbb{R}^3$ with smooth $\Sigma = \partial E$ such that

$$H_\Sigma^s(x) := \int_{\mathbb{R}^3} \frac{\chi_E(y) - \chi_{E^c}(y)}{|x-y|^{3+s}}\, dy = 0, \quad \text{for all} \quad x \in \Sigma \tag{2.1}$$

where $0 < s < 1$, $1 - s$ is small and the integral is understood in a principal value sense sense.

We look for $E$ in the form of a solid of revolution around the $x_3$-axis. More precisely, let us represent points in space by $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, and denote $r = |x'|$. We shall construct a first approximation for $E$ of the form

$$E_0 = \{\ x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : |x'| < R \text{ or } |x'| \ge R, |x_3| > f(x)\ \}, \tag{2.2}$$

where $f$ is a positive and increasing function on $[R, \infty)$.

From now on we let $\varepsilon = 1 - s$. As we will demonstrate later, for an appropriate class of sets $E$ equation (2.1) formally resembles

$$-2H_\Sigma(x) + \frac{\varepsilon}{|x_3|} = 0. \tag{2.3}$$

We will obtain the surface $\Sigma$ and the corresponding set $E$ by first constructing an initial surface $\Sigma_0 = \partial E_0$ that is an approximate solution of (2.3) and then perturbing it.

For the construction of $\Sigma_0$ we take the standard catenoid parametrized as

$$|x_3| = f_C(r), \quad r = |x'| \ge 1,$$

where

$$f_C(r) = \log(r + \sqrt{r^2 - 1}), \quad r \ge 1. \tag{2.4}$$

If we describe $\Sigma = \partial E$ with $E$ as in (2.2) and assume that for $r$ large $f'(r)$ is small, then equation (2.3) is approximated by

$$\Delta f = \frac{\varepsilon}{f}. \tag{2.5}$$

This motivates us to define $f_\varepsilon(r)$ as solution of the initial value problem

$$\begin{cases} f_\varepsilon'' + \dfrac{1}{r} f_\varepsilon' = \dfrac{\varepsilon}{f_\varepsilon}, \quad r > \varepsilon^{-\frac12} \\ f_\varepsilon(\varepsilon^{-\frac12}) = f_C(\varepsilon^{-\frac12}), \quad f_\varepsilon'(\varepsilon^{-\frac12}) = f_C'(\varepsilon^{-\frac12}). \end{cases} \tag{2.6}$$

Let

$$F_\varepsilon(r) := f_C(r) + \eta(r - \varepsilon^{-\frac12})(f_\varepsilon(r) - f_C(r)), \quad r \ge 1, \tag{2.7}$$

where $\eta \in C^\infty(\mathbb{R})$ is a cut-off function with

$$\eta(t) = 0 \quad \text{for } t < 0, \quad \eta(t) = 1 \quad \text{for } t > 1. \tag{2.8}$$

We define the surface $\Sigma_0$ by

$$\Sigma_0 = \{|x_3| = F_\varepsilon(r), r \ge 1\}. \tag{2.9}$$

Then

$$\Sigma_0 = \partial E_0, \quad E_0 = \{r < 1, \text{ or } r \ge 1 \text{ and } |x_3| \ge F_\varepsilon(r)\}.$$

Next we perturb the surface $\Sigma_0$ in the normal direction. For this, let $\nu_{\Sigma_0}(x)$ be the unit normal vector field on $\Sigma_0$ such that $\nu_3(x)x_3 \ge 0$. We consider a function $h$ defined on $\Sigma_0$, and define

$$\Sigma_h = \{x + h(x)\nu_{\Sigma_0}(x) \ / \ x \in \Sigma_0\}.$$

If $h$ is small in a suitable norm, then $\Sigma_h$ is an embedded surface that can be written as $\Sigma_h = \partial E_h$ for a set $E_h$ that is close to $E_0$. We can expand, for a point $x \in \Sigma_0$ and $x_h = x + h(x)\nu_{\Sigma_0}(x)$:

$$H^s_{\Sigma_h}(x_h) = H^s_{\Sigma_0}(x) + 2\mathcal{J}^s_{\Sigma_0}(h)(x) + N(h)(x), \tag{2.10}$$

where $\mathcal{J}^s_{\Sigma_0}$ is the *nonlocal Jacobi operator* given by

$$\mathcal{J}^s_{\Sigma_0}(h)(x) = \int_{\Sigma_0} \frac{h(y) - h(x)}{|x-y|^{3+s}} dy + h(x) \int_{\Sigma_0} \frac{\langle \nu_{\Sigma_0}(x) - \nu_{\Sigma_0}(y), \nu_{\Sigma_0}(x)\rangle}{|x-y|^{3+s}} dy,$$

for $x \in \Sigma_0$, and $N(h)$ is defined by equality (2.10).

The objective is then to find $h$ such that

$$H^s_{\Sigma_0} + 2\mathcal{J}^s_{\Sigma_0}(h) + N(h) = 0. \tag{2.11}$$

We note that, assuming $h$ is smooth and bounded,

$$\text{p.v.} \int_{\Sigma_0} \frac{h(y) - h(x)}{|x-y|^{3+s}} dy = \frac{1}{\varepsilon}\frac{\pi}{2} \Delta_{\Sigma_0} h(x) + O(1)$$

as $\varepsilon \to 0$, where $\Delta_{\Sigma_0}$ is the Laplace-Beltrami operator on $\Sigma_0$ (see Lemma A.2). Therefore it is more convenient to rewrite (2.11) as

$$\varepsilon H^s_{\Sigma_0} + 2\varepsilon \mathcal{J}^s_{\Sigma_0}(h) + \varepsilon N(h) = 0 \quad \text{in } \Sigma_0.$$

It is natural to expect that $h$ has linear growth, and therefore we will work with weighted Hölder norms allowing such behavior. For $0 < \alpha < 1$ and $\gamma \in \mathbb{R}$, we define norms for functions defined on $\Sigma_0$ or $\mathbb{R}^2$ as follows:

$$[f]_{\gamma,\alpha} = \sup_{x \ne y} \min(1 + |x|, 1 + |y|)^{\gamma+\alpha} \frac{|f(x) - f(y)|}{|x-y|^\alpha},$$
$$\|f\|_{\gamma,\alpha} = \|(1+|x|)^\gamma f\|_{L^\infty} + [f]_{\gamma,\alpha},$$

and

$$\|h\|_* = \|(1+|x|)^{-1} h\|_{L^\infty} + \|\nabla h\|_{L^\infty} + \|(1+|x|) D^2 h\|_{L^\infty} + [D^2 h]_{1,\alpha}. \tag{2.12}$$

Then we look for a solution $h$ of (2.11) with $\|h\|_* < \infty$ and measure $\varepsilon \mathcal{J}^s_{\Sigma_0}(h)$ in the norm

$$\|f\|_{1-\varepsilon,\alpha+\varepsilon} = \|(1+|x|)^{1-\varepsilon} f\|_{L^\infty} + [f]_{1-\varepsilon,\alpha+\varepsilon} \tag{2.13}$$

More explicitly,

$$\|f\|_{1-\varepsilon,\alpha+\varepsilon} = \|(1+|x|)^{1-\varepsilon} f\|_{L^\infty} + \sup_{x\neq y} \min(1+|x|, 1+|y|)^{1+\alpha} \frac{|f(x)-f(y)|}{|x-y|^{\alpha+\varepsilon}}.$$

An outline of the proof of Theorem 1 is the following. In Section 4, using estimates for $f_\varepsilon$ obtained in Section 3, we will prove:

**Proposition 2.1.** *For $\varepsilon > 0$ sufficiently small we have*

$$\|\varepsilon H^s_{\Sigma_0}\|_{1-\varepsilon,\alpha+\varepsilon} \le \frac{C\varepsilon^{\frac{1}{2}}}{|\log \varepsilon|}. \tag{2.14}$$

The next result is about invertibility of the operator $\varepsilon \mathcal{J}^s_{\Sigma_0}$ on a weighted Hölder space.

**Proposition 2.2.** *There is a linear operator that to a function $f$ on $\Sigma_0$ such that $f$ is radially symmetric and symmetric with respect to $x_3 = 0$ with $\|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty$, gives a solution $\phi$ of*

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi) = f \quad \text{in } \Sigma_0.$$

*Moreover $\phi$ has the same symmetries as $f$ and*

$$\|\phi\|_* \le C\|f\|_{1-\varepsilon,\alpha+\varepsilon}.$$

The proof is given in Section 7, based on preliminaries in Sections 5 and 6.

In Section 8 we obtain the estimate

**Proposition 2.3.** *There is $C$ independent of $\varepsilon > 0$ small such that for $\|h_i\|_* \le \sigma_0 \varepsilon^{\frac{1}{2}}$, $i = 1, 2$ we have*

$$\varepsilon\|N(h_1) - N(h_2)\|_{1-\varepsilon,\alpha+\varepsilon} \le C\varepsilon^{-\frac{1}{2}}(\|h_1\|_* + \|h_2\|_*)\|h_1 - h_2\|_*. \tag{2.15}$$

Here $\sigma_0 > 0$ is small and fixed.

With these results we can give a

*Proof of Theorem 1.* We need a solution $h$ to (2.11) which we look for in the Banach space

$$X = \{h \in C^{2,\alpha}_{loc}(\Sigma_0),\ \|h\|_* < \infty\},$$

with norm $\|\ \|_*$. Consider also the Banach space

$$Y = \{f \in C^{\alpha+\varepsilon}_{loc},\ \|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty\},$$

with norm $\|\ \|_{1-\varepsilon,\alpha+\varepsilon}$. In both spaces we restrict functions to be axially symmetric and symmetric with respect to $x_3 = 0$.

Let $T$ be the linear operator constructed in Proposition 2.2. Then we reformulate (2.11) as

$$2h = A(h) := T(-\varepsilon H^s_{\Sigma_0} - \varepsilon N(h)).$$

We claim that for $\varepsilon > 0$ small, $A$ is a contraction on the ball

$$B = \{h \in X : \|h\|_* \le M\frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}\},$$

if we choose $M$ large. Indeed, for $h \in B$, by (2.14) and (2.15)

$$\begin{aligned}\|A(h)\|_* &\le C\|\varepsilon H^s_{\Sigma_0}\|_{1-\varepsilon,\alpha+\varepsilon} + C\|\varepsilon N(h)\|_{1-\varepsilon,\alpha+\varepsilon} \\ &\le \frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}\left(C + \frac{M^2}{|\log\varepsilon|}\right) \le M\frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|},\end{aligned}$$

if we take $M = 2C$ then let $\varepsilon > 0$ be small. Next, for $h_1$, $h_2 \in B$,

$$\|A(h_1) - A(h_2)\|_* \le C\varepsilon^{-\frac{1}{2}}(\|h_1\|_* + \|h_2\|_*)\|h_1 - h_2\|_*.$$

But $\varepsilon^{-\frac{1}{2}}(\|h_1\|_* + \|h_2\|_*) \le \frac{C}{|\log\varepsilon|}$ and so $A$ is a contraction on $B$ for $\varepsilon > 0$ small. $\square$

## 3. The ODE of the initial approximation

The purpose of this section is to analyze the solution $f_\varepsilon(r)$ of (2.6), which is used in the construction of the initial approximation. Thanks to (2.4) we have

$$\begin{cases} f(\varepsilon^{-\frac{1}{2}}) = C(\varepsilon^{-\frac{1}{2}}) = \frac{1}{2}|\log\varepsilon| + \log 2 + O(\varepsilon) \\ f'(\varepsilon^{-\frac{1}{2}}) = C'(\varepsilon^{-\frac{1}{2}}) = \sqrt{\varepsilon}(1+O(\varepsilon)). \end{cases} \tag{3.1}$$

Note that $f_\varepsilon'(r) \geq 0$ so in particular

$$f_\varepsilon(r) \geq f_\varepsilon(\varepsilon^{-\frac{1}{2}}) \quad \text{for all } r \geq r^{-\frac{1}{2}}. \tag{3.2}$$

**Lemma 3.1.** *We have*

$$C_1|\log\varepsilon| \leq |f_\varepsilon(r)| \leq C_2|\log\varepsilon|, \quad |f_\varepsilon'(r)| \leq C\varepsilon^{\frac{1}{2}} \tag{3.3}$$

$$|f_\varepsilon''(r)| \leq \frac{C}{r^2} + \frac{C\varepsilon}{|\log\varepsilon|^2}$$

*for* $\varepsilon^{-\frac{1}{2}} \leq r \leq |\log\varepsilon|\varepsilon^{-\frac{1}{2}}$.

*Proof.* We make the change of variables

$$f_\varepsilon(r) = |\log\varepsilon|\tilde{f}(\varepsilon^{\frac{1}{2}}r),$$

and then $\tilde{f}$ satisfies

$$\Delta\tilde{f} = \frac{1}{|\log\varepsilon|^2\tilde{f}}, \tag{3.4}$$

for $r \geq 1$, with initial conditions

$$\tilde{f}(1) = \frac{1}{2} + O(\frac{1}{|\log\varepsilon|}), \qquad \tilde{f}'(1) = \frac{1+O(\varepsilon)}{|\log\varepsilon|}. \tag{3.5}$$

Integrating once (3.4) we get

$$r\tilde{f}'(r) - \tilde{f}'(1) = \frac{1}{|\log\varepsilon|^2}\int_1^r \frac{s}{\tilde{f}(s)}\,ds \tag{3.6}$$

for $r \geq 1$. By (3.2)

$$\tilde{f}(r) \geq \frac{1}{2} + O(\frac{1}{|\log\varepsilon|}) \quad \text{for } r \geq 1. \tag{3.7}$$

Therefore from (3.6) and (3.7) we obtain

$$\tilde{f}'(r) \leq \frac{1}{r}\left(\frac{C}{|\log\varepsilon|} + \frac{Cr^2}{|\log\varepsilon|^2}\right) \quad \text{for } r \geq 1.$$

This implies

$$\tilde{f}'(r) \leq \frac{C}{|\log\varepsilon|}, \quad \text{for } 1 \leq r \leq |\log\varepsilon|,$$

and using (3.5) also

$$\tilde{f}(r) \leq C, \quad \text{for } 1 \leq r \leq |\log\varepsilon|.$$

To estimate $f_\varepsilon''$ we note that

$$\begin{aligned} |\tilde{f}_\varepsilon''(r)| &\leq \frac{1}{r}|\tilde{f}'(r)| + \frac{1}{|\log\varepsilon|^2\tilde{f}} \\ &\leq \frac{C}{r^2} + \frac{C}{|\log\varepsilon|^2} \quad \text{for } r \geq 1. \end{aligned}$$

□

We study now the asymptotic behavior of $f_\varepsilon(r)$ as $r \to \infty$. For this let us write

$$f_\varepsilon(r) = |\log\varepsilon| f_0^{(\varepsilon)}\Big(\frac{\varepsilon^{\frac12}}{|\log\varepsilon|} r\Big), \qquad \text{for } r \geq \frac{1}{|\log\varepsilon|}, \tag{3.8}$$

for a new function $f_0^{(\varepsilon)}$. Then $f_0^{(\varepsilon)}$ satisfies

$$\Delta f_0^{(\varepsilon)} = \frac{1}{f_0^{(\varepsilon)}} \quad \text{for } r \geq \frac{1}{|\log\varepsilon|}$$

and from (3.1)

$$\begin{aligned} f_0^{(\varepsilon)}\Big(\frac{1}{|\log\varepsilon|}\Big) &= \frac12 + \frac{\log 2}{|\log\varepsilon|} + O\Big(\frac{\varepsilon}{|\log\varepsilon|}\Big) \\ [f_0^{(\varepsilon)}]'\Big(\frac{1}{|\log\varepsilon|}\Big) &= 1 + O(\varepsilon), \end{aligned}$$

as $\varepsilon \to 0$.

**Lemma 3.2.** *For any $r_0 > 0$ and $\varepsilon > 0$ small there is $C$ such that*

$$\begin{aligned} &|f_0^{(\varepsilon)}(r) - r| \leq C, \qquad |[f_0^{(\varepsilon)}]'(r) - 1| \leq \frac{C}{r}, \\ &|[f_0^{(\varepsilon)}]''(r)| \leq \frac{C}{r} \end{aligned}$$

*for all $r \geq r_0$.*

*Proof.* Let us introduce the Emden-Fowler change of variables

$$f_0^{(\varepsilon)}(r) = r\psi_\varepsilon(t), \qquad \text{where } r = e^t \tag{3.9}$$

for $t \geq -\log|\log\varepsilon|$. Then $\psi_\varepsilon(t) > 0$ and

$$\psi_\varepsilon'' + 2\psi_\varepsilon' + \psi_\varepsilon = \frac{1}{\psi_\varepsilon} \quad \text{for } t \geq -\log|\log\varepsilon|. \tag{3.10}$$

Let

$$G_\varepsilon(t) = \frac12(\psi_\varepsilon')^2 + \frac12\psi_\varepsilon^2 - \log\psi_\varepsilon - \frac12$$

and note that

$$G_\varepsilon'(t) = -2(\psi_\varepsilon')^2 \leq 0. \tag{3.11}$$

Using (3.3) we see that $\psi_\varepsilon(0) = O(1)$ and $\psi_\varepsilon'(0) = O(1)$ as $\varepsilon \to 0$ and this implies that $G_\varepsilon(0) = O(1)$ as $\varepsilon \to 0$. Then by (3.11) $G_\varepsilon(t) \leq C$ for all $t \geq 0$ and all $\varepsilon > 0$ small. This implies that

$$0 < a \leq \psi_\varepsilon(t) \leq b < \infty, \quad |\psi_\varepsilon'(t)| \leq C \quad \text{for all } t \geq 0, \tag{3.12}$$

and all $\varepsilon > 0$ small, for some uniform constants $0 < a < b$ and $C > 0$.

From (3.11)

$$\int_0^t \psi_\varepsilon'(s)^2\, ds = 2G_\varepsilon(0) - 2G_\varepsilon(t) \leq C$$

with $C$ independent of $\varepsilon$ and $t \geq 0$. From this we see that

$$\int_0^\infty \psi_\varepsilon^2(s)\, ds \leq C \tag{3.13}$$

with $C$ independent of $\varepsilon$. Using interpolation estimates (or elliptic estimates) for the equation for $Z_\varepsilon = \psi_\varepsilon'$:

$$Z_\varepsilon'' + 2Z_\varepsilon' + Z_\varepsilon\Big(1 + \frac{1}{\psi_\varepsilon^2}\Big) = 0$$

we have

$$|\psi_\varepsilon'(t)| = |Z_\varepsilon(t)| \leq C\Big(\int_{t-1}^{t+1} Z_\varepsilon(s)^2\, ds\Big)^{1/2} \to 0$$

as $t \to \infty$, by (3.13). We claim the convergence is uniform and exponential. To see this, define

$$E_{2,\varepsilon} = \frac{1}{2}(\psi_\varepsilon'')^2 + \frac{1}{2}(\psi_\varepsilon')^2\Big(1 + \frac{1}{\psi_\varepsilon^2}\Big).$$

Then

$$E_{2,\varepsilon}' = -2(\psi_\varepsilon'')^2 - \Big(\frac{\psi_\varepsilon'}{\psi}\Big)^3.$$

For $\alpha > 0$ to be fixed later on consider

$$\tilde G_\varepsilon = \alpha E_\varepsilon + E_{2,\varepsilon}.$$

Then

$$\tilde G_\varepsilon' = -2(\psi_\varepsilon'')^2 - \Big(\frac{\psi_\varepsilon'}{\psi}\Big)^3 - 2\alpha(\psi_\varepsilon')^2.$$

But by (3.12)

$$-\Big(\frac{\psi_\varepsilon'}{\psi}\Big)^3 \le C(\psi_\varepsilon')^2$$

so that

$$\tilde G_\varepsilon' \le -2(\psi_\varepsilon'')^2 - (2\alpha - C)(\psi_\varepsilon')^2.$$

At this point we choose $\alpha$ so that $2\alpha = C$. We then obtain

$$\tilde G_\varepsilon' \le -(\psi_\varepsilon'')^2 - (\psi_\varepsilon')^2. \tag{3.14}$$

Using (3.10) we note that for some $A > 0$

$$\Big(\frac{1}{\psi}_\varepsilon - \psi_\varepsilon\Big)^2 \le 2((\psi_\varepsilon'')^2 + A(\psi_\varepsilon')^2). \tag{3.15}$$

Using again (3.12)

$$\begin{aligned}\tilde G_\varepsilon &= \alpha\left[\frac{1}{2}(\psi_\varepsilon')^2 + \frac{1}{2}\psi_\varepsilon^2 - \log\psi_\varepsilon - \frac{1}{2}\right] + \frac{1}{2}(\psi_\varepsilon'')^2 + \frac{1}{2}(\psi_\varepsilon)'^2\Big(1 + \frac{1}{\psi_\varepsilon^2}\Big).\\ &\le C\left((\psi_\varepsilon'')^2 + (\psi_\varepsilon')^2 + \Big(\frac{1}{\psi}_\varepsilon - \psi_\varepsilon\Big)^2\right)\end{aligned}$$

Combining (3.14), (3.15) and the last estimate we see that

$$\tilde G_\varepsilon \le -C\tilde G_\varepsilon'.$$

This implies that

$$\tilde G_\varepsilon(t) \le Ce^{-\delta t} \quad \text{for all } t \ge 0,$$

for some constants $C$, $\delta > 0$ independent of $\varepsilon > 0$ small. From this we obtain

$$|\psi_\varepsilon'(t)| + |\psi_\varepsilon(t) - 1| \le Ce^{-\delta t/2}, \quad \text{for all } t \ge 0.$$

Then, after a fixed $t_1$ independent of $\varepsilon$, the point $\psi_\varepsilon(t_1), \psi_\varepsilon'(t_1)$ is sufficiently close to $(1,0)$. Let

$$v_1 = \frac{1}{\psi}, \quad v_2 = \psi + \psi'.$$

Then (3.10) is equivalent to

$$\begin{aligned} v_1' &= v_1 - v_1^2 v_2 \\ v_2' &= v_1 + v_2. \end{aligned} \tag{3.16}$$

For $t_1$ sufficiently large the point $(v_1(t_1), v_2(t_1))$ is sufficiently close to $(1,1)$, which is a hyperbolic stationary point of (3.16). The eigenvalues of the linearization at $(1,1)$ are $-1 \pm i$ so that by applying a $C^1$ conjugacy to the linearization at $(1,1)$ we obtain

$$|(v_1(t), v_2(t)) - (1,1)| \le Ce^{-t} \quad \text{for all } t \ge t_1.$$

This implies

$$|\psi_\varepsilon'(t)| + |\psi_\varepsilon(t) - 1| \le Ce^{-t}, \quad \text{for all } t \ge 0, \tag{3.17}$$

For the function $f_0^{(\varepsilon)}$ we find

$$|f_0^{(\varepsilon)}(r) - r| \le C, \qquad |[f_0^{(\varepsilon)}]'(r) - 1| \le \frac{C}{r}$$

for all $r \ge r_0$, for any $r_0 > 0$ fixed. □

**Corollary 3.1.** *We have the following properties of $F_\varepsilon$:*

$$\begin{aligned}
&F_\varepsilon(r) = f_C(r) = \log(r + \sqrt{r^2-1}) = \log(2r) + O(r^{-2}), && 1 \le r \le \varepsilon^{-\frac12},\\
&C_1|\log\varepsilon| \le F_\varepsilon(r) \le C_2|\log\varepsilon|, && \varepsilon^{-\frac12} \le r \le \delta|\log\varepsilon|\varepsilon^{-\frac12},\\
&F_\varepsilon(r) = \varepsilon^{\frac12} r + O(|\log\varepsilon|), && r \ge \delta|\log\varepsilon|\varepsilon^{-\frac12},
\end{aligned}$$

$$\begin{aligned}
&F_\varepsilon'(r) = C'(r) = \frac{1}{r} + O(r^{-3}), && 1 \le r \le \varepsilon^{-\frac12},\\
&F_\varepsilon'(r) = O(\varepsilon^{\frac12}), && \varepsilon^{-\frac12} \le r \le \delta|\log\varepsilon|\varepsilon^{-\frac12},\\
&F_\varepsilon'(r) = \varepsilon^{\frac12}(1 + O(\frac{|\log\varepsilon|}{\varepsilon^{1/2} r})), && r \ge \delta|\log\varepsilon|\varepsilon^{-\frac12},
\end{aligned}$$

$$\begin{aligned}
&F_\varepsilon''(r) = C''(r) = -\frac{1}{r^2} + O(r^{-4}), && 1 \le r \le \varepsilon^{-\frac12},\\
&F_\varepsilon''(r) = O(\frac{1}{r^2} + \frac{\varepsilon}{|\log\varepsilon|}), && \varepsilon^{-\frac12} \le r \le \delta|\log\varepsilon|\varepsilon^{-\frac12},\\
&F_\varepsilon''(r) = O(\frac{\varepsilon^{\frac12}}{r}), && r \ge \delta|\log\varepsilon|\varepsilon^{-\frac12}.
\end{aligned}$$

$$\begin{aligned}
&F_\varepsilon'''(r) = C'''(r) = 2\frac{1}{r^3} + O(r^{-5}), && 1 \le r \le \varepsilon^{-\frac12},\\
&F_\varepsilon'''(r) = O(\frac{\varepsilon^{\frac12}}{r^2}), && \varepsilon^{-\frac12} \le r \le \delta|\log\varepsilon|\varepsilon^{-\frac12},\\
&F_\varepsilon'''(r) = O(\frac{\varepsilon^{\frac12}}{r^2}), && r \ge \delta|\log\varepsilon|\varepsilon^{-\frac12}.
\end{aligned}$$

*Proof.* The estimates for $F_\varepsilon$, and first and second derivatives follow from the Lemmas 3.1 and 3.2. To estimate the third derivate we can differentiate the equation and use the previous estimates. □

It will be useful for later purposes to have also estimates for the elements in the linearization of (3.8). Namely consider

$$\Delta z + \frac{1}{(f_0^{(\varepsilon)})^2(r)} z = 0, \quad \text{for } r \ge \frac{1}{|\log\varepsilon|}. \tag{3.18}$$

The function

$$\tilde z_1(r) = f_0^{(\varepsilon)} - r[f_0^{(\varepsilon)}]'(r) \tag{3.19}$$

satisfies (3.18), since the equation (3.8) is invariant by the scaling $f_\lambda(r) = \frac{1}{\lambda} f(\lambda r)$, $\lambda > 0$. We may construct a second independent solution $\tilde z_2$ of (3.18) by solving this equation with initial conditions

$$\tilde z_2(r_0) = -\tilde z_1'(r_0), \qquad \tilde z_2'(r_0) = \tilde z_1(r_0).$$

Here $r_0 > 0$ is fixed.

**Lemma 3.3.** *Fix $r_0 > 0$. We have*

$$|\tilde z_i(r)| \le C, \quad |\tilde z_i'(r)| \le \frac{C}{r}$$

*for all $r \ge r_0$, $i = 1, 2$.*

*Proof.* In terms of $\psi$ defined in (3.9), we may write

$$\tilde z_1(r) = -r\psi'(\log(r))$$

so that the boundedness of $\tilde z_1$ is consequence of (3.17). For $\tilde z_2$, we may consider the equation

$$\phi'' + 2\phi' + 2\phi = g, \quad \text{for } t \geq \log(r_0)$$

with kernel given by $\zeta_1(t) = e^{-t}\cos(t)$, $\zeta_2(t) = e^{-t}\sin(t)$. Then we may express $\tilde z_2$ as a perturbation of the correct linear combination of $\zeta_1$, $\zeta_2$.

□

## 4. Approximate equation and error

The main result in this section is the proof of Proposition 2.1, namely the estimate

$$\|\varepsilon H^s_{\Sigma_0}\|_{1-\varepsilon,\alpha+\varepsilon} \leq \frac{C\varepsilon^{\frac12}}{|\log\varepsilon|}.$$

For for $x \in \Sigma_0$ we compute $H^s_{\Sigma_0}(x)$ by splitting

$$H^s_{\Sigma_0}(x) = \int_{\mathbb{R}^3} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dy = I_i + I_o, \tag{4.1}$$

where

$$I_i = \int_{C_R(x)} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dy, \qquad I_o = \int_{C_R(x)^c} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dy,$$

are inner and outer contributions respectively. The inner part is the integral on a cylinder $C_R(x)$ of radius $R$ centered at $x$ and the outer contribution the rest. We take $R$ as a function of $x \in \Sigma_0$, $x = (x', F_\varepsilon(x'))$, defined by

$$R = (1 - \eta(|x'| - R_0))R_1 + \eta(|x'| - R_0)F_\varepsilon(|x'|) \tag{4.2}$$

where $R_0 > 0$ is fixed large, $R_1 > 0$ is a small constant and $\eta$ is as in (2.8).

To define the cylinder, let $\Pi_1$, $\Pi_2$ be tangent vectors to $\Sigma_0$ at $x$, orthogonal and of length 1, and $\nu_{\Sigma_0}$ be the unit normal vector to $\Sigma_0$ oriented such that $\nu_{\Sigma_0}(x)x_3 > 0$. Introduce coordinates $(t_1, t_2, t_3)$ in $\mathbb{R}^3$ by

$$(t_1, t_2, t_3) \mapsto t_1\Pi_1 + t_2\Pi_2 + t_3\nu_{\Sigma_0}.$$

Define the cylinder of center $x$, radius $R$ and base plane the plane generated by $\Pi_1$, $\Pi_2$ as

$$C_R(x) = \{x + t_1\Pi_1 + t_2\Pi_2 + t_3\nu_{\Sigma_0}(x) : t_1^2 + t_2^2 < R^2, |t_3| < R\}.$$

For the computation of the inner integral, we represent the surface $\Sigma_0$ near $x$ as the graph over its tangent plane at $x$. More precisely, if $R_1 > 0$ in (4.2) is chosen small and $\|h\|_*$ is small, there is a function $g = g_x : B_R(0) \subset \mathbb{R}^2$ to $\mathbb{R}$ of class $C^{2,\alpha}$ such that

$$\Sigma_0 \cap C_R(x) = \{x + \Pi t + \nu_{\Sigma_0} g(t) : |t| < R\}, \tag{4.3}$$

where $t = (t_1, t_2)$ and

$$\Pi = [\Pi_1, \Pi_2].$$

Then

$$g(0) = 0, \quad \nabla g(0) = 0, \quad \Delta g(0) = 2H_{\Sigma_0}(x),$$

where $H_{\Sigma_0}$ is the mean curvature of $\Sigma_0$ at $x$.

In the following statements we use the notation

$$[v]_{\alpha,D} = \sup_{x,y\in D,\, x\neq y} \frac{|v(x) - v(y)|}{|x-y|^\alpha}.$$

**Lemma 4.1.** *For $x \in \Sigma_0$ and $R = R(x)$ given by (4.2) we have*

$$I_i = -2\pi \frac{H_{\Sigma_0}(x) R^\varepsilon}{\varepsilon} + Rest_1, \tag{4.4}$$

*where*

$$|Rest_1| \le C[D^2 g]_{\alpha, B_R(0)} R^{1+\alpha-s} + C\|D^2 g\|^3_{L^\infty(B_R(0))} R^{3-s}. \tag{4.5}$$

*Here $C$ remains bounded as $s \to 1$ (i.e. $\varepsilon \to 0$).*

The main contribution from the outer integral is given in the next result.

**Lemma 4.2.** *For $x = (x', F_\varepsilon(x')) \in \Sigma_0$ and $R = R(x)$ given by (4.2) we have*

$$|I_o| \le \frac{C}{R^{1-\varepsilon}}, \tag{4.6}$$

*and if $|x'| \ge \varepsilon^{-\frac{1}{2}}$,*

$$I_o = \frac{\pi}{R^{1-\varepsilon}} \left(1 + O(\varepsilon^{\frac{1}{2}})\right). \tag{4.7}$$

By (4.4) and (4.7) we see that the equation $H^s_{\Sigma_0}(x) = 0$ takes the form

$$-2H_{\Sigma_0}(x) + \frac{\varepsilon}{R} \approx 0,$$

which motivates (2.3).

**Lemma 4.3.** *Let $x \in \Sigma_0$, and write $x = (x', F_\varepsilon(x'))$, $r = |x'|$. There is $\delta_0 > 0$ and $g : B_\rho(0) \to \mathbb{R}$ of class $C^{2,\alpha}$ such that*

$$\Sigma_0 \cap C_\rho(x) = \{x + \Pi t + \nu g(t) : |t| < \rho\}.$$

*where $\rho = \delta_0 r$. In particular $g$ is well defined in $B_R(0)$ where $R$ is defined in (4.2). Moreover $g$ satisfies*

$$\|g\|_{L^\infty(B_R(0))} \le \begin{cases} C\varepsilon^{\frac{3}{2}} r & \text{if } r \ge \delta |\log\varepsilon| \varepsilon^{-\frac{1}{2}} \\ C\frac{\varepsilon^{\frac{1}{2}}|\log\varepsilon|}{r} & \text{if } \varepsilon^{-\frac{1}{2}} \le r \le \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}} \\ C\frac{\log(r)^2}{r^2} & \text{if } r \le \varepsilon^{-\frac{1}{2}} \end{cases}$$

$$\|Dg\|_{L^\infty(B_R(0))} \le \begin{cases} C\varepsilon^{\frac{1}{2}} & \text{if } r \ge \varepsilon^{-\frac{1}{2}} \\ \frac{C}{r} & \text{if } R_0 \le r \le \varepsilon^{-\frac{1}{2}} \end{cases}$$

$$\|D^2 g\|_{B_R(0)} \le \begin{cases} \frac{C\varepsilon^{\frac{1}{2}}}{r} & \text{if } r \ge \varepsilon^{-\frac{1}{2}} \\ \frac{C}{r^2} & \text{if } r \le \varepsilon^{-\frac{1}{2}} \end{cases}$$

$$[D^2 g]_{\alpha, B_R} \le \begin{cases} \frac{C\varepsilon^{\frac{1}{2}}}{r^{1+\alpha}} & \text{if } r \ge \varepsilon^{-\frac{1}{2}} \\ \frac{C}{r^{2+\alpha}} & \text{if } r \le \varepsilon^{-\frac{1}{2}}. \end{cases} \tag{4.8}$$

(Proof in Appendix C).

*Proof of Lemma 4.1.* We compute

$$I_i = \int_{C_R(x)} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}} \, dy = -2 \int_{|t|<R} \int_0^{g(t)} \frac{1}{(|t|^2 + t_3^2)^{\frac{4-\varepsilon}{2}}} dt_3 \, dt,$$

expanding

$$\int_0^z \frac{1}{(|t|^2 + t_3^2)^{\frac{4-\varepsilon}{2}}} dt_3 = \frac{z}{|t|^{4-\varepsilon}} - (4-\varepsilon) z^2 \int_0^1 (1-\tau) \frac{\tau z}{(|t|^2 + (\tau z)^2)^{\frac{6-\varepsilon}{2}}} \, d\tau.$$

Then

$$I_i = I_{i,1} + I_{i,2} + I_{i,3}$$

where

$$I_{i,1} = -2\int_{|t|<R} \frac{\frac{1}{2}D^2g(0)[t^2]}{|t|^{4-\varepsilon}}\,dt$$

$$I_{i,2} = -2\int_{|t|<R} \frac{g(t)-\frac{1}{2}D^2g(0)[t^2]}{|t|^{4-\varepsilon}}\,dt$$

$$I_{i,3} = 2(4-\varepsilon)\int_{|t|<R} g(t)^2\int_0^1(1-\tau)\frac{\tau g(t)}{(|t|^2+(\tau g(t))^2)^{\frac{6-\varepsilon}{2}}}\,d\tau\,dt,$$

and $D^2g$ denotes the Hessian matrix of $g$. Then

$$I_{i,1} = -\pi\frac{\Delta g(0)R^\varepsilon}{\varepsilon} = -2\pi\frac{H(x)R^\varepsilon}{\varepsilon}.$$

We estimate

$$\begin{aligned} |I_{i,2}| &\le 2\int_{|t|<R}\frac{|g(t)-\frac{1}{2}D^2g(0)[t^2]|}{|t|^{4-\varepsilon}}\,dt \\ &\le C[D^2g]_{B_R(0),\alpha}\int_{|t|<R,t\in\mathbb{R}^2}|t|^{\alpha-2+\varepsilon}dt \le C[D^2g]_{B_R(0),\alpha}R^{\alpha+\varepsilon}. \end{aligned} \tag{4.9}$$

Using $|g(t)| \le \|D^2g\|_{L^\infty(B_R(0))}|t|^2$, we can bound $I_{i,3}$

$$|I_{i,3}| \le C\|D^2g\|_{L^\infty}^3\int_{|t|<R,t\in\mathbb{R}^2}|t|^\varepsilon dt \le C\|D^2g\|_{L^\infty}^3R^{2+\varepsilon}. \tag{4.10}$$

This proves (4.5). □

*Proof of Lemma 4.2.* Let $x\in\Sigma_0$, $x=(x',F_\varepsilon(x'))$. We change variables $y=Rz$ and write $\tilde{x}_R = x/R$

$$\int_{C_R(x)^c}\frac{\chi_{E_0}(y)-\chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\,dy = \frac{1}{R^{1-\varepsilon}}\int_{C_1(\tilde{x}_R)^c}\frac{\chi_{E_0/R}(z)-\chi_{E_0^c/R}(z)}{|\tilde{x}_R-z|^{4-\varepsilon}}\,dz,$$

where $C_1(\tilde{x}_R)$ denotes the cylinder of radius 1 centered at $\tilde{x}_R$ and base plane given by the tangent plane to $\partial E_0/R$ at $\tilde{x}_R$. Then (4.6) follows since

$$\left|\int_{C_1(\tilde{x}_R)^c}\frac{\chi_{E_0/R}(z)-\chi_{E_0^c/R}(z)}{|\tilde{x}_R-z|^{4-\varepsilon}}\,dz\right| \le C.$$

To obtain the second estimate we first note that for any $\delta_0>0$ fixed,

$$\left|\int_{|\tilde{x}_R-z|\ge\delta_0\varepsilon^{-\frac{1}{2}}}\frac{\chi_{E_0/R}(z)-\chi_{E_0^c/R}(z)}{|\tilde{x}_R-z|^{4-\varepsilon}}\,dz\right| \le C\varepsilon^{\frac{1}{2}},$$

and therefore we need to prove

$$\left|\int_{C_1(\tilde{x}_R)^c,|\tilde{x}_R-z|\le\delta_0\varepsilon^{-\frac{1}{2}}}\frac{\chi_{E_0/R}(z)-\chi_{E_0^c/R}(z)}{|\tilde{x}_R-z|^{4-\varepsilon}}\,dz-\pi\right| \le C\varepsilon^{\frac{1}{2}}.$$

We note that

$$\int_{C_1(\tilde{x}_R)^c,|z-\tilde{x}_R|\le\delta_0\varepsilon^{-\frac{1}{2}}}\frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\,dz = \pi+O(\varepsilon^{\frac{1}{2}}).$$

(here $z=(z',z_3)$, $z'\in\mathbb{R}^2$, $e_3=(0,0,1)$). Indeed,

$$\begin{aligned} &\int_{C_1(\tilde{x}_R)^c,|z-\tilde{x}_R|\le\delta_0\varepsilon^{-\frac{1}{2}}}\frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\,dz \\ &= \int_{|z-\tilde{x}_R|>1,|z-\tilde{x}_R|\le\delta_0\varepsilon^{-\frac{1}{2}}}\frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\,dz \end{aligned}$$

since by symmetry the difference of the two integrals is zero. Since

$$\int_{|z-\tilde{x}_R|\geq \delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\, dz = O(\varepsilon^{\frac{1}{2}})$$

we get

$$\begin{aligned}&\int_{C_1(\tilde{x}_R)^c,|z-\tilde{x}_R|\leq \delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\, dz\\ &= \int_{|z-\tilde{x}_R|>1} \frac{\chi_{[|z_3|>1]}-\chi_{[|z_3|<1]}}{|z-\tilde{x}_R|^{4-\varepsilon}}\, dz + O(\varepsilon^{\frac{1}{2}})\\ &= \pi + O(\varepsilon^{\frac{1}{2}}).\end{aligned}$$

Therefore

$$\begin{aligned}&\left|\int_{C_1(\tilde{X}_R)^c,|\tilde{X}_R-Z|\leq \delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{E_0/R}(Z)-\chi_{E_0^c/R}(Z)}{|\tilde{X}_R-Z|^{4-\varepsilon}}\, dZ - \pi\right|\\ &\leq \left|\int_{C_1(\tilde{X}_R)^c,|\tilde{X}_R-Z|\leq \delta_0\varepsilon^{-\frac{1}{2}}} \frac{\chi_{E_0/R}(Z)-\chi_{[|z_3|>1]}+\chi_{[|z_3|<1]}-\chi_{E_0^c/R}(Z))}{|\tilde{X}_R-Z|^{4-\varepsilon}}\, dZ\right| + C\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Note that the point $\tilde{x}_R$ has the form $\tilde{x}_R = (\frac{x'}{R}, 1)$. Inside the region $C_1(\tilde{x}_R)^c \cap \{z : |\tilde{x}_R - z| \leq \delta_0\varepsilon^{-\frac{1}{2}}\}$, $\partial E_0$ can be represented by

$$|z_3| = \frac{1}{R}F_\varepsilon(R|z'|)$$

By Corollary 3.1 we have

$$|\frac{d}{dr}(\frac{1}{R}F_\varepsilon(Rr))| \leq C\varepsilon^{\frac{1}{2}},$$

in $C_1(\tilde{x}_R)^c \cap \{z : |\tilde{x}_R - z| \leq \delta_0\varepsilon^{-\frac{1}{2}}\}$. Let us consider the upper part, namely $C_1(\tilde{x}_R)^c \cap \{z : |\tilde{x}_R - z| \leq \delta_0\varepsilon^{-\frac{1}{2}}\} \cap \{z_3 > 0\}$. Inside this region, the symmetric difference of the two sets $E_0/R$ and $|z_3| > 1$ is contained in the cone

$$\tilde{x}_R + \{(z', z_3) \in \mathbb{R}^2 \times \mathbb{R} : |z'| \leq \delta_0\varepsilon^{-\frac{1}{2}}, |z_3| \leq C\varepsilon^{\frac{1}{2}}|z'|\}.$$

Therefore we can estimate

$$\begin{aligned}&\left|\int_{C_1(\tilde{x}_R)^c,|\tilde{x}_R-z|\leq \delta_0\varepsilon^{-\frac{1}{2}},z_3>0} \frac{\chi_{E_0/R}(z)-\chi_{[|z_3|>1]}+\chi_{[|z_3|<1]}-\chi_{E_0^c/R}(z))}{|\tilde{x}_R-z|^{4-\varepsilon}}\, dz\right|\\ &\leq \int_{\frac{1}{10}\leq|z'|\leq \delta_0\varepsilon^{-\frac{1}{2}},|z_3|\leq C\varepsilon^{\frac{1}{2}}|z|} \frac{1}{|z|^{4-\varepsilon}} dZ \leq C\varepsilon^{\frac{1}{2}}.\end{aligned}$$

The integral over $C_1(\tilde{x}_R)^c \cap \{z : |\tilde{x}_R - z| \leq \delta_0\varepsilon^{-\frac{1}{2}}\} \cap \{z_3 < 0\}$ can be handled similarly. $\square$

*Proof of Proposition 2.1.* Let $x \in \Sigma_0$, $x = (x', F_\varepsilon(x'))$ where $|x'| \geq 1$. Let $R = R(x)$ be given by (4.2).

By (4.1), (4.4) we can write

$$\varepsilon H^s_{\Sigma_0}(x) = -2\pi H_{\Sigma_0} R^\varepsilon + \varepsilon Rest_1 + \varepsilon I_o.$$

Since $\Sigma_0$ is a minimal surface for $r = |x| \leq \varepsilon^{-\frac{1}{2}}$, we have

$$\varepsilon H^s_{\Sigma_0}(x) = E_1 + E_2 + E_3 + E_4 + E_5,$$

where

$$
\begin{aligned}
E_1 &= \pi R^\varepsilon \eta_\varepsilon(-2H_{\Sigma_0} + \frac{\varepsilon}{R})\\
E_2 &= -2\varepsilon \int_{|t|<R} \frac{g(t) - \frac{1}{2}D^2 g(0)[t^2]}{|t|^{4-\varepsilon}}\, dt\\
E_3 &= \varepsilon 2(4-\varepsilon) \int_{|t|<R} g(t)^2 \int_0^1 (1-\tau) \frac{\tau g(t)}{(|t|^2 + (\tau g(t))^2)^{\frac{5+s}{2}}}\, d\tau\, dt\\
E_4 &= \varepsilon I_o (1-\eta_\varepsilon)\\
E_5 &= (\varepsilon I_o - \frac{\pi\varepsilon}{R^s})\eta_\varepsilon,
\end{aligned}
$$

and $\eta_\varepsilon(r) = \eta(r - \varepsilon^{-\frac{1}{2}})$ with $\eta$ is the cut-off function (2.8). Here $g$ is a function such that we have the representation of $\Sigma_0$ near $X$ as the graph of $g$ over the tangent plane of $\Sigma_0$ at $X$, as in (4.3).

We start with $E_1$. For $r \geq \varepsilon^{-\frac{1}{2}} + 1$, $F_\varepsilon$ satisfies $\Delta F_\varepsilon = \frac{\varepsilon}{F_\varepsilon}$, so

$$
E_1 = \pi F_\varepsilon^\varepsilon \left( \Delta F_\varepsilon (1 - \frac{1}{\sqrt{1 + (F_\varepsilon')^2}}) + \frac{(F_\varepsilon')^2 F_\varepsilon''}{(1 + (F_\varepsilon')^2)^{3/2}} \right).
$$

But for this range $F_\varepsilon'(r) = O(\varepsilon^{\frac{1}{2}})$, $F_\varepsilon''(r) = O(\frac{\varepsilon^{\frac{1}{2}}}{r})$, $F_\varepsilon(r) \leq C\varepsilon^{\frac{1}{2}} r$ if $r \geq \delta\varepsilon^{-\frac{1}{2}}|\log\varepsilon|$ and $F_\varepsilon(r) \leq C|\log\varepsilon|$ if $\varepsilon^{-\frac{1}{2}} r \leq \delta\varepsilon^{-\frac{1}{2}}|\log\varepsilon|$ , so

$$
\sup_{r \geq \varepsilon^{-1/2}+1} r^{1-\varepsilon}|E_1| = O(\varepsilon^{\frac{3}{2}}), \quad \text{as } \varepsilon \to 0.
$$

For $r \in [\varepsilon^{-\frac{1}{2}}, \varepsilon^{-\frac{1}{2}} + 1]$ we have $\Delta f_\varepsilon = O(\frac{\varepsilon}{|\log\varepsilon|})$, $\Delta f_C = O(\varepsilon^2)$, and so $(f_\varepsilon - f_C)' = O(\frac{\varepsilon}{|\log\varepsilon|})$, $f_\varepsilon - f_C = O(\frac{\varepsilon}{|\log\varepsilon|})$ in this region. Then for these $r$

$$
\begin{aligned}
-\Delta F_\varepsilon + \frac{\varepsilon}{F_\varepsilon} &= -\eta_\varepsilon \frac{\varepsilon}{f_\varepsilon} + \frac{\varepsilon}{\eta_\varepsilon f_\varepsilon + (1-\eta_\varepsilon) f_C} - (1-\eta_\varepsilon)\Delta f_C - 2\eta_\varepsilon'(f_\varepsilon - f_C)' - \Delta\eta_\varepsilon (f_\varepsilon - f_C)\\
&= O(\frac{\varepsilon}{|\log\varepsilon|}).
\end{aligned}
$$

It follows that

$$
\sup_{r \in [\varepsilon^{-\frac{1}{2}}, \varepsilon^{-\frac{1}{2}}+1]} r^{1-\varepsilon}|E_1| = O(\frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|}).
$$

To estimate the Hölder part of the norm, i.e. $[E_1]_{1-\varepsilon,\alpha+\varepsilon}$, it is enough to show that

$$
\sup_{r \geq \varepsilon^{-1/2}} r^{2-\varepsilon}|E_1'(r)| \leq C\frac{\varepsilon^{\frac{1}{2}}}{|\log\varepsilon|},
$$

and the computation is analogous to the previous one.

We estimate $E_2$. By (4.9), we need to estimate

$$
\varepsilon \| [D^2 g]_{B_R(0),\alpha} R^{\alpha+\varepsilon} \|_{1-\varepsilon,\alpha+\varepsilon}
$$

where $R = R(x)$, $g = g_x$, $x = (x', F_\varepsilon(x')) \in \Sigma_0$. In the regime $r = |x'| \geq \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$ we have

$$
R = F_\varepsilon(r) \leq C\varepsilon^{\frac{1}{2}} r, \tag{4.11}
$$

and by (4.8)

$$
[D^2 g]_{\alpha, B_R(0)} \leq \frac{C\varepsilon^{\frac{1}{2}}}{r^{1+\alpha}} \tag{4.12}
$$

Therefore

$$
\sup_{r \geq \delta|\log\varepsilon|\varepsilon^{-1/2}} r^{1-\varepsilon} \left( \varepsilon [D^2 g]_{\alpha, B_R(0)} R^{\alpha+\varepsilon} \right) \leq C\varepsilon^{\frac{3+\alpha}{2}}.
$$

In the region $\varepsilon^{-\frac{1}{2}} \leq r \leq \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$ we have

$$R = F_\varepsilon(r) = O(|\log\varepsilon|)$$

and (4.12) still holds. Hence

$$\sup_{\varepsilon^{-1/2}\leq r\leq\delta|\log\varepsilon|\varepsilon^{-1/2}} r^{1-\varepsilon}\left(\varepsilon[D^2g]_{\alpha,B_R(0)}R^{\alpha+\varepsilon}\right) \leq C\varepsilon^{\frac{3+\alpha}{2}}|\log\varepsilon|^\alpha.$$

Finally for $r \leq \varepsilon^{-\frac{1}{2}}$, $R = \log(r) + O(1)$ and

$$[D^2g]_{\alpha,B_R(0)} \leq \frac{C}{r^{2+\alpha}}$$

so

$$\sup_{r\leq\varepsilon^{-\frac{1}{2}}} r^{1-\varepsilon}\left(\varepsilon[D^2g]_{B_R(0),\alpha}R^{\alpha+\varepsilon}\right) \leq C\varepsilon.$$

It follows that

$$\||x|^{1-\varepsilon}E_2\|_{L^\infty} \leq C\varepsilon.$$

We estimate the $C^\alpha$ norm of $E_2$. For this let $x_1 = (x_1', F_\varepsilon(x_1'))$, $x_2 = (x_2', F_\varepsilon(x_2')) \in \Sigma_0$, $R_i = R(x_i)$, and $g_i : B_{R_i} \to \mathbb{R}$ be such that $\Sigma_0$ can be represented as a graph of $g_i$ over its tangent plane at $x_i$.

We can assume that $|x_1| \leq |x_2|$ and $R_1 \leq R_2$ (if $R_2 \leq R_1$ the argument is the same). We can also assume $|x_1 - x_2| \leq \frac{1}{10}|x_1|$.

Let us write

$$E_1(x_1) - E_2(x_2) = E_{1,1} + E_{1,2},$$

where

$$E_{1,1} = \varepsilon\int_{|t|<R_1} \frac{g_1(t) - \frac{1}{2}D^2g_1(0)[t^2] - (g_2(t) - \frac{1}{2}D^2g_2(0)[t^2])}{|t|^{4-\varepsilon}}\,dt$$

$$E_{1,2} = -\varepsilon\int_{R_1<|t|<R_2} \frac{g_2(t) - \frac{1}{2}D^2g_2(0)[t^2]}{|t|^{4-\varepsilon}}\,dt.$$

Assume $|x_1-x_2| \leq R_1$. Then note that by the same computation as in Lemma 4.1 and writing $\bar{R} = |x_1-x_2|$,

$$\begin{aligned}&\left|\int_{|t|\leq\bar{R}} \frac{g_1(t) - \frac{1}{2}D^2g_1(0)[t^2] - (g_2(t) - \frac{1}{2}D^2g_2(0)[t^2])}{|t|^{4-\varepsilon}}\,dt\right|\\ &\leq C([D^2g_1]_{\alpha,B_{\bar{R}}(0)} + [D^2g_2]_{\alpha,B_{\bar{R}}(0)})\bar{R}^{1+\alpha-s}\\ &\leq C\frac{\varepsilon^{\frac{3}{2}}}{|x_1|^{1+\alpha}}|x_1 - x_2|^{\alpha+\varepsilon},\end{aligned}$$

where we have used (4.8). Let us estimate the integral over $\bar{R} \leq |t| \leq R_1$. For this note that from Appendix C

$$\|D^2(g_1 - g_2)\|_{L^\infty(B_{R_1})} \leq \frac{C\varepsilon^{\frac{1}{2}}}{|x_1|^2}|x_1 - x_2|$$

if $|x_1| \geq \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$. In this case we see that

$$\begin{aligned}&\left|\int_{\bar{R}\leq|t|\leq R_1} \frac{g_{1,0}(t) - \frac{1}{2}D^2g_1(0)[t^2] - (g_2(t) - \frac{1}{2}D^2g_2(0)[t^2])}{|t|^{4-\varepsilon}}\,dt\right|\\ &\leq \frac{C\varepsilon^{\frac{1}{2}}}{|x_1|^2}|x_1 - x_2|\int_{\bar{R}\leq|t|\leq R_1} |t|^{-1-s}\,dt\\ &\leq \frac{C\varepsilon^{\frac{1}{2}}}{|x_1|^2}|x_1 - x_2|^{\alpha+\varepsilon}\int_{\bar{R}\leq|t|\leq R_1} |t|^{-s-\alpha-\varepsilon}\,dt\\ &\leq \frac{C\varepsilon^{\frac{1}{2}}}{|x_1|^2}|x_1 - x_2|^{\alpha+\varepsilon}R_1^{2-s-\alpha-\varepsilon} = \frac{C\varepsilon^{\frac{1}{2}}}{|x_1|^2}|x_1 - x_2|^{\alpha+\varepsilon}R_1^{1-\alpha}.\end{aligned}$$

Then recalling that $R_1 = O(\varepsilon^{\frac12}|X_1|)$

$$\begin{aligned}&\varepsilon \left| \int_{\bar R \le |t| \le R_1} \frac{g_1(t) - \frac12 D^2 g_1(0)[t^2] - (g_2(t) - \frac12 D^2 g_2(0)[t^2])}{|t|^{4-\varepsilon}}\, dt \right| \\ &\le C \frac{\varepsilon^{\frac{4-\alpha}{2}}}{|x_1|^{1+\alpha}} |x_1 - x_2|^{\alpha+\varepsilon}.\end{aligned}$$

Therefore

$$|E_{1,1}| \le C \frac{\varepsilon^{\frac32}}{|x_1|^{1+\alpha}} |x_1 - x_2|^{\alpha+\varepsilon},$$

if $|x_1| \ge \delta |\log \varepsilon| \varepsilon^{-\frac12}$. The other cases can be handled similarly.

For the second term we have

$$\left| \int_{R_1 < |t| < R_2} \frac{g_2(t) - \frac12 D^2 g_2(0)[t^2]}{|t|^{4-\varepsilon}}\, dt \right| \le C [D^2 g_2]_{\alpha, B_{R_2}(0)} (R_2^{1+\alpha-s} - R_1^{1+\alpha-s}).$$

But we can estimate

$$|R_2 - R_1| \le C \frac{\varepsilon^{\frac12}}{|x_1|^{\alpha-1}} |x_1 - x_2|^\alpha$$

if $|x_1| \ge \varepsilon^{-\frac12}$, and

$$|R_2 - R_1| \le C \frac{|x_1 - x_2|^\alpha}{|x_1|^\alpha}$$

if $|x_1| \le \varepsilon^{-\frac12}$. Summarizing

$$|E_2(x_1) - E_2(x_2)| \le C\varepsilon \frac{|x_1 - x_2|^{\alpha+\varepsilon}}{|x_1|^{1+\alpha}}.$$

Let us consider $E_3$. By (4.10)

$$|E_3| \le C\varepsilon \|D^2 g\|^3_{L^\infty(B_R)} R^{3-s}.$$

In the region $|x| \ge \delta |\log \varepsilon| \varepsilon^{-\frac12}$ we use (4.11) and

$$\|D^2 g\|_{L^\infty(B_R(0))} \le \frac{C\varepsilon^{\frac12}}{|x|}$$

to obtain

$$\sup_{|x| \ge \delta |\log \varepsilon| \varepsilon^{-\frac12}} |x|^{1-\varepsilon} \left( \varepsilon \|D^2 g\|^3_{L^\infty(B_R(0))} R^{2+\varepsilon} \right) \le C \varepsilon^{\frac72}.$$

When $\varepsilon^{-\frac12} \le |x| \le \delta |\log\varepsilon| \varepsilon^{-\frac12}$ we get

$$\sup_{\varepsilon^{-\frac12} \le |x| \le \delta |\log \varepsilon| \varepsilon^{-\frac12}} |x|^{1-\varepsilon} \left( \varepsilon \|D^2 g\|^3_{L^\infty(B_R(0))} R^{2+\varepsilon} \right) \le C \varepsilon^{\frac72} |\log \varepsilon|^2.$$

Also

$$\sup_{|x| \le \varepsilon^{-\frac12}} |x|^{1-\varepsilon} \left( \varepsilon \|D^2 g\|^3_{L^\infty(B_R(0))} R^{2+\varepsilon} \right) \le C\varepsilon.$$

Similar computations as before show that if $|x_1| \le |x_2|$ and $|x_1 - x_2| \le \frac{1}{10}|x_1|$ then

$$|E_3(x_1) - E_3(x_2)| \le C\varepsilon \frac{|x_1 - x_2|^{\alpha+\varepsilon}}{|x_1|^{1+\alpha}}.$$

To estimate $E_4 = \varepsilon I_o(1 - \eta_\varepsilon)$ we use (4.6):

$$|\varepsilon I_o(1 - \eta_\varepsilon)| \le \frac{C\varepsilon}{R^{1-\varepsilon}},$$

and note that it is supported in $r \leq \varepsilon^{-\frac{1}{2}} + 1$. But in this range $R = \log(|x'|) + O(1)$ and this implies

$$\sup_x |x|^{1-\varepsilon} |E_4| \leq C\varepsilon \sup_{|x| \leq \varepsilon^{-\frac{1}{2}}} \frac{|x|^{1-\varepsilon}}{|\log(|x|)|} \leq \frac{C\varepsilon^{\frac{1}{2}}}{|\log \varepsilon|}.$$

To estimate the Hölder norm of $E_4$, we actually claim that

$$|D_x E_4(x)| \leq \frac{C}{|x| \log(|x|)^2}. \tag{4.13}$$

To obtain (4.13), we write $x = (x', F_\varepsilon(x'))$ and write $x' = (x_1, x_2)$. We compute

$$D_{x_i} \int_{C_R(x)^c} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dy = B_1 + B_2 + B_3$$

with

$$B_1 = -(4-\varepsilon) \int_{C_R(x)^c} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{5+s}} \langle x - y, D_{x_i} x \rangle\, dy$$

$$B_2 = -\int_{\partial C_R(x)} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}} \langle \nu, \frac{y-x}{|y-x|} \rangle\, dy D_{x_i} R$$

$$B_3 = -\int_{\partial C_R(x)} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}} \nu_i\, dy,$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit exterior normal to $C_R(x)$. But $D_{x_i} x = e_i + (0, 0, D_{x_i} F_\varepsilon)$ and so $B_1$ can be combined with $B_3$. Indeed

$$B_1 = B_{1,1} + B_{1,2}$$

where

$$B_{1,1} = -(4-\varepsilon) \int_{C_R(x)^c} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{5+s}} \langle x - y, e_i \rangle\, dy$$

$$B_{1,2} = -(4-\varepsilon) \int_{C_R(x)^c} \frac{\chi_{E_h}(y) - \chi_{E_h^c}(y)}{|x-y|^{5+s}} \langle x - y, (0, 0, D_{x_i} F_\varepsilon) \rangle\, dY.$$

But

$$B_{1,1} = -\int_{C_R(x)^c} D_{y_i} \frac{\chi_{E_0}(y) - \chi_{E_0^c}(y)}{|x-y|^{4-\varepsilon}}\, dY$$

$$= 2\int_{\partial E_h \setminus C_R(x)} \frac{1}{|x-y|^{4-\varepsilon}} \nu_i\, dy + \int_{\partial C_R(x)} \frac{\chi_{E_h}(y) - \chi_{E_h^c}(y)}{|x-y|^{4-\varepsilon}} \nu_i\, dy,$$

where we have used the unit normal $\nu$ pointing up on $\partial E_0$, and the exterior unit normal to $C_R(x)$. Therefore

$$B_1 + B_2 + B_3 = 2\int_{\partial E_0 \setminus C_R(x)} \frac{1}{|x-y|^{4-\varepsilon}} \nu_i\, dy + B_{1,2} + B_2.$$

We now estimate

$$\int_{\partial E_0 \setminus C_R(x)} \frac{1}{|x-y|^{4-\varepsilon}} \nu_i\, dy = \int_{\partial E_0 \cap B_\rho(x) \setminus C_R(x)} \ldots + \int_{\partial E_0 \setminus B_\rho(x)} \ldots$$

where we take $\rho = |x|/100$. For the outside integral we have

$$\left| \int_{\partial E_0 \setminus B_\rho(x)} \frac{1}{|x-y|^{4-\varepsilon}} \nu_i\, dy \right| \leq C\rho^{-1-s} \leq C|x|^{-1-s}.$$

For the inner region, we observe that $|\nu_i(y)| \leq \frac{C}{|y|} \leq \frac{C}{|x|}$ and so

$$\left| \int_{\partial E_0 \cap B_\rho(x) \setminus C_R(x)} \frac{1}{|x-y|^{4-\varepsilon}} \nu_i\, dy \right| \leq \frac{C}{|x| \log(|x|)^{2-\varepsilon}}.$$

For $B_2$ we also get

$$|B_2| \le \frac{C}{|x|\log(|x|)^{2-\varepsilon}}$$

since $D_{x_i} R = O(1/|x_0|)$ while the integral is $O(\frac{C}{\log(|x_0|)^2})$. Hence

$$|B| \le \frac{C}{|x_0|\log(|x_0|)^2}.$$

and combined with the estimate for $A$ yields (4.13).

Using (4.13) we can estimate the Hölder norm of $E_4$. For this let $x_1, x_2 \in \Sigma_0$, $x_i = (x_i', F_\varepsilon(x_i'))$, $R_i = R(x_i)$. We can assume that $|x_1| \le |x_2|$, $R_1 \le R_2$ and also $|x_1 - x_2| \le \frac{1}{10}|x_1|$.

Then

$$\begin{aligned}|E_4(x_1) - E_4(x_2)| &\le C\varepsilon \frac{|x_{0,1}|^{1+\alpha}}{|x_{0,1}|\log(|x_{0,1}|)^2}|x_{0,1} - x_{0,2}| \\ &\le C\varepsilon \frac{|x_{0,1}|}{\log(|x_{0,1}|)^2}|x_{0,1} - x_{0,2}|^\alpha \\ &\le C\frac{\varepsilon^{\frac12}}{|\log\varepsilon|}|x_{0,1} - x_{0,2}|^\alpha.\end{aligned}$$

Therefore

$$\|E_4\|_{1-\varepsilon,\alpha+\varepsilon} \le \frac{C\varepsilon^{\frac12}}{|\log\varepsilon|}.$$

To estimate $E_5 = \varepsilon(I_o - \frac{\pi}{R^s})\eta_\varepsilon$ we use (4.7) to obtain

$$|E_5| \le C\frac{\varepsilon^{\frac32}}{R^s}.$$

Since $R = F_\varepsilon(x') = \varepsilon^{\frac12} r + O(|\log\varepsilon|)$ for $r \ge \delta|\log\varepsilon|\varepsilon^{-\frac12}$, where $r = |x'|$, then

$$\sup_{r\ge\delta|\log\varepsilon|\varepsilon^{-\frac12}} r^{1-\varepsilon}|E_5(r)| \le C\varepsilon.$$

Also

$$\sup_{\varepsilon^{-\frac12}\le r\le\delta|\log\varepsilon|\varepsilon^{-\frac12}} r^{1-\varepsilon}|E_5| \le C \sup_{\varepsilon^{-\frac12}\le r\le\delta|\log\varepsilon|\varepsilon^{-\frac12}} \frac{\varepsilon r^{1-\varepsilon}}{|\log\varepsilon|} \le \frac{C\varepsilon^{\frac12}}{|\log\varepsilon|}.$$

We estimate the Hölder estimate for $E_5$. Let us write $x = (x', F_\varepsilon(x'))$, $x' = (x_1, x_2)$, $r = |x'|$. We claim that

$$|\frac{d}{dx_i}E_5| \le C\frac{\varepsilon}{r^{2-\varepsilon}} \quad \text{for } r \ge \delta|\log\varepsilon|\varepsilon^{-\frac12}. \tag{4.14}$$

As in the proof of Lemma 4.2 we can rewrite $E_5$ as

$$E_5 = \frac{\varepsilon}{R^{1-\varepsilon}}\eta_\varepsilon J,$$

where

$$J = \int_{C_1(\tilde x_R)^c} \frac{\chi_{E_0/R}(z) - \chi_{E_0^c/R}(z)}{|\tilde x_R - z|^{4-\varepsilon}}\,dz,$$

$\tilde x_R = x/R$ and $C_1(\tilde x_R)$ is the cylinder of radius 1 centered at $\tilde x_R$ and base plane given by the tangent plane to $\partial E_0/R$ at $\tilde x_R$. Then

$$\frac{d}{dx_i}E_5 = D_1 + D_2 + D_3$$

where

$$D_1 = -\frac{\varepsilon(1-\varepsilon)}{R^{2-\varepsilon}}\frac{dR}{dx_i}\eta_\varepsilon J, \qquad D_2 = \frac{\varepsilon}{R^{1-\varepsilon}}\eta'_\varepsilon \frac{dr}{dx_i}J,$$
$$D_3 = \frac{\varepsilon}{R^{1-\varepsilon}}\eta_\varepsilon \frac{dJ}{dx_i}.$$

Let us estimate $D_3$. By a translation

$$J = \int_{C_x^c} \frac{\chi_{E_x} - \chi_{E_x^c}}{|z|^{4-\varepsilon}}\, dz$$

where $C_x$ is the cylinder centered at the origin, with base a unit disk on a plane parallel to the tangent plane to $\Sigma_0$ at $x$, and unit height, and $E_x = (E_0 - x)/R$.

We can write

$$\begin{aligned}\frac{dJ}{dx_i} &= -2\int_{\partial E_x\setminus C_x} \frac{1}{|z|^{4-\varepsilon}}\nu(z)\cdot(\frac{1}{R^2}\frac{dR}{dx_i}z + \frac{1}{R}\frac{dx}{dx_i})\, dz\\ &\quad + \int_{\partial C_x\cap E_x} \frac{1}{|z|^{4-\varepsilon}}\nu(z)\cdot(\frac{1}{R^2}\frac{dR}{dx_i}z + \frac{1}{R}\frac{dx}{dx_i})\, dz\\ &\quad - \int_{\partial C_x\cap E_x^c} \frac{1}{|z|^{4-\varepsilon}}\nu(z)\cdot(\frac{1}{R^2}\frac{dR}{dx_i}z + \frac{1}{R}\frac{dx}{dx_i})\, dz,\end{aligned}$$

where for points on $\partial E_x$ $\nu$ represents the unit normal vector pointing into $E_0$, and on $\partial C_x$, $\nu$ points oust side of $C_x$. This follows from the transport theorem in the form

$$\frac{d}{dt}\int_{T_t(U)} f(y)\, dy = \int_{\partial T_t(U)} f(y)\nu(y)\cdot(D_t T_t)(T_t^{-1}(y))\, dy,$$

where $\nu$ points to the exterior of $T_t(U)$. In our case $E_x = T_{x'}(E_0)$ where $T_{x'}(y) = \frac{1}{R}(y-x)$, $x = (x', F_\varepsilon(x'))$. Hence $D_{x_i}T_{x'}(T_{x'}^{-1}(z)) = -\frac{1}{R^2}z\frac{dR}{dx_i} - \frac{1}{R}\frac{dx}{dx_i}$.

We claim that for $|x'| \geq \varepsilon^{-\frac{1}{2}}$ we have:

$$\frac{dJ}{dx_i} = O(\frac{1}{r^2}), \tag{4.15}$$

$r = |x'|$. Indeed, we compute with detail the case when $|x'| \geq \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$. For these $x'$, $R = \varepsilon^{\frac{1}{2}}r + O(|\log\varepsilon|)$, $\frac{dR}{dx_i} = O(\varepsilon^{\frac{1}{2}})$. Then

$$\begin{aligned}&\left|\int_{\partial E_x\setminus C_x\cap[z_3>-1]\cap[|z|\geq\varepsilon^{-\frac{1}{2}}/100]} \frac{\nu(z)\cdot z}{|z|^{4-\varepsilon}}\, dz\right|\\ &\leq \int_{\partial E_x\setminus C_x\cap[z_3>-1]\cap[|z|\geq\varepsilon^{-\frac{1}{2}}/100]} \frac{1}{|z|^{4-\varepsilon}}\, dz = O(\varepsilon).\end{aligned}$$

Inside the ball $|z| \leq \varepsilon^{-\frac{1}{2}}/100$, we have $\nu(z)\cdot z = O(\varepsilon^{\frac{1}{2}})|z|$. Then

$$\int_{\partial E_x\setminus C_x\cap[z_3>-1]\cap[|z|\leq\varepsilon^{-\frac{1}{2}}/100]} \frac{\nu(z)\cdot z}{|z|^{4-\varepsilon}}\, dz = O(\varepsilon^{\frac{1}{2}}).$$

The estimate in the lower half are similar, and therefore

$$\frac{1}{R^2}\frac{dR}{dx_i}\int_{\partial E_x\setminus C_x} \frac{1}{|z|^{4-\varepsilon}}\nu(z)\cdot z dz = O(\frac{1}{r^2})$$

where $r = |x'|$.

In the upper half we have

$$\begin{aligned}\left|\int_{\partial E_x\setminus C_x\cap[z_3>-1]\cap[|z|\geq\varepsilon^{-1/2}/100} \frac{\frac{dx}{dx_i}\cdot\nu(z)}{|z|^{4-\varepsilon}}\, dz\right| &\leq C\int_{\partial E_x\setminus C_x\cap[z_3>-1]\cap[|z|\geq\varepsilon^{-1/2}/100} \frac{dz}{|z|^{4-\varepsilon}}\\ &= O(\varepsilon).\end{aligned}$$

For the integral over $|z| \le \varepsilon^{-\frac{1}{2}}/100$, notice that before the change of variables $y \mapsto z = (y-x)/R$, we have

$$\left| \nu_{\Sigma_0}(y) \cdot \frac{dx}{dx_i}(x') \right| \le C \frac{\varepsilon^{\frac{1}{2}}}{|x|} |y-x|$$

for $y \in \Sigma_0$, $|y-x| \le |x|/100$, since $\nu_{\Sigma_0}(y) \cdot \frac{dx}{dx_i}(x')$ vanishes at $y = x$ and has derivative of order $\frac{\varepsilon^{\frac{1}{2}}}{|x|}$. After the change of variables this implies

$$|\frac{dx}{dx_i} \cdot \nu(z)| \le C \frac{\varepsilon^{\frac{1}{2}}}{r} |z|.$$

Therefore

$$\int_{\partial E_x \setminus C_x \cap [z_3 > -1] \cap [|z| \le \varepsilon^{\frac{1}{2}}/100} \frac{\frac{dx}{dx_i} \cdot \nu(z)}{|z|^{4-\varepsilon}} \, dz = O(\frac{\varepsilon^{\frac{1}{2}}}{r}).$$

The estimate in the lower half are similar, and therefore

$$\frac{1}{R} \int_{\partial E_x \setminus C_x} \frac{1}{|z|^{4-\varepsilon}} \nu(z) \cdot \frac{dx}{dx_i} \, dz = O(\frac{1}{r^2}).$$

For the last 2 terms in $\frac{dJ}{dx_i}$ we observe that

$$\int_{\partial C_x \cap E_x} \frac{1}{|z|^{4-\varepsilon}} \nu(z) \cdot z \, dz - \int_{\partial C_x \cap E_x^c} \frac{1}{|z|^{4-\varepsilon}} \nu(z) \cdot z \, dz = O(\varepsilon^{\frac{1}{2}})$$

since most of the integral cancels by symmetry, except a region of area $O(\varepsilon^{\frac{1}{2}})$ and similarly

$$\int_{\partial C_x \cap E_x} \frac{1}{|z|^{4-\varepsilon}} \nu(z) \cdot \frac{dx}{dx_i} \, dz - \int_{\partial C_x \cap E_x^c} \frac{1}{|z|^{4-\varepsilon}} \nu(z) \cdot \frac{dx}{dx_i} \, dz = O(\frac{\varepsilon^{\frac{1}{2}}}{r}).$$

This implies and the previous estimates imply the claim (4.15) (the range $\varepsilon^{-\frac{1}{2}} \le r \le \delta \varepsilon^{-\frac{1}{2}} |\log \varepsilon|$ is analogous).

The estimates for $D_1$, and $D_2$ are analogous, and since $R \approx \varepsilon^{\frac{1}{2}} |x_0|$ we obtain (4.14).

Using (4.14), we can show, as was done before, that

$$|E_5(x_1) - E_5(x_2)| \le C \frac{\varepsilon^{\frac{1}{2}}}{|\log \varepsilon|} |x_{0,1} - x_{0,2}|^\alpha.$$

$x_1, x_2 \in \Sigma_0$, with $x_i = (x_i', F_\varepsilon(x_i'))$, $|x_i'| \ge \varepsilon^{-\frac{1}{2}}$, and $|x_1| \le |x_2|$, $|x_1 - x_2| \le \frac{1}{10}|x_1|$. It follows that

$$\|E_5\|_{1-\varepsilon, \alpha+\varepsilon} \le C \frac{\varepsilon^{\frac{1}{2}}}{|\log \varepsilon|}.$$

□

## 5. Limit problem in $\Sigma_0$

We want to build a right inverse for the operator

$$L_0(h) = \Delta h + \frac{\varepsilon}{F_\varepsilon(r)^2} \eta_\varepsilon(r) h,$$

which arises as the linearization of the approximate problem (2.5). Here $\eta_\varepsilon$ is any family continuous cut-off functions with $\eta_\varepsilon(r) = 0$ for $r \le \varepsilon^{-\frac{1}{2}}$ and $\eta_\varepsilon(r) = 1$ for $r \ge \delta |\log \varepsilon| \varepsilon^{-\frac{1}{2}}$, where $\delta > 0$ is a sufficiently small number.

We then consider the equation

$$L_0(\phi) = g, \quad \text{in } \mathbb{R}^2, \tag{5.1}$$

and work in the class of radial functions.

**Proposition 5.1.** *Let $1 \le \gamma < 2$. If $\varepsilon > 0$ is small there is $C > 0$ such that for $g$ radially symmetric with $\|(1+|x|)^\gamma g\|_{L^\infty} < +\infty$ there exists a radially symmetric solution of (5.1) $\phi = T(g)$ with $\|(1+|x|)^{\gamma-2}\phi\|_{L^\infty} < +\infty$ that defines a linear operator of $g$ with*

$$\||x|^{\gamma-2}\phi\|_{L^\infty} \ \le \ C\,\|(1+|x|)^\gamma g\|_{L^\infty},$$

*and $\phi(0) = 0$.*

*Proof.* Since all functions are radial, we have to solve

$$\phi'' + \frac{1}{r}\phi' + \frac{\varepsilon}{F_\varepsilon(r)^2}\eta_\varepsilon(r)\phi = g, \quad r > 0. \tag{5.2}$$

We solve this ODE with initial condition $\phi(0) = \phi'(0) = 0$. For $r \le \varepsilon^{-\frac{1}{2}}$ we obtain directly

$$|\phi(r)| \le C r^{2-\gamma}\|(1+|x|)^\gamma g\|_{L^\infty}, \quad \forall r \ge 0.$$

We also have

$$|\phi(\varepsilon^{-\frac{1}{2}}+1)| \le C\varepsilon^{\frac{\gamma-2}{2}}\|(1+|x|)^\gamma g\|_{L^\infty}, \tag{5.3}$$

$$|\phi'(\varepsilon^{-\frac{1}{2}}+1)| \le C\varepsilon^{\frac{\gamma-1}{2}}\|(1+|x|)^\gamma g\|_{L^\infty}. \tag{5.4}$$

Let us estimate $\phi(r)$ for $r \ge \varepsilon^{-\frac{1}{2}}+1$. First we deal with the region $\varepsilon^{-\frac{1}{2}}+1 \le r \le \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$, where $\delta > 0$ is to be fixed later on. Let us rewrite (5.2) as

$$\phi_{rr} + \frac{1}{r}\phi_r = \tilde{g}, \quad \text{for } \varepsilon^{-\frac{1}{2}}+1 \le r \le \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$$

where

$$\tilde{g} = g - \frac{\varepsilon}{F_\varepsilon(r)^2}\eta_\varepsilon(r)\phi,$$

and let $r_0 = \varepsilon^{-\frac{1}{2}}+1$. Integrating we find

$$\phi(r) = \phi(r_0) + r_0\phi'(r_0)\log(\frac{r}{r_0}) + \int_{r_0}^r \frac{1}{t}\int_{r_0}^t \tau\tilde{g}(\tau)\,d\tau dt. \tag{5.5}$$

Let us introduce the norm

$$\|\phi\| = \sup_{r\in I} r^{\gamma-2}|\phi(r)|,$$

where $I = [\varepsilon^{-\frac{1}{2}}+1, \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}]$. We have from (3.3)

$$\Big|\frac{\varepsilon}{F_\varepsilon(r)^2}\Big| \le \frac{C\varepsilon}{|\log\varepsilon|^2}, \quad \text{for } \varepsilon^{-\frac{1}{2}}+1 \le r \le \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}. \tag{5.6}$$

Using formulas (5.3), (5.4),(5.5), (5.6) we find

$$\|\phi\| \le C\delta^2\|\phi\| + C\|(1+|x|)^\gamma g\|_{L^\infty}.$$

Then we can choose $\delta > 0$ small so that for all $\varepsilon > 0$ small we find

$$\|\phi\| \le C\|(1+|x|)^\gamma g\|_{L^\infty}.$$

This is the desired estimate in the region $\varepsilon^{-\frac{1}{2}} \le r \le \delta\varepsilon^{-\frac{1}{2}}|\log\varepsilon|$.

Let us consider the range $r \ge r_1$ where $r_1 = \delta|\log\varepsilon|\varepsilon^{-\frac{1}{2}}$. Then by the previous step we have

$$|\phi(r_1)| \le C r_1^{2-\gamma}\|(1+|x|)^\gamma g\|_{L^\infty}, \quad |\phi'(r_1)| \le C r_1^{1-\gamma}\|(1+|x|)^\gamma g\|_{L^\infty}.$$

We write the solution $\phi$ in terms of elements in the kernel of the linear operator $\Delta + \frac{\varepsilon}{f_\varepsilon^2}$, where $f_\varepsilon$ is defined in (2.6). Let $\tilde{z}_i$ be the functions introduced in (3.19) and

$$z_i(r) = \tilde{z}_i\Big(\frac{\varepsilon^{\frac{1}{2}}r}{|\log\varepsilon|}\Big), \quad r \ge \frac{\delta|\log\varepsilon|}{\varepsilon^{\frac{1}{2}}}.$$

By Lemma 3.3 we have

$$|z_i(r)| \le C, \qquad |z_i'(r)| \le \frac{C}{r}, \quad r \ge r_1. \tag{5.7}$$

We write now

$$\phi(r) = c_1 z_1(r) + c_2 z_2(r) + \phi_0(r), \qquad r \ge r_1, \tag{5.8}$$

where $c_1$, $c_2$ are determined so that

$$\phi(r_1) = c_1 z_1(r_1) + c_2 z_2(r_1), \quad \phi'(r_1) = c_1 z_1'(r_1) + c_2 z_2'(r_1)$$

and

$$\phi_0(r) = -z_1(r) \int_{r_1}^r \frac{z_2(s)h(s)}{W(s)}\, ds + z_2(r) \int_{r_1}^r \frac{z_1(s)h(s)}{W(s)}\, ds.$$

Here $W = z_1' z_2 - z_1 z_2'$ is the Wronskian. Then $W = \frac{c}{r}$ for some $c$ and using (5.7) gives $c = O(1)$. Also by (5.7) we see that

$$|c_1| + |c_2| \le C r_1^{2-\gamma} \|(1+|x|)^\gamma g\|_{L^\infty}.$$

Using the estimates (5.7) we obtain

$$\sup_{r \ge r_1} r^{\gamma-2} |\phi_0(r)||| \le C \|(1+|x|)^\gamma g\|_{L^\infty}.$$

Therefore (5.8) yields

$$\sup_{r \ge r_1} r^{\gamma-2} |\phi(r)| \le C \|(1+|x|)^\gamma g\|_{L^\infty},$$

which is the desired estimate □

## 6. Fractional exterior problem

In this section we will construct a linear bounded operator that maps $f$ defined on $\Sigma_0$ to $\phi$ defined also on $\Sigma_0$ with the property

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi)(x) = f(x) \quad \text{for } x \in \Sigma_0,\ |x| \ge R, \tag{6.1}$$

where $R > 0$ will be a large fixed constant.

**Proposition 6.1.** *If $R$ is fixed large, there is a linear operator $f \mapsto \phi$ defined for radial, symmetric functions $f$ on $\Sigma_0$ with $\|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty$, such that $\phi$ is radial, symmetric, satisfies (6.1) and*

$$\|\phi\|_* \le C \|f\|_{1-\varepsilon,\alpha+\varepsilon}.$$

Here the norms $\| \ \|_*$ and $\| \ \|_{1-\varepsilon,\alpha+\varepsilon}$ are the ones defined in (2.12), (2.13).

We will also need a version of this result for right hand sides with fast decay. Let $0 < \tau < 1$.

**Proposition 6.2.** *If $R$ is fixed large, there is a linear operator $f \mapsto \phi$ defined for $f$ radial, symmetric and $\||x|^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)} < \infty$, such that $\phi$ is symmetric, satisfies (6.1) and*

$$\||x|^\tau \phi\|_{L^\infty(\Sigma_0)} \le C \||x|^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)}.$$

In order to prove Propositions 6.1 and 6.2 we study first

$$L_\varepsilon(\phi) + W_\varepsilon(r)\phi = f \quad \text{in } \mathbb{R}^2, \tag{6.2}$$

where

$$L_\varepsilon(\phi)(x) = \varepsilon \frac{2}{\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{\phi(y) - \phi(x)}{|x-y|^{4-\varepsilon}}\, dy, \tag{6.3}$$

and

$$W_\varepsilon(r) = \frac{\varepsilon}{F_\varepsilon(r)^{2-\varepsilon}} \eta_\varepsilon(r), \quad r = |x|$$

where

$$\eta_\varepsilon(r) = \eta(\varepsilon^{-\frac{1}{2}} r - 1) \tag{6.4}$$

and $\eta$ is a smooth cut-off function with $\eta(t) = 1$ for $t \ge 1$ and $\eta(t) = 0$ for $t \le 0$.

We start with a version of Proposition 6.1 for problem (6.2).

**Lemma 6.1.** *There is a linear operator that given a radial function $f$ in $\mathbb{R}^2$ such that $\|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty$ produces a radial solution $\phi$ of (6.2) with the property*

$$\|\phi\|_* \le C\|f\|_{1-\varepsilon,\alpha+\varepsilon}. \tag{6.5}$$

Then norms are the ones defined in (2.12), (2.13) in the context of functions defined on $\mathbb{R}^2$.

For smooth bounded functions $h$, $L_\varepsilon(h)$ has the expansion

$$L_\varepsilon(h) = \Delta h(x) + O(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

so equation (6.2) can be considered a perturbation of

$$\Delta h + W(x)h = g \quad \text{in } \mathbb{R}^2.$$

where

$$W(x) = \frac{\varepsilon}{F_\varepsilon(x)^2}\eta_\varepsilon(x)$$

The next lemma is a standard estimate for convolutions.

**Lemma 6.2.** *Assume $\gamma, \beta < 2$, $\gamma + \beta > 2$. Let $\|(1+|x|)^\gamma f\|_{L^\infty} < \infty$. Then*

$$\left|\int_{\mathbb{R}^2} \frac{1}{|x-y|^\beta} f(y)\,dy\right| \le C\|(1+|x|)^\gamma f\|_{L^\infty}(1+|x|)^{2-\beta-\gamma}.$$

**Lemma 6.3.** *Let $g$ be radial with $\|(1+|x|)^{\gamma-\varepsilon} g\|_{L^\infty} < \infty$ where $\gamma \in (1,2)$. Then for $\varepsilon > 0$ small (6.2) has a radial solution $h$ depending linearly on $g$ with $h(0) = 0$. Moreover*

$$\|(1+|x|)^{\gamma-2} h\|_{L^\infty} \le C\|(1+|x|)^{\gamma-\varepsilon} g\|_{L^\infty}.$$

*Proof.* Instead of looking directly for a solution of (6.2) we will solve

$$D_r h(x) = c_{2,\varepsilon}\,\text{p.v.}\int_{\mathbb{R}^2} \frac{|x| - \langle y, \frac{x}{|x|}\rangle}{|x-y|^{2+\varepsilon}}(W_\varepsilon h - g)\,dy, \tag{6.6}$$

for a radial function $h$ with $h(0) = 0$. Here $D_r$ is the radial derivative. In (6.6) the integral converges if $\|(1+|x|)^{\gamma-\varepsilon}(W_\varepsilon h - g)\|_{L^\infty} < \infty$ by Lemma 6.2.

Equation (6.6) is equivalent to

$$D_r h - A_\varepsilon(h) = B_\varepsilon(g) \tag{6.7}$$

where

$$A_\varepsilon(h)(x) = c_{2,\varepsilon}\,\text{p.v.}\int_{\mathbb{R}^2} \frac{|x| - \langle y, \frac{x}{|x|}\rangle}{|x-y|^{2+\varepsilon}} W_\varepsilon(y)h(y)\,dy$$

$$B_\varepsilon(g)(x) = -c_{2,\varepsilon}\,\text{p.v.}\int_{\mathbb{R}^2} \frac{|x| - \langle y, \frac{x}{|x|}\rangle}{|x-y|^{2+\varepsilon}} g(y)\,dy.$$

Let $A_0$ be the operator

$$A_0(h)(x) = c_2\,\text{p.v.}\int_{\mathbb{R}^2} \frac{|x| - \langle y, \frac{x}{|x|}\rangle}{|x-y|^2} W(y)h(y)\,dy.$$

Then (6.7) is equivalent to

$$D_r h - A_0(h) = A_\varepsilon(h) - A_0(h) + B_\varepsilon(g). \tag{6.8}$$

We claim that given $\psi$ radial in $\mathbb{R}^2$ with $\|(1+r)^{\gamma-1}\psi\|_{L^\infty} < \infty$ we can find a radial solution $h$ to

$$D_r h - A_0(h) = \psi \tag{6.9}$$

satisfying $h(0) = 0$ and

$$\|(1+r)^{\gamma-1}h'\|_{L^\infty} + \|r^{\gamma-2}h\|_{L^\infty} \le C\|(1+r)^{\gamma-1}\psi\|_{L^\infty}. \tag{6.10}$$

Indeed, we need to solve

$$h'(r) + \frac{1}{r}\int_0^r W(s)h(s)s\,ds = \psi(r) \quad \text{for all } r > 0.$$

Let

$$\tilde\psi(r) = \int_0^r \psi(s)\,ds, \quad \tilde h(r) = h(r) - \tilde\psi(r).$$

Then we look for $\tilde h$ satisfying

$$\tilde h'(r) + \frac{1}{r}\int_0^r W(s)\tilde h(s)s\,ds = -\frac{1}{r}\int_0^r W(s)\tilde\psi(s)s\,ds$$

which we write as

$$\Delta\tilde h + W(r)\tilde h(r) = W(r)\tilde\psi(r), \quad 0 < r < \infty.$$

We solve this equation using Proposition 5.1 and obtain

$$\|(1+r^{\gamma-1}\tilde h'\|_{L^\infty} + \|r^{\gamma-2}\tilde h\|_{L^\infty} \le C\|(1+r)^{\gamma-2}\tilde\psi\|_{L^\infty}.$$

Then $h = \tilde h + \tilde\psi$ satisfies (6.9), $h(0) = 0$ and estimate (6.10).

Let $T$ denote the operator that to a radial function $\psi \in L^\infty(\mathbb{R}^2)$ gives the radial solution $h$ to (6.9) just constructed, and note that by (6.10)

$$\|T(\psi)\|_a \le C\|(1+r)^{\gamma-1}\psi\|_{L^\infty}. \tag{6.11}$$

where

$$\|\varphi\|_a = \||x|^{\gamma-2}\varphi\|_{L^\infty} + \|(1+|x|)^{\gamma-1}\nabla\varphi\|_{L^\infty}.$$

We rewrite (6.8) as

$$h = T(A_\varepsilon(h) - A_0(h) + B_\varepsilon(g)) \tag{6.12}$$

in the space $X = \{h \in W^{1,\infty}_{loc}(\mathbb{R}^2) : h \text{ is radial}, \|h\|_a < \infty\}$ with norm $\|\ \|_a$.

We solve (6.12) by the contraction mapping principle. Consider the difference

$$\int_{\mathbb{R}^2}\left(\frac{x_i - y_i}{|x-y|^2}W(y) - \frac{x_i-y_i}{|x-y|^{2+\varepsilon}}W_\varepsilon(y)\right)\varphi(y)dy$$

where we assume that $\varphi$ is radial and $\|\varphi\|_a < \infty$. Let

$$D = \left\{y : \max\left(\frac{|x-y|}{|y|}, \frac{|y|}{|x-y|}\right) \le \varepsilon^{-m}\right\}$$

where $0 < m < 1$ is fixed. Let us estimate the integral outside $D$. Then we can estimate separately

$$\int_{D^c}\frac{1}{|x-y|}\frac{1}{|y|^2}\eta_\varepsilon(y)\varphi(y)dy, \quad \int_{D^c}\frac{1}{|x-y|^{1+\varepsilon}}\frac{1}{|y|^{2-\varepsilon}}\eta_\varepsilon(y)\varphi(y)dy,$$

respectively, since $\varepsilon^{\frac12}r \le CF_\varepsilon(r)$ for all $r \ge \varepsilon^{-\frac12}$. First, for the integral over the region $|y| \ge |x-y|\varepsilon^{-m}$ note that this condition is equivalent to $|y-(1+\delta)x|^2 \le (\delta+\delta^2)|x|^2$ where $\delta = O(\varepsilon^{2m})$ as $\varepsilon \to 0$. So, for $y$ in this region $|y| \sim |x|$ and hence

$$\begin{aligned}&\Big|\int_{\{|y|\ge\varepsilon^{-m}|x-y|\}}\frac{1}{|x-y|^{1+\varepsilon}}\frac{1}{|y|^{2-\varepsilon}}\eta_\varepsilon(y)\varphi(y)dy\Big| \\ &\qquad \le \|\varphi\|_a\frac{C}{(1+|x|)^{1-\varepsilon}}\int_{|y-x|\le C\delta^{1/2}|x|}\frac{1}{|x-y|^{1+\varepsilon}}\,dy \\ &\qquad \le \|\varphi\|_a\frac{C}{(1+|x|)^{1-\varepsilon}}(\delta^{1/2}|x|)^{1-\varepsilon} \le o(1)\|\varphi\|_a,\end{aligned}$$

as $\varepsilon \to 0$. Similarly

$$\Big|\int_{\{|y|\ge\varepsilon^{-m}|x-y|\}}\frac{1}{|x-y|}\frac{1}{|y|^2}\eta_\varepsilon(y)\varphi(y)dy\Big| \le o(1)\|\varphi\|_a,$$

as $\varepsilon \to 0$.

Next, for the integral over the region $|y| \le \varepsilon^m |x-y|$, note that this condition is equivalent to $|y+\delta x|^2 \le (\delta+\delta^2)|x|^2$ with $\delta = O(\varepsilon^{2m})$ as $\varepsilon \to 0$. In this region $|x-y| \sim |x|$ and then we estimate

$$\int_{\{|y|\le \varepsilon^m|x-y|\}} \frac{1}{|x-y|^{1+\varepsilon}} \frac{1}{|y|^{2-\varepsilon}} \eta_\varepsilon(y)\varphi(y)dy \le \frac{C\|\varphi\|_a}{(1+|x|)^{1+\varepsilon}} \int_{|y|\le C\delta^{1/2}|x|} \frac{1}{|y|^{1-\varepsilon}}dy$$
$$\le C\|\varphi\|_a \delta^{\frac{n-1+\varepsilon}{2}} \le o(1)\|\varphi\|_a.$$

Similarly

$$\int_{\{|y|\le \varepsilon^m|x-y|\}} \frac{1}{|x-y|^{n-1}} \frac{1}{|y|^2} \eta_\varepsilon(y)\varphi(y)dy \le o(1)\|\varphi\|_a.$$

Next consider the integrals inside $D$. For this we let

$$A_1 = \{y\in\mathbb{R}^2 : |y| \ge 2|x|\} \cap D$$
$$A_2 = \{y\in\mathbb{R}^2 : |y| \le 2|x|, |x-y| \ge |x|/2\} \cap D$$
$$A_3 = \{y\in\mathbb{R}^2 : |x-y| \le |x|/2\} \cap D.$$

We have now to estimate

$$I = \int_D \left( \frac{x_i-y_i}{|x-y|^2} W(y) - \frac{x_i-y_i}{|x-y|^{2+\varepsilon}} W_\varepsilon(y) \right) \varphi(y)dy$$
$$= \int_D \frac{x_i-y_i}{|x-y|^2} \left( 1 - \frac{F_\varepsilon(y)^\varepsilon}{|x-y|^\varepsilon} \right) g(y)dy,$$

where

$$g(y) = \frac{\varepsilon\eta_\varepsilon(y)}{F_\varepsilon(y)^2}\varphi(y).$$

Note that

$$\|(1+|x|)^\gamma g\|_{L^\infty} \le C\|\varphi\|_a.$$

Inside $D$ we have $1-\frac{F_\varepsilon(y)^\varepsilon}{|x-y|^\varepsilon} = O(\varepsilon|\log\varepsilon|)$. We assume $|x|\ge 10$. In $A_1$, $|x-y|\sim|y|$ so

$$\left| \int_{A_1} \frac{x_i-y_i}{|x-y|^2} \left( 1 - \frac{F_\varepsilon(y)^\varepsilon}{|x-y|^\varepsilon} \right) g(y)dy \right| \le C\varepsilon|\log\varepsilon|\,\|\varphi\|_a \int_{|y|\ge 2|x|} \frac{1}{|y|^{1+\gamma}}dy$$
$$\le C\varepsilon|\log\varepsilon|\,\|\varphi\|_a |x|^{1-\gamma}.$$

For the integral in $A_2$ note that $|x-y|\sim|x|$ and hence

$$\left| \int_{A_2} \frac{x_i-y_i}{|x-y|^2} \left( 1 - \frac{F_\varepsilon(y)^\varepsilon}{|x-y|^\varepsilon} \right) g(y) \right| \le C\varepsilon|\log\varepsilon|\,\|\varphi\|_a |x|^{-1} \int_{A_2} \frac{1}{|y|^\gamma}\,dy$$
$$\le C\varepsilon|\log\varepsilon|\,\|\varphi\|_a |x|^{1-\gamma}.$$

For $y\in A_3$ we have $|y|\sim|x|$ and therefore

$$\left| \int_{A_3} \frac{x_i-y_i}{|x-y|^2} \left( 1 - \frac{F_\varepsilon(y)^\varepsilon}{|x-y|^\varepsilon} \right) g(y)\,dy \right| \le C\varepsilon|\log\varepsilon|\,\|\varphi\|_a |x|^{-\gamma} \int_{|y-x|\le|x|/2} \frac{1}{|x-y|}\,dy$$
$$\le C\varepsilon|\log\varepsilon|\,\|\varphi\|_a |x|^{1-\gamma}.$$

Using the previous calculation we see that the map from $X$ to itself given by $T(A_\varepsilon(h)-A_0(h)+B_\varepsilon(g))$ is a contraction for $\varepsilon>0$ small, and hence has a unique fixed point. This fixed point satisfies

$$\|h\|_a \le C\|T(B_\varepsilon(g))\|_a \le C\|(1+r)^{\gamma-1}B_\varepsilon(g)\|_{L^\infty}$$

by (6.11). Using then Lemma 6.2

$$\|h\|_a \le C\|(1+|x|)^{\gamma-\varepsilon}g\|_{L^\infty}.$$

We need to verify that $h$ solves also (6.2). We define

$$\tilde{w}_k(x) = \tilde{c}_{n,\varepsilon} \int_{\mathbb{R}^2} \frac{1}{|x-y|^\varepsilon} (\frac{\eta_\varepsilon(y)}{|y|^{2-\varepsilon}}h - g)\eta_k(y)\,dy.$$

and

$$w_k(x) = \tilde{w}_k(x) - \tilde{w}_k(0)$$

where $\eta_k$ is a sequence of smooth cut-off functions with support in $B_{k^2}$, $\eta_k = 1$ in $B_k$ and $|D\eta_k| \le 1/k^2$. Hence $w_k$ are well defined and satisfy

$$\varepsilon\,\mathrm{p.v.}\int_{\mathbb{R}^2} \frac{w_k(y)-w_k(x)}{|x-y|^{4-\varepsilon}}\,dy = (g - \frac{\eta_\varepsilon(x)}{|x|^{2-\varepsilon}}h)\eta_k \quad \text{in } \mathbb{R}^n.$$

By Lemma 6.2

$$\begin{aligned} |x|^{\gamma-1}|D_{x_i}w_k(x)| &\le C\|(1+|x|)^{\gamma-\varepsilon}(g - \frac{\eta_\varepsilon(x)}{|x|^{2-\varepsilon}}h)\eta_k\|_{L^\infty} \\ &\le C\|(1+|x|)^{\gamma-\varepsilon}g\|_{L^\infty}. \end{aligned}$$

Then up to subsequence $w_k \to w$ uniformly on compact sets of $\mathbb{R}^2$, $\|(1+|x|)^{\gamma-1}Dw\|_\infty \le C\|(1+|x|)^{\gamma-\varepsilon}g\|_{L^\infty}$, and $w$ satisfies

$$\varepsilon\,\mathrm{p.v.}\int_{\mathbb{R}^2} \frac{w(y)-w(x)}{|x-y|^{n+2-\varepsilon}}\,dy = g - \frac{\eta_\varepsilon(x)}{|x|^{2-\varepsilon}}h \quad \text{in } \mathbb{R}^n. \tag{6.13}$$

From this equation

$$D_{x_i}w(x) = c_{n,\varepsilon}\int_{\mathbb{R}^n} \frac{x_i-y_i}{|x-y|^{n+\varepsilon}}(\frac{\eta_\varepsilon(y)}{|y|^{2-\varepsilon}}h - g)\,dy = D_{x_i}h(x).$$

Hence $w$ and $h$ differ by a constant and from (6.13) we see that $h$ solves (6.2).

□

*Proof of Lemma 6.1.* The proof is based on the following apriori estimate for radial solutions $h$ of (6.2) such that $\||x|^{-1}h\|_{L^\infty} < \infty$:

$$\||x|^{-1}h\|_{L^\infty} \le C\|(1+|x|)^{1-\varepsilon}g\|_{L^\infty}, \tag{6.14}$$

and we claim it holds if $\varepsilon > 0$ is sufficiently small.

We argue by contradiction, assuming that there are sequences $\varepsilon_i \to 0$, radial functions $g_i$, $h_i$ solving (6.2) and satisfying

$$\||x|^{-1}h_i\|_{L^\infty} = 1, \quad \|(1+|x|)^{1-\varepsilon_i}g_i\|_{L^\infty} \to 0 \tag{6.15}$$

as $i \to \infty$. Let $x_i \in \mathbb{R}^2$ be such that

$$(1+|x_i|)^{-1}|h_i(x_i)| \ge \frac{1}{2}.$$

Assume first that $x_i$ remains bounded and, up to a subsequence $x_i \to x$ as $i \to \infty$. The bounds (6.15) and standard estimates for $L_\varepsilon$, uniform as $\varepsilon \to 0$, show that $h_i$ is bounded in $C^{1,\alpha}_{loc}$. Therefore passing to a subsequence we find $h_i \to h$ locally uniformly in $\mathbb{R}^2$. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$. Multiplying (6.2) by $\varphi$ and integrating we find

$$\int_{\mathbb{R}^2} h_i L_{\varepsilon_i}(\varphi) + W_{\varepsilon_i}h_i\varphi_i = \int_{\mathbb{R}^2} g_i\varphi.$$

Taking the limit we find that $h$ is harmonic in $\mathbb{R}^2$. But also $|h(x)| \ge \frac{1}{2}$, $h$ is radial and $|h(r)| \le r$ for all $r \ge 0$, which is impossible.

Suppose that $x_i$ is unbounded so that up to subsequence $r_i = |x_i| \to \infty$ as $i \to \infty$. Let

$$\tilde{h}_i(x) = \frac{1}{r_i}h(r_ix), \quad \tilde{g}_i(x) = r_i^{1-\varepsilon_i}g(r_ix)$$

so that

$$L_{\varepsilon_i}(\tilde{h}_i) + W_i(x)\tilde{h}_i = \tilde{g}_i \quad \text{in } \mathbb{R}^2,$$

where

$$W_i(x) = \frac{\varepsilon_i\eta_{\varepsilon_i}(r_ix)r_i^{2-\varepsilon_i}}{F_{\varepsilon_i}(r_ix)^{2-\varepsilon_i}}$$

Also

$$\||x|^{-1}\tilde{h}_i\|_{L^\infty} = 1, \quad \||x|^{1-\varepsilon_i}\tilde{g}_i\|_{L^\infty} \to 0$$

as $i \to \infty$. Up to subsequence $\tilde{h}_i \to h$ locally uniformly in $\mathbb{R}^2$ and $x_i/r_i \to \hat{x}$. Moreover $|h(\hat{x})| \geq \frac{1}{2}$.

If $\varepsilon_i^{-\frac{1}{2}}|\log \varepsilon_i| r_i^{-1} \to \infty$ as $i \to \infty$ then $W_i(x) \to 0$ uniformly on compact sets and we reach a contradiction as before.

If $\varepsilon_i^{-\frac{1}{2}}|\log \varepsilon_i| r_i^{-1} \to R_0$, then $W_i(x) \to W(x)$ uniformly on compact sets where $W(x)$ is bounded for $|x| \leq R_0$ and $W(x) = \frac{1}{|x|^2}$ for $|x| \geq R_0$. Then $h$ solves

$$\Delta h + W h = 0 \quad \text{in } \mathbb{R}^2$$

with $|h(r)| \leq r$ for all $r \geq 0$. This implies $h \equiv 0$, a contradiction.

Finally, if $\varepsilon_i^{-\frac{1}{2}}|\log \varepsilon_i| r_i^{-1} \to 0$, then $h$ satisfies

$$\Delta h + \frac{1}{|x|^2} h = 0 \quad \text{in } \mathbb{R}^2 \setminus \{0\}$$

with $|h(r)| \leq r$ for all $r > 0$. Again this implies that $h$ is trivial.

Existence of a solution to (6.2) can be deduced from the solvability obtained in Lemma 6.3 and the apriori estimate (6.14), with an approximation argument. Namely, let $g$ be radial with $\|(1+|x|)^{1-\varepsilon} g\|_{L^\infty} < \infty$ and $\eta$ be a smooth cut-off function with $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$. Thanks to Lemma 6.3 there is a radial solution $h_n$ of (6.2) with right hand side $g\eta(x/n)$. By (6.14) we have $\|(1+|x|)^{-1} h_n\|_{L^\infty} \leq C$ and by standard estimates $h_n$ is bounded is $C^{1,\alpha}_{loc}$. Up to subsequence $h_n$ converges to a solution $h$ satisfying

$$\|(1+|x|)^{-1} h\|_{L^\infty} \leq C \|(1+|x|)^{1-\varepsilon} g\|_{L^\infty}.$$

Finally estimate (6.5) follows from a standard scaling argument and Schauder estimates for $L_\varepsilon$, which is $(-\Delta)^{\frac{1+s}{2}}$ up to constant, and which are uniform as $\varepsilon \to 1$. □

Next we give a result analogous to Lemma 6.1 but for functions with fast decay.

**Lemma 6.4.** *There is a linear operator that given a radial function $f$ in $\mathbb{R}^2$ such that $\|(1+|x|)^{2+\tau-\varepsilon} f\|_{L^\infty} < \infty$ produces a solution $\phi$ of (6.2) with the property*

$$\||x|^\tau \phi\|_{L^\infty} \leq C \|(1+|x|)^{2+\tau-\varepsilon} f\|_{L^\infty}. \tag{6.16}$$

*Proof.* Let $Y$ denote the space of radial functions in $\mathbb{R}^2$ satisfying $\||x|^\tau \phi\|_{L^\infty} < \infty$. We claim there exists $\phi \in Y$ that depends linearly on $f$ satisfying

$$\nabla \phi(x) = c_{2,\varepsilon} \int_{\mathbb{R}^2} \left( \frac{x-y}{|x-y|^{2+\varepsilon}} - \frac{x}{|x|^{2+\varepsilon}} \right) \left( f(y) - \frac{\eta_\varepsilon(|y|)}{|y|^{2-\varepsilon}} \phi(y) \right) dy \tag{6.17}$$

and the estimate (6.16). This function is the desired solution. Here $c_{2,\varepsilon} \to \frac{1}{2\pi}$ as $\varepsilon \to 0$.

Similar to Lemma 6.2 we have the following estimate. Assume $0 < \beta < 2$, $2 < \gamma < 3$ and $\gamma + \beta > 2$. Let $\|(1+|x|)^\gamma f\|_{L^\infty} < \infty$. Then

$$\left| \int_{\mathbb{R}^2} \left( \frac{x-y}{|x-y|^{\beta+1}} - \frac{x}{|x|^{\beta+1}} \right) f(y)\, dy \right| \leq C \|(1+|x|)^\gamma f\|_{L^\infty} |x|^{2-\beta-\gamma}.$$

Using this estimate with $\beta = 1 + \varepsilon$ we see that the integral (6.17) is well defined if $\|(1+|x|)^{2+\tau-\varepsilon} f\|_\infty < \infty$ and $\phi \in Y$.

We treat (6.17) as a perturbation of the case $\varepsilon = 0$. So first we consider the equation

$$\Delta \phi + \frac{\eta_\varepsilon}{r^2} \phi = f \quad \text{in } \mathbb{R}^2$$

with $\eta_\varepsilon$ as in (6.4), for which we want to construct a solution such that

$$\||x|^\tau \phi\|_{L^\infty(\mathbb{R}^2)} \leq \|(1+|x|)^{2+\tau} f\|_{L^\infty(\mathbb{R}^2)}. \tag{6.18}$$

For $r \geq \varepsilon^{-\frac{1}{2}} + 1$ the equation is given by

$$\frac{1}{r}(r\phi')' + \frac{1}{r^2}\phi = f, \quad r \geq \varepsilon^{-\frac{1}{2}},$$

hence we take $\phi$ of the form

$$\phi(r) = \cos(\log(r)) \int_r^\infty \sin(\log(t)) t f(t) dt - \sin(\log(r)) \int_r^\infty \cos(\log(t)) t f(t) dt$$

for $r \geq \varepsilon^{-\frac{1}{2}} + 1$. From this formula we get directly

$$\sup_{r \geq \varepsilon^{-\frac{1}{2}}} r^\tau |\phi(r)| \leq \|r^{2+\tau} f\|_{L^\infty}.$$

For $0 < r \leq \varepsilon^{-\frac{1}{2}} + 1$ we define $\phi$ as the unique solution of the equation

$$\frac{1}{r}(r\phi')' + \frac{\eta_\varepsilon(r)}{r^2} = f, \quad r \leq \varepsilon^{-\frac{1}{2}} + 1,$$

with initial conditions at $\varepsilon^{-\frac{1}{2}} + 1$ to make $\phi$ a global solution for $r \in (0, \infty)$. Note that

$$\phi(\varepsilon^{-\frac{1}{2}}) = O(\varepsilon^{\frac{\tau}{2}}), \quad \phi'(\varepsilon^{-\frac{1}{2}}) = O(\varepsilon^{\frac{1+\tau}{2}}).$$

Let $r_0 = \varepsilon^{-\frac{1}{2}}$. Then for $r \leq r_0$ we can represent

$$\phi(r) = c_1 + c_2 \log(\frac{r}{r_0}) + \int_r^{r_0} \frac{1}{s} \int_s^{r_0} t f(t) dt ds,$$

where $c_1, c_2$ have to satisfy

$$c_1 = \phi(r_0) = O(\varepsilon^{\frac{\tau}{2}}), \quad c_2 = r_0 \phi'(r_0) = O(\varepsilon^{\frac{\tau}{2}}).$$

With this formula we can verify (6.18). The previous solution satisfies

$$\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \left( f(y) - \frac{\eta_\varepsilon(|y|)}{|y|^2} \phi(y) \right) dy + A \log|x| + B$$

where $A$, $B$ depend on $f$ and are such that $\phi(x) \to 0$ as $|x| \to \infty$. Therefore for the gradient we have

$$\begin{aligned} \nabla\phi(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \left( f(y) - \frac{\eta_\varepsilon(|y|)}{|y|^2} \phi(y) \right) dy + A \frac{x}{|x|^2} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left( \frac{x-y}{|x-y|^2} - \frac{x}{|x|^2} \right) \left( f(y) - \frac{\eta_\varepsilon(|y|)}{|y|^2} \phi(y) \right) dy. \end{aligned} \tag{6.19}$$

Let $\phi = T(f)$ denote the operator that associates the function $\nabla\phi$ constructed above, so that in particular (6.18) and (6.19) hold. To find a solution of (6.17) it then suffices to find $\phi \in Y$ such that

$$\nabla\phi = T(B_\varepsilon(f) + A_0(\phi) - A_\varepsilon(\phi))$$

where the operators $B_\varepsilon$, $A_0$, $A_\varepsilon$ are defined as

$$\begin{aligned} B_\varepsilon(f)(x) &= c_{2,\varepsilon} \int_{\mathbb{R}^2} \left( \frac{x-y}{|x-y|^{2+\varepsilon}} - \frac{x}{|x|^{2+\varepsilon}} \right) f(y)\, dy \\ A_\varepsilon(\phi)(x) &= c_{2,\varepsilon} \int_{\mathbb{R}^2} \left( \frac{x-y}{|x-y|^{2+\varepsilon}} - \frac{x}{|x|^{2+\varepsilon}} \right) \frac{\eta_\varepsilon(|y|)}{|y|^{2-\varepsilon}} \phi(y)\, dy \\ A_0(\phi)(x) &= c_{2,\varepsilon} \int_{\mathbb{R}^2} \left( \frac{x-y}{|x-y|^2} - \frac{x}{|x|^2} \right) \frac{\eta_\varepsilon(|y|)}{|y|^2} \phi(y)\, dy, \end{aligned}$$

and $\phi$ is defined from $\nabla\phi$ by integration such that $\lim_{|x|\to\infty} \phi(x) = 0$ (here all functions are radial). Similarly as in Lemma 6.3 we can show that for $\varepsilon > 0$ small the map from $Y$ to $Y$ given by $\phi \mapsto T(B_\varepsilon(f) + A_0(\phi) - A_\varepsilon(\phi))$ is a contraction.

□

For the proof of Proposition 6.1 we need an estimate of

$$a_\varepsilon(x) = \varepsilon \int_{\Sigma_0} \frac{1 - \langle \nu_{\Sigma_0}(y), \nu_{\Sigma_0}(y) \rangle}{|x-y|^{4-\varepsilon}} dy.$$

**Lemma 6.5.** *Let $x = (x', F_\varepsilon(x')) \in \Sigma_0$. Then*

$$a_\varepsilon(x) = \pi|A_{\Sigma_0}|^2|x'|^\varepsilon + O(\frac{\varepsilon}{(1+|x|)^{2-\varepsilon}}) + O(\frac{\varepsilon}{\log(|x|)^{2-\varepsilon}})\chi_{|x|\le\varepsilon^{-\frac12}}$$
$$+ \pi\frac{\varepsilon}{F_\varepsilon(x')^{2-\varepsilon}}(1+o(1))\chi_{|x|\ge\varepsilon^{-\frac12}},$$

*where $|A_{\Sigma_0}|$ is the norm of the second fundamental form of $\Sigma_0$ and $O()$, $o()$ are uniform $x$ as $\varepsilon \to 0$.*

*Proof.* Let $R_1 > 0$ and write

$$a_\varepsilon = a_\varepsilon^+ + a_\varepsilon^-,$$

where

$$a_\varepsilon^\pm(x) = \varepsilon\int_{\Sigma_0^\pm}\frac{1-\langle\nu_{\Sigma_0}(y),\nu_{\Sigma_0}(x)\rangle}{|x-y|^{4-\varepsilon}}dy,$$

and $\Sigma_0^\pm$ is $\Sigma_0$ intersected with $x_3 \ge 0$ or $x_3 \le 0$ respectively. Let us split

$$a_\varepsilon^+ = a_{1,\varepsilon}^+ + a_{2,\varepsilon}^+,$$

where

$$a_{1,\varepsilon}^+ = \varepsilon\int_{\Sigma_0^+\cap C_{R_1}(x)}\frac{1-\langle\nu_{\Sigma_0}(y),\nu_{\Sigma_0}(x)\rangle}{|x-y|^{4-\varepsilon}}dy,$$
$$a_{2,\varepsilon}^+ = \varepsilon\int_{\Sigma_0^+\setminus C_{R_1}(x)}\frac{1-\langle\nu_{\Sigma_0}(y),\nu_{\Sigma_0}(x)\rangle}{|x-y|^{4-\varepsilon}}dy,$$

and $C_{R_1}(x)$ is the cylinder with base the disk of radius $R_1$ on the tangent plane to $\Sigma_0$ at $x$, and height $R_1$, which will be chosen later depending on $x$. Let $g : B_{R_1}(0) \subset \mathbb{R}^2 \to \mathbb{R}$ be such that $\Sigma_0$ can be described as the graph of $g$ over the tangent plane at $X$. Then

$$a_{1,\varepsilon}^+ = \varepsilon\int_{|t|\le R_1}\frac{\sqrt{1+|\nabla g|^2}-1}{(|t|^2+g(t)^2)^{\frac{4-\varepsilon}{2}}}\,dt.$$

A calculation gives

$$a_{1,\varepsilon}^+(x) = \pi|A_{\Sigma_0}(X)|^2R_1^\varepsilon + O(\varepsilon[D^2g]_{\alpha,B_{R_1}}R_1^{\alpha+\varepsilon}).$$

We choose now $R_1$ as follows. Recall that $x = (x', F_\varepsilon(x'))$ and $|x| \sim |x'|$. If $|x'| \le 100$ we take $R_1 > 0$ a fixed small constant so that the representation of $\Sigma_0 \cap C_{R_1}(x)$ by a graph is possible. If $|x'| > 100$, we take $R = \delta|x'|$ with $\delta > 0$ a small positive constant. By estimates (4.8)

$$[D^2g]_{\alpha,B_{R_1}} \le \begin{cases} \frac{C\varepsilon^{\frac12}}{|x'|^{1+\alpha}} & \text{if } |x'| \ge \varepsilon^{-\frac12} \\ \frac{C}{|x'|^{2+\alpha}} & \text{if } |x'| \le \varepsilon^{-\frac12}. \end{cases}$$

and therefore

—— X – x

$$a_{1,\varepsilon}^+ = \pi|A_{\Sigma_0}|^2|x'|^\varepsilon + O(\frac{\varepsilon}{(1+|x|)^{2-\varepsilon}})$$

as $\varepsilon \to 0$. On the other hand a direct estimate gives

$$a_{2,\varepsilon}^+ = O(\frac{\varepsilon}{(1+|x|)^{2-\varepsilon}}).$$

Therefore

$$a_\varepsilon^+ = \pi|A_{\Sigma_0}|^2|x|^\varepsilon + O(\frac{\varepsilon}{(1+|x|)^{2-\varepsilon}}).$$

We can write explicitly

$$a_\varepsilon^-(x) = \varepsilon\int_{|y|\ge1}\frac{\sqrt{1+|\nabla F_\varepsilon(y)|^2}+\frac{1-F_\varepsilon'(x)F_\varepsilon'(y)}{\sqrt{1+|\nabla F_\varepsilon(x)|^2}}}{(|x-y|^2+(F_\varepsilon(x)+F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}}\,dy$$

For $10 \le |x| \le \varepsilon^{-\frac{1}{2}}$ we estimate

$$
\begin{aligned}
|a_\varepsilon^-(x)| &\le \varepsilon C \int_{|y|\ge 1} \frac{1}{(|x-y|^2 + F_\varepsilon(x)^2)^{\frac{4-\varepsilon}{2}}}\, dy \\
&\le \frac{C\varepsilon}{F_\varepsilon(x)^{2-\varepsilon}} \le \frac{C\varepsilon}{\log(x)^{2-\varepsilon}}.
\end{aligned}
$$

For $|x| \ge \varepsilon^{-\frac{1}{2}}$ we split $a_\varepsilon^- = a_{1,\varepsilon}^- + a_{2,\varepsilon}^-$ where

$$
\begin{aligned}
a_{1,\varepsilon}^-(x) &= \varepsilon \int_{\Sigma_0^- \cap \tilde C_{R_2}(x,0)} \frac{1 - \langle \nu_{\Sigma_0}(Y), \nu_{\Sigma_0}(X)\rangle}{|X-Y|^{4-\varepsilon}} dY, \\
a_{2,\varepsilon}^-(x) &= \varepsilon \int_{\Sigma_0^- \setminus \tilde C_{R_2}(x,0)} \frac{1 - \langle \nu_{\Sigma_0}(Y), \nu_{\Sigma_0}(X)\rangle}{|X-Y|^{4-\varepsilon}} dY,
\end{aligned}
$$

where $\tilde C_{R_2}(X)$ is the cylinder with base a disk of radius $R_2$ on $\mathbb{R}^2$ centered at $(x,0)$. We choose $R_2 = |\log\varepsilon|^{-\frac{1}{2}}|x|$. Then

$$
\begin{aligned}
|a_{2,\varepsilon}^-(x)| &= \varepsilon C \int_{|y-x|\ge R} \frac{1}{(|x-y|^2 + F_\varepsilon(x)^2)^{\frac{4-\varepsilon}{2}}}\, dy \\
&\le \frac{C\varepsilon |\log\varepsilon|}{|x|^{2-\varepsilon}}.
\end{aligned}
$$

For $|x| \ge \varepsilon^{-\frac{1}{2}}$ and $|y-x| \le R_2$, $F_\varepsilon'(x) = O(\varepsilon^{\frac{1}{2}})$ and $F_\varepsilon(y) = F_\varepsilon(x) + O(\varepsilon^{\frac{1}{2}} R)$ so

$$
\begin{aligned}
a_\varepsilon^-(x) &= \varepsilon \int_{|x-y|\le R} \frac{2 + O(\varepsilon)}{(|x-y|^2 + (F_\varepsilon(x) + F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}}\, dy \\
&= \pi \frac{\varepsilon}{F_\varepsilon(x)^{2-\varepsilon}} (1 + o(1))
\end{aligned}
$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$.

□

*Proof of Propositions 6.1 and 6.2.* The idea is to reduce problem (6.1) to one in $\mathbb{R}^2$. Suppose that $\phi$ is a radial function on $\Sigma_0$, symmetric with respect to $x_3 = 0$ vanishing in $B_{2R}(0)$. Here $R > 0$ is large and fixed, to be chosen later. Since $\phi$ is symmetric with respect to $x_3 = 0$, we can define $\tilde\phi$ globally in $\mathbb{R}^2$ by

$$
\tilde\phi(x) = \phi(x, \pm F_\varepsilon(x)), \quad |x| \ge R,
$$

and $\tilde\phi = 0$ in $B_R(0)$. Let $C_R$ be the cylinder

$$
C_R = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2\}.
$$

Then, for $X \in \Sigma_0$ of the form $X = (x, F_\varepsilon(x))$ with $|x| \ge R$, we have

$$
\begin{aligned}
&\text{p.v.} \int_{\Sigma_0 \setminus C_R} \frac{\phi(Y) - \phi(X)}{|Y-X|^{4-\varepsilon}}\, dY \\
&= \text{p.v.} \int_{\mathbb{R}^2 \setminus B_R} \frac{\tilde\phi(y) - \tilde\phi(x)}{(|x-y|^2 + (F_\varepsilon(x) - F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}} \sqrt{1 + |\nabla F_\varepsilon(y)|^2}\, dy \\
&\quad + \int_{\mathbb{R}^2 \setminus B_R} \frac{\tilde\phi(y) - \tilde\phi(x)}{(|x-y|^2 + (F_\varepsilon(x) + F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}} \sqrt{1 + |\nabla F_\varepsilon(y)|^2}\, dy
\end{aligned}
$$

Then we find for $|X| \ge R$, $X = (x, F_\varepsilon(x))$,

$$
\text{p.v.} \int_{\Sigma_0} \frac{\phi(Y) - \phi(X)}{|Y-X|^{4-\varepsilon}}\, dY = \text{p.v.} \int_{\mathbb{R}^2} \frac{\tilde\phi(y) - \tilde\phi(x)}{|y-x|^{4-\varepsilon}}\, dy + b(x)\tilde\phi(x) + B_1(\tilde\phi)(x)
$$

where

$$b(x) = \int_{B_R} \frac{1}{|x-y|^{4-\varepsilon}}\,dy - \int_{\Sigma_0\cap C_R} \frac{1}{|(x,F_\varepsilon(x))-Y|^{4-\varepsilon}}\,dY$$

$$B_1(\tilde\phi)(x) = \int_{\mathbb{R}^2\setminus B_R} \left(\tilde\phi(y)-\tilde\phi(x)\right)\left(\frac{\sqrt{1+|\nabla F_\varepsilon(y)|^2}}{(|x-y|^2+(F_\varepsilon(x)-F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}} - \frac{1}{|x-y|^{4-\varepsilon}}\right) dy$$
$$+ \int_{\mathbb{R}^2\setminus B_R} \frac{\tilde\phi(y)-\tilde\phi(x)}{(|x-y|^2+(F_\varepsilon(x)+F_\varepsilon(y))^2)^{\frac{4-\varepsilon}{2}}}\sqrt{1+|\nabla F_\varepsilon(y)|^2}\,dy.$$

Let

$$a_\varepsilon(X) = \varepsilon\int_{\Sigma_0} \frac{1-\langle \nu_{\Sigma_0}(Y),\nu_{\Sigma_0}(X)\rangle}{|X-Y|^{3+s}}dY.$$

Then (6.1) reads as

$$L_\varepsilon(\tilde\phi) + \frac{\eta_\varepsilon}{|x|^{2-\varepsilon}}\tilde\phi(x) + \varepsilon B_1(\tilde\phi)(x) + (\varepsilon b(x) + a_\varepsilon - \frac{\eta_\varepsilon}{|x|^{2-\varepsilon}})\tilde\phi(x) = \tilde f(x) \tag{6.20}$$

where $\tilde f(x) = f(x,F_\varepsilon(x))$ and $L_\varepsilon$ is the operator (6.3). We look for $\tilde\phi$ of the form $\tilde\phi = \eta\varphi$, where $\eta$ is a smooth radial cut-off function such that $\eta(x)=1$ for $|x|\ge 3R$ and $\eta(x)=0$ for $|x|\le 2R$. Then we ask that $\varphi$ solves

$$L_\varepsilon(\varphi) + \frac{\eta_\varepsilon}{|x|^{1-s}}\varphi + \varepsilon B_2(\varphi) + \eta(\varepsilon b(x) + a_\varepsilon - \frac{\eta_\varepsilon}{|x|^{1-s}})\varphi = \tilde f(x) \quad \text{in } \mathbb{R}^2, \tag{6.21}$$

where

$$B_2(\varphi)(x) = \varepsilon\tilde\eta(x)\int_{\mathbb{R}^2}\varphi(y)\frac{\eta(y)-\eta(x)}{|x-y|^{4-\varepsilon}}\,dy + \varepsilon\tilde\eta(x)B_1[\eta\varphi](x),$$

and where $\tilde\eta$ is another radial smooth cut-off function such that $\tilde\eta(x)=1$ for $|x|\ge 5R$, $\tilde\eta(x)=0$ for $|x|\le 4R$. If $\varphi$ solves (6.21), then $\tilde\phi=\eta\varphi$ will satisfy (6.20) for $|x|\ge 5R$. Let $T$ denote the operator constructed in Lemma 6.1, so that $\phi = T(f)$ is a radial solution to (6.2) satisfying the estimate (6.5). Then we rewrite (6.21) as the fixed point problem

$$\varphi = T(-\varepsilon B_2(\varphi) - \eta(\varepsilon b(x) + a_\varepsilon - \frac{\eta_\varepsilon}{|x|^{1-s}})\varphi + \tilde f).$$

We can apply the contraction mapping principle by the following estimates

$$\|\varepsilon B_2(\varphi)\|_{1-\varepsilon,\alpha} \le o(1)\|\varphi\|_*$$
$$\|\eta(\varepsilon b(x) + a_\varepsilon - \frac{\eta_\varepsilon}{|x|^{2-\varepsilon}})\varphi\|_{1-\varepsilon,\alpha} \le o(1)\|\varphi\|_*$$

where $o(1)\to 0$ as $\varepsilon\to 0$ and $R\to\infty$, which can be proved using Lemma 6.5.

The proof of Proposition 6.2 follows the same lines as the one of Proposition 6.1. □

## 7. Linear theory

The purpose here is to construct a linear operator $f\mapsto\phi$ which gives a solution to the problem

$$\varepsilon\mathcal{J}^s_{\Sigma_0}(\phi) = f \quad \text{in } \Sigma_0, \tag{7.1}$$

where $\mathcal{J}^s_{\Sigma_0}$ is the nonlocal Jacobi operator

$$\mathcal{J}^s_{\Sigma_0}(\phi)(x) = \text{p.v.}\int_{\Sigma_0}\frac{\phi(y)-\phi(x)}{|x-y|^{4-\varepsilon}}\,dy + \phi(x)\int_{\Sigma_0}\frac{(\nu(x)-\nu(y))\cdot\nu(x)}{|x-y|^{4-\varepsilon}}dy,$$

and $\Sigma_0$ is the surface defined in (2.9).

The main result is stated in Proposition 2.2, which we recall: there is a linear operator that to a function $f$ on $\Sigma_0$ such that $f$ is radially symmetric and symmetric with respect to $x_3=0$ with $\|f\|_{1-\varepsilon,\alpha+\varepsilon}<\infty$, gives a solution $\phi$ of (7.1). Moreover

$$\|\phi\|_* \le C\|f\|_{1-\varepsilon,\alpha+\varepsilon}.$$

The norms $\|\ \|_{1-\varepsilon,\alpha+\varepsilon}$ and $\|\ \|_*$ are defined in (2.13), (2.12).

As $\varepsilon \to 0$, $\Sigma_0$ approaches the standard catenoid $\mathcal{C}$ on compact sets, which can be described by the parametrization

$$y \in \mathbb{R} \mapsto \big(\sqrt{1+y^2}\cos(\theta), \sqrt{1+y^2}\sin(\theta), \log(y+\sqrt{1+y^2})\big)$$

with $y \in \mathbb{R}$, $\theta \in [0, 2\pi]$. Hence for smooth bounded $\phi$ we have

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi) \to \frac{\pi}{2}(\Delta_{\mathcal{C}}\phi + |A|^2\phi)$$

uniformly over compact sets as $\varepsilon \to 0$, where $\Delta_{\mathcal{C}}$ is the Laplace-Beltrami operator and $|A|$ the norm of the second fundamental form of $\mathcal{C}$ (see Lemmas A.2 and A.4).

Let us recall the standard nondegeneracy property of the Jacobi operator $\Delta_{\mathcal{C}} + |A|^2$ on the catenoid. Linearly independent elements in its kernel are the functions

$$Z_1(y) = \frac{y}{\sqrt{y^2+1}}, \quad Z_2(y) = -1 + \frac{y}{\sqrt{y^2+1}}\log(y+\sqrt{y^2+1}). \tag{7.2}$$

The knowledge of these elements in the kernel of $\Delta_{\mathcal{C}} + |A|^2$ immediately yields

**Lemma 7.1.** *If $\phi$ is a bounded axially symmetric solution of $\Delta_{\mathcal{C}}\phi + |A|^2\phi = 0$ in $\mathcal{C}$ then $\phi = cZ_1$ for some $c \in \mathbb{R}$.*

Let

$$a_\varepsilon(x) = \varepsilon \int_{\Sigma_0} \frac{1 - \langle \nu_{\Sigma_0}(y), \nu_{\Sigma_0}(x)\rangle}{|x-y|^{3+s}} dy$$

and

$$b_\varepsilon(x) = a_\varepsilon(x)\eta_\varepsilon(x)$$

where $\eta_\varepsilon$ is smooth, radial, $\eta(x) = 0$ for $|x| \geq \varepsilon^{-\frac{1}{2}} + 1$, and $\eta(x) = 1$ for $|x| \leq \varepsilon^{-\frac{1}{2}}$.

Let us write

$$L_\varepsilon(\phi)(x) = \varepsilon \,\text{p.v.} \int_{\Sigma_0} \frac{\phi(y) - \phi(x)}{|x-y|^{4-\varepsilon}} dy$$

and consider the equation

$$L_\varepsilon(\phi) + b_\varepsilon(x)\phi = f \quad \text{in } \Sigma_0. \tag{7.3}$$

We will consider from now only right hand sides $f : \Sigma_0 \to \mathbb{R}$ which are symmetric with respect to the plane $x_3 = 0$, and symmetric solutions $\phi$.

Let $0 < \tau < 1$.

**Proposition 7.1.** *For $\varepsilon > 0$ small there is a linear operator that takes $f$ symmetric with respect to $x_3$ with $\|y^{2+\tau-\varepsilon} f\|_{L^\infty} < \infty$ to a a symmetric bounded solution $\phi$ of (7.3). Moreover*

$$\|\phi\|_{L^\infty} \leq C\|y^{2+\tau-\varepsilon} f\|_{L^\infty},$$

$$\|(1+|y|)^{1+\tau}\nabla\phi\|_{L^\infty} \leq C\|y^{2+\tau-\varepsilon} f\|_{L^\infty}, \tag{7.4}$$

*and $\lim_{|x|\to\infty}\phi(x)$ exists.*

The counterpart of this result for the Jacobi operator $\Delta_{\mathcal{C}} + |A|^2$, without assuming any symmetry on $f$ or $\phi$ is: if $\||y|^{2+\tau} f\|_{L^\infty} < \infty$ and $\int_{\mathcal{C}} f Z_1 = 0$, there is a bounded solution $\phi$ of

$$\Delta_{\mathcal{C}}\phi + |A|^2\phi = f \quad \text{in } \mathcal{C},$$

and this solution is unique except a constant times $Z_1$. Moreover $\phi$ has limits at both ends, which have to coincide. In the nonlocal setting, to simplify we work with functions that are symmetric with respect to $x_3$, so in some sense the condition $\int_{\mathcal{C}} f Z_1 = 0$ is automatic.

For the existence part in Proposition 7.1 we study the truncated problem

$$\begin{cases} L_\varepsilon(\phi) + b_\varepsilon\phi = f & \text{in } \Sigma_0 \cap B_R(0) \\ \phi = 0 & \text{on } \Sigma_0 \setminus B_R(0) \end{cases} \tag{7.5}$$

Let

$$\sigma = \frac{1+s}{2} = 1 - \frac{\varepsilon}{2}.$$

Given in $f \in L^2(\Sigma_0 \cap B_R(0))$ there is a weak solution $\phi \in H^\sigma(\Sigma_0)$ of

$$\begin{cases} -L_\varepsilon(\phi) = f \quad \text{in } \Sigma_0 \cap B_R(0) \\ \phi = 0 \quad \text{on } \Sigma_0 \setminus B_R(0) \end{cases}$$

By weak solution we mean $\phi \in H^\sigma(\Sigma_0)$, $\phi = 0$ on $\Sigma_0 \setminus B_R(0)$ and

$$\int_{\Sigma_0}\int_{\Sigma_0} \frac{(\phi(y)-\phi(x))(\varphi(y)-\varphi(x))}{|x-y|^{2+2\sigma}}\, dydx = \int_{\Sigma_0} f(x)\varphi(x)\, dx$$

for all $\varphi \in H^\sigma(\Sigma_0)$ with $\varphi = 0$ in $\Sigma_0 \setminus B_R(0)$. This solution can be found by minimizing the functional

$$\frac{1}{4}\int_{\Sigma_0}\int_{\Sigma_0} \frac{(\phi(y)-\phi(x))^2}{|x-y|^{2+2\sigma}}\, dydx - \int_{\Sigma_0} f(x)\phi(x)\, dx$$

over the space $\{\phi \in H^\sigma(\Sigma_0) : \phi = 0 \quad \text{on } \Sigma_0 \setminus B_R(0)\}$. For $f$ locally bounded and $\varepsilon > 0$ small ($\sigma$ is close to 1), the solution belongs to $C^{1,\alpha}_{loc}$.

First we establish an apriori estimate for solutions of (7.5).

**Lemma 7.2.** *Suppose $f$ is symmetric and $\||y|^{2+\tau-\varepsilon} f\|_{L^\infty} < \infty$. There are $\varepsilon_0, R_0, C > 0$ such that for $0 < \varepsilon \le \varepsilon_0$, $R \ge R_0$, and any symmetric solution $\phi$ of (7.5) we have*

$$\|\phi\|_{L^\infty} \le C\||y|^{2+\tau-\varepsilon} f\|_{L^\infty}.$$

*Proof.* If the conclusion fails, there are sequences $\varepsilon_n \to 0$, $R_n \to \infty$, $\phi_n$ solving (7.5) for some $f_n$ such that

$$\|\phi_n\|_{L^\infty} = 1, \quad \||y|^{2+\tau-\varepsilon_n} f_n\|_{L^\infty} \to 0$$

as $n \to \infty$. We show that for any $\rho > 0$ fixed

$$\sup_{\Sigma_0 \cap B_\rho(0)} |\phi_n| \to 0 \quad \text{as } n \to \infty.$$

If not, then passing to a subsequence, for some $x_n \in \Sigma_0 \cap B_\rho(0)$,

$$|\phi_n(x_n)| \ge \delta > 0.$$

By standard estimates, $\phi_n$ is bounded in $C^\alpha_{loc}$. Hence by passing to a new subsequence, $\phi_n \to \phi$ locally uniformly as $n \to \infty$. We pass to the limit in the weak formulation and obtain a bounded symmetric solution $\phi \not\equiv 0$ of

$$\Delta_{\mathcal{C}}\phi + |A|^2\phi = 0 \quad \text{in } \mathcal{C}.$$

But by Lemma 7.1 the only bounded solution is $cZ_1$, which is odd. Hence $\phi \equiv 0$ and this is a contradiction.

We claim that

$$\|\phi_n\|_{L^\infty(\Sigma_0 \cap B_{R_n}(0))} \to 0$$

as $n \to \infty$, which is a contradiction.

Indeed, let $w = 1 - \delta|y|^{-\tau}$. One can check that

$$L_{\varepsilon_n}(w) \le -c_{\varepsilon_n}\delta|y|^{-\tau-2+\varepsilon_n}$$

for $|y| \ge \bar{R}$ where $\bar{R}$ is large and fixed and $c_{\varepsilon_n}$ converges to a positive constant as $\varepsilon_n \to 0$. Next we choose $\delta > 0$ such that $\inf_{\Sigma_0 \cap B_{\bar{R}}(0))} w > 0$. We claim that

$$\phi_n \le C(\|\phi\|_{L^\infty(\Sigma_0 \cap B_{\bar{R}}(0))} + \||y|^{\tau+2-\varepsilon_n} f_n\|_{L^\infty})w \tag{7.6}$$

in $\Sigma_0 \cap (B_{R_n}(0) \setminus B_{\bar{R}}(0))$. Note that (7.6) holds for $C$ large depending on $\phi_n$ because $\phi_n$ is bounded. The claim is that this holds for $C = C_0$ with

$$C_0 = \max\left(2(\inf_{\Sigma_0 \cap B_{\bar{R}}(0))} w)^{-1}, \sup \frac{|f_n|}{c_{\varepsilon_n}\delta|y|^{-\tau-2-+\varepsilon_n}}\right)$$

The comparison can be done by sliding. ∎

Using the Fredholm alternative, we deduce the following result.

**Lemma 7.3.** *Suppose $f$ is symmetric and $\||y|^{2+\tau-\varepsilon}f\|_{L^\infty}<\infty$. For $0<\varepsilon\le\varepsilon_0$ and $R\ge R_0$ there is a unique symmetric solution $\phi$ of (7.5).*

*Proof of Proposition 7.1.* We fix $0<\varepsilon\le\varepsilon_0$ for $R\ge R_0$ and let $\phi_R$ be the solution of (7.5). Then for a sequence $R_j\to\infty$, $\phi=\lim_{j\to\infty}\phi_{R_j}$ exists and is a solution of (7.3). Estimate (7.4) is obtained by scaling and the gradient estimates of Caffarelli and Silvestre [7]. Finally $\lim_{|x|\to\infty}\phi(x)$ exists because of (7.4). □

We need a solvability theory with a constraint on the right hand side so that the solution decays. For this we consider the equation

$$L_\varepsilon(\phi)+b_\varepsilon\phi=f-cZ_2\eta_1\quad\text{in }\Sigma_0,\tag{7.7}$$

where $\eta_1$ is a smooth radial symmetric cut-off function on $\Sigma_0$, such that $\eta_1(x)=1$ for $|x|\le A_1$, $\eta_1(x)=0$ for $|x|\ge A_1+1$ and $A_1$ is a fixed large constant. The function $Z_2\eta_1$ in the right hand side can be replaced by any $f_0$ with $f_0(x)=O(|x|^{-2-\tau+\varepsilon})$, $\int_{\Sigma_0}f_0Z_2\ne0$.

**Proposition 7.2.** *There is $\varepsilon_0>0$ such that for all $0<\varepsilon\le\varepsilon_0$ and any $f$ symmetric with respect to $x_3$ with $\||y|^{2+\tau-\varepsilon}f\|_{L^\infty}<\infty$ there is a unique solution $\phi$, $c$ of (7.7) such that $\phi$ is symmetric and $\||y|^\tau\phi\|_{L^\infty}<\infty$. Moreover*

$$\||y|^\tau\phi\|_{L^\infty}+|c|\le C\||y|^{2+\tau-\varepsilon}f\|_{L^\infty}.$$

*Proof.* First we prove existence. For this we let $\phi_0$ be the solution of (7.3) constructed in Proposition 7.1 with right hand side $Z_2\eta_1$. Then $\lim_{|x|\to\infty}\phi_0(x)=\Lambda_\varepsilon$ exists. We claim that $\Lambda_\varepsilon>0$ stays bounded and bounded away from 0 as $\varepsilon\to0$. To prove this, let $Z_2$ be given as in (7.2). Multiply (7.3) by $Z_0\eta_R$ where $\eta_R(x)=\eta(x/R)$ and $\eta$ is a radial, symmetric, smooth cut-off function such that $\eta(x)=1$ for $|x|\le1$, $\eta(x)=0$ for $|x|\ge2$. Then for $R\ge A_1+1$ we find

$$\varepsilon\int_{\Sigma_0}\phi_0(x)\int_{\Sigma_0}Z_2(y)\frac{\eta_R(y)-\eta_R(x)}{|x-y|^{4-\varepsilon}}\,dy\,dx+\int_{\Sigma_0}\phi_0\eta_Rg_0=\int_{\Sigma_0}Z_2^2\eta_1$$

where

$$g_0=L_\varepsilon(Z_2)+b_\varepsilon Z_2.$$

Let us consider the first term

$$\varepsilon\int_{\Sigma_0(\varepsilon)}\phi_0(x)\int_{\Sigma_0(\varepsilon)}Z_2(y)\frac{\eta_R(y)-\eta_R(x)}{|x-y|^{4-\varepsilon}}\,dy\,dx=I_1-I_2$$

with

$$\begin{aligned}I_1&=\frac12\varepsilon\int_{\Sigma_0}\int_{\Sigma_0}\phi_0(x)\frac{(Z_2(y)-Z_2(x))(\eta_R(y)-\eta_R(x))}{|x-y|^{4-\varepsilon}}\,dydx\\ I_2&=\frac12\varepsilon\int_{\Sigma_0}\int_{\Sigma_0}Z_2(y)\frac{(\phi_0(y)-\phi_0(x))(\eta_R(y)-\eta_R(x))}{|x-y|^{4-\varepsilon}}\,dydx.\end{aligned}$$

Since $\phi_0(x)=\Lambda_\varepsilon+O(R^{-1-\tau})$ for $|x|\ge R/4$ it is possible to show that

$$I_1=a_\varepsilon\Lambda_\varepsilon+o(1)$$

where $a_\varepsilon>0$ remains bounded and bounded away from 0 and $o(1)\to0$ as as $\varepsilon\to0$ and $R\to\infty$ with $R^\varepsilon\to1$. Indeed, consider the regions $R_1=\{x\in\Sigma_0:|x|\le R/2\}$, $R_2=\{R/2\le|x|\le4R\}$, $R_3=\{|x|\ge4R\}$. Then

$$\varepsilon\int_{x\in R_j}\int_{y\in R_j}\phi_0(x)\frac{(Z_2(y)-Z_2(x))(\eta_R(y)-\eta_R(x))}{|x-y|^{4-\varepsilon}}\,dydx=0$$

for $j = 1, 3$. We have

$$\begin{aligned}&\left|\varepsilon \int_{x\in R_1}\int_{y\in R_2} \phi_0(x)\frac{(Z_2(y)-Z_2(x))(\eta_R(y)-\eta_R(x))}{|x-y|^{4-\varepsilon}}\,dydx\right|\\&\le 2\varepsilon\|\phi_0\|_{L^\infty}\log(R)\int_{|x|\le R/4}\int_{|y|\ge R/2}\frac{1}{|x-y|^{4-\varepsilon}}\,dydx\\&\le C\varepsilon\log(R)R^{-\varepsilon}.\end{aligned}$$

and

$$\begin{aligned}&\varepsilon\int_{x\in R_2}\int_{y\in R_2}\phi_0(x)\frac{(Z_2(y)-Z_2(x))(\eta_R(y)-\eta_R(x))}{|x-y|^{4-\varepsilon}}\,dydx\\&=(\Lambda_\varepsilon+O(R^{-1-\tau}))\varepsilon\int_{x\in R_2}\int_{y\in R_2}\frac{(Z_2(y)-Z_2(x))(\eta_R(y)-\eta_R(x))}{|x-y|^{4-\varepsilon}}\,dydx\\&=(\Lambda_\varepsilon+O(R^{-1-\tau}))R^\varepsilon\pi(1+O(\varepsilon)+O(R^{-1})).\end{aligned}$$

In the last integral we have rescaled by $R$ and used the expansion for $Z_2$. Other terms in $I_1$ can be handled similarly. Also, similar calculations show that $I_2 \to 0$.

Now let $\hat\phi$ be the solution of (7.3) constructed in Proposition 7.1 with right hand side $f$. Let $\ell = \lim_{|x|\to\infty}\hat\phi(x)$, which exists by Proposition 7.1. Then $\phi = \hat\phi - \frac{\ell}{\Lambda_\varepsilon}\phi_0$ satisfies

$$\Lambda_\varepsilon(\phi) + b_\varepsilon\phi = f - \frac{\ell}{\Lambda_\varepsilon}Z_2\eta_1.$$

Moreover we have the estimates $|\ell| \le C\||y|^{-2-\tau}f\|_{L^\infty}$ and

$$\||y|^\tau\phi\|_{L^\infty} \le C\||y|^{-2-\tau}f\|_{L^\infty}$$

by (7.4).

Let us prove uniqueness. Suppose that for a sequence $\varepsilon_n \to 0$ there is a nontrivial solution $\phi_n$, $c_n$ of (7.7) with $f = 0$. We can assume

$$\||y|^\tau\phi\|_{L^\infty} = 1. \tag{7.8}$$

To estimate $c_n$, we test equation (7.7) with $Z_2\eta_n$ where $\eta_n$ is a smooth cut-off function such that $\eta_n(r) = 1$ for $r \le R_n$ and $\eta_n(r) = 0$ for $r \ge 2R_n$, with

$$R_n \to \infty \quad\text{and}\quad R_n\varepsilon_n^{\frac12} \to 0. \tag{7.9}$$

We get

$$\begin{aligned}&\varepsilon_n\int_{\Sigma_0}\phi_n(x)\int_{\Sigma_0}Z_2(y)\frac{\eta_n(y)-\eta_n(x)}{|x-y|^{4-\varepsilon_n}}\,dy\,dx + \int_{\Sigma_0}\phi_n\eta_n g_n\\&= -c_n\int_{\Sigma_0}b_{\varepsilon_n}Z_2\eta_n,\end{aligned}$$

where

$$g_n = L_{\varepsilon_n}(Z_2) + b_{\varepsilon_n}Z_2.$$

We claim that

$$\varepsilon_n\int_{\Sigma_0(\varepsilon_n)}\phi_n(x)\int_{\Sigma_0(\varepsilon_n)}Z_2(y)\frac{\eta_n(y)-\eta_n(x)}{|x-y|^{4-\varepsilon_n}}\,dy\,dx \to 0$$

as $n\to\infty$. Indeed

$$\begin{aligned}&\varepsilon_n\int_{\Sigma_0(\varepsilon_n)}\phi_n(x)\int_{\Sigma_0(\varepsilon_n)}Z_2(y)\frac{\eta_n(y)-\eta_n(x)}{|x-y|^{4-\varepsilon_n}}\,dy\,dx\\&\quad=\int_{\Sigma_0}\phi_nZ_2L_{\varepsilon_n}(\eta_n) + \varepsilon_n\int_{\Sigma_0}\phi_n(x)\int_{\Sigma_0}\frac{(Z_2(y)-Z_2(x))(\eta_n(y)-\eta_n(x))}{|x-y|^{4-\varepsilon_n}}\,dy\,dx.\end{aligned}$$

By calculation

$$L_{\varepsilon_n}(\eta_n)(x)=\begin{cases}O(R_n^{\varepsilon_n-2}) & \text{if } |x|\le 10R_n\\ O(\frac{\varepsilon_n R_n^{4-\varepsilon_n}}{|x|^{4-\varepsilon_n}}) & \text{if } |x|\ge 10R_n\end{cases}$$

Then by (7.8)

$$\begin{aligned}\left|\int_{\Sigma_0}\phi_n Z_2 L_{\varepsilon_n}(\eta_n)\right| &\le CR_n^{\varepsilon_n-2}\int_{B_{R_n}\subset\mathbb{R}^2}|x|^{-\tau}\log(2+|x|)\\ &\quad+C\varepsilon_n R_n^{4-\varepsilon_n}\int_{B_{R_n}^c\subset\mathbb{R}^2}|x|^{\varepsilon_n-4-\tau}\log(2+|x|)\\ &\le CR_n^{-\tau}\log(R_n)+C\varepsilon_n R_n^{2-\varepsilon_n-\tau}\log(R_n)\to 0\end{aligned}$$

as $n\to\infty$ by (7.9). Similarly

$$\varepsilon_n\int_{\Sigma_0}\frac{(Z_2(y)-Z_2(x))(\eta_n(y)-\eta_n(x))}{|x-y|^{4-\varepsilon_n}}\,dy=\begin{cases}O(R_n^{\varepsilon_n-2}) & \text{if } |x|\le 10R_n\\ O(\frac{\varepsilon_n R_n^{4-\varepsilon_n}}{|x_n|^{4-\varepsilon_n}}\log(\frac{|x|}{R_n})) & \text{if } |x|\ge 10R_n.\end{cases}$$

This implies

$$\left|\varepsilon_n\int_{\Sigma_0}\phi_n(x)\int_{\Sigma_0}\frac{(Z_2(y)-Z_2(x))(\eta_n(y)-\eta_n(x))}{|x-y|^{4-\varepsilon_n}}\,dy\,dx\right|\to 0$$

as $n\to\infty$ as before.

We also have

$$\int_{\Sigma_0}\phi_n(y)\eta_n(y)g_n(y)\,dy\to 0 \tag{7.10}$$

as $n\to\infty$. Indeed, $g_n=L_{\varepsilon_n}(Z_2)+b_{\varepsilon_n}Z_2\to\frac{\pi}{2}(\Delta_{\mathcal{C}}+|A|^2)Z_2=0$ uniformly on compact sets (Lemmas A.2 and A.4), so for any fixed $\rho>0$

$$\int_{\Sigma_0\cap B_\rho}\phi_n(y)\eta_n(y)g_n(y)\,dy\to 0 \tag{7.11}$$

as $n\to\infty$. For the integral in $\Sigma_0\setminus B_\rho$, we note that

$$|L_{\varepsilon_n}(Z_2)(x)=O(\log(|x|)|x|^{\varepsilon_n-4})$$

and by Lemma 6.5

$$b_\varepsilon(x)=\begin{cases}\pi|A_{\Sigma_0}|^2|x|^\varepsilon+O(\frac{\varepsilon}{\log(|x|)^{2-\varepsilon}}) & \text{for } |x|\le\varepsilon^{-\frac12}+1,\\ 0 & \text{for } |x|\ge\varepsilon^{-\frac12}+1.\end{cases}$$

In the region $|x|\le\varepsilon^{-\frac12}$, $\Sigma_0$ is the catenoid and hence

$$|A_{\Sigma_0}|^2=O(|x|^{-4}).$$

This implies that for $|x|\le\varepsilon^{-\frac12}$

$$|b_{\varepsilon_n}(x)|\le C|x|^{\varepsilon-4}+C\frac{\varepsilon_n}{\log(|x|)^{2-\varepsilon_n}}$$

It follows that

$$|g_n(x)|\le C\log(|x|)|x|^{\varepsilon-4}+C\frac{\varepsilon_n}{\log(|x|)^{1-\varepsilon_n}}$$

Hence

$$\left|\int_{\Sigma_0\setminus B_\rho}\phi_n(y)\eta_n(y)g_n(y)\,dy\right|\le C\rho^{-2-\tau+\varepsilon_n}\log(\rho)+C\varepsilon_n R_n^{2-\tau}.$$

Using this and (7.11) we deduce the claim (7.10).

It follows that

$$c_n\to 0\quad\text{as } n\to\infty.$$

As in Lemma 7.2, $\phi_n \to 0$ uniformly on compact sets. Then by (7.8) there is a point $x_n \in \Sigma_0$ such that

$$(1+|x_n|)^\tau |\phi_n(x_n)| \geq \frac{1}{2}.$$

and $|x_n| \to \infty$. By scaling and translating we obtain a non-trivial $\phi$ satisfying

$$\Delta\phi = 0 \quad \text{in } \mathbb{R}^2 \setminus \{0\}$$

with

$$|\phi(x)| \leq C|x|^{-\tau},$$

which is impossible. □

Next we establish an a priori estimate for decaying solutions of (7.1). We do not expect solutions of this problem to decay, but that this will be the case if $f$ satisfies a constraint. For this reason, instead of (7.1) we consider a projected equation

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi) = f - cf_0 \quad \text{in } \Sigma_0. \tag{7.12}$$

where $f_0$ is an appropriate function. For $f_0$ we can take almost any smooth function with compact support, but it will be important that

$$\int_{\Sigma_0} f_0 Z_2 \neq 0,$$

and that we have a solution $\phi_0$ with $\|\phi_0\|_* < \infty$ of

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_0) = f_0 \quad \text{in } \Sigma_0.$$

One possibility to achieve this is the following. Let $R>0$ the number given in Proposition 6.1. For $\rho > R$ let $\eta_\rho(x) = \eta(x/\rho)$ where $\eta$ is a smooth radial cut-off function in $\mathbb{R}^3$, such that $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$. Let $f_\rho = Z_2\eta_\rho$ and $\phi_\rho$ be the function constructed in Proposition 6.1. We recall that it satisfies

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_\rho)(X) = f_\rho(X) \quad \text{for } X \in \Sigma_0, \ |X| \geq R,$$

and the estimate

$$\|\phi_\rho\|_* \leq C\|f_\rho\|_{1-\varepsilon,\alpha+\varepsilon}.$$

Note that

$$\|f_\rho\|_{1-\varepsilon,\alpha+\varepsilon} \leq C\rho\log(\rho).$$

and that since $f_\rho$ is smooth, $\phi_\rho$ is also smooth. Using elliptic estimates we deduce that $\|\phi_\rho\|_{C^{2,\alpha}(B_R)} \leq C\rho\log(\rho)$. Let

$$\tilde{f}_\rho = \varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_\rho).$$

Then

$$\int_{\Sigma_0} \tilde{f}_\rho Z_2 = \int_{\Sigma_0 \cap B_R} \varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_\rho) Z_2 + \int_{\Sigma_0 \setminus B_R} Z_2^2 \eta_\rho.$$

Since

$$\int_{\Sigma_0 \cap B_R} \varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_\rho) Z_2 = O(\rho\log(\rho)), \qquad \int_{\Sigma_0 \setminus B_R} Z_2^2 \eta_\rho = c\rho^2 \log(\rho)^2(1+o(1))$$

as $\rho \to \infty$, where $c>0$, we find that for $\rho > 0$ large

$$\int_{\Sigma_0} \tilde{f}_\rho Z_2 \neq 0.$$

We fix $\rho$ large and take

$$\phi_0 = \phi_\rho, \qquad f_0 = \tilde{f}_\rho. \tag{7.13}$$

**Lemma 7.4.** *Assume $\||x|^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)} < \infty$ and $\phi$, $c$ is a solution of (7.12) such that $\||x|^\tau\phi\|_{L^\infty(\Sigma_0)} < \infty$. If $\varepsilon$ is small enough, then there is $C$ independent of $f$, $\phi$, $c$ such that*

$$\||x|^\tau\phi\|_{L^\infty(\Sigma_0)} + |c| \leq C\||x|^{2+\tau-\varepsilon} f\|_{L^\infty(\Sigma_0)}.$$

*Proof.* Assume by contradiction that there are sequences $\varepsilon_n \to 0$, $\phi_n$, $c_n$ solving (7.12) with right hand side $f_n$ such that

$$\|(1+|x|)^\tau \phi_n\|_{L^\infty(\Sigma_0)} = 1, \quad \|(1+|x|)^{2+\tau-\varepsilon_n} f_n\|_{L^\infty(\Sigma_0)} \to 0$$

as $n \to \infty$. Recall that $\Sigma_0 = \Sigma_0(\varepsilon_n)$.

To estimate $c_n$, let $Z_2$ be given as in (7.2). We test equation (7.12) with $Z_2\eta_n$ where $\eta_n$ is a smooth cut-off function such that $\eta_n(r) = 1$ for $r \le R_n$ and $\eta_n(r) = 0$ for $r \ge 2R_n$, with $R_n \to \infty$ and

$$R_n << \varepsilon_n^{-\frac{1}{2}}.$$

We get

$$\begin{aligned}
&\varepsilon_n \int_{\Sigma_0(\varepsilon_n)} \phi_n(x) \int_{\Sigma_0(\varepsilon_n)} Z_2(y) \frac{\eta_n(y)-\eta_n(x)}{|x-y|^{4-\varepsilon_n}}\, dy\, dx + \int_{\Sigma_0(\varepsilon_n)} \phi_n(y)\eta_n(y)\mathcal{J}_{\Sigma_0}(Z_2)(y)\, dy \\
&= \int_{\Sigma_0(\varepsilon_n)} f_n Z_2 \eta_n - c_n \int_{\Sigma_0(\varepsilon_n)} f_0 Z_2 \eta_n.
\end{aligned}$$

By a calculation

$$\varepsilon_n \int_{\Sigma_0(\varepsilon_n)} \phi_n(x) \int_{\Sigma_0(\varepsilon_n)} Z_2(y) \frac{\eta_n(y)-\eta_n(x)}{|x-y|^{4-\varepsilon_n}}\, dy\, dx \to 0$$

as $n \to \infty$, and

$$\int_{\Sigma_0(\varepsilon_n)} \phi_n(y)\eta_n(y)\mathcal{J}_{\Sigma_0}[Z_2](y)\, dy \to 0$$

as $n \to \infty$. It follows that

$$c_n \to 0 \quad \text{as } n \to \infty.$$

There is a point $x_n \in \Sigma_0(\varepsilon_n)$ such that

$$(1+|x_n|)^\tau |\phi_n(x_n)| \ge \frac{1}{2}.$$

If $x_n$ remains bounded, then up to subsequence $\phi_n \to \phi$ uniformly on compact sets of the catenoid $\mathcal{C}$ and $\phi$ is a nontrivial solution of

$$\Delta_{\mathcal{C}}\phi + |A|^2\phi = 0 \quad \text{on } \mathcal{C}$$

with $|\phi(x)| \le (1+|x|)^{-\tau}$. By Lemma 7.1 $\phi$ must be zero, a contradiction.

Hence $x_n$ is unbounded. By scaling and translating we obtain a non-trivial $\phi$ satisfying

$$\Delta\phi + \frac{\tilde\eta}{r^2}\phi = 0 \quad \text{in } \mathbb{R}^2$$

with

$$|\phi(x)| \le C|x|^{-\tau},$$

where $0 \le \tilde\eta \le 1$ is a radial, non-decreasing function such that $\tilde\eta = 1$ for all $|x| \ge m$, where $m \ge 0$. For $r \ge m$ we get

$$\phi(r) = a\cos(\log(r)) + b\sin(\log(r))$$

but then $a = b = 0$, so $\phi \equiv 0$, a contradiction. □

*Proof of Proposition 2.2.* We want to solve (7.1) where $f$ is radial and symmetric such that $\|f\|_{1-\varepsilon,\alpha+\varepsilon} < \infty$. First we reduce the problem to one where the right hand side has fast decay. Let $\bar\phi = \bar\phi(f)$ be the function constructed in Proposition 6.1 with right hand side $f$, namely $\bar\phi$ satisfies

$$\varepsilon\mathcal{J}^s_{\Sigma_0}(\bar\phi)(X) = f \quad X \in \Sigma_0, |X| \ge R$$

where $R > 0$ is fixed in this proposition. Then we look for $\phi$ of the form $\phi = \phi_1 + \eta\bar\phi$ where $\eta \in C^\infty(\mathbb{R}^2)$ is a cut-off function such $\eta(x) = 1$ for $|x| \ge 2R$, $\eta(x) = 0$ for $|x| \le R$. The function $\phi_1$ then needs to satisfy

$$\varepsilon\mathcal{J}^s_{\Sigma_0}(\phi_1) = f_1 \quad \text{in } \Sigma_0$$

where

$$f_1(x) = (1-\eta(x))f(x) - \varepsilon \int_{\Sigma_0} \bar\phi(y) \frac{\eta(y)-\eta(x)}{|y-x|^{4-\varepsilon}}\, dy.$$

Since the second term decays like $|x|^{-4+\varepsilon}$ as $|x|\to\infty$, $f_1$ has fast decay, meaning $\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty(\Sigma_0)}<\infty$.

In the sequel, we assume that $f$ is symmetric, radial with $\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty(\Sigma_0)}<\infty$. First, we claim that it is possible to find a solution $\phi$, $c$ to (7.12), which depends linearly on $f$ and such that

$$\|(1+|x|)^{\tau}\phi\|_{L^\infty}+|c|\le C\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty}.$$

We construct this solution by looking for it in the form

$$\phi=\varphi+\eta_0\psi$$

and we ask that

$$L_\varepsilon(\varphi)+b_\varepsilon\varphi=-[L_\varepsilon,\eta_0](\psi)+(1-\eta_0)f+cf_0 \quad \text{in } \Sigma_0 \tag{7.14}$$

$$L_\varepsilon(\psi)+a_\varepsilon\psi=-a_\varepsilon(1-\eta_\varepsilon)\varphi+f \quad \text{in } \Sigma_0\setminus B_R(0) \tag{7.15}$$

Here

$$[L_\varepsilon,\eta](\psi)=L_\varepsilon(\eta_0\psi)-\eta_0L_\varepsilon(\psi)=\varepsilon\,\text{p.v.}\int_{\Sigma_0}\psi(y)\frac{\eta_0(y)-\eta_0(x)}{|x-y|^{4-\varepsilon}}\,dy,$$

and $R$ is the same as in Proposition 6.2. The smooth cut-off functions, $\eta_0$ and $\eta_\varepsilon$ are radial in $\mathbb{R}^3$ and such that

$$\eta_0(x)=0 \text{ for } |x|\le R, \qquad \eta_0(x)=1 \text{ for } |x|\ge 2R,$$
$$\eta_\varepsilon(x)=1 \text{ for } |x|\le \varepsilon^{-\frac12}, \qquad \eta_\varepsilon(x)=0 \text{ for } |x|\ge \varepsilon^{-\frac12}+1.$$

We rewrite this system as a fixed point problem as follows. Let $Y$ be the space $Y=\{\varphi\in L^\infty(\Sigma_0):\|(1+|x|)^\tau\varphi\|_{L^\infty}<\infty\}$ with the norm $\|\varphi\|_Y=\|(1+|x|)^\tau\varphi\|_{L^\infty}$. Given $\varphi\in Y$ we solve (7.15) using Proposition 6.2 and obtain a solution $\psi=\psi(\varphi)$. With this $\psi$ we solve now problem (7.14) using Proposition 7.2 and obtain a solution $\tilde\varphi=\tilde\varphi(\varphi)\in Y$. Let $T(\varphi)=\tilde\varphi(\varphi)$ denote the operator defined in this way, so that $T:Y\to Y$ is an affine linear operator.

We claim that $T$ is compact. Assume that $\varphi_n$ is a bounded sequence in $Y$, and let $\psi_n$ be the corresponding solution of (7.15). By Proposition 6.2 $\|\psi_n\|_Y\le C$. Let $\tilde\varphi_n$, $c_n$ be the solution of (7.14) with $\psi$ replaced by $\psi_n$ and $c$ by $c_n$. We claim that up to subsequence $\tilde\varphi_n$ converges in $Y$. By standard regularity $\tilde\varphi_n$ is bounded in $C^{1,\alpha}_{loc}(\Sigma_0)$ (any $0<\alpha<1$). Then for a subsequence (denoted the same), $\tilde\varphi_n\to\tilde\varphi$ uniformly on compact sets of $\Sigma_0$ as $n\to\infty$. Let $\tau'\in(\tau,1)$. Then note that $[L_\varepsilon,\eta][\psi_n]$ and $(1-\eta_0)f+c_nf_0$ have fast decay uniform in $\varepsilon$, more precisely

$$\|(1+|x|)^{2+\tau'-\varepsilon}(-[L_\varepsilon,\eta_0](\psi_n)+(1-\eta_0)f+c_nf_0)\|_{L^\infty}\le C.$$

By Proposition 7.2

$$\|(1+|x|)^{\tau'}\tilde\varphi_n\|_{L^\infty}\le C$$

and hence also $\|(1+|x|)^{\tau'}\tilde\varphi\|_{L^\infty}<\infty$. It follows that for any $r>0$

$$\limsup_{n\to\infty}\sup_{\Sigma_0\cap B_r(0)}(1+|x|)^\tau|\tilde\varphi_n-\varphi|=0$$

$$\limsup_{n\to\infty}\sup_{\Sigma_0\setminus B_r(0)}(1+|x|)^\tau|\tilde\varphi_n-\varphi|\le Cr^{\tau-\tau'},$$

so that $\limsup_{n\to\infty}\|\tilde\varphi_n-\varphi\|_Y\le Cr^{\tau-\tau'}$. Since $r$ is arbitrary, $\|\tilde\varphi_n-\tilde\varphi\|_Y\to0$ as $n\to\infty$. This proves that $T$ is compact. By Lemma 7.4 and the Fredholm alternative there is a unique solution of the system (7.14), (7.15) and hence we find a unique solution $\phi$ to (7.12). Moreover

$$\|(1+|x|)^\tau\phi\|_{L^\infty}+|c|\le C\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty},$$

by Lemma 7.4.

Finally, we solve (7.1) when $\|(1+|x|)^{2+\tau-\varepsilon}f\|_{L^\infty}<\infty$. For this let $\phi_0$ be be defined by (7.13). We look now for a solution $\phi$ of (7.1) of the form $\phi=\phi_1+\alpha\phi_0$, where we want $\phi_1$ to have fast decay. Then (7.1) is equivalent to

$$\varepsilon\mathcal{J}^s_{\Sigma_0}(\phi_1)=f-\alpha f_0.$$

Given $\alpha \in \mathbb{R}$, by the previous results we know that there exists $c_1 = c_1(\alpha)$ and $\phi_1 = \phi_1(\alpha)$ of fast decay solving

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(\phi_1) = f - (\alpha + c_1(\alpha)) f_0.$$

We claim that it is possible to choose $\alpha$ such that $c_1(\alpha) = 0$. For this, consider the function $Z_2$ of (7.2) and $\eta$ a smooth cut-off function on $\Sigma_0$ such that $\eta(x) = 1$ for $|x| \le \tilde{R}$ and $\eta(x) = 0$ for $|x| \ge 2\tilde{R}$ with $\tilde{R}$ such that $\tilde{R} \to \infty$ and $\varepsilon \tilde{R}^2 \log(\tilde{R}) \to 0$. By the same calculation as in Proposition 7.2 we get

$$\begin{aligned} &\varepsilon \int_{\Sigma_0} \phi_1(x) \int_{\Sigma_0} Z_2(y) \frac{\eta(y) - \eta(x)}{|x-y|^{4-\varepsilon}} \, dy \, dx + \int_{\Sigma_0} \phi_1(y) \eta(y) \mathcal{J}_{\Sigma_0}(Z_2)(y) \, dy \\ &= \int_{\Sigma_0} f Z_2 \eta - (\alpha + c_1(\alpha)) \int_{\Sigma_0} f_0 Z_2 \eta. \end{aligned} \tag{7.16}$$

For the first 2 terms, we have

$$\begin{aligned} \left| \varepsilon \int_{\Sigma_0} \phi_1(x) \int_{\Sigma_0} Z_2(y) \frac{\eta(y) - \eta(x)}{|x-y|^{4-\varepsilon}} \, dy \, dx \right| &= o(1) \| (1+|x|)^\tau \phi_1 \|_{L^\infty} \\ &\le o(1) (\| (1+|x|)^{2+\tau-\varepsilon} f \|_{L^\infty} + |\alpha|) \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\Sigma_0} \phi_1(y) \eta(y) \mathcal{J}_{\Sigma_0}(Z_2)(y) \, dy \right| &= o(1) \| (1+|x|)^\tau \phi_1 \|_{L^\infty} \\ &\le o(1) (\| (1+|x|)^{2+\tau-\varepsilon} f \|_{L^\infty} + |\alpha|) \end{aligned}$$

where $o(1) \to 0$ as $\tilde{R} \to \infty$ and $\varepsilon \to 0$. Then the equation (7.16) for $\alpha$ is uniquely solvable if $\varepsilon$ is small. $\square$

## 8. The nonlinear term

Consider $h_1.h_2$ defined on $\Sigma_0$ with $\|h_i\|_* \le \sigma_0 \varepsilon^{\frac{1}{2}}$, where $\sigma_0 > 0$ is a small constant. The main result in this section is the following estimate stated in Proposition 2.3:

$$\varepsilon \| N(h_1) - N(h_2) \|_{1-\varepsilon, \alpha+\varepsilon} \le C \varepsilon^{-\frac{1}{2}} (\|h_1\|_* + \|h_2\|_*) \|h_1 - h_2\|_*.$$

Note the "extra" $\varepsilon^{-\frac{1}{2}}$ in the left hand side.

We rewrite the fractional mean curvature in the following way. For a point $x = (x', F_\varepsilon(x')) \in \Sigma_0$ let $x_h = x + \nu_{\Sigma_0}(x) h(x)$ and let $L_h(x)$ denote the half space defined by

$$L_h(x) = \{ y \in \mathbb{R}^3 : \langle y - x_h, \nu_{\Sigma_h}(x_h) \rangle \ge 0 \},$$

where $\nu_{\Sigma_h}$ is the unit normal vector to $\partial E_h$ pointing into $E_h$. Then

$$H^s_{E_h}(x_h) = 2 \int_{\mathbb{R}^3} \frac{\chi_{E_h}(y) - \chi_{L_h(x)}(y)}{|x_h - y|^{3+s}} \, dy$$

which has the advantage that the integral is convergent.

To compute the previous integral restricted to a ball around $x$, let us represent $\Sigma_h$ near this point as a graph over the tangent plane to $\Sigma_0$ at $X$. We start with $r, \theta$ polar coordinates for $x \in \mathbb{R}^2$, i.e. $x = (r\cos\theta, r\sin\theta)$ and let $\hat{r} = \frac{x'}{r} = (\cos\theta, \sin\theta)^T$, $\hat{\theta} = (-\sin\theta, \cos\theta)^T$. Given a point $x \in \Sigma_0$, $x = (x', F_\varepsilon(x'))$ we let

$$\Pi_1(x) = \frac{1}{\sqrt{1 + F_\varepsilon'(x')^2}} \begin{bmatrix} \hat{r} \\ F_\varepsilon'(x') \end{bmatrix}, \quad \Pi_2(x) = \begin{bmatrix} \hat{\theta} \\ 0 \end{bmatrix} \in \mathbb{R}^3, \tag{8.1}$$

$$\Pi = [\Pi_1, \Pi_2].$$

The unit normal vector to $\Sigma_0$ at $X$ pointing up is then given by

$$\nu_{\Sigma_0}(X) = \frac{1}{\sqrt{1 + F_\varepsilon'(x')^2}} \begin{bmatrix} -F_\varepsilon'(x') \hat{r} \\ 1 \end{bmatrix}. \tag{8.2}$$

Then we consider coordinates $t = (t_1, t_2)$ and $t_3$ defined by

$$(t_1, t_2, t_3) \mapsto \Pi_1(x) t_1 + \Pi_2(x) t_2 + \nu_{\Sigma_0}(x) t_3.$$

Let

$$R_x = \delta|x|$$

where $\delta > 0$ is a small fixed constant, and let us define $t_0 = t_0(x)$ such that $\Pi(x)t_0$ is the orthogonal projection of $x$ onto the plane generated by $\Pi_1(x)$, $\Pi_2(x)$.

Using the implicit function theorem (see Appendix C), given $h$ on $\Sigma_0$ with $\|h\|_* \le \sigma_0\varepsilon^{\frac{1}{2}}$, we can represent $\partial E_h$ near $x_h = x + \nu_{\Sigma_0}(x)h(x)$ as

$$\Pi(x)t + \nu_{\Sigma_0}(x)g_h(t), \quad |t - t_0(x)| \le 2R_x$$

where $g_h$ is of class $C^{2,\alpha}$ in the ball $B_{4R_x}(t_0(x))$. We call $G_x$ the operator defined by

$$g_h = G_x(h). \tag{8.3}$$

Let

$$\eta_x(t,t_3) = \eta(\frac{|t-t_0(x)|}{R_x})\eta(\frac{100|t_3|}{\varepsilon^{\frac{1}{2}}|x|}) \tag{8.4}$$

where $\eta \in C^\infty(\mathbb{R})$ is such that $\eta(s) = 1$ for $s \le 1$ and $\eta(s) = 0$ for $s \ge 2$. We also require $\eta' \le 0$.

Let us write

$$H^s_{\partial E_h}(x_h) = H_i(h)(x) + H_o(h)(x)$$

where

$$H_i(h)(x_h) = 2\int_{\mathbb{R}^3} \eta_x(y - x_h)\frac{\chi_{E_h}(y) - \chi_{L_h(x)}(y)}{|x_h - y|^{3+s}}\,dy$$

$$H_o(h)(x_h) = 2\int_{\mathbb{R}^3} (1 - \eta_x(y - x_h))\frac{\chi_{E_h}(y) - \chi_{L_h(x)}(y)}{|x_h - y|^{3+s}}\,dy.$$

Let us explain the choice of cut-off function (8.4). For this, let us write

$$D_{R_x}(x) = \{\Pi(x)t + x : t \in \mathbb{R}^2,\ |t - t_0(x)| < R_x\},$$

which is a 2-dimensional disk on the tangent plane to $\Sigma_0$ at $x$, centered at $x$, and of radius $R_x = \delta|x|$. Let us call

$$C(x) = \{\Pi(x)t + t_3\nu_{\Sigma_0}(x) + x : t \in \mathbb{R}^2,\ |t - t_0(x)| < R_x,\ |t_3| < \frac{\varepsilon^{\frac{1}{2}}|x|}{100}\},$$

the cylinder with base the disk $D_{R_x}$ and height $\varepsilon^{\frac{1}{2}}|x|/100$, and

$$\tilde{C}(x) = \{\Pi(x)t + t_3\nu_{\Sigma_0}(x) + x : t \in \mathbb{R}^2,\ |t - t_0(x)| < 2R_x,\ |t_3| < \frac{\varepsilon^{\frac{1}{2}}|x|}{50}\},$$

which is a similar cylinder with twice the radius and height. The cut-off function (8.4) is zero outside the $\tilde{C}(x)$, while it is one on $C(x)$. Since we assume $\|h\|_* \le \sigma_0\varepsilon^{\frac{1}{2}}$, we have $\|Dg_h\|_{L^\infty} = O(\varepsilon^{\frac{1}{2}})$ and then the set $\Sigma_h$ separates from $\Sigma_0$ in the $\nu_{\Sigma_0}(x)$ direction an amount bounded by $O(\varepsilon^{\frac{1}{2}}2R_x) = O(\delta\varepsilon^{\frac{1}{2}}|x|)$ over the disk $D_{2R_x}(x)$. By choosing $\delta << 100$ we achieve that the parts of $\Sigma_h$ and the plane $\partial L_h$ inside $\tilde{C}(x)$ are in fact contained in a cylinder with base $D_{2R_x}(x)$ but height $O(\delta\varepsilon^{\frac{1}{2}}|x|)$, which is much small than the height of $C(x)$.

We expand $H_i$, $H_0$

$$H_i(h)(x_h) = H_i(0)(x) + H_i'(0)(h)(x) + N_i(h)(x)$$

$$H_o(h)(x_h) = H_o(0)(x) + H_o'(0)(h)(x) + N_o(h)(x).$$

Estimate (2.15) will follow from similar estimates of $N_o(h)$ and $N_i(h)$, which we state in the next lemmas.

**Lemma 8.1.** *There is $C$ independent of $\varepsilon > 0$ small such that for $\|h_i\|_* \le \sigma_0\varepsilon^{\frac{1}{2}}$, $i = 1,2$ we have*

$$\|N_i(h_1) - N_i(h_2)\|_{1-\varepsilon,\alpha+\varepsilon} \le \frac{C}{\varepsilon}(\|h_1\|_* + \|h_2\|_*)\|h_1 - h_2\|_*.$$

**Lemma 8.2.** *There is $C$ independent of $\varepsilon > 0$ small such that for $\|h_i\|_* \le \sigma_0\varepsilon^{\frac{1}{2}}$, $i = 1,2$ we have*

$$\|N_o(h_1) - N_o(h_2)\|_{1-\varepsilon,\alpha+\varepsilon} \le \frac{C}{\varepsilon^{\frac{3}{2}}}(\|h_1\|_* + \|h_2\|_*)\|h_1 - h_2\|_*.$$

For the integral involved in $H_i$ we can write

$$H_i(h)(x_h) = 2\int_{B_{2R_x}(0)} \frac{\eta(\frac{|t|}{R_x})}{|t|^{3-\varepsilon}} \left( \psi(\frac{\nabla g_h(t_0(x))t}{|t|}) - \psi(\frac{g_h(t+t_0(x)) - g_h(t_0(x))}{|t|}) \right) dt$$

where

$$\psi(s) = \int_0^s \frac{d\tau}{(1+\tau^2)^{\frac{4-\varepsilon}{2}}}.$$

For a given $C^{2,\alpha}$ function $g$ defined on $B_{2R_x}(t_0(x))$ let

$$\tilde H_x(g) = 2\int_{B_{2R_x}(0)} \frac{\eta(\frac{|t|}{R_x})}{|t|^{3-\varepsilon}} \left( \psi(\frac{\nabla g(t_0(x))t}{|t|}) - \psi(\frac{g(t+t_0(x)) - g(t_0(x))}{|t|}) \right) dt$$

so that

$$H_i(h) = \tilde H_x(G_x(h)),$$

where $G_x$ is the operator defined in (8.3).

For the expansion of $\tilde H_X$ it will be convenient to rewrite it as

$$\tilde H_X(g) = 2\int_0^1 \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}} \psi'\left(A_t(g)\right) B(g)\, dz dt,$$

where

$$\begin{aligned} A_t(g)(X,z) &= t\frac{g(z+t_0(X)) - g(t_0(X))}{|z|} + (1-t)\frac{\nabla g(t_0(X))z}{|z|}, \\ B(g)(X,z) &= \frac{g(z+t_0(X)) - g(t_0(X)) - \nabla g(t_0(X))z}{|z|}. \end{aligned}$$

Note that

$$DH_i(h)[h_1] = D\tilde H_X(G_X(h))[DG_X(h)[h_1]], \tag{8.5}$$

$$\begin{aligned} D^2H_i(h)[h_1,h_2] &= D^2\tilde H_X(G_X(h))[DG_X(h)[h_1], DG_X(h)[h_2]] \\ &\quad + D\tilde H_X(G_X(h))[D^2G_X(h)[h_1,h_2]]. \end{aligned} \tag{8.6}$$

and

$$\begin{aligned} D\tilde H_X(g)[g_1] &= \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}} \big[\psi''(A_t(g)(X,z))A_t(g_1)(X,z)B(g)(X,z) \\ &\quad + \psi'(A_t(g)(X,z))B(g_1)(X,z)\big]\, dz, \\ D^2\tilde H_X&(g)[g_1,g_2] \\ = \int_0^1 &\int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}} \big[\psi'''(A_t(g)(X,z))B(g)(X,z)A_t(g_1)(X,z)A_t(g_2)(X,z) \\ &\quad + \psi''(A_t(g)(X,z))A_t(g_1)(X,z)B(g_2)(X,z) \\ &\quad + \psi''(A_t(g)(X,z))A_t(g_2)(X,z)B(g_1)(X,z)\big]\, dz dt. \end{aligned}$$

For later computations we will need the following properties of $DG_X$, $D^2G_X$.

**Lemma 8.3.** *Let $\|h\|_*, \|h_1\|_*, \|h_2\|_* \le \sigma_0 \varepsilon^{\frac12}$, $X \in \Sigma_0$ and*

$$g = G_X(h), \quad g_i = DG_X(h)[h_i] \quad i=1,2, \quad \hat g = D^2G_X(h)[h_1,h_2].$$

*Then*

$$\|G_X(h)\|_b \le C$$

*where*

$$\|g\|_b = |X|^{-1}\|g\|_{L^\infty(B_X)} + \|\nabla g\|_{L^\infty(B_X)} + |X|\|D^2 g\|_{L^\infty(B_X)} + |X|^{1+\alpha}[D^2 g]_{\alpha,B_X}. \tag{8.7}$$

*and $B_X = B_{2R_X}(t_0(X))$. Also, for $z \in B_X$:*

$$|A_t(g)(X,z)| \le C\|h\|_* \tag{8.8}$$

$$|B(g)(X,z)| \le C\frac{\|h\|_*}{|X|}|z|, \tag{8.9}$$

$$|A_t(g_i)(X,z)| \le C\|h_i\|_* \tag{8.10}$$

$$|B(g_i)(X,z)| \le C\frac{\|h_i\|_*}{|X|}|z|. \tag{8.11}$$

We leave the proof of these estimate for the appendix.

**Lemma 8.4.** *Let $h, h_1, h_2$ be defined on $\Sigma_0$ with $\|h\|_*, \|h_i\|_* \le \sigma_0\varepsilon^{\frac{1}{2}}$. Let $X \in \Sigma_0$ and*

$$g = G_X(h), \quad g_i = DG_X(h)[h_i] \quad i = 1,2, \quad \hat{g} = D^2G_X(h)[h_1,h_2].$$

*Then*

$$\varepsilon|D\tilde{H}_X(g)[\hat{g}](X)| \le \frac{C}{|X|^{1-\varepsilon}}\|h_1\|_*\|h_2\|_*$$

$$\varepsilon\left|D^2\tilde{H}(g)[g_1,g_2](X)\right| \le \frac{C}{|X|^{1-\varepsilon}}\|h_1\|_*\|h_2\|_*.$$

*Proof.* Let us start with the first term in $D\tilde{H}_X(g)[g_1]$. Using (8.8), (8.10)

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}}\psi''(A_t(g))A_t(g_1)B(g)\,dz\right| &\le \|\psi''\|_{L^\infty}\|A_t(g_1)\|_{L^\infty}\int_{B_{2R_X}(0)}\frac{1}{|z|^{3-\varepsilon}}|B(g)|\,dz\\ &\le C\|h_1\|_*\int_{B_{2R_X}(0)}\frac{1}{|z|^{3-\varepsilon}}|B(g)|\,dz.\end{aligned}$$

Then by (8.9)

$$\begin{aligned}\int_{B_{2R_X}(0)}\frac{1}{|z|^{3-\varepsilon}}|B(g)|\,dz &\le \frac{\|h\|_*}{|X|}\int_{B_{2R_X}(0)}\frac{1}{|z|^{2-\varepsilon}}\,dz\\ &\le \frac{C}{|X|}\|h\|_*\frac{R_X^\varepsilon}{\varepsilon} \le \frac{C}{\varepsilon|X|^{1-\varepsilon}}\|h\|_*.\end{aligned}$$

Therefore

$$\left|\int_{\mathbb{R}^2}\frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}}\psi''(A_t(g))A_t(g_1)B(g)\,dz\right| \le \frac{C}{\varepsilon^|X|^{1-\varepsilon}}\|h_1\|_*.$$

For the second term observe that

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}}\psi'(A_t(g))B(g_1)\,dz\right| &\le C\int_{B_{2R_X}(0)}|A_t(g)B(g_1)|\,dz\\ &\le \frac{C}{\varepsilon|X|^{1-\varepsilon}}\|g_1\|_b,\end{aligned}$$

which is obtained using (8.8) and (8.11).

For the first term in $D^2\tilde{H}_X(g)[g_1,g_2]$, we have, using (8.9) and (8.10),

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}}\psi'''(A_t(g))A_t(g_1)A_t(g_2)B(g)\,dz\right|\\ &\le \|\psi'''\|_{L^\infty}\|A_t(g_1)\|_{L^\infty}\|A_t(g_2)\|_{L^\infty}\int_{B_{2R_X}(0)}\frac{1}{|z|^{3-\varepsilon}}|B(g)|\,dz\\ &\le \frac{C}{\varepsilon|X|^{1-\varepsilon}}\|h_1\|_*\|h_2\|_*.\end{aligned}$$

Similarly, for the second and third terms

$$\left| \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_X})}{|z|^{3-\varepsilon}} \psi''(A_t(g)) A_t(g_1) B(g_2)\, dz \right| \le \|\psi''\|_{L^\infty} \|A_t(g_1)\|_{L^\infty} \int_{B_{2R_X}(0)} \frac{1}{|z|^{3-\varepsilon}} |B(g_2)|\, dz$$
$$\le \frac{C}{\varepsilon |X|^{1-\varepsilon}} \|h_1\|_* \|h_2\|_*.$$

□

Now we deal with the Hölder part of the norm $\| \ \|_{1-\varepsilon,\alpha+\varepsilon}$.

**Lemma 8.5.** *Let $X_1 = (x_1, F_\varepsilon(x_1))$, $X_2 = (x_2, F_\varepsilon(x_2)) \in \Sigma_0$, be such that $|X_1| \le |X_2|$ and $|X_1 - X_2| \le \frac{1}{10}|X_1|$. Let*

$$g_{X_j} = G_{X_j}(h) \quad j = 1, 2$$
$$g_{i,X_j} = DG_{X_j}(h_0)[h_i] \quad i, j = 1, 2.$$

*Then*

$$|D^2 \tilde{H}_{X_1}(g_{X_1})[g_{1,X_1}, g_{2,X_1}] - D^2 \tilde{H}_{X_2}(g_{X_2})[g_{1,X_2}, g_{2,X_2}]|$$
$$\le \frac{C}{\varepsilon} (\|h_1\|_* + \|h_2\|_*) \|h_1 - h_2\|_* \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}. \tag{8.12}$$

*Proof.* Thanks to (8.5) and (8.6), to prove (8.12) it is enough to show

$$|D^2 \tilde{H}_{X_1}(g_{X_1})[g_{1,X_1}, g_{2,X_1}] - D^2 \tilde{H}_{X_2}(g_{X_2})[g_{1,X_2}, g_{2,X_2}]|$$
$$\le \frac{C}{\varepsilon} \|h_1\|_* \|h_2\|_* \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}, \tag{8.13}$$

and

$$|D\tilde{H}_{X_1}(g_{X_1})[D^2 G_{X_1}(h)[h_1, h_2]] - D\tilde{H}_{X_2}(g_{X_2})[D^2 G_{X_2}(h)[h_1, h_2]]|$$
$$\le \frac{C}{\varepsilon} \|h_1\|_* \|h_2\|_* \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}. \tag{8.14}$$

Let us show (8.13). For this, write

$$D^2 \tilde{H}_{X_j}(g_{X_j})[g_{1,X_j}, g_{2,X_j}] = A_1(X_j) + A_2(X_j) + A_3(X_j)$$

where

$$A_1(X_j) = \int_0^1 \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_j}})}{|z|^{3-\varepsilon}} \psi'''(A_t(g_{X_j})(X_j, z)) B(g_{X_j})(X_j, z) A_t(g_{1,X_j})(X_j, z) A_t(g_{2,X_j})(X_j, z)\, dz dt,$$

$$A_2(X_j) = \int_0^1 \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_j}})}{|z|^{3-\varepsilon}} \psi''(A_t(g_{X_j})(X_j, z)) A_t(g_{1,X_j})(X_j, z) B(g_{2,X_j})(X_j, z)\, dz dt,$$

$$A_3(X_j) = \int_0^1 \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_j}})}{|z|^{3-\varepsilon}} \psi''(A_t(g_{X_j})(X_j, z)) A_t(g_{2,X_j})(X_j, z) B(g_{1,X_j})(X_j, z)\, dz dt.$$

Let us estimate the difference

$$A_1(X_j) - A_1(X_j) = \int_0^1 (B_1 + B_2 + B_3 + B_4 + B_5)\, dt$$

where

$$B_1 = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_1}}) - \eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} \psi'''(A_t(g_{X_1})(X_1, z)) B(g_{X_1})(X_1, z)$$
$$\cdot A_t(g_{1,X_1})(X_1, z) A_t(g_{2,X_1})(X_1, z)\, dz,$$

$$B_2 = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} \big(\psi'''(A_t(g_{X_1})(X_1,z)) - \psi'''(A_t(g_{X_2})(X_2,z))\big) B(g_{X_1})(X_1,z) \cdot A_t(g_{1,X_1})(X_1,z) A_t(g_{2,X_1})(X_1,z)\, dz,$$

$$B_3 = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} \Big) \psi'''(A_t(g_{X_2})(X_2,z)) \big(B(g_{X_1})(X_1,z) - B(g_{X_2})(X_2,z)\big) \cdot A_t(g_{1,X_1})(X_1,z) A_t(g_{2,X_1})(X_1,z)\, dz,$$

$$B_4 = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} \Big) \psi'''(A_t(g_{X_2})(X_2,z)) B(g_{X_2})(X_2,z) \big(A_t(g_{1,X_1})(X_1,z) - A_t(g_{1,X_2})(X_2,z)\big) \cdot A_t(g_{2,X_1})(X_1,z)\, dz,$$

$$B_5 = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} \Big) \psi'''(A_t(g_{X_2})(X_2,z)) B(g_{X_2})(X_2,z) A_t(g_{1,X_2})(X_2,z) \cdot \Big(A_t(g_{2,X_1})(X_1,z) - A_t(g_{2,X_2})(X_2,z)\Big)\, dz.$$

We estimate $B_1$:

$$|B_1| \le \|\psi'''\|_{L^\infty} \|A_t(g_1)\|_{L^\infty} \|A_t(g_2)\|_{L^\infty} \int_{B_{2R_{X_2}}(0)} \frac{\eta(\frac{|z|}{R_{X_2}}) - \eta(\frac{|z|}{R_{X_1}})}{|z|^{3-\varepsilon}} |B(g)|\, dz$$

where we have used $\eta' \le 0$ and $R_2 \ge R_1$. Thanks to (8.10), (8.9) we find

$$\begin{aligned}
|B_1| &\le C \frac{\|h_1\|_* \|h_2\|_*}{|X_1} \int_{B_{2R_{X_2}}(0)} \frac{\eta(\frac{|z|}{R_{X_2}}) - \eta(\frac{|z|}{R_{X_1}})}{|z|^{2-\varepsilon}}\, dz \\
&\le \frac{C}{\varepsilon |X_1|} \|h_1\|_* \|h_2\|_* (R_{X_2}^\varepsilon - R_{X_1}^\varepsilon) \\
&\le \frac{C|X_1 - X_2|^{\alpha+\varepsilon}}{\varepsilon |X_1|^{1+\alpha}} \|h_1\|_b \|h_2\|_b.
\end{aligned}$$

Let us consider $B_2$. Using (8.10) we get

$$|B_2| \le C \|h_1\|_* \|h_2\|_* \int_{B_{2R_{X_2}}(0)} |A_t(g_{X_1})(X_1,z) - A_t(g_{X_2})(X_2,z)| |B(g)(X_1,z)|\, dz.$$

For $z \in B_{2R_{X_2}}(0)$ we have

$$\begin{aligned}
&|A_t(g_{X_1})(X_1,z) - A_t(g_{X_2})(X_2,z)| \\
&\le |A_t(g_{X_1})(X_1,z) - A_t(g_{X_1})(X_2,z)| + |A_t(g_{X_1})(X_2,z) - A_t(g_{X_2})(X_2,z)|.
\end{aligned}$$

For the first term

$$\begin{aligned}
|A_t(g_{X_1})(X_1,z) - A_t(g_{X_1})(X_2,z)| &\le \|g''_{X_1}\|_{L^\infty(B_{2R_{X_2}}(t_0(X_1)))} |t_0(X_1) - t_0(X_2)| \\
&\le C \frac{\|g_{X_1}\|_b}{|X_1|} |X_1 - X_2|.
\end{aligned}$$

For the second term we have

$$|A_t(g_{X_1})(X_2,z) - A_t(g_{X_2})(X_2,z)| \le \|g_{X_1} - g_{X_2}\|_b \le C \frac{|X_1 - X_2|}{|X_1|},$$

where the last inequality follow from ... Therefore for $z \in B_{2R_{X_2}}(0)$ • make ref

$$|A_t(g_{X_1})(X_1,z) - A_t(g_{X_2})(X_2,z)| \le \frac{C}{|X_1|} |X_1 - X_2|.$$

This combined with (8.9) gives

$$\begin{aligned}
&\int_{B_{2R_{X_2}}(0)} |A_t(g_{X_1})(X_1,z) - A_t(g_{X_2})(X_2,z)||B(g)(X_1,z)|\,dz \\
&\le C\frac{|X_1-X_2|}{|X_1|^2}\int_{B_{2R_{X_2}}(0)} |z|^{\varepsilon-2}\,dz \\
&\le C\frac{|X_1-X_2|}{\varepsilon|X_1|^2}R_{X_2}^{\varepsilon} \le \frac{C|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}},
\end{aligned}$$

and therefore

$$|B_2| \le C\|h_1\|_*\|h_2\|_*\frac{|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.$$

For $B_3$ we proceed as follows:

$$|B_3| \le \|\psi'''\|_{L^\infty}\|g_1\|_b\|g_2\|_b\int_{B_{2R_{X_2}}(0)} \frac{|B(g_{X_1})(X_1,z)-B(g_{X_2})(X_2,z)|}{|z|^{3-\varepsilon}}\,dz \tag{8.15}$$

Let $R = 10|X_1-X_2|$ and assume that $R \le \frac{1}{2}\min(|X_1|,|X_2|)$. We split

$$\begin{aligned}
&|B(g_{X_1})(X_1,z) - B(g_{X_2})(X_2,z)| \\
&\le |B(g_{X_1})(X_1,z) - B(g_{X_1})(X_2,z)| + |B(g_{X_1})(X_2,z) - B(g_{X_2})(X_2,z)|.
\end{aligned}$$

For the first term we have

$$\begin{aligned}
&\int_{B_R(0)} \frac{|B(g_{X_1})(X_1,z)-B(g_{X_1})(X_2,z)|}{|z|^{3-\varepsilon}}\,dz \\
&\le \int_0^1 (1-\tau)\int_{B_R(0)} \frac{|g''_{X_1}(t_0(X_1)+\tau z)[z^2] - g''_{X_1}(t_0(X_2)+\tau z)[z^2]|}{|z|^{4-\varepsilon}}\,dz\,d\tau \\
&\le C|t_0(X_1)-t_0(X_2)|^\alpha [g''_{X_1}]_{\alpha,B_R(0)}\frac{R^\varepsilon}{\varepsilon} \\
&\le \frac{C}{\varepsilon}\frac{|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}\|g_{X_1}\|_b.
\end{aligned}$$

We next estimate the integral in $B_{2R_{X_2}}(0)\setminus B_R(0)$ and for this we compute for $z \in B_{2R_{X_2}}(0)\setminus B_R(0)$,

$$\begin{aligned}
&|z|(B(g_{X_1})(X_1,z) - B(g_{X_1})(X_2,z)) \\
&= g_{X_1}(t_0(X_1)+z) - g_{X_1}((t_0(X_1)) - \nabla g_{X_1}((t_0(X_1))z \\
&\quad - [g_{X_1}((t_0(X_2)+z) - g_{X_1}(t_0(X_2)) - \nabla g_{X_1}(t_0(X_2))z] \\
&= \int_0^1 (\nabla g_{X_1}(x_\tau+z) - \nabla g_{X_1}(x_\tau) - g''_{X_1}(x_\tau)z)(t_0(X_1)-t_0(X_2))\,d\tau \\
&= \int_0^1\int_0^1 (g''_{X_1}(x_\tau+\rho z)z - g''_{X_1}(x_\tau)z)(t_0(X_1)-t_0(X_2))\,d\rho d\tau,
\end{aligned}$$

where $x_\tau = \tau t_0(X_1) + (1-\tau)t_0(X_2)$. Then

$$\begin{aligned}
|z||B(g_{X_1})(X_1,z) - B(g_{X_1})(X_2,z)| &\le [g''_{X_1}]_{\alpha,B_{\bar R}(\bar x)}|z|^{1+\alpha}|t_0(X_1)-t_0(X_2)| \\
&\le C\|g_{X_1}\|_b|X_1|^{-1-\alpha}|z|^{1+\alpha}|X_1-X_2|.
\end{aligned}$$

Integrating

$$\begin{aligned}
&\int_{B_{2R_{X_2}}(0)\setminus B_R(0)} \frac{|B(g_{X_1})(X_1,z)-B(g_{X_1})(X_2,z)|}{|z|^{3-\varepsilon}}\,dz \\
&\qquad \le C\|g_{X_1}\|_*|X_1|^{-1-\alpha}|X_1-X_2|R^{\alpha+\varepsilon-1} \\
&\qquad \le C\|g_{X_1}\|_b|X_1|^{-1-\alpha}|X_1-X_2|^{\alpha+\varepsilon}.
\end{aligned}$$

To estimate

$$\int_{B_{2R_{X_2}}(0)} \frac{|B(g_{X_1})(X_2,z) - B(g_{X_2})(X_2,z)|}{|z|^{3-\varepsilon}}\, dz$$

we observe that

$$\begin{aligned}|B(g_{X_1})(X_2,z)-&B(g_{X_2})(X_2,z)| \\ &\le |z| \int_0^1 (1-\tau)|g''_{X_1}(t_0(X_2)+\tau z) - g''_{X_2}(t_0(X_2)+\tau z)|\, dz \\ &\le |z|^{1+\alpha} \frac{\|g_{X_1} - g_{X_2}\|_b}{|X_1|^{1+\alpha}} \le \frac{C|z|^{1+\alpha}}{|X_1|^{2+\alpha}} |X_1 - X_2|.\end{aligned}$$

Integrating we find

$$\int_{B_{R_{X_2}}(0)} \frac{|B(g_{X_1})(X_2,z) - B(g_{X_2})(X_2,z)|}{|z|^{3-\varepsilon}}\, dz \le \frac{C}{|X_1|^{1+\alpha}} |X_1 - X_2|^{\alpha+\varepsilon}.$$

This shows that

$$|B_3| \le \frac{C}{\varepsilon} \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}} \|g_1\|_b \|g_2\|_b,$$

The estimates of $B_4$ and $B_5$ are similar and we omit the details. This proves the estimate

$$|A_1(X_1) - A_1(X_2)| \le \frac{C}{\varepsilon} \|h_1\|_* \|h_2\|_* \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.$$

Let us estimate the difference

$$A_2(X_1) - A_2(X_2) = \int_0^1 (B_1 + B_2 + B_3 + B_4)\, dt$$

with

$$B_1 = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_1}}) - \eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} \psi''(A_t(g_{X_1})(X_1,z)) A_t(g_{1,X_1})(X_1,z) B(g_{2,X_1})(X_1,z)\, dz$$

$$\begin{aligned}B_2 = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} &\big(\psi''(A_t(g_{X_1})(X_1,z)) - \psi''(A_t(g_{X_2})(X_2,z))\big) A_t(g_{1,X_1})(X_1,z) \\ &\cdot B(g_{2,X_1})(X_1,z)\, dz\end{aligned}$$

$$\begin{aligned}B_3 = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} &\psi''(A_t(g_{X_2})(X_2,z)) \big(A_t(g_{1,X_1})(X_1,z) - A_t(g_{1,X_2})(X_2,z)\big) \\ &\cdot B(g_{2,X_1})(X_1,z)\, dz\end{aligned}$$

$$\begin{aligned}B_4 = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} &\psi''(A_t(g_{X_2})(X_2,z)) A_t(g_{1,X_2})(X_2,z) \\ &\cdot \big(B(g_{2,X_1})(X_1,z) - B(g_{2,X_2})(X_2,z)\big)\, dz\end{aligned}$$

The terms $B_1$, $B_2$, $B_3$ are similar as before and we have

$$\begin{aligned}|B_1| + |B_2| + |B_3| &\le \frac{C}{\varepsilon} \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}} \|g_1\|_b \|g_2\|_b \\ &\le \frac{C}{\varepsilon} \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}} \|h_1\|_* \|h_2\|_*.\end{aligned}$$

Let us focus on

$$|B_4| \le C\|h_1\|_* \int_{B_{2R_{X_2}}(0)} \frac{|B(g_{2,X_1})(X_1,z) - B(g_{2,X_2})(X_2,z)|}{|z|^{3-\varepsilon}}\, dz. \tag{8.16}$$

The difference with the estimate for (8.15) is that now we cannot control $\|g_{2,X_1}\|_b$ only assuming $\|h\|_*$ bounded, since $g_{2,X_1}$ involves derivatives of $h$.

We proceed by the following observation. We will see that $g_{2,X_j}$ can be decomposed in 2 parts, one of them being regular enough to perform the previous calculations, and the second part having a special form. A model for the second part is

$$D_t\tilde h(t+t_0(X_j))b(X_j,t+t_0(X_j))$$

where $\tilde h$ is $h$ composed with an appropriate change of variables (this in reality also depends on $X_j$, but for to explain the idea here we will omit this dependence), and

$$b(X_j,t_0(X_j)) = 0.$$

Let us see what we get if we assume for the moment that

$$g_{2,X_j} = D_t\tilde h(t+t_0(X_j))b(X_j,t+t_0(X_j)).$$

Then we have

$$B(g_{2,X_j})(X_j,z) = \frac{1}{|z|}\big[D_t\tilde h(z+t_0(X_j))b(X_j,z+t_0(X_j)) - D_t\tilde h(t_0(X_j))D_tb(X_j,t_0(X_j))\big]$$

and so

$$|B(g_{2,X_1})(X_1,z) - B(g_{2,X_2})(X_2,z)| \le A + B$$

where

$$\begin{aligned} A &= \frac{1}{|z|}\Big|(D_t\tilde h(z+t_0(X_1)) - D_t\tilde h(t_0(X_1)))b(X_1,z+t_0(X_1)) \\ &\qquad - (D_t\tilde h(z+t_0(X_2)) - D_t\tilde h(t_0(X_2)))b(X_2,z+t_0(X_2))\Big| \\ B &= \frac{1}{|z|}\Big|D_t\tilde h(t_0(X_1)))(D_tb(X_1,t_0(X_1))z - b(X_1,t_0(X_1))) \\ &\qquad - D_t\tilde h(t_0(X_2)))(D_tb(X_2,t_0(X_2))z - b(X_2,t_0(X_2)))\Big|. \end{aligned}$$

For $A$ we write

$$A \le A_1 + A_2$$

where

$$\begin{aligned} A_1 &= \frac{1}{|z|}\big|D_t\tilde h(z+t_0(X_1)) - D_t\tilde h(t_0(X_1)) \\ &\qquad - (D_t\tilde h(z+t_0(X_2)) - D_t\tilde h(t_0(X_2)))\big||b(X_1,z+t_0(X_1))| \\ A_2 &= \frac{1}{|z|}\big|(D_t\tilde h(z+t_0(X_2)) - D_t\tilde h(t_0(X_2)))(b(X_1,z+t_0(X_1)) - b(X_2,z+t_0(X_2)))\big| \end{aligned}$$

For the first term we split the integral in $B_R(0)$ and outside, where $R = 10|X_1 - X_2|$ and we assume $R \le \frac{1}{10}|X_1|$. For $z \in B_R(0)$ we estimate

$$\begin{aligned} A_1 &\le \frac{1}{|z|}\int_0^1 \big|D_{tt}\tilde h(\tau z+t_0(X_1)) - D_{tt}\tilde h(\tau z+t_0(X_2))\big|\, d\tau|z||b(X_1,z+t_0(X_1))| \\ &\le C\|\frac{b(X_1,z+t_0(X_1))}{|z|}\|_{L^\infty}\frac{|X_1-X_2|^\alpha|z|}{|X_1|^{1+\alpha}}, \end{aligned}$$

where we have used

$$\|\frac{b(X_1,z+t_0(X_1))}{|z|}\|_{L^\infty} < \infty \tag{8.17}$$

and the norm is computed in a ball $B_{4R_{X_1}}(0)$. Therefore

$$\begin{aligned}\int_{B_R(0)} \frac{1}{|z|^{3-\varepsilon}} A_1\, dz &\leq C \frac{|X_1 - X_2|^\alpha}{|X_1|^{1+\alpha}} \int_{B_R(0)} |z|^{\varepsilon-2}\, dz \\ &= C \frac{|X_1 - X_2|^\alpha}{|X_1|^{1+\alpha}} \frac{R^\varepsilon}{\varepsilon} \\ &= \frac{C}{\varepsilon} \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.\end{aligned}$$

For the integral outside $B_R(0)$ we estimate

$$A_1 \leq C \int_0^1 \left| D_{tt}\tilde{h}(z + t_0(X_\tau)) - D_{tt}\tilde{h}(t_0(X_\tau)) \right| d\tau |X_1 - X_2| \| \frac{b(X_1, z + t_0(X_1))}{|z|} \|_{L^\infty} |z|,$$

where $\{X_\tau : \tau \in [0,1]\}$ denotes a path joining $X_1$ to $X_2$, with $|\frac{d}{d\tau} X_\tau| \leq C|X_1 - X_2|$. Hence

$$A_1 \leq C \| \frac{b(X_1, z + t_0(X_1))}{|z|} \|_{L^\infty} \frac{|X_1 - X_2|}{|X_1|^{1+\alpha}} |z|^\alpha.$$

Integrating,

$$\begin{aligned}\int_{B_{2R_{X_2}}(0) \setminus B_R(0)} \frac{1}{|z|^{3-\varepsilon}} A_1\, dz &\leq C \frac{|X_1 - X_2|}{|X_1|^{1+\alpha}} \int_{B_{2R_{X_2}}(0) \setminus B_R(0)} |z|^{\varepsilon-3+\alpha}\, dz \\ &\leq C \frac{|X_1 - X_2|}{|X_1|^{1+\alpha}} R^{\varepsilon-1+\alpha} \\ &= C \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.\end{aligned}$$

The estimate for $A_2$ works directly by using

$$|b(X_1, z + t_0(X_1)) - b(X_2, z + t_0(X_2))| \leq C \frac{|z||X_1 - X_2|}{|X_1|^2}$$

and there is no need to split the integral.

For $B$ we estimate as

$$B \leq B_1 + B_2$$

where

$$\begin{aligned}B_1 &= \frac{1}{|z|} \left| (D_t\tilde{h}(t_0(X_1)) - D_t\tilde{h}(t_0(X_2)))(D_t b(X_1, t_0(X_1))z - b(X_1, z + t_0(X_1))) \right| \\ B_2 &= \frac{1}{|z|} \left| D_t\tilde{h}(t_0(X_2))(D_t b(X_1, t_0(X_1))z - b(X_1, z + t_0(X_1)) \right. \\ &\qquad \left. - (D_t b(X_2, t_0(X_2))z - b(X_2, z + t_0(X_2)))) \right|.\end{aligned}$$

Using

$$\left| D_t b(X_1, t_0(X_1))z - b(X_1, z + t_0(X_1)) \right| \leq C \frac{|z|^2}{|X_1|} \tag{8.18}$$

we get

$$\begin{aligned}B_1 &\leq C \frac{1}{|z|} \frac{|X_1 - X_2|}{|X_1|} \left| D_t b(X_1, t_0(X_1))z - b(X_1, z + t_0(X_1)) \right| \\ &\leq C \frac{1}{|z|} \frac{|X_1 - X_2|}{|X_1|} \frac{|z|^2}{|X_1|}\end{aligned}$$

and then

$$\int_{B_{2R_{X_2}}(0)} \frac{1}{|z|^{3-\varepsilon}} B_1\, dz \leq C \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.$$

For $B_2$, let $R = 10|X_1 - X_2|$ and we assume $R \le \frac{1}{10}|X_1|$. Then, using

$$\begin{aligned}&\big|D_t b(X_1, t_0(X_1))z - b(X_1, z + t_0(X_1)) \\ &- (D_t b(X_2, t_0(X_2))z - b(X_2, z + t_0(X_2)))\big| \le C\frac{|X_1 - X_2|^\alpha |z|^2}{|X_1|^{1+\alpha}},\end{aligned} \tag{8.19}$$

we have

$$B_2 \le \frac{C}{|z|}\frac{|X_1 - X_2|^\alpha |z|^2}{|X_1|^{1+\alpha}}$$

and we obtain

$$\int_{B_R(0)} \frac{1}{|z|^{3-\varepsilon}} B_2 \, dz \le C\frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.$$

To estimate the integral outside $B_R(0)$ we use

$$\begin{aligned}&\big|D_t b(X_1, t_0(X_1))z - b(X_1, z + t_0(X_1)) \\ &- (D_t b(X_2, t_0(X_2))z - b(X_2, z + t_0(X_2)))\big| \le C\frac{|X_1 - X_2||z|^{1+\alpha}}{|X_1|^{1+\alpha}},\end{aligned} \tag{8.20}$$

and we get

$$\begin{aligned}\int_{B_{2R_{X_2}}(0)\setminus B_R(0)} \frac{1}{|z|^{3-\varepsilon}} B_2 \, dz &\le C\frac{|X_1 - X_2|}{|X_1|^{1+\alpha}} \int_{B_{2R_{X_2}}(0)\setminus B_R(0)} \frac{1}{|z|^{3-\varepsilon}} |z|^\alpha \, dz \\ &\le \frac{C}{\varepsilon}\frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}\end{aligned}$$

Let us verify the assertions on $b$ made in (8.17)–(8.20). The function $b(X, z + t_0(X_0))$ is given at main order by

$$\begin{aligned}&b(X, t + t_0(X)) \\ &= \nu^{(i)}(x)\nu^{(j)}(x + r_0 y(x, \frac{t - t_0(X)}{r_0})) - \nu^{(j)}(x)\nu^{(i)}(x + r_0 y(x, \frac{t - t_0(X)}{r_0})),\end{aligned} \tag{8.21}$$

where $\nu^{(i)}$ are the components of the unit normal vector to $\Sigma_0$, which we consider as functions of $x$ with $X = (x, F_\varepsilon(x))$, $r_0 = \delta|X|$, and $y$ is a change of variables from variables $(t_1, t_2)$ parametrizing the tangent plane to $\Sigma_0$ at $X$ to $\mathbb{R}^2$. It has a bounded $C^{2,\alpha}$ norm. Then

$$|b(X, t + t_0(X))| \le C\|\nabla\nu\|_{L^\infty}|t|$$

but $r_0 = \delta|X|$ and $\|\nabla\nu\|_{L^\infty} = O(\frac{\varepsilon^{1/2}}{|X|})$, and this implies (8.17).

To prove (8.18) note that

$$\big|D_t b(X, t_0(X))z - b(X, z + t_0(X))\big| \le \|D_{tt} b(X, \cdot)\|_{L^\infty}|z|^2,$$

where the $L^\infty$ norm is in ball of center $x$ and radius $O(\delta|X|)$. By using formula (8.21) we get $\|D_{tt} b(X_1, \cdot)\|_{L^\infty} = \frac{\varepsilon^{\frac{1}{2}}}{|X|^2}$ and we obtain (8.18).

Estimates (8.19) and (8.20) can be prove similarly.

The complete argument is given next. Thanks to (C.11) we can decompose

$$g_{2,X_j} = \bar{g}_{2,X_j} + \tilde{g}_{2,X_j}$$

where $\bar{g}_{2,X_j}$ can be chosen so that it does not involve derivatives of $h$, and satisfies the estimate

$$\|\bar{g}_{2,X_j}\|_b \le C\|h_2\|_*.$$

Then one can prove as was done for (8.15):

$$\begin{aligned}\int_{B_{2R_{X_2}}(0)} \frac{|B(\bar g_{2,X_1})(X_1,z)-B(\bar g_{2,X_2})(X_2,z)|}{|z|^{3-\varepsilon}}\,dz &\le \frac{C}{\varepsilon}\frac{|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}\|\bar g_{2,X_1}\|_b\\ &\le \frac{C}{\varepsilon}\frac{|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}\|h_2\|_*.\end{aligned}$$

For $\tilde g_{2,X_j}$ we claim that the same estimate holds, that is, we claim that

$$\int_{B_{2R_{X_2}}(0)} \frac{|B(\tilde g_{2,X_1})(X_1,z)-B(\tilde g_{2,X_2})(X_2,z)|}{|z|^{3-\varepsilon}}\,dz \le \frac{C}{\varepsilon}\frac{|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}\|h_2\|_*. \tag{8.22}$$

To prove this, we use representation

$$\tilde g_{2,X_j}(t) = \tilde h_2(X_j,t)Q_0(X_j,t,\tilde h(X_j,t),D_t\tilde h(X_j,t)), \tag{8.23}$$

where the functions $\tilde h_2(X_j,t)$, $\tilde h(X_j,t)$ are obtained from $h_2$ and $h$ through a change of variables:

$$\tilde h(X_j,t) = h(y_{X_j}(t)), \quad \tilde h_2(X_j,t) = h_2(y_{X_j}(t)),$$

as in the Appendix C. Here, for a given $X\in\Sigma_0$, $Q_0 = Q_0(X,t,h,\xi)$, is defined for $t\in\mathbb{R}^2$, $|t-t_0(X)|\le 4R_X$, $h\in\mathbb{R}$, $\xi\in\mathbb{R}^2$ and is explicitly written in (C.12). It has the properties

$$Q_0(X_j,t_0(X_j),h,\xi)=0$$

$$D_hQ_0(X_j,t_0(X_j),h,\xi)=0$$

$$D_{\xi_k}Q_0(X_j,t_0(X_j),h,\xi)=0.$$

Let us define

$$Q(X,t+t_0(X),\xi) = Q_0(X,t+t_0(X),\tilde h(X,z),\xi)$$

so that

$$\tilde g_{2,X_j}(t) = \tilde h_2(X_j,t)Q(X_j,t+t_0(X_j),D_t\tilde h(X_j,t))).$$

Now we write

$$\begin{aligned}B(\tilde g_{2,X_j})(X_j,z) &= \frac{\tilde g_{2,X_j}(z+t_0(X_j))-\tilde g_{2,X_j}(t_0(X_j))-\nabla\tilde g_{2,X_j}(t_0(X_j))z}{|z|}\\ &= \frac{\tilde h_2(X_j,z+t_0(X_j))Q(X_j,z+t_0(X_j),\ldots)-\tilde h_2(X_j,t_0(X_j))D_tQ(X_j,t_0(X_j),\ldots)z}{|z|}\\ &= \frac{(\tilde h_2(X_j,z+t_0(X_j))-\tilde h_2(X_j,t_0(X_j)))Q(X_j,z+t_0(X_j),\ldots)}{|z|}\\ &\quad+\frac{\tilde h_2(X_j,t_0(X_j))(Q(X_j,z+t_0(X_j),\ldots)-D_tQ(X_j,t_0(X_j),\ldots)z)}{|z|},\end{aligned}$$

where we are using the notation

$$\begin{aligned}Q(X_j,z+t_0(X_j),\ldots) &= Q(X_j,z+t_0(X_j),D_t\tilde h(X_j,z+t_0(X_j)))\\ Q(X_j,t_0(X_j),\ldots) &= Q(X_j,t_0(X_j),D_t\tilde h(X_j,t_0(X_j))).\end{aligned}$$

We have to estimate

$$B(\tilde g_{2,X_1})(X_1,z)-B(\tilde g_{2,X_1})(X_1,z) = D_1+D_2+D_3$$

where

$$
\begin{aligned}
D_1 &= \frac{(\tilde h_2(X_1, z+t_0(X_1)) - \tilde h_2(X_1,t_0(X_1)))Q(X_1,z+t_0(X_1),\ldots)}{|z|}\\
&\quad - \frac{(\tilde h_2(X_2, z+t_0(X_2)) - \tilde h_2(X_2,t_0(X_2)))Q(X_2,z+t_0(X_2),\ldots)}{|z|}\\
D_2 &= (\tilde h_2(X_1,t_0(X_1)) - \tilde h_2(X_2,t_0(X_2)))\frac{Q(X_1,z+t_0(X_1),\ldots) - D_tQ(X_1,t_0(X_1),\ldots)z}{|z|}\\
D_3 &= \tilde h_{2,X_2}(t_0(X_2))\Big[\frac{Q(X_1,z+t_0(X_1),\ldots) - D_tQ(X_1,t_0(X_1),\ldots)z)}{|z|}\\
&\qquad - \frac{Q_{X_2}(z+t_0(X_2),\ldots) - D_tQ_{X_2}(t_0(X_2),\ldots)z}{|z|}\Big].
\end{aligned}
$$

The estimate

$$
\int_{B_{2R_{X_2}}(0)} \frac{1}{|z|^{3-\varepsilon}}(|D_1|+|D_2|)\,z \le C\frac{|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}
$$

can be proved in the same way as before, since for $D_1$ difference quotients of $Q$ involve only difference quotients of $D_t\tilde h$ which can be controlled by $\|h\|_*$, and for $D_2$ we need only to consider difference quotients of $\tilde h$.

The estimate of $D_3$ is more delicate, and we proceed with detail. We further split

$$
D_3 = \tilde h_{2,X_2}(t_0(X_2))(D_{3,a}+D_{3,b})
$$

where

$$
\begin{aligned}
D_{3,a} &= \frac{1}{|z|}\big[Q(X_1,z+t_0(X_1),D_t\tilde h(X_1,z+t_0(X_1))) - Q(X_1,z+t_0(X_1),D_t\tilde h(X_1,t_0(X_1)))\\
&\qquad - \big(Q(X_2,z+t_0(X_2),D_t\tilde h(X_2,z+t_0(X_2))) - Q(X_2,z+t_0(X_2),D_t\tilde h(X_2,t_0(X_2)))\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
D_{3,b} &= \frac{1}{|z|}\big[Q(X_1,z+t_0(X_1),D_t\tilde h(X_1,t_0(X_1))) - D_tQ(X_1 t_0(X_1),D_t\tilde h(X_1,t_0(X_1)))\\
&\qquad - \big(Q(X_2,z+t_0(X_2),D_t\tilde h(X_2,t_0(X_2))) - D_tQ(X_2,t_0(X_2),D_t\tilde h(X_2,t_0(X_2)))\big)\big]-
\end{aligned}
$$

To estimate the integral of $\frac{1}{|z|^{3-\varepsilon}}|D_{3,a}|$ over $B_{R_{X_2}}(0)$ we divide the region of integration in $B_R(0)$ and $B_{R_{X_2}}(0)\setminus B_R(0)$, where $R = 10|X_1-X_2|$. To estimate the integral inside $B_R(0)$ we compute

$$
\begin{aligned}
D_{3,a} &= \frac{1}{|z|}\int_0^1 \frac{d}{d\tau}\big[Q(X_1,z+t_0(X_1),D_t\tilde h(X_1,\tau z+t_0(X_1)))\\
&\qquad - Q(X_2,z+t_0(X_2),D_t\tilde h(X_2,\tau z+t_0(X_2)))\big]\,d\tau\\
&= \frac{1}{|z|}\int_0^1 \big[D_\xi Q(X_1,z+t_0(X_1),D_t\tilde h(X_1,\tau z+t_0(X_1)))D_{tt}\tilde h(X_1,\tau z+t_0(X_1))\\
&\qquad - D_\xi Q(X_2,z+t_0(X_2),D_t\tilde h(X_2,\tau z+t_0(X_2)))D_{tt}\tilde h(X_2,\tau z+t_0(X_2))\big]\,d\tau.
\end{aligned}
$$

From this

$$
|D_{3,a}| \le D_{3,a,1} + D_{3,a,2}
$$

where

$$
\begin{aligned}
D_{3,a,1} &= \frac{1}{|z|}\int_0^1 \big|\big[D_\xi Q(X_1,z+t_0(X_1),D_t\tilde h(X_1,\tau z+t_0(X_1)))\\
&\qquad - D_\xi Q(X_2,z+t_0(X_2),D_t\tilde h(X_2,\tau z+t_0(X_2)))\big]D_{tt}\tilde h(X_1,\tau z+t_0(X_1))\big|\,d\tau\\
D_{3,a,2} &= \int_0^1 \big|D_\xi Q(X_2,z+t_0(X_2),D_t\tilde h(X_2,\tau z+t_0(X_2))))\\
&\qquad \cdot\big[D_{tt}\tilde h(X_1,\tau z+t_0(X_1)) - D_{tt}\tilde h(X_2,\tau z+t_0(X_2))\big]\big|\,d\tau
\end{aligned}
$$

Using regularity of $Q(X,t,h,\xi)$ with respect to $X$ and that we we have control of $D^2\tilde h$ we have

$$\begin{aligned}\big|D_\xi Q(X_1,z+t_0(X_1),D_t\tilde h(X_1,\tau z+t_0(X_1)))\\ - D_\xi Q(X_2,z+t_0(X_2),D_t\tilde h(X_2,\tau z+t_0(X_2)))\big| \le C\frac{|z||X_1-X_2|}{|X_1|}.\end{aligned}$$

Using this and that $|D^2_{tt}\tilde h|\le \|h\|_*/|X_1|$ we find

$$\begin{aligned}&\int_{B_{2R_{X_2}}(0)}\frac{1}{|z|^{4-\varepsilon}}|(D_\xi Q(X_1,z+t_0(X_1),\ldots)-D_\xi Q(X_2,z+t_0(X_2),\ldots))D_{tt}\tilde h(X_1,\tau z+t_0(X_1))z|\,dz\\ &\le C\frac{|X_1-X_2|}{|X_1|^2}\int_{B_{R_{X_2}}(0)}\frac{1}{|z|^{2-\varepsilon}}\,dz\le\frac{C}{\varepsilon}\frac{|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.\end{aligned}$$

For the second term we use

$$|D_\xi Q(X_2,z+t_0(X_2),\ldots)|\le C|z|$$

and

$$|D_{tt}\tilde h(X_1,\tau z+t_0(X_1))-D_{tt}\tilde h(X_2,\tau z+t_0(X_1))|\le C\frac{|X_1-X_2|^\alpha}{|X_1|^{1+\alpha}}.$$

Then we obtain

$$\begin{aligned}&\int_{B_R(0)}\frac{1}{|z|^{4-\varepsilon}}|D_\xi Q(X_2,z+t_0(X_2),\ldots)(D_{tt}\tilde h(X_1,\tau z+t_0(X_1))z-D_{tt}\tilde h(X_2,\tau z+t_0(X_1))z)|\,dz\\ &\le\frac{C}{\varepsilon}\frac{|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.\end{aligned}$$

Therefore we get

$$\int_{B_R(0)}\frac{1}{|z|^{3-\varepsilon}}|D_{3,a}|\,dz\le\frac{C}{\varepsilon}\frac{|X_1-X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.$$

Let us proceed with the estimate of the integral of $\frac{1}{|z|^{3-\varepsilon}}|D_{3,a}|$ over the region $B_{R_{X_2}}(0)\setminus B_R(0)$.

Recall that the points $X_j$ have the form $X_j=(x_j,F_\varepsilon(x_j))$, $j=1,2$. For for $\tau\in[0,1]$ we let $X_\tau=(x_\tau,F_\varepsilon(x_\tau))$ where $x_\tau=\tau x_1+(1-\tau)x_2$. We compute

$$\begin{aligned}D_{3,a}=\frac{1}{|z|}\int_0^1\frac{d}{d\tau}\big[Q(X_\tau,z+t_0(X_\tau),D_t\tilde h(X_\tau,z+t_0(X_\tau)))\\ - Q(X_\tau,z+t_0(X_\tau),D_t\tilde h(X_\tau,t_0(X_\tau)))\big]\,d\tau\end{aligned}$$

and so

$$|D_{3,a}|\le\int_0^1(\bar D_{3,a,1}+\bar D_{3,a,2}+\bar D_{3,a,3}+\bar D_{3,a,4})\,d\tau$$

where

$$\begin{aligned}\bar D_{3,a,1}=\frac{|X_1-X_2|}{|z|}\big|D_XQ(X_\tau,z+t_0(X_\tau),D_t\tilde h(X_\tau,z+t_0(X_\tau)))\\ - D_XQ(X_\tau,z+t_0(X_\tau),D_t\tilde h(X_\tau,t_0(X_\tau)))\big|\end{aligned}$$

$$\begin{aligned}\bar D_{3,a,2}=\frac{|X_1-X_2|}{|z|}|(D_tQ(X_\tau,z+t_0(X_\tau),D_t\tilde h(X_\tau,z+t_0(X_\tau)))\\ - D_tQ(X_\tau,z+t_0(X_\tau),D_t\tilde h(X_\tau,t_0(X_\tau))))D_Xt_0(X_\tau)|\end{aligned}$$

$$\begin{aligned}\bar D_{3,a,3}=\frac{|X_1-X_2|}{|z|}\big|\big[D_\xi Q(X_\tau,z+t_0(X_\tau),D_t\tilde h(X_\tau,z+t_0(X_\tau)))\\ - D_\xi Q(X_\tau,z+t_0(X_\tau),D_t\tilde h(X_\tau,t_0(X_\tau)))\big]D^2_{X,t}\tilde h(X_\tau,z+t_0(X_\tau))\big|\end{aligned}$$

$$\bar{D}_{3,a,4} = \frac{|X_1 - X_2|}{|z|} \big| D_\xi Q(X_\tau, z + t_0(X_\tau), D_t \tilde{h}(X_\tau, t_0(X_\tau))) \\ \cdot \big[ D^2_{X,t} \tilde{h}(X_\tau, z + t_0(X_\tau)) - D^2_{X,t} \tilde{h}(X_\tau, t_0(X_\tau)) \big] \big|$$

$$\bar{D}_{3,a,5} = \frac{|X_1 - X_2|}{|z|} \big| \big[ D_\xi Q(X_\tau, z + t_0(X_\tau), D_t \tilde{h}(X_\tau, z + t_0(X_\tau))) \\ - D_\xi Q(X_\tau, z + t_0(X_\tau), D_t \tilde{h}(X_\tau, t_0(X_\tau))) \big] D^2_{tt} \tilde{h}(X_\tau, z + t_0(X_\tau)) D_X t_0(X_\tau) |$$

$$\bar{D}_{3,a,6} = \frac{|X_1 - X_2|}{|z|} \big| D_\xi Q(X_\tau, z + t_0(X_\tau), D_t \tilde{h}(X_\tau, t_0(X_\tau))) \\ \big[ D^2_{tt} \tilde{h}(X_\tau, z + t_0(X_\tau)) - D^2_{tt} \tilde{h}(X_\tau, t_0(X_\tau)) \big] D_X t_0(X_\tau).$$

The most delicate terms are the ones involving differences of second derivatives of $\tilde{h}$. For example, for $\bar{D}_{3,a,6}$ we use

$$|D_\xi Q(X_\tau, z + t_0(X_\tau), D_t \tilde{h}(X_\tau, t_0(X_\tau)))| \le C|z|$$

and

$$|D^2_{tt} \tilde{h}(X_\tau, z + t_0(X_\tau)) - D^2_{tt} \tilde{h}(X_\tau, t_0(X_\tau))| \le C \frac{|z|^\alpha}{|X_1|^{1+\alpha}}$$

and obtain

$$\int_{B_{2R_{X_2}}(0)} \frac{1}{|z|^{3-\varepsilon}} \bar{D}_{3,a,6} \, dz \le C \frac{|X_1 - X_2|}{|X_1|^{1+\alpha}} \int_{B_{2R_{X_2}}(0)} |z|^{\alpha - 3 + \varepsilon} \, dz \\ \le C \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.$$

Other terms in $D_{3,a}$ are estimated similarly and we find

$$\int_{B_{2R_{X_2}}(0) \setminus B_R(0)} \frac{1}{|z|^{3-\varepsilon}} |D_{3,a}| \, dz \le \frac{C}{\varepsilon} \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.$$

All other terms can be handled with analogous computations, and this establishes (8.13).

Let us prove now (8.14). Let

$$\hat{g}_{X_j} = D^2 G_{X_j}(h)[h_1, h_2].$$

We claim that

$$|D\tilde{H}_{X_1}(g_{X_1})[\hat{g}_{X_1}] - D\tilde{H}_{X_2}(g_{X_2})[\hat{g}_{X_2}]| \le \frac{C}{\varepsilon} \|h_1\|_* \|h_2\|_* \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.$$

For this we write

$$D\tilde{H}_{X_j}(g_{X_j})[\hat{g}_{X_j}] = A_4(X_j) + A_5(X_j)$$

where

$$A_4(X_j) = \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_j}})}{|z|^{3-\varepsilon}} \psi''(A_t(g_{X_j})(X_j, z)) A_t(\hat{g}_{X_j})(X_j, z) B(g_{X_j})(X_j, z) \, dz$$

$$A_5(X_j) = \int_0^1 \frac{\eta(\frac{|z|}{R_{X_j}})}{|z|^{3-\varepsilon}} \psi'(A_t(g_{X_j})(X_j, z)) B(\hat{g}_{X_j})(X_j, z) \, dz.$$

The most delicate difference is

$$A_5(X_1) - A_5(X_2) = \int_0^1 (B_1 + B_2 + B_3) \, dt$$

where

$$
\begin{aligned}
B_1 &= \int_{\mathbb{R}^2} \frac{\eta(\frac{|z|}{R_{X_1}}) - \eta(\frac{|z|}{R_{X_2}})}{|z|^{3-\varepsilon}} \psi'(A_t(g_{X_1})(X_1,z)) B(\hat g_{X_1})(X_1,z)\,dz,\\
B_2 &= \int_{B_{R_{X_2}}(0)} \frac{1}{|z|^{3-\varepsilon}} \big(\psi'(A_t(g_{X_1})(X_1,z)) - \psi'(A_t(g_{X_2})(X_2,z))\big) B(\hat g_{X_1})(X_1,z)\,dz,\\
B_3 &= \int_{B_{R_{X_2}}(0)} \frac{1}{|z|^{3-\varepsilon}} \psi'(A_t(g_{X_2})(X_2,z)) \big(B(\hat g_{X_1})(X_1,z) - B(\hat g_{X_2})(X_2,z)\big)\,dz.
\end{aligned}
$$

Let us focus on the most delicate term, $B_3$. It can be estimated as follows:

$$
|B_3| \le \int_{B_{R_{X_2}}(0)} \frac{1}{|z|^{3-\varepsilon}} |B(\hat g_{X_1})(X_1,z) - B(\hat g_{X_2})(X_2,z)|\,dz
$$

and we claim that

$$
\int_{B_{R_{X_2}}(0)} \frac{1}{|z|^{3-\varepsilon}} |B(\hat g_{X_1})(X_1,z) - B(\hat g_{X_2})(X_2,z)|\,dz \le \frac{C}{\varepsilon} \|h_1\|_* \|h_2\|_* \frac{|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}.
$$

The computation involves a similar difficulty as in the estimate of (8.16), except that now the functions $\hat g_{X_j}$ involve up to second derivatives of $h$. They can also be decomposed as follows

$$
\hat g_{X_j} = \hat g_{0,X_j} + \hat g_{1,X_j} + \hat g_{2,X_j}.
$$

Since $B$ is linear we have to estimate

$$
\int_{B_{R_{X_2}}(0)} \frac{1}{|z|^{3-\varepsilon}} |B(\hat g_{k,X_1})(X_1,z) - B(\hat g_{k,X_2})(X_2,z)|\,dz \tag{8.24}
$$

for $k = 0,1,2$. For $\hat g_{0,X_j}$ we have

$$
\|\hat g_{0,X_j}\|_b \le C \|h_1\|_* \|h_2\|_*. \tag{8.25}
$$

and so we can estimate the integral as we did for (8.15), using (8.25). For $\hat g_{1,X_j}$ we have the same properties as for $\tilde g_{2,X_j}$ in (8.23) and so the estimate of the integral can be done in the same way as in the proof of (8.22).

Let us show that (8.24) holds for $k = 2$. This function has the form:

$$
\hat g_{2,X_j} = D_{tt}\tilde h(X_j, t + t_0(X_j)) b(X_j, t + t_0(X_j)),
$$

with $b$ now satisfying

$$
b(X_j, t + t_0(X_j)) = O(\frac{|t|^2}{|X_j|}).
$$

Let us sketch the computations, assuming no dependence on the first variables in $h$ and $b$, and $t_0(X) = X$ i.e.

$$
B(\hat g_{2,X_j})(X_j,z) = \frac{D_{tt}\tilde h(z + X_j) b(z)}{|z|}
$$

and so

$$
\begin{aligned}
&B(\hat g_{2,X_1})(X_1,z) - B(\hat g_{2,X_2})(X_2,z)\\
&= \frac{1}{|z|} \big[D_{tt}\tilde h(z + X_1) b(z) - D_{tt}\tilde h(z + X_2) b(z)\big]
\end{aligned}
$$

Note that the functions

$$
\left| \frac{1}{|z|^{4-\varepsilon}} D_{tt}\tilde h(z + X_j) b(z) \right| \le C \frac{1}{|z|^{2-\varepsilon}}
$$

are integrable.

We have to estimate

$$\int_{B_{R_{X_2}}(0)} \frac{1}{|z|^{4-\varepsilon}} \left[D_{tt}\tilde{h}(z+X_1)b(z) - D_{tt}\tilde{h}(z+X_2)b(z)\right] dz$$
$$= \int_{B_{R_{X_2}}(0)} \frac{1}{|z|^{4-\varepsilon}} D_{tt}\tilde{h}(z+X_1)b(z)\, dz - \int_{B_{R_{X_2}}(0)} \frac{1}{|z|^{4-\varepsilon}} D_{tt}\tilde{h}(z+X_2)b(z)\, dz$$

Let $\bar{R} = 10|X_1 - X_2|$ and $\bar{X}$ be the middle point between $X_1$ and $X_2$: $\bar{X} = \frac{1}{2}(X_1 + X_2)$ in this simplified calculation.

Then

$$\begin{aligned}
&\left| \int_{B_R(0)} \frac{1}{|z|^{4-\varepsilon}} D_{tt}\tilde{h}(z+X_1)b(z)\, dz - \int_{B_R(0)} \frac{1}{|z|^{4-\varepsilon}} D_{tt}\tilde{h}(z+X_2)b(z)\, dz \right| \\
&\le \int_{B_R(0)} \frac{1}{|z|^{4-\varepsilon}} |(D_{tt}\tilde{h}(z+X_1) - D_{tt}\tilde{h}(z+X_2))b(z)|\, dz \\
&\le C[D^2\tilde{h}]_\alpha |X_1 - X_2|^\alpha \sup \frac{|b(z)|}{|z|^2} \int_{B_R(0)} \frac{1}{|z|^{2-\varepsilon}}\, dz \\
&\le C[D^2\tilde{h}]_\alpha |X_1 - X_2|^\alpha \sup \frac{|b(z)|}{|z|^2} \frac{R^\varepsilon}{\varepsilon} \\
&\le \frac{C|X_1 - X_2|^{\alpha+\varepsilon}}{|X_1|^{1+\alpha}}
\end{aligned}$$

and we need here

$$[D^2\tilde{h}]_\alpha \le \frac{C}{|X_1|^{1+\alpha}}, quad \sup \frac{|b(z)|}{|z|^2} \le C.$$

The remaining part is

$$\begin{aligned}
(a) &= \int_{B_{R_{X_2}}(0)\setminus B_R(0)} \frac{1}{|z|^{4-\varepsilon}} D_{tt}\tilde{h}(z+X_1)b(z)\, dz - \int_{B_{R_{X_2}}(0)\setminus B_R(0)} \frac{1}{|z|^{4-\varepsilon}} D_{tt}\tilde{h}(z+X_2)b(z)\, dz \\
&= \int_{B_{R_{X_2}}(0)\setminus B_R(0)} \frac{1}{|z|^{4-\varepsilon}} (D_{tt}\tilde{h}(z+X_1) - D_{tt}\tilde{h}(\bar{X}))b(z)\, dz \\
&\quad - \int_{B_{R_{X_2}}(0)\setminus B_R(0)} \frac{1}{|z|^{4-\varepsilon}} (D_{tt}\tilde{h}(z+X_2) - D_{tt}\tilde{h}(\bar{X}))b(z)\, dz \\
&= \int_{B_{R_{X_2}}(X_1)\setminus B_R(X_1)} \frac{1}{|z-X_1|^{4-\varepsilon}} (D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X}))b(z-X_1)\, dz \\
&\quad - \int_{B_{R_{X_2}}(X_2)\setminus B_R(X_2)} \frac{1}{|z-X_2|^{4-\varepsilon}} (D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X}))b(z-X_2)\, dz,
\end{aligned}$$

where we have added and subtracted

$$D_{tt}\tilde{h}(\bar{X}) \int_{B_{R_{X_2}}(0)\setminus B_R(0)} \frac{1}{|z|^{4-\varepsilon}} b(z)\, dz.$$

Let us decompose

$$
\begin{aligned}
(a) = &\int_A \frac{1}{|z|^{2-\varepsilon}} |D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X})| \left| \frac{b(z-X_1)}{|z-X_1|^{4-\varepsilon}} - \frac{b(z-X_2)}{|z-X_2|^{4-\varepsilon}} \right| dz \\
&+ \int_{R_1} \frac{1}{|z-X_1|^{4-\varepsilon}} (D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X})) b(z-X_1)\, dz \\
&- \int_{R_2} \frac{1}{|z-X_2|^{4-\varepsilon}} (D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X})) b(z-X_2)\, dz, \\
&+ \int_{R_3} \frac{1}{|z-X_1|^{4-\varepsilon}} (D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X})) b(z-X_1)\, dz \\
&- \int_{R_4} \frac{1}{|z-X_2|^{4-\varepsilon}} (D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X})) b(z-X_2)\, dz.
\end{aligned}
$$

where

$$
\begin{aligned}
A &= [B_{R_{X_2}}(X_1) \setminus B_R(X_1)] \cap [B_{R_{X_2}}(X_2) \setminus B_R(X_2)] \\
R_1 &= B_{R_{X_2}}(X_1) \setminus B_{R_{X_2}}(X_2) \\
R_2 &= B_{R_{X_2}}(X_2) \setminus B_{R_{X_2}}(X_1) \\
R_3 &= B_R(X_2) \setminus B_R(X_1) \\
R_4 &= B_R(X_1) \setminus B_R(X_2).
\end{aligned}
$$

Note that

$$
B_{R/2}(\bar{X}) \subset A \subset B_{2R_{X_2}}(\bar{X}).
$$

We estimate

$$
\begin{aligned}
&\int_A \frac{1}{|z|^{2-\varepsilon}} |D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X})| \left| \frac{b(z-X_1)}{|z-X_1|^{4-\varepsilon}} - \frac{b(z-X_2)}{|z-X_2|^{4-\varepsilon}} \right| dz \\
&\le C[D^2\tilde{h}]_\alpha \int_A |z-\bar{X}|^\alpha \left| \frac{b(z-X_1)}{|z-X_1|^{4-\varepsilon}} - \frac{b(z-X_2)}{|z-X_2|^{4-\varepsilon}} \right| dz \\
&\le C[D^2\tilde{h}]_\alpha \int_0^1 \int_A |z-\bar{X}|^\alpha \left| \frac{d}{dt} \frac{b(z-X_t)}{|z-X_t|^{4-\varepsilon}} \right| dz dt
\end{aligned}
$$

where $X_t = tX_2 + (1-t)X_1$. Assuming

$$
\begin{aligned}
\sup \frac{b(z)}{|z|^2} &\le C \\
\sup \frac{|\nabla b(z)|}{|z|} &\le C
\end{aligned}
$$

we get

$$
\left| \frac{b(z-X_1)}{|z-X_1|^{4-\varepsilon}} - \frac{b(z-X_2)}{|z-X_2|^{4-\varepsilon}} \right| \le C \frac{|X_1-X_2|}{|z-X_t|^{3-\varepsilon}}.
$$

Then

$$
\begin{aligned}
&\int_A \frac{1}{|z|^{2-\varepsilon}} |D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X})| \left| \frac{b(z-X_1)}{|z-X_1|^{4-\varepsilon}} - \frac{b(z-X_2)}{|z-X_2|^{4-\varepsilon}} \right| dz \\
&\le C[D^2\tilde{h}]_\alpha |X_1-X_2| \int_0^1 \int_A \frac{|z-\bar{X}|^\alpha}{|z-X_t|^{3-\varepsilon}}\, dz dt \\
&\le C[D^2\tilde{h}]_\alpha |X_1-X_2| \int_{B_{R/2}(\bar{X})^c} |z-X_t|^{\alpha+\varepsilon-3}\, dz \\
&\le C[D^2\tilde{h}]_\alpha |X_1-X_2| R^{\alpha+\varepsilon-1} \\
&\le C[D^2\tilde{h}]_\alpha |X_1-X_2|^{\alpha+\varepsilon}.
\end{aligned}
$$

Now we estimate

$$
\begin{aligned}
&\left|\int_{R_1} \frac{1}{|z-X_1|^{4-\varepsilon}}(D_{tt}\tilde{h}(z) - D_{tt}\tilde{h}(\bar{X}))b(z-X_1)\,dz\right| \\
&\leq C[D^2\tilde{h}]_\alpha \int_{B_{R_{X_2}+|X_1-X_2|}(X_1)\setminus B_{R_{X_2}-|X_1-X_2|}(X_1)} |z-X_1|^{\alpha+\varepsilon-2}\,dz \\
&\leq C[D^2\tilde{h}]_\alpha((R_{X_2}+|X_1-X_2|)^{\alpha+\varepsilon} - (R_{X_2}-|X_1-X_2|)^{\alpha+\varepsilon}) \\
&\leq C[D^2\tilde{h}]_\alpha R_{X_2}^{\alpha+\varepsilon-1}|X_1-X_2| \\
&\leq C[D^2\tilde{h}]_\alpha |X_1-X_2|^{\alpha+\varepsilon}.
\end{aligned}
$$

The estimate of $R_2$, $R_3$ and $R_4$ are analogous.

□

*Proof of Lemma 8.1.* Write

$$
\begin{aligned}
N_i(h_1) - N_i(h_2) &= H_i(h_1) - H_i(h_2) - DH_i(0)[h_1-h_2] \\
&= \int_0^1 (DH_i(th_1+(1-t)h_2)[h_1-h_2] - DH_i(0)[h_1-h_2])\,dt \\
&= \int_0^1\int_0^1 D^2H_i(s(th_1+(1-t)h_2))[h_1-h_2, th_1+(1-t)h_2]\,dsdt
\end{aligned}
$$

Using Lemma 8.4 we get

$$
|N_i(h_1)(X) - N_i(h_2)(X)| \leq \frac{C}{\varepsilon|X|^{1-\varepsilon}}\|h_1-h_2\|_*(\|h_1\|_* + \|h_2\|_*).
$$

By Lemma 8.5, if $|X_1 - X_2| \leq \frac{1}{10}\min(|X_1|, |X_2)$,

$$
\begin{aligned}
&|N_i(h_1)(X_1) - N_i(h_2)(X_1) - (N_i(h_1)(X_2) - N_i(h_2)(X_2))| \\
&\leq \frac{C}{\varepsilon}\frac{|X_1-X_2|^{\alpha+\varepsilon}}{\min(|X_1|, |X_2)^{1+\alpha}}\|h_1-h_2\|_*(\|h_1\|_* + \|h_2\|_*).
\end{aligned}
$$

□

*Proof of Lemma 8.2.* By a direct and long computation we obtain

$$
\varepsilon|D^2H_o(h)[h_1,h_2](x)| \leq \frac{C}{\varepsilon^{\frac12}|x|^{1-\varepsilon}}\|h_1\|_*\|h_2\|_*
$$

for $x \in \Sigma_0$, and if $x_1, x_2 \in \Sigma_0$, $|x_1 - x_2| \leq \frac{1}{10}|x_1|$, then

$$
\varepsilon|D^2H_o(h)[h_1,h_2](x_1) - D^2H_o(h)[h_1,h_2](x_2)| \leq C\frac{|x_1-x_2|^{\alpha+\varepsilon}}{\varepsilon^{\frac12}|x_1|^{1+\alpha}}\|h_1\|_*\|h_2\|_*.
$$

Then the lemma follows as in the proof of Lemma 8.1.

□

## 9. Layered fractional minimal surfaces

*Proof of Theorem 2.* The proof is essentially the same as for Theorem 1. This time we look for a set $E \subseteq \mathbb{R}^3$ of the form

$$
E = \{(x', x_3) :\in \mathbb{R}^3 : |x_3| > f(x')\},
$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a positive radially symmetric function. We take as a first approximation

$$
E_0 = \{(x', x_3) :\in \mathbb{R}^3 : |x_3| > f_\varepsilon(x')\},
$$

where $f_\varepsilon$ is the unique radial solution to

$$
\Delta f_\varepsilon = \frac{\varepsilon}{f_\varepsilon}, \quad f_\varepsilon > 0, \quad \text{in } \mathbb{R}^2, \tag{9.1}
$$

with $f_\varepsilon(0) = 1$. Then $f_\varepsilon(x) = f_1(\varepsilon^{\frac{1}{2}} x)$ where $f_1$ is the radial solution of $\Delta f = \frac{1}{f}$ with $f_1(0) = 1$. The same analysis of Section 3 applies to show that $f_1(r) = r + O(1)$ as $r \to \infty$ and one obtains the same estimates for $f_\varepsilon$ as for $F_\varepsilon$. This leads to the estimate

$$\|\varepsilon H^s_{\Sigma_0}\|_{1-\varepsilon,\alpha+\varepsilon} \le C\varepsilon.$$

As before, we construct the surface $\Sigma$ and the corresponding set $E$ by perturbing the surface $\Sigma_0$ in the normal direction $\nu_{\Sigma_0}$ (it could also be done using vertical perturbations). That is, for a function $h$ defined on $\Sigma_0$ (small with a suitable norm) we let

$$\Sigma_h = \{x + h(x)\nu_{\Sigma_0}(x) \ / \ x \in \Sigma_0\}.$$

As before, we are led to find $h$ such that

$$H^s_{\Sigma_0} + 2\mathcal{J}^s_{\Sigma_0}(h) + N(h) = 0.$$

We solve for $h$ in this equation using the contraction mapping principle, employing the same norms as in (2.12), (2.13). The solvability of the linearized problem

$$\varepsilon \mathcal{J}^s_{\Sigma_0}(h) = f \quad \text{in } \Sigma_0$$

in weighted Hölder space and the estimates for $N(h)$ are very similar to the ones in Theorem 1. □

We can also construct axially symmetric solutions with multiple layers. Suppose that

$$f_1 > f_2 > \ldots > f_k,$$

are radially symmetric functions on $\mathbb{R}^n$ and consider the surface $\Sigma$ defined by

$$\Sigma = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^1 : x_{n+1} = f_i(x), \text{ for some } i\}.$$

We claim it is possible to choose $f_i$ such that this surface is $s$-minimal for $s$ close to 1.

We will not give a detailed proof of this statement, but only derive formally the form of the elliptic system that plays the role of the equation (9.1) for the case of two layers and mention a few of its properties.

For the derivation of the system, we assume that the functions $f_i$ have small gradient, a condition that a posteriori is verified. Note that the surface $\Sigma$ is the boundary of the region $E$ given by

$$E = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : f_j(x) > x_{n+1} > f_{j+1}(x) \text{ for } j \text{ even}, \ j \in \{0, \ldots, k\}\},$$

with the convention $f_0 = \infty$, $f_{k+1} = -\infty$.

Consider a point $X = (x, x_{n+1})$ with $x_{n+1} = f_i(x)$. We split the integral

$$\int_{\mathbb{R}^{n+1}\setminus B_R(X)} \frac{\chi_E(Y) - \chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\, dY = \int_{y\in\mathbb{R}^n}\int_{y_{n+1} > f_{i-1}(y)} \frac{\chi_E(Y) - \chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\, dY$$
$$+ \int_{y\in\mathbb{R}^n}\int_{y_{n+1} < f_{i+1}(y)} \frac{\chi_E(Y) - \chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\, dY + \int_{y\in\mathbb{R}^n}\int_{f_{i-1}(y) > y_{n+1} > f_{i+1}(y)} \frac{\chi_E(Y) - \chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\, dY.$$

Remark that for $b \ge a > f_i(x)$

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_a^b \frac{1}{(|y-x|^2 + (y_{n+1} - f_i(x))^2)^{\frac{n+1+s}{2}}}\, dy_{n+1}\, dy \\
&= const \int_{a-f_i(x)}^{b-f_i(x)} \frac{1}{|t|^{1+s}}\, dt = const\Big(\frac{1}{(a-f_i(x))^s} - \frac{1}{(b-f_i(x))^s}\Big) \\
&\approx const\Big(\frac{1}{a-f_i(x)} - \frac{1}{b-f_i(x)}\Big)
\end{aligned}$$

and for $f_i(x) > b \ge a$

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_a^b \frac{1}{(|y-x|^2 + (y_{n+1} - f_i(x))^2)^{\frac{n+1+s}{2}}}\, dy_{n+1}\, dy \\
&= const\Big(\frac{1}{f_i(x)-b} - \frac{1}{f_i(x)-a}\Big),
\end{aligned}$$

where $const > 0$.

By decomposing into a ball and its complement and assuming for instance $f_{i-1} - f_i \geq f_i - f_{i+1}$, we have

$$
\begin{aligned}
&\int_{y\in\mathbb{R}^n}\int_{f_{i-1}(y)>y_{n+1}>f_{i+1}(y)} \frac{\chi_E(Y)-\chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\,dY \\
&\approx (-1)^i\frac{\Delta f_i}{1-s} + (-1)^{i-1}\int_{\mathbb{R}^n}\int_{2f_i(y)-f_{i+1}(y)}^{f_{i-1}(y)} \frac{1}{(|y-x|^2+(y_{n+1}-f_i(x))^2)^{\frac{n+1+s}{2}}}\,dy_{n+1}\,dy \\
&\approx (-1)^i\frac{\Delta f_i}{1-s} + (-1)^{i-1}\int_{\mathbb{R}^n}\int_{2f_i(x)-f_{i+1}(x)}^{f_{i-1}(x)} \frac{1}{(|y-x|^2+(y_{n+1}-f_i(x))^2)^{\frac{n+1+s}{2}}}\,dy_{n+1}\,dy \\
&\approx (-1)^i\frac{\Delta f_i}{1-s} + (-1)^{i-1}const\left(-\frac{1}{f_{i-1}(x)-f_i(x)} + \frac{1}{f_i(x)-f_{i+1}(x)}\right)
\end{aligned}
\tag{9.2}
$$

The case $f_{i-1} - f_i \leq f_i - f_{i+1}$ leads to the same formula.

We compute

$$
\begin{aligned}
&\int_{y\in\mathbb{R}^n}\int_{y_{n+1}>f_{i-1}(y)} \frac{\chi_E(Y)-\chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\,dY \\
&= \sum_{j=0}^{i-2}(-1)^j\int_{\mathbb{R}^n}\int_{f_{j+1}(y)}^{f_j(y)} \frac{1}{(|y-x|^2+(y_{n+1}-f_i(x))^2)^{\frac{n+1+s}{2}}}\,dy_{n+1}\,dy \\
&\approx \sum_{j=0}^{i-2}(-1)^j\int_{\mathbb{R}^n}\int_{f_{j+1}(x)}^{f_j(x)} \frac{1}{(|y-x|^2+(y_{n+1}-f_i(x))^2)^{\frac{n+1+s}{2}}}\,dy_{n+1}\,dy \\
&\approx const\sum_{j=0}^{i-2}(-1)^j\left(\frac{1}{f_{j+1}(x)-f_i(x)} - \frac{1}{f_j(x)-f_i(x)}\right)
\end{aligned}
\tag{9.3}
$$

Similarly

$$
\begin{aligned}
&\int_{y\in\mathbb{R}^n}\int_{y_{n+1}<f_{i+1}(x)} \frac{\chi_E(Y)-\chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\,dY \\
&= \sum_{j=i+1}^{k}(-1)^j\int_{\mathbb{R}^n}\int_{f_{j+1}(x)}^{f_j(x)} \frac{1}{(|y-x|^2+(y_{n+1}-f_i(x))^2)^{\frac{n+1+s}{2}}}\,dy_{n+1}\,dy \\
&\approx const\sum_{j=i+1}^{k}(-1)^j\left(-\frac{1}{f_j(x)-f_i(x)} + \frac{1}{f_{j+1}(x)-f_i(x)}\right)
\end{aligned}
\tag{9.4}
$$

Adding (9.2), (9.3) and (9.4) we find

$$
0 \approx (-1)^i\frac{\Delta f_i}{1-s} + 2const(-1)^{i-1}\sum_{j\neq i}\frac{(-1)^{i+j}}{f_i-f_j}
$$

and hence we are lead to

$$
\Delta f_i = 2const\varepsilon\sum_{j\neq i}\frac{(-1)^{i+j+1}}{f_i-f_j}. \tag{9.5}
$$

To be able to carry out the construction of a solution with multiple layers we need a solution of (9.5), and we show next how to find a certain family. For this we shall work with $\varepsilon = 1$, that is, we consider now

$$
\Delta f_i = 2\sum_{j\neq i}\frac{(-1)^{i+j+1}}{f_i-f_j}. \tag{9.6}
$$

We look for a solution of the form

$$f_i = a_i f_0, \quad \Delta f_0 = \frac{1}{f_0}. \tag{9.7}$$

Then the $a_i$ have to satisfy

$$a_i = 2 \sum_{j \neq i} \frac{(-1)^{i+j+1}}{a_i - a_j} \tag{9.8}$$

Note that $\sum_{i=1}^k f_i$ is harmonic and radially symmetric, so it is constant. Since $\sum f_i = f_0 \sum a_i$ is a constant we must have $\sum a_i = 0$.

A solution of the system (9.8) can be obtained by minimization of

$$E(a_1, \dots, a_k) = \frac{1}{2} \sum_{i=1}^k a_i^2 + \sum_{i,j: i \neq j} (-1)^{i+j} \log(|a_i - a_j|)$$

subject to

$$\sum_{i=1}^k a_i = 0.$$

Let

$$\Lambda = \{(a_1, \dots, a_k) \in \mathbb{R}^k : \ a_1 > a_2 > \dots > a_k, \ a_j = -a_{k-j+1} \ \forall j \in \{1, \dots, k\}\}.$$

**Proposition 9.1.** *The minimum of $E$ over $\Lambda$ is achieved.*

*Proof.* Consider a sequence $a^{(i)} \in \Lambda$ such that $E(a^{(i)}) \to \inf_\Lambda E$ as $i \to \infty$. We claim that $a^{(i)}$ remains bounded and

$$\liminf_{i \to \infty} \min_{j=1,\dots,k-1} a_j^{(i)} - a_{j+1}^{(i)} > 0. \tag{9.9}$$

To prove these claims, for $j \in \{1, \dots, k-1\}$ let $x_j^{(i)} = a_j^{(i)} - a_{j+1}^{(i)} > 0$. Then

$$E(a^{(i)}) = \frac{1}{2} \sum_{j=1}^k (a_j^{(i)})^2 - 2 \sum_{j=1}^{k-1} \log(x_j^{(i)}) + 2 \sum_{j=1}^{k-2} \log(x_j^{(i)} + x_{j+1}^{(i)}) +$$

$$+ \dots + (-1)^k 2 \log(x_1^{(i)} + \dots + x_k^{(i)}).$$

Consider $l$ an odd integer in $\{1, \dots, k-1\}$ and $j_l \in \{1 \dots k-l\}$. Then

$$\sum_{j=1}^{k-(l+1)} \log(x_j^{(i)} + \dots + x_{j+l}^{(i)}) \geq \sum_{j=1, j \neq j_l}^{k-l} \log(x_j^{(i)} + \dots + x_{j+l-1}^{(i)}), \tag{9.10}$$

since this is equivalent to

$$\prod_{j=1}^{k-(l+1)} (x_j^{(i)} + \dots + x_{j+l}^{(i)}) \geq \prod_{j=1, j \neq j_l}^{k-l} (x_j^{(i)} + \dots + x_{j+l-1}^{(i)}).$$

The product in the right hand side is always present as a term on the left hand side, while the other terms in the left hand side are positive.

From (9.10) we have

$$-\sum_{j=1}^{k-l} \log(x_j^{(i)} + \dots + x_{j+l-1}^{(i)}) + \sum_{j=1}^{k-(l+1)} \log(x_j^{(i)} + \dots + x_{j+l}^{(i)}) \geq -\log(x_{j_l}^{(i)} + \dots x_{j_l+l-1}^{(i)}).$$

Suppose that $k$ is odd and let $m = k-2$ be the largest odd integer that is less or equal than $k-1$. Then $j_m$ is either 1 or 2 and

$$E(a^{(i)}) \geq \frac{1}{2}\sum_{j=1}^{k}(a_j^{(i)})^2 - 2\log(x_{j_1}^{(i)}) - 2\log(x_{j_3}^{(i)} + x_{j_3+1}^{(i)}) - \dots \\ - 2\log(x_{j_m}^{(i)} + \dots + x_{j_m+m-1}^{(i)}). \tag{9.11}$$

The last term is $\log(x_1^{(i)} + \dots + x_{k-2}^{(i)}) = \log(a_1^{(i)} - a_{k-1}^{(i)})$ or $\log(x_2^{(i)} + \dots + x_{k-1}^{(i)}) = \log(a_2^{(i)} - a_k^{(i)})$ depending on whether $j_m = 1$ or $j_m = 2$. In any case both terms are equal by the symmetry. Then we obtain

$$C \geq E(a^{(i)}) \geq \frac{1}{2}\sum_{j=1}^{k}(a_j^{(i)})^2 - (m+1)\log(a_1^{(i)} - a_{k-1}^{(i)})$$

and we deduce that $a^{(i)}$ remains bounded as $i \to \infty$.

In the case that $k$ is even, let $m = k-1$. Then $j_m = 1$

$$E(a^{(i)}) \geq \frac{1}{2}\sum_{j=1}^{k}(a_j^{(i)})^2 - 2\log(x_{j_1}^{(i)}) - 2\log(x_{j_3}^{(i)} + x_{j_3+1}^{(i)}) - \dots \\ - 2\log(x_1^{(i)} + \dots + x_{k-1}^{(i)}) \tag{9.12}$$

and the last term is $\log(a_1^{(i)} - a_k^{(i)})$. Again from $E(a^{(i)}) \leq C$ we see that $a^{(i)}$ remains bounded as $i \to \infty$.

Using now that $a^{(i)}$ remains bounded as $i \to \infty$, and (9.11) or (9.12) we obtain (9.9). Once we have established that $a^{(i)}$ is bounded and (9.9) it is direct that up to subsequence $a^{(i)}$ converges as $i \to \infty$ to a minimizer of $E$ over $\Lambda$. $\square$

There is however a further restriction on a solution $a_i$ to (9.8) that we need to impose for our method to work, and it is related to the linearization of the system (9.6) around a solution of the form (9.7) . Indeed, the linearized operator around the approximate solution (9.7) is given by

$$\Delta\phi_i - 2\sum_{j\neq i}(-1)^{i+j}\frac{\phi_i - \phi_j}{(f_i - f_j)^2}.$$

Let us write this operator acting on the vector $\Phi = (\phi_1, \dots, \phi_k)$ as

$$\Delta\Phi + \frac{1}{f_0^2}A\Phi$$

where $A = (a_{ij})$ has entries

$$a_{ij} = \begin{cases} 2\frac{(-1)^{i+j}}{(a_i - a_j)^2} & \text{if } i \neq j \\ -2\sum_{k\neq i}\frac{(-1)^{i+k}}{(a_i - a_k)^2} & \text{if } i = j \end{cases}$$

Note that $f_0 \sim r$ as $r \to \infty$, so the linearized operator is asymptotic to

$$\Delta\Phi + \frac{1}{r^2}A\Phi,$$

as $r \to \infty$.

As done before, a natural space to find the solution $\Phi$ should involve norms allowing linear growth. We see that it is possible to find such solutions for a given right hand side of the form $\sim 1/r$ if the matrix $A$ has no eigenvalue equal to $-1$, since otherwise, $\Phi(r) = rv$ with $v$ an eigenvector of $A$ associated to eigenvalue 1 would be in the kernel of the operator.

We note that

$$D^2_{a_i,a_k}E = \begin{cases} 2(-1)^{i+k}\frac{1}{(a_i - a_k)^2} & \text{if } i \neq k \\ 1 - 2\sum_{j\neq i}(-1)^{i+j}\frac{1}{(a_i - a_j)^2} & \text{if } i = k, \end{cases}$$

so that

$$D^2E = I + A.$$

At a local minimum of $E$, $D^2E \geq 0$ which means that eigenvalues of $A$ are greater or equal than $-1$. If $(a_i, \ldots, a_k)$ is a non degenerate local minimum of $E$ then $D^2E > 0$ and the eigenvalues of $A$ are greater than $-1$.

## 10. Existence of $s$-Lawson cones

*Proof of Theorem 3.*

Let us write

$$E_\alpha = \{x = (y, z) \, : \, y \in \mathbb{R}^m, \, z \in \mathbb{R}^n, \, |z| > \alpha |y| \, \}, \tag{10.1}$$

so that $C_\alpha = \partial E_\alpha$.

**Existence.** We fix $N$, $m$, $n$ with $N = m + n$, $n \leq m$ and also fix $0 < s < 1$. If $m = n$ then $C_1$ is a minimal cone, since (1.1) is satisfied by symmetry. So we concentrate next on the case $n < m$.

Before proceeding we remark that for a cone $C_\alpha$ the quantity appearing in (1.1) has a fixed sign for all $p \in C_\alpha$, $p \neq 0$, since by rotation we can always assume that $p = rp_\alpha$ for some $r > 0$ where

$$p_\alpha = \frac{1}{\sqrt{1+\alpha^2}} (e_1^{(m)}, \alpha e_1^{(n)})$$

with

$$e_1^{(m)} = (1, 0, \ldots, 0) \in \mathbb{R}^m \tag{10.2}$$

and similarly for $e_1^{(n)}$. Then we observe that

$$\text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x - rp_\alpha|^{N+s}} \, dx = \frac{1}{r^s} \text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x - p_\alpha|^{N+s}} \, dx.$$

Let us define

$$H(\alpha) = \text{p.v.} \int_{\mathbb{R}^N} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x - p_\alpha|^{N+s}} \, dx \tag{10.3}$$

and note that it is a continuous function of $\alpha \in (0, \infty)$.

**Claim 1.** We have

$$H(1) \leq 0. \tag{10.4}$$

Indeed, write $y \in \mathbb{R}^m$ as $y = (y_1, y_2)$ with $y_1 \in \mathbb{R}^n$ and $y_2 \in \mathbb{R}^{m-n}$. Abbreviating $e_1 = e_1^{(n)} = (1, 0, \ldots, 0) \in \mathbb{R}^n$ we rewrite

$$\begin{aligned} H(1) &= \lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus B(p_1, \delta)} \frac{\chi_{E_1}(x) - \chi_{E_1^c}(x)}{|x - p_1|^{N+s}} \, dx \\ &= \lim_{\delta \to 0} \int_{A_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2)^{\frac{N+s}{2}}} \\ &\quad - \lim_{\delta \to 0} \int_{B_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2)^{\frac{N+s}{2}}}, \end{aligned}$$

where

$$\begin{aligned} A_\delta &= \{|z|^2 > |y_1|^2 + |y_2|^2, \ |y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2 > \delta^2\} \\ B_\delta &= \{|z|^2 < |y_1|^2 + |y_2|^2, \ |y_1 - \frac{1}{\sqrt{2}} e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}} e_1|^2 > \delta^2\}. \end{aligned}$$

But the first integral can be rewritten as

$$\int_{A_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2)^{\frac{N+s}{2}}}$$
$$= \int_{\tilde A_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2)^{\frac{N+s}{2}}}$$

where

$$\tilde A_\delta = \{|y_1|^2 > |z|^2 + |y_2|^2,\ |y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2 > \delta^2\}$$

(we just have exchanged $y_1$ by $z$ and noted that the integrand is symmetric in these variables). But $\tilde A_\delta \subset B_\delta$ and so

$$\int_{\mathbb{R}^N \setminus B(p_1,\delta)} \frac{\chi_{E_1}(x) - \chi_{E_1^c}(x)}{|x-p_1|^{N+s}}\,dx$$
$$= -\int_{B_\delta \setminus \tilde A_\delta} \frac{1}{(|y_1 - \frac{1}{\sqrt{2}}e_1|^2 + |y_2|^2 + |z - \frac{1}{\sqrt{2}}e_1|^2)^{\frac{N+s}{2}}} \le 0.$$

This shows the validity of (10.4).

**Claim 2.** We have

$$H(\alpha) \to +\infty \quad \text{as } \alpha \to 0. \tag{10.5}$$

Let $0<\delta<1/2$ be fixed and write

$$H(\alpha) = I_\alpha + J_\alpha$$

where

$$I_\alpha = \int_{\mathbb{R}^N \setminus B(p_\alpha,\delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x-p_\alpha|^{N+s}}\,dx$$
$$J_\alpha = \text{p.v.} \int_{B(p_\alpha,\delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x-p_\alpha|^{N+s}}\,dx.$$

With $\delta$ fixed

$$\lim_{\alpha\to 0} I_\alpha = \int_{\mathbb{R}^N \setminus B(p_\alpha,\delta)} \frac{1}{|x-p_0|^{N+s}}\,dx > 0. \tag{10.6}$$

For $J_\alpha$ we make a change of variables $x = \alpha\tilde x + p_\alpha$ and obtain

$$J_\alpha = \text{p.v.} \int_{B(p_\alpha,\delta)} \frac{\chi_{E_\alpha}(x) - \chi_{E_\alpha^c}(x)}{|x-p_\alpha|^{N+s}}\,dx = \frac{1}{\alpha^s}\text{p.v.} \int_{B(0,\delta/\alpha)} \frac{\chi_{F_\alpha}(\tilde x) - \chi_{F_\alpha^c}(\tilde x)}{|\tilde x|^{N+s}} d\tilde x \tag{10.7}$$

where $F_\alpha = \frac{1}{\alpha}(E_\alpha - p_\alpha)$. But

$$\text{p.v.} \int_{B(0,\delta/\alpha)} \frac{\chi_{F_\alpha}(\tilde x) - \chi_{F_\alpha^c}(\tilde x)}{|\tilde x|^{N+s}} d\tilde x \to \text{p.v} \int_{\mathbb{R}^N} \frac{\chi_{F_0}(x) - \chi_{F_0^c}(x)}{|x|^{N+s}} dx$$

as $\alpha \to 0$ where $F_0 = \{x = (y,z) : y \in \mathbb{R}^m, z \in \mathbb{R}^n,\ |z + e_1^{(n)}| > 1\}$. But writing $z = (z_1,\ldots,z_n)$ we see that

$$\text{p.v} \int_{\mathbb{R}^N} \frac{\chi_{F_0}(x) - \chi_{F_0^c}(x)}{|x|^{N+s}} dx \ge \text{p.v} \int_{\mathbb{R}^N} \frac{\chi_{[z_1>0 \text{ or } z_1<-2]} - \chi_{[-2<z_1<0]}}{|x|^{N+s}} dx$$
$$\ge \int_{\mathbb{R}^N} \frac{\chi_{[\ |z_1|>2\ ]}}{|x|^{N+s}} dx$$

and this number is positive. This and (10.7) show that $J_\alpha \to +\infty$ as $\alpha \to 0$ and combined with (10.6) we obtain the desired conclusion.

By (10.4), (10.5) and continuity we obtain the existence of $\alpha \in (0,1]$ such that $H(\alpha) = 0$.

**Uniqueness.** Consider 2 cones $C_{\alpha_1}$, $C_{\alpha_2}$ with $\alpha_1 > \alpha_2 > 0$, associated to solid cones $E_{\alpha_1}$ and $E_{\alpha_2}$. We claim that there is a rotation $R$ so that $R(E_{\alpha_1}) \subset E_{\alpha_2}$ (strictly) and that

$$H(\alpha_1) = \text{p.v.} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_{R(E_{\alpha_1})}(x) - \chi_{R(E_{\alpha_1})^c}(x)}{|x - p_{\alpha_2}|^{N+s}}\, dx.$$

Note that the denominator in the integrand is the same that appears in (10.3) for $\alpha_2$ and then

$$\begin{aligned} H(\alpha_1) &= \text{p.v.} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_{R(E_{\alpha_1})}(x) - \chi_{R(E_{\alpha_1})^c}(x)}{|x - p_{\alpha_2}|^{N+s}}\, dx \\ &< \text{p.v.} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_{E_{\alpha_2}}(x) - \chi_{E_{\alpha_2}^c}(x)}{|x - p_{\alpha_2}|^{N+s}}\, dx = H(\alpha_2). \end{aligned} \tag{10.8}$$

This shows that $H(\alpha)$ is decreasing in $\alpha$ and hence the uniqueness. To construct the rotation let us write as before $x = (y, z) \in \mathbb{R}^N$, with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, and $y = (y_1, y_2)$ with $y_1 \in \mathbb{R}^n$, $y_2 \in \mathbb{R}^{m-n}$ (we assume alway $n \le m$). Let us write the vector $(y_1, z)$ in spherical coordinates of $\mathbb{R}^{2n}$ as follows

$$y_1 = \rho \begin{bmatrix} \cos(\varphi_1) \\ \sin(\varphi_1)\cos(\varphi_2) \\ \sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3) \\ \vdots \\ \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_{n-1})\cos(\varphi_n) \end{bmatrix}$$

$$z = \rho \begin{bmatrix} \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_n)\cos(\varphi_{n+1}) \\ \vdots \\ \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_{2n-2})\cos(\varphi_{2n-1}) \\ \sin(\varphi_1)\sin(\varphi_2)\sin(\varphi_3)\ldots\sin(\varphi_{2n-2})\sin(\varphi_{2n-1}) \end{bmatrix}$$

where $\rho > 0$, $\varphi_{2n-1} \in [0, 2\pi)$, $\varphi_j \in [0, \pi]$ for $j = 1, \ldots, 2n-2$. Then

$$|z|^2 = \rho^2 \sin(\varphi_1)^2 \sin(\varphi_2)^2 \ldots \sin(\varphi_n)^2, \quad |y_1|^2 + |z|^2 = \rho^2.$$

The equation for the solid cone $E_{\alpha_i}$, namely $|z| > \alpha_i |y|$, can be rewritten as

$$\rho^2 \sin(\varphi_1)^2 \sin(\varphi_2)^2 \ldots \sin(\varphi_n)^2 > \alpha_i^2 (|y_1|^2 + |y_2|^2).$$

Adding $\alpha_i^2 |z|^2$ to both sides this is equivalent to

$$\sin(\varphi_1)^2 \sin(\varphi_2)^2 \ldots \sin(\varphi_n)^2 > \sin(\beta_i)^2 \left(1 + \frac{|y_2|^2}{\rho^2}\right)$$

where $\beta_i = \arctan(\alpha_i)$. We let $\theta = \beta_1 - \beta_2 \in (0, \pi/2)$, and define the rotated cone $R_\theta(E_{\alpha_1})$ by the equation

$$\sin(\varphi_1 + \theta)^2 \sin(\varphi_2)^2 \ldots \sin(\varphi_n)^2 > \sin(\beta_1)^2 \left(1 + \frac{|y_2|^2}{\rho^2}\right).$$

We want to show that $R_\theta(E_{\alpha_1}) \subset E_{\alpha_2}$. To do so, it suffices to prove that for any given $t \ge 1$, if $\varphi$ satisfies the inequality $|\sin(\varphi + \theta)| > \sin(\beta_1) t$ then it also satisfies $|\sin(\varphi)| > \sin(\beta_2) t$. This in turn can be proved from the inequality

$$\arccos(\sin(\beta_1) t) + \theta < \arccos(\sin(\beta_2) t)$$

for $1 < t \le \frac{1}{\sin(\beta_1)}$. For $t = 1$ we have equality by definition of $\theta$. The inequality for $1 < t \le \frac{1}{\sin(\beta_1)}$ can be checked by computing a derivative with respect to $t$. The strict inequality in (10.8) is because $R(E_{\alpha_1}) \subset E_{\alpha_2}$ strictly. $\square$

## 11. Stability and instability

We consider the nonlocal minimal cone $C_m^n(s) = \partial E_\alpha$ where $E_\alpha$ is defined in (10.1) and $\alpha$ is the one of Theorem 3. For $0 \le s < 1$ we obtain a characterization of their stability in terms of constants that depend on $m$, $n$ and $s$. For the case $s = 0$ we consider the limiting cone with parameter $\alpha_0$ given in Proposition 11.2 below. Note that in the case $s = 0$ the limiting Jacobi operator $\mathcal{J}^0_{C_{\alpha_0}}$ is well defined for smooth functions with compact support.

For brevity, in this section we write $\Sigma = C_m^n(s)$.

11.1. **Characterization of stability.** Recall that

$$\mathcal{J}^s_\Sigma[\phi](x) = \text{p.v.} \int_\Sigma \frac{\phi(y) - \phi(x)}{|y-x|^{N+s}} dy + \phi(x) \int_\Sigma \frac{1 - \langle \nu(x), \nu(y) \rangle}{|x-y|^{N+s}} dy$$

for $\phi \in C_0^\infty(\Sigma \setminus \{0\})$. Let us rewrite this operator in the form

$$\mathcal{J}^s_\Sigma[\phi](x) = \text{p.v.} \int_\Sigma \frac{\phi(y) - \phi(x)}{|x-y|^{N+s}} dy + \frac{A_0(m,n,s)^2}{|x|^{1+s}} \phi(x)$$

where

$$A_0(m,n,s)^2 = \int_\Sigma \frac{\langle \nu(\hat p) - \nu(x), \nu(\hat p) \rangle}{|\hat p - x|^{N+s}} dx \ge 0$$

and this integral is evaluated at any $\hat p \in \Sigma$ with $|\hat p| = 1$. We can think of $\mathcal{J}^s_\Sigma$ as analogous to the fractional Hardy operator

$$-(-\Delta)^{\frac{1+s}{2}} \phi + \frac{c}{|x|^{1+s}} \phi \quad \text{in } \mathbb{R}^{N-1},$$

for which positivity is related to a fractional Hardy inequality with best constant, see Herbst [15]. This suggests that the positivity of $\mathcal{J}_\Sigma$ is related to the existence of $\beta$ in an appropriate range such that $\mathcal{J}^s_\Sigma[|x|^{-\beta}] \le 0$, and it turns out that the best choice of $\beta$ is $\beta = \frac{N-2-s}{2}$. This motivates the definition

$$H(m,n,s) = \text{p.v.} \int_\Sigma \frac{1 - |y|^{-\frac{N-2-s}{2}}}{|\hat p - y|^{N+s}} dy$$

where $\hat p \in \Sigma$ is any point with $|\hat p| = 1$.

We have then the following Hardy inequality with best constant:

**Proposition 11.1.** *For any $\phi \in C_0^\infty(\Sigma \setminus \{0\})$ we have*

$$H(m,n,s) \int_\Sigma \frac{\phi(x)^2}{|x|^{1+s}} dx \le \frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} dx dy \tag{11.1}$$

*and $H(m,n,s)$ is the best possible constant in this inequality.*

As a result we have:

**Corollary 11.1.** *The cone $C_m^n(s)$ is stable if and only if $H(m,n,s) \ge A_0(m,n,s)^2$.*

Other related fractional Hardy inequalities have appeared in the literature, see for instance [3, 12].

*Proof of Proposition 11.1.* Let us write $H = H(m,n,s)$ for simplicity. To prove the validity of (11.1) let $w(x) = |x|^{-\beta}$ with $\beta = \frac{N-2-s}{2}$ so that from the definition of $H$ and homogeneity we have

$$\text{p.v.} \int_\Sigma \frac{w(y) - w(x)}{|y-x|^{N+s}} dy + \frac{H}{|x|^{1+s}} w(x) = 0 \quad \text{for all } x \in \Sigma \setminus \{0\}.$$

Now the same argument as in the proof of corollary B.1 shows that

$$\frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} dx dy = \int_\Sigma \frac{H}{|x|^{1+s}} \phi(x)^2 dx + \frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\psi(x) - \psi(y))^2 w(x) w(y)}{|x-y|^{N+s}} dx dy. \tag{11.2}$$

for all $\phi \in C_0^\infty(\Sigma \setminus \{0\})$ with $\psi = \frac{\phi}{w} \in C_0^\infty(\Sigma \setminus \{0\})$

Now let us show that $H$ is the best possible constant in (11.1). Assume that

$$\tilde{H} \int_\Sigma \frac{\phi(x)^2}{|x|^{1+s}} dx \le \frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} dx dy$$

for all $\phi \in C_0^\infty(\Sigma \setminus \{0\})$. Using (11.2) and letting $\phi = w\psi$ with $\psi \in\in C_0^\infty(\Sigma \setminus \{0\})$ we then have

$$\begin{aligned} \tilde{H} \int_\Sigma \frac{w(x)^2 \psi(x)^2}{|x|^{1+s}} dx \le H & \int_\Sigma \frac{w(x)^2 \psi(x)^2}{|x|^{1+s}} dx \\ & + \frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\psi(x) - \psi(y))^2 w(x) w(y)}{|x-y|^{N+s}} dx dy. \end{aligned}$$

For $R > 3$ let $\psi_R : \Sigma \to [0,1]$ be a radial function such that $\psi_R(x) = 0$ for $|x| \le 1$, $\psi_R(x) = 1$ for $2 \le |x| \le 2R$, $\psi_R(x) = 0$ for $|x| \ge 3R$. We also require $|\nabla \psi_R(x)| \le C$ for $|x| \le 3$, $|\nabla \psi_R(x)| \le C/R$ for $2R \le |x| \le 3R$. We claim that

$$a_0 \log(R) - C \le \int_\Sigma \frac{w(x)^2 \psi_R(x)^2}{|x|^{1+s}} dx \le a_0 \log(R) + C \tag{11.3}$$

where $a_0 > 0$, $C > 0$ are independent of $R$, while

$$\left| \int_\Sigma \int_\Sigma \frac{(\psi_R(x) - \psi_R(y))^2 w(x) w(y)}{|x-y|^{N+s}} dx dy \right| \le C. \tag{11.4}$$

Letting then $R \to \infty$ we deduce that $\tilde{H} \le H$.

To prove the upper bound in (11.3) let us write points in $\Sigma$ as $x = (y, z)$, with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$. Let us write $y = r\omega_1$, $z = r\omega_2$, with $r > 0$, $\omega_1 \in S^{m-1}$, $\omega_2 \in S^{n-1}$ and use spherical coordinates $(\theta_1, \ldots, \theta_{m-1})$ and $(\varphi_1, \ldots, \varphi_{n-1})$ for $\omega_1$ and $\omega_2$ as in (11.6) and (11.7) . We assume here that $m \ge n \ge 2$. In the remaining cases the computations are similar. Then we have

$$\int_\Sigma \frac{w(x)^2 \psi_R(x)^2}{|x|^{1+s}} dx \le a_0 \int_1^{4R} \frac{1}{r^{N-2-s}} \frac{1}{r^{1+s}} r^{N-2} dr \le a_0 \log(R) + C$$

where

$$a_0 = \sqrt{1 + \alpha^2} A_{m-1} A_{n-1}$$

and $A_k$ denotes the area of the sphere $S^k \subseteq \mathbb{R}^{k+1}$ and is given by

$$A_k = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma(\frac{k+1}{2})}. \tag{11.5}$$

The lower bound in (11.3) is similar.

To obtain (11.4) we split $\Sigma$ into the regions $R_1 = \{x : |x| \le 3\}$, $R_2 = \{x : 3 \le x \le R\}$, $R_3 = \{x : R \le |x| \le 4R\}$ and $R_4 = \{x : |x| \ge 4R\}$ and let

$$I_{i,j} = \int_{x \in R_i} \int_{y \in R_j} \frac{(\psi_R(x) - \psi_R(y))^2 w(x) w(y)}{|x-y|^{N+s}} dx dy.$$

Then $I_{i,j} = I_{j,i}$ and $I_{j,j} = 0$ for $j = 2, 4$. Moreoover $I_{1,1} = O(1)$ since the region of integration is bounded and $\psi_R$ is uniformly Lipschitz.

Estimate of $I_{1,2}$: We bound $w(x) \le C$ for $|x| \ge 1$ and then

$$|I_{1,2}| \le C \int_{y \in R_2} \frac{w(y)}{|p-y|^{N+s}} dy \le C \int_2^R \frac{1}{r^{\frac{N-2-s}{2}}} \frac{1}{r^{N+s}} r^{N-2} dr \le C,$$

where $p \in \Sigma$ is fixed with $|p| = 2$.

By the same argument $I_{1,3} = O(1)$ and $I_{1,4} = O(1)$ as $R \to \infty$.

Estimate of $I_{2,3}$: for $y \in R_3$, $w(y) \le CR^{-\frac{N-2-s}{2}}$, so

$$|I_{2,3}| \le CR^{-\frac{N-2-s}{2}} \int_{x\in R_2} \frac{1}{|x|^{\frac{N-2-s}{2}}} \int_{y\in R_3} \frac{(\psi_R(x)-\psi_R(y))^2}{|x-y|^{N+s}} dydx$$
$$\le CR^{-\frac{N-2-s}{2}} \frac{Vol(R_3)}{R^{N+s}} \int_{x\in R_2} \frac{1}{|x|^{\frac{N-2-s}{2}}} dx \le C.$$

Estimate of $I_{2,4}$:

$$|I_{2,4}| \le C \int_{x\in R_2} \frac{1}{|x|^{\frac{N-2-s}{2}}} \int_{y\in R_4} \frac{1}{|x-y|^{N+s}} \frac{1}{|y|^{\frac{N-2-s}{2}}} dydx.$$

By scaling

$$\int_{y\in R_4} \frac{1}{|x-y|^{N+s}} \frac{1}{|y|^{\frac{N-2-s}{2}}} dy \le CR^{-\frac{N}{2}-\frac{s}{2}} \quad \text{for } x \in R_2,$$

so that

$$|I_{2,4}| \le CR^{-\frac{N}{2}-\frac{s}{2}} \int_{x\in R_2} \frac{1}{|x|^{\frac{N-2-s}{2}}} dx \le C.$$

To estimate $I_{3,3}$ we use $|\psi_R(x)-\psi_R(y)| \le \frac{C}{R}|x-y$ for $x,y \in R_3$, which yields

$$|I_{3,3}| \le \frac{C}{R^2} \frac{1}{R^{N-2-s}} \int_{x,y\in R_3} \frac{1}{|x-y|^{N+s-2}} dydx.$$

The integral is finite and by scaling we see that is bounded by $CR^{N-s}$, so that

$$|I_{3,3}| \le C.$$

Estimate of $I_{3,4}$:

$$|I_{3,4}| \le CR^{-\frac{N-2-s}{2}} \int_{x\in R_3} \int_{y\in R_4} \frac{1}{|x-y|^{N+s}} \frac{1}{|y|^{\frac{N-2-s}{2}}} dydx.$$

By scaling

$$\int_{y\in R_4} \frac{1}{|x-y|^{N+s}} \frac{1}{|y|^{\frac{N-2-s}{2}}} dy \le \frac{C}{|x|^{\frac{N+s}{2}}}$$

for $x \in R_3$. Therefore

$$|I_{3,4}| \le CR^{-\frac{N-2-s}{2}} \int_{x\in R_3} \frac{1}{|x|^{\frac{N+s}{2}}} dx \le C.$$

This concludes the proof of (11.4). □

11.2. **Minimal cones for** $s = 0$**.** Here we derive the limiting value $\alpha_0 = \lim_{s\to 0} \alpha_s$ where $\alpha_s$ is such that $C_{\alpha_s}$ is an $s$-minimal cone.

**Proposition 11.2.** *Assume that $n \le m$ in (10.1), $N = m + n$. The number $\alpha_0$ is the unique solution to*

$$\int_\alpha^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{N}{2}}} dt - \int_0^\alpha \frac{t^{n-1}}{(1+t^2)^{\frac{N}{2}}} dt = 0.$$

*Proof.* We write $x = (y,z) \in \mathbb{R}^N$ with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$. Let us assume in the rest of the proof that $n \ge 2$. The case $n = 1$ is similar. We evaluate the integral in (1.1) for the point $p = (e_1^{(m)}, \alpha e_1^{(n)})$ using spherical coordinates for $y = r\omega_1$ and $z = \rho\omega_2$ where $r, \rho > 0$ and

$$\omega_1 = \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1)\cos(\theta_2) \\ \vdots \\ \sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{m-2})\cos(\theta_{m-1}) \\ \sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{m-2})\sin(\theta_{m-1}) \end{bmatrix} \tag{11.6}$$

$$\omega_2 = \begin{bmatrix} \cos(\varphi_1) \\ \sin(\varphi_1)\cos(\varphi_2) \\ \vdots \\ \sin(\varphi_1)\sin(\varphi_2)\dots\sin(\varphi_{n-2})\cos(\varphi_{n-1}) \\ \sin(\varphi_1)\sin(\varphi_2)\dots\sin(\varphi_{n-2})\sin(\varphi_{n-1}) \end{bmatrix}, \tag{11.7}$$

where $\theta_j \in [0,\pi]$ for $j = 1,\dots,m-2$, $\theta_{m-1} \in [0,2\pi]$, $\varphi_j \in [0,\pi]$ for $j = 1,\dots,n-2$, $\varphi_{n-1} \in [0,2\pi]$. Then

$$|(y,z) - (e_1^{(m)}, \alpha e_1^{(n)})|^2 = r^2 + 1 - 2r\cos(\theta_1) + \rho^2 + \alpha^2 - 2\rho\alpha\cos(\varphi_1).$$

Assuming that $\alpha = \alpha_s > 0$ is such that $C_{\alpha_s}$ is an $s$-minimal cone, (1.1) yields the following equation for $\alpha$

$$\text{p.v.} \int_0^\infty r^{m-1}(A_{\alpha,s}(r) - B_{\alpha,s}(r))dr = 0 \tag{11.8}$$

where

$$A_{\alpha,s}(r) = \int_{r\alpha}^\infty \int_0^\pi \int_0^\pi \frac{\rho^{n-1}\sin(\theta_1)^{m-2}\sin(\varphi_1)^{n-2}}{(r^2+1-2r\cos(\theta_1)+\rho^2+\alpha^2-2\rho\alpha\cos(\varphi_1))^{\frac{N+s}{2}}} d\theta_1 d\varphi_1 d\rho$$

$$B_{\alpha,s}(r) = \int_0^{r\alpha} \int_0^\pi \int_0^\pi \frac{\rho^{n-1}\sin(\theta_1)^{m-2}\sin(\varphi_1)^{n-2}}{(r^2+1-2r\cos(\theta_1)+\rho^2+\alpha^2-2\rho\alpha\cos(\varphi_1))^{\frac{N+s}{2}}} d\theta_1 d\varphi_1 d\rho,$$

which are well defined for $r \neq 1$. Setting $\rho = rt$ we get

$$\begin{aligned} &A_{\alpha,s}(r) \\ &= r^{-m-s} \int_\alpha^\infty \int_0^\pi \int_0^\pi \frac{t^{n-1}\sin(\varphi_1)^{m-2}\sin(\theta_1)^{n-2}}{(1+\frac{1}{r^2}-\frac{2}{r}\cos(\theta_1)+t^2+\frac{\alpha^2}{r^2}-\frac{2}{r}t\alpha\cos(\varphi_1))^{\frac{N+s}{2}}} d\theta_1 d\varphi_1 dt \\ &= c_{m,n} r^{-m-s} \int_\alpha^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{N+s}{2}}} dt + O(r^{-m-s-1}) \end{aligned}$$

as $r \to \infty$ and this is uniform in $s$ for $s > 0$ small. Here $c_{m,n} > 0$ is some constant. Similarly

$$B_{\alpha,s}(r) = c_{m,n} r^{-m-s} \int_0^\alpha \frac{t^{n-1}}{(1+t^2)^{\frac{N+s}{2}}} dt + O(r^{-m-s-1})$$

Then (11.8) takes the form

$$0 = \int_0^2 \dots dr + \int_2^\infty \dots dr = O(1) + C_s(\alpha)\int_2^\infty r^{-1-s} dr = O(1) + \frac{2^{-s}}{s} C_s(\alpha)$$

where

$$C_s(\alpha) = \int_\alpha^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{N+s}{2}}} dt - \int_0^\alpha \frac{t^{n-1}}{(1+t^2)^{\frac{N+s}{2}}} dt$$

and $O(1)$ is uniform as $s \to 0$, because $0 < \alpha_s \le 1$ by Theorem 3, and the only singularity in (11.8) occurs at $r = 1$. This implies that $\alpha_0 = \lim_{s\to 0} \alpha_s$ has to satisfy $C_0(\alpha_0) = 0$. □

11.3. **Proof of Theorem 4.** In what follows we will obtain expressions for $H(m,n,s)$ and $A_0(m,n,s)^2$ for $m \ge 2$, $n \ge 1$, $0 \le s < 1$. We always assume $m \ge n$. For the sake of generality, we will compute

$$C(m,n,s,\beta) = \text{p.v.} \int_\Sigma \frac{1-|x|^{-\beta}}{|\hat{p}-x|^{N+s}} dx$$

where $\hat{p} \in \Sigma$, $|\hat{p}| = 1$, and $\beta \in (0, N-2-s)$, so that $H(m,n,s) = C(m,n,s,\frac{N-2-s}{2})$.

Let $x = (y,z) \in \Sigma$, with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$. For simplicity in the next formulas we take $p = (e_1^{(m)}, \alpha e_2^{(n)})$ (see the notation in (10.2)), and $h(y,z) = |y|^{-\beta}$, so that

$$C(m,n,s,\beta) = (1+\alpha^2)^{\frac{1+s}{2}} \text{p.v.} \int_\Sigma \frac{h(p)-h(x)}{|p-x|^{N+s}} dx.$$

**Computation of $C(m,1,s,\beta)$.** Write $y = r\omega_1$, $z = \pm\alpha r$, with $r > 0$, $\omega_1 \in S^{m-1}$. Let us use the notation $\Sigma_\alpha^+ = \Sigma \cap [z > 0]$, $\Sigma_\alpha^- = \Sigma \cap [z < 0]$. Using polar coordinates $(\theta_1, \ldots, \theta_{m-1})$ for $\omega_1$ as in (11.6) we have

$$|x-p|^2 = |r\theta_1 - e_1^{(m)}|^2 + \alpha^2 |r\theta_1 - e_1^{(m)}|^2 = r^2 + 1 - 2r\cos(\theta_1) + \alpha^2 (r-1)^2,$$

for $x \in \Sigma_\alpha^+$ and

$$|x-p|^2 = |r\theta_1 - e_1^{(m)}|^2 + \alpha^2 |r\theta_1 - e_1^{(m)}|^2 = r^2 + 1 - 2r\cos(\theta_1) + \alpha^2 (r+1)^2,$$

for $x \in \Sigma_\alpha^-$. Hence, with $h(y,z) = |y|^{-\beta}$

$$\text{p.v.} \int_\Sigma \frac{h(p)-h(x)}{|x-p|^{N+s}} dx = \sqrt{1+\alpha^2} A_{m-2} \text{p.v.} \int_0^\infty (1 - r^{-\beta})(I_+(r) + I_-(r)) r^{N-2} dr \tag{11.9}$$

where

$$I_+(r) = \int_0^\pi \frac{\sin(\theta_1)^{m-2}}{(r^2+1-2r\cos(\theta_1) + \alpha^2(r-1)^2)^{\frac{N+s}{2}}} d\theta_1$$
$$I_-(r) = \frac{\sin(\theta_1)^{m-2}}{(r^2+1-2r\cos(\theta_1) + \alpha^2(r+1)^2)^{\frac{N+s}{2}}} d\theta_1,$$

and $A_{m-2}$ is defined in (11.5) for $m \geq 2$. From (11.9) we obtain

$$C(m,1,s,\beta) = (1+\alpha^2)^{\frac{3+s}{2}} A_{m-2} \int_0^1 (r^{N-2} - r^{N-2-\beta} + r^s - r^{\beta+s})(I_+(r) + I_-(r)) dr. \tag{11.10}$$

**Computation of $A_0(m,1,s)^2$.** Let $x = (r\theta_1, \pm\alpha r)$, $p = (e_1^{(n)}, \alpha)$ so that

$$\nu(x) = \frac{(-\alpha\omega_1, \pm 1)}{\sqrt{1+\alpha^2}}, \quad \nu(p) = \frac{(-\alpha e_1^{(n)}, 1)}{\sqrt{1+\alpha^2}},$$

and hence

$$\int_\Sigma \frac{1 - \langle \nu(x), \nu(p)\rangle}{|p-x|^{N+s}} dx = \sqrt{1+\alpha^2} A_{m-2} \int_0^\infty (J_+(r) + J_-(r)) r^{N-2} dr$$
$$= \sqrt{1+\alpha^2} A_{m-2} \int_0^1 (r^{N-2} + r^s)(J_+(r) + J_-(r)) dr,$$

where

$$J_+(r) = \frac{\alpha^2}{1+\alpha^2} \int_0^\pi \frac{(1-\cos(\theta_1))\sin(\theta_1)^{m-2}}{(r^2+1-2\cos(\theta_1) + \alpha^2(r-1)^2)^{\frac{N+s}{2}}} d\theta_1$$
$$J_-(r) = \frac{1}{1+\alpha^2} \int_0^\pi \frac{[2 + \alpha^2 - \alpha^2\cos(\theta_1))\sin(\theta_1)^{m-2}}{(r^2+1-2r\cos(\theta_1) + \alpha^2(r+1)^2)^{\frac{N+s}{2}}} d\theta_1$$

Therefore we find

$$A_0(m,1,s)^2 = (1+\alpha^2)^{\frac{3+s}{2}} A_{m-2} \int_0^1 (r^{N-2} + r^s)(J_+(r) + J_-(r)) dr.$$

**Computation of $C(m,n,s,\beta)$ for $n \geq 2$.** Write $y = r\omega_1$, $z = r\omega_2$, with $r > 0$, $\omega_1 \in S^{m-1}$, $\omega_2 \in S^{n-1}$ and let us use spherical coordinates $(\theta_1, \ldots, \theta_{m-1})$ and $(\varphi_1, \ldots, \varphi_{n-1})$ for $\omega_1$ and $\omega_2$ as in (11.6) and (11.7). Recalling that $p = (e_1^{(m)}, \alpha e_2^{(n)})$, we have

$$|x-p|^2 = |r\theta_1 - e_1^{(m)}|^2 + |r\theta_1 - e_1^{(m)}|^2 = r^2 + 1 - 2r\cos(\theta_1) + \alpha^2(r^2 + 1 - 2r\cos(\varphi_1)).$$

Hence, with $h(y,z) = |y|^{-\beta}$

$$\text{p.v.} \int_\Sigma \frac{h(p)-h(x)}{|x-p|^N} dx = \sqrt{1+\alpha^2} A_{m-2} A_{n-2} \text{p.v.} \int_0^\infty (1 - r^{-\beta}) I(r) r^{N-2} dr$$
$$= \sqrt{1+\alpha^2} A_{m-2} A_{n-2} \int_0^1 (r^{N-2} - r^{N-2-\beta} + r^s - r^{\beta+s}) I(r) dr$$

| | | $n$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $m$ | | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | $H$ | 0.8140 | 1.0679 | | | | | |
| | $A_0^2$ | 3.2669 | 2.3015 | | | | | |
| 3 | $H$ | 1.1978 | 1.2346 | 0.3926 | | | | |
| | $A_0^2$ | 2.5984 | 1.7918 | 0.4463 | | | | |
| 4 | $H$ | 1.3968 | 1.3649 | 0.4477 | 0.1613 | | | |
| | $A_0^2$ | 2.0413 | 1.5534 | 0.4288 | 0.1356 | | | |
| 5 | $H$ | 1.5117 | 1.4570 | 0.4895 | 0.1845 | 0.06978 | | |
| | $A_0^2$ | 1.7332 | 1.3981 | 0.4118 | 0.1398 | 0.04849 | | |
| 6 | $H$ | 1.5833 | 1.5231 | 0.5215 | 0.2031 | 0.08013 | 0.03113 | |
| | $A_0^2$ | 1.5318 | 1.2841 | 0.3955 | 0.1412 | 0.05173 | 0.01885 | |
| 7 | $H$ | 1.6303 | 1.5719 | 0.5465 | 0.2182 | 0.08885 | 0.03583 | 0.01416 |
| | $A_0^2$ | 1.3872 | 1.1951 | 0.3802 | 0.1409 | 0.05381 | 0.02051 | 0.007704 |

TABLE 1. Values of $H(m,n,0)$ and $A_0(m,n,0)^2$ divided by $(1+\alpha^2)^{\frac{3+s}{2}} A_{m-2}A_{n-2}$

where

$$I(r) = \int_0^\pi \int_0^\pi \frac{\sin(\theta_1)^{m-2}\sin(\varphi_1)^{n-2}}{(r^2+1-2r\cos(\theta_1)+\alpha^2(r^2+1-2r\cos(\varphi_1)))^{\frac{N+s}{2}}} d\theta_1 d\varphi_1.$$

We find then that

$$C(m,n,s,\beta) = (1+\alpha)^{\frac{3+s}{2}} A_{m-2}A_{n-2} \int_0^1 (r^{N-2} - r^{N-2-\beta} + r^s - r^{\beta+s}) I(r) dr. \tag{11.11}$$

**Computation of $A_0(m,n,s)^2$ for $n \geq 2$.** Similarly as before we have, for $x = (r\omega_1, \alpha r\omega_2) \in \Sigma$, and $p = (e_1^{(m)}, \alpha e_2^{(n)})$:

$$\nu(x) = \frac{(-\alpha\omega_1, \omega_2)}{\sqrt{1+\alpha^2}}, \quad \nu(p) = \frac{(-\alpha e_1^{(n)}, 1)}{\sqrt{1+\alpha^2}}.$$

Hence

$$\begin{aligned}\int_\Sigma \frac{1-\langle \nu(x), \nu(p)\rangle}{|p-x|^{N+s}} dx &= \sqrt{1+\alpha^2} A_{m-2}A_{n-2} \int_0^\infty r^{N-2} J(r) dr \\ &= \sqrt{1+\alpha^2} A_{m-2}A_{n-2} \int_0^1 (r^{N-2} + r^s) J(r) dr\end{aligned}$$

where

$$J(r) = \frac{1}{1+\alpha^2} \int_0^\pi \int_0^\pi \frac{(1+\alpha^2-\alpha^2\cos(\theta_1)-\cos(\varphi_1))\sin(\theta_1)^{m-2}\sin(\varphi_1)^{n-2}}{(r^2+1-2r\cos(\theta_1)+\alpha^2(r^2+1-2r\cos(\varphi_1))^{\frac{N+s}{2}}} d\theta_1 d\varphi_1.$$

We finally obtain

$$A_0(m,n,s)^2 = (1+\alpha^2)^{\frac{3+s}{2}} A_{m-2}A_{n-2} \int_0^1 (r^{N-2} + r^s) J(r) dr.$$

In table 1 we show the values obtained for $H(m,n,0)$ and $A_0(m,n,0)^2$, divided by $(1+\alpha^2)^{\frac{3+s}{2}} A_{m-2}A_{n-2}$, from numerical approximation of the integrals. From these results we can say that for $s=0$, $\Sigma$ is stable if $n+m=7$ and unstable if $n+m \leq 6$. The same holds for $s>0$ close to zero by continuity of the values with respect to $s$.

**Remark 11.1.** *We see from formulas (11.10) and (11.11) that $C(m,n,s,\beta)$ is symmetric with respect to $\frac{N-2-s}{2}$ and is maximized for $\beta = \frac{N-2-s}{2}$.*

**Remark 11.2.** *In table 2 we give some numerical values of $\alpha$, $H(m,n,s)$ and $A_0(m,n,s)^2$ divided by $(1+\alpha^2)^{\frac{3+s}{2}}A_{m-2}A_{n-2}$ for $m=4$, $n=3$, which show how in this dimension stability depends on $s$. One may conjecture that there is $s_0$ such that the cone is stable for $0\le s\le s_0$ and unstable for $s_0<s<1$.*

## Appendix A. Asymptotics

We prove convergence of geometric fractional quantities as $s\to 1$ ($\varepsilon = 1-s\to 0$). Let $\Sigma\subset\mathbb{R}^{n+1}$ be a smooth embedded hyper surface.

**Lemma A.1.** *Assume $\Sigma=\partial E$. Then for any $X\in\Sigma$*

$$(1-s)\int_{\mathbb{R}^{n+1}}\frac{\chi_E(Y)-\chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\,dY = -H_\Sigma(X)n\omega_n + O(1-s),$$

*as $s\to 1$, where $H_\Sigma(X)=\frac{\kappa_1+\ldots+\kappa_n}{n}$ is the mean curvature of $\Sigma$ at $X$ and $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.*

*Proof.* Let us fix $R>0$ and $X\in\Sigma$ and assume $X=0$ for simplicity. Let $\Sigma_R$ be $\Sigma$ intersected with the cylinder $B_R(0)\times(-R,R)$, $B_R(0)\subset\mathbb{R}^n$. After rotation, we describe $\Sigma_R$ as the graph of $g:B_R(0)\to\mathbb{R}$ with

$$g(0)=0,\quad Dg(0)=0,$$

and assume $E$ lies above $\Sigma_R$.

Note that

$$\int_{(B_R(0)\times(-R,R))^c}\frac{\chi_E(Y)-\chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\,dY = O(1)$$

as $s\to 1$. We compute

$$I=\int_{B_R(0)\times(-R,R)}\frac{\chi_E(Y)-\chi_{E^c}(Y)}{|X-Y|^{n+1+s}}\,dY = -2\int_{B_R\subset\mathbb{R}^n}\int_0^{g(t)}\frac{1}{(|t|^2+t_3^2)^{\frac{n+1+s}{2}}}dt_3\,dt,$$

expanding

$$\int_0^z\frac{1}{(|t|^2+t_3^2)^{\frac{n+1+s}{2}}}dt_3 = \frac{z}{|t|^{3+s}} - (n+1+s)z^2\int_0^1(1-\tau)\frac{\tau z}{(|t|^2+(\tau z)^2)^{\frac{n+3+s}{2}}}\,d\tau.$$

Then

$$I=I_1+I_2+I_3$$

where

$$I_1=-2\int_{|t|<R}\frac{\frac12 D^2g(0)[t^2]}{|t|^{n+1+s}}\,dt,\qquad I_2=-2\int_{|t|<R}\frac{g(t)-\frac12 D^2g(0)[t^2]}{|t|^{n+1+s}}\,dt,$$

$$I_3=2(3+s)\int_{|t|<R}g(t)^2\int_0^1(1-\tau)\frac{\tau g(t)}{(|t|^2+(\tau g(t))^2)^{\frac{n+3+s}{2}}}\,d\tau\,dt,$$

where $D^2g$ denotes the Hessian matrix of $g$. Then

$$I_1=-\frac{\omega_n\Delta g(0)R^{1-s}}{1-s} = -n\omega_n\frac{H_\Sigma(X)R^{1-s}}{(1-s)}.$$

For the other terms we have $I_2=O(1)$ and $I_3=O(1)$ as $s\to 1$.

□

For the next results we assume that there is $C$ such that for all $0<s<1$ and $X\in\Sigma$

$$\int_{Y\in\Sigma,|Y-X|\ge 1}\frac{1}{|X-Y|^{n+1+s}}\,dY\le C.$$

**Lemma A.2.** *If $h$ is $C^{2,\alpha}(\Sigma)$ and bounded,*

$$(1-s)p.v.\int_{\Sigma}\frac{h(Y)-h(X)}{|X-Y|^{n+1+s}}dY=\frac{\omega_n}{2}\Delta_{\Sigma}h(X)+O(1-s),$$

*as $s\to 1$, where $\Delta_\Sigma$ is the Laplace-Beltrami operator on $\Sigma$ and $\omega_n=\frac{area(S^{n-1})}{n}$ is the volume of the unit ball in $\mathbb{R}^n$.*

For the proof we use the following computation.

**Lemma A.3.** *If $\phi\in C^{2,\alpha}(\overline{B}_R(0))$,*

$$(1-s)\int_{B_R\subset\mathbb{R}^n}\frac{\phi(t)-\phi(0)}{|t|^{n+1+s}}\,dt=\frac{\omega_n}{2}\Delta\phi(0)+O(1-s), \tag{A.1}$$

*as $s\to 1$.*

*Proof.* We expand

$$\phi(t)=\phi(0)+D\phi(0)t+\frac{1}{2}D^2\phi(0)[t^2]+O(|t|^{2+\alpha})$$

as $t\to 0$ and compute

$$\begin{aligned}\int_{B_R}\frac{\phi(t)-\phi(0)}{|t|^{n+1+s}}\,dt&=\frac{1}{2}\int_{B_R}\frac{D^2\phi(0)[t^2]}{|t|^{n+1+s}}\,dt+O(1)\\&=\frac{1}{2}\frac{area(S^{n-1})}{n}\frac{R^{1-s}}{1-s}\Delta\phi(0)+O(1)\end{aligned}$$

as $s\to 1$. $\square$

*Proof of Lemma A.2.* Let us fix $R>0$ and $X\in\Sigma$ and assume $X=0$ for simplicity. Let $\Sigma_R$ be $\Sigma$ intersected with the cylinder $B_R(0)\times(-R,R)$, $B_R(0)\subset\mathbb{R}^n$. After rotation, we describe $\Sigma_R$ as the graph of $g:B_R(0)\to\mathbb{R}$ with

$$g(0)=0,\quad Dg(0)=0.$$

Then

$$\int_{\Sigma_R^c}\frac{h(Y)-h(X)}{|X-Y|^{n+1+s}}dY=O(1)$$

as $s\to 1$. We have

$$\int_{\Sigma_R}\frac{h(Y)-h(X)}{|X-Y|^{n+1+s}}dY=\int_{B_R(0)}\frac{h(g(t))-h(g(0))}{(g(t)^2+|t|^2)^{\frac{n+1+s}{2}}}\sqrt{1+|Dg(t)|^2}\,dt$$

The previous lemma also holds if $\phi$ depends on $s$ and $\phi_s\to\phi$ in $C^{2,\alpha}$ as $s\to 1$. We apply (A.1) to

$$\phi_s(t)=\frac{h(g(t))-h(g(0))}{(\frac{g(t)^2}{|t|^2}+1)^{\frac{n+1+s}{2}}}\sqrt{1+|Dg(t)|^2}$$

and note that $\phi_s\to\phi$ as $s\to 1$, where

$$\phi(t)=\frac{h(g(t))-h(g(0))}{(\frac{g(t)^2}{|t|^2}+1)^{n+2}}\sqrt{1+|Dg(t)|^2}$$

and

$$\Delta\phi(0)=\sum_{i=1}^{n}D_i(h\circ g)(0)=\Delta_\Sigma h(0).$$

$\square$

**Lemma A.4.** *Let $\nu$ be smooth choice of normal vector $\nu$ on $\Sigma$. Then*

$$(1-s)\int_\Sigma \frac{(\nu(x)-\nu(y))\cdot\nu(x)}{|x-y|^{n+1+s}}dy = \frac{\omega_n}{2}|A(x)|^2 + O(1)$$

*as $s\to 0$, where $|A(x)|^2$ is the norm squared of the second fundamental form at $x$, i.e. $\sum_{i=1}^n \kappa_i^2$, where $\kappa_1$, $\ldots$, $\kappa_n$ are the principal curvatures at $x$.*

*Proof.* We apply Lemma A.2 with $h(y)=\nu(y)\cdot\nu(x)-1$ and use that

$$\Delta_\Sigma h(x) = -|A(x)|^2.$$

□

## Appendix B. The Jacobi operator

In this section we prove formula (1.5) and derive the formula for the nonlocal Jacobi operator (1.6).

Let $E\subset\mathbb{R}^N$ be an open set with smooth boundary and $\Omega$ be a bounded open set. Let $\nu$ be the unit normal vector field of $\Sigma=\partial E$ pointing to the exterior of $E$. Given $h\in C_0^\infty(\Omega\cap\Sigma)$ and $t$ small, let $E_{th}$ be the set whose boundary $\partial E_{th}$ is parametrized as

$$\partial E_{th} = \{x+th(x)\nu(x)\ /\ x\in\partial E\},$$

with exterior normal vector close to $\nu$.

**Proposition B.1.** *For $h\in C_0^\infty(\Omega\cap\Sigma)$*

$$\frac{d^2}{dt^2}Per_s(E_{th},\Omega)\Big|_{t=0} \;=\; -2\int_\Sigma \mathcal{J}^s_\Sigma[h]\,h - \int_\Sigma h^2 H H^s_\Sigma, \tag{B.1}$$

*where $\mathcal{J}^s_\Sigma$ is the nonlocal Jacobi operator defined in (1.6), $H$ is the classical mean curvature of $\Sigma$ and $H^s_\Sigma$ is the nonlocal mean curvature defined in (1.1).*

In case that $\Sigma$ is a nonlocal minimal surface in $\Omega$ we obtain formula (1.5). Another related formula is the following.

**Proposition B.2.** *Let $\Sigma_{th}=\partial E_{th}$. For $p\in\Sigma$ fixed let $p_t=p+th(p)\nu(p)\in\Sigma_{th}$. Then for $h\in C^\infty(\Sigma)\cap L^\infty(\Sigma)$*

$$\frac{d}{dt}H^s_{\Sigma_{th}}(p_t)\Big|_{t=0} = 2\mathcal{J}^s_\Sigma[h](p). \tag{B.2}$$

A consequence of proposition B.2 is that entire nonlocal minimal graphs are stable.

**Corollary B.1.** *Suppose that $\Sigma=\partial E$ with*

$$E=\{(x',F(x'))\in\mathbb{R}^N : x'\in\mathbb{R}^{N-1}\}$$

*is a nonlocal minimal surface. Then*

$$-\int_\Sigma \mathcal{J}^s_\Sigma[h]\,h\,\geq 0 \quad \text{for all} \quad h\in C_0^\infty(\Sigma). \tag{B.3}$$

*Proof of Proposition B.1.* Let

$$K_\delta(z) = \frac{1}{|z|^{N+s}}\eta_\delta(z)$$

where $\eta_\delta(x)=\eta(x/\delta)$ $(\delta>0)$ and $\eta\in C^\infty(\mathbb{R}^N)$ is a radially symmetric cut-off function with $\eta(x)=1$ for $|x|\geq 2$, $\eta(x)=0$ for $|x|\leq 1$.

Consider

$$Per_{s,\delta}(E_{th},\Omega) = \int_{E_{th}\cap\Omega}\int_{\mathbb{R}^N\setminus E_{th}} K_\delta(x-y)\,dy\,dx + \int_{E_{th}\setminus\Omega}\int_{\Omega\setminus E_{th}} K_\delta(x-y)dydx. \tag{B.4}$$

We will show that $\frac{d^2}{dt^2}Per_{s,\delta}(E_{th},\Omega)$ approaches a certain limit $D_2(t)$ as $\delta \to 0$, uniformly for $t$ in a neighborhood of 0 and that

$$D_2(0) = -2\int_\Sigma \mathcal{J}^s_\Sigma[h]\,h - \int_\Sigma h^2 H H^s_\Sigma.$$

First we need some extensions of $\nu$ and $h$ to $\mathbb{R}^N$. To define them, let $K \subset \Sigma$ be the support of $h$ and $U_0$ be an open bounded neighborhood of $K$ such that for any $x \in U_0$, the closest point $\hat{x} \in \Sigma$ to $x$ is unique and defines a smooth function of $x$. We also take $U_0$ smaller if necessary as to have $\overline{U}_0 \subset \Omega$. Let $\tilde{\nu} : \mathbb{R}^N \to \mathbb{R}^N$ be a globally defined smooth unit vector field such that $\tilde{\nu}(x) = \nu(\hat{x})$ for $x \in U_0$. We also extend $h$ to $\tilde{h} : \mathbb{R}^N \to \mathbb{R}$ such that it is smooth with compact support contained in $\Omega$ and $\tilde{h}(x) = h(\hat{x})$ for $x \in U_0$. From now one we omit the tildes ($\tilde{}$) in the definitions of the extensions of $\nu$ and $h$. For $t$ small $\bar{x} \mapsto \bar{x} + th(\bar{x})\nu(\bar{x})$ is a global diffeomorphism in $\mathbb{R}^N$. Let us write

$$u(\bar{x}) = h(\bar{x})\nu(\bar{x}) \quad \text{for } \bar{x} \in \mathbb{R}^N,$$

$$\nu = (\nu^1, \ldots, \nu^N), \quad u = (u^1, \ldots, u^N)$$

and let

$$J_t(\bar{x}) = J_{id+tu}(\bar{x})$$

be the Jacobian determinant of $id + tu$.

We change variables

$$x = \bar{x} + tu(\bar{x}), \quad y = \bar{y} + tu(\bar{y}),$$

in (B.4)

$$\begin{aligned} Per_{s,\delta}(E_{th},\Omega) = &\int_{E\cap\phi_t(\Omega)}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)J_t(\bar{x})J_t(\bar{y})d\bar{y}d\bar{x}, \\ &+ \int_{E\setminus\phi_t(\Omega)}\int_{\phi_t(\Omega)\setminus E} K_\delta(x-y)J_t(\bar{y})d\bar{y}d\bar{x}, \end{aligned}$$

where $\phi_t$ is the inverse of the map $\bar{x} \mapsto \bar{x} + tu(\bar{x})$.

Differentiating with respect to $t$:

$$\begin{aligned} \frac{d}{dt}Per_{s,\delta}(E_{th},\Omega) = &\int_{E\cap\phi_t(\Omega)}\int_{\mathbb{R}^N\setminus E} \Big[\nabla K_\delta(x-y)(u(\bar{x})-u(\bar{y}))J_t(\bar{x})J_t(\bar{y}) \\ &+ K_\delta(x-y)(J_t'(\bar{x})J_t(\bar{y}) + J_t(\bar{x})J_t'(\bar{y}))\Big]d\bar{y}d\bar{x} \\ &+ \int_{E\setminus\phi_t(\Omega)}\int_{\phi_t(\Omega)\setminus E} \Big[\nabla K_\delta(x-y)(u(\bar{x})-u(\bar{y}))J_t(\bar{x})J_t(\bar{y}) \\ &+ K_\delta(x-y)(J_t'(\bar{x})J_t(\bar{y}) + J_t(\bar{x})J_t'(\bar{y}))\Big]d\bar{y}d\bar{x}, \end{aligned}$$

where

$$J_t'(\bar{x}) = \frac{d}{dt}J_t(\bar{x}).$$

Note that there are no integrals on $\partial\phi_t(\Omega)$ for $t$ small because $u$ vanishes in a neighborhood of $\partial\Omega$.

Since the integrands in $\frac{d}{dt}Per_{s,\delta}(E_{th},\Omega)$ have compact support contained in $\phi_t(\Omega)$ ($t$ small), we can write

$$\begin{aligned} \frac{d}{dt}Per_{s,\delta}(E_{th},\Omega) = \int_E\int_{\mathbb{R}^N\setminus E} &\Big[\nabla K_\delta(x-y)(u(\bar{x})-u(\bar{y}))J_t(\bar{x})J_t(\bar{y}) \\ &+ K_\delta(x-y)(J_t'(\bar{x})J_t(\bar{y}) + J_t(\bar{x})J_t'(\bar{y}))\Big]d\bar{y}d\bar{x}. \end{aligned}$$

Differentiating once more

$$\frac{d^2}{dt^2}Per_{s,\delta}(E_{th},\Omega) = A(\delta,t) + B(\delta,t) + C(\delta,t)$$

where

$$
\begin{aligned}
A(\delta,t) &= \int_E \int_{\mathbb{R}^N\setminus E} D^2K_\delta(x-y)(u(\bar x)-u(\bar y))(u(\bar x)-u(\bar y))J_t(\bar x)J_t(\bar y)d\bar y d\bar x\\
B(\delta,t) &= 2\int_E \int_{\mathbb{R}^N\setminus E} \nabla K_\delta(x-y)(u(\bar x)-u(\bar y))(J_t'(\bar x)J_t(\bar y)+J_t(\bar x)J_t'(\bar y))d\bar y d\bar x\\
C(\delta,t) &= \int_E \int_{\mathbb{R}^N\setminus E} K_\delta(x-y)(J_t''(\bar x)J_t(\bar y)+2J_t'(\bar x)J_t'(\bar y)+J_t(\bar x)J_t''(\bar y))d\bar y d\bar x.
\end{aligned}
$$

We claim that $A(\delta,t)$, $B(\delta,t)$ and $C(\delta,t)$ converge as $\delta\to 0$ for uniformly for $t$ near 0, to limit expressions $A(0,t)$, $B(0,t)$ and $C(0,t)$, which are the same as above replacing $\delta$ by 0, and that the integrals appearing in $A(0,t)$, $B(0,t)$ and $C(0,t)$ are well defined. Indeed, we can estimate

$$
|A(\delta,t)-A(0,t)| \le C\int_{x\in E\cap K_0}\int_{y\in E^c,|x-y|\le 2\delta} \frac{1}{|x-y|^{N+s}}\,dy\,dx,
$$

where $K_0$ is a fixed bounded set. For $x\in E\cap K_0$ we see that

$$
\int_{y\in E^c,|x-y|\le 2\delta} \frac{1}{|x-y|^{N+s}}\,dy \le \frac{C}{dist(x,E^c)^s},
$$

and therefore

$$
|A(\delta,t)-A(0,t)| \le C \le C\int_{x\in E\cap K_0,\ dist(x,E^c)\le 2\delta} \frac{1}{dist(x,E^c)^s}\,dx \le C\delta^{1-s}.
$$

The differences $B(\delta,t)-B(0,t)$, $C(\delta,t)-C(0,t)$ can be estimated similarly. This shows that

$$
\frac{d^2}{dt^2}Per_s(E_{th},\Omega)\Big|_{t=0} = \lim_{\delta\to0}\frac{d^2}{dt^2}Per_{s,\delta}(E_{th},\Omega)\Big|_{t=0} = \lim_{\delta\to0} A(\delta,0)+B(\delta,0)+C(\delta,0).
$$

In what follows we will evaluate $A(\delta,0)+B(\delta,0)+C(\delta,0)$. At $t=0$ we have

$$
\begin{aligned}
A(\delta,0) &= \int_E\int_{\mathbb{R}^N\setminus E} D_{x_ix_j}K_\delta(x-y)(u^i(x)-u^i(y))(u^j(x)-u^j(y))\,dy\,dx\\
&= A_{11}+A_{12}+A_{21}+A_{22}
\end{aligned}
$$

where

$$
\begin{aligned}
A_{11} &= \int_E\int_{\mathbb{R}^N\setminus E} D_{x_ix_j}K_\delta(x-y)u^i(x)u^j(x)\,dy\,dx\\
A_{12} &= -\int_E\int_{\mathbb{R}^N\setminus E} D_{x_ix_j}K_\delta(x-y)u^i(x)u^j(y)\,dy\,dx\\
A_{21} &= -\int_E\int_{\mathbb{R}^N\setminus E} D_{x_ix_j}K_\delta(x-y)u^i(y)u^j(x)\,dy\,dx\\
A_{22} &= \int_E\int_{\mathbb{R}^N\setminus E} D_{x_ix_j}K_\delta(x-y)u^i(y)u^j(y)\,dy\,dx.
\end{aligned}
$$

Let us also write

$$
\begin{aligned}
B(\delta,0) &= 2\int_E\int_{\mathbb{R}^N\setminus E} D_{x_j}K_\delta(x-y)(u^j(x)-u^j(y))(\mathrm{div}\,(u)(x)+\mathrm{div}\,(u)(y))\,dy\,dx\\
&= B_{11}+B_{12}+B_{21}+B_{22},
\end{aligned}
$$

where

$$B_{11} = 2\int_E \int_{\mathbb{R}^N\setminus E} D_{x_j} K_\delta(x-y) u^j(x) \mathrm{div}\,(u)(x)\, dy\, dx$$
$$B_{12} = 2\int_E \int_{\mathbb{R}^N\setminus E} D_{x_j} K_\delta(x-y) u^j(x) \mathrm{div}\,(u)(y)\, dy\, dx$$
$$B_{21} = -2\int_E \int_{\mathbb{R}^N\setminus E} D_{x_j} K_\delta(x-y) u^j(y) \mathrm{div}\,(u)(x)\, dy\, dx$$
$$B_{22} = 2\int_E \int_{\mathbb{R}^N\setminus E} D_{y_j} K_\delta(x-y) u^j(y) \mathrm{div}\,(u)(y)\, dy\, dx,$$

and

$$C(\delta, 0) = C_1 + C_2 + C_3,$$

where

$$C_1 = \int_E \int_{\mathbb{R}^N\setminus E} K_\delta(x-y) \Big[\mathrm{div}\,(u)(x)^2 - tr(Du(x)^2)\Big]\, dy\, dx$$
$$C_2 = \int_E \int_{\mathbb{R}^N\setminus E} K_\delta(x-y) \Big[\mathrm{div}\,(u)(y)^2 - tr(Du(y)^2)\Big]\, dy\, dx$$
$$C_3 = 2\int_E \int_{\mathbb{R}^N\setminus E} K_\delta(x-y) \mathrm{div}\,(u)(x) \mathrm{div}\,(u)(y)\, dy\, dx.$$

We compute

$$\begin{aligned} A_{11} &= \int_E \int_{\mathbb{R}^N\setminus E} D_{x_i}\Big[D_{x_j} K_\delta(x-y) u^i(x) u^j(x)\Big]\, dy\, dx \\ &\quad - \int_E \int_{\mathbb{R}^N\setminus E} D_{x_j} K_\delta(x-y) D_{x_i}\Big[u^i(x) u^j(x)\Big]\, dy\, dx \\ &= \int_{\partial E} \int_{\mathbb{R}^N\setminus E} D_{x_j} K_\delta(x-y) u^i(x) u^j(x) \nu^i(x)\, dy\, dx \\ &\quad - \int_E \int_{\mathbb{R}^N\setminus E} D_{x_j} K_\delta(x-y) \Big[D_{x_i} u^i(x) u^j(x) + u^i(x) D_{x_i} u^j(x)\Big]\, dy\, dx. \end{aligned}$$

Therefore

$$\begin{aligned} A_{11} + B_{11} &= \int_{\partial E} \int_{\mathbb{R}^N\setminus E} D_{x_j} K_\delta(x-y) u^i(x) u^j(x) \nu^i(x)\, dy\, dx \\ &\quad + \int_E \int_{\mathbb{R}^N\setminus E} D_{x_j} K_\delta(x-y) \Big[D_{x_i} u^i(x) u^j(x) - u^i(x) D_{x_i} u^j(x)\Big]\, dy\, dx. \end{aligned}$$

We express the first term as

$$\begin{aligned} &\int_{\partial E} \int_{\mathbb{R}^N\setminus E} D_{x_j} K_\delta(x-y) u^i(x) u^j(x) \nu^i(x)\, dy\, dx \\ &\qquad = -\int_{\partial E} \int_{\mathbb{R}^N\setminus E} D_{y_j} K_\delta(x-y) u^i(x) u^j(x) \nu^i(x)\, dy\, dx \\ &\qquad = \int_{\partial E} \int_{\partial E} K_\delta(x-y) u^i(x) u^j(x) \nu^i(x) \nu^j(y)\, dy\, dx \\ &\qquad = \int_{\partial E} \int_{\partial E} K_\delta(x-y) h(x)^2 \nu(x) \nu(y)\, dy\, dx. \end{aligned}$$

For the second term of $A_{11}+B_{11}$ let us write

$$\begin{aligned}
&\int_E\int_{\mathbb{R}^N\setminus E} D_{x_j}K_\delta(x-y)D_{x_i}u^i(x)u^j(x)\,dy\,dx\\
&\qquad=\int_E\int_{\mathbb{R}^N\setminus E} D_{x_j}\Big[K_\delta(x-y)D_{x_i}u^i(x)u^j(x)\Big]\,dy\,dx\\
&\qquad\quad-\int_E\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)D_{x_j}\Big[D_{x_i}u^i(x)u^j(x)\Big]\,dy\,dx\\
&\qquad=\int_{\partial E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)D_{x_i}u^i(x)u^j(x)\nu^j(x)\,dy\,dx\\
&\qquad\quad-\int_E\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\Big[D_{x_jx_i}u^i(x)u^j(x)+\operatorname{div}(u)(x)^2\Big]\,dy\,dx.
\end{aligned}$$

The third term of $A_{11}+B_{11}$ is

$$\begin{aligned}
&-\int_E\int_{\mathbb{R}^N\setminus E} D_{x_j}K_\delta(x-y)u^i(x)D_{x_i}u^j(x)\,dy\,dx\\
&\qquad=-\int_E\int_{\mathbb{R}^N\setminus E} D_{x_j}\Big[K_\delta(x-y)u^i(x)D_{x_i}u^j(x)\Big]\,dy\,dx\\
&\qquad\quad+\int_E\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)D_{x_j}\Big[u^i(x)D_{x_i}u^j(x)\Big]\,dy\,dx\\
&\qquad=-\int_{\partial E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)u^i(x)D_{x_i}u^j(x)\nu^j(x)\,dy\,dx\\
&\qquad\quad+\int_E\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\Big[D_{x_j}u^i(x)D_{x_i}u^j(x)+u^i(x)D_{x_jx_i}u^j(x)\Big]\,dy\,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
A_{11}+B_{11}&=\int_{\partial E}\int_{\partial E} K_\delta(x-y)h(x)^2\nu(x)\nu(y)\,dy\,dx\\
&\quad+\int_{\partial E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\Big[D_{x_i}u^i(x)u^j(x)\nu^j(x)-u^i(x)D_{x_i}u^j(x)\nu^j(x)\Big]\,dy\,dx\\
&\quad+\int_E\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\Big[D_{x_j}u^i(x)D_{x_i}u^j(x)-\operatorname{div}(u)(x)^2\Big]\,dy\,dx,
\end{aligned}$$

so that

$$\begin{aligned}
A_{11}+B_{11}+C_1&=\int_{\partial E}\int_{\partial E} K_\delta(x-y)h(x)^2\nu(x)\nu(y)\,dy\,dx\\
&\quad+\int_{\partial E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\Big[D_{x_i}u^i(x)u^j(x)\nu^j(x)-u^i(x)D_{x_i}u^j(x)\nu^j(x)\Big]\,dy\,dx.
\end{aligned}$$

But using $u=\nu h$ and $\operatorname{div}(\nu)=H$ where $H$ is the mean curvature of $\partial E$ we have

$$D_{x_i}u^i(x)u^j(x)\nu^j(x)-u^i(x)D_{x_i}u^j(x)\nu^j(x)=h(x)^2H(x)$$

and therefore

$$A_{11}+B_{11}+C_1=\int_{\partial E}\int_{\partial E} K_\delta(x-y)h(x)^2\nu(x)\nu(y)\,dy\,dx+\int_{\partial E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)h(x)^2H(x).$$

In a similar way, we have

$$
\begin{aligned}
A_{22}+B_{22}+C_2 &= \int_{\partial E}\int_{\partial E} K_\delta(x-y)h(y)^2\nu(x)\nu(y)\,dy\,dx \\
&\quad - \int_{E}\int_{\partial E} K_\delta(x-y)\Big[D_{y_i}u^i(y)u^j(y)\nu^j(y) - u^i(y)D_{y_i}u^j(y)\nu^j(y)\Big]\,dy\,dx \\
&= \int_{\partial E}\int_{\partial E} K_\delta(x-y)h(y)^2\nu(x)\nu(y)\,dy\,dx - \int_{E}\int_{\partial E} K_\delta(x-y)h(y)^2 H(y)\,dy\,dx.
\end{aligned}
$$

Further calculations show that

$$
\begin{aligned}
A_{12} = &- \int_{\partial E}\int_{\partial E} K_\delta(x-y)h(x)h(y)\,dydx \\
&- \int_{\partial E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\mathrm{div}\,(u)(y)u^i(x)\nu^i(x)\,dy\,dx \\
&+ \int_{E}\int_{\partial E} K_\delta(x-y)\mathrm{div}\,(u)(x)u^i(y)\nu^i(y)\,dy\,dx \\
&+ \int_{E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\mathrm{div}\,(u)(x)\mathrm{div}\,(u)(y)\,dy\,dx,
\end{aligned}
$$

$$
\begin{aligned}
A_{21} = &- \int_{\partial E}\int_{\partial E} K_\delta(x-y)h(x)h(y)\,dydx \\
&- \int_{\partial E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\mathrm{div}\,(u)(y)u^j(x)\nu^j(x)\,dy\,dx \\
&+ \int_{E}\int_{\partial E} K_\delta(x-y)\mathrm{div}\,(u)(x)u^i(y)\nu^i(y)\,dy\,dx \\
&+ \int_{E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\mathrm{div}\,(u)(x)div(u)(y)\,dy\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
B_{12}+B_{21} = 2&\int_{\partial E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\mathrm{div}\,(u)(y)u^j(x)\nu^j(x)\,dy\,dx \\
&- 2\int_{E}\int_{\partial E} K_\delta(x-y)\mathrm{div}\,(u)(x)u^j(y)\nu^j(y)\,dy\,dx \\
&- 4\int_{E}\int_{\mathbb{R}^N\setminus E} K_\delta(x-y)\mathrm{div}\,(u)(x)div(u)(y)\,dy\,dx,
\end{aligned}
$$

so that

$$
A_{12}+A_{21}+B_{12}+B_{21}+C_3 = -2\int_{\partial E}\int_{\partial E} K_\delta(x-y)h(x)h(y)\,dydx.
$$

Therefore

$$
\begin{aligned}
\frac{d^2}{dt^2}Per_{s,\delta}(E_{th},\Omega)\Big|_{t=0} = 2&\int_{\partial E}\int_{\partial E} K_\delta(x-y)h(x)^2(\nu(x)\nu(y)-1)\,dy\,dx \\
&- 2\int_{\partial E} h(x)\int_{\partial E} K_\delta(x-y)(h(y)-h(x))\,dydx \\
&- \int_{\partial E} h(x)^2 H(x)\int_{\mathbb{R}^N}(\chi_E(y)-\chi_{E^c}(y))K_\delta(x-y)\,dy\,dx.
\end{aligned}
$$

Taking the limit as $\delta \to 0$ we find (B.1). $\square$

*Proof of Proposition B.2.* Let $\nu_t(x)$ denote the unit normal vector to $\partial E_t$ at $x \in \partial E_t$ pointing out of $E_t$. Note that $\nu(x) = \nu_0(x)$. Let $L_t$ be the half space defined by $L_t = \{x : \langle x - p_t, \nu_t(p_t)\rangle > 0\}$. Then

$$H^s_{\Sigma_{th}}(p_t) = \int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x) - \chi_{E^c}(x) + \chi_{L_t^c}(x)}{|x - p_t|^{N+s}}\, dx \tag{B.5}$$

since the function $1 - 2\chi_{L_t}$ has zero principal value. Note that the integral in (B.5) is well defined and

$$H^s_{\Sigma_{th}}(p_t) = 2 \int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x)}{|x - p_t|^{N+s}}\, dx.$$

For $\delta > 0$ let $\eta \in C^\infty(\mathbb{R}^N)$ be a radially symmetric cut-off function with $\eta(x) = 1$ for $|x| \geq 2$, $\eta(x) = 0$ for $|x| \leq 1$. Define $\eta_\delta(x) = \eta(x/\delta)$ and write

$$\int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x)}{|x - p_t|^{N+s}}\, dx = f_\delta(t) + g_\delta(t)$$

where

$$f_\delta(t) = \int_{\mathbb{R}^N} \frac{\chi_{E_t}(x) - \chi_{L_t}(x)}{|x - p_t|^{N+s}} \eta_\delta(x - p_t)\, dx$$

and $g_\delta(t)$ is the rest. Then it is direct that $f_\delta$ is differentiable and

$$\begin{aligned}
f'_\delta(0) = &\int_{\partial E} \frac{h(x)}{|x-p|^{N+s}} \eta_\delta(x - p) \\
&- \int_{\partial L_0} \frac{h(p)\langle \nu(p), \nu(p)\rangle - \langle x - p, \frac{\partial \nu_t(p_t)}{\partial t}|_{t=0}\rangle}{|x-p|^{N+s}} \eta_\delta(x-p) \\
&+ (N+s)h(p) \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{L_0}(x)}{|x-p|^{N+s+2}} \langle x - p, \nu(p)\rangle \eta_\delta(x-p) dx \\
&- h(p) \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{L_0}(x)}{|x-p|^{N+s}} \langle \nabla \eta_\delta(x-p), \nu(p)\rangle dx.
\end{aligned}$$

We integrate the third term by parts

$$\begin{aligned}
&(N+s) \int_{\mathbb{R}^N} \frac{\chi_E(x) - \chi_{L_0}(x)}{|x-p|^{N+s+2}} \langle x-p, \nu(p)\rangle \eta_\delta(x-p) dx \\
&= - \int_{\mathbb{R}^N} (\chi_E(x) - \chi_{L_0}(x)) \langle \nabla \frac{1}{|x-p|^{N+s}}, \nu(p)\rangle \eta_\delta(x-p) dx \\
&= - \int_{\partial E} \frac{\langle \nu(x), \nu(p)\rangle}{|x-p|^{N+s}} \eta_\delta(x-p) + \int_{\partial L_0} \frac{\langle \nu(p), \nu(p)\rangle}{|x-p|^{N+s}} \eta_\delta(x-p) \\
&\quad + \int_{\mathbb{R}^N} \frac{\chi_{E(x)} - \chi_{L_0}(x)}{|x-p|^{N+s}} \langle \nabla \eta_\delta(x-p), \nu(p)\rangle dx.
\end{aligned}$$

Since $\eta_\delta$ is radially symmetric,

$$\int_{\partial L_0} \frac{\langle x - p, \frac{\partial \nu_t(p_t)}{\partial t}|_{t=0}\rangle}{|x-p|^{N+s}} \eta_\delta(x-p)\, dx = 0$$

and then

$$f'_\delta(0) = \int_{\partial E} \frac{h(x)}{|x-p|^{N+s}} \eta_\delta(x-p) dx - h(p) \int_{\partial E} \frac{\langle \nu(x), \nu(p)\rangle}{|x-p|^{N+s}} \eta_\delta(x-p) dx,$$

which we write as

$$f'_\delta(0) = \int_{\partial E} \frac{h(x) - h(p)}{|x-p|^{N+s}} \eta_\delta(x-p)\, dx + h(p) \int_{\partial E} \frac{1 - \langle \nu(x), \nu(p)\rangle}{|x-p|^{N+s}} \eta_\delta(x-p)\, dx.$$

We claim that $g_\delta'(t) \to 0$ as $\delta \to 0$, uniformly for $t$ in a neighborhood of 0. Indeed, in a neighborhood of $p_t$ we can represent $\partial E_t$ as a graph of a function $G_t$ over $L_t \cap B(p_t, 2\delta)$, with $G_t$ defined in a neighborhood of 0 in $\mathbb{R}^{N-1}$, $G_t(0) = 0$, $\nabla_{y'} G_t(0) = 0$ and smooth in all its variables (we write $y' \in \mathbb{R}^{N-1}$). Then $g_\delta(t)$ becomes

$$g_\delta(t) = \int_{|y'|<2\delta} \int_0^{G_t(y')} \frac{1}{(|y'|^2 + y_N^2)^{\frac{N+s}{2}}} (1 - \eta_\delta(y', y_N)) dy_N dy'$$

so that

$$g_\delta'(t) = \int_{|y'|<2\delta} \frac{1}{(|y'|^2 + G_t(y')^2)^{\frac{N+s}{2}}} \frac{\partial G_t}{\partial t}(y')(1 - \eta_\delta(y', y_N)) dy'.$$

But $|G_t(y')| \le K|y'|^2$ and $|\frac{\partial G_t}{\partial t}(y')| \le K|y'|^2$, so

$$g_\delta'(t) \le C\delta^{1-s}.$$

Therefore

$$\begin{aligned} \frac{d}{dt} H^s_{\Sigma_{th}}(p_t)\Big|_{t=0} = 2 \lim_{\delta \to 0} \Bigg[ &\int_{\partial E} \frac{h(x) - h(p)}{|x-p|^{N+s}} \eta_\delta(x-p) dx \\ &+ h(p) \int_{\partial E} \frac{1 - \langle \nu(x), \nu(p) \rangle}{|x-p|^{N+s}} \eta_\delta(x-p) dx \Bigg]. \end{aligned}$$

Letting $\delta \to 0$ we find (B.2). □

*Proof of Corollary B.1.* The same argument as in the proof of Proposition B.2 shows that if $F : \Sigma \to \mathbb{R}^N$ is a smooth bounded vector field and we let $E_t$ be the set whose boundary $\Sigma_t = \partial E_t$ is parametrized as

$$\partial E_{th} = \{x + tF(x) \ / \ x \in \partial E\},$$

with exterior normal vector close to $\nu$, then

$$\frac{d}{dt} H^s_{\Sigma_t}(p_t)\Big|_{t=0} = 2\mathcal{J}^s_\Sigma[\langle F, \nu \rangle](p),$$

where $p_t = p + tF(p)$. Taking as $F(x) = e_N = (0, \dots, 0, 1)$ we conclude that $w = \langle \nu, e_N \rangle$ is a positive function satisfying

$$\mathcal{J}^s_\Sigma[w](x) = 0 \quad \text{for all } x \in \Sigma.$$

More explicitly

$$\text{p.v.} \int_\Sigma \frac{w(y) - w(x)}{|y-x|^{N+s}} dy + w(x)A(x) = 0 \quad \text{for all } x \in \Sigma, \tag{B.6}$$

where

$$A(x) = \int_\Sigma \frac{\langle \nu(x) - \nu(y), \nu(x) \rangle}{|x-y|^{N+s}} dy.$$

As in the classical setting we can show that $\Sigma$ is stable in the sense that (B.3) holds. Let $\phi \in C_0^\infty(\Sigma)$ and observe that

$$\frac{1}{2} \int_\Sigma \int_\Sigma \frac{(\phi(x) - \phi(y))^2}{|x-y|^{N+s}} dxdy = \int_\Sigma \int_\Sigma \frac{(\phi(x) - \phi(y))\phi(x)}{|x-y|^{N+s}} dxdy.$$

Write $\phi = w\psi$ with $\psi \in C_0^\infty(\Sigma)$. Then

$$\begin{aligned} \int_\Sigma \int_\Sigma \frac{(\phi(x) - \phi(y))\phi(x)}{|x-y|^{N+s}} dxdy = &\int_\Sigma \int_\Sigma \frac{(w(x) - w(y))w(x)\psi(x)^2}{|x-y|^{N+s}} dxdy \\ &+ \int_\Sigma \int_\Sigma \frac{(\psi(x) - \psi(y))w(x)w(y)\psi(x)}{|x-y|^{N+s}} dxdy. \end{aligned} \tag{B.7}$$

Multiplying (B.6) by $w\psi^2$ and integrating we get

$$\int_\Sigma \int_\Sigma \frac{(w(x) - w(y))w(x)\psi(x)^2}{|x-y|^{N+s}} dxdy = \int_\Sigma A(x)w(x)^2\psi(x)^2 dx = \int_\Sigma A(x)\phi(x)^2 dx. \tag{B.8}$$

For the second term in (B.7) we observe that

$$\int_\Sigma\int_\Sigma \frac{(\psi(x)-\psi(y))w(x)w(y)\psi(x)}{|x-y|^{N+s}}dxdy = \frac{1}{2}\int_\Sigma\int_\Sigma \frac{(\psi(x)-\psi(y))^2 w(x)w(y)}{|x-y|^{N+s}}dxdy. \tag{B.9}$$

Therefore, combining (B.7), (B.8), (B.9) we obtain

$$\begin{aligned}\frac{1}{2}\int_\Sigma\int_\Sigma \frac{(\phi(x)-\phi(y))^2}{|x-y|^{N+s}}dxdy &= \int_\Sigma A(x)\phi(x)^2dx\\ &\quad+\frac{1}{2}\int_\Sigma\int_\Sigma \frac{(\psi(x)-\psi(y))^2 w(x)w(y)}{|x-y|^{N+s}}dxdy.\end{aligned}$$

and tis shows (B.3). □

## Appendix C. Graph representation

Let $r,\theta$ be polar coordinates for $x\in\mathbb{R}^2$, i.e. $x=(r\cos\theta, r\sin\theta)$. Then we define $\hat r = \frac{x}{r} = (\cos\theta,\sin\theta)^T$, $\hat\theta = (-\sin\theta,\cos\theta)^T$. Given a point $X\in\Sigma_0$, $X=(x,F_\varepsilon(x))$ we let $\Pi_1(X)$, $\Pi_2(X)$ and $\nu_{\Sigma_0}(X)$ be tangent and normal vector to $\Sigma_0$ at $X$ as defined in (8.1), (8.2) and let $\Pi=[\Pi_1,\Pi_2]$. Then we consider coordinates $t=(t_1,t_2)$ and $t_3$ defined by

$$(t_1,t_2,t_3)\mapsto \Pi_1(X)t_1+\Pi_2(X)t_2+\nu_{\Sigma_0}(X)t_3.$$

Let

$$R_X=\delta|X|$$

where $\delta>0$ is a small fixed constant.

Given $h$ on $\Sigma_0$ with $\|h\|_*\le \sigma_0\varepsilon^{\frac12}$, we can represent $\partial E_h$ near $X_h = X+\nu_{\Sigma_0}(X)h(X)$ as

$$\Pi(X)t+\nu_{\Sigma_0}(X)g(t),\quad |t-t_0(X)|\le 2R_X$$

where $g$ is of class $C^{2,\alpha}$ in the ball $B_{2R_X}(t_0(X))$, with $t_0=t_0(X)$ such that $\Pi(X)t_0$ is the orthogonal projection of $X$ onto the plane generated by $\Pi_1(X)$, $\Pi_2(X)$. We call $G_X$ the operator defined by

$$g_h = G_X(h).$$

To get the correct dependence of the various functions on $|X|$, let $r_0=|x|$. Let us change variables

$$y = x+r_0B\bar y,\quad t=r_0\bar t,\quad g=r_0\bar g \tag{C.1}$$

where the $2\times2$ matrix $B$ is given by

$$B=[\hat r,\hat\theta]$$

(and depends on $X$), so that the equation takes the form

$$0=\Pi(X)\bar t+\bar g\nu_{\Sigma_0}(X)-\begin{bmatrix}\frac{1}{r_0}x+B\bar y\\ r_0^{-1}F_\varepsilon(x+r_0B\bar y)\end{bmatrix}-\frac{1}{r_0}h(x+r_0B\bar y)\nu_{\Sigma_0}(x+r_0B\bar y).$$

To simplify notation we will omit the bars in $\bar t$, $\bar y$, $\bar g$ and let $\Phi=(y,g)$.

We search for a function $\Phi(t)=(y(t),g(t))$, $y(t)\in\mathbb{R}^2$, $g(t)\in\mathbb{R}$ that solves

$$\mathcal{F}(\Phi,X,h)=0 \tag{C.2}$$

where

$$\begin{aligned}\mathcal{F}(\Phi,X,h)(t) &= \Pi(X)t+g(t)\nu_{\Sigma_0}(X)-\begin{bmatrix}\frac{1}{r_0}x+By(t)\\ r_0^{-1}F_\varepsilon(x+r_0By(t))\end{bmatrix}\\ &\quad-\frac{1}{r_0}h(x+r_0By(t))\nu_{\Sigma_0}(x+r_0By(t)).\end{aligned}$$

We search for functions $y,g$ defined in a ball $B_{\delta_0}(t_0(X))$, where $\delta_0>$ is some small fixed number. By shifting $t$ to $t-t_0(X)$ we will assume $t_0(X)=0$.

Let $X$ be a Banach space of functions over $\overline{B}_{\delta_0}(0)\subset\mathbb{R}^2$ with values in $\mathbb{R}^3$. We will take later $X$ either $C^1$, $C^2$ or $C^{2,\alpha}$. Let $B_{\delta_1}(\Phi_0)\subset X$ be the open ball of radius $\delta_1>0$ centered at the function

$$\Phi_0=(y_0,0)$$

where $y_0(t) = t$. Note that $\mathcal{F}(\cdot, X, h)$ maps $\overline{B}_{\delta_1}(\Phi_0)$ into $X$. We intend to show that if $\delta_0$ is fixed small, $\delta_1$ is small depending on $\varepsilon$, and $\|h\|_*$, then there is a unique solution $\Phi \in \overline{B}_{\delta_1}(\Phi_0)$ of $\mathcal{F}(\Phi, X, h) = 0$.

For this we need to construct a bounded left inverse for $D_\Phi \mathcal{F}(\Phi_0, X, h)$. We have, for $\Phi = (y, g)$

$$D_\Phi \mathcal{F}(\Phi, X, h) \tag{C.3}$$
$$= \begin{bmatrix} -B_{1,1} - D_{y_1}\nu_{\Sigma_0}^{(1)} h - \nu_{\Sigma_0}^{(1)} D_{y_1} h & -B_{1,2} - D_{y_2}\nu_{\Sigma_0}^{(1)} h - \nu_{\Sigma_0}^{(1)} D_{y_2} h & \nu_{\Sigma_0}^{(1)}(X) \\ -B_{2,1} - D_{y_1}\nu_{\Sigma_0}^{(2)} h - \nu_{\Sigma_0}^{(2)} D_{y_1} h & -B_{2,2} - D_{y_2}\nu_{\Sigma_0}^{(2)} h - \nu_{\Sigma_0}^{(2)} D_{y_2} h & \nu_{\Sigma_0}^{(2)}(X) \\ -D_{y_1} F_\varepsilon - D_{y_1}\nu_{\Sigma_0}^{(3)} h - \nu_{\Sigma_0}^{(3)} D_{y_1} h & -D_{y_2} F_\varepsilon - D_{y_2}\nu_{\Sigma_0}^{(3)} h - \nu_{\Sigma_0}^{(3)} D_{y_2} h & \nu_{\Sigma_0}^{(3)}(X) \end{bmatrix},$$

where $h$, $\nu_{\Sigma_0}$, $F_\varepsilon$ are evaluated at $x + r_0 B y(t)$ when it is not explicitly written to depend on $X$ (third column). We write $\nu_{\Sigma_0}^{(i)}$ the $i$-th component $\nu_{\Sigma_0}$.

We take

$$A = \begin{bmatrix} -B^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

as a simple approximation of the inverse of $D_\Phi \mathcal{F}(\Phi_0, X, h)$. We claim that

$$\|A(\mathcal{F}(\Phi_1, X, h) - \mathcal{F}(\Phi_2, X, h)) - (\Phi_1 - \Phi_2)\|_X \leq L\|\Phi_1 - \Phi_2\|_X \tag{C.4}$$

for $\Phi_1, \Phi_2 \in \overline{B}_{\delta_1}(\Phi_0)$, where $0 < L < 1$ and that

$$\|A\mathcal{F}(\Phi_0, X, h)\|_X \leq (1 - L)\delta_1. \tag{C.5}$$

With (C.4), (C.5) we conclude from the contraction mapping principle, applied to

$$T(\Phi) = \Phi - A\mathcal{F}(\Phi, X, h) \tag{C.6}$$

that there is a unique $\Phi \in \overline{B}_{\delta_1}(\Phi_0)$ such that $\mathcal{F}(\Phi, X, h) = 0$.

To prove estimates (C.4), (C.5) we always assume $\|h\|_* \leq \sigma_0 \varepsilon^{\frac{1}{2}}$.

We consider first the case $r_0 \geq \delta \varepsilon^{-\frac{1}{2}} |\log \varepsilon|$. Let us proceed with (C.4) and $\| \ \|_X = \| \ \|_{C^1}$. Let $\Phi_1 = (y_1, g_1)$, $\Phi_2 = (y_2, g_2) \in \overline{B}_{\delta_1}(\Phi_0)$. Then we claim that

$$\|A(\mathcal{F}(\Phi_1, X, h) - \mathcal{F}(\Phi_2, X, h)) - (\Phi_1 - \Phi_2)\|_{C^1} \leq \varepsilon^{\frac{1}{2}} \|\Phi_1 - \Phi_2\|_{C^1}. \tag{C.7}$$

Indeed

$$A(\mathcal{F}(\Phi_1, X, h) - \mathcal{F}(\Phi_2, X, h)) - (\Phi_1 - \Phi_2) = D_1 + D_2 + D_3.$$

We estimate the norm of

$$D_1 = (g_1 - g_2)\left(A\nu_{\Sigma_h} - e_3\right),$$

where $e_3 = (0, 0, 1)$. By Corollary 3.1 $|A\nu_{\Sigma_h} - e_3| \leq C\varepsilon^{\frac{1}{2}}$ so

$$\|D_1\|_{C^1} \leq C\varepsilon^{\frac{1}{2}} \|\Phi_1 - \Phi_1\|_{C^1}.$$

Next,

$$D_2 = -\begin{bmatrix} 0 \\ r_0^{-1}(F_\varepsilon(x + r_0 B y_1(t)) - F_\varepsilon(x + r_0 B y_2(t))) \end{bmatrix}$$

and using Corollary 3.1

$$\|D_2\|_{C^1} \leq C\varepsilon^{\frac{1}{2}} \|\Phi_1 - \Phi_2\|_{C^1}.$$

Finally

$$D_3 = -\frac{1}{r_0}\left(h(x + r_0 B y_1(t))\nu_{\Sigma_0}(x + r_0 B y_1(t)) - h(x + r_0 B y_2(t))\nu_{\Sigma_0}(x + r_0 B y_2(t))\right)$$

so

$$\sup_{|t| \leq \delta_0} |D_3| \leq C\varepsilon^{\frac{1}{2}} \|\Phi_1 - \Phi_2\|_{C^1}.$$

We write the derivative as

$$D_tD_3 = (Dh(x + r_0By_1(t)) - Dh(x + r_0By_2(t)))Dy_1(t) \\ + Dh(x + r_0By_2(t))B(Dy_1(t) - Dy_2(t)).$$

Since $\|h\|_* \le \sigma_0\varepsilon^{\frac{1}{2}}$ and $\|\ \|_*$ is weighted $C^{2,\alpha}$ norm we have

$$\sup_{|t|\le\delta_0} |(Dh(x + r_0By_1(t)) - Dh(x + r_0By_2(t)))Dy_1(t)| \\ \le \|D^2h\|_{L^\infty}\|y_1 - y_2\|_{C^1}\|y_1\|_{C^1} \le \|h\|_*\|y_1 - y_2\|_{C^1} \le C\varepsilon^{\frac{1}{2}}\|\Phi_1 - \Phi_2\|_{C^1}.$$

The other term in $D_tD_3$ is estimated as

$$\sup_{|t|\le\delta_0} |Dh(x + r_0By_2(t))|(B(Dy_1(t) - Dy_2(t))| \le C\varepsilon^{\frac{1}{2}}\|\Phi_1 - \Phi_2\|_{C^1}.$$

Therefore

$$\|D_3\|_{C^1} \le C\varepsilon^{\frac{1}{2}}\|\Phi_1 - \Phi_2\|_{C^1},$$

and this proves (C.7).

Regarding (C.5), we have

$$A\mathcal{F}(\Phi_0, X, h) \\ = -\frac{1}{r_0}A\begin{bmatrix} 0 \\ F_\varepsilon(x_0) - F_\varepsilon(x_0 + r_0Bt)\end{bmatrix} + A\left(\Pi(X) - \begin{bmatrix} B \\ 0\end{bmatrix}\right)t \\ - \frac{1}{r_0}h(x_0 + r_0Bt)\nu_{\Sigma_0}(x_0 + r_0Bt),$$

and we see that

$$\|A\mathcal{F}(\Phi_0, X_0, h)\|_{C^1} \le C\varepsilon^{\frac{1}{2}}.$$

Then (C.4), (C.5) hold with $C^1$ norm and $\delta_1 = C\varepsilon^{\frac{1}{2}}$. We conclude that there is a unique $\Phi$ with $\|\Phi - \Phi_0\|_{C^1(\overline{B}_{\delta_0}(0))} \le C\varepsilon^{\frac{1}{2}}$ such that $\mathcal{F}(\Phi, X, h) = 0$.

We can get also estimates for $\Phi$ in $C^{2,\alpha}$. For this we claim that for $\Phi_1, \Phi_2 \in C^{2,\alpha}(\overline{B}_{\delta_0}(0))$:

$$\|D^2\{A(\mathcal{F}(\Phi_1, X_0, h) - \mathcal{F}(\Phi_2, X_0, h)) - (\Phi_1 - \Phi_2)\}\|_{C^0} \\ \le C\varepsilon^{\frac{1}{2}}(\|\Phi_1\|^2_{C^1}\|\Phi_1 - \Phi_2\|^\alpha_{C^0} + \|\Phi_1\|_{C^1}\|\Phi_1 - \Phi_2\|_{C^1} + \|D^2\Phi_1\|_{C^0}\|\Phi_1 - \Phi_2\|_{C^0} \\ + \|D^2(\Phi_1 - \Phi_2)\|_{C^0}).$$

Let us consider $\Phi_1, \Phi_2$ with $\|\Phi_i - \Phi_0\|_{C^1} \le C\varepsilon^{\frac{1}{2}}$ so $\|\Phi_i\|_{C^1} \le C$. Then we can simplify the above estimate to

$$\|D^2\{A(\mathcal{F}(\Phi_1, X_0, h) - \mathcal{F}(\Phi_2, X_0, h)) - (\Phi_1 - \Phi_2)\}\|_{C^0} \\ \le C\varepsilon^{\frac{1}{2}}(\|\Phi_1 - \Phi_2\|^\alpha_{C^1} + \|D^2\Phi_1\|_{C^0}\|\Phi_1 - \Phi_2\|_{C^0} + \|D^2(\Phi_1 - \Phi_2)\|_{C^0}). \tag{C.8}$$

In a similar way, assuming $\|\Phi_i\|_{C^1} \le C$,

$$[D^2\{A(\mathcal{F}(\Phi_1, X_0, h) - \mathcal{F}(\Phi_2, X_0, h)) - (\Phi_1 - \Phi_2)\}]_{\alpha,B_{\delta_0}} \\ \le C\varepsilon^{\frac{1}{2}}([D^2(\Phi_1 - \Phi_2)]_{\alpha,B_{\delta_0}} + 1 + \|D^2\Phi_1\|_{C^0} + \|D^2(\Phi_1 - \Phi_2)\|_{C^0}).$$

Let $T$ be the operator defined by (C.6) and $\Phi_k$ the sequence defined by

$$\Phi_{k+1} = T(\Phi_k), \Phi_0 = (y_0, 0). \tag{C.9}$$

As shown before $\Phi_{k+1}$ is a Cauchy sequence in $\overline{B}_{\delta_1}(\Phi_0)$ with $C^1$ topology. Using (C.8) we get

$$\|D^2\Phi_{k+1}\|_{C^0} \le \|D^2(T(\Phi_k) - T(\Phi_0))\|_{C^0} + \|D^2T(\Phi_0)\|_{C^0} \\ \le C\varepsilon^{\frac{1}{2}}(\|D^2\Phi_k\|_{C^0} + 1).$$

Iterating this inequality shows that $\|D^2\Phi_k\|_{C^0}$ remains bounded as $k \to \infty$. Similarly

$$[D^2T(\Phi_{k+1})]_{\alpha,B_{\delta_0}} \le C\varepsilon^{\frac{1}{2}}([D^2T(\Phi_k)]_{\alpha,B_{\delta_0}} + 1)$$

and iterating this shows that $[D^2T(\Phi_k)]_{\alpha,B_{\delta_0}}$ remains bounded. Therefore the fixed point $\Phi$ actually satisfies $\Phi\in C^{2,\alpha}(\overline{B}_{\delta_0})$. Again using (C.8) and (C.9) we find actually

$$\|\Phi-\Phi_0\|_{C^{2,\alpha}}\le C\varepsilon^{\frac{1}{2}}.$$

*Proof of Lemma 8.3.* Estimate (8.8) follows from the definition and the mean value formula.

Let us prove (8.9):

$$g(z+t_0(X))-g(t_0(X))-\nabla g(t_0(X))z=\int_0^1(1-\tau)g''(t_0(X)+\tau z)[z^2]\,d\tau,$$

so that

$$|B(g)|\le\|g''\|_{L^\infty(B_{2R_X}(t_0(X)))}|z|\le\frac{\|g\|_b}{|X|}|z|\quad\text{in } B_{2R_X}(0).$$

To prove the estimates for $g_i=DG_X(h)[h_i]$ we give first and expression for this function. Next we compute $g_i=DG_X(h)[h_i]$. For this we write $h(y,s)=h(y)+sh_i(y)$, and let us write $\Phi'=\frac{\partial}{\partial s}\Phi$, where $\Phi=(y,g)$. We use the scaled variables as defined in (C.1) and find

$$D_\Phi\mathcal{F}\Phi'=h_i(x+r_0By(t))\nu_{\Sigma_0}(x+r_0By(t))$$

where $D_\Phi\mathcal{F}$ is given in (C.3) and is evaluated at $\Phi,X,h$. From this formula we get

$$g_i(t)=h_i(x+r_0By(t))\frac{m}{D},\tag{C.10}$$

where

$$m=m_0+hm_1$$

and $D$ is the determinant of $D_\Phi\mathcal{F}$ and can be written as

$$D=D_0+hD_1+D_{y_1}hD_2+D_{y_2}hD_3.$$

The functions $D_i$, $m_0$, $m_1$, have the following expressions:

$$D_0=\nu_{\Sigma_0}^{(1)}(X)(B_{21}D_{y_2}F-B_{22}D_{y_1}F)-\nu_{\Sigma_0}^{(2)}(X)(B_{11}D_{y_2}F-B_{12}D_{y_1}F)+\nu_{\Sigma_0}^{(3)}(X)$$

$$\begin{aligned}D_1=\nu_{\Sigma_0}^{(1)}(X)\big[&B_{21}D_{y_2}\nu_{\Sigma_0}^{(3)}+D_{y_2}FD_{y_1}\nu_{\Sigma_0}^{(2)}+D_{y_1}\nu_{\Sigma_0}^{(2)}D_{y_2}\nu_{\Sigma_0}^{(3)}\\&-D_{y_1}FD_{y_2}\nu_{\Sigma_0}^{(2)}-B_{22}D_{y_1}\nu_{\Sigma_0}^{(3)}-D_{y_1}\nu_{\Sigma_0}^{(3)}D_{y_2}\nu_{\Sigma_0}^{(2)}\big]\\-\nu_{\Sigma_0}^{(2)}(X)\big[&B_{11}D_{y_2}\nu_{\Sigma_0}^{(3)}+D_{y_1}\nu_{\Sigma_0}^{(1)}D_{y_2}F+D_{y_1}\nu_{\Sigma_0}^{(1)}D_{y_2}\nu_{\Sigma_0}^{(3)}\\&-D_{y_1}FD_{y_2}\nu_{\Sigma_0}^{(1)}-B_{12}D_{y_1}\nu_{\Sigma_0}^{(3)}-D_{y_1}\nu_{\Sigma_0}^{(3)}D_{y_2}\nu_{\Sigma_0}^{(1)}\big]\\+\nu_{\Sigma_0}^{(3)}(X)\big[&B_{11}D_{y_2}\nu_{\Sigma_0}^{(2)}+B_{22}D_{y_1}\nu_{\Sigma_0}^{(1)}+D_{y_1}\nu_{\Sigma_0}^{(1)}D_{y_2}\nu_{\Sigma_0}^{(2)}\\&-B_{21}D_{y_2}\nu_{\Sigma_0}^{(1)}-B_{12}D_{y_1}\nu_{\Sigma_0}^{(2)}-D_{y_1}\nu_{\Sigma_0}^{(2)}D_{y_2}\nu_{\Sigma_0}^{(1)}\big],\end{aligned}$$

where all functions are evaluated at $x+r_0By(t)$ if not explicitly written;

$$\begin{aligned}D_2&=D_{y_2}F(\nu_{\Sigma_0}^{(1)}(X)\nu_{\Sigma_0}^{(2)}-\nu_{\Sigma_0}^{(2)}(X)\nu_{\Sigma_0}^{(1)})+B_{22}(\nu_{\Sigma_0}^{(3)}(X)\nu_{\Sigma_0}^{(1)}-\nu_{\Sigma_0}^{(1)}(X)\nu_{\Sigma_0}^{(3)})\\&\quad+B_{12}(\nu_{\Sigma_0}^{(2)}(X)\nu_{\Sigma_0}^{(3)}-\nu_{\Sigma_0}^{(3)}(X)\nu_{\Sigma_0}^{(2)})\\D_3&=-D_{y_1}F(\nu_{\Sigma_0}^{(1)}(X)\nu_{\Sigma_0}^{(2)}-\nu_{\Sigma_0}^{(2)}(X)\nu_{\Sigma_0}^{(1)})+B_{11}(\nu_{\Sigma_0}^{(3)}(X)\nu_{\Sigma_0}^{(2)}-\nu_{\Sigma_0}^{(2)}(X)\nu_{\Sigma_0}^{(3)})\\&\quad+B_{21}(\nu_{\Sigma_0}^{(1)}(X)\nu_{\Sigma_0}^{(3)}-\nu_{\Sigma_0}^{(3)}(X)\nu_{\Sigma_0}^{(1)}).\end{aligned}$$

For $m_0$, $m_1$ we have a similar expressions

$$m_0=\nu_{\Sigma_0}^{(1)}(B_{21}D_{y_2}F-B_{22}D_{y_1}F)-\nu_{\Sigma_0}^{(2)}(B_{11}D_{y_2}F-B_{12}D_{y_1}F)+\nu_{\Sigma_0}^{(3)}$$

$$
\begin{aligned}
m_1 = \nu_{\Sigma_0}^{(1)} \big[ & B_{21} D_{y_2} \nu_{\Sigma_0}^{(3)} + D_{y_2} F D_{y_1} \nu_{\Sigma_0}^{(2)} + D_{y_1} \nu_{\Sigma_0}^{(2)} D_{y_2} \nu_{\Sigma_0}^{(3)} \\
& - D_{y_1} F D_{y_2} \nu_{\Sigma_0}^{(2)} - B_{22} D_{y_1} \nu_{\Sigma_0}^{(3)} - D_{y_1} \nu_{\Sigma_0}^{(3)} D_{y_2} \nu_{\Sigma_0}^{(2)} \big] \\
- \nu_{\Sigma_0}^{(2)} \big[ & B_{11} D_{y_2} \nu_{\Sigma_0}^{(3)} + D_{y_1} \nu_{\Sigma_0}^{(1)} D_{y_2} F + D_{y_1} \nu_{\Sigma_0}^{(1)} D_{y_2} \nu_{\Sigma_0}^{(3)} \\
& - D_{y_1} F D_{y_2} \nu_{\Sigma_0}^{(1)} - B_{12} D_{y_1} \nu_{\Sigma_0}^{(3)} - D_{y_1} \nu_{\Sigma_0}^{(3)} D_{y_2} \nu_{\Sigma_0}^{(1)} \big] \\
+ \nu_{\Sigma_0}^{(3)} \big[ & B_{11} D_{y_2} \nu_{\Sigma_0}^{(2)} + B_{22} D_{y_1} \nu_{\Sigma_0}^{(1)} + D_{y_1} \nu_{\Sigma_0}^{(1)} D_{y_2} \nu_{\Sigma_0}^{(2)} \\
& - B_{21} D_{y_2} \nu_{\Sigma_0}^{(1)} - B_{12} D_{y_1} \nu_{\Sigma_0}^{(2)} - D_{y_1} \nu_{\Sigma_0}^{(2)} D_{y_2} \nu_{\Sigma_0}^{(1)} \big].
\end{aligned}
$$

Let us rewrite (C.10) as

$$
g_i = \bar g_i + \tilde g_i \tag{C.11}
$$

where

$$
\begin{aligned}
\bar g_i &= h_i(x + r_0 B y(t)) \\
\tilde g_i(t) &= h_i(x + r_0 B y(t)) \frac{(m_0 - D_0) + (m_1 - D_1) h - D_{y_1} h D_2 - D_{y_2} h D_3}{D_0 + h D_1 + D_{y_1} h D_2 + D_{y_2} h D_3}.
\end{aligned}
$$

These expressions imply the following estimate (after changing variables back from (C.1)):

$$
\|\bar g_i\|_b \le C \|h_i\|_*
$$

where $\| \ \|_b$ is the norm (8.7). Therefore

$$
|B(\bar g_i)(X,z)| \le C \frac{\|h_i\|_*}{|X|} |z|.
$$

Moreover we can write $\tilde g_i$ as

$$
\tilde g_i(t) = h_i(x + r_0 B y(t)) Q(X, t, h, D_t h)
$$

where

$$
Q(X,t,h,\xi) = \frac{(m_0 - D_0) + (m_1 - D_1) h - \xi_1 D_2 - \xi_2 D_3}{D_0 + h D_1 + \xi_1 D_2 + \xi_2 D_3}. \tag{C.12}
$$

Let us use the notation

$$
\begin{aligned}
\tilde h(t) &= h(x + r_0 B y(t)), \quad \tilde h_i(t) = h_i(x + r_0 B y(t)), \\
\tilde Q(t,\xi) &= Q(X, t, \tilde h(t), \xi)
\end{aligned}
$$

so that

$$
\tilde g_i(t) = \tilde h_i(t) \tilde Q(t, D_t \tilde h(t)).
$$

Observe that $\tilde Q(t_0(X), \xi) = 0$, $D_\xi \tilde Q(t_0(X), \xi) = 0$. Then we have

$$
\begin{aligned}
|B(\tilde g_i)(X,z)| &= \frac{1}{|z|} \Big| \tilde h_i(z + t_0(X)) \tilde Q(z + t_0(X), D_t \tilde h(z + t_0(X))) \\
&\qquad - \tilde h_i(t_0(X)) D_t \tilde Q(t_0(X), D_t \tilde h(t_0(X))) z \Big| \\
&\le A_1 + A_2 + A_3
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \frac{1}{|z|} \Big| (\tilde h_i(z + t_0(X)) - \tilde h_i(t_0(X))) \tilde Q(z + t_0(X), D_t \tilde h(z + t_0(X))) \Big| \\
A_2 &= \frac{1}{|z|} |\tilde h_i(t_0(X))| \Big| \tilde Q(z + t_0(X), D_t \tilde h(z + t_0(X))) - \tilde Q(z + t_0(X), D_t \tilde h(t_0(X))) \Big| \\
A_3 &= \frac{1}{|z|} |\tilde h_i(t_0(X))| \Big| \tilde Q(z + t_0(X), D_t \tilde h(t_0(X))) - D_t \tilde Q(t_0(X), D_t \tilde h(t_0(X))) z \Big|.
\end{aligned}
$$

We then have for $z \in B_{2R_X}(t_0)$

$$A_1 \le C\|\tilde{h}_i\|_{B_{2R_X}(t_0)}|z| \le C\|h_i\|_* \frac{|z|}{|X|}.$$

For $A_2$

$$\begin{aligned} A_2 &\le \frac{1}{|z|}\int_0^1 |D_\xi \tilde{Q}(z+t_0(X), D_t\tilde{h}(\tau z + t_0(X)))||D_{tt}\tilde{h}(\tau z + t_0(X))|\, d\tau |z| \\ &\le C\|h_i\|_* \frac{|z|}{|X|} \end{aligned}$$

since $|D_\xi \tilde{Q}(z+t_0(X), D_t\tilde{h}(\tau z + t_0(X)))| \le C|z|$ for this range of arguments. Finally also

$$A_3 \le C\|h_i\|_* \frac{|z|}{|X|}$$

because

$$\left|\tilde{Q}(z+t_0(X), D_t\tilde{h}(t_0(X))) - D_t\tilde{Q}(t_0(X), D_t\tilde{h}(t_0(X)))z\right| \le \frac{|z|^2}{|X|}$$

in this range of argument. This establishes (8.11).

The estimate (8.8) and (8.10) are direct since the expression $A_t$ involves only one derivative the function where it is applied to, and we have control of one derivative of $g_i$ directly from (C.11). □

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