MULTIPLE-END SOLUTIONS TO THE ALLEN-CAHN EQUATION IN $\mathbb{R}^2$

MANUEL DEL PINO, MICHAŁ KOWALCZYK, FRANK PACARD, AND JUNCHENG WEI

Abstract. We construct new class of entire solutions of the Allen-Cahn equation $\Delta u+(1-u^2)u=0$, in $\mathbb{R}^2(\sim\mathbb{C})$. Given $k\geq 1$, we find a family of solutions whose zero level sets are, away from a compact set, asymptotic to $2k$ straight lines (which we call the "ends"). These solutions have the property that there exists $\theta_0<\theta_1<\ldots<\theta_{2k}=\theta_0+2\pi$ such that $\lim_{r\to+\infty}u(re^{i\theta})=(-1)^j$ uniformly in θ on compacts of (θ_j,θ_{j+1}), for $j=0,\ldots 2k-1$, they complement the solutions with dihedral symmetry which have been obtained in [14].

1. Introduction and statement of main results

This paper deals with the construction of a new class of entire solutions for the semilinear elliptic equation in $\mathbb{R}^N$,

$$\Delta u+(1-u^2)\,u=0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

known as the Allen-Cahn equation. This problem has its origin in the *gradient theory of phase transitions* [1], a model in which two distinct phases (represented by the values $u=\pm 1$) try to coexist in a domain Ω while minimizing their interaction which is proportional to the $(N-1)$- dimensional volume of the interface. Idealizing the phase as a regular function which takes values close to ± 1 in most of the domain, except for a narrow transition layer of width ε, one defines the Allen-Cahn energy,

$$J_\varepsilon(u)=\frac{\varepsilon}{2}\int_\Omega|\nabla u|^2\,dx+\frac{1}{4\varepsilon}\int_\Omega(1-u^2)^2\,dx\,.$$

whose critical points satisfy the Euler-Lagrange equation

$$\varepsilon^2\Delta u+(1-u^2)\,u=0 \quad \text{in} \quad \Omega. \tag{1.2}$$

Replacing $u(x)$ by $u(\varepsilon x)$ we obtain the equation

$$\Delta u+(1-u^2)\,u=0 \quad \text{in} \quad \varepsilon^{-1}\,\Omega. \tag{1.3}$$

Therefore, equation (1.1) appears as the limit problem in the blow up analysis of (1.2) as ε tends to 0. The relation between interfaces of least volume and critical points of J_ε was first established by Modica in [23]. Let us briefly recall the main results in this direction : If u_ε is a family of *local minimizers* of J_ε for which

$$\sup_{\varepsilon>0}J_\varepsilon(u_\varepsilon)<+\infty, \tag{1.4}$$

Date: November 17, 2008.

1991 *Mathematics Subject Classification.* 35J25, 35J20, 35B33, 35B40.

Key words and phrases. De Giorgi conjecture, Allen-Cahn equation, minimal graph, counterexample, infinite-dimensional Liapunov-Schmidt reduction.

then, up to a subsequence, u_ε converges in L^1 to $\chi_\Lambda - \chi_{\Lambda^c}$, where $\partial\Lambda$ has minimal perimeter. Here χ_Λ (resp. χ_{Λ^c}) is the characteristic function of the set Λ (resp. $\Lambda^c = \Omega - \Lambda$). Moreover, $J_\varepsilon(u_\varepsilon) \to c\,\mathcal{H}^{N-1}(\partial\Lambda)$ where c is a universal constant.

For *critical points* u_ε of J_ε which satisfy (1.4), a related assertion is proven in [17]. In this case, the convergence of the interface holds with certain integer multiplicity to take into account the possibility of multiple transition layers converging to the same minimal hypersurface.

These results enlighten a strong link between solutions to equation (1.1) and the theory of minimal hypersurfaces, which has been widely explored in the literature. For example, solutions concentrating along non-degenerate, minimal hypersurfaces of a compact manifold were found in [25] (see also [19]). As far as multiple transition layers are concerned, given a minimal hypersurface Γ (subject to some additional property on the sign of the potential of the Jacobi operator about Γ, which holds on manifolds with positive Ricci curvature) and given an integer $k \geq 1$, solutions of (1.2) with multiple transitions near Γ were built in [27] (see [11] for the 2-dimensional case, and [9] for the euclidean case), in such a way that $J_\varepsilon(u_\varepsilon) \to k\,c\,\mathcal{H}^{N-1}(\Gamma)$.

As already mentioned, entire solutions of (1.1) arise as limits of blown up solutions of (1.2). Conversely, given a solution u of (1.1), the scalings $u_\varepsilon(x) = u(x/\varepsilon)$, satisfy equation (1.2) on any domain Ω and condition (1.4) is equivalent to the existence of a constant $C > 0$ such that

$$I_R(u) := \frac{1}{2}\int_{B_R(0)} |\nabla u|^2\,dx + \frac{1}{4}\int_{B_R(0)} (1-u^2)^2\,dx \,\leq\, C R^{N-1} \quad \text{for all } R > 0. \tag{1.5}$$

Entire solutions with this type of energy growth are thus of special importance in further understanding the links between the Allen-Cahn equation and theory of minimal hypersurfaces.

This paper deals with the construction of a new, rather unexpected class of entire solutions of equation (1.1) satisfying the energy growth condition (1.5). Recall that a basic solution satisfying (1.5) is the following : Let H denote the unique solution to the problem

$$H'' + (1 - H^2)\,H = 0, \quad \text{with} \quad H(\pm\infty) = \pm 1 \quad \text{and} \quad H(0) = 0\,. \tag{1.6}$$

Then, for all $b \in \mathbb{R}^N$ with $|b| = 1$ and for all $a \in \mathbb{R}$, the function $u(x) = H(b \cdot x + a)$ solves (1.1) and satisfies condition (1.5). A celebrated conjecture due to De Giorgi states that these solutions are the only ones that are bounded and monotone in one direction for dimension $N \leq 8$. Let us recall that monotonicity is related to the fact that solutions u are local minimizers.

In dimensions $N = 2, 3$ De Giorgi's conjecture has been proven in [13, 3] and (under some extra assumption) in the remaining dimensions in [26]. Establishing that condition (1.5) holds is a key element in the proof of De Giorgi's conjecture in dimensions $N = 2, 3$. When $N = 2$, the monotonicity assumption can even be replaced by a weaker stability assumption [15]. Finally, counterexamples in dimension $N \geq 9$ have recently been built in [10], using the existence of non trivial minimal graphs in higher dimensions.

In light of these results, it is natural to study the set of entire solutions to (1.1) satisfying (1.5). The functions $u(x) = H(b \cdot x + a)$ are obvious solutions.

In dimension $N = 2$, nontrivial examples (whose nodal set is the union of two perpendicular lines) were built in [5] using the following strategy : A positive solution to (1.1) in the quadrant $\{x = (x_1, x_2) \, : \, x_1, x_2 > 0\}$ with zero boundary conditions is built by constructing appropriate super and subsolutions. This solution is then extended by odd reflections to yield a solution u_2 to (1.1) in entire $\mathbb{R}^2$. Function u_2 is a solution of (1.1), whose 0-level set is the union of the two axis, satisfies property (1.5). It can easily be generalized to obtain solutions with dihedral symmetry by considering the corresponding solution within the sector $\{x = (r\, cos\theta, r \sin\theta) \, : \, r > 0\, , \theta \in (0, \pi/k)\}$, see [14], and extending it by k consecutive reflections to yield a solution u_k. The zero level set of u_k is constituted outside any ball by $2k$ infinite half lines with dihedral symmetry. To our knowledge, no other nontrivial examples of solutions satisfying (1.5) are known in dimension $N = 2$ (up to the action of rigid motions).

We will assume from now on that the dimension is equal to $N = 2$.

Definition 1. *We will say that u, solution of (1.1), has* $2k$-ends *if, away from a compact, its nodal set is given by $2k$ connected curves which are asymptotic to $2k$ infinite half lines $b_j \cdot x + a_j = 0$, $j = 1, \ldots, 2k$ (for some choice of $b_j \in \mathbb{R}^2$, $|b_j| = 1$ and $a_j \in \mathbb{R}$) and if, along these curves, the solution is asymptotic to $\eta_j\, H(b_j \cdot x + a_j)$, with $\eta_j = \pm 1$.*

Given any $k \geq 1$, we prove in this paper that the space of $2k$-ended solutions of (1.1) is not empty and in fact contains a smooth finite dimensional family of non congruent solutions. In a forthcoming paper [6] we show that the solutions we construct in the present paper belong to some smooth $2k$-parameter family of $2k$-ended solutions of (1.1).

Given $\alpha > 0$ small enough, our main result states the existence of a a function u_α, solution of (1.1) in $\mathbb{R}^2$, which can be described as follows :

$$u_\alpha(x) = \sum_{j=1}^{k} (-1)^{j+1} H\big(x_1 - f_{\alpha j}(x_2)\big) + \sigma_k + o(1),$$

with $x = (x_1, x_2)$ and

$$\sigma_k = -\frac{1}{2}((-1)^k + 1)\,,$$

where H is the solution of (1.6). The functions $f_{\alpha j}(z)$ are even and satisfy

$$f_{\alpha j}(z) \, = \, \alpha\, a_j\, |z| + b_{j\alpha} + O(e^{-a\,\alpha\,|z|})\,,$$

as $|z| \to +\infty$. Here $a_1 < a_2 < \cdots < a_k$ do not depend on α while $b_{j\alpha}$ are constants depending on both j and α, and finally $a > 0$.

To state our result in precise way, we need to introduce the *Toda system*

$$c\, f_j'' - \left(e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}\right) = 0, \tag{1.7}$$

for $j = 1, \ldots, k$, where the constant $c > 0$ is given by

$$c = \frac{\displaystyle\int_{\mathbb{R}} (H')^2\, dz}{3 \displaystyle\int_{\mathbb{R}} (1 - H^2) H' e^{-\sqrt{2} z}\, dz}.$$

Here $'$ denotes the derivative with respect to the variable z and the solutions are supposed to be defined on $\mathbb{R}$. We agree that $f_0 \equiv -\infty$ while $f_{k+1} \equiv +\infty$.

Given the symmetries of the system it is natural to look for solutions of (1.7) which are even, namely

$$f_j(z) = f_j(-z) \quad \text{for all } z \in \mathbb{R}. \tag{1.8}$$

As will be discussed later, given initial conditions

$$f_j(0) = x_{0j}, \quad \text{and} \quad f_j'(0) = 0, \tag{1.9}$$

for $j = 1, \dots k$, with

$$x_{0j} < x_{0j+1}, \tag{1.10}$$

the system (1.7) has a unique solution $\mathbf{f} = (f_1, \dots, f_k)$ which is defined for all $z \in \mathbb{R}$ and which satisfies (1.8). Moreover, thanks to (1.10), we have

$$f_j(z) < f_{j+1}(z), \quad \forall z \in \mathbb{R}\,. \tag{1.11}$$

Finally, for all $j = 1, \dots, k$,

$$f_j(z) = a_j|z| + b_j + O(e^{-\theta_j|z|}) \tag{1.12}$$

as $|z|$ tends to $+\infty$, for some $a_j > 0$, $b_j \in \mathbb{R}$ and $\theta_j > 0$. We will show also that there exists $\vartheta > 0$ depending only on the initial conditions such that

$$a_{j+1} - a_j > \vartheta, \quad \text{and} \quad \theta_j > \vartheta\,.$$

Let us observe that, for each $\alpha > 0$, the vector function $\mathbf{f}_\alpha$, whose components are defined by

$$f_{\alpha j}(z) = f_j(\alpha z) - \frac{1}{\sqrt{2}}\Big(j - \frac{k+1}{2}\Big)\log\alpha\,, \tag{1.13}$$

corresponds to a solution of (1.7) which satisfies (1.8), (1.11) and

$$f'_{\alpha j}(+\infty) \;=\; a_j\,\alpha\,, \tag{1.14}$$

for all $j = 1, \dots, k$.

Granted the above we have the :

Theorem 1.1. *For all $\alpha > 0$ sufficiently small, there exists u_α a solution of equation (1.1) satisfying*

$$u_\alpha(x_1, x_2) = u_\alpha(x_1, -x_2) \quad \textit{for all} \quad (x_1, x_2) \in \mathbb{R}^2$$

and which has the form

$$u_\alpha(x_1, x_2) = \sum_{j=1}^{k}(-1)^{j+1}H\big(x_1 - f_{\alpha j}(x_2)\big) + \sigma_k + o(1), \qquad \sigma_k = -\frac{1}{2}(1 + (-1)^k).$$

Here $o(1) \to 0$ as $\alpha \to 0$ uniformly as $|x|$ tends to $+\infty$.

In other words, to each symmetric solution of the Toda system, we have found a one parameter family of $2k$-ended solutions of (1.1) which depend on a small parameter $\alpha > 0$. The *ends* of the solutions we construct have slopes of order α, and, as α tends to 0, they converge on compacts to the straight line $x_1 = 0$.

Our result raises some interesting questions :

(i) - The classification of entire solutions of (1.1) with growth condition (1.5) remains an important and unexplored problem. In dimension $N = 2$, we believe that these solutions are precisely the solutions with finitely many ends. In addition, there is strong evidence that these solutions correspond to the solutions of (1.1) with finite Morse index (presumably, in dimension $N = 2$, their Morse index is equal to $k - 1$ if the solution has $2k$ ends).

(ii) - Still in dimension $N = 2$, the understanding of the moduli space of all $2k$-ended solutions is far from complete : the result in Theorem 1.1 (see also [5]) implies that this space is non empty and contained smooth families of solutions. Moreover, the result of [6] shows that this moduli space has formal dimension equal to $2k$ (the formal dimension is the dimension of the moduli space close to any non degenerate solution). Let us also mention that some *balancing conditions* on the directions of the ends is available (see [14]), it states that the sum of the unit vectors of the ends (oriented toward the ends) has to be 0.

(iii) - It is tempting to conjecture that the solution u_k (whose nodal set has dihedral symmetry) and the solutions given in Theorem 1.1 belong to the same connected component of the moduli space of $2k$-ended solutions.

(iv) - Observe that the solutions we construct still have some symmetry since they are invariant under the reflexion with respect to the x_1 axis. Nevertheless, using Theorem 1.1 together with the result of [6], it is possible to prove (in dimension $N = 2$) the existence of solutions of 1.1 which have no symmetry provided the number of ends is larger than or equal to 6. We believe that all solutions with 2 ends are given (up to the action of rigid motion) by H while solutions with 4 ends are symmetric with respect to reflection through two perpendicular lines.

These questions hint towards a program generalizing De Giorgi's conjecture, on the classification finite Morse index entire solutions of (1.1).

Let us briefly comment on the proof of Theorem 1.1. The method is based on an infinite dimensional form of Lyapunov Schmidt reduction, conceptually related with the method in [8], see also [20, 21]. The proof uses some ideas of the proof of the existence of entire solutions to nonlinear Schrödinger type equations have been recently built in [7] and [22].

2. The approximate solution

2.1. **Model for the transition layers.** In order to define the approximate solution we first need to describe possible locations of its transition layers. Let us recall that there exists a unique solution to the Toda system (1.7) with the initial data (1.9). According to the results in [18], assuming (1.10) such solution satisfies (1.14) and (1.11). Let us consider fixed, smooth functions

$$f_{j\alpha}(z) = f_j(\alpha z) + \sqrt{2}\Big(j - \frac{k+1}{2}\Big)\log\frac{1}{\alpha}, \quad j = 1, \dots, k \tag{2.1}$$

such that

$$\begin{aligned} & f_j(z) = a_j z + b_j + \tilde{f}_j(z), \quad |z| \gg 1, \\ & \text{where } a_j - a_{j-1} > \theta > 0, \quad \text{and } \|\tilde{f}_j e^{\theta|z|}\|_{C^4(\mathbb{R})} < M, \end{aligned} \tag{2.2}$$

with some fixed constants $\theta > 0$ and $M > 0$. From now on we will assume that f_j's are solutions to the Toda system (1.7) and that conditions (1.8)–(1.12) hold.

This will guarantee that all our estimates below (with constants independent on α) are valid if we choose α small while keeping θ, M fixed. It is convenient to denote $f_0 = f_{0\alpha} = -\infty$ and $f_{k+1} = f_{k+1\alpha} = \infty$.

Let

$$\gamma_{j\alpha} = \{(x,z) \mid x = f_{j\alpha}(z)\} \subset \mathbb{R}^2,$$

We observe that the signed curvature of $\gamma_{j\alpha}$ is:

$$\varkappa_{j\alpha}(z) = -\frac{\alpha^2 f_j''}{(1+\alpha^2(f_j')^2)^{3/2}},$$

and the radius of curvature is

$$R_{j\alpha}(z) = \frac{1}{|\varkappa_{j\alpha}(z)|} \geq M\alpha^{-2} e^{\theta\alpha|z|}, \quad |z| \gg 1. \tag{2.3}$$

This means that Fermi coordinates around $\gamma_{j\alpha}$ are well defined in a neighborhood

$$N_{j\alpha} = \bigcup_{z\in\mathbb{R}} \big\{(x,z) \mid \operatorname{dist}(\gamma_{j\alpha},(x,z)) < \frac{1}{4} R_{j\alpha}(z)\big\},$$

where $R_{j\alpha}$ satisfies (2.3). We observe that from (2.2) and (2.3) it follows that

$$\gamma_{n\alpha} \subset N_{j\alpha}, \quad \forall n, j = 1,\ldots,k. \tag{2.4}$$

To define Fermi coordinates precisely let $n_{j\alpha}(z)$ be the unit normal to $\gamma_{j\alpha}$, where we fix the orientation on $\gamma_{j\alpha}$ in such a way that $n_{j\alpha}\cdot(1,0) \geq 0$ (so that $(n_{j\alpha}, \tau_{j\alpha})$, where $\tau_{j\alpha}$ is he unit tangent, is positively oriented). We have

$$n_{j\alpha}(z) = \frac{\big(1, -\alpha f_j'(\alpha z)\big)}{\sqrt{1+\alpha^2\big(f_j'(\alpha z)\big)^2}}, \tag{2.5}$$

Then, given $(x,z) \in N_{j\alpha}$, the Fermi coordinates $(\mathtt{x}_j, \mathtt{z}_j)$ are defined by:

$$(x,z) = (f_{j\alpha}(\mathtt{z}_j), \mathtt{z}_j) + \mathtt{x}_j n_{j\alpha}(\mathtt{z}_j). \tag{2.6}$$

The Fermi coordinate $\mathtt{x}_j$ associated with a fixed curve $\gamma_{j\alpha}$ represents the signed distance to $\gamma_{j\alpha}$. In what follows we will need the following elementary fact:

$$\mathtt{x}_n = \big(1+O(\alpha^2)\big)\mathtt{x}_j + \alpha(a_n - a_j)\mathtt{z}_n - \sqrt{2}(n-j)\log\frac{1}{\alpha} + O(1), \tag{2.7}$$

where $O(\cdot)$ terms are uniform for α small. Notice that (2.7) follows directly from the definition of the Fermi coordinates and formulas (2.1). We will derive an expression of Δ in the Fermi of every $f_{j\alpha}$. Using (2.6) and denoting $X = (x,z)$ we have:

$$\begin{aligned} X_{\mathtt{x}_j} &= n_{j\alpha}, \\ X_{\mathtt{z}_j} &= \big(\alpha f_j', 1\big) + \mathtt{x}_j n_{j\alpha,\mathtt{z}_j}, \end{aligned}$$

where $n_{j\alpha}$ is defined in (2.5). From this we get the following expression for the metric

$$\mathtt{g}_j = \begin{pmatrix} 1 & 0 \\ 0 & \mathtt{g}_{j,22} \end{pmatrix}, \quad \mathtt{g}_{j,22} = X_{\mathtt{z}_j}\cdot X_{\mathtt{z}_j},$$

where

$$X_{\mathtt{z}_j}\cdot X_{\mathtt{z}_j} = [1+\alpha^2(f_j')^2]\Big[1+2\mathtt{x}_j\frac{\varkappa_{j\alpha}}{\sqrt{1+\alpha^2(f_j')^2}}\Big] + \mathtt{x}_j^2 n_{j\alpha,\mathtt{z}_j}\cdot n_{j\alpha,\mathtt{z}_j},$$

$$n_{j\alpha,\mathtt{z}_j} = \alpha^2\frac{(\alpha f_j' f_j'', f_j'')}{\big(1+\alpha^2(f_j')^2\big)^{3/2}}$$

For future references we observe that

$$\mathtt{g}_{j,22} = 1 + O(\alpha^2) + O(\alpha^2|\mathtt{x}_j|e^{-\theta\alpha|\mathtt{z}_j|}) + O(\alpha^4|\mathtt{x}_j|^2 e^{-2\theta\alpha|\mathtt{z}_j|}), \quad \text{in } N_{j\alpha}.$$

Also we notice that $\det(\mathtt{g}_j) = \mathtt{g}_{j,22}$ and $g_j^{22} = \frac{1}{\det \mathtt{g}_j}$. Then we have in $N_{j\alpha}$:

$$\begin{aligned}\Delta_{x,z} &= \frac{1}{\sqrt{\det(\mathtt{g}_j)}}\partial_{\mathtt{x}_j}(\sqrt{\det(\mathtt{g}_j)}\partial_{\mathtt{x}_j}) + \frac{1}{\sqrt{\det(\mathtt{g}_j)}}\partial_{\mathtt{z}_j}(\sqrt{\det(\mathtt{g}_j)}\mathtt{g}_j^{22}\partial_{\mathtt{z}_j}) \\ &= \Delta_{\mathtt{x}_j,\mathtt{z}_j} - \alpha^2 f_j''\partial_{\mathtt{x}_j} + \alpha^2 B_{1j}(\mathtt{x}_j,\mathtt{z}_j) + \alpha^3 B_{2j}(\mathtt{x}_j,\mathtt{z}_j),\end{aligned} \tag{2.8}$$

where B_{jm}, $m=1,2$ are second order differential operators:

$$\begin{aligned} B_{1j}(\mathtt{x}_j,\mathtt{z}_j) &= -\frac{1}{2}\big[(f_j')^2 - 2\mathtt{x}_j f_j'' + \alpha^2\mathtt{x}_j^2(f_j'')^2\big]\partial_{\mathtt{z}_j\mathtt{z}_j} \\ B_{2j}(\mathtt{x}_j,\mathtt{z}_j) &= O(e^{-\theta\alpha|z|})\partial_{\mathtt{x}_j} + e^{-\theta\alpha|z|}O(1+\alpha|\mathtt{x}_j|)\partial_{\mathtt{z}_j} \\ &\quad + O(1+|\mathtt{x}_j|e^{-\theta\alpha|z|})\partial_{\mathtt{z}_j\mathtt{z}_j}\end{aligned} \tag{2.9}$$

2.2. **The ansatz.** In the sequel we will often make use of the following formulas for the heteroclinic H:

$$\begin{aligned} H(x)\,-\,1 &= \,-A_0\,e^{-\sqrt{2}\,|x|}\,+\,O\big(e^{-2\sqrt{2}\,|x|}\big), \quad x>1, \\ H(x)\,+\,1 &= \,A_0\,e^{-\sqrt{2}\,|x|}\,+\,O\big(e^{-2\sqrt{2}\,|x|}\big), \quad x<-1, \\ H'(x) &= \,\sqrt{2}\,A_0\,e^{-\sqrt{2}\,|x|}\,+\,O\big(e^{-2\sqrt{2}\,|x|}\big), \quad |x|>1,\end{aligned} \tag{2.10}$$

Let $h_j(z)$, $j=1,\dots,k$ be given even functions of z such that

$$\|h_j' e^{\theta|z|}\|_{C^1(\mathbb{R})} + \|h_j\|_\infty \le \bar\delta, \tag{2.11}$$

where $\bar\delta$ is a small constant such that

$$0<\bar\delta<\frac{\alpha^2}{4}\min R_{j\alpha}(z), \quad j=1,\dots,k.$$

As we will see later on in reality we have $\bar\delta = o(1)$, as $\alpha\to 0$. We will set $h_{j\alpha}(z) = h_j(\alpha z)$. Let $\eta(t)$ be a cutoff function such that $\eta(t)=1$, $|t|\le 1$, $\eta(t)=0$, $|t|>2$. Then given $f_{j\alpha}$, $j=1,\dots,k$, with f_j satisfying (2.2) we define:

$$H_j(x,z) = \begin{cases} (-1)^{j+1}\eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\big[H(\mathtt{x}_j - h_{j\alpha}(z)) - 1\big] + (-1)^{j+1}, & \mathtt{x}_j>0, \\ (-1)^{j+1}\eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\big[H(\mathtt{x}_j - h_{j\alpha}(z)) + 1\big] - (-1)^{j+1}, & \mathtt{x}_j<0 \end{cases} \tag{2.12}$$

Finally we define a multiple-end approximate solution of (1.1) to be

$$w(x,z) = \sum_{j=1}^k H_j(x,z) + \sigma_k, \qquad \sigma_k = \frac{1}{2}((-1)^{k+1}-1). \tag{2.13}$$

In the sequel we will use weighted norms for functions defined in $\mathbb{R}^2$. To define them we let $\sigma > 0$ to be a fixed constant. We set:

$$\|\phi\|_{\sigma,\theta\alpha,*} := \Big\| \Big(\sum_{j=1}^{k} e^{-\sigma|x-f_{j\alpha}(z)|-\theta\alpha|z|} \Big)^{-1} \phi \Big\|_\infty. \tag{2.14}$$

Constant θ is in general the same one as in (2.2). A more precise upper bound on the constant σ will be determined later on. Let us observe that in the set where the Fermi coordinate for a fixed j is well defined, say in $N_{j\alpha}$ we have that:

$$\begin{aligned} x - f_{j\alpha} &= \frac{\mathtt{x}_j}{\sqrt{1+\alpha^2(f_j'(z))^2}}, \\ z &= \mathtt{z}_j - \frac{\alpha \mathtt{x}_j f_j'(\alpha z)}{\sqrt{1+\alpha^2(f_j'(z))^2}}. \end{aligned} \tag{2.15}$$

It follows that for the exponential weights in (2.14) we have:

$$Ce^{-\sigma'|\mathtt{x}_j|-\theta\alpha|\mathtt{z}_j|} \le e^{-\sigma|x-f_{j\alpha}(z)|-\theta\alpha|z|} \le Ce^{-\sigma''|\mathtt{x}_j|-\theta\alpha|\mathtt{z}_j|}, \tag{2.16}$$

with $\sigma'' < \sigma < \sigma'$. We get also from (2.15) that in $N_{j\alpha}$ we have

$$\frac{\partial z}{\partial \mathtt{z}_j} = 1 + O(\alpha^2|\mathtt{x}_j|e^{-\alpha\theta|z|}). \tag{2.17}$$

Our next goal is to compute the size of the error, namely:

$$S(w) \equiv -\Delta w - w(1-w^2),$$

in the norm $\|\cdot\|_{\sigma,\theta,*}$. We have:

$$\Delta_{x,z} w = \sum_{j=1}^{k} \Delta_{x,z} H_j,$$

and, for a fixed j,

$$\begin{aligned} \Delta_{x,z} H_j = (-1)^{j+1}\eta\Big(\frac{\mathtt{x}_j}{2R_{j\alpha}}\Big)\Delta_{x,z} H(\mathtt{x}_j - h_{j\alpha}(z)) \\ + (-1)^{j+1}\nabla_{x,z} H(\mathtt{x}_j - h_{j\alpha}(z)) \cdot \nabla_{x,z}\eta\Big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\Big) \\ + (-1)^{j+1}[H(\mathtt{x}_j - h_{j\alpha}(z)) \pm 1]\Delta_{x,z}\eta\Big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\Big). \end{aligned}$$

Using (2.8) we get

$$\begin{aligned} \Delta_{x,z} H(\mathtt{x}_j - h_{j\alpha}(z)) = H''\big(1+\alpha^2(h_j')^2\big) - \alpha^2(h_j'' + f_j'')H' \\ + \alpha^2 B_1(\mathtt{x}_j,\mathtt{z}_j)[H] + \alpha^3 B_2(\mathtt{x}_j,\mathtt{z}_j)[H], \end{aligned} \tag{2.18}$$

where $H = H\big(\mathtt{x}_j - h_{j\alpha}(z)\big)$, $h_{j\alpha}(z) = h_j(\alpha z)$. We have

$$\begin{aligned} |B_1(\mathtt{x}_j,\mathtt{z}_j)[H]| &\le C\alpha^2\big[|(f_j')^2 + |f_j''|\big]\big[|(h_j')^2 + |h_j''|\big]e^{-\frac{\sqrt{2}}{2}|\mathtt{x}_j|} \\ &\le C\alpha^2 e^{-\theta\alpha|\mathtt{z}_j|}e^{-\frac{\sqrt{2}}{2}|\mathtt{x}_j|}, \end{aligned} \tag{2.19}$$

with similar estimate for $B_2(\mathtt{x}_j, \mathtt{z}_j)[H]$. We also observe that

$$\begin{aligned}
&\Big|\nabla_{x,z} H(\mathtt{x}_j - h_{j\alpha}(\mathtt{z}_j)) \cdot \nabla_{x,z}\eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\Big| + \Big|[H(\mathtt{x}_j - h_{j\alpha}(\mathtt{z}_j)) \pm 1]\Delta_{x,z}\eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\Big| \\
&\leq C R_{j\alpha}^{-1} e^{-\sqrt{2}|\mathtt{x}_j|}\Big|\eta'\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\Big| \\
&\quad \leq C\alpha^3 e^{-\theta\alpha|\mathtt{z}_j|} e^{-\frac{\sqrt{2}}{2}|\mathtt{x}_j|}.
\end{aligned} \tag{2.20}$$

For convenience we will denote $F(w) = w(1-w^2)$. Now we will compute

$$w(1-w^2) = F(w) = F\big(\sum_{j=1}^{k} H_j + \sigma_k\big).$$

For every fixed j, $1 \leq j \leq k$, we consider the following set

$$A_j = \Big\{(x,z) \in \mathbb{R}^2 \Big| \operatorname{dist}\big(\gamma_{j\alpha}, (x,z)\big) \leq \min_{n=j\pm 1}\big\{\operatorname{dist}\big(\gamma_{n\alpha}, (x,z)\big)\big\}\Big\}. \tag{2.21}$$

We recall here that formally we have set $\gamma_{0\alpha} = -\infty$, $\gamma_{k+1\alpha} = \infty$.

For $(x,z) \in A_j$, we write

$$\begin{aligned}
F(w) &= F(H_j) + F'(H_j)(w - H_j) + \frac{1}{2}F''(H_j)(w-H_j)^2 + \max_{n\neq j} O(e^{-3\sqrt{2}\,|\mathtt{x}_n|}) \\
&= \sum_{n=1}^{k} F(H_n) + \Big[F'(H_j)(w-H_j) - \sum_{n\neq j} F(H_n)\Big] \\
&\qquad + \frac{1}{2}F''(H_j)(w-H_j)^2 + \max_{n\neq j} O(e^{-3\sqrt{2}\,|\mathtt{x}_n|}).
\end{aligned}$$

Following similar computations in [9], we obtain, for $(x,z) \in A_j$, $j = 1, \ldots, k$

$$\begin{aligned}
F(w) = &\sum_{n=1}^{k} F(H_n) + \frac{1}{2}F''(H_j)(w-H_j)^2 + 3(1-H_j^2)(w-H_j) \\
&- \frac{1}{2}\sum_{n\neq j} F''(\sigma_{jn})(\sigma_{jn} - H_n)^2 + \max_{n\neq j} O(e^{-3\sqrt{2}\,|\mathtt{x}_n|}).
\end{aligned} \tag{2.22}$$

In the above formula we define σ_{jn} as follows: if j is even, $\sigma_{jn} = (-1)^n$ for $n < j$ and $\sigma_{jn} = (-1)^{n+1}$ for $n > j$; if j is odd, $\sigma_{jn} = (-1)^{n+1}$ for $n < j$ and $\sigma_{jn} = (-1)^n$

for $n > j$. We have in A_j:

(2.23)
$$\begin{aligned}\Delta_{x,z}H_j + F(H_j) = &(-1)^{j+1}\eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\big[\Delta_{x,z}H(\mathtt{x}_j - h_\alpha(z)) + F\big(H(\mathtt{x}_j - h_\alpha(z))\big)\big]\\
&+ \eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\big[F(H_j) - (-1)^{j+1}F\big(H(\mathtt{x}_j - h_\alpha(z))\big)\big]\\
&+ \big[1 - \eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\big]F(H_j)\\
&+ (-1)^{j+1}\nabla_{x,z}H(\mathtt{x}_j - h_{j\alpha}(z))\cdot\nabla_{x,z}\eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\\
&+ (-1)^{j+1}[H(\mathtt{x}_j - h_{j\alpha}(z)) \pm 1]\Delta_{x,z}\eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big)\\
= &S_{j1} + S_{j2} + S_{j3} + S_{j4} + S_{j5}.\end{aligned}$$

Using (2.18)–(2.19) we get:

$$|S_{j1}| \le C\alpha^2 e^{-\frac{\sqrt{2}}{2}|\mathtt{x}_j| - \theta\alpha|\mathtt{z}_j|}. \tag{2.24}$$

To estimate S_{2j} we observe that when $\eta\big(\frac{4\mathtt{x}_j}{R_{j\alpha}}\big) = 1$ then

$$F(H_j) - (-1)^{j+1}F\big(H(\mathtt{x}_j - h_{j\alpha}(z))\big) = 0,$$

hence

$$\begin{aligned}|S_{2j}| &\le Ce^{-\frac{\sqrt{2}}{2}|\mathtt{x}_j|}e^{-\frac{\sqrt{2}}{4}R_{j\alpha}(\mathtt{z}_j)}\\
&\le C\alpha^3 e^{-\frac{\sqrt{2}}{2}|\mathtt{x}_j| - \theta\alpha|\mathtt{z}_j|}.\end{aligned} \tag{2.25}$$

Similar estimate holds for S_{3j}, and also by (2.20), for S_{4j}, S_{5j}.

By an analogous argument one can show that for if $n \neq j$ then, still in A_j we have

$$|\Delta_{x,z}H_n + F(H_n)| \le C\alpha^2 e^{-\frac{\sqrt{2}}{2}|\mathtt{x}_n| - \theta\alpha|\mathtt{z}_n|} \tag{2.26}$$

Now, let us consider a typical term involved in (2.22), which is of the form:

$$|F''(H_j)(w - H_j)^2| \le C\sum_{n\neq j} e^{-2\sqrt{2}|\mathtt{x}_n|}\chi_{A_j}.$$

Using (2.7) we get that for each fixed $n \neq j$, again assuming that $(x,z) \in A_j$, we have

$$|\mathtt{x}_n| \ge \alpha|a_n - a_j||\mathtt{z}_n| + \frac{\sqrt{2}}{2}\log\frac{1}{\alpha} + O(1),$$

hence, taking $\sigma \in (0,1)$

$$\begin{aligned}e^{-2\sqrt{2}|\mathtt{x}_n|}\chi_{A_j} &\le Ce^{-2\sqrt{2}\sigma|\mathtt{x}_n|}e^{-2\sqrt{2}(1-\sigma)|\mathtt{x}_n|}\\
&\le C\alpha^{2(1-\sigma)}e^{-2\sqrt{2}\sigma|\mathtt{x}_n| - 2\sqrt{2}(1-\sigma)\theta\alpha|\mathtt{z}_n|}.\end{aligned} \tag{2.27}$$

Similar estimate holds for other terms involved in (2.22). Since we can write

$$-S(w) = \Delta w + F(w) = \sum_{j=1}^{k}[\Delta w + F(w)]\chi_{A_j},$$

therefore combining (2.23)–(2.27) and using (2.16) we get the following:

Lemma 2.1. *Let w be the aproximate solution defined in (2.13) and let $S(w) = -\Delta w - F(w)$. Then we have:*

$$\|S(w)\|_{\sigma,\theta\alpha,*} \le C\alpha^{2(1-\sigma)}, \tag{2.28}$$

where $\|\cdot\|_{\sigma,\theta\alpha,*}$ is the norm defined in (2.14) and $\sigma \in (0, \frac{2\sqrt{2}-1}{2\sqrt{2}})$. In addition if $f_{j\alpha}, h_{j\alpha}$ are even functions of z then so is w and $S(w)$.

2.3. **Setting up the infinite dimensional reduction.** We look for a solution of (1.1) in the form:

$$u = w + \phi,$$

where ϕ is a "small" perturbation of w. Substituting in (1.1) we obtain to the following problem for ϕ:

$$\mathtt{L}(\phi) = S(w) + N(\phi), \quad N(\phi) = -F(w+\phi) + F(w) + F'(w)\phi, \tag{2.29}$$

and

$$\mathtt{L}(\phi) = \Delta\phi + F'(w)\phi, \quad F'(w) = 1 - 3w^2. \tag{2.30}$$

As is well know, the linear operator $\mathtt{L}$ will have in general unbounded (as $\alpha \to 0$) inverse and to remedy this situation we will use the so called Lyapunov-Schmidt reduction scheme. To be more precise let ρ be a cutoff function such that $\rho(t) = 1$, $|t| < \frac{\sqrt{2}}{2}(1 - 2^{-5})$ and $\rho(t) = 0$, $|t| > \frac{\sqrt{2}}{2}(1 - 2^{-6})$, and let η be a cut-ff function such that $\eta(t) = 1$, $|t| < \frac{\sqrt{2}}{2}(1 - 2^{-7})$ and $\eta(t) = 0$, $|t| > \frac{\sqrt{2}}{2}(1 - 2^{-8})$. We define:

$$\eta_j(x,z) = \eta\Big(\frac{|x - \mathtt{f}_{j\alpha}|}{|\log\alpha|}\Big), \quad \rho_j(x,z) = \rho\Big(\frac{|x - \mathtt{f}_{j\alpha}|}{|\log\alpha|}\Big) \quad j = 1, \ldots, k, \tag{2.31}$$

where $\mathtt{f}_{j\alpha} = f_{j\alpha} + h_{j\alpha}$. Then, instead of (2.29), we will solve a projected problem for ϕ, namely:

$$\begin{aligned} &\mathtt{L}(\phi) = S(w) + N(\phi) + \sum_{j=1}^{k} c_j(z) H'(x - \mathtt{f}_{j\alpha})\eta_j, \quad \text{in } \mathbb{R}^2, \\ &\int_{\mathbb{R}} \phi(x,z) H'(x - \mathtt{f}_{j\alpha})\rho_j \, dx = 0, \quad \forall z \in \mathbb{R}, \quad j = 1, \ldots k. \end{aligned} \tag{2.32}$$

The choices of the cutoff functions above seem at this point somewhat unnatural and it is dictated by purely technical reasons that will become apparent later on.

Above we have introduced new unknowns $c_j = c_j(z)$ that are needed in order that the extra orthogonality conditions be satisfied. Remembering that our ansatz w depends on k functions-paramaters h_j, see (2.11), we conclude that (1.1) is reduced to first solving (2.32) for given h_j's and then to adjusting functions-parameters in such a way that

$$c_j(z) = 0, \quad j = 1, \ldots, k. \tag{2.33}$$

In the next section we will develop the necessary linear theory to deal with (2.32) after which we will solve the reduced problem (2.33).

3. Linear theory

3.1. **Linearized operator for a single interface.** In this section we will consider the basic linearized operator. Let

$$L_0(\phi) = \phi_{xx} + F'(H)\phi, \quad F'(H) = 1 - 3H^2.$$

First, we recall that L_0 has one dimensional kernel $L_0(H_x) = 0$. Second, let $\rho \in L^2(\mathbb{R})$ be a fixed, nonnegative, even function. Then, there exists $\nu_0 > 0$, such that for any $\phi \in H^1(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} H_x \phi \rho \, dx = 0,$$

we have

$$\int_{\mathbb{R}} \left[\frac{1}{2}|\phi_x|^2 + F'(H)\phi^2\right] dx \geq \nu_0 \int_{\mathbb{R}} \phi^2 \, dx. \tag{3.34}$$

As a consequence of these facts we observe that problem

$$L_0(\phi) - \xi^2\phi = h, \tag{3.35}$$

is uniquely solvable whenever $\xi \neq 0$ for $h \in L^2(\mathbb{R})$. Actually, rather standard argument, using comparison principle and the fact that L_0 is of the form

$$L_0(\phi) = \phi_{xx} - 2\phi + 3(1 - H^2(x))\phi, \quad |1 - H^2(x)| \leq Ce^{-\sqrt{2}|x|},$$

can be used to show that whenever h is for instance a compactly supported function then the solution of (3.35) is an exponentially decaying function.

Let us consider operator

$$L(\phi) = L_0(\phi) + \phi_{zz},$$

defined in the whole plane $(x, z) \in \mathbb{R}^2$. Equation $L(\phi) = 0$, has obviously a bounded solution $\phi = H_x$. Our first result shows that converse is also true.

Lemma 3.1. *Let ϕ be a bounded solution of the problem*

$$L(\phi) = 0 \quad \textit{in } \mathbb{R}^2. \tag{3.36}$$

Then $\phi(x, z) = aH_x$, for certain $a \in \mathbb{R}$.

Proof. Let assume that ϕ is a bounded function that satisfies

$$\phi_{zz} + \phi_{xx} + F'(H)\phi = 0. \tag{3.37}$$

Let us consider the Fourier transform of $\phi(x, z)$ in the z variable, $\hat{\phi}(x, \xi)$ which is by definition the distribution defined as

$$\langle\hat{\phi}(x, \cdot), \mu\rangle_{\mathbb{R}} = \langle\phi(x, \cdot), \hat{\mu}\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \phi(x, \xi)\hat{\mu}(\xi) d\xi,$$

where $\mu(\xi)$ is any smooth rapidly decreasing function. Let us consider a smooth rapidly decreasing function of the two variables $\psi(x, \xi)$. Then from equation (3.37) we find

$$\int_{\mathbb{R}} \langle\hat{\phi}(x, \cdot), \psi_{xx} - \xi^2\psi + F'(H)\psi\rangle_{\mathbb{R}} \, dx = 0.$$

Let $\varphi(x)$ and $\mu(\xi)$ be smooth and compactly supported functions such that

$$\{0\} \cap \operatorname{supp}(\mu) = \emptyset. \tag{3.38}$$

Then we can solve the equation

$$\psi_{xx} - \xi^2\psi + F'(H)\psi = \mu(\xi)\varphi(x), \quad x \in \mathbb{R},$$

uniquely for a smooth, rapidly decreasing function $\psi(x,\xi)$ such that $\psi(x,\xi) = 0$ whenever $\xi \notin \operatorname{supp}(\mu)$. We conclude that

$$\int_{\mathbb{R}} \langle \hat{\phi}(x,\cdot), \mu\rangle_{\mathbb{R}}\, \varphi(x)\, dx = 0,$$

so that for all $x \in \mathbb{R}$, $\langle \hat{\phi}(x,\cdot), \mu\rangle_{\mathbb{R}} = 0$, whenever (3.38) holds. In other words

$$\operatorname{supp}(\hat{\phi}(x,\cdot)) \subset \{0\}.$$

It follows that $\hat{\phi}(x,\cdot)$ is a linear combination (with coefficients depending on x) of derivatives up to a finite order of Dirac masses supported in $\{0\}$. Taking inverse Fourier transform, we get that

$$\phi(x,z) = p(x,z),$$

where p is a polynomial in z with coefficients depending on x. Since ϕ is bounded these polynomial is of zero order, namely $p(x,z) \equiv p(x)$, and the bounded function p must satisfy the equation

$$L_0(p) = 0,$$

from where the desired result thus follows. □

Let $B(\phi)$ be an operator of the form

$$B(\phi) = b_1\partial_{xx}\phi + b_2\partial_{xz}\phi + b_3\partial_x\phi + b_4\partial_z\phi + b_5\phi,$$

where the coefficients b_i are small functions. In the sequel we will denote $\mathbf{b} = (b_1,\ldots,b_5)$ and assume that

$$\|\mathbf{b}\| \equiv \sum_{j=1}^{5} \|b_j\|_\infty + \|\nabla b_1\|_\infty + \|\nabla b_2\|_\infty < \delta_0, \tag{3.39}$$

where the small number δ_0 will be subsequently fixed. The linear theory used in this paper is based on a priori estimates for the solutions of the following problems

$$B(\phi) + L(\phi) = h, \quad \text{in } \mathbb{R}^2. \tag{3.40}$$

The results of Lemma 3.1 imply that such estimates without imposing extra conditions on ϕ may not exist. The form of the bounded solutions of $L(\phi) = 0$ suggests the following orthogonality condition:

$$\int_{\mathbb{R}} \phi(x,z) H_x(x) d\mu(x) = 0, \quad \forall z \in \mathbb{R}, \tag{3.41}$$

where $d\mu(x)$ is a fixed measure in $\mathbb{R}$ absolutely continuous with respect to the Lebesque measure. In the sequel we will in particular consider $d\mu(x) = \chi(x)\, dx$ where χ is a compactly supported cut-off function (c.f (2.31)), however our next result applies for a general $d\mu(x)$ as well. With these restrictions imposed we have the following result concerning a priori estimates for this problem.

Lemma 3.2. *There exist constants δ_0 and C such that if the bound (3.39) holds and $h \in L^\infty(\mathbb{R}^2)$, then any bounded solution ϕ of problem (3.40)-(3.41) satisfies*

$$\|\phi\|_\infty \le C\|h\|_\infty.$$

Proof. Proof of the Lemma is similar to the argument in the proof of Lemma 2.2 [7]. We will argue by contradiction. Assuming the opposite means that there are sequences b_j^n, ϕ_n, h_n such that

$$\sum_{j=1}^{5} \|b_j^n\|_\infty + \|\nabla b_1^n\|_\infty + \|\nabla b_2^n\|_\infty \to 0,$$
$$\|\phi_n\|_\infty = 1, \quad \|h_n\|_\infty \to 0,$$

and

$$B_n(\phi_n) + L(\phi_n) = h_n, \quad \text{in } \mathbb{R}^2, \tag{3.42}$$

$$\int_{\mathbb{R}} \phi_n(x,z) H_x(x) d\mu(x) = 0, \quad \text{for all } z \in \mathbb{R}. \tag{3.43}$$

Here

$$B_n(\phi) = b_1^n \partial_{xx}\phi + b_2^n \partial_{xz}\phi + b_3^n \partial_x \phi + b_4^n \partial_z \phi + b_5^n \phi.$$

Let us assume that $(x_n, z_n) \in \mathbb{R}^2$ is such that

$$|\phi_n(x_n, z_n)| \to 1.$$

We claim that the sequence $\{x_n\}$ is bounded. Indeed, if not, using the fact that $L\phi = \Delta\phi - 2\phi + O(e^{-c|x|})\phi$ and employing elliptic estimates we find that the sequence of functions

$$\tilde\phi_n(x,z) = \phi_n(x_n + x, z_n + z),$$

converges, up to a subsequence, locally uniformly to a solution $\tilde\phi$ of the equation

$$\Delta\tilde\phi - 2\tilde\phi = 0, \quad \text{in } \mathbb{R}^2,$$

whose absolute value attains its maximum at $(0,0)$, This implies $\tilde\phi \equiv 0$, so that $\{x_n\}$ is indeed bounded. Let now

$$\tilde\phi_n(x,z) = \phi_n(x, z_n + z).$$

Then $\tilde\phi_n$ converges uniformly over compacts to a bounded, nontrivial solution $\tilde\phi$ of

$$L(\tilde\phi) = 0 \quad \text{in } \mathbb{R}^2,$$

$$\int_{\mathbb{R}} \tilde\phi(x,z) H_x(x) d\mu(x) = 0, \quad \text{for all } z \in \mathbb{R}.$$

Lemma 3.1 then implies $\tilde\phi \equiv 0$, a contradiction and the proof is concluded. □

Using Lemma 3.2 we can also find a priori estimates with norms involving exponential weights. Let us consider the norm

$$\|\phi\|_{\sigma,a} \equiv \|e^{\sigma|x| + a|z|}\phi\|_\infty.$$

where numbers $\sigma, a \geq 0$ are fixed and will be subsequently adjusted. In the case $a = 0$ we have the following a priori estimates.

Corollary 3.1. *There are numbers C and δ_0 as in Lemma 3.2 for which, if $\|h\|_{\sigma,0} < +\infty$, $\sigma \in [0, \sqrt{2})$, then a bounded solution ϕ of (3.40)–(3.41) satisfies*

$$\|\phi\|_{\sigma,0} + \|\nabla\phi\|_{\sigma,0} \leq C\|h\|_{\sigma,0}. \tag{3.44}$$

Proof. Besides some obvious changes in the proof, Corollary 3.1 follows from the same argument as Corollary 2.1 in [7]. Again we concentrate on estimates for the problem (3.40)–(3.41). We already know that

$$\|\phi\|_\infty \le C\|h\|_{\sigma,0}.$$

We set $\tilde\phi = \phi\|h\|_{\sigma,0}^{-1}$. Then we have

$$(L+B)(\tilde\phi) = \tilde h, \quad \text{where } \|\tilde h\|_{\sigma,0} \le 1,$$

and also $\|\tilde\phi\|_\infty \le C$. Let us fix a number $R_0 > 0$ such that for $x > R_0$ we have

$$1 - H^2(x) < \frac{2-\sigma^2}{6},$$

which is always possible since $1 - H^2(x) = O(e^{-\sqrt{2}|x|})$. For an arbitrary number $\rho > 0$ let us set

$$\bar\phi(x,z) = \rho[\cosh(z/2) + e^{\sigma x}] + Me^{-\sigma x},$$

where M is to be chosen. Then we find that, reducing δ_0 in (3.39) if necessary,

$$(L+B)(\bar\phi) \le -\frac{M(2-\sigma^2)}{4}e^{-\sigma x}, \quad \text{for } x > R_0.$$

Thus

$$(L+B)(\bar\phi) \le \tilde h, \quad \text{for } x > R_0,$$

if

$$\frac{M(2-\sigma^2)}{4} \ge \|\tilde h\|_{\sigma,0} = 1.$$

If we also also assume

$$Me^{-\sigma R_0} \ge \|\tilde\phi\|_\infty,$$

we conclude from maximum principle that $\tilde\phi \le \bar\phi$. Letting $\rho \to 0$ we then get by fixing M,

$$\tilde\phi \le Me^{-\sigma x}, \quad \text{for } x > 0,$$

hence

$$\phi \le M\|h\|_{\sigma,0}e^{-\sigma x}, \quad \text{for } x > 0.$$

In a similar way we obtain the lower bound

$$\phi \ge -M\|h\|_{\sigma,0}e^{-\sigma x}, \quad \text{for } x > 0.$$

Finally, the same argument for $x < 0$ yields

$$\|\phi\|_{\sigma,0} \le C\|h\|_{\sigma,0},$$

while from local elliptic estimates we find

$$\|\nabla\phi\|_{\sigma,0} \le C\|h\|_{\sigma,0},$$

and the proof is concluded. □

When $a > 0$ in the definition of the norm $\|\cdot\|_{\sigma,a}$ then we have the following a priori estimates.

Corollary 3.2. *There are numbers C, δ_0 as in Lemma 3.2, and $a_0 > 0$ for which, if $\|h\|_{\sigma,a} < +\infty$, $\sigma \in (0,\sqrt{2})$, $a \in [0,a_0)$, then a bounded solution ϕ to problem (3.40)-(3.41) satisfies*

$$\|\phi\|_{\sigma,a} + \|\nabla\phi\|_{\sigma,a} \le C_\sigma\|h\|_{\sigma,a}.$$

Proof. We already know that

$$\|\phi\|_{\sigma,0} + \|\nabla\phi\|_{\sigma,0} \le C\|h\|_{\sigma,a}.$$

Then we may write

$$\psi(z) = \int_{\mathbb{R}} \phi^2(x,z)\,dx,$$

and differentiate twice weakly to get

$$\psi''(z) = 2\int_{\mathbb{R}} \phi_z^2\,dx + 2\int_{\mathbb{R}} \phi_{zz}\phi\,dx.$$

We have

$$\int_{\mathbb{R}} \phi_{zz}\phi\,dx = \int_{\mathbb{R}} \phi_x^2\,dx + \int_{\mathbb{R}} F'(H)\phi^2\,dx - \int_{\mathbb{R}} B(\phi)\phi + \int_{\mathbb{R}} h\phi. \tag{3.45}$$

Integrating by parts once in x we find

$$\begin{aligned} \Big|\int_{\mathbb{R}} B(\phi)\phi\Big| &= \Big|\int_{\mathbb{R}} [-(b_1\phi)_x\phi_x - (b_2\phi)_x\phi_z + b_3\phi_x\phi + b_4\phi_z\phi + b_5\phi^2]\Big| \\ &\le C\delta_0 \int_{\mathbb{R}} (\phi_z^2 + \phi_x^2 + \phi^2)\,dx. \end{aligned} \tag{3.46}$$

Because of the orthogonality condition (3.41) we also have that for a certain $\nu_0 > 0$,

$$\int_{\mathbb{R}} \phi_x^2\,dx + \int_{\mathbb{R}} F'(H)\phi^2\,dx \ge \nu_0 \int_{\mathbb{R}} (\phi_x^2 + \phi^2)\,dx.$$

Hence, reducing δ_0 if necessary, we find that for a certain constant $C > 0$

$$\psi''(z) \ge \frac{\nu_0}{4}\psi(z) - C\int_{\mathbb{R}} h^2(x,z)\,dx,$$

so that

$$-\psi''(z) + \frac{\nu_0}{4}\psi(z) \le \frac{C}{\sigma} e^{-2a|z|}\|h\|_{\sigma,a}^2.$$

Since we also know that ψ is bounded by:

$$|\psi(z)| \le \frac{C}{\sigma}\|h\|_{\sigma,0}^2,$$

we can use a barrier of the form $\psi^+(z) = M\|h\|_{\sigma,a}^2 e^{-2az} + \rho e^{2az}$, with M sufficiently large and $\rho > 0$ arbitrary, to get the bound $0 \le \psi \le \psi^+$ for $z \ge 0$ and any $a < \frac{\sqrt{\nu_0}}{4} \equiv a_0$. A similar argument can be used for $z < 0$. Letting $\rho \to 0$ we get then

$$\int_{\mathbb{R}} \phi^2(x,z)\,dx \le C_\sigma e^{-2a|z|}\|h\|_{\sigma,a}^2, \quad a < a_0.$$

Elliptic estimates yield that for R_0 fixed and large

$$|\phi(x,z)| \le C_\sigma e^{-a|z|}\|h\|_{\sigma,a} \quad \text{for } |x| < R_0.$$

The corresponding estimate in the complementary region can be found by barriers. For instance in the quadrant $\{x > R_0, z > 0\}$ we may consider a barrier of the form

$$\bar\phi(x,z) = M\|h\|_{\sigma,a} e^{-(\sigma x + az)} + \rho e^{\frac{x}{2} + \frac{z}{2}},$$

with $\rho > 0$ arbitrarily small. Fixing M depending on R_0 we find the desired estimate for $\|\phi\|_{\sigma,a}$ letting $\rho \to 0$. Arguing similarly in the remaining quadrants is similar. The corresponding bound for $\|\nabla\phi\|_{\sigma,a}$ is then deduced from local elliptic estimates. This concludes the proof. $\square$

Notice that for a general right hand side h equation of the form $L(\phi)+B(\phi)=h$ with the orthogonality conditions imposed as above does not have a solution. On the other hand the problem

$$L(\phi)+B(\phi)=h+c(z)H_x, \quad \text{in } \mathbb{R}^2, \tag{3.47}$$

under orthogonality conditions

$$\int_{\mathbb{R}} \phi(x,z)H_x(x)d\mu(x)=0, \quad \text{for all } z\in\mathbb{R}. \tag{3.48}$$

has a solution in the sense that for given h one can find (ϕ,c) satisfying (3.47)–(3.48).

Corollary 3.3. *There exist $C>0$ and $\delta_0>0$ as in Lemma 3.2, and $\eta_0>0$ and $a_0>0$ for which, if $\|h\|_{\sigma,\alpha}<+\infty$, $\sigma\in(0,\sqrt{2})$, $a\in[0,a_0)$, and $d\mu(x)=\rho(x)\,dx$ is such that*

$$\int_{\mathbb{R}} e^{-\sigma|x|}[|H_x|\rho_{xx}|+2|H_{xx}||\rho_x|]\,dx<\eta_0, \tag{3.49}$$

then a bounded solution ϕ to problem (3.47)-(3.48) satisfies

$$\|\phi\|_{\sigma,a}+\|\nabla\phi\|_{\sigma,a}\le C\|h\|_{\sigma,a}. \tag{3.50}$$

Moreover we have

$$|c(z)|\le C\|h\|_{\sigma,a}e^{-a|z|}. \tag{3.51}$$

Proof. To find a priori estimate (3.50) we have to find bounds for the function $c(z)$. Testing equation (3.47) against H_x and integrating with respect to $d\mu(x)$ we get

$$\int_{\mathbb{R}}\phi_{zz}H_x\,d\mu(x)+\int_{\mathbb{R}}L_0(\phi)H_x\,d\mu(x)+\int_{\mathbb{R}}B(\phi)H_x\,d\mu(x)=\int_{\mathbb{R}}hH_x\,d\mu(x) + c(z)\int_{\mathbb{R}}H_x^2\,d\mu(x).$$

Let us assume that $\|h\|_{\sigma,0}<+\infty$ and that ϕ is a bounded solution. Integrating by parts and using $L_0(H_x)=0$ and the orthogonality condition (3.48) we get

$$c(z)\int_{\mathbb{R}}H_x^2\rho\,dx=\int_{\mathbb{R}}B(\phi)H_x\rho\,dx+\int_{\mathbb{R}}\phi(2w_{xx}\rho_x+H_x\rho_{xx})\,dx-\int_{\mathbb{R}}hH_x\,d\mu(x). \tag{3.52}$$

To estimate term $\int_{\mathbb{R}}B(\phi)H_x\rho\,dx$ we use an argument similar to that of (3.46), and to estimate $\int_{\mathbb{R}}\phi(H_x\rho)_{xx}\,dx$ we use (3.49) to get

$$|c(z)|\le C\int_{\mathbb{R}}|hH_x|+C(\delta_0+\eta_0)(\|\nabla\phi\|_{\sigma,0}+\|\phi\|_{\sigma,0}).$$

From the a priori estimates (Corollary 3.1) applied to (3.47) we know that

$$\|\nabla\phi\|_{\sigma,0}+\|\phi\|_{\sigma,0}\le C(\|h\|_{\sigma,0}+\|c\|_\infty),$$

since

$$\|H_xe^{\sigma|x|}\|_\infty\le C,$$

for $\sigma\in[0,\sqrt{2})$. Thus, reducing δ_0,η_0 if necessary, we find

$$\|c\|_\infty\le C\|h\|_{\sigma,0},$$

and the estimate

$$\|\nabla\phi\|_{\sigma,0}+\|\phi\|_{\sigma,0}\le C\|h\|_{\sigma,0}.$$

follows.

Finally, if we additionally have $\|h\|_{\sigma,a} < +\infty$, we obtain that

$$\int_{\mathbb{R}} |hH_x| \leq C\|h\|_{\sigma,a} e^{-a|z|}.$$

The same procedure as above and the a priori estimates found in Corollary 3.2 then yield

$$|c(z)| \leq C\|h\|_{\sigma,a} e^{-a|z|},$$

from where the relation (3.50) immediately follows. □

Concerning the existence of bounded solutions of (3.47)–(3.48) we have:

Proposition 3.1. *There exists numbers $C > 0$, $\delta_0 > 0$, $\eta_0 > 0$ such that whenever bounds (3.39), (3.49) hold, then given h with $\|h\|_{\sigma,a} < +\infty$, $\sigma \in (0, \sqrt{2})$, $a \in [0, a_0)$, there exists a unique bounded solution $\phi = T(h)$ to problem (3.47)-(3.48) which defines a bounded linear operator of h in the sense that*

$$\|\nabla\phi\|_{\sigma,a} + \|\phi\|_{\sigma,a} \leq C\|h\|_{\sigma,a}.$$

Proof. We will first consider solvability of the following problem

$$(L + B)(\phi) = h, \quad \text{in } \mathbb{R}^2, \tag{3.53}$$

in the space V, where $\psi \in V$ if $\|\psi\|_{\sigma,0} < \infty$, $\sigma \in (0, \sqrt{2})$ and

$$\int_{\mathbb{R}} \psi(x, z) H_x(x)\rho(x)\, dx = 0, \quad \text{for all } z \in \mathbb{R}, \tag{3.54}$$

where the density $\rho(x)$ satisfies the hypothesis of Corollary 3.3. We claim that given $h \in V$ there exists a unique solution ϕ of (3.53) in V. We will argue by approximations. Let us replace h by the function $h(x, z)\chi_{(-R,R)}(z)$ extended $2R$-periodically to the whole plane. With this right hand side we can give to the problem (3.53) a weak formulation in the subspace of $H^1_R \subset H^1(\mathbb{R}^2)$ of functions that are $2R$-periodic in z. To be more precise let

$$[\psi, \eta] = \int_{-\infty}^{\infty}\int_{-R}^{R} \nabla\psi \cdot \nabla\eta\, dzdx + \int_{-\infty}^{\infty}\int_{-R}^{R} \psi\eta\, dzdx.$$

By W we will denote the subspace of functions in H^1_R that satisfy (3.54). Then (3.53) can be written in the form

$$-\left[(A + K)(\phi), \psi\right] = \int_{-\infty}^{\infty}\int_{-R}^{R} h\psi, \quad \psi \in W, \tag{3.55}$$

where $A : W \to W$ is defined for $\psi \in W$ by

$$\begin{aligned}[A(\phi), \psi] = &\int_{-\infty}^{\infty}\int_{-R}^{R} (\nabla\phi\nabla\psi + b_1\phi_x\psi_x + b_2\phi_x\psi_z)\, dzdx \\ &+ \int_{-\infty}^{\infty}\int_{-R}^{R} (2 - b_5)\phi\psi\, dzdx,\end{aligned}$$

and $K : W \to W$ is a linear operator defined by

$$[K(\phi), \psi] = \int_{-\infty}^{\infty}\int_{-R}^{R}[(b_{1x} + b_{2z} - b_3)\phi_x - b_4\phi_z]\psi\, dzdx - \int_{-\infty}^{\infty}\int_{-R}^{R}\phi\psi[2 + F'(H)]\, dzdx.$$

Using (3.39), (3.49), and the fact that $\|2+F'(H)\|_{\sqrt{2},0} < \infty$ one can show that the operator A is invertible and the operator K is compact.

From Fredholm alternative and Lemma 3.1 we find a solution of (3.55), which in addition can be extended periodically to a unique solution $\phi \in V$, of (3.53) with h replaced by $h(x,z)\chi_{(-R,R)}(z)$. Letting $R \to +\infty$ and using the uniform a priori estimates valid for the approximations completes the proof of the claim.

The existence of a solution to (3.47)–(3.48) as well as the rest of the Proposition follows from this claim. Indeed, given h such that $\|h\|_{\sigma,0} < \infty$ by $\Pi_V(h)$ we will denote the orthogonal projection of h onto V (in the sense of $L^2(\rho\, dx)$ as indicated by (3.54)). Using the claim we can solve then the following problem

$$(L + B)(\phi) = \Pi_V(h).$$

Now we only need to chose function $c(z)$ such that

$$(I - \Pi_V)[(L + B)(\phi)] = (I - \Pi_V)(h) + c(z)H_x.$$

This ends the proof. □

3.2. **Linear theory for multiple interfaces.** We want to develop a theory similar to that in the previous section now for the operator

$$\mathtt{L}(\phi) = \Delta\phi + F'(w)\phi.$$

Let us recall the definition of the weighted norm which we will use (2.14):

$$\|\phi\|_{\sigma,\theta\alpha,*} := \Big\|\Big(\sum_{j=1}^{k} e^{-\sigma|x-f_{j\alpha}(z)|-\theta\alpha|z|}\Big)^{-1}\phi\Big\|_{\infty}.$$

We will search for a bounded left inverse for a projected problem for the operator $\mathtt{L}$ in the space of functions whose $\|\cdot\|_{\sigma,\theta\alpha,*}$ norm is finite. In section 2.1 we have introduced approximate locations of the transitions layers: $\gamma_{j\alpha} = \{x = f_{j\alpha}(z) + h_{j\alpha}(z) \mid z \in \mathbb{R}\}$, $j = 1,\dots,k$ where functions $f_{j\alpha}$, $h_{j\alpha}$ satisfy, respectively, (2.2) and (2.11). For convenience we will set

$$\mathtt{f}_{j\alpha}(z) = f_{j\alpha}(z) + h_{j\alpha}(z).$$

Furthermore we will denote:

$$\mathtt{H}_j(x,z) = H\big(x - \mathtt{f}_{j\alpha}(z)\big),$$
$$\mathtt{H}_{j,x}(x,z) = H'\big(x - \mathtt{f}_{j\alpha}(z)\big).$$

Thus we consider the problem

$$\mathtt{L}(\phi) = h + \sum_{j=1}^{k} c_j(z)\eta_j(x,z)\mathtt{H}_{j,x}(x,z), \quad \text{in } \mathbb{R}^2, \tag{3.56}$$

where now we *do not assume* necessarily orthogonality conditions on ϕ. Above we have introduced cut-off functions η_j , which are defined as folllows: let $\eta_a^b(s)$ be a smooth function with $\eta_a^b(s) = 1$ for $|s| < a$ and $= 0$ for $|s| > b$, where $0 < a < b < 1$. Then, with $d_* = \log \frac{1}{\alpha}$, we set

$$\eta_j(x,z) = \eta_a^b\Big(\frac{|x - \mathtt{f}_{j\alpha}|}{d_*}\Big), \quad a = \frac{\sqrt{2}}{2}(1-2^{-7}), \quad b = \frac{\sqrt{2}}{2}(1-2^{-8}). \tag{3.57}$$

In the sequel we will also use other cut-off functions:

$$\begin{aligned} \rho_j(x,z) &= \eta_a^b\Big(\frac{|x - \mathtt{f}_{j\alpha}|}{d_*}\Big), \quad a = \frac{\sqrt{2}}{2}(1-2^{-5}), \quad b = \frac{\sqrt{2}}{2}(1-2^{-6}), \\ \eta_j^-(x,z) &= \eta_a^b\Big(\frac{|x - \mathtt{f}_{j\alpha}|}{d_*}\Big), \quad a = \frac{\sqrt{2}}{2}(1-2^{-6}), \quad b = \frac{\sqrt{2}}{2}(1-2^{-7}), \\ \eta_j^+(x,z) &= \eta_a^b\Big(\frac{|x - \mathtt{f}_{j\alpha}|}{d_*}\Big), \quad a = \frac{\sqrt{2}}{2}(1-2^{-8}), \quad b = \frac{\sqrt{2}}{2}(1-2^{-9}). \end{aligned} \tag{3.58}$$

We will prove the following result:

Proposition 3.2. *There exist positive constants α_0, σ_0, $\delta_0 > 0$ such that if (3.39) is satisfied, and if $\alpha \in [0, \alpha_0)$, $\sigma \in (0, \sigma_0)$, and conditions (2.1)-(2.2), (2.11) hold, then problem (3.56) has a solution $\phi = \mathtt{T}(h)$ which defines a linear operator of h with $\|h\|_{\sigma,\theta\alpha,*} < +\infty$ and satisfies the estimate*

$$\|\phi\|_{\sigma,\theta\alpha,*} + \|\nabla\phi\|_{\sigma,\theta\alpha,*} \le C_\sigma \|h\|_{\sigma,\theta\alpha,*}.$$

In addition function ϕ satisfies the following orthogonality conditions

$$\int_{\mathbb{R}} \phi(x,z)\mathtt{H}_{j,x}(x,z)(x,z)\rho_j(x,z)\,dx = 0, \quad \textit{for all } z \in \mathbb{R}. \tag{3.59}$$

Besides, the coefficients $c_j(z)$ in (3.56) can be estimated as

$$\sum_{j=1}^k |c_j(z)| \le C\|h\|_{\sigma,\theta\alpha,*} e^{-\theta\alpha|z|}. \tag{3.60}$$

Proof. The main idea in the proof of this proposition is to decompose problem (3.56) into *interior* problems that can be handled with the help of the theory developed in the previous section and an *exterior* problem and then *glue* the solutions of the subproblems.

From the definition of the functions η_j, $\eta_j^\pm$ we have

$$\eta_j\eta_j^- = \eta_j^-, \quad \eta_j^+\eta_j = \eta_j. \tag{3.61}$$

We search for a solution of (3.56) of the form

$$\phi = \sum_{j=1}^k \eta_j\phi_j + \psi.$$

Substituting this expression into equation (3.56) and arranging terms we find

$$\sum_{j=1}^k \eta_j[\Delta\phi_j + F'(w)\phi_j - c_j(z)\mathtt{H}_{j,x} - h] + \Big[\Delta\psi + F'(w)\psi - \Big(1 - \sum_{j=1}^k \eta_j\Big)h - \sum_{j=1}^k (2\nabla\eta_j\nabla\phi_j + \Delta\eta_j\phi_j)\Big] = 0.$$

We will denote

$$h_j = \eta_j^+ h, \quad r_j = \eta_j^+ [F'(\mathtt{H}_j) - F'(w)].$$

Let us observe that in the support of η_j we have, using (3.61),

$$\eta_j h_j = \eta_j h, \quad \eta_j r_j = \eta_j [F'(\mathtt{H}_j) - F'(w)],$$

Then we find a solution to problem (3.56) if we solve the following linear system of equations

$$\Delta \phi_j + F'(\mathtt{H}_j)\phi_j + r_j \phi_j = h_j - 3(1-w^2)\eta_j^- \psi + c_j(z)\mathtt{H}_{j,x}, \tag{3.62}$$

in $\mathbb{R}^2$, for $j = 1, \ldots, k$, and

$$\Delta \psi - \Big[2 - 3\Big(1 - \sum_{j=1}^k \eta_j^-\Big)(1-w^2)\Big]\psi = \Big(1 - \sum_{j=1}^k \eta_j\Big)h - \sum_{j=1}^k (2\nabla\eta_j \nabla\phi_j + \Delta\eta_j \phi_j), \tag{3.63}$$

in $\mathbb{R}^2$. To solve equations (3.62) we denote $\tilde\phi_j = \phi_j + \eta_j^- \psi$ and use (3.62)–(3.63) to write the equation for $\tilde\phi_j$

$$\begin{aligned} \Delta \tilde\phi_j + F'(\mathtt{H}_j)\tilde\phi_j + r_j \tilde\phi_j = h_j + \eta_j^- \psi \Big[& F'(\mathtt{H}_j) - 3(1-w^2) + r_j + 2 \\ & - 3\Big(1 - \sum_{m=1}^k \eta_m^-\Big)(1-w^2)\Big] \\ & - 2\nabla\psi\nabla\eta_j^- - \Delta\eta_j^- \psi + c_j \mathtt{H}_{j,x}. \end{aligned} \tag{3.64}$$

We observe that equation (3.63) written in terms of $\tilde\phi_j$ has form

$$\Delta \psi - \Big[2 - 3\Big(1 - \sum_{j=1}^k \eta_j^-\Big)(1-w^2)\Big]\psi = \Big(1 - \sum_{j=1}^k \eta_j\Big)h - \sum_{j=1}^k (2\nabla\eta_j \nabla\tilde\phi_j + \Delta\eta_j \tilde\phi_j), \tag{3.65}$$

since for instance $\nabla\eta_j \nabla(\eta_j^- \psi) \equiv 0$.

Let us denote

$$L_j(\phi) = \Delta\phi + F'(\mathtt{H}_j)\phi,$$

and consider first the auxiliary problem

$$L_j(\phi) = h + c(z)\mathtt{H}_{j,x}, \quad \text{in } \mathbb{R}^2, \tag{3.66}$$

under orthogonality condition

$$\int_{\mathbb{R}} \phi(x,z)\mathtt{H}_{j,x}(x,z)\rho_j(x,z)\,dx = 0, \quad \text{for all } z \in \mathbb{R}. \tag{3.67}$$

For future references we observe that

$$\rho_j \eta_j^- = \rho_j, \quad \rho_j(1-\eta_j^-) \equiv 0. \tag{3.68}$$

We want to solve (3.66)–(3.67) using Proposition 3.1. To this end we consider the natural change of coordinates

$$x \mapsto \mathtt{x} \equiv x - \mathtt{f}_{j\alpha}, \quad z \mapsto \mathtt{z}.$$

and set

$$\phi(x,z) = \tilde\phi(\mathtt{x},\mathtt{z}), \quad h(x,z) = \tilde h(\mathtt{x},\mathtt{z}).$$

Direct computation then shows that problem (3.66)-(3.67) is equivalent to

$$L(\tilde\phi) + B_j(\tilde\phi) = \tilde h + c(\mathtt{z})H'(\mathtt{x}), \quad \text{in } \mathbb{R}^2. \tag{3.69}$$

under orthogonality conditions

$$\int_{\mathbb{R}} \tilde\phi(\mathtt{x},\mathtt{z})H'(\mathtt{x})\rho(\mathtt{x})\,d\mathtt{x} = 0, \quad \text{for all } \mathtt{z}\in\mathbb{R}, \tag{3.70}$$

where

$$\rho(\mathtt{x}) = \eta_a^b\Big(\frac{\mathtt{x}}{d^*}\Big), \quad a = \frac{\sqrt2}{2}(1-2^{-5}), b = \frac{\sqrt2}{2}(1-2^{-6}).$$

Here

$$L(\tilde\phi) = \tilde\phi_{\mathtt{zz}} + \tilde\phi_{\mathtt{xx}} + F'(H)\tilde\phi,$$

and

$$\begin{aligned} &B_j(\tilde\phi) = \Big(\frac{\partial\mathtt{x}}{\partial z}\Big)^2\tilde\phi_{\mathtt{xx}} + 2\Big(\frac{\partial\mathtt{x}}{\partial z}\Big)\tilde\phi_{\mathtt{xz}} + \Big(\frac{\partial^2\mathtt{x}}{\partial z^2}\Big)\tilde\phi_{\mathtt{x}},\\ &\frac{\partial\mathtt{x}}{\partial z} = -\frac{d}{d\mathtt{z}}\mathtt{f}_{j\alpha}(\mathtt{z}), \quad \frac{\partial^2\mathtt{x}}{\partial z^2} = -\frac{d^2}{d\mathtt{z}^2}\mathtt{f}_{j\alpha}(\mathtt{z}). \end{aligned} \tag{3.71}$$

The operator B_j satisfies the assumptions of Proposition 3.1, since from (2.2), (2.11) we have, using the notation of (3.39) and denoting the vector of the coefficients of B_j by $\mathbf{b}_j$,

$$\|\mathbf{b}_j\| \le C\big(\big\|\frac{d}{d\mathtt{z}}\mathtt{f}_{j\alpha}\big\|_\infty + \big\|\frac{d^2}{d\mathtt{z}^2}\mathtt{f}_{j\alpha}\big\|_\infty\big) \le C\alpha. \tag{3.72}$$

Problem (3.69)–(3.70) has a unique bounded solution $\tilde\phi = \tilde T_j(\tilde\phi)$ where $\tilde T_j$ is the linear operator T predicted by the proposition for $B = B_j$. Besides, if $\|\tilde h\|_{\sigma,\alpha} < +\infty$ then we have the estimate

$$\|\nabla_{\mathtt{x},\mathtt{z}}\tilde\phi\|_{\sigma,\theta\alpha} + \|\tilde\phi\|_{\sigma,\theta\alpha} \le C\|\tilde h\|_{\sigma,\theta\alpha}.$$

Going back to original variables we see then that there is a unique solution to (3.66)-(3.67), $\phi = T_j(h)$ where T_j is a linear operator. In addition we have the estimate

$$\|\nabla\phi\|_{\sigma,\theta\alpha,j} + \|\phi\|_{\sigma,\theta\alpha,j} \le C\|h\|_{\sigma,\theta\alpha,j},$$

where

$$\|\phi\|_{\sigma,\theta\alpha,j} \;=\; \|\, e^{\sigma|x-\mathtt{f}_{j\alpha}(z)|+\theta\alpha|z|}\,\phi\,\|_\infty. \tag{3.73}$$

Moreover, c is estimated using (3.51) by

$$|c(z)| \le C\|h\|_{\sigma,\theta\alpha,j}e^{-\theta_0\alpha|z|}. \tag{3.74}$$

Let us observe that given ψ we can recast the equations (3.62) for $\tilde\phi_j$ as a system of the form

(3.75)

$$\begin{aligned} &\tilde\phi_j + T_j(r_j\tilde\phi_j) = T_j\big(h_j + g_j(\psi)\big), \text{where}\\ &\quad g_j(\psi) = \eta_j^-\psi\Big[F'(\mathtt{H}_j) - 3(1-w^2) + r_j + 2 - 3\Big(1-\sum_{m=1}^k\eta_m^-\Big)(1-w^2)\Big]\\ &\qquad\quad - 2\nabla\psi\nabla\eta_j^- - \Delta\eta_j^-\psi,\\ &\qquad\qquad\qquad j = 1,\dots,k, \end{aligned}$$

We will solve next equation (3.65) for ψ as a linear operator

$$\psi = \Psi(\Phi, h),$$

where Φ denotes the k-tuple $\Phi = (\tilde{\phi}_1, \ldots, \tilde{\phi}_k)$. To this end let us consider first the problem

$$\Delta\psi - (2 - \Theta)\psi = g \quad \text{in } \mathbb{R}^2. \tag{3.76}$$

where

$$\Theta = 3\Big(1 - \sum_{j=1}^{k} \eta_j^-\Big)(1 - w^2).$$

Observe that if the number d_* is large enough then Θ is uniformly small, indeed $\Theta = o(1)$ as $\alpha \to 0$. Let us assume that g satisfies

$$|g(x,z)| \le A \sum_{j=1}^{k} e^{-\mu|x - \mathtt{f}_{j\alpha}(z)| - \theta\alpha|z|},$$

for some $0 \le \mu < \sqrt{2}$. Then, given that if d_* is sufficiently large then number θ is small, and also that (2.2), (2.11) holds, the use of barriers and elliptic estimates proves that this problem has a unique bounded solution with

$$|\nabla\psi(x,z)| + |\psi(x,z)| \le C \sum_{j=1}^{k} e^{-\mu|x - f_j(z)| - \theta\alpha|z|}.$$

Thus if we take

$$g = (1 - \sum_{j=1}^{k} \eta_j)h - \sum_{j=1}^{k} (2\nabla\eta_j \nabla\tilde{\phi}_j + \Delta\eta_j \tilde{\phi}_j),$$

we clearly have that

$$\begin{aligned} |g(x,z)| \le & C\Big[\|h\|_{\sigma,\theta\alpha,*} + o(1) \sum_{j=1}^{k} \big(\|\tilde{\phi}_j\|_{\sigma,\theta\alpha,j} + \|\nabla\tilde{\phi}_j\|_{\sigma,\theta\alpha,j}\big)\Big] \\ & \times \sum_{j=1}^{k} e^{-\mu|x - f_j(z)| - \theta\alpha|z|}, \end{aligned}$$

and hence equation (3.65) has a unique bounded solution

$$\psi = \Psi(\Phi, h),$$

which defines a linear operator in its argument and satisfies the estimate

$$\begin{aligned} |\Psi(\Phi,h)| \le & C\Big[\|h\|_{\sigma,\theta\alpha,*} + o(1) \sum_{j=1}^{k} \big(\|\tilde{\phi}_j\|_{\sigma,\theta\alpha,j} + \|\nabla\tilde{\phi}_j\|_{\sigma,\theta\alpha,j}\big)\Big] \\ & \times \sum_{j=1}^{k} e^{-\mu|x - f_j(z)| - \theta\alpha|z|}. \end{aligned} \tag{3.77}$$

In addition, we find that

$$\begin{aligned} &\|\Psi(\Phi,h)\|_{\sigma,\theta\alpha,*}+\|\nabla\Psi(\Phi,h)\|_{\sigma,\theta\alpha,*}\\ &\qquad\le C\Big[\|h\|_{\sigma,\theta\alpha,*}+o(1)\sum_{j=1}^{k}\big(\|\tilde\phi_j\|_{\sigma,\theta\alpha,j}+\|\nabla\tilde\phi_j\|_{\sigma,\theta\alpha,j}\big)\Big]. \end{aligned} \tag{3.78}$$

Now we have the ingredients to solve the full system (3.64)-(3.65). Accordingly to (3.75) we obtain a solution if we solve the following system in $\Phi=(\tilde\phi_1,\dots,\tilde\phi_k)$

$$\tilde\phi_j+T_j\big(r_j\tilde\phi_j-g_j(\Psi(\Phi,0))\big)=T_j\big(h_j+g_j(\Psi(0,h))\big),\quad j=1,\dots,k, \tag{3.79}$$

where functions $g_j(\Phi)$ are defined in (3.75). We consider this system defined in the space X of all C^1 functions Φ such that the norm

$$\|\Phi\|_X:=\sum_{j=1}^{k}\|\nabla\tilde\phi_j\|_{\sigma,\theta\alpha,j}+\|\tilde\phi_j\|_{\sigma,\theta\alpha,j},$$

is finite. System (3.79) can be written as

$$\Phi+\mathtt{A}(\Phi)=\mathtt{B}(h),$$

where $\mathtt{A}$ and $\mathtt{B}$ are linear operators. Thanks to the estimates derived for the operators T_j and the bound (3.77) we see that

$$\|\mathtt{B}(h)\|_X\le C\|h\|_{\sigma,\theta\alpha,*}.$$

On the other hand we have that

$$\|\mathtt{A}(\Phi)\|_X\le C\Big[\sum_{j=1}^{k}\|g_j(\Psi(\Phi,0))\|_{\sigma,\theta\alpha,j}+\sum_{j=1}\|r_j\tilde\phi_j\|_{\sigma,\theta\alpha,j}\Big].$$

Using estimates (3.77) and (3.78) we find

$$\sum_{j=1}^{k}\|g_j(\Psi(\Phi,0))\|_{\sigma,\theta\alpha,j}\le o(1)\|\Phi\|_X. \tag{3.80}$$

From the definition of r_j and (2.15) we get $\|r_j\|_\infty=o(1)$ which implies

$$\sum_{j=1}^{k}\|r_j\tilde\phi_j\|_{\sigma,\theta\alpha,j}\le o(1)\|\Phi\|_X.$$

Summarizing the last estimates we obtain

$$\|\mathtt{A}(\Phi)\|_X\le o(1)\|\Phi\|_X,$$

hence the operator $\mathtt{A}$ is a uniformly small operator in the norm $\|\cdot\|_X$ provided that α is sufficiently small. We conclude that system (3.79) has a unique solution $\Phi=\Phi(h)$, which in addition is a linear operator of h such that

$$\|\Phi(h)\|_X\le C\|h\|_{\sigma,\theta\alpha,*}.$$

Thus we get a solution to problem (3.56) by setting

$$\phi=\sum_{j=1}^{k}\eta_j\tilde\phi_j(h)+\Big(1-\sum\eta_j^-\Big)\Psi(\Phi(h),h). \tag{3.81}$$

Using (3.68) we get $\rho_j\phi=\rho_j\tilde\phi_j$ hence from and (3.67) we obtain (3.59). Estimate (3.60) follows directly from (3.74). The proof of the proposition is complete. □

4. The nonlinear projected problem

4.1. **Solvability.** Let us recall (see section 2.3) that our goal is to find a solution of the problem

$$
\begin{aligned}
&\mathtt{L}(\phi) = S(w) + N(\phi) + \sum_{j=1}^{k} c_j(z)\mathtt{H}_{j,x}\eta_j, \quad \text{in } \mathbb{R}^2, \\
&\int_{\mathbb{R}} \phi(x,z)\mathtt{H}_{j,x}\rho_j \, dx = 0, \quad \forall z \in \mathbb{R}, \quad j = 1, \dots k.
\end{aligned} \tag{4.1}
$$

As we have shown already in Lemma 2.1:

$$
\|S(w)\|_{\sigma,\theta\alpha,*} \leq C\alpha^{2(1-\sigma)}, \tag{4.2}
$$

where θ is as in (2.2), (2.11) and $\sigma \in (0, \frac{2\sqrt{2}-1}{2\sqrt{2}})$. We will establish next that the nonlinear problem (4.1) is solvable with similar estimates to those obtained in Proposition in 3.2 for problem (3.56) with $h = S(w)$, namely we show the following result.

Proposition 4.1. *There exist positive numbers $\alpha_0, \delta_0, \sigma_0$, such that for any number $\sigma \in (0, \sigma_0)$, $\alpha \in [0, \alpha_0)$ and any $\mathtt{f}$, satisfying constraints (2.2)-(2.11), Problem (4.1) has a solution ϕ with $\|\phi\|_{\sigma,\theta\alpha,*} \leq C_{\sigma'}\alpha^{2-2\sigma'}$, where $\sigma' \in (\sigma, \frac{2\sqrt{2}-1}{2\sqrt{2}})$ such that*

$$
\int_{\mathbb{R}} \phi(x,z)\mathtt{H}_{j,x}(x,z)\rho_j(x,z) \, dx = 0. \tag{4.3}
$$

The coefficients c_j can be estimated as follows:

$$
\sum_{j=1}^{k} |c_j(z)| \leq C_{\sigma'}\alpha^{2-2\sigma'} e^{-\theta\alpha|z|}. \tag{4.4}
$$

In addition, if functions $f_{j\alpha}, h_{j\alpha}$ are even functions of z then so is ϕ.

Proof. Let us observe that we obtain a solution of the problem (4.1) if we solve the fixed point problem

$$
\phi = \mathtt{T}\big(S(w) - N(\phi)\big) := \mathcal{M}(\phi), \tag{4.5}
$$

where $\mathtt{T}$ is the operator found in Proposition 3.2. Assume that $\|\phi_j\|_{\sigma,\theta\alpha,*} < 1$, $j = 1, 2$. We have that

$$
|\, N(\phi_2) - N(\phi_1)\,| \leq C(|\phi_1| + |\phi_2|)\,|\phi_2 - \phi_1|,
$$

and hence

$$
\|\, N(\phi_2) - N(\phi_1)\,\|_{\sigma,\theta\alpha,*} \;\leq\; C(\,\|\phi_1\|_{\sigma,\theta\alpha,*} + \|\phi_2\|_{\sigma,\theta\alpha,*})\,\|\phi_2 - \phi_1\|_{\sigma,\theta\alpha,*}, \tag{4.6}
$$

in particular

$$
\|\, N(\phi)\,\|_{\sigma,\theta\alpha,*} \;\leq\; \|\phi\|^2_{\sigma,\theta\alpha,*}.
$$

Then, from (4.2), the following holds: for each fixed $\sigma' \in (\sigma, \frac{2\sqrt{2}-1}{2\sqrt{2}})$ there exists a number $\nu > 0$ such that for all small α the operator $\mathcal{M}$ is a contraction mapping in a region of the form

$$
\mathtt{B} = \{\phi \;/\; \|\phi\|_{\sigma,\theta\alpha,*} \leq \nu\alpha^{2-2\sigma'}\},
$$

and hence a solution of the fixed point problem (4.5) in $\mathtt{B}$ exists. Furthermore ϕ solves (4.1), and by (3.59) we find that ϕ satisfies (4.3). The proof of the proposition is complete. □

4.2. **Lipschitz dependence on the parameters.** Now we will consider the the dependence of the solution ϕ of (4.1) on the function-parameters $h_{j\alpha}$, $j = 1, \dots, k$ (we recall that functions $f_{j\alpha}$ are fixed). For convenience we will denote

$$\mathtt{f}_\alpha = (f_{1\alpha}, \dots, f_{k\alpha}), \quad \mathtt{h}_\alpha = (h_{1\alpha}, \dots, h_{k\alpha}).$$

More specifically we are interested in establishing the Lipschitz character of ϕ as a function of variables $(\mathtt{h}''_\alpha, \mathtt{h}'_\alpha, \mathtt{h}_\alpha)$. We will begin with the observation that the error term $S(w)$ can be written as follows:

$$\begin{aligned} S(w) &= \sum_{j=1}^k \Delta_{x,z} H_j + F'(w) \\ &= E_1(\mathtt{h}''_\alpha, \mathtt{h}'_\alpha, \mathtt{h}_\alpha) + E_2(\mathtt{h}_\alpha), \end{aligned}$$

where

$$E_1 = \sum_{j=1}^k [\Delta_{x,z} H_j + F(H_j)] = \sum_{j=1}^k E_{1j}(h''_{j\alpha}, h_{j\alpha'}, h_{j\alpha}).$$

Observe that functions E_{1j} depend linearly on $h''_{j\alpha}$ and are quadratic functions of $h'_{j\alpha}$ (see (2.8)-(2.9) and (2.18)–(2.19)). Another observation we make is that to establish the Lipschitz dependence of ϕ it suffices to vary one parameter at a time. Thus we will consider functions h_j such that

$$\|h''_j e^{\theta|z|}\|_\infty + \|h'_j e^{\theta|z|}\|_\infty + \|h_j\|_\infty < \bar{\delta}, \quad j = 1, \dots, k. \tag{4.7}$$

We will denote $h_{j\alpha}(z) = h_j(\alpha z)$, $\mathtt{h}_\alpha = (h_{1\alpha}, \dots, h_{k\alpha})$, etc. Also we will denote

$$\frac{d}{dz} h_{j\alpha} = h'_{j\alpha} = \alpha h'_j(\alpha z), \quad \frac{d^2}{dz^2} h_{j\alpha} = h''_{j\alpha} = \alpha^2 h''_j(\alpha z).$$

We observe that the error terms E_1, and E_2 can be thought of as functions of $\mathtt{h}''_\alpha, \mathtt{h}'_\alpha, \mathtt{h}_\alpha$, and the same applies for the solution ϕ of (4.1). Thus we will further write:

$$E_1 = E_1(\mathtt{h}''_\alpha, \mathtt{h}'_\alpha, \mathtt{h}_\alpha), \quad E_2 = E_2(\mathtt{h}_\alpha), \quad \phi = \phi(\mathtt{h}''_\alpha, \mathtt{h}'_\alpha, \mathtt{h}_\alpha),$$

etc., whenever convenient. Given $\mathtt{h}_\alpha$ function ϕ solves the following problem (c.f. (4.1)):

$$\begin{aligned} &\mathtt{L}_{\mathtt{h}_\alpha}(\phi) = E_1(\mathtt{h}''_\alpha, \mathtt{h}'_\alpha, \mathtt{h}_\alpha) + E_2(\mathtt{h}_\alpha) + N(\phi) + \sum_{j=1}^k c_j(z) \mathtt{H}_{j,x} \eta_j, \quad \text{in } \mathbb{R}^2, \\ &\int_{\mathbb{R}} \phi(x,z) \mathtt{H}_{j,x} \rho_j \, dx = 0, \quad \forall z \in \mathbb{R}, \quad j = 1, \dots k, \end{aligned} \tag{4.8}$$

where $\mathtt{f}_{j\alpha} = f_{j\alpha} + h_{j\alpha}$, $\mathtt{H}_{j,x}(x,z) = H'\big(x - \mathtt{f}_{j\alpha}(z)\big)$ and

$$\mathtt{L}_{\mathtt{h}_\alpha} = \Delta + F'\big(w(\mathtt{h}_\alpha)\big).$$

The solution of this problem can be obtained by the argument of Section 4.1. In fact the analog of Proposition 4.1 yields $\|\phi\|_{\sigma,\theta\alpha,*} \le C\alpha^{2(1-\sigma')}$, where $\sigma' \in (\sigma, \frac{2\sqrt{2}-1}{2\sqrt{2}})$.

We will consider functions $\mathtt{h}_{\alpha i}$, $i = 1, 2$ such that

$$\alpha^2\|\mathtt{h}_{\alpha 1} - \mathtt{h}_{\alpha 2}\|_\infty + \alpha\|\mathtt{h}'_{\alpha 1} - \mathtt{h}'_{\alpha 2}\|_{\theta\alpha} + \|\mathtt{h}''_{\alpha 1} - \mathtt{h}''_{\alpha 2}\|_{\theta\alpha} \le \bar{\delta}\alpha^2, \tag{4.9}$$

where $\bar{\delta} > 0$ is a fixed small constant (c.f. (2.11)). For convenience we will denote

$$|\!|\!|\mathtt{h}_\alpha|\!|\!|_{\theta\alpha} = \alpha^2\|\mathtt{h}_\alpha\|_\infty + \alpha\|\mathtt{h}'_\alpha\|_{\theta\alpha} + \|\mathtt{h}''_\alpha\|_{\theta\alpha}$$

Lemma 4.1. *Under the assumptions (4.9) we have the following estimates:*

$$\|E_1(\mathtt{h}''_{\alpha 1}, \mathtt{h}'_{\alpha 1}, \mathtt{h}_{\alpha 1}) - E_1(\mathtt{h}''_{\alpha 1}, \mathtt{h}'_{\alpha 1}, \mathtt{h}_{\alpha 1})\|_{\sigma\theta\alpha,*} \le C\alpha^{-2\sigma'}|\!|\!|\mathtt{h}_{\alpha 1} - \mathtt{h}_{\alpha 2}|\!|\!|_{\theta\alpha} \tag{4.10}$$

$$\|E_2(\mathtt{h}_{\alpha 1}) - E_2(\mathtt{h}_{\alpha 2})\|_{\sigma,\theta\alpha,*} \le C\alpha^{2(1-\sigma')}\|\mathtt{h}_{\alpha 1} - \mathtt{h}_{\alpha 2}\|_\infty \tag{4.11}$$

The proof of this Lemma follows the lines of the proof of Lemma 2.1 and is omitted here.

In what follows we will emphasize the dependence of ϕ and c_j on parameters by writing

$$\begin{aligned} \phi^{(i)} &= \phi(\mathtt{h}''_{\alpha i}, \mathtt{h}'_{\alpha i}, \mathtt{h}_{\alpha i}), \\ c_j^{(i)} &= c_j^{(i)}(\mathtt{h}''_{\alpha i}, \mathtt{h}'_{\alpha i}, \mathtt{h}_{\alpha i}), \text{ etc.} \end{aligned} \tag{4.12}$$

Proposition 4.2. *Let ϕ, be the solution of (4.8). Then functions ϕ and c_j for $j = 1, \dots, k$, are continuous with respect to the parameters $\mathtt{h}''_\alpha, \mathtt{h}'_\alpha, \mathtt{h}_\alpha$. Moreover assuming (4.9), with the notation (4.12) we have the following estimates*

$$\|\phi^{(1)} - \phi^{(2)}\|_{\sigma,\theta_0\alpha,*} + \|\nabla(\phi^{(1)} - \nabla\phi^{(2)})\|_{\sigma,\theta_0\alpha,*} \le C\alpha^{-2\sigma'}|\!|\!|\mathtt{h}_{\alpha 1} - \mathtt{h}_{\alpha 2}|\!|\!|_{\theta_0\alpha} \tag{4.13}$$

where $\sigma' \in (\sigma, \frac{2\sqrt{2}-1}{2\sqrt{2}})$.

Proof. With the notations in (4.12) we have that $\phi^{(i)}$, $i = 1, 2$ is the solution of the following problem in $\mathbb{R}^2$:

$$\mathtt{L}_{\mathtt{h}_{\alpha i}}(\phi^{(i)}) = E_1^{(i)} + E_2^{(i)} - N(\phi^{(i)}) + \sum_{j=1}^k c_j^{(i)}(z)\mathtt{H}_{j,x}^{(i)}\eta_j^{(i)}. \tag{4.14}$$

In terms of the operator $\mathtt{T}^{(i)}$ defined in Proposition 3.2, we can write $\phi^{(i)}$ as the solution of the fixed point problem:

$$\phi^{(i)} = \mathtt{T}^{(i)}\big(S(w^{(i)}) - N(\phi^{(i)})\big). \tag{4.15}$$

In order to estimate the difference $\phi^{(1)} - \phi^{(2)}$ we need first to analyze the Lispchitz dependence of the operator $\mathtt{T}^{(i)}$ on the function parameters $\mathtt{h}_{\alpha i}$ and their derivatives. Let h be a given function satisfying $\|h\|_{\sigma,\theta_0\alpha,*} < +\infty$. Let us denote

$$\varphi^{(i)} = \mathtt{T}^{(i)}(h),$$

Using the same notation as in the proof of Proposition 3.2 we get

$$\varphi^{(i)} = \sum_{j=1}^k \eta_j^{(i)}\tilde{\varphi}_j^{(i)}(h) + \Big(1 - \sum \eta_j^{(i)-}\Big)\Psi^{(i)}(\Phi^{(i)}(h), h). \tag{4.16}$$

where functions $\tilde{\varphi}_j^{(i)}$ solve system (3.79), and $\Phi^{(i)} = (\tilde{\varphi}_1^{(i)}, \dots, \tilde{\varphi}_k^{(i)})$. Clearly the Lipschitz dependence of $\mathtt{T}^{(i)}$ on the parameters depends on the Lipschitz dependence of the operator $\Psi^{(i)}$ as well as the functions $\tilde{\varphi}_j^{(i)}$, $j = 1, \dots, k$. This is the issue to which we turn our attention now. Let us denote

$$\Theta^{(i)} = 3\Big(1 - \sum_{j=1}^{k} \eta_j^{(i)-}\Big)\big(1 - (w^{(i)})^2\big),$$

and

$$g^{(i)} = (1 - \sum_{j=1}^{k} \eta_j^{(i)})h - \sum_{j=1}^{k} (2\nabla\eta_j^{(i)}\nabla\tilde{\varphi}_j^{(i)} + \Delta\eta_j^{(i)}\tilde{\varphi}_j^{(i)}),$$

Then $\Psi^{(i)}(\Phi^{(i)}, h)$ is defined by (see 3.76)

$$\Delta\Psi^{(i)} - (2 - \Theta^{(i)})\Psi^{(i)} = g^{(i)} \quad \text{in } \mathbb{R}^2. \tag{4.17}$$

Writing the difference as $\bar{\Psi} = \Psi^{(1)} - \Psi^{(2)}$ we get:

$$\Delta\bar{\Psi} - 2\bar{\Psi} = \Theta^{(1)}\Psi^{(1)} - \Theta^{(2)}\Psi^{(2)} + g^{(1)} - g^{(2)}.$$

If $q(s)$ denotes a smooth real function and if we set $q_j^{(i)} = q(x - f_{j\alpha} - h_{j\alpha}^{(i)})$ (as suggested by the form of a typical term involved in the expression for $g^{(i)}$) then:

$$|q^{(1)} - q^{(2)}| \le \|\mathtt{h}_{\alpha 1} - \mathtt{h}_{\alpha 2}\|_\infty \|\nabla q\|_\infty. \tag{4.18}$$

From this it follows:

$$\begin{aligned} \|g^{(1)} - g^{(2)}\|_{\sigma,\theta\alpha,*} &\le C d_*^{-1} \|\mathtt{h}_{\alpha 1} - \mathtt{h}_{j\alpha 2}\|_\infty \\ &\qquad \times \Big(\|h\|_{\sigma,\theta,*} + d_*^{-1}[\|\tilde{\varphi}^{(1)}\|_{\sigma,\theta\alpha,*} + \|\nabla\tilde{\varphi}^{(1)}\|_{\sigma,\theta\alpha,*}]\Big) \\ &\quad + C d_*^{-1}\big[\|\tilde{\varphi}^{(1)} - \tilde{\varphi}^{(2)}\|_{\sigma,\theta\alpha,*} + \|\nabla\tilde{\varphi}^{(1)} - \nabla\tilde{\varphi}^{(2)}\|_{\sigma,\theta\alpha,*}\big] \end{aligned} \tag{4.19}$$

(see (3.73) for the definition of $\|\cdot\|_{\sigma,\theta\alpha,j}$.) In a similar manner we can estimate:

$$\begin{aligned} \|\Theta^{(1)}\Psi^{(1)} - \Theta^{(2)}\Psi^{(2)}\|_{\sigma,\theta\alpha,*} &\le \|\Theta^{(1)} - \Theta^{(2)}\|_\infty \|\Psi^{(1)}\|_{\sigma,\theta\alpha,*} \\ &\quad + \|\Theta^{(2)}\|_\infty \|\Psi^{(1)} - \Psi^{(2)}\|_{\sigma,\theta\alpha,*} \\ &\le o(1)\|\mathtt{h}_{\alpha 1} - \mathtt{h}_{\alpha 1}\|_\infty \|\Psi^{(1)}\|_{\sigma,\theta\alpha,*} + o(1)\|\bar{\Psi}\|_{\sigma,\theta\alpha,*} \end{aligned} \tag{4.20}$$

Using then a standard comparison argument and (3.78) we get

$$\begin{aligned} \|\bar{\Psi}\|_{\sigma,\theta\alpha,*} + \|\nabla\bar{\Psi}\|_{\sigma,\theta\alpha,*} &\le o(1)\|\mathtt{h}_{\alpha 1} - \mathtt{h}_{\alpha 2}\|_\infty \\ &\qquad \times \Big(\|h\|_{\sigma,\theta\alpha,*} + d_*^{-1}[\|\tilde{\varphi}^{(1)}\|_{\sigma,\theta\alpha,*} + \|\nabla\tilde{\varphi}^{(1)}\|_{\sigma,\theta\alpha,*}]\Big) \\ &\quad + o(1)\big[\|\tilde{\varphi}^{(1)} - \tilde{\varphi}^{(2)}\|_{\sigma,\theta\alpha,*} + \|\nabla\tilde{\varphi}^{(1)} - \nabla\tilde{\varphi}^{(2)}\|_{\sigma,\theta\alpha,*}\big] \end{aligned} \tag{4.21}$$

Let us now recall (c.f. (3.75)) that functions $\tilde\varphi_j^{(i)}$ that appear in formula (4.16) satisfy

$$
\begin{aligned}
\tilde\varphi_j^{(i)} + T_j^{(i)}(r_j^{(i)}\tilde\varphi_j^{(i)}) &= T_j^{(i)}\big(h_j^{(i)} + g_j^{(i)}(\psi^{(i)})\big), \text{where}\\
g_j^{(i)}(\psi^{(i)}) &= \eta_j^{-(i)}\psi^{(i)}\Big[F'(\mathtt{H}_j^{(i)}) - 3(1-(w^{(i)})^2) + r_j^{(i)} + 2\\
&\qquad - 3\Big(1-\sum_{m=1}^k \eta_m^{-(i)}\Big)(1-(w^{(i)})^2)\Big]\\
&\qquad - 2\nabla\psi^{(i)}\nabla\eta_j^{-(i)} - \Delta\eta_j^{-(i)}\psi^{(i)},\\
&\qquad j=1,\dots,k, \quad i=1,2,
\end{aligned}
\tag{4.22}
$$

and

$$
h_j^{(i)} = \eta_j^{+(i)}h, \quad r_j^{(i)} = \eta_j^{+(i)}[F'(\mathtt{H}_j^{(i)}) - F'(w^{(i)})], \quad \psi^{(i)} = \Psi^{(i)}(\Phi^{(i)}(h),h).
$$

As it can be seen from (4.20)–(4.22) we need to analyze the Lipschitz character of the operators $T_j^{(i)}$, where we say that $T_j^{(i)}(h) = \varphi_j^{(i)}$, if $\varphi_j^{(i)}$ is a unique solution of (3.66)–(3.67). Using the linear operators $L_j^{(i)}$, where

$$
L_j^{(i)}(\varphi_j^{(i)}) = \Delta\varphi_j^{(i)} + F'\big(H(x-f_{j\alpha}-h_{j\alpha}^{(i)})\big)\varphi_j^{(i)},
$$

we can write:

$$
L_j^{(1)}(\varphi_j^{(1)} - \varphi_j^{(2)}) = c_j^{(1)}\mathtt{H}_{j,x}^{(1)} - c_j^{(2)}\mathtt{H}_{j,x}^{(2)} + (L_j^{(2)} - L_j^{(1)})\varphi_j^{(2)}.
$$

To use the theory developed for the a priori estimates for the operator $L_j^{(1)}$ we denote

$$
a_j = \frac{\int_{\mathbb{R}}(\varphi_j^{(2)} - \varphi_j^{(1)})\mathtt{H}_{j,x}^{(1)}\rho_j^{(1)}\,dx}{\int_{\mathbb{R}}\big(\mathtt{H}_{j,x}^{(1)}\big)^2\rho_j^{(1)}\,dx},
$$

so that function $\bar\varphi_j = \varphi_j^{(1)} - \varphi_j^{(2)} + a_j H_{j,x}^{(1)}$ satisfies:

$$
\begin{aligned}
L_j^{(1)}\bar\varphi_j &= -c_j^{(2)}(\mathtt{H}_{j,x}^{(2)} - \mathtt{H}_{j,x}^{(1)}) + (L_j^{(2)} - L_j^{(1)})\varphi_j^{(2)} + L_j^{(1)}(a_j\mathtt{H}_{j,x}^{(1)})\\
&\qquad + (c_j^{(1)} - c_j^{(2)})\mathtt{H}_{j,x}^{(1)},\\
\int_{\mathbb{R}} &\bar\varphi_j\mathtt{H}_{j,x}^{(1)}\rho_j^{(1)}\,dx = 0.
\end{aligned}
\tag{4.23}
$$

Notice that the first equation in (4.23) can be written in the form

$$
L_j^{(1)}\bar\varphi_j = \bar h_j + \bar c_j\mathtt{H}_{j,x}^{(1)},
$$

where

$$
\begin{aligned}
\bar h_j &= -c_j^{(2)}(\mathtt{H}_{j,x}^{(2)} - \mathtt{H}_{j,x}^{(1)}) + (L_j^{(2)} - L_j^{(1)})\varphi_j^{(2)} + a_j L_j^{(1)}(\mathtt{H}_{j,x}^{(1)}) + 2a_{j,z}\mathtt{H}_{j,xz}^{(1)},\\
\bar c_j &= (c_j^{(1)} - c_j^{(2)} + a_{j,zz}).
\end{aligned}
$$

As a consequence, function $\bar\varphi_j$, as a unique solution of (4.23), can be estimated in terms of the norm of $\bar h_j$ only. Using (3.60) we have

$$
\|c_j^{(2)}(\mathtt{H}_{j,x}^{(2)} - \mathtt{H}_{j,x}^{(1)})\|_{\sigma,\theta\alpha,j} \le C\|\mathtt{h}_{\alpha 1} - \mathtt{h}_{\alpha 2}\|_\infty\|h\|_{\sigma,\theta\alpha,j}.
$$

(See (3.73) for the definition of $\|\cdot\|_{\sigma,\theta_0\alpha,j}$). Similarly we find

$$\|(L_j^{(2)}-L_j^{(1)})\varphi_j^{(2)}\|_{\sigma,\theta\alpha,j}\le C\|\mathtt{h}_{\alpha 1}-\mathtt{h}_{\alpha 2}\|_\infty\|h\|_{\sigma,\theta\alpha,j}.$$

In order to estimate the last two term in (4.23) we observe:

$$\begin{aligned}&\|a_jL_j^{(1)}(\mathtt{H}_{j,x}^{(1)})\|_{\theta\alpha}+\|a_{j,z}\mathtt{H}_{j,xz}^{(1)}\|_{\theta\alpha}\\&\le C(\|\varphi_j^{(2)}-\varphi_j^{(1)}\|_{\sigma,\theta\alpha,j}+\|\nabla(\varphi_j^{(2)}-\varphi_j^{(1)})\|_{\sigma,\theta\alpha,j})\\&\quad\times\big(\|\frac{d}{dz}\mathtt{f}_{j\alpha}\|_{\theta\alpha}+\|\frac{d^2}{dz^2}\mathtt{f}_{j\alpha}\|_{\theta\alpha}\big)\\&\le C\alpha(\|\varphi_j^{(2)}-\varphi_j^{(1)}\|_{\sigma,\theta\alpha,j}+\|\nabla(\varphi_j^{(2)}-\varphi_j^{(1)})\|_{\sigma,\theta\alpha,j}).\end{aligned}$$

Summarizing we get,

$$\begin{aligned}\|\bar\varphi_j\|_{\sigma,\theta\alpha,j}\le{}& C\|\mathtt{h}_{\alpha 1}-\mathtt{h}_{\alpha 2}\|_{\theta\alpha}\|h\|_{\sigma,\theta\alpha,j}\\&+C\alpha(\|\varphi_j^{(2)}-\varphi_j^{(1)}\|_{\sigma,\theta\alpha,j}+\|\nabla(\varphi_j^{(2)}-\varphi_j^{(1)})\|_{\sigma,\theta\alpha,j}).\end{aligned}\tag{4.24}$$

For future references we also observe that, using the orthogonality condition,

$$\|a_j\|_{\theta\alpha}\le C\|\mathtt{h}_{\alpha 1}-\mathtt{h}_{\alpha 2}\|_\infty\|h\|_{\sigma,\theta\alpha,j}.\tag{4.25}$$

Combining (4.24)–(4.25) we find:

$$\begin{aligned}\|\varphi_j^{(2)}-\varphi_j^{(1)}\|_{\sigma,\theta\alpha,j}\le{}& C\|\mathtt{h}_{\alpha 1}-\mathtt{h}_{\alpha 2}\|_\infty\|h\|_{\sigma,\theta\alpha,j}\\&+C\alpha(\|\varphi_j^{(2)}-\varphi_j^{(1)}\|_{\sigma,\theta\alpha,j}+\|\nabla(\varphi_j^{(2)}-\varphi_j^{(1)})\|_{\sigma,\theta\alpha,j}).\end{aligned}\tag{4.26}$$

An estimate similar to (4.26) holds for $\nabla(\varphi_j^{(2)}-\varphi_j^{(1)})$ and thus we get (by definition $\varphi_j^{(i)}=T_j^{(i)}(h)$):

$$\begin{aligned}&\|T_j^{(1)}(h)-T_j^{(2)}(h)\|_{\sigma,\theta\alpha,j}+\|\nabla(T_j^{(1)}(h)-T_j^{(2)}(h))\|_{\sigma,\theta\alpha,j}\\&\quad\le C\|\mathtt{h}_{\alpha 1}-\mathtt{h}_{\alpha 2}\|_\infty\|h\|_{\sigma,\theta\alpha,j}.\end{aligned}\tag{4.27}$$

Going back to formula (4.14) and using the above we derive at the end the following estimate for the function $\varphi^{(1)}-\varphi^{(2)}$:

$$\begin{aligned}&\|\varphi^{(1)}-\varphi^{(2)}\|_{\sigma,\theta\alpha,*}+\|\nabla(\varphi^{(1)}-\varphi^{(2)})\|_{\sigma,\theta\alpha,*}\\&\quad\le C\|\mathtt{h}_{\alpha 1}-\mathtt{h}_{\alpha 2}\|_\infty(\|h_j^{(1)}\|_{\sigma,\theta\alpha,*}+\|h_j^{(2)}\|_{\sigma,\theta\alpha,*})\\&\qquad+C\|h_j^{(1)}-h_j^{(2)}\|_{\sigma,\theta\alpha,*}.\end{aligned}\tag{4.28}$$

Now we use (4.28), (4.15), Proposition 3.2 and Lemma 4.1 to conclude estimate (4.13). This ends the proof of the proposition. □

5. The final step of the Lyapunov-Schmidt procedure

5.1. **Derivation of the reduced problem.** In this section we will derive a second order differential system for the location of the interfaces which will guarantee that

$$c_j(z)=0,\quad j=1,\dots,k,\tag{5.1}$$

where c_j's are defined in Proposition 4.1. In order to determine these functions we multiply the first equation in (2.32) by $\rho_j\mathtt{H}_{j,x}$ and integrate in x to get

(5.2)

$$-\int_{\mathbb R}S(w)\rho_j\mathtt{H}_{j,x}\,dx+\int_{\mathbb R}\mathtt{L}(\phi)\rho_j\mathtt{H}_{j,x}\,dx-\int_{\mathbb R}N(\phi)\rho_j\mathtt{H}_{j,x}\,dx=c_j(z)\int_{\mathbb R}\mathtt{H}_{j,x}^2\rho_j\,dx,$$

where we have used the fact that $\rho_j\eta_{j'} = \delta_{jj'}\rho_j$, $j, j' = 1, \ldots, k$. As we will see later on that leading order term in the expansion of $c_j(z)$ as a function of the parameter α is

$$\Pi_j = -\int_{\mathbb{R}} S(w)\rho_j \mathtt{H}_{j,x}\, dx.$$

Now we will compute this term. Let us notice first that for a fixed $1 \le j \le k$:

(5.3)

$$\begin{aligned}
S(w) = {} & \Delta_{x,z}H_j + F(H_j) \\
& + 3(1-H_j^2)(w-H_j) \\
& + \sum_{n\neq j}[\Delta_{x,z}H_n + F(H_n)] \\
& + \frac{1}{2}F''(H_j)(w-H_j)^2 - \frac{1}{2}\sum_{n\neq j}F''(\sigma_{jn})(\sigma_{jn}-H_n)^2 + \max_{n\neq j} O(e^{-3\sqrt{2}\,|\mathtt{x}_n|}) \\
= {} & \tilde{S}_{j1}(w) + \tilde{S}_{j2}(w) + \tilde{S}_{j3}(w) + \tilde{S}_{j4}(w).
\end{aligned}$$

Accordingly we can write,

$$\begin{aligned}
\Pi_j &= -\int_{\mathbb{R}} \tilde{S}_{j1}(w)\rho_j\mathtt{H}_{j,x}\, dx - \int_{\mathbb{R}} \tilde{S}_{j2}(w)\rho_j\mathtt{H}_{j,x}\, dx - \int_{\mathbb{R}} \tilde{S}_{j3}(w)\rho_j\mathtt{H}_{j,x}\, dx \\
&\quad - \int_{\mathbb{R}} \tilde{S}_{j4}(w)\rho_j\mathtt{H}_{j,x}\, dx \\
&= \tilde{\Pi}_{j1} + \tilde{\Pi}_{j2} + \tilde{\Pi}_{j3} + \tilde{\Pi}_{j4}.
\end{aligned}$$

Using (2.18) we get

$$\begin{aligned}
\tilde{\Pi}_{j1} &= -\alpha^2\int_R [H''(h'_{j\alpha})^2 - (h''_j + f''_{j\alpha})H']\mathtt{H}_{j,x}\rho_j\, dx \\
&\quad - \alpha^2\int_R B_1(\mathtt{x}_j,\mathtt{z}_j)[H]\mathtt{H}_{j,x}\rho_j\, dx - \alpha^3\int_R B_2(\mathtt{x}_j,\mathtt{z}_j)[H]\mathtt{H}_{j,x}\rho_j\, dx,
\end{aligned}$$

where $H = H\big(\mathtt{x}_j - h_{j\alpha}(z)\big)$, $h_{j\alpha}(z) = h_j(\alpha z)$. From the definition of the Fermi coordinates $(\mathtt{x}_j, \mathtt{z}_j)$ in (2.6) we find

$$|x - f_{j\alpha}(z) - \mathtt{x}_j| \le C\alpha^2|\mathtt{x}_j|,$$

hence,

$$|H'(\mathtt{x}_j - h_{j\alpha}) - H'(x - f_{j\alpha} - h_{j\alpha})| \le C\alpha^2|\mathtt{x}_j|e^{-\sqrt{2}|\mathtt{x}_j|}. \tag{5.4}$$

Replacing now $H'(\mathtt{x}_j - h_{j\alpha})$ in in the formula for $\tilde{\Pi}_{j1}$ by $H'((x - f_{j\alpha} - h_{j\alpha}) = \mathtt{H}_{j,x}$ we get:

$$\tilde{\Pi}_{j1} = c_{0,\alpha}(h''_{j\alpha} + f''_{j\alpha}) + \alpha^2 M_{1j}(\mathtt{f}_\alpha, \mathtt{h}_\alpha), \tag{5.5}$$

where $\mathtt{f}_\alpha(z) = \mathtt{f}(\alpha z)$, $\mathtt{h}_\alpha(z) = \mathtt{h}(\alpha z)$ and $\mathtt{f}_\alpha = (f_{1\alpha}, \ldots, f_{k\alpha})$, $\mathtt{h}_\alpha = (h_{1\alpha}, \ldots, h_{k\alpha})$, and we have denoted:

$$c_{0,\alpha} = \int_{\mathbb{R}} (H')^2\rho\Big(\frac{x}{|\log\alpha|}\Big)\, dx,$$

Since

$$\rho(t) = \eta_a^b(t), \quad a = \frac{\sqrt{2}}{2}(1-2^{-5}), \quad b = \frac{\sqrt{2}}{2}(1-2^{-6}),$$

therefore

$$c_{0,\alpha} = c_0\big(1 + O(\alpha^{2-\mu})\big), \quad c_0 = \int_{\mathbb{R}} (H')^2\, dx, \quad \frac{1}{2} > \mu > \frac{1}{2^5}. \tag{5.6}$$

Function M_{1j} defined above satisfies:

$$\|M_{1j}\|_{\theta_0\alpha} \le C(\|\mathtt{h}'_\alpha\|^2_{\theta_0\alpha} + \|\mathtt{h}''_\alpha\|_{\theta_0\alpha}),$$

where constants C depends on function $\mathtt{f}_\alpha$. Furthermore we get

$$\tilde\Pi_{j2} = -c_{1,\alpha}(e^{-\sqrt{2}(\mathtt{f}_{j\alpha}-\mathtt{f}_{j-1\alpha})} - e^{-\sqrt{2}(\mathtt{f}_{j+1\alpha}-\mathtt{f}_{j\alpha})}) + P_{1j}(\mathtt{f}_\alpha, \mathtt{h}_\alpha), \tag{5.7}$$

where

$$c_{1,\alpha} = c_1\big(1 + O(\alpha^{1-\mu})\big), \quad c_1 = 3\int_{\mathbb{R}} (1-H^2)H'e^{-\sqrt{2}x}\, dx,$$

and we recall that $\mathtt{f}_{j\alpha} = f_{j\alpha} + h_{j\alpha}$. Function P_{1j} can be estimated as follows:

$$|P_{1j}(\mathtt{f}_\alpha, \mathtt{h}_\alpha)| \le C\alpha^2 \max_{n\neq j} e^{-\sqrt{2}|\mathtt{f}_{n\alpha}-\mathtt{f}_{j\alpha}|}.$$

Similarly, we get

$$\begin{aligned} \|\tilde\Pi_{j3}\|_{\theta\alpha} &\le C\alpha(\alpha^2 + \|\mathtt{h}'_\alpha\|^2_{\theta\alpha} + \|\mathtt{h}''_\alpha\|_{\theta\alpha}) \\ |\tilde\Pi_{j4}| &\le C\max_{n\neq j} e^{-2\sqrt{2}|\mathtt{f}_{n\alpha}-\mathtt{f}_{j\alpha}|} \le C\alpha^2 \max_{n\neq j} e^{-\sqrt{2}|\mathtt{f}_{n\alpha}-\mathtt{f}_{j\alpha}|}. \end{aligned}$$

Now we will consider

$$\begin{aligned} Q_1 \equiv \int_{\mathbb{R}} \mathtt{L}(\phi)\rho_j \mathtt{H}_{j,x}\, dx &= \int_{\mathbb{R}} [\phi_{xx} + F'(\mathtt{H}_j)\phi]\rho_j\mathtt{H}_{j,x}\, dx \\ &\quad + \int_{\mathbb{R}} \phi[F'(w) - F'(\mathtt{H}_j)]\rho_j\mathtt{H}_{j,x}\, dx \\ &\quad + \int_{\mathbb{R}} [-2\phi_z(\rho_j\mathtt{H}_{j,x})_z - \phi(\rho_j\mathtt{H}_{j,x})_{zz}]\, dx \\ &= Q_{j1} + Q_{j2} + Q_{j3}, \end{aligned} \tag{5.8}$$

where we have made use of the orthogonality condition satisfied be ϕ. After an integration by parts in the expression for Q_{j1} we get

$$Q_{j1} = \int_{\mathbb{R}} \phi(2\rho_{j,x}\mathtt{H}_{j,xx} + \rho_{j,xx}\mathtt{H}_{j,x})\, dx.$$

If σ in the definition of the $\|\cdot\|_{\sigma,\theta\alpha,*}$ norm is chosen sufficiently small and σ' is taken close to σ then from Proposition 4.1 it follows:

$$\|Q_{1j}\|_{\theta\alpha} \le C\alpha^{2+\mu'}, \quad 3/4 > \mu' > 1/2. \tag{5.9}$$

Likewise we get:

$$\begin{aligned} \|Q_{2j}\|_{\theta\alpha} &\le C\alpha^{2+\mu'}, \\ \|Q_{3j}\|_{\theta\alpha} &\le C\alpha^{2+\mu'}(1 + \|\mathtt{h}_\alpha\|_{\theta\alpha}). \end{aligned} \tag{5.10}$$

Finally we get:

$$\|R_j\|_{\theta\alpha} \le C\alpha^{2+\mu'}, \quad \text{where } R_j = \int_{\mathbb{R}} N(\phi)\rho_j\mathtt{H}_{j,x}\, dx. \tag{5.11}$$

Summarizing these calculations and also taking into account the the results of the previous section we get:

Lemma 5.1. *The reduced problem*

$$c_j(z)=0,\quad j=1,\dots,k,$$

is equivalent to the following system of differential equations for functions $\mathtt{f}_\alpha=(f_{1\alpha},\dots,f_{k\alpha})$ *and* $\mathtt{h}_\alpha=(h_{1\alpha},\dots,h_{k,\alpha})$*:*

$$\begin{aligned}c(h''_{j\alpha}+f''_{j\alpha})-(e^{-\sqrt{2}[f_{j\alpha}-f_{j-1\alpha}+h_{j\alpha}-h_{j-1\alpha}]}-e^{-\sqrt{2}[f_{j+1\alpha}-f_{j\alpha}+h_{j+1\alpha}-h_{j\alpha}]})\\=\mathcal{M}(\mathtt{f}_\alpha,\mathtt{h}_\alpha),\end{aligned}\tag{5.12}$$

where $c=c_0/c_1$ *and (nonlinear and nonlocal) function* $\mathcal{M}(\mathtt{f}_\alpha,\mathtt{h}_\alpha)$ *satisfies:*

$$\|\mathcal{M}(\mathtt{f}_\alpha,\mathtt{h}_\alpha)\|_{\theta\alpha}\leq C\alpha^{\mu'}(\alpha^2+\|\mathtt{h}_\alpha\|_{\theta\alpha}),\quad \frac{3}{4}>\mu'>\frac{1}{2}.\tag{5.13}$$

Moreover, if $\mathtt{f}_\alpha$, $\mathtt{h}_\alpha$ *are even functions of the variable* z *so is* $\mathcal{M}(\mathtt{f}_\alpha,\mathtt{h}_\alpha)$*. In addition* $\mathcal{M}(\mathtt{f}_\alpha,\mathtt{h}_\alpha)$ *is a Lipschitz function with respect to the second variable and we have:*

$$\|\mathcal{M}(\mathtt{f}_\alpha,\mathtt{h}^{(1)}_\alpha)-\mathcal{M}(\mathtt{f}_\alpha,\mathtt{h}^{(2)}_\alpha)\|_{\theta\alpha}\leq C\alpha^{\mu'}\|\mathtt{h}^{(1)}_\alpha-\mathtt{h}^{(2)}_\alpha\|_{\theta\alpha}.\tag{5.14}$$

5.2. **The Toda system and its linearization.** We will briefly outline the theory necessary for solving (5.12).

As suggested by the form of (5.12), first we will consider the Toda system for even functions f_j, $j=1,\dots,k$:

$$cf''_j-(e^{-\sqrt{2}(f_j-f_{j-1})}-e^{-\sqrt{2}(f_{j+1}-f_j)})=0,\quad j=1,\dots,k.\tag{5.15}$$

together with the initial conditions

$$\begin{cases}f_j(0)=x_{0j},\\ f'_j(0)=0,\end{cases}\qquad j=1,\dots,k\tag{5.16}$$

It is convenient to consider our problem in a slightly more general framework then that of the system (5.15)–(5.16). Thus for given functions $q_j(t),p_j(t)$, $j=1,\dots,k$ such that

$$\sum_{j=1}^k q_j=\sum_{j=1}^k p_j=0,$$

we define the Hamiltonian

$$H=\sum_{j=1}^k\frac{p_j^2}{2}+V,\quad V=\sum_{j=1}^{k-1}e^{(q_j-q_{j+1})}.$$

We consider the following Toda system

$$\begin{aligned}\frac{dq_j}{dt}&=p_j,\\ \frac{dp_j}{dt}&=-\frac{\partial H}{\partial q_j}\\ q_j(0)&=q_{0j},\quad p_j(0)=0\qquad j=1,\dots,k.\end{aligned}\tag{5.17}$$

Solutions to (5.17) are of course even. Observe that also that the location of their center of mass remains fixed. Thus to mode out translations we will assume that

$$\sum_{j=1}^k q_{0j}=0.\tag{5.18}$$

We will now give a more precise of these solutions and in particular their asymptotic behavior as $z \to \pm\infty$. To this end we will often make use of classical results of Konstant [18] (c.f. [24]) and in particular we will use the explicit formula for the solutions of (5.17) (see formula (7.7.10) in [18]).

We will first introduce some notation. Given numbers $w_1, \dots, w_k \in \mathbb{R}$ such that

$$\sum_{j=1}^{k} w_j = 0, \quad \text{and } w_j > w_{j+1}, \quad j = 1, \dots, k \tag{5.19}$$

we define the matrix

$$\mathbf{w}_0 = \operatorname{diag}(w_1, \dots, w_k).$$

Furthermore, given numbers $g_1, \dots, g_k \in \mathbb{R}$ such that

$$\prod_{j=1}^{k} g_j = 1, \quad \text{and } g_j > 0, \quad j = 1, \dots, k, \tag{5.20}$$

we define the matrix

$$\mathbf{g}_0 = \operatorname{diag}(g_1, \dots, g_k).$$

The matrices $\mathbf{w}_0$ and $\mathbf{g}_0$ can be parameterized by introducing the following two sets of parameters

$$c_j = w_j - w_{j+1}, \quad d_j = \log g_{j+1} - \log g_j, \quad j = 1, \dots, k. \tag{5.21}$$

Furthermore, we define functions $\Phi_j(\mathbf{g}_0, \mathbf{w}_0; t)$, $t \in \mathbb{R}$, $j = 0, \dots, k$, by

$$\begin{aligned}
&\Phi_0 = \Phi_k \equiv 1\\
&\Phi_j(\mathbf{g}_0, \mathbf{w}_0; t) =\\
&(-1)^{j(k-j)} \sum_{1 \le i_i < \dots < i_j \le k} r_{i_1 \dots i_j}(\mathbf{w}_0) g_{i_1} \dots g_{i_j} \exp[-t(w_{i_1} + \dots + w_{i_j})],
\end{aligned} \tag{5.22}$$

where $r_{i_1 \dots i_j}(\mathbf{w}_0)$ are rational functions of the entries of the matrix $\mathbf{w}_0$. It is proven in [18] that all solutions of (5.17) are of the form

$$q_j(t) = \log \Phi_{j-1}(\mathbf{g}_0, \mathbf{w}_0; t) - \log \Phi_j(\mathbf{g}_0, \mathbf{w}_0; t), \quad j = 1, \dots, k, \tag{5.23}$$

Namely, given initial conditions in (5.17) there exist matrices $\mathbf{w}_0$ and $\mathbf{g}_0$ satisfying (5.19)–(5.20). According to Theorem 7.7.2 of [18], it holds

$$q_j^{'}(+\infty) = w_{k+1-j}, \quad q_j^{'}(-\infty) = w_j, \quad j = 1, ..., k. \tag{5.24}$$

We introduce variables

$$u_j = q_j - q_{j+1}. \tag{5.25}$$

In terms of $\mathbf{u} = (u_1, \dots, u_{k-1})$ system (5.17) becomes

$$\begin{aligned}
&\mathbf{u}'' - M e^{\mathbf{u}} = 0,\\
&u_j(0) = q_{0j} - q_{0j+1}, \quad u_j'(0) = 0, \quad j = 1, \dots, k-1,
\end{aligned} \tag{5.26}$$

where

$$M = \begin{pmatrix} 2 & -1 & 0 \cdots & 0 \\ -1 & 2 & -1 \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 2 & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}, \quad e^{-\mathbf{u}} = \begin{pmatrix} e^{u_1} \\ \vdots \\ e^{u_{k-1}} \end{pmatrix}.$$

As a consequence of (5.22) all solutions to (5.26) are given by

$$\begin{aligned} u_j(t) = q_j(t) - q_{j+1}(t) &= -2\log\Phi_j(\mathbf{g}_0, \mathbf{w}_0; t) + \log\Phi_{j-1}(\mathbf{g}_0, \mathbf{w}_0; t) \\ &\quad + \log\Phi_{j+1}(\mathbf{g}_0, \mathbf{w}_0; t). \end{aligned} \tag{5.27}$$

Our next result is the following:

Lemma 5.2. *Let* $\mathbf{w}_0$ *be such that*

$$\min_{j=1,\dots,k-1}(w_j - w_{j+1}) = \vartheta > 0. \tag{5.28}$$

Then there holds

$$u_j(t) = \begin{cases} -c_{k-j}t - d_{k-j} + \tau_j^+(\mathbf{c}) + O(e^{-\vartheta|t|}), \text{ as } t \to +\infty, & j = 1, \dots, k-1, \\ c_j t + d_j + \tau_j^-(\mathbf{c}) + O(e^{-\vartheta|t|}), \text{ as } t \to -\infty, & j = 1, \dots, k-1, \end{cases} \tag{5.29}$$

where $\tau_j^\pm(\mathbf{c})$ *are smooth functions of the vector* $\mathbf{c} = (c_1, \dots, c_{k-1})$.

Proof. This Lemma has been proven in [8]. We include a proof here for completeness. Let q_j, $j = 1, \dots, k$ be a solution of the system (5.17) depending on the (matrix valued) parameters $\mathbf{w}_0$, $\mathbf{g}_0$ and defined in (5.23). We need to study the asymptotic behavior of $\Phi_j(\mathbf{w}_0, \mathbf{g}_0; t)$ as $t \to \pm\infty$ with the entries of $\mathbf{w}_0$ satisfying (5.28) and still undetermined $\mathbf{g}_0$.

By (5.22) and (5.19), we get that as $t \to -\infty$

$$\Phi_j = (-1)^{j(k-j)} r_{1\dots j}(\mathbf{w}_0) g_1 \dots g_j e^{-(w_1+\dots+w_j)t}(1 + O(e^{-(w_j - w_{j-1})t}),$$

hence

$$\frac{\Phi_{j+1}\Phi_{j-1}}{\Phi_j^2} = \frac{g_{j+1} r_{1\dots(j-1)}(\mathbf{w}_0) r_{1\dots(j+1)}(\mathbf{w}_0) e^{-c_j t}}{g_j r_{1\dots j}^2(\mathbf{w}_0)}(1 + O(e^{-\vartheta|t|})). \tag{5.30}$$

It follows that as $t \to -\infty$

$$\begin{aligned} u_j(t) &= \log\left(\frac{\Phi_{j+1}\Phi_{j-1}}{\Phi_j^2}\right) \\ &= -c_j t + \log\left(\frac{g_{j+1} r_{1\dots(j-1)}(\mathbf{w}_0) r_{1\dots(j+1)}(\mathbf{w}_0)}{g_j r_{1\dots j}^2(\mathbf{w}_0)}\right) + O(e^{-\vartheta|t|}) && (5.31) \\ &= c_j t + d_j + \tau_j^-(\mathbf{c}) + O(e^{-\vartheta|t|}), && (5.32) \end{aligned}$$

where

$$\tau_j^-(\mathbf{c}) = \log\left(\frac{r_{1\dots(j-1)}(\mathbf{w}_0) r_{1\dots(j+1)}(\mathbf{w}_0)}{r_{1\dots j}^2(\mathbf{w}_0)}\right).$$

Similarly, as $t \to +\infty$ we get

$$\begin{aligned} u_j(t) &= \log\left(\frac{\Phi_{j+1}\Phi_{j-1}}{\Phi_j^2}\right) \\ &= -c_{k-j}t - d_{k-j} + \tau_j^+ + O(e^{-\vartheta|t|}), && (5.33) \end{aligned}$$

where

$$\tau_j^+(\mathbf{c}) = \log\left(\frac{r_{k+2-j\dots k}(\mathbf{w}_0) r_{k-j\dots k}(\mathbf{w}_0)}{r_{k+1-j\dots k}^2(\mathbf{w}_0)}\right).$$

This ends the proof. □

To find a family of solutions parameterized by α starting from a solution of (5.17) we use functions u_j and set

$$u_{j\alpha}(z) = u_j(\alpha z) - 2\log\frac{1}{\alpha} - \log\frac{c}{\sqrt{2}}. \tag{5.34}$$

Then functions $f_{j\alpha}(z)$ are obtained from the relations

$$\begin{gathered} u_{j\alpha}(z) = \sqrt{2}\big(f_{j\alpha}(z) - f_{j+1\alpha}(z)\big), \\ \sum_{j=1}^{k} f_{j\alpha}(z) = 0. \end{gathered} \tag{5.35}$$

Observe that as a consequence we get that there exist w_j, g_j, $j = 1, \dots, k$ such that (5.19)–(5.20) holds, that

$$\min_{j=1,\dots,k}(w_j - w_{j+1}) = \vartheta > 0,$$

and functions $f_{j\alpha}$ satisfy

$$\begin{aligned} &\|f''_{j\alpha} e^{\vartheta\alpha|z|}\|_\infty \le C\alpha^2, \\ &f'_{j\alpha}(\infty) = \beta_j = f'_{j\alpha}(-\infty), \quad \text{where } \beta_{j+1} - \beta_j = (w_j - w_{j+1})\alpha > \vartheta\alpha, \\ &f_{j\alpha}(z) - f_{j-1\alpha}(z) \ge \sqrt{2}\log\frac{1}{\alpha} + \frac{1}{\sqrt{2}}\log\frac{c}{\sqrt{2}}. \end{aligned} \tag{5.36}$$

In this case we take $\theta = \frac{1}{4}\vartheta$. Relations (5.36) are easily seen to be consistent with assumptions (2.1)–((2.2).

Next we will continue with preliminaries needed to solve (5.13). We will study the linearization of the system (5.13) around the solution $\mathtt{f}_\alpha$ of the Toda system defined in (5.35) and (5.36). We will always assume that $\alpha > 0$ is small and $\theta > 0$ is a fixed constant. We are lead to the following linear system

$$\begin{gathered} \vec{\phi}'' - \begin{pmatrix} 2e^{u_{1\alpha}} & -e^{u_{2\alpha}} & 0\cdots & 0 \\ -e^{u_{1\alpha}} & 2e^{u_{2\alpha}} & -e^{u_{3\alpha}}\cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 2e^{u_{k-2\alpha}} & -e^{u_{k-1\alpha}} \\ 0 & \cdots & -e^{u_{k-2\alpha}} & 2e^{u_{k-1\alpha}} \end{pmatrix} \vec{\phi}^T = \vec{p}, \\ \vec{\phi} = (\phi_1, \dots, \phi_{k-1}), \quad \vec{p} = (p_1, \dots, p_{k-1}), \end{gathered} \tag{5.37}$$

where $\vec{p}$ is an even function such that

$$\|\vec{p}\|_{\theta\alpha} \le C\alpha^{2+\mu'}. \tag{5.38}$$

We will analyze the solvability of this problem in the space of even C^2 functions $\vec{\phi}$ such that

$$\|\vec{\phi}\|_{\theta\alpha} < \infty. \tag{5.39}$$

Lemma 5.3. *Let us assume that $\vec{p}$ is an even function of z and that (5.38) holds. Then problem (5.37) has a bounded, even solution $\vec{\phi} = \mathcal{R}[\vec{p}]$. Moreover we have*

$$\|\mathcal{R}[\vec{p}]''\|_{\theta\alpha} \le C\|\vec{p}\|_{\theta\alpha}, \tag{5.40}$$

$$\|\mathcal{R}[\vec{p}_1] - \mathcal{R}[\vec{p}_2]\|_{\theta\alpha} \le C\|\vec{p}_1 - \vec{p}_2\|_{\theta\alpha}. \tag{5.41}$$

Proof. We first observe that

$$g_j \frac{\partial u_{m\alpha}}{\partial g_j} = \alpha \begin{cases} 1, & j = m+1, \quad z \to \infty, \\ -1, & j = m, \quad z \to \infty \\ 1, & j = k+2-m, \quad z \to -\infty, \\ -1, & j = k+1-m, \quad z \to -\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Hence by a transformation we can find a set of linearly independent solutions to the homogenous version of (5.37)

$$\psi_{1j\alpha}(z) = \begin{cases} \alpha \vec{e}_j, & z \to \infty, \\ -\alpha \vec{e}_j, & z \to -\infty. \end{cases}$$

Notice that functions $\psi_{1j\alpha}$ are odd. Similarly, considering derivatives of $u_{m\alpha}$ with respect to w_j we can find solutions of (5.37), $\psi_{2j}(z)$, $j = 1, \ldots, k-1$ which are even and such that

$$\psi_{2j\alpha}(t) = \alpha \vec{e}_j |z| + O(1).$$

Functions $\{(\psi_{1j}(z), \psi'_{1j}(z)), (\psi_{2j}(z), \psi'_{2j}(z))\}$ form a fundamental set for the system (5.37). Let us denote the fundamental matrix of the system (5.37) described above by $\Psi_\alpha(z)$ and the right hand side of the transformed system by $\vec{q}$. Let us denote also (with some abuse) by $\vec{\phi}$ the solution we are after. We observe that as $z \to \infty$, matrices $\Psi_\alpha, \Psi_\alpha^{-1}$ are block matrices of the form

(5.42)
$$\Psi_\alpha(z) = \begin{pmatrix} \alpha I + o(1) & \alpha z I + O(1) \\ o(1) & \alpha I + o(1) \end{pmatrix}, \quad \Psi_\alpha^{-1} = \begin{pmatrix} \alpha I + o(1) & -\alpha z I + O(1) \\ o(1) & \alpha I + o(1) \end{pmatrix}.$$

Let us denote these blocks by $\Psi_{mn\alpha}$, $\Psi^{-1}_{mn\alpha}$, $m, n = 1, 2$, respectively. Then, from variation of parameters formula we get that the solution of our problem has form

(5.43)
$$\begin{aligned} (\vec{\phi}(z), \vec{\phi}'(z)) = {} & \Psi_\alpha(z) \cdot \int_0^z \Psi_\alpha^{-1}(s) \cdot (\mathbf{0}, \vec{q}(s))^T \, ds \\ & - \big(\Psi_{12\alpha}(z) \int_0^\infty \Psi^{-1}_{22\alpha}(s) \vec{q}(s) \, ds, \Psi_{22\alpha}(z) \int_0^\infty \Psi^{-1}_{22\alpha}(s) \vec{q}(s) \, ds\big). \end{aligned}$$

The key fact which allows us to conclude that the solution given in (5.43) is indeed bounded as $|z| \to \infty$ is that

$$\int_{-\infty}^{\infty} \Psi^{-1}_{22\alpha}(s) \vec{q}(s) \, ds = 0.$$

This follows from the evenness of $\vec{p}$ and hence of $\vec{q}$ and the oddness of $\Psi^{-1}_{22\alpha}$. Using now (5.43) we can directly estimate

$$\|\vec{\phi}\|_{\theta\alpha} \le C \|\vec{q}\|_{\theta\alpha},$$

from which we infer (5.40). Formula (5.41) follows from the variation of parameters formula as well. The proof of the lemma is complete. □

5.3. **Resolution of the reduced problem.** Now we are in position to solve problem (5.12). First let us fix the solution of the Toda system given by the formula (5.35). Further let us denote

$$\begin{aligned}\mathcal{N}_j(\mathtt{h}_\alpha) = &(e^{-\sqrt{2}[f_{j\alpha}-f_{j-1\alpha}+h_{j\alpha}-h_{j-1\alpha}]} - e^{-\sqrt{2}[f_{j+1\alpha}-f_{j\alpha}+h_{j+1\alpha}-h_{j\alpha}]})\\ &-(e^{-\sqrt{2}[f_{j\alpha}-f_{j-1\alpha}]} - e^{-\sqrt{2}[f_{j+1\alpha}-f_{j\alpha}]})\\ &-\sqrt{2}[e^{-\sqrt{2}[f_{j\alpha}-f_{j-1\alpha}]}(h_{j\alpha}-h_{j-1\alpha}) - e^{-\sqrt{2}[f_{j+1\alpha}-f_{j\alpha}]}(h_{j+1\alpha}-h_{j\alpha})],\end{aligned}$$

and set $\mathcal{N} = (\mathcal{N}_1, \dots, \mathcal{N}_k)$. Then problem (5.12) can be set up as a fixed point problem for $\mathtt{h}_\alpha$ in a ball

$$B(D\alpha^{2+\mu'}) = \{\mathtt{h}_\alpha \mid \|\mathtt{h}_\alpha\|_{\theta\alpha} < D\alpha^{2+\mu'}\},$$

where D is a large constant depending on the constant C appearing in the estimate (5.13). Indeed assuming that $\mathtt{h}_\alpha \in B(D\alpha^{2+\mu'})$ we get easily

$$\|\mathcal{N}(\mathtt{h}_\alpha)\|_{\theta\alpha} \le C\alpha^{2+2\mu'}.$$

Problem (5.12) after linearization around the fixed solution of the Toda system $\mathtt{f}_\alpha$ can be put in terms described in previous section by setting

$$\begin{aligned}&u_{j\alpha} = \sqrt{2}\big(f_{j\alpha}(z) - f_{j+1\alpha}(z)\big),\\ &\phi_{j\alpha} = \sqrt{2}\big(h_{j\alpha}(z) - h_{j+1\alpha}(z)\big), \quad \vec{\phi_\alpha} = (\phi_{1\alpha}, \dots, \phi_{k\alpha}),\\ &p_{j\alpha} = \mathcal{M}_j + \mathcal{N}_j - \mathcal{M}_{j+1} - \mathcal{N}_{j+1}, \quad \vec{p_\alpha} = (p_{1\alpha}, \dots, p_{k\alpha}).\end{aligned}$$

Using then the operator $\mathcal{R}$ defined in Lemma 5.3 we get the solution of (5.12) by a standard fixed point argument. This completes the construction of the solution of (1.1) described in the statement of Theorem 1.1.

Acknowledgments: This work has been partly supported by chilean research grants Fondecyt 1070389, 1050311, FONDAP, Nucleus Millennium grant P04-069-F, an ECOS-CONICYT contract C05E05 and an Earmarked Grant from RGC of Hong Kong.

References

[1] S. Allen and J. W. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening,* Acta. Metall. 27 (1979), 1084–1095.

[2] M.T. Barlow, R.F. Bass and C. Gui, *The Liouville property and a conjecture of De Giorgi*, Comm. Pure Appl. Math. 53 (8) (2000), 1007-1038.

[3] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi,* Journal Amer. Math. Soc. 13 (2000), 725–739.

[4] H. Berestycki, F. Hamel and R. Monneau, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. 103 (2000), 375-396.

[5] H. Dang, P.C. Fife and L.A. Peletier, *Saddle solutions of the bistable diffusion equation*, Z. Angew. Math. Phys. Vol. 43 (1992).

[6] M. del Pino, M. Kowalczyk and F. Pacard, *Moduli space theory for some class of solutions to the Allen-Cahn equation in the plane*, preprint (2008).

[7] M. del Pino, M. Kowalczyk, F. Pacard and J. Wei, *The Toda system and multiple-end solutions of autonomous planar elliptic problems*, preprint (2007).

[8] M. del Pino, M. Kowalczyk and J. Wei, *Concentration on curves for nonlinear Schrödinger equations*, Comm. Pure Appl. Math. 70 (2007), 113-146.

[9] M. del Pino, M. Kowalczyk and J. Wei, *The Toda system and clustering interfaces in the Allen-Cahn equation*, Arch. Rat. Mech. Anal. 190 (2008), 141-187.

[10] M. del Pino, M. Kowalczyk and J. Wei, *A conjecture by De Giorgi in large dimensions.* Preprint 2008.

[11] M. del Pino, M. Kowalczyk, J. Wei and J. Yang *Interface foliation of a positively curved manifold near a closed geodesic*. Preprint (2008).

[12] A. Farina, *Symmetry for solutions of semilinear elliptic equations in $\mathbb{R}^N$ and related conjectures*, Ricerche Mat. 48(suppl.) (1999), 129–154.

[13] N. Ghoussoub and C. Gui, *On a conjecture of de Giorgi and some related problems*, Math. Ann. Vol. 311 (1998), 481-491.

[14] C. Gui, *Hamiltonian identities for elliptic partial differential equations.* J. Funct. Anal. 254 (2008), no. 4, 904-933.

[15] C. Gui, *Allen Cahn equation and its generalizations.* Preprint 2008.

[16] R. V. Kohn and P. Sternberg, *Local minimizers and singular perturbations*, Proc. Royal Soc. Edinburgh **111 A** (1989), 69–84.

[17] J. E. Hutchinson and Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. 10 (2000), 49–84.

[18] B. Konstant, *The solution to a generalized Toda lattice and Representation Theory*, Adv. Math. **34** (1979), 195–338.

[19] M. Kowalczyk, *On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions*, Ann. Mat. Pura Appl. **184** 4(2005), no. 1, 17–52.

[20] A. Malchiodi and M. Montenegro, *Boundary concentration phenomena for a singularly perturbed elliptic problem.* Comm. Pure Appl. Math. 55 (2002), no. 12, 1507–1568.

[21] A. Malchiodi and M. Montenegro, *Multidimensional boundary layers for a singularly perturbed Neumann problem.* Duke Math. J. 124 (2004), no. 1, 105–143.

[22] A. Malchiodi, *Some new entire solutions of semilinear elliptic equations on $\mathbb{R}^n$.* Preprint.

[23] L. Modica, *Convergence to minimal surfaces problem and global solutions of $\Delta u = 2(u^3 - u)$.* Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 223–244, Pitagora, Bologna, (1979).

[24] J. Moser, *Finitely many mass points on the line under the influence of an exponential potential–an integrable system.* Dynamical systems, theory and applications (Rencontres, BattelleRes. Inst., Seattle, Wash., 1974), pp. 467–497. Lecture Notes in Phys., Vol. 38, Springer, Berlin, 1975.

[25] F. Pacard and M. Ritoré, *From the constant mean curvature hypersurfaces to the gradient theory of phase transitions*, J. Differential Geom. 64 (2003), no. 3, 356–423.

[26] O. Savin, *Regularity of flat level sets in phase transitions.* To appear in Ann. of Math.

[27] J. Wei and J. Yang, *Interface foliation of a minimal submanifold on Riemannian manifolds with positive Ricci curvature.* Preprint (2008).

M. del Pino - Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.

E-mail address: `delpino@dim.uchile.cl`

M. Kowalczyk - Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.

E-mail address: `kowalczy@dim.uchile.cl`

Franck Pacard - Université Paris Est and Institut Universitaire de France.

E-mail address: `pacard@univ-paris12.fr`

J. Wei - Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong.

E-mail address: `wei@math.cuhk.edu.hk`