JOURNAL OF FUNCTIONAL ANALYSIS **157**, 292–325 (1998)
ARTICLE NO. FU983271

On Conformal Deformations of Metrics on S^n

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Received November 17, 1997; accepted January 19, 1998

On S^n, there is a naturally metric defined nth order conformal invariant operator P_n. Associated with this operator is a so-called Q-curvature quantity. When two metrics are pointwise conformally related, their associated operators, together with their Q-curvatures, satisfy the natural differential equations. This paper is devoted to the question of which function can be a Q-curvature candidate. This is the so-called *prescribing Q-curvature problem*. Our main result is that if Q is positive, nondegenerate and the naturally defined mapping associated with Q has nonzero degree, then our problem has a solution. This is the natural generalization of prescribing Gaussian curvature on S^2 into S^n. © 1998 Academic Press

Key Words: Conformally invariant operators; Q_n-curvature; higher order elliptic differential equations.

1. INTRODUCTION

On a general Riemannian manifold M with metric g, a metrically defined operator A is said to be *conformally invariant* if, under the conformal change in metric $g_w = e^{2w}g$, the pair of corresponding operators A_w and A are related by

$$A_w(\varphi) = e^{-bw}A(e^{aw}\varphi) \tag{1.1}$$

for all $\varphi \in C^\infty(M)$ and some constants a and b.

One such well-known second-order conformally invariant operator is the conformal Laplacian which is closely related to the Yamabe problem and,

0022-1236/98 $25.00

more generally, to the problem of prescribing scalar curvature: *Given a smooth positive function K defined on a compact Riemannian manifold* (M, g_0) *of dimension* $n \geqslant 2$, *does there exist a metric g conformal to* g_0 *for which K is the scalar curvature of the new metric g*?

If $g = e^{2u} g_0$ for $n = 2$ or $g = u^{4/(n-2)} g_0$ for $n \geqslant 3$, our problem is reduced to finding solutions to the following nonlinear elliptic equations:

$$\Delta_{g_0} u + K e^{2u} = k_0 \tag{1.2}$$

for $n = 2$, or

$$\begin{cases} \dfrac{4(n-1)}{n-2} \Delta_{g_0} u + K u^{(n+2)/(n-2)} = k_0 u \\ u > 0 \qquad \text{on } M \end{cases} \tag{1.3}$$

for $n \geqslant 3$. (Here Δ_{g_0} denotes the Laplace–Beltrami operator of (M, g_0), k_0 is the Gaussian curvature of g_0 when $n = 2$ and the scalar curvature of g_0 when $n \geqslant 3$.)

The problem of determining which K admits a solution to (1.2) (or (1.3)) has been studied extensively. See [1, 5, 18] and the references therein.

In search for a higher order conformally invariant operator, Paneitz [16] discovered an interesting 4th-order operator on a compact 4-manifold

$$P_4 \varphi = \Delta^2 \varphi + \Delta(\tfrac{2}{3} RI - 2\, Ric)\, d\varphi$$

where δ denotes the divergence, d the differential, and Ric the Ricci curvature of the metric g. Under the conformal change $g_w = e^{2w} g$, P_4 undergoes the transformation $(P_4)_w = e^{-4w} P_4$ (i.e., $a = 0, b = 4$ in (1.1)). See [2, 4, 8, 10, 11] for a discussion of general properties of Paneitz operators.

On a general compact manifold of dimension n, the existence of such an operator P_n with $(P_n)_w = e^{-nw} P_n$ for even dimension is established in [12]. However P_n's form is known explicitly only for Euclidean space R^n with standard metric ($P_n = (-\Delta)^{n/2}$) and hence only for the sphere S^n with standard metric g_0. The explicit formula for P_n on S^n which appears in [2] and [3] is

$$P_n = \begin{cases} \displaystyle\prod_{k=1}^{(n-2)/2} (-\Delta + k(n-k-1)), & \text{for } n \text{ even}, \\ \displaystyle\left(-\Delta + \left(\frac{n-1}{2}\right)^2\right)^{1/2} \prod_{k=0}^{(n-3)/2} (-\Delta + k(n-k-1)), & \text{for } n \text{ odd}. \end{cases}$$

Analogous to the second-order case there exists some naturally defined curvature invariance Q_n of order n which, under the conformal change of metric $g_w = e^{2w} g_0$, is related to $P_n w$ through the following differential equation

$$P_n w + (Q_n)_0 = (Q_n)_w \, e^{nw} \qquad \text{on } M. \tag{1.4}$$

Stimulated by the problem of the prescribing Gaussian curvature on S^2, we pose the following prescribing Q_n-curvature problem on S^n: *Given a smooth function Q on S^n, find a conformal metric $g_w = e^{2w} g_0$ for which $(Q_n)_w = Q$.*

We remark that there is a similar problem for general compact Riemannian manifolds. But since, in this case, the explicit expression for the operator P_n is unknown, we will not address the general prescribing Q_n curvature problem.

Clearly the above question is equivalent to finding a solution of the differential equation

$$P_n w + (n-1)! = Q e^{nw} \qquad \text{on } S^n. \tag{1.5}$$

The purpose of this paper is to determine for which Q Eq. (1.5) admits a solution. By simple integration (1.5) on S^n, we observe that Q must be positive somewhere on S^n. Thus without loss of generality, we restrict ourselves to the case where $Q > 0$ on S^n. We then observe that the well-known Kazdan–Warner obstruction holds (see Lemma 2.4 or [8]);

$$\int_{S^n} \langle \nabla Q, \nabla x_j \rangle \, e^{nw} \, d\sigma = 0, \qquad j = 1, ..., n+1. \tag{1.6}$$

Thus functions of the form $Q = \psi \circ x_j$, where ψ is any monotonic function defined on $[-1, 1]$, do not admit solutions. Finally, motivated by the prescribing Gaussian curvature case, we expect that the conditions $Q > 0$ and (1.6) are insufficient to solve Eq. (1.5). We hope to return to this point in the future.

To state our main result, we define a map G associated to the function Q by using the action of the conformal group of S^n. As in [5], we consider the following set of conformal transformations of $S^n (n \geqslant 2)$: given $x \in S^n$, $t \geqslant 1$, using y as the stereographic projection from $S^n - \{x\}$ (where x is the north pole) to the equatorial plane $y_1, y_2, ..., y_n$. Let $\varphi_{x,t}$ be the conformal map of S^n given by $\varphi_{x,t}(y) = ty$. The totality of all such conformal transformations comprises a set which is diffeomorphic to the unit ball B^{n+1} in $\mathbb{R}^{n+1}$, with the identity transformation identified with the origin in B^{n+1} and $\phi_{x,t} \leftrightarrow ((t-1)/t)\, x = p \in B^{n+1}$ in general. We construct the map $G \colon B^{n+1} \to \mathbb{R}^{n+1}$ by setting

$$G(p) = G\left(\frac{t-1}{t} x\right) = \fint_{S^n} (Q \circ \varphi_{x,t}) \cdot \vec{x} \, d\sigma$$

For large values of t, the asymptotic behavior of $G(p)$ is determined by the leading coefficient of the Taylor series development of Q near the point $-x$. In general, $G(p)$ is non-zero for large values of t if the low order Taylor series coefficients at $-P$ are suitably non-degenerate. In particular, if the function Q satisfies the following non-degeneracy condition

$$\Delta Q(x) \neq 0 \qquad \text{whenever} \quad \nabla Q(x) = 0, \qquad (nd)$$

the map G does not vanish for large values of t (so that deg $(G, B^{n+1}, 0)$ is well defined).

The following is the main result of this paper.

Main Theorem. *On S^n, suppose $Q > 0$ is a smooth function satisfying the non-degeneracy condition* (nd) *and deg* $(G, B^{n+1}, 0) \neq 0$, *then equation* (1.5) *has a solution.*

Remark. There are similar results for the problem of prescribing scalar curvature problem on S^n. In [6] (where they studied Eq. (1.2) on S^2) and [1] (where they studied Eq. (1.3) on S^3), it is assumed that the curvature function K is positive, has only isolated non-degenerate critical points and in addition satisfies $\Delta K(Q) \neq 0$ at critical points, as well as the index count condition:

$$\sum_{Q \text{ critical},\, \Delta K(Q) < 0} (-1)^{ind(Q)} \neq (-1)^n.$$

We point out that these conditions alone are insufficient to ensure a solution to the problem of prescribing scalar curvature in general dimension n. Our question has a solution under these conditions alone, yet we do not know what the real reasons are.

Our main theorem here was motivated by the results of [5] and [7] where they generalized the above results of [6] and [1]. Under a similar condition to (nd) on the curvature function K and a similar degree condition, they proved the existence of solutions for Eq. (1.2) on S^2 and Eq. (1.3) on S^3.

In the remaining part of this section, we outline our proof of the main theorem. We first introduce some notation. Let

$$\mathscr{S} \equiv \left\{ w \in H^{n/2,\,2}(S^n) \,\middle|\, \fint_{S^n} e^{nw} x_j \, d\sigma = 0,\ j = 1, 2, ..., n+1 \right\}$$

and

$$\mathscr{S}_0 \equiv \left\{ w \in \mathscr{S} \,\middle|\, \fint_{S^n} e^{nw} \, d\sigma = 1 \right\}.$$

Our proof is divided into two parts. In the first part, we derive a perturbation result. Given $x \in S^n$, $t \in [1, \infty)$, let $p = ((t-1)/t)\, x \in B^{n+1}$. For each Q_n-curvature candidate Q, we consider the new candidate $Q_p = Q \circ \varphi_p$ and the functional

$$F_p[w] = \log \fint_{S^n} Q_p e^{nw}\, d\sigma - \frac{n}{2(n-1)!} S_n[w]. \tag{1.7}$$

where $S_n[w] = \fint_{S^n} (P_n w)\, w d\sigma + 2(n-1)! \fint_{S^n} w d\sigma$.

Let

$$\mathcal{M}_p = \sup_{w \in \mathcal{S}_0} F_p[w].$$

Under the condition that $\varepsilon_Q = \| Q - (n-1)! \|_\infty$ is very small, we show that $\mathcal{M}_p$ is achieved by an extremal function w_p. The Euler equation for w_p is written as

$$P_n w + (n-1)! = (Q_p - \vec{\Lambda}_p \cdot \vec{x})\, e^{nw}. \tag{1.8}$$

We show that, given $p \in B^{n+1}$, w_p is uniquely determined and w_p, as well as the Langrange multiplier $\vec{\Lambda}_p$, vary continuously in p. Hence, we may consider $\vec{\Lambda}: B^{n+1} \to \mathbb{R}^{n+1}$ as a continuous map. We will show that, as $t \to \infty$ (or equivalently $r = (t-1)/t \to 1$), $\vec{\Lambda}$ restricted to φB_r^{n+1} has the same degree as $G|_{\varphi B_r^{n+1}}$, provided there is a neighborhood of φB_r^{n+1} where $\vec{\Lambda}$ has no zero. Therefore, under the hypothesis of the theorem we have deg $(0, \vec{\Lambda}_{\varphi B_r^{n+1}}, 0) \neq 0$ and hence the Langrange multiplier $\vec{\Lambda}_p$ must vanish for some p, $|p| < r$. By a simple conformal transformation this means that the original Eq. (1.5) has a solution when $\varepsilon_Q = \| Q - (n-1)! \|_\infty$ is very small.

In the second part, we use a continuity method. We join the curvature function Q to the constant function $Q_0 = (n-1)!$ by one parameter family of functions

$$Q_s = sQ + (1-s)\, Q_0 \qquad (0 \leqslant s \leqslant 1)$$

and consider the family of differential equations

$$(Q_s) \quad P_n w + (n-1)! = Q_s e^{nw}.$$

We show that under the hypothesis of nondegeneracy (nd), all solutions of the Eq. (Q_s) are uniformly bounded by a constant independent of s and Q_s. This provides a continuity argument needed to verify the invariance of the Leray–Schauder degree as one moves along the parameter s in the continuity scheme. A topological degree argument then completes the proof of the main theorem.

We briefly outline the organization of the paper. In Section 2, we prove an improved Beckner inequality which allows the rest of our argument to follow. In Section 3, we obtain a priori estimate for solutions of Eq. (1.5) with Q satisfying condition (nd) by a blow up argument and Kazdan–Warner obstruction. In Section 4, we finish the first part of our proof—the perturbation argument. Finally in Section 5 we complete the proof of the main theorem by a continuity argument.

2. IMPROVED BECKNER INEQUALITY

In this section, we set up some basic facts about the solutions of Eq. (1.5). Let $(x_1, x_2, ..., x_{n+1})$ denote the ambient coordinates of S^n. Denote

$$\mathscr{S} \equiv \left\{ w \in H^{n/2,2}(S^n) \,\middle|\, \fint_{S^n} e^{nw} x_j \, d\sigma = 0,\ j = 0, 1, 2, \cdots, n+1 \right\} \tag{2.1}$$

$$\mathscr{S}_0 \equiv \left\{ w \in \mathscr{S} \,\middle|\, \fint_{S^n} e^{nw} \, d\sigma = 1 \right\} \tag{2.2}$$

LEMMA 2.1. *Given $w \in H^{n/2,2}(S^n)$ satisfying* (1.5), *there exists a conformal transform $\varphi = \varphi_{p,t}$ of S^n for some $p \in S^n$, $t \in [1, +\infty)$ such that $e^{2v} g_0 = \varphi^*(e^{2w} g_0)$ with $v \in \mathscr{S}$. In addition, v satisfies the equation*

$$-P_n v + (Q \circ \varphi) e^{nv} = (n-1)! \qquad \text{on } S^n. \tag{2.3}$$

Proof. The first statement follows by the fixed-point theorem. The details of this argument can be found in the proof of Lemma 1 of [8] or [20]. The second statement can be verified by a change of variable argument and noticing that P_n is conformally invariant. We leave these to reader. ∎

LEMMA 2.2. *Denote* $S_n[w] = \langle P_n w, w \rangle + 2(n-1)! \fint_{S^n} w \, d\sigma$, *where* $\langle P_n w, w \rangle = \fint_{S^n} (P_n w) w \, d\sigma$. *Then $S_n[w]$ is a conformally invariant quantity in the sense that if v and w are related as in Lemma* 2.1, *then $S_n[v] = S_n[w]$.*

Proof. This statement was proved by Chang and Yang in their fundamental work [9]. See step 1 in their proof of Theorem 4.1. ∎

LEMMA 2.3 (*Beckner's Inequality* [3, 9]). *We always have*

$$\fint_{S^n} e^{nw} \, d\sigma \leqslant \exp\left\{ \frac{n}{2(n-1)!} S_n[w] \right\} \qquad \text{for all} \quad w \in H^{n/2,2}(S^n) \tag{2.4}$$

with equality if and only if $g_0 = \varphi^*(e^{2w} g_0)$ *for some conformal transformation* φ *of* S^n, *i.e., if and only if* $w = 1/n \log(\det(\varphi_*))$.

Proof. See [3, 9] for details. ∎

LEMMA 2.4 (Kazdan–Warner Condition). *Let* M *be a compact Riemannian manifold of dimension* n *without boundary. Let* P_m *be a well-defined conformally invariant operator on* M *and let* Q_n *be the certain quantity for which* (1.4) *holds for any two conformally related metric* $g = e^{2w} g_0$. *If* X *is a conformal vector field, then the quantity* Q *associated with the metric* g *satisfies the condition*

$$\fint_M \langle \nabla Q, X \rangle \, e^{nw} \, d\sigma_0 = \fint_M \langle \nabla Q_n, X \rangle \, d\sigma_0, \tag{2.5}$$

where $d\sigma_0$ *is the volume form with respect to the metric* g_0.

Proof. If X is a conformal vector field on a compact Riemannian manifold (M, g_0) without boundary, then $L_X g_0 = 2w g_0$ for some function w. In fact, w has to be $div_{g_0} X/n$, where n is the dimension of the manifold. By conformal invariance, P_m satisfies

$$P_m \left(X + \frac{n-m}{2} w \right) f = \left(X \frac{n+m}{2} w \right) P_m f, \tag{2.6}$$

for all smooth function f on M. Applying this to the constant function 1, we get

$$\frac{n-m}{2} P_m(w) = \left(X + \frac{n+m}{2} w \right) \frac{n-m}{2} Q.$$

Here we have used the following convention on what Q is. $P_m = \sum a_k \nabla^k + ((n-m)/2) Q$, so that $P_m 1 = ((n-m)/2) Q$. This gives

$$P_m w = \left(X + \frac{n-mw}{2} w \right) Q = \left(X + \frac{n+m}{2n} div \, X \right) Q \tag{2.7}$$

in dimension other than 1, 2, m.

For dimension m, one uses the trick of checking that the aforementioned relation (2.6), divided by $(n-m)/2$ still holds, then argues that the calculation takes place in differential polynomials with coefficients rational in n, so one is entitled to cancel the factor $(n-m)/2$. In any event, we get

$$P_m(div_{g_0} X) = n(X \cdot Q_0 + div_{g_0} X Q_0) \tag{2.8}$$

in dimension $n = m$.

Now let $g = e^{2u} g_0$ be a metric conformally related to go. Then Q_g and Q_0 satisfies the relation

$$Q_g = e^{-nu}(-P_n u + Q_0). \tag{2.9}$$

If ϕ is a conformal transformation, then

$$Q_g \circ \phi = Q_{\phi^*(e^{2u} g_0)} = Q_{e^{2w} g_0}, \tag{2.10}$$

with $w = u \circ \phi + 1/n \log \det(\phi_*)$.

We evaluate the derivative for the flow $(\xi_t)_{t \in R}$ of a conformal vector field at $t = 0$. Clearly, we have

$$\frac{d}{dt}(Q_g \circ \xi_t)\bigg|_{t=0} \quad X \cdot Q_g. \tag{2.11}$$

On the other hand, we also have

$$\begin{aligned} \frac{d}{dt}(Q_{e^{2w_t} g_0})|_{t=0} = e^{-nu} \bigg[& -P_n \left(X \cdot u + \frac{1}{n} div_{g_0} X \right) \\ & - n(-P_n u + Q_0) \left(X \cdot u + \frac{1}{n} div_{g_0} X \right) \bigg]. \end{aligned} \tag{2.12}$$

Combining (2.10), (2.11), and (2.12) we get

$$\begin{aligned} X \cdot Q_g = e^{-nu} \bigg[& -P_n \left(X \cdot u + \frac{1}{n} div_{g_0} X \right) \\ & - n(-P_n u + Q_0) \left(X \cdot u + \frac{1}{n} div_{g_0} X \right) \bigg]. \end{aligned} \tag{2.13}$$

Since M is compact, we can integrate this identity against $d\sigma_g$, the volume element of g. (Recall that $d\sigma_g = e^{nu}\, d\sigma_0$.) We get

$$\fint_M X \cdot Q_g \, d\sigma_g = n \fint_M X \cdot u P_n u \, d\sigma_0 + \fint_M X \cdot Q_0 \, d\sigma_0. \tag{2.14}$$

That the first integral on the right side of (2.14) is zero can be seen from the conformal invariance of the integral

$$\fint_M (P_n u)\, u \, d\sigma_0. \quad \blacksquare$$

COROLLARY 2.5. *If w satisfies equation* (1.5), *then it satisfies the condition*

$$\fint_{S^n} \langle \nabla Q, \nabla x_j \rangle \, e^{nw} \, d\sigma = 0, \qquad \text{for all} \quad j = 1, 2, ..., n+1. \tag{2.15}$$

Proof. Letting $M = S^n$, $X = \nabla x_j$ and recalling that $Q_0 = (n-1)!$ is a constant, the corollary follows. ∎

We shall prove the following theorem, the main result of the present section.

THEOREM 2.6. *There exists a constant $a < 1$ such that*

$$\log \fint_{S^n} e^{nw} \, d\sigma_0 \leqslant \frac{n}{2(n-1)!} \left[a \langle P_n w, w \rangle + 2(n-1)! \fint_{S^n} w \, d\sigma_0 \right] \tag{2.16}$$

for all $w \in S$.

Proof. Let us consider for each $a \leqslant 1$, the functional

$$J_a(w) = \log \fint_{S^n} e^{nw} \, d\sigma_0 - \frac{n}{2(n-1)!} \left(a \langle P_n w, w \rangle + 2(n-1)! \fint_{S^n} w \, d\sigma_0 \right) \tag{2.17}$$

and let $\mathcal{M}_a = \sup_{w \in \mathcal{S}} J_a(w)$. Then by Lemma 4.6 of [8], for each $a > 1/2$, $\mathcal{M}_a$ is achieved by some function $w_a \in \mathcal{S}_0$ which satisfies:

For each $\eta > 0$, there exists a constant C_η with the following property:

$$\langle P_n w_a, w_a \rangle \leqslant C_\eta \qquad \text{for} \quad 1 \geqslant a \geqslant \tfrac{1}{2} + \eta. \tag{2.18}$$

$$-a P_n w_a + (n-1)! \, e^{nw_a} = (n-1)! + \sum_{j=1}^{n+1} (\alpha_j^a x_j) \, e^{nw_a} \qquad \text{on } S^n \tag{2.19}$$

for some constants α_j^a, $j = 1, 2, ..., n+1$.

We claim that

$$w_a \equiv 0 \qquad \text{for } a \text{ sufficiently close to 1.} \tag{2.20}$$

It is clear that our theorem follows from (2.20). Therefore, we only need to show (2.20). To this end, we divide our proof into several steps.

Step 1. In this part, we show that all constants α_j^a are zero. This can be done by our Corollary 2.5 above. In fact, for $a \leqslant 1$, we rewrite Eq. (2.19) as

$$-P_n w_a + Q e^{nw_a} = (n-1)!, \tag{2.21}$$

where $Q = 1/a((n-1)! - \sum_{k=1}^{n+1} \alpha_k^a x_k) - (1/a - 1)(n-1)!\, e^{-nw_a}$. Applying (2.5) to (2.21), we get

$$0 = \fint_{S^n} \langle \nabla Q, \nabla x_j \rangle \, e^{nw_a} \, d\sigma_0$$
$$= \frac{1}{a} \fint_{S^n} \sum_{k=1}^{n+1} \alpha_k^a \langle \nabla x_k, \nabla x_j \rangle \, e^{nw_a} \, d\sigma_0$$
$$+ n\left(\frac{1}{a} - 1\right)(n-1)! \fint_{S^n} \langle \nabla w_a, \nabla x_j \rangle \, d\sigma_0. \tag{2.22}$$

By integrating by parts, using identity (2.8) and the fact that ∇x_j is a conformal vector, we can rewrite the second term as

$$\fint_{S^n} \langle \nabla w_a, \nabla x_j \rangle \, d\sigma_0 = -\fint_{S^n} w_a (div\, \nabla x_j) \, d\sigma_0$$
$$= -\frac{1}{n!} \fint_{S^n} P_n (div\, \nabla x_j) \, w_a \, d\sigma_0$$
$$= -\frac{1}{n!} \fint_{S^n} (div\, \nabla x_j) \, P_n w_a \, d\sigma_0$$
$$= -\frac{1}{an!} \fint_{S^n} div\, \nabla x_j [-(n-1)!]$$
$$- \sum_{k=1}^{n+1} \alpha_k^a x_k e^{nw_a} - (n-1)!\, e^{nw_a} \, d\sigma_0$$
$$= \frac{1}{a(n-1)!} \fint_{S^n} x_j \sum_{k=1}^{n+1} \alpha_k^a x_k e^{nw_a} \, d\sigma_0, \tag{2.23}$$

since $div\, \nabla x_j = -n x_j$.

Plugging (2.23) into (2.22) we get

$$\frac{1}{a} \fint_{S^n} \langle \nabla x_j, \sum_{k=1}^{n+1} \alpha_k^a \nabla x_k \rangle \, e^{nw_a} \, d\sigma_0 = n \frac{1}{a}\left(1 - \frac{1}{a}\right) \fint_{S^n} x_j \sum_{k=1}^{n+1} \alpha_k^a x_k e^{nw_a} \, d\sigma_0. \tag{2.24}$$

Multiplying both sides of (2.24) by α_j^a and summing from $j=1$ to $j = n + l$, we get

$$\frac{1}{a} \fint_{S^n} \left| \sum_{k=1}^{n+1} \alpha_k^a \nabla x_k \right| e^{nw_a} \, d\sigma_0 = \frac{n}{a}\left(1 - \frac{1}{a}\right) \fint_{S^n} \left(\sum_{k=1}^{n+1} \alpha_k^a x_k \right)^2 e^{nw_a} \, d\sigma_0. \tag{2.25}$$

When $a<1$, the left hand side of (2.25) is always positive while the right hand side is always negative (or zero when $a=1$) unless $\sum_{k=1}^{n+1} \alpha_k^a x_k \equiv 0$, i.e., $\alpha_k^a = 0$ for all $k=1, 2, ..., n+1$.

Step 2. Applying Step 1 to Eq. (2.19), we have that w_a $(a \leqslant 1)$ satisfies

$$-aP_n w_a + (n-1)!\, e^{nw_a} = (n-1)!. \tag{2.26}$$

We now derive some pointwise estimates for w_a.

CLAIM 1. *w_a satisfies*

$$\fint_{S^n} e^{2n(w_a - \fint_{S^n} w_a\, d\sigma_0}\, d\sigma_0 = 1 + o(1) \qquad \textit{as} \quad a \to 1. \tag{2.27}$$

Proof of Claim 1. Assuming the contrary, there will be an $\varepsilon > 0$ and a sequence $a_k \to 1$ with

$$v_k = w_{a_k} - \fint_{S^n} w_{a_k}\, d\sigma_0$$

satisfying

$$\fint_{S^n} e^{2nv_k}\, dv_0 \geqslant 1 + \varepsilon$$

as $k \to \infty$. From (2.18), there is some $v \in H^{n/2,2}(S^n)$ with $v_k \to v$ weakly in $H^{n/2,2}$. Thus $\fint_{S^n} e^{cv_k}\, d\sigma_0 \to \fint_{S^n} e^{cv}\, d\sigma_0$ for any real number c. Also $v_k \in \mathscr{S}$ implies that $v \in \mathscr{S}$. Thus

$$\begin{aligned}
J(v) = J_1(v) &= \log \fint_{S^n} e^{nv}\, d\sigma_0 - \frac{n}{2(n-1)!} S_n[v] \\
&\geqslant \limsup_k J_1(v_k) \\
&= \limsup_k \left(J_{a_k}(v_k) - (1-a_k) \frac{n}{2(n-1)!} \langle P_n v_k, v_k \rangle \right) \\
&= \limsup_k \left(\mathscr{M}_{a_k} - (1-a_k) \frac{n}{2(n-1)!} \langle P_n v_k, v_k \rangle \right) \\
&\geqslant 0.
\end{aligned} \tag{2.28}$$

On the other hand $J_1(v) \leqslant \mathscr{M}_1 = 0$ by Beckner's inequality. Thus $J_1(v) = 0$ and hence v satisfies the equation

$$-P_n v + (n-1)! \frac{e^{nv}}{⨍_{S^n} e^{nv}\, d\sigma_0} = (n-1)!.$$

This together with the fact that $v \in \mathscr{S}$ with $⨍_{S^n} v\, d\sigma_0 = 0$ implies $v \equiv 0$, which contradicts our assumption that

$$⨍_{S^n} e^{2nv}\, d\sigma_0 = \lim_k ⨍_{S^n} e^{2nv_k}\, d\sigma_0 \geqslant 1 + \varepsilon$$

and hence establishes Claim 1.

Claim 2. *$⨍_{S^n} w_a\, d\sigma_0 = o(1)$ as $a \to 1$.*

Proof of Claim 2: We know that $⨍_{S^n} e^{nw_a}\, d\sigma_0 = ⨍_{S^n} d\sigma_0$ by Eq. (2.26). By Hölder's inequality and the convexity of the exponential function, we have $⨍_{S^n} w_a\, d\sigma_0 \leqslant 0$ and $⨍_{S^n} e^{2nw_a}\, d\sigma_0 \geqslant ⨍_{S^n} d\sigma_0$. Therefore, by Claim 1, we have

$$1 \leqslant (1 + o(1))\, e^{2n ⨍_{S^n} w_a\, d\sigma_0} \leqslant 1 + o(1), \tag{2.29}$$

from which Claim 2 follows.

Claim 3. *Actually $w_a(x) = o(1)$ as $a \to 1$.*

Proof of Claim 3. This is routine by combining Claims 1, 2 and Green's identity for P_n (see Lemma 4.8 of [8]).

Step 3. Set $v_a = w_a - ⨍_{S^n} w_a\, d\sigma_0$. By Claims 2 and 3 above, we easily see that

$$\frac{e^{nv_a} - 1}{nv_a} = 1 + o(1) \qquad \text{as} \quad a \to 1. \tag{2.30}$$

We also know that $⨍_{S^n} v_a\, d\sigma_0 = 0$ and $⨍_{S^n} v_a x_j\, d\sigma_0 = 0$ for all $j = 1, 2, ..., n+1$ which can be seen from Eq. (2.26) since $w_a \in \mathscr{S}$. However the second eigenvalue of operator P_n is $(n+1)!$. Therefore we have

$$
\begin{aligned}
(n+1)! \fint_{S^n} v_a^2 \, d\sigma_0 &\leqslant \fint_{S^n} v_a P_n v_a \, d\sigma_0 \\
&= \frac{(n-1)!}{a} \fint_{S^n} (e^{nw_a} - 1)\, v_a \, d\sigma_0 \\
&= \frac{(n-1)!}{a} e^{n \fint_{S^n} w_a \, d\sigma_0} \fint_{S^n} (e^{nv_a} - 1)\, v_a \, d\sigma_0 \\
&= \frac{n!(1+o(1))}{a} e^{n \fint_{S^n} w_a \, d\sigma_0} \fint_{S^n} v_a^2 \, d\sigma_0 \qquad (2.31)
\end{aligned}
$$

Thus as $a \to 1$, $\fint_{S^n} v_a^2 \, d\sigma_0 = 0$, i.e., $v_a \equiv 0$ as $a \to 1$. But by definition, $w_a = \fint_{S^n} w_a \, d\sigma_0$ as $a \to 1$. From (2.26), we have $w_a \equiv 0$ as $a \to 1$, which finishes the proof of Claim (2.20) and hence Theorem 2.5. ∎

3. A PRIORI ESTIMATES ON S^N

In this section, we prove the following

THEOREM 3.1. (a) *Suppose w_k is a sequence of functions satisfying Eq.* (1.5) *with* $0 < m \leqslant Q \leqslant M$. *Then there exists some constant* $C_1 = C_1(m, M) > 0$ *with* $|S_n[w_k]| \leqslant C_1$.

(b) *Suppose Q is a smooth function on S^n satisfying the non-degeneracy condition* (nd) *with* $0 < m \leqslant Q \leqslant M$. *Then there exists a constant*

$$C_2 = C_2(M, m, \min\{|\Delta Q(x)| : \nabla Q(x) = 0\}) > 0$$

such that for all functions w satisfying Eq. (1.5), $|w| \leqslant C_2$.

Proof. The proof of part (a) is given in [8] Theorem 5.3. Since we need a stronger version of this result, we prove the following

LEMMA 3.2. *Suppose $w \in \mathcal{S}$ is a function with Q-curvature Q satisfying* $0 < m \leqslant Q \leqslant M$. *Then*

$$\langle w, P_n w \rangle \leqslant C(m, M)$$

and

$$\|w\|_\infty \leqslant C(m, M), \; \|\nabla^k w\|_\infty \leqslant C(m, M) \qquad \textit{for} \quad k \leqslant n-1.$$

Proof of Lemma 3.2. Since P_n is divergence free, by integrating Eq. (1.5), we have

$$\fint_{S^n} Qe^{nw}\, d\sigma = (n-1)!.$$

Hence,

$$\frac{(n-1)!}{M} \leqslant \fint_{S^n} e^{nw}\, d\sigma \leqslant \frac{(n-1)!}{m}. \tag{3.1}$$

Denote $\tilde{w} = w - 1/n \log \int_{S^n} e^{nw}\, d\sigma$. Then $\tilde{w} \in \mathscr{S}_0$ and we can apply Theorem 2.6 to conclude that

$$\begin{aligned}
&\frac{n(1-a)}{2(n-1)!} \fint_{S^n} \langle w, P_n w\rangle\, d\sigma \\
&\quad = \frac{n(1-a)}{2(n-1)!} \fint_{S^n} \langle \tilde{w}, P_n \tilde{w}\rangle\, d\sigma \\
&\quad = \frac{n}{2(n-1)!} \left[S_n[\tilde{w}] - \left(a \fint_{S^n} \langle \tilde{w}, P_n \tilde{w}\rangle\, d\sigma + 2(n-1)! \fint_{S^n} \tilde{w}\, d\sigma \right) \right] \\
&\quad \leqslant \frac{n}{2(n-1)!}\, S_n[\tilde{w}] = \frac{n}{2(n-1)!} \left(S_n[w] - \frac{2(n-1)!}{n} \log \fint_{S^n} e^{nw}\, d\sigma \right) \\
&\quad \leqslant C(m, M)
\end{aligned} \tag{3.2}$$

by Theorem 5.3 [8] and above fact (3.1).

It follows that

$$\fint_{S^n} \langle w, P_n w\rangle\, d\sigma \leqslant C(m, M). \tag{3.3}$$

Thus we have the estimate

$$\begin{aligned}
\left| 2(n-1)! \fint_{S^n} w\, d\sigma \right| &\leqslant \left[|S_n[w]| + \frac{2(n-1)!}{n} \fint_{S^n} \langle w, P_n w\rangle\, d\sigma \right] \\
&\leqslant C(m, M).
\end{aligned} \tag{3.4}$$

Notice that for any $p > 1$, we may then apply Beckner's inequality to conclude

$$\begin{aligned}
\fint_{S^n} e^{pw}\, d\sigma &\leqslant \exp\left\{ \frac{n}{2(n-1)!} \left[\frac{p^2}{n^2} \fint_{S^n} \langle w, P_n w\rangle\, d\sigma + \frac{2(n-1)!}{n}\, p \fint_{S^n} w\, d\sigma \right] \right\} \\
&\leqslant C(m, M, P).
\end{aligned} \tag{3.5}$$

It then follows from Green's identity that

$$\left| -w(x) + \fint_{S^n} w \, d\sigma \right| = \left| \fint_{S^n} G(x, y)(P_n w)(y) \, d\sigma_y \right|$$
$$\leqslant \left(\int_{S^n} |G(x, y)|^2 \, d\sigma_y \right)^{1/2} \left(\int_{S^n} (Qe^{nw} - 1)^2 \, d\sigma_y \right)^{1/2} \tag{3.6}$$

where $G(x, \cdot)$ is the Green's function for the operator P_n on S^n with pole at x. The last inequality follows from (3.5) by choosing $p = 2n$. Since $\nabla_x^k G(x, y)$ is L^2 integrable for all $k \leqslant n-2$ with respect to y, the same argument establishes the remaining estimates stated in Lemma 3.2. This finishes the proof of Lemma 3.2. ∎

Proof of Theorem 3.1 (*continued*). We now prove part (b) of our Theorem 3.1. We will prove the result by contradiction.

Given $Q > 0$ satisfying the nondegeneracy condition (nd), suppose the statement of Theorem 3.1(b) does not hold. Then there exists a sequence w_k satisfying

$$P_n w_k + (n-1)! = Qe^{nw_k} \qquad \text{on } S^n \tag{3.7}$$

with $\max_{S^n} w_k \to +\infty$. Applying Lemma 2.1, we got a sequence of conformal transformations $\varphi_k = \varphi_{x_k, t_k}$ with $e^{2v_k} g_0 = \varphi_k^*(e^{2w_k} g_0)$, $v_k \in \mathscr{S}$ satisfying

$$P_n v_k + (n-1)! = Q \circ \varphi_k e^{nv_k} \qquad \text{on } S^n.$$

Applying Lemma 3.2, we have $\|v_k\|_\infty \leqslant C(m, M)$ and $\fint_{S^n} \langle v_k, P_n v_k \rangle \, d\sigma \leqslant C(m, M)$. Hence, we may conclude that some subsequence of $t_k \to \infty$. For if not, i.e., $t_k \leqslant t_0$ for all k, for some t_0, then $w_k \circ \varphi_k = v_k - 1/n \log \det(d\varphi_k)$ is uniformly bounded, which contradicts our assumption that $\max_{S^n} w_k \to \infty$. Thus, after passing to a subsequence, we may assume that $t_k \to \infty$, $x_k \to x_0 \in S^n$ and $v_k \to v_\infty$ in $C^{n-1, \alpha}$ for some $\alpha \in (0, 1)$. The last fact follows from the pointwise estimates on v_k and the equation above along with the Sobolev Imbedding Theorem. Notice that $Q \circ \varphi_k \to Q(x_0)$ uniformly on compact subsets of $S^n \backslash \{-x_0\}$ and hence v_∞ satisfies

$$P_n v_\infty + (n-1)! = Q(x_0) \, e^{nv_\infty}, \tag{3.8}$$

at least weakly on $S^n \backslash \{-x_0\}$. But after applying standard arguments from elliptic theory, one sees that in fact v_∞ satisfies (3.8) on all of S^n. By the uniqueness of solutions of (3.8) belonging to $\mathscr{S}$ [9], [14] and [19], we conclude that $v_\infty = -1/n \log Q(x_0)$. Normalizing v_k (by rotating x_k to x_0

and adding a suitable constant), we may assume that $Q(x_0)=(n-1)!$ and that v_k satisfies

$$P_n v_k+(n-1)!=Q\circ\varphi_k e^{nv_k} \tag{3.9}$$

with $\varphi_k=\varphi_{x_0,\,t_k}$. Also, by our estimates in Lemma 3.2, we have

$$\|v_k\|_\infty=o(1) \qquad \text{as} \quad k\to\infty \tag{3.10}$$

$$\|\nabla^k v_k\|_\infty=o(1) \qquad \text{as} \quad k\to\infty \qquad \text{for} \quad k=1,2,...,n-2. \tag{3.11}$$

Applying the Kazdan–Warner condition (Corollary 2.5) to (3.9), we have

$$⨍_{S^n}\langle\nabla(Q\circ\varphi_k),\nabla x_j\rangle\, e^{nv_k}\,d\sigma=0, \qquad j=1,2,...,n+1 \tag{3.12}$$

where $\varphi_k=\varphi_{x_0,\,t_k}$ and $t_k\to\infty$. Therefore, the conclusion of Theorem 3.1(b) can be obtained by showing that (3.12) contradicts the non-degenerate assumption (nd). To see this, we denote the left hand side of (3.12) by A_k and using integration by parts, we rewrite A_k as the sum of two other terms B_k, C_k, i.e.,

$$A_k=B_k+C_k$$

where

$$B_k^j=⨍_{S^n}\langle\nabla(Q\circ\varphi_k),\nabla x_j\rangle\,d\sigma=n⨍_{S^n}(Q\circ\varphi_k-(n-1)!)\,x_j\,d\sigma$$

and

$$C_k^j=n⨍_{S^n}(Q\circ\varphi_k-(n-1)!)\,\vec{x}(e^{nv_k}-1)\,d\sigma$$

$$-n⨍_{S^n}(Q\circ\varphi_k-(n-1)!)\langle\nabla x_j,\nabla v_k\rangle\,e^{nv_k}\,d\sigma,$$

where $j=1,2,...,n+1$.

We now estimate B_k^j and C_k^j. We use the stereographic projection coordinates of S^n to compute $\vec{B}_k$ and $\vec{C}_k$ in terms of the Taylor series expansion of Q. To do this, we denote $x=(x_1,x_2,...,x_{n+1})\in S^n$ and let $y=(y_1,y_2,...,y_n)$ be the stereographic projection from S^n to the equatorial hyperplane R^n sending the north pole $N=(0,0,...,0,1)$ to ∞. We can also identify the point x_0 as the north pole N. Thus $x_i=2y_i/(1+|y|^2)$ for

$i=1, 2, ..., n$ and $x_{n+1}=(|y|^2-1)/(|y|^2+1)$. We assume that the Taylor series expansion of Q around N is given by

$$Q(x_1, x_2, ..., x_{n+1}) = Q(y_1, y_2, ..., y_n)$$
$$= Q(N) + \sum_{i=1}^{n} a_i y_i + \sum_{i,\, j=1}^{n} b_{ij} y_i y_j + o(|y|^2). \qquad (3.13)$$

and (3.13) holds in the neighborhood $\tilde{D}=\{ y \in R^n, |y| \geqslant M\}$ of N for some $M>0$ large. Notice in this notation, $\varphi_k(y)=t_k y$. Denote $D_k=\{ y \in R^n \,|y| \geqslant M/t_k\}$, then $\varphi_k(D_k)=\tilde{D}$. To estimate B_k, we let $d\sigma_y=(1/\omega)(r^{n-1}\, dr/(1+r^2)^n)\, d\theta$ denote the volume form, then

$$\int_{R^n \backslash D_k} d\sigma_y = n \int_0^{M/t_k} \frac{r^{n-1}}{(1+r^2)^n}\, dr \leqslant n \int_0^{M/t_k} r^{n-1}\, dr = \left(\frac{M}{t_k}\right)^n = O\left(\frac{1}{t_k^n}\right)$$

as $t_k \to \infty$.

Thus

$$\vec{B}_k = n \fint_{S^n} (Q \circ \varphi_k - (n-1)!)\, \vec{x}\, d\sigma$$
$$= n \int_{D_k} (Q \circ \varphi_k - (n-1)!)\, \vec{y}\, d\sigma_y + O\left(\frac{1}{t_k^n}\right).$$

Next, notice that by circular symmetry,

$$\int_{D_k} x_i(t_k y)\, x_j(y)\, d\sigma_y = 0 \qquad \text{if} \quad i \neq j, \quad 1 \leqslant i, \quad j \leqslant n+1.$$

Hence

$$B_k^j = n \int_{D_k} a_j x_j(t_k y)\, x_j(y)\, d\sigma_y$$
$$+ n \int_{D_k} \left(\sum_{i,\, l=1}^{n} b_{il} x_i(t_k y)\, x_l(t_k y) \right) x_j(y)\, d\sigma_y + E_k^j + O\left(\frac{1}{t_k^n}\right)$$
$$\text{for} \quad j=1, 2, ..., n$$

where

$$E_k^j = O\left(\int_{D_k} \left(\frac{|t_k y|}{1+|t_k y|^2} \right)^3 |x_j(y)|\, d\sigma_y \right), \qquad j=1, 2, ..., n+1$$

and

$$B_k^{n+1} = n \int_{D_k} \left(\sum_{j,\, l=1}^{n} b_{jl} x_j(t_k y)\, x_l(t_k y) \right) x_{n+1}(y)\, d\sigma_y + E_k^{n+1} + O\left(\frac{1}{t_k^n}\right)$$

Then by direct calculation, we have

$$\int_{D_k} x_i(t_k y)\, x_i(y)\, d\sigma_y \sim \frac{1}{t_k} \qquad \text{as} \quad t_k \to \infty$$

and

$$\int_{D_k} x_i(t_k y)\, x_l(t_k y)\, x_j(y)\, d\sigma_y = \begin{cases} 0, & \text{if} \quad 1 \leqslant i, \quad j, l \leqslant n, \\ 0, & \text{if} \quad j = n+1, \quad i \neq l, \\ C \dfrac{1}{t_k^2}, & \text{if} \quad j = n+1, \quad 1 \leqslant i = l \leqslant n. \end{cases}$$

for some constant C.

Moreover,

$$|E_k^j| = O\left(\frac{1}{t_k^2}\right) \qquad \text{if} \quad 1 \leqslant j \leqslant n;$$

$$|E_k^{n+1}| = \begin{cases} O\left(\dfrac{1}{t_k^3} \log t_k\right) & \text{when} \quad n = 3, \\ O\left(\dfrac{1}{t_k^3}\right) & \text{when} \quad n \geqslant 4. \end{cases}$$

Thus,

$$B_k^j = c a_j \frac{1}{t_k} + O\left(\frac{1}{t_k^2}\right), \qquad \text{for} \quad j = 1, \ldots, n \qquad \text{when} \quad n \geqslant 3;$$

$$B_k^{n+1} = \begin{cases} c' \left(\sum_{i=1}^{n} b_{ii} \right) \dfrac{1}{t_k^2} + O\left(\dfrac{1}{t_k^3} \log t_k\right) & \text{when} \quad n \geqslant 3; \\ c' \left(\sum_{i=1}^{n} b_{ii} \right) \dfrac{1}{t_k^2} + O\left(\dfrac{1}{t_k^3}\right) & \text{when} \quad n > 3. \end{cases}$$

In the above formulas, the constants c and c' depend only on n and M. Using the same argument, we can also conclude that

$$C_k^i = o\left(\frac{|a|}{t_k}\right) + 0\left(\frac{1}{t_k^2}\right), \qquad 1 \leqslant i \leqslant n+1$$

where $|a| = \sum_{i=1}^{n} |a_i|$. Combining above relations, we obtain:

$$a_i = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant n \qquad \text{and} \qquad \sum_{i=1}^{n} b_{ii} = 0.$$

That is,

$$\nabla Q(x_0) = 0 \qquad \text{and} \qquad \Delta Q(x_0) = 0.$$

This finishes the proof of Theorem 3.1. ▮

4. THE MAP Λ

Again we begin by setting some notation. Given $x \in S^n$, $t \in [1, \infty)$, let $p = ((t-1)/t)\, x \in B^{n+1}$. For each Q-curvature candidate Q, we consider the new candidate $Q_p = Q \circ \varphi_p$ and the functional

$$F_p[w] = \log \fint_{S^n} Q_p e^{nw}\, d\sigma - \frac{n}{2(n-1)!} S_n[w]. \tag{4.1}$$

Let

$$\mathcal{M}_p = \sup_{w \in \mathcal{S}_0} F_p[w].$$

If $\mathcal{M}_p$ is achieved by an extremal function w_p, the Euler equation is written as

$$P_n w = (n-1)! = (Q_p - \vec{\Lambda} \cdot \vec{x})\, e^{nw}. \tag{4.2}$$

Comparing this with (1.5), we see that the Q-curvature Q_{w_p}, of the new metric $e^{2w_p} g_0$ is given by

$$Q_{w_p} = (Q_p - \vec{\Lambda} \cdot \vec{x}). \tag{4.3}$$

We now state the first result in this section.

PROPOSITION 4.1. *There exists a constant* $\varepsilon'(n)$ *such that, if* $\varepsilon_Q = \|Q-(n-1)!\|_\infty \leqslant \varepsilon'(n)$, *then* $\mathcal{M}_p$ *is achieved at a conformal factor* w_p *with Lagrange multiplier* $\vec{\Lambda}_p$ *satisfying*

$$\|w_p\|_\infty \leqslant O(\varepsilon_Q) \qquad \text{and} \qquad \|\nabla^k w_p\|_\infty \leqslant O(\varepsilon_Q) \qquad \text{with} \quad k \leqslant n-1,$$

and

$$\|\vec{\Lambda}_p\|_\infty \leqslant O(\varepsilon_Q).$$

Proof. Since the proof is very long, we divide it several parts.

Part A. Given $w \in \mathcal{S}$, by Theorem 2.6, we have

$$F_p[w] \leqslant \log(\max Q) - \left((1-a) \fint_{S^n} \langle w, P_n w\rangle \, d\sigma\right) \frac{n}{2(n-1)!}. \tag{4.4}$$

But since $w \in \mathcal{S}$, $\int_{S^n} \langle w, P_n w\rangle \, d\sigma \geqslant 0$. Thus

$$\log \fint_{S^n} Q_p \, d\sigma = F_p[0] \leqslant \mathcal{M}_p \leqslant \log(\max Q). \tag{4.5}$$

Moreover, for a sequence of $w_k \in \mathcal{S}$ with $F_p[w_k] \to \mathcal{M}_p$, we have, by (4.4)

$$\begin{aligned} \frac{n(1-a)}{2(n-1)!} \fint_{S^n} \langle w_k, P_n w_k\rangle \, d\sigma &\leqslant \log(\max Q) - (\mathcal{M}_p - \varepsilon_k) \\ &\leqslant \log(\max Q) - \log \fint_{S^n} Q_p \, d\sigma + \varepsilon_k \end{aligned} \tag{4.6}$$

for some $\varepsilon_k \to 0$. Thus if we normalize as $\tilde{w}_k = w_k - \fint_{S^n} w_k \, d\sigma$, then $\tilde{w}_k$ is uniformly bounded in $H^{n/2,2}$. A standard argument then indicates that $\tilde{w}_k \to \tilde{w}_p$ weakly in $H^{n/2,2}$ with $F_p[\tilde{w}_p] = \mathcal{M}_p$. The regularity of $\tilde{w}_p$ here is easy by elliptic theory. Since the functional F_p is scale invariant, we may assume, after some rescaling, that w_p of $\tilde{w}_p$ satisfies $\fint_{S^n} Q_p e^{nw_p} \, d\sigma = (n-1)!$ with $F_p[w_p] = \mathcal{M}_p$.

Part B. Set

$$\varepsilon_p = \log \fint_{S^n} Q_p e^{nw_p} \, d\sigma - \log\left[Q(x_0) \fint_{S^n} e^{nw_p \, d\sigma}\right], \tag{4.7}$$

$$\delta_p = \log \fint_{S^n} Q_p \, d\sigma - \log Q(x_0). \tag{4.8}$$

Then we have

$$\begin{aligned}|\delta_p| &= \left|\log \fint_{S^n} Q_p\, d\sigma - \log Q(x_0)\right| \\ &= \left|\log \left[\fint_{S^n} Q_p\, d\sigma - Q(x_0) + Q(x_0)\right] - \log Q(x_0)\right| \\ &\leqslant \frac{1}{\min Q}\left|\fint_{S^n} [Q_p(y) - Q(x_0)]\, d\sigma_y\right| \\ &= O(\|Q_p - Q(x_0)\|_{L^2}). \end{aligned} \tag{4.9}$$

Setting $\tilde{w}_p = w_p - \fint_{S^n} w_p\, d\sigma$, we have

$$\begin{aligned}\log \fint_{S^n} Q_p\, d\sigma &= F_p[0] \leqslant F_p[w_p] = F_p[\tilde{w}_p] \\ &= \log \fint_{S^n} Q_p e^{n\tilde{w}_p}\, d\sigma - \frac{n}{2(n-1)!}\fint_{S^n} \langle \tilde{w}_p, P_n \tilde{w}_p\rangle\, d\sigma. \end{aligned} \tag{4.10}$$

It follows from (4.10) that

$$\min Q \leqslant \fint_{S^n} Q_p\, d\sigma \leqslant \fint_{S^n} Q_p e^{n\tilde{w}_p}\, d\sigma \leqslant \max Q \fint_{S^n} e^{n\tilde{w}_p}\, d\sigma. \tag{4.11}$$

Notice that, by Beckner's inequality,

$$\begin{aligned}\frac{n}{2(n-1)!} S_n[w_p] + \log Q(x_0) &= \log Q(x_0) - (\mathscr{M}_p - \log(n-1)!) \\ &\leqslant \log Q(x_0) - \log \fint_{S^n} Q_p\, d\sigma + \log(n-1)! \end{aligned} \tag{4.12}$$

and

$$\begin{aligned}F_p[w_p] &= \log \fint_{S^n} Q_p e^{nw_p}\, d\sigma - \frac{n}{2(n-1)!} S_n[w_p] \\ &= \log\left(Q(x_0) \fint_{S^n} e^{nw_p}\, d\sigma\right) - \frac{n}{2(n-1)!} S_n[w_p] \\ &\quad + \log \fint_{S^n} (Q_p - Q(x_0))\, e^{nw_p}\, d\sigma \\ &\leqslant \log Q(x_0) + \log \fint_{S^n} e^{nw_p}\, d\sigma - \log\left[Q(_0) \fint_{S^n} e^{nw_p}\, d\sigma\right]. \end{aligned} \tag{4.13}$$

It follows from (4.12) and (4.13) that

$$\frac{n}{2(n-1)!} S_n[w_p] + \log\left[\frac{Q(x_0)}{(n-1)!}\right]$$
$$\geqslant \log Q(x_O) \fint_{S^n} e^{nw_p}\, d\sigma - \log \fint_{S^n} Q_p e^{nw_p}\, d\sigma. \tag{4.14}$$

Combining (4.12) and (4.14) we get

$$-\varepsilon_p \leqslant \frac{n}{2(n-1)!} S_n[w_p] + \log\left[\frac{Q(x_0)}{(n-1)!}\right] \leqslant -\delta_p, \tag{4.15}$$

and

$$\begin{aligned} |\varepsilon_p| &= \log \fint_{S^n} Q_p e^{nw_p}\, d\sigma - \log Q(x_0) \fint_{S^n} e^{nw_p}\, d\sigma \Big| \\ &= \log \fint_{S^n} Q_p e^{n\tilde{w}_p}\, d\sigma - \log \left[Q(x_0) \fint_{S^n} e^{n\tilde{w}_p}\, d\sigma \right]\Big| \\ &= O(\|Q_p - Q(x_0)\|_{L^2}). \end{aligned} \tag{4.16}$$

Therefore we have shown that

$$S_n[w_p] + \log \frac{Q(x_0)}{(n-1!)} \leqslant C \, \|Q_p - Q(x_0)\|_{L^2} \tag{4.17}$$

for all $t \geqslant 1$ and some constant $C > 0$ (depending on max Q, min Q and a where a is defined in Theorem 2.6).

Part C. Applying (4.17) and the fact that

$$\int_{S^n} Q_p e^{nw_p}\, d\sigma = (n-1)!,$$

we conclude that

$$\frac{n(1-a)}{2(n-1)!} \fint_{S^n} \langle w_p, P_n w_p \rangle \, d\sigma \leqslant \varepsilon_p + O(\|Q_p - Q(x_0)\|_{L^2}). \tag{4.18}$$

It follows that

$$\fint_{S^n} \langle w_p, P_n w_p \rangle \, d\sigma = O(\|Q_p - Q(x_0)\|_{L^2}). \tag{4.19}$$

Part D. From Part A, w_p is an extremal solution of F_p for each t, $(p=t-1/tx)$ and we have

$$P_n w_p + (n-1)! = \left[Q_p - \sum_{j=1}^{n+1} \lambda_p^j x_j \right] e^{n w_p} \tag{4.20}$$

for some constants λ_p^j, $1 \leqslant j \leqslant n+1$.

Applying the Kazdan–Warner condition, we obtain

$$\fint_{S^n} \langle \nabla Q_p, \nabla x_i \rangle \, d\sigma = \sum_{j=1}^{n+1} \lambda_p^j \fint_{S^n} e^{n w_p} \langle \nabla x_j, \nabla x_i \rangle \, d\sigma \tag{4.21}$$

for each $i = 1, 2, ..., n+1$.

Denoting

$$\vec{\Lambda}_p = (\lambda_p^1, \lambda_p^2, ..., \lambda_p^{n+1}),$$

$$\vec{A}_p = \fint_{S^n} e^{n w_p} \langle \nabla Q_p, \nabla \vec{x} \rangle \, d\sigma,$$

and

$$C_p = (C_{ij}^p)_{(n+1) \times (n+1)}$$

where

$$C_{ij}^p = \fint_{S^n} e^{n w_p} \langle \nabla x_i, \nabla x_j \rangle \, d\sigma,$$

we can rewrite the Kazdan–Warner condition as

$$\vec{A}_p = C_p \vec{\Lambda}_p,$$

or equivalently,

$$\vec{\Lambda}_p = C_p^{-1} \vec{A}_p. \tag{4.22}$$

Part E. Since $S_n[w_p] \geqslant 0$ by Beckner's inequality,

$$-\fint_{S^n} w_p \, d\sigma \leqslant \frac{1}{2(n-1)!} \fint_{S^n} \langle w_p, P_n w_p \rangle \, d\sigma.$$

Hence, we have the following estimate:

$$
\begin{aligned}
&\fint_{S^n} e^{-nw_p}\langle \vec{x}, \vec{x}\rangle \, d\sigma \\
&\qquad \leqslant \left(\fint_{S^n} e^{-2nw_p}\, d\sigma\right)^{1/2} \left(\fint_{S^n} \langle \vec{x}\vec{x}\rangle \, d\sigma\right)^{1/2} \\
&\qquad \leqslant C\left(\fint_{S^n} e^{-2nw_p}\, d\sigma\right)^{1/2} \\
&\qquad \leqslant C \exp \frac{n}{(n-1)!}\left[\fint_{S^n} \langle w_p, P_n w_p\rangle \, d\sigma - (n-1)! \fint_{S^n} w_p \, d\sigma\right] \\
&\qquad \leqslant C \exp \frac{3n}{2(n-1)!}\left[\fint_{S^n} \langle w_p, P_n w_p\rangle \, d\sigma\right] \\
&\qquad \leqslant C \exp \frac{3n}{2(1-a)(n-1)!}\, S_n[w_p].
\end{aligned}
$$

Notice that, from (4.17) and Part C, $S_n[w_p] \leqslant C(m, M)$. Thus there exists a constant $C = C(m, M) > 0$ such that

$$
\fint_{S^n} e^{-nw_p}\langle \vec{x}, \vec{x}\rangle \, d\sigma \leqslant C. \tag{4.23}
$$

This implies that

$$
\begin{aligned}
\langle C_p \vec{a}, \vec{a}\rangle \, d\sigma &= \fint_{S^n} e^{nw_p} \left|\sum a_i x_i\right|^2 d\sigma \\
&= \int_{S^n} e^{nw_p}\langle \vec{x}, \vec{x}\rangle \, d\sigma \qquad \text{with} \quad \vec{x} = \sum a_i x_i \\
&\geqslant \left(\int_{S^n} \langle \vec{x}, \vec{x}\rangle \, d\sigma\right)^{1/2} \left(\fint_{S^n} e^{-nw_p}\langle \vec{x}, \vec{x}\rangle \, d\sigma\right)^{-1} \\
&\geqslant \frac{1}{C(n+1)^2} > 0.
\end{aligned} \tag{4.24}
$$

Part F.

$$\|\vec{A}_p\|^2 \leqslant 2 \sum_{j=1}^{n+1} \left(\fint_{S^n} |\langle \nabla Q_p, \nabla x_j \rangle|^2 \, d\sigma \right) \left(\fint_{S^n} (e^{nw_p} - 1)^2 \, d\sigma \right)$$
$$+ 2n^2 \sum_{j=1}^{n+1} \left| \fint_{S^n} Q_p - (n-1)!) \, x_j \, d\sigma \right|^2$$
$$\leqslant C \fint_{S^n} (e^{nw_p} - 1)^2 \, d\sigma + 2n^2 O(\|Q_p - (n-1)!\|_{L^2})$$

since

$$\fint_{S^n} |\langle \nabla Q_p, \nabla x_j \rangle|^2 \, d\sigma \leqslant \sqrt{n} \left(\fint_{S^n} |\nabla Q|^n \, d\sigma \right)^{2/n}$$
$$= \sqrt{n} \left(\fint_{S^n} |\nabla Q|^n \, d\sigma \right)^{2/n} \leqslant \sqrt{n} \, \mathbb{C}$$

and by (4.11),

$$\fint_{S^n} e^{nw_p} \, d\sigma \to \qquad \text{if} \quad \|Q - (n-1)!\|_\infty \to 0.$$

Also using Beckner's inequality and Parts B and C, we have

$$\fint_{S^n} e^{2nw_p} \, d\sigma \leqslant \exp \left[\frac{2n}{(n-1)!} \left(\fint_{S^n} \langle w_p, P_n w_p \rangle \, d\sigma + (n-1)! \fint_{s^n} w_p \, d\sigma \right) \right]$$
$$\leqslant C(\varepsilon_Q)$$

where $C(\varepsilon_Q)$ is of order e^ε when $\|Q - (n-1)!\|_\infty \leqslant \varepsilon$. Hence

$$\|\vec{A}_p\|^2 \leqslant C(\varepsilon_Q) \tag{4.25}$$

with $C(\varepsilon_Q) = O(\varepsilon_Q)$ when $\|Q - (n-1)!\|_\infty \leqslant \varepsilon_Q$.

Combining (4.22), (4.25), and Part E, we obtain

$$\|\vec{A}_p\|_\infty^2 \leqslant C(\varepsilon_Q). \tag{4.26}$$

Part G. The rest of the proof of Proposition 4.1 follows from the proof of Lemma 3.2. This completes the proof of Proposition 4.1. ∎

Proposition 4.1 gives us a natural map $\Lambda : p \in B^{n+1} \to \Lambda_p \in R^{n+1}$. The next proposition demonstrates that Λ is a well-defined map.

PROPOSITION 4.2. *For $\varepsilon_Q = \|Q-(n-1)!\|_\infty$ sufficiently small, the functional F_p has a unique maximum in the class $\mathscr{S}_0$ which we denote by w_p. The map $p \to w_p$ is, in fact, continuous from B^{n+1} to $\mathscr{S}$.*

Proof. To verify uniqueness, we assume to the contrary that there is a $p \in B$ where F_p has two distinct maxima w_0 and w_1. Join w_0 to w_1 by a one-parameter family of conformal factors w_t which satisfy $e^{nw_t} = te^{nw_0} + (1-t)\, e^{nw_1}$. For each t, let

$$\dot{w}_t = \frac{1}{dt}\, w_t = -\frac{1}{n}\, e^{-nw_t}(e^{nw_0} - e^{nw_1})$$

so that

$$\ddot{w}_t = \frac{d\dot{w}_t}{dt} = -\frac{1}{n}\, e^{-nw_t}(e^{nw_0} - e^{nw_1})(n\dot{w}_t) = -n\dot{w}_t^2.$$

It follows from $w_t \in \mathscr{S}$ that

$$⨍_{S^n} e^{nw_t}\dot{w}_t x_j \, d\sigma = 0$$

and hence we have

$$\left| ⨍_{S^n} \dot{w}_t x_j \, d\sigma \right| = \left| \int_{S^n} (1 - e^{nw_t})\, \dot{w}_t x_j \, d\sigma \right| = O(\varepsilon_Q \, \|\dot{w}_t\|_{L^2})$$

by Proposition 4.1.

Resolving $\dot{w}_t$ into $\psi + x_t$ where x_t is the orthogonal projection of $\dot{w}_t$ (with respect to the standard metric) onto the first order spherical harmonic functions, we find that $\|x_t\|_{L^2} = O(\varepsilon_Q \, \|\dot{w}_t\|_{L^2})$ and that

$$\begin{aligned}(1 + O(\varepsilon_Q)) ⨍_{S^n} \langle \dot{w}_t, P_n \dot{w}_t \rangle \, d\sigma &= ⨍_{S^n} \langle \psi, P_n \psi \rangle \, d\sigma \geqslant (n+1)! ⨍_{S^n} \psi^2 \, d\sigma \\ &= ((n+1)! - O(\varepsilon_Q)) ⨍_{S^n} \dot{w}_t^2 \, d\sigma\end{aligned}$$

and

$$\begin{aligned}(1 + O(\varepsilon_Q)) ⨍_{S^n} |\nabla \dot{w}_t|^2 \, d\sigma &= ⨍_{S^n} \langle \psi, (-\Delta)\, \psi \rangle \, d\sigma \geqslant 2(n-1) ⨍_{S^n} \psi^2 \, d\sigma \\ &= (2(n-1) - O(\varepsilon_Q)) ⨍_{S^n} \dot{w}_t^2 \, d\sigma.\end{aligned}$$

Here we have used the fact that the second eigenvalue of P_n is $(n+1)!$. Now we consider the function

$$g(t) = F_p[w_t]. \tag{4.27}$$

Clearly $g(t)$ is twice differentiable functions and we have

$$g'(t) = DF_p[w_t](\dot{w}_t) \tag{4.28}$$

and

$$\begin{aligned}
g''(t) &= D^2F_p(w_t)(\dot{w}_t, \dot{w}_t) + DF_p(w_t)(\ddot{w}_t) \\
&= n\left(\frac{⨍_{S^n} Q_p e^{nw_t}\ddot{w}_t \, d\sigma}{⨍_{S^n} Q_p e^{nw_t} \, d\sigma}\right) \\
&\quad - \frac{n}{(n-1)!}\left(⨍_{S_n} \langle \ddot{w}_t, P_n w_t\rangle \, d\sigma + (n-1)! ⨍_{S^n} \ddot{w}_t \, d\sigma\right) \\
&\quad + n\left\{\frac{⨍_{S^n} Q_p e^{nw_t}\dot{w}_t^2 \, d\sigma}{⨍_{S^n} Q_p e^{nw_t} \, d\sigma} - \left(\frac{⨍_{S^n} Q_p e^{nw_t}\dot{w}_t \, d\sigma}{⨍_{S^n} Q_p e^{nw_t} \, d\sigma)}\right)^2\right\} \\
&\quad - \frac{n}{(n-1)!} ⨍_{S^n} \langle \dot{w}_t, P_n \dot{w}_t\rangle \, d\sigma \\
&= (n-n^2)\left[\frac{⨍_{S^n} Q_p e^{nw_t}\dot{w}_t^2 \, d\sigma}{⨍_{S^n} Q_p e^{nw_t} \, d\sigma}\right] \\
&\quad + \frac{n^2}{(n-1)!}\left[⨍_{S^n} \langle \dot{w}_t^2, P_n w_t\rangle \, d\sigma + (n-1)! \int_{S^n} \dot{w}_t^2 \, d\sigma\right] \\
&\quad - n\left(\frac{⨍_{S^n} Q_p e^{nw_t}\dot{w}_t \, d\sigma}{⨍_{S^n} Q_p e^{nw_t}}\right)^2 - \frac{n}{(n-1)!} ⨍_{S^n} \langle \dot{w}_t, P_n \dot{w}_t\rangle \, d\sigma \\
&= \frac{n^2}{(n-1)!} ⨍_{S^n} ((n-1)! - Q_p e^{nw_t}) \, \dot{w}_t^2 \, d\sigma \\
&\quad + \frac{n^2}{(n-1)!}\left[⨍_{S^n} \langle \nabla \dot{w}_t^2, P'_n w_t\rangle \, d\sigma - \frac{1}{n} ⨍_{S^n} \langle \dot{w}_t, P_n \dot{w}_t\rangle \, d\sigma\right] \\
&\quad - \frac{n}{((n-1)!)^2}\left[\left(⨍_{S^n} Q_p e^{nw_t}\dot{w}_t \, d\sigma\right)^2 - (n-1)! ⨍_{S^n} Q_p e^{nw_t}\dot{w}_t^2 \, d\sigma\right],
\end{aligned} \tag{4.29}$$

where we have used the fact that $⨍_{S^n} Q_p e^{nw_t}\, d\sigma = (n-1)!$ and that

$$P_n' = \begin{cases} \prod_{k=1}^{(n-2)/2} (-\Delta + k(n-k+1)), & \text{if } n \text{ is even} \\ \left(-\Delta + \left(\frac{n-1}{2}\right)^2\right)^{1/2} \prod_{k=1}^{(n-3)/2} (-\Delta + k(n-k+1)), & \text{if } n \text{ is odd.} \end{cases}$$

Since

$$(n-1)! ⨍_{S^n} Q_p e^{nw_t} \dot{w}_t^2\, d\sigma - \left(⨍_{S^n} Q_p e^{nw_t} \dot{w}_t\, d\sigma\right)^2$$
$$= (((n-1)!)^2 + O(\varepsilon_Q)) ⨍_{S^n} \dot{w}_t^2\, d\sigma, \tag{4.30}$$

by Proposition 4.1, the equation for w_0 and w_1 and the fact that $\|\nabla P_n' w_t\|_\infty = O(\varepsilon_Q)$, we obtain

$$g''(t) = \frac{n^2}{(n-1)!} O(\varepsilon_Q) ⨍_{S^n} \dot{w}_t^2\, d\sigma + n(1 + O(\varepsilon_Q)) ⨍_{S^n} \dot{w}_t^2\, d\sigma$$
$$- \frac{n^2}{(n-1)!} ⨍_{S^n} \langle \dot{w}_t, P_n \dot{w}_t \rangle\, d\sigma$$
$$\leqslant (n + O(\varepsilon_Q)) ⨍_{S^n} \dot{w}_t^2\, d\sigma - \frac{n}{(n-1)!} \frac{(n+1)! - O(\varepsilon_Q)}{1 = O(\varepsilon_Q)} ⨍_{S^n} \dot{w}_t^2\, d\sigma$$
$$\leqslant n(-n(n+1) + 1 - O(\varepsilon_Q)) ⨍_{S^n} \dot{w}_t^2\, d\sigma \leqslant 0. \tag{4.31}$$

This means that $g(t)$ is a concave function, which contradicts the assumption that $g(0) = g(1)$ are both maxima of g unless $w_0 = w_1$.

Now a simple calculation shows that

$$DF_p[w](\varphi) = n \frac{⨍_{S^n} Q_p e^{nw} \varphi\, d\sigma}{⨍_{S^n} Q_p e^{nw}\, d\sigma}$$
$$- \frac{n}{(n-1)!} ⨍_{S^n} [(w P_n \varphi) + (n-1)!\, \varphi]\, d\sigma. \tag{4.32}$$

Then the second derivative $D^2 F_p$ is $D(DF_p)$, where we view DF_p as a map $H^{n/2,2} \to L(H^{n/2,2}, R)$. First we calculate a directional derivative of DF_p in the direction ψ at w:

$$D_\psi[DF_p][w](\varphi) = \frac{d}{dt}\bigg|_{t=0} \{DF_p(w+t\psi))(\varphi)\}$$
$$= n^2 \left[\frac{⨍_{S^n} Q_p e^{nw}\varphi\psi \, d\sigma}{⨍_{S^n} Q_p e^{nw} \, d\sigma} - \frac{⨍_{S^n} Q_p e^{nw}\varphi \, d\sigma ⨍_{S^n} Q_p e^{nw}\psi \, d\sigma}{(⨍_{S^n} Q_p e^{nw} \, d\sigma)^2} \right]$$
$$- \frac{n}{(n-1)!} \left[⨍_{S^n} \langle \psi, P_n\varphi \rangle \, d\sigma \right]. \tag{4.33}$$

Writing $D_\psi(DF_p)[w](\varphi) = D^2F_p(\psi, \varphi)$, we observe that the Beckner's inequality implies that the map

$$w \to D^2F_p(\cdot, \cdot) \in L^2(H^{n/2,2} \times H^{n/2,2}, R) \tag{4.34}$$

is continuous, so that F_p is a C^2 functional.

Therefore, if w_p is the unique maximum of F_p, then for any $\varphi \in T_w(\mathscr{S}_0)$, we have

$$D^2F_p[w_p](\varphi, \varphi) = -\frac{n}{(n-1)!} ⨍_{S^n} \langle \varphi, P_n\varphi \rangle \, d\sigma$$
$$+ n^2 \left[\frac{⨍_{S^n} Q_p e^{nw_p}\varphi^2 \, d\sigma}{⨍_{S^n} Q_p e^{nw_p} \, d\sigma} - \frac{⨍_{S^n} Q_p e^{nw_p}\varphi \, d\sigma)^2}{⨍_{S^n} Q_p e^{nw_p} \, d\sigma)^2} \right]$$
$$= -\frac{n}{(n-1)!} ⨍_{S^n} \langle \varphi, P_n\varphi \rangle \, d\sigma$$
$$+ n^2 \left[⨍_{S^n} \frac{Q_p}{(n-1)!} e^{nw_p}\varphi^2 \, d\sigma - \left(⨍_{S^n} \frac{Q_p}{n-1)!} e^{nw_p}\varphi \, d\sigma \right)^2 \right]$$
$$= (n^2 + O(\varepsilon_Q)) ⨍_{S^n} \varphi^2 \, d\sigma - \frac{n}{(n-1)!} ⨍_{S^n} \langle \varphi, P_n\varphi \rangle \, d\sigma$$
$$\leqslant \left[\frac{n^2 + O(\varepsilon_Q)}{(n+1)!} - \frac{n}{(n-1)!} \right] ⨍_{S^n} \langle \varphi, P_n\varphi \rangle \, d\sigma$$
$$= \frac{n}{(n+1)!} \left[\frac{1}{n+1} - 1 + O(\varepsilon_Q) \right] ⨍_{S^n} \langle \varphi, P_n\varphi \rangle \, d\sigma. \tag{4.35}$$

Since $D^2F_p[w_p](\varphi, \varphi)$ is the quadratic form associated with the linear transform $D(DF_p)[w_p]$, the map $w \to DF_p[w_p]$ has nonsingular derivatives at w_p. We now recall the following implicit function theorem [15, 59].

Implicit Function Theorem. *Let X, Y, Z be Banach spaces and f a continuous mapping of an open set $U \subset X \times Y \to Z$. Assume that f has a Frechet derivative with respect to x, $f_x(x, y)$, which is also continuous in U.*

Let $(x_0, y_0) \in U$ and $f(x_0, y_0) = 0$. If $A = f_x(x_0, y_0)$ is an isomorphism of X onto Z, then there is a ball $B_r(y_0)$ and a unique continuous map $w: B_r(y_0) \to X$ such that $w(y_0) = x_0$ and $f(w(y), y) = 0$.

We apply the theorem to the situation $X = \mathscr{S}_0$, $Y =$ parameter space B^{n+1}, $Z = H^{n/2, 2}$ and $f(w, p_0) = DF_{p_0}[w] \in H^{n/2, 2}$ (by duality). Take $x_0 = w_{p_0}$ the unique maximum for the functional $F_{p_0} = \log \fint_{S^n} Q_p e^{nw_{p_0}} \, d\sigma - n/(2(n-1)!) \, S_n[w_{p_0}]$. Then the conditions of the theorem are satisfied, and we obtain a continuous branch of critical points w_p of the functional F_p in the fixed space $\mathscr{S}_0$ for p sufficiently close to p_0. But since the second derivative $D^2 F_p$ is continuous, it follows that the nearby w_p are local maxima of F_p and satisfy the same conditions in Proposition 4.1. Hence the same argument in the uniqueness assertion can be applied to show that w_p must be the unique maximum for the functional F_p for p sufficiently close to p_0. This proves Proposition 4.2.

Under the assumption that the extremal solutions w_p of the parametric problems F_p have non-vanishing Lagrange multipliers Λ_p, we want to compare Λ_p with G_p for p sufficiently near boundary point x of B_1^{n+1}. (Recall that $p = ((t-1)/t) \, x$. It follows from the equation

$$P_n w + (n-1)! = (Q_p - \Lambda_p \cdot \vec{x}) \, e^{nw} \tag{4.36}$$

and the Kazdan–Warner condition

$$\fint_{S^n} \langle \nabla (Q_p - \Lambda_p \cdot \vec{x}), \nabla x_j \rangle \, e^{nw} \, d\sigma = 0 \tag{4.37}$$

that

$$\sum_{j=1}^{n+1} \Lambda_j \fint_{S^n} \langle \nabla x_j \nabla x_i \rangle \, d\sigma = \fint_{S^n} \langle \nabla Q_p, \langle \nabla x_i \rangle \, d\sigma. \tag{4.38}$$

Adopting the notation from Part D of the Proof of Proposition 4.1, we rewrite this as

$$C_p \Lambda_p = \vec{A}_p. \quad \blacksquare \tag{4.39}$$

Hence we have the following:

Proposition 4.3. (i) *$\lambda_p = 0$ if and only if $\vec{A}_p = 0$.*

(ii) *$deg(\Lambda_p, \{(p, t) \mid t = t_0\}, 0) = deg(\vec{A}_p, \{(p, t) \mid t = t_0\}, 0)$.*

(iii) *If we define another map $G: p \in B^{n+1} \to R^{n+1}$ by*

$$G\left(p = \frac{t-1}{t}\right) = \fint_{S^n} (Q \circ \varphi_p) \, \vec{x} \, d\sigma, \tag{4.40}$$

and if Q is non-degenerate of order $\alpha \leqslant n$ at p, we have

$$G(x, t) \cdot \vec{A}(x, t) \geqslant 0. \tag{4.41}$$

Proof. (i) and (ii) are clearly true. (iii) follows from the same argument as in the P_2 case in [7]. ∎

5. PROOF OF THE MAIN THEOREM

In this section we apply the a priori estimates developed in the previous sections to give the basic existence result. To apply the a priori estimates, we use the Leray–Schauder degree theory (as developed in Nirenberg's Courant Lecture notes [15] on nonlinear functional analysis) to prove the main theorem stated in the introduction.

To set up the continuity method, we join the curvature function Q to the constant function $Q_0 = (n-1)!$ by an one parameter family of functions

$$Q_s = sQ + (1-s)\, Q_0 \tag{5.1}$$

and consider the family of differential equations

$$P_n w + (n-1)! = Q_s e^{nw}. \tag{5.2}$$

For any $s_0 > 0$, there is a uniform bound for the C^2 norm of the function Q_s, as well as a uniform positive lower bound for $|\Delta Q_s(x)|$ at all critical points x of the function Q_s for $s \in [s_0, 1]$. Thus according to the a priori estimates of Theorem 3.1, all solutions of the Eq. (5.2) satisfy a uniform bound

$$\|w\|_{n,\alpha} \leqslant C, \qquad \text{for all} \quad 0 < \alpha < 1. \tag{5.3}$$

We rewrite the differential Eq. (5.2) in the form

$$w + P_n^{-1} \Psi_s[w] = 0 \tag{5.4}$$

where $\Psi[w] = (n-1)! - Q_s e^{nw}$. Let Ω_c be the open set in $X = \{w \in \mathscr{C}^{n,\alpha}(S^n), ⨍_{S^n} Q e^{nw}\, d\sigma = (n-1)!\}$:

$$\Omega_c = \{w \in X, \|w\|_{n,\alpha} < C\}. \tag{5.5}$$

In Ω_c, define the map $\psi_s(w) = w + P_n^{-1} \Psi_s[w]$. Since $P_n^{-1} \Psi_s$, is a Fredholm map: $\Omega_c \to \mathscr{C}^{n,\alpha}(S^n)$ and is continuous in s and $0 \notin \psi_s(\partial \Omega_c)$ for $s \geqslant s_0$, we see that $deg(\psi_s, \Omega_c, 0)$ is defined and independent of s for $s \geqslant s_0$.

For S_0 sufficiently small, Q_{s_0} is close in C^2 to the constant function $(n-1)!$. For such a Q- curvature function Q_s, we carried out the perturbation argument given in the last section and by Proposition 4.3, we have $deg(\Lambda, B^{n+1}, 0) = deg(G, B^{n+1}, 0)$. Therefore, to finish the proof of our main theorem, it suffices to show that $deg(\Lambda, B^{n+1}, 0) = deg(\psi_s, B^{n+1}, 0)$ for some sufficiently small positive s_0.

If s_0 is sufficient small, each zero of the map ψ_{s_0} is contained in the set $B = \{w_{x,t}, x \in S^n \text{ and } t \geqslant 1\}$, which is homeomorphic to the unit ball $B^{n+1} \subset R^{n+1}$, according to our discussion in the previous section. First we notice that, by a simple transversality argument of the continuity of degree under small perturbation, we may assume that ψ_{s_0} and Λ have only isolated non-degenerate zeros so that the corresponding degrees are actually sums of local degrees of zeros of the corresponding maps. We recall that the local degree of ψ_s, at an isolated zero w_0 is given by taking a neighborhood O of w_0 where $0 \notin \psi_{s_0}(\partial O)$ and then taking an approximation k_ε of $P_n^{-1} \Psi_{s_0}$ mapping into a finite dimensional subspace Y of X so that $\psi_{s_0, \varepsilon}(w) = w - k_\varepsilon(w) \neq 0$ on ∂O. Now consider the map

$$\psi_{s,\varepsilon}|_{Y \cap \bar{O}} \colon Y \cap \bar{O} \to Y.$$

We have

$$deg(\psi_s, O, 0) = deg(\psi_{s,\varepsilon}|_{Y \cap \bar{O}}, Y \cap \bar{O}, 0). \tag{5.6}$$

In our problem, the natural space Y we can take is the linear space of $E_1 \oplus E_2 \oplus \cdots \oplus E_m$, where E_k denotes the space of the kth order spherical harmonic functions. To study the local degree of ψ_s at w_0, it will be convenient to transform w_0 so that $\tilde{w}_0 = w_0 \circ \varphi_0 + 1/n \log(det(d\varphi_0)) \in \mathscr{S}_0$. For any $w \in B = \{w_{x,t} \mid x \in S^n, t \geqslant 1\}$, let $T_0 w = w \circ \varphi + 1/n \log(det(d\varphi_0))$ and if $\bar{Q} = Q \circ \varphi_0$, we see that

$$P_n w + (n-1)!\ Qe^{nw} \tag{5.7}$$

if and only if

$$P_n(T_0 w) + (n-1)! = \bar{Q} e^{nT_0 w}. \tag{5.8}$$

Hence if we set $\check{\psi}_s \circ T_0^{-1}$, $\bar{O}T_0(0)$, we have

$$deg(\psi_s, O, 0) = deg(\check{\psi}_s, \bar{O}, 0).$$

Thus, without loss of generality, we may assume a given solution w_0 belongs to the symmetric class $\mathscr{S}$ when we calculate its local degree. So suppose $w_0 - P_n^{-1} \Psi_s[w_0] = 0$. The linearization of the map ψ_s around w_0

is given by $\psi'_s(w_0)[v] = v - nP_n^{-1}(Q_s e^{nw_0} v)$ when $\|Q_s - (n-1)!\|_\infty < \varepsilon$. Thus we have $\|e^{nw_0} - 1\|_\infty \leqslant C_\varepsilon$, so that the linearization is approximated by

$$\psi'_{s_0}(w_0)[v] = v - n!\, P_n^{-1}(v) + O(\varepsilon + C_\varepsilon)\, \|P_n^{-1} v\|. \tag{5.9}$$

Since w_0 is the unique element in both B and $\mathscr{S}$, $span\{T_{w_0}(B), T_{w_0}(\mathscr{S})\} = L^2$. Now if Q_0 denotes a constant function, then $Y \cap T_0(\mathscr{S}) = E_2 \oplus E_3 \oplus \cdots \oplus E_m$. Our estimate $w_0\|_\infty \leqslant C_\varepsilon$ implies that $Y \cap T_{w_0}(\mathscr{S}) = E_1 \oplus E_2 \oplus \cdots \oplus E_m \oplus V$, where $v \in V$ implies that $\|v\|_n < \delta(m, \varepsilon)$ and $\delta(m, E) \to 0$ as $m \to \infty$ or $\varepsilon \to 0$. Calculating $\psi_s(w_0)$ in the direction of an element $v \in Y \cap T_{w_0}(\mathscr{S})$ is relatively straightforward. Since the tangent space $T_{w_0}(B)$ is transverse to the spaces E_k for each $k \neq 1$, we want to use Eq. (4.36) to compute the derivative ψ'_s in the direction $T_{w_0} B$. To this end, we have

$$\psi_s(w_{x,t}) = P_n^{-1}(\Lambda \cdot (\vec{x} \circ \varphi_{x,t}^{-1})\, e^{nw_{x,t}}). \tag{5.10}$$

Hence in the direction $T_{w_0} B$, we have $\psi'_s(w_0) = \Lambda'(w_0) \cdot P_n^{-1}(\vec{x} e^{nw_0})$. Next observe that we can find a basis for $T_{w_0}(B)$ consisting of $\{\beta_i : i = 1, 2, ..., n+1\}$ where $\beta_i = x_1 + e_i + \varepsilon_i$ with e_i bounded and contained in the span $E_2 \oplus E_3 \oplus \cdots \oplus E_m$ and $|\varepsilon_i| \to 0$ as $\varepsilon \to 0$. Thus $x_i = \beta_i - e_i - \varepsilon_i$, so that we can express the derivative of ψ_s in terms of a matrix with respect to the natural decomposition $Y = E_1 \oplus E_2 \oplus \cdots \oplus E_m$ to be a small perturbation of the following matrix:

$$\begin{pmatrix} \Lambda' & 0 & 0 & \cdots & 0 \\ x & 1 - \dfrac{n!}{(n+1)!} & 0 & \cdots & 0 \\ x & 0 & 1 - \dfrac{n!}{a_3} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x & 0 & 0 & \cdots & 1 - \dfrac{n!}{a_m} \end{pmatrix} \tag{5.11}$$

where a_m are the mth eigenvalues of P_n (we do not count the repeated eigenvalues). This finishes the proof of the main theorem.

ACKNOWLEDGMENTS

Special thanks are due to Professor Thomas P. Branson for enlightening discussions on Lemma 2.4. We also thank the referee for carefully reading the manuscript and many helpful suggestions. Finally we thank Professor Jay Kovats for polishing the English of the paper. The research of the first author is supported by an Earmarked Grant from RGC of Hong Kong.

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