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Spikes for the Two-Dimensional Gierer-Meinhardt System: The Weak Coupling Case

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Summary. In this paper, we rigorously prove the existence and stability of multiple-peaked patterns that are far from spatial homogeneity for the singularly perturbed Gierer-Meinhardt system in a two-dimensional domain. The Green's function, together with its derivatives, is linked to the peak locations and to the $o(1)$ eigenvalues, which vanish in the limit. On the other hand two nonlocal eigenvalue problems (NLEPs), one of which is new, are related to the $O(1)$ eigenvalues. Under some geometric condition on the peak locations, we establish a threshold behavior: If the inhibitor diffusivity exceeds the threshold, then we get instability; if it lies below, then we get stability.

Key words. Pattern formation, mathematical biology, singular perturbation, weak coupling

MSC numbers. Primary 35B45; Secondary 35J40

1. Introduction

Morphogenesis is the development of an organism from a single cell. This complex process can be understood by dividing it into several elementary steps, such as the change of cell shapes, cell to cell interaction, growth, and cell movement. One of the most important of these elementary steps is the formation of a spatial pattern of cell structure, starting from an almost homogeneous cell distribution.

Turing, in his pioneering work in 1952 [29], proposed that a patterned distribution of two chemical substances, called the morphogens, could trigger the emergence of such a cell structure. He also gives the following explanation for the formation of the morphogenetic pattern: It is assumed that one of the morphogens, the activator, diffuses slowly and the other, the inhibitor, diffuses much faster. In the mathematical framework of a coupled system of reaction-diffusion equations with very different diffusion coefficients, he shows by linear stability analysis that the homogeneous state may be unstable. In particular, a small perturbation of spatially homogeneous initial data may evolve to a stable spatially complex pattern of the morphogens.

Since the work of Turing, a lot of models have been proposed and analyzed to explore more fully this phenomenon, which is now called Turing instability, and its implications for the understanding of various patterns. One of the most famous of these models is the Gierer-Meinhardt system [8], [19]. In two dimensions after rescaling and considering a special case, it can be stated as follows:

$$(GM) \qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + \dfrac{A^2}{H}, & A > 0, \quad \text{in } \Omega, \\ \tau H_t = D\Delta H - H + A^2, & H > 0, \quad \text{in } \Omega, \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial H}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

The unknowns $A = A(x,t)$ and $H = H(x,t)$ represent the concentrations of the activator and inhibitor at a point $x \in \Omega \subset R^2$ and at a time $t > 0$; $\Delta := \sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^2; Ω is a bounded and smooth domain in R^2; $\nu = \nu(x)$ is the outer normal at $x \in \partial\Omega$. Throughout this paper, we assume that

$\epsilon \ll 1$, $\quad \epsilon$ does not depend on x or t,
$\tau \geq 0$ is a fixed constant which does not depend on $x, t,$ or ϵ,
$D > 0$ does not depend on x or t but may depend on ϵ,
$D \ll e^{\frac{\delta}{\epsilon}}$, $\quad$ where $\delta > 0$ is a small constant which is independent of $\epsilon > 0$.

In this paper, we further assume $D \to \infty$ as $\epsilon \to 0$ (and call this the **weak coupling** case).

Numerical studies by Meinhardt [19] and more recently by Holloway [12] and Maini and McInerney [18] have revealed that when ϵ is small and D is finite, (GM) seems to have stable stationary states with the property that the activator is mainly concentrated in K peaks which are each placed near K different points in Ω whose locations satisfy suitable conditions. Moreover, as $\epsilon \to 0$, the pattern exhibits a **"point condensation phenomenon."** By this we mean that these peaks become narrower and narrower and eventually shrink to the set of points itself. In fact, their spatial extension is of the order $O(\epsilon)$. We also say that the spike solutions "concentrate" at the set of points. Furthermore, we remark that the maximum value of activator and inhibitor, respectively, diverges to $+\infty$.

Although it has been observed numerically that these patterns are stable, giving a rigorous proof of these facts has been an open problem. Namely, how can one rigorously construct these solutions? Where are the peaks located? Are these solutions stable?

In this paper we solve these questions. We explicitly give a rigorous construction of K-peaked stationary states by using the powerful method of Liapunov-Schmidt reduction. This enables us to reduce the infinite-dimensional problem of finding an equilibrium state to (GM) to the finite-dimensional problem of locating the K points at which the spikes concentrate. We give a sufficient condition for the locations of these points in terms of the derivatives of Green's function.

Furthermore, concerning stability, one has to study separately the eigenvalues of the order $O(1)$, which are called "large eigenvalues," and the eigenvalues of the order $o(1)$, which are called "small eigenvalues." We show that the small eigenvalues are related to the derivatives of Green's function and to the spike locations. Suppose these small eigenvalues all have negative real parts and that τ is large or $K > 1$, then the following result holds true and is the main contribution of this paper:

For $\epsilon \ll 1$ there are stability thresholds

$$D_1(\epsilon) > D_2(\epsilon) > D_3(\epsilon) > \cdots > D_K(\epsilon) > \cdots,$$

such that

if $\lim_{\epsilon\to 0} \frac{D_K(\epsilon)}{D} > 1$, then the K-peaked solution is stable,

and if $\lim_{\epsilon\to 0} \frac{D_K(\epsilon)}{D} < 1$, then the K-peaked solution is unstable.

Furthermore, we will show that

$$D_K(\epsilon) = \frac{|\Omega|}{2\pi K} \log \frac{1}{\epsilon} \quad \text{as } \epsilon \to 0.$$

In particular, if

$$\lim_{\epsilon\to 0} \frac{D}{\log \frac{1}{\epsilon}} = 0 \quad \text{as } \epsilon \to 0,$$

then, for every positive integer K, the K-peaked solution is stable for ϵ small enough. This recovers our earlier result in the **strong coupling** case, [40].

We now describe the results of the paper in detail.

We first introduce a Green's function G_0 which we need to formulate our main results. Let $G_0(x, \xi)$ be the Green's function given by

$$\begin{cases} \Delta G_0(x,\xi) - \dfrac{1}{|\Omega|} + \delta_\xi(x) = 0 & \text{in } \Omega, \\ \displaystyle\int_\Omega G_0(x,\xi)\,dx = 0, & \\ \dfrac{\partial G_0(x,\xi)}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and let

$$H_0(x,\xi) = \frac{1}{2\pi} \log \frac{1}{|x-\xi|} - G_0(x,\xi) \tag{1.2}$$

be the regular part of $G_0(x, \xi)$.

Denote $\mathbf{P} \in \Omega^K$, where $\mathbf{P}$ is arranged such that

$$\mathbf{P} = (P_1, P_2, \ldots, P_K),$$

with

$$P_i = (P_{i,1}, P_{i,2}) \quad \text{for } i = 1, 2, \ldots, K.$$

For the rest of the paper we assume that $\mathbf{P} \in \overline{\Lambda_\delta}$, where for $\delta > 0$ we define

$$\Lambda_\delta = \{(P_1, P_2, \ldots, P_K) \in \Omega^K : |P_i - P_j| > 4\delta \quad \text{for } i \neq j, \\ \text{and} \quad d(P_i, \partial\Omega) > 4\delta \quad \text{for } i = 1, 2 \ldots, K\}. \tag{1.3}$$

For $\mathbf{P} \in \overline{\Lambda_\delta}$, we define,

$$F(\mathbf{P}) = \sum_{k=1}^{K} H_0(P_k, P_k) - \sum_{i,j=1,\ldots,K, i\neq j} G_0(P_i, P_j), \tag{1.4}$$

and

$$M(\mathbf{P}) = (\nabla^2_{\mathbf{P}} F(\mathbf{P})). \tag{1.5}$$

Here $M(\mathbf{P})$ is a $(2K)\times(2K)$ matrix with components $\frac{\partial^2 F(\mathbf{P})}{\partial P_{i,j} \partial P_{k,l}}$, $i, k = 1, \ldots, K$, $j, l = 1, 2$ (recall that $P_{i,j}$ is the j-th component of P_i).

Note that $F(\mathbf{P}) \in C^\infty(\overline{\Lambda_\delta})$.

Set

$$D = \frac{1}{\beta^2}, \qquad \eta_\epsilon := \frac{\beta^2 |\Omega|}{2\pi} \log \frac{1}{\epsilon}. \tag{1.6}$$

Then $D \to +\infty$ is equivalent to $\beta \to 0$.

The stationary system for (GM) is the following system of elliptic equations:

$$\begin{cases} \epsilon^2 \Delta A - A + \dfrac{A^2}{H} = 0, & A > 0, \quad \text{in } \Omega, \\ \dfrac{1}{\beta^2} \Delta H - H + A^2 = 0, & H > 0, \quad \text{in } \Omega, \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial H}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Our first theorem concerns the existence of $K-$peaked solutions.

Theorem 1.1. *Let $\mathbf{P}^0 = (P_1^0, P_2^0, \ldots, P_K^0) \in \overline{\Lambda_\delta}$ be a nondegenerate critical point of $F(\mathbf{P})$ (defined by (1.4)). Moreover, we assume that the following technical condition holds:*

$$\text{if } K > 1, \quad \text{then} \quad \lim_{\epsilon \to 0} \eta_\epsilon \neq K, \tag{1.8}$$

where η_ϵ is defined by (1.6).

Then for ϵ sufficiently small and $D = \frac{1}{\beta^2}$ sufficiently large, problem (1.7) has a solution (A_ϵ, H_ϵ) with the following properties:

(1) $A_\epsilon(x) = \xi_\epsilon(\sum_{j=1}^K w(\frac{x-P_j^\epsilon}{\epsilon}) + O(k(\epsilon, \beta)))$ uniformly for $x \in \bar{\Omega}$. Here w is the unique solution of the problem

$$\begin{cases} \Delta w - w + w^2 = 0, & w > 0, \quad \text{in } R^2, \\ w(0) = \max_{y \in R^2} w(y), & w(y) \to 0, \quad \text{as } |y| \to \infty, \end{cases} \tag{1.9}$$

$$\xi_\epsilon = \begin{cases} \dfrac{1}{K}\dfrac{|\Omega|}{\epsilon^2\int_{R^2} w^2(y)\,dy}, & \textit{if } \eta_\epsilon \to 0, \\ \dfrac{1}{\eta_\epsilon}\dfrac{|\Omega|}{\epsilon^2\int_{R^2} w^2(y)\,dy}, & \textit{if } \eta_\epsilon \to \infty, \\ \dfrac{1}{K+\eta_0}\dfrac{|\Omega|}{\epsilon^2\int_{R^2} w^2(y)\,dy}, & \textit{if } \eta_\epsilon \to \eta_0, \end{cases} \tag{1.10}$$

and

$$k(\epsilon, \beta) := \epsilon^2 \xi_\epsilon \beta^2. \tag{1.11}$$

(By (1.10), $k(\epsilon,\beta) = O(\min\{\frac{1}{\log\frac{1}{\epsilon}}, \beta^2\})$*.)*

Furthermore, $P_j^\epsilon \to P_j^0$ *as* $\epsilon \to 0$ *for* $j = 1, \ldots, K$*.*

(2) $H_\epsilon(x) = \xi_\epsilon(1 + O(k(\epsilon,\beta)))$ *uniformly for* $x \in \bar{\Omega}$*.*

Remark. (1.1) Condition (1.8) in Theorem 1.1 is a technical condition that is needed for the Liapunov-Schmidt reduction process. In Appendix A we will explain how it arises.

For existence and uniqueness of the solutions of (1.9), we refer to [9] and [16]. We also recall that

$$w(y) \sim |y|^{-1/2} e^{-|y|} \quad \text{as } |y| \to \infty. \tag{1.12}$$

Next we study the stability and instability of the K-peaked solutions constructed in Theorem 1.1. To this end, we need to study the following eigenvalue problem,

$$\mathcal{L}_\epsilon \begin{pmatrix} \phi_\epsilon \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + 2\dfrac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \dfrac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon \\ \dfrac{1}{\tau}\left(\dfrac{1}{\beta^2}\Delta\psi_\epsilon - \psi_\epsilon + 2A_\epsilon\phi_\epsilon\right) \end{pmatrix} = \lambda_\epsilon \begin{pmatrix} \phi_\epsilon \\ \psi_\epsilon \end{pmatrix}, \tag{1.13}$$

where (A_ϵ, H_ϵ) is the solution constructed in Theorem 1.1 and $\lambda_\epsilon \in \mathcal{C}$ – the set of complex numbers.

We say that (A_ϵ, H_ϵ) is **linearly stable** if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ lies in the left half-plane $\{\lambda \in \mathcal{C}\colon \text{Re}(\lambda) < 0\}$. (A_ϵ, H_ϵ) is called **linearly unstable** if there exists an eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ with $\text{Re}(\lambda_\epsilon) > 0$. (From now on, we use the notations *linearly stable* and *linearly unstable* as defined above.)

Our second main result, which is on stability, is stated as follows.

Theorem 1.2. *Let* $\mathbf{P}^0 \in \overline{\Lambda_\delta}$ *be a nondegenerate critical point of* $F(\mathbf{P})$*, and for* ϵ *sufficiently small and* $D = \frac{1}{\beta^2}$ *sufficiently large, let* (A_ϵ, H_ϵ) *be the* $K-$*peaked solutions constructed in Theorem 1.1 whose peaks approach* $\mathbf{P}^0$*.*

Assume (1.8) holds and further that

$(*)$ $\qquad \mathbf{P}^0$ *is a nondegenerate local maximum point of* $F(\mathbf{P})$*.*

Then we have

Case 1. $\eta_\epsilon \to 0$ *(i.e.,* $\frac{2\pi D}{|\Omega|} \gg \log \frac{1}{\epsilon}$*).*

If $K = 1$*, then there exists a unique* $\tau_1 > 0$ *such that for* $\tau < \tau_1$*,* (A_ϵ, H_ϵ) *is linearly stable, while for* $\tau > \tau_1$*,* (A_ϵ, H_ϵ) *is linearly unstable.*

If $K > 1$*,* (A_ϵ, H_ϵ) *is linearly unstable for any* $\tau \geq 0$*.*

Case 2. $\eta_\epsilon \to +\infty$ *(i.e.,* $\frac{2\pi D}{|\Omega|} \ll \log \frac{1}{\epsilon}$*).*

(A_ϵ, H_ϵ) *is linearly stable for any* $\tau > 0$*.*

Case 3. $\eta_\epsilon \to \eta_0 \in (0, +\infty)$ *(i.e.,* $\frac{2\pi D}{|\Omega|} \sim \frac{1}{\eta_0} \log \frac{1}{\epsilon}$*).*

If $K > 1$ *and* $\eta_0 < K$*, then* (A_ϵ, H_ϵ) *is linearly unstable for any* $\tau > 0$*.*

If $\eta_0 > K$*, then there exist* $0 < \tau_2 \leq \tau_3$ *such that* (A_ϵ, H_ϵ) *is linearly stable for* $\tau < \tau_2$ *and* $\tau > \tau_3$*.*

If $K = 1$*,* $\eta_0 < 1$*, then there exist* $0 < \tau_4 \leq \tau_5$ *such that* (A_ϵ, H_ϵ) *is linearly stable for* $\tau < \tau_4$ *and linearly unstable for* $\tau > \tau_5$*.*

The statement of Theorem 1.2 is rather long. Let us therefore explain the results by the following remarks.

Remarks. (1.2) Assuming that condition (*) holds, then for ϵ small the stability behavior of (A_ϵ, H_ϵ) can be summarized in the following table:

	Case 1	Case 2	Case 3 ($\eta_0 < K$)	Case 3 ($\eta_0 > K$)
$K = 1,\ \tau$ small	stable	stable	stable	stable
$K = 1,\ \tau$ finite	?	stable	?	?
$K = 1,\ \tau$ large	unstable	stable	unstable	stable
$K > 1,\ \tau$ small	unstable	stable	unstable	stable
$K > 1,\ \tau$ finite	unstable	stable	unstable	?
$K > 1,\ \tau$ large	unstable	stable	unstable	stable

(1.3) The condition (*) on the locations $\mathbf{P}^0$ arises in the study of small ($o(1)$) eigenvalues. For any bounded smooth domain Ω, the functional $F(\mathbf{P})$, defined by (1.4), always admits a global maximum at some $\mathbf{P}^0 \in \overline{\Lambda_\delta}$ (for some small $\delta > 0$). The proof of this fact is similar to that in the appendix in [40]. We believe that in **generic** domains, this global maximum point $\mathbf{P}^0$ is nondegenerate.

It is an interesting open question to numerically compute the critical points of $F(\mathbf{P})$ and link them explicitly to the geometry of the domain Ω.

We believe that for other types of critical points of $F(\mathbf{P})$, such as saddle points, the solution constructed in Theorem 1.1 should be linearly unstable. We are not able to prove this at the moment, since the operator $\mathcal{L}_\epsilon$ is **not self-adjoint.**

(1.4) Case 1 and Case 3 with $\eta_0 < K$ resemble the **shadow system**, and Case 2 and Case 3 with $\eta_0 > K$ are similar to the **strong coupling** case. Theorem 1.2 contains a new result even in the shadow system case: For the limiting nonlocal eigenvalue problem (NLEP), we have shown the uniqueness of Hopf bifurcation at τ_1 (Lemma 2.4); compare [24], [34]. Note that our τ is **fixed**. If we allow τ to vary with respect to ϵ, we conjecture that there is a unique $\tau_1(\epsilon) = \tau_1 + o(1)$ such that Hopf bifurcation occurs for $\mathcal{L}_\epsilon$.

(1.5) We conjecture that in Case 3, $\tau_2 = \tau_3$. This will imply that for any $\tau \geq 0$ and $\eta_0 > K$, multiple spikes are stable, provided condition (*) is satisfied. (It is possible to obtain explicit values for τ_2 and τ_3. See the Remark 2.2 after the proof of Theorem 2.5.)

(1.6) Roughly speaking, assuming that condition (*) holds and that $K \geq 1$ or τ is large, then for $\epsilon \ll 1$, $D_K(\epsilon) = \frac{|\Omega|}{2\pi K} \log \frac{1}{\epsilon}$ is the critical threshold for the asymptotic behavior of the diffusion coefficient of the inhibitor which determines the stability of K-peaked solutions. Thus we have established a result which is similar as in the one-dimensional case, [14], [41]. In [14] the case when τ is small is studied by a matched asymptotic analysis approach. A rigorous proof of the results of [14] is contained in [41]. A dynamics approach that covers the case of general $\tau \geq 0$ but is restricted to the whole R^1 or to periodic boundary conditions is contained in [6]. However, in higher dimensions the analysis is very different because it has to reflect the geometry of the domain, which is trivial for an interval on the real line (where the peaks are placed equidistantly).

Let us recall the result in the one-dimensional case. It is shown ([14]) for $K \geq 2$ that the critical thresholds $D_K(\epsilon) = D_K$ are in leading order independent of ϵ. Moreover, the critical thresholds arise in the computation of the small eigenvalues. Here in R^2, $D_K(\epsilon) \to +\infty$ as $\epsilon \to 0$. Furthermore, $D_K(\epsilon)$ is obtained in the study of the large eigenvalues. Since these thresholds are independent of the peak locations, they can be studied without considering higher-order terms of the equilibrium.

(1.7) We have obtained the leading order asymptotics for the critical threshold $D_K(\epsilon)$ which is the order $\log \frac{1}{\epsilon}$. This is true if we take ϵ sufficiently small. In practice, it will be very useful to obtain the next order term in the asymptotic expansion of $D_K(\epsilon)$, which we believe should be $O(1)$.

We now comment on some related work.

Generally speaking, system (1.7) is quite difficult to solve because it has neither a **variational structure** nor **a priori estimates**. One way to study (1.7) is to examine the so-called **shadow system**. Namely, we let $D \to +\infty$ first. It is known (see [15], [20], [27]) that the study of the shadow system amounts to the study of the following single equation for $p = 2$:

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0, & u > 0, \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

Equation (1.14) has a variational structure and has been studied by numerous authors. It is known that equation (1.14) has both boundary spike solutions and interior spike solutions. For the existence of boundary spike solutions, see [1], [10], [21], [22], [23], [32], [37], [38], and the references therein. For the existence of interior spike solutions, please see [11], [26], [31], [33], and the references therein. For the stability of spike solutions, please see [2], [13], [24], [25], [34], [35]. For dynamics, we refer to [3].

Now we describe some previous results for the two-dimensional strong coupling case, i.e., for finite $D \sim 1$. In [39], we constructed single interior spike solutions to (1.7) (without loss of generality, we assumed that $D = 1$). Then in [40] we continued that study: After constructing interior K-peaked solutions, we also proved that they are stable for $\tau = 0$ provided that the limiting peaks $\mathbf{P}^0 = (P_1^0, \ldots, P_K^0)$ are a nondegenerate local maximum point of the following functional:

$$F_1(\mathbf{P}) = \sum_{k=1}^{K} H_1(P_k, P_k) - \sum_{i,j=1,\ldots,K, i \neq j} G_1(P_i, P_j), \tag{1.15}$$

where $G_1(P, x)$ is Green's function of $-\Delta + 1$ under the Neumann boundary condition, i.e., G_1 satisfies

$$\begin{cases} -\Delta G_1 + G_1 = \delta_P & \text{in } \Omega, \\ \dfrac{\partial G_1}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here δ_P is the Dirac delta distribution at a point P, and

$$H_1(P, x) = K_1(|x - P|) - G_1(P, x),$$

where $K_1(|x|) = \frac{1}{2\pi} \log \frac{1}{|x|}$ is the fundamental solution of $-\Delta + 1$ in R^2 with singularity at 0.

Therefore for any finite $D \sim 1$, the stability of $K-$peaked solutions does not depend on D but on the peak locations only.

In the case of boundary spikes for the weak coupling case, the boundary mean curvature may interact with the Green's function. We will study this effect in a forthcoming paper.

Finally we remark that some of the results of Theorem 1.1 and Theorem 1.2 may be extended to the following generalized Gierer-Meinhardt system:

$$\text{(Generalized GM)} \qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + \dfrac{A^p}{H^q}, & A > 0, \quad \text{in } \Omega, \\ \tau H_t = D\Delta H - H + \dfrac{A^r}{H^s}, & H > 0, \quad \text{in } \Omega, \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial H}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where the exponents (p, q, r, s) satisfy the following conditions:

$$p > 1, \qquad q > 0, \qquad r > 0, \qquad s \geq 0, \qquad \frac{qr}{(p-1)(s+1)} > 1.$$

For example, the existence result Theorem 1.1 can be applied to the above system without any technical difficulty. For the stability result Theorem 1.2, there should be some restrictions on the (p, q, r, s). See [4], [24], [25], [36], and [42] for related studies on NLEPs. We shall leave this to further investigations.

Other work on concentrated solutions for reaction-diffusion systems includes [5], [28], [30], and the survey [20].

The structure of the paper is as follows:

Preliminaries:
- Section 2: Study of Two NLEPs
- Section 3: Calculations on the Heights of the Peaks

Existence: Proof of Theorem 1.1:
- Section 4: Reduction to Finite Dimensions
- Section 5: Solving the Reduced Problem

Stability: Proof of Theorem 1.2:
- Section 6: Study of Large Eigenvalues
- Section 7: Study of Small Eigenvalues

The proof of the invertibility of the linearized operator is delayed to Appendix A.

Throughout the paper, $C > 0$ is a generic constant that is independent of ϵ and β and may change from line to line and δ is a very small but fixed constant. We always assume that $\mathbf{P}, \mathbf{P}^0 \in \overline{\Lambda_\delta}$, where $\overline{\Lambda_\delta}$ was defined in (1.3), and that $|\mathbf{P} - \mathbf{P}^0| < 4\delta$. To simplify our notation, we use $e.s.t.$ to denote exponentially small terms in the corresponding norms; more precisely, $e.s.t. = O(e^{-\delta/\epsilon})$. The notation $A(\epsilon) \sim B(\epsilon)$ means that $\lim_{\epsilon\to 0} \frac{A(\epsilon)}{B(\epsilon)} = c_0 > 0$, for some positive number c_0.

2. Preliminaries I: Some Properties of w and the Study of Two Nonlocal Eigenvalue Problems (NLEPs)

Let w be the unique solution of (1.9). In this section, we study some properties of w as well as two NLEPs.

Let

$$L_0\phi = \Delta\phi - \phi + 2w\phi, \qquad \phi \in H^2(R^2). \tag{2.1}$$

We first recall the following well-known result:

Lemma 2.1. *The eigenvalue problem*

$$L_0\phi = \mu\phi, \qquad \phi \in H^2(R^2), \tag{2.2}$$

admits the following set of eigenvalues

$$\mu_1 > 0, \qquad \mu_2 = \mu_3 = 0, \qquad \mu_4 < 0, \dots. \tag{2.3}$$

The eigenfunction Φ_0 corresponding to μ_1 can be made positive and radially symmetric; the space of eigenfunctions corresponding to the eigenvalue 0 is

$$K_0 := span\ \left\{ \frac{\partial w}{\partial y_j},\ j = 1, 2 \right\}. \tag{2.4}$$

Proof. This lemma follows from Theorem 2.1 of [17] and Lemma C of [22]. □

Next, we consider the following two nonlocal eigenvalue problems:

$$L\phi := \Delta\phi - \phi + 2w\phi - \gamma \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0\phi, \qquad \phi \in H^2(R^2), \tag{2.5}$$

where either (a) $\gamma = \dfrac{\mu}{1+\tau\lambda_0}$, where $\mu > 0,\ \tau \geq 0$, or

(b) $\gamma = \dfrac{2(K + \eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)}$, where $\eta_0 > 0,\ \tau \geq 0$.

Case (a) will be studied in Theorem 2.2 and Case (b) in Theorem 2.5.

Problem (2.5) plays the key role in the study of large eigenvalues (Section 6 below). It is here that the critical stability thresholds arise.

We consider case (a) first:

Theorem 2.2. *Let* $\gamma = \frac{\mu}{1+\tau\lambda_0}$ *where* $\mu > 0$, $\tau \geq 0$, *and let* L *be defined by (2.5).*

(1) Suppose that $\mu > 1$. *Then there exists a unique* $\tau = \tau_1 > 0$ *such that for* $\tau < \tau_1$, *(2.5) admits a positive eigenvalue, and for* $\tau > \tau_1$, *all nonzero eigenvalues of problem (2.5) satisfy* $Re(\lambda) < 0$. *At* $\tau = \tau_1$, L *has a Hopf bifurcation.*

(2) Suppose that $\mu < 1$. *Then* L *admits a positive eigenvalue* $\lambda_0 > 0$.

Proof of Theorem 2.2. Theorem 2.2 will be proved by two lemmas below.

Lemma 2.3. *If* $\mu < 1$, *then* L *has a positive eigenvalue* $\lambda_0 > 0$.

Proof. By arguments similar to [4] or [42], we may assume that ϕ is a radially symmetric function, namely, $\phi \in H_r^2(R^2) = \{u \in H^2(R^2) | u = u(|y|)\}$. Let L_0 be given by (2.1). Then by Lemma 2.1, L_0 is invertible in $H_r^2(R^2)$. Let us denote the inverse as L_0^{-1}. By Lemma 2.1, L_0 has a unique positive eigenvalue μ_1. It is easy to see that $\lambda_0 \neq \mu_1$ since $\int_{R^2} w\Phi_0 > 0$.

Then $\lambda_0 > 0$ is an eigenvalue of (2.5) if and only if it satisfies the following algebraic equation:

$$\int_{R^2} w^2 = \frac{\mu}{1+\tau\lambda_0} \int_{R^2} [((L_0 - \lambda_0)^{-1} w^2) w]. \tag{2.6}$$

Equation (2.6) can be simplified further to the following:

$$\rho(\lambda_0) := ((\mu - 1) - \tau\lambda_0) \int_{R^2} w^2 + \mu\lambda_0 \int_{R^2} [((L_0 - \lambda_0)^{-1} w) w] = 0. \tag{2.7}$$

Note that $\rho(0) = (\mu - 1) \int_{R^2} w^2 < 0$. On the other hand, as $\lambda_0 \to \mu_1$, $\lambda_0 < \mu_1$, we have $\int_{R^2} ((L_0 - \lambda_0)^{-1} w) w \to +\infty$, and hence $\rho(\lambda_0) \to +\infty$. By continuity, there exists a $\lambda_0 \in (0, \mu_1)$ such that $\rho(\lambda_0) = 0$. Such a positive λ_0 will be an eigenvalue of L. □

Next we consider the case $\mu > 1$. As in [4], we may consider radially symmetric functions only. By Theorem 1.4 of [34], for $\tau = 0$ (and by perturbation, for τ small), all eigenvalues lie on the left half-plane. By [4], for τ large, there exist unstable eigenvalues.

Note that the eigenvalues will not cross through zero: In fact, if $\lambda_0 = 0$, then we have

$$L_0\phi - \mu \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = 0,$$

which implies that

$$L_0 \left(\phi - \mu \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w \right) = 0,$$

and hence, by Lemma 2.1,

$$\phi - \mu \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w \in K_0.$$

This is impossible since ϕ is radially symmetric and $\phi \neq cw$ for all $c \in R$.

Thus there must be a point τ_1 at which L has a Hopf bifurcation, i.e., L has a purely imaginary eigenvalue $\alpha = \sqrt{-1}\alpha_I$. To prove Theorem 2.2 (1), all we need to show is that τ_1 is unique. That is,

Lemma 2.4. *Let $\mu > 1$. Then there exists a unique $\tau_1 > 0$ such that L has a Hopf bifurcation.*

Proof. Let $\lambda_0 = \sqrt{-1}\alpha_I$ be an eigenvalue of L. Without loss of generality, we may assume that $\alpha_I > 0$. (Note that $-\sqrt{-1}\alpha_I$ is also an eigenvalue of L.) Let $\phi_0 = (L_0 - \sqrt{-1}\alpha_I)^{-1}w^2$. Then (2.5) becomes

$$\frac{\int_{R^2} w\phi_0}{\int_{R^2} w^2} = \frac{1+\tau\sqrt{-1}\alpha_I}{\mu}. \tag{2.8}$$

Let $\phi_0 = \phi_0^R + \sqrt{-1}\phi_0^I$. Then from (2.8), we obtain the two equations

$$\frac{\int_{R^2} w\phi_0^R}{\int_{R^2} w^2} = \frac{1}{\mu}, \tag{2.9}$$

$$\frac{\int_{R^2} w\phi_0^I}{\int_{R^2} w^2} = \frac{\tau\alpha_I}{\mu}. \tag{2.10}$$

Note that (2.9) is independent of τ.

Let us now compute $\int_{R^2} w\phi_0^R$. Observe that (ϕ_0^R, ϕ_0^I) satisfies

$$L_0\phi_0^R = w^2 - \alpha_I\phi_0^I, \qquad L_0\phi_0^I = \alpha_I\phi_0^R.$$

So $\phi_0^R = \alpha_I^{-1}L_0\phi_0^I$ and

$$\phi_0^I = \alpha_I(L_0^2+\alpha_I^2)^{-1}w^2, \qquad \phi_0^R = L_0(L_0^2+\alpha_I^2)^{-1}w^2. \tag{2.11}$$

Substituting (2.11) into (2.9) and (2.10), we obtain

$$\frac{\int_{R^2}[wL_0(L_0^2+\alpha_I^2)^{-1}w^2]}{\int_{R^2} w^2} = \frac{1}{\mu}, \tag{2.12}$$

$$\frac{\int_{R^2}[w(L_0^2+\alpha_I^2)^{-1}w^2]}{\int_{R^2} w^2} = \frac{\tau}{\mu}. \tag{2.13}$$

Let $h(\alpha_I) = \frac{\int_{R^2} wL_0(L_0^2+\alpha_I^2)^{-1}w^2}{\int_{R^2} w^2}$. Then integration by parts gives $h(\alpha_I) = \frac{\int_{R^2} w^2(L_0^2+\alpha_I^2)^{-1}w^2}{\int_{R^2} w^2}$. Note that $h'(\alpha_I) = -2\alpha_I\frac{\int_{R^2} w^2(L_0^2+\alpha_I^2)^{-2}w^2}{\int_{R^2} w^2} < 0$. So since

$$h(0) = \frac{\int_{R^2} w(L_0^{-1}w^2)}{\int_{R^2} w^2} = 1,$$

$h(\alpha_I) \to 0$ as $\alpha_I \to \infty$, and $\mu > 1$, there exists a unique $\alpha_I > 0$ such that (2.12) holds. Substituting this unique α_I into (2.13), we obtain a unique $\tau = \tau_1 > 0$.

Lemma 2.4 is thus proved. □

Proof. Theorem 2.2 now follows from Lemma 2.3 and Lemma 2.4. □

Remark. (2.1) Theorem 2.2 is true in R^N, $N \le 4$. The existence of a Hopf bifurcation has been studied in [4], [24], [25], [42]. Here we have proved the **uniqueness** of such a Hopf bifurcation, which is new and interesting in its own right.

Finally we study case (b), namely the following NLEP:

$$\Delta\phi - \phi + 2w\phi - \frac{2(K+\eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)} \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0\phi, \qquad \phi \in H^2(R^2), \quad (2.14)$$

where $0 < \eta_0 < +\infty$ and $0 \le \tau < +\infty$.

Then we have

Theorem 2.5. *(1) If $\eta_0 < K$, then for τ small, problem (2.14) is stable, while for τ large it is unstable.*

(2) If $\eta_0 > K$, then there exists $0 < \tau_2 \le \tau_3$ such that problem (2.14) is stable for $\tau < \tau_2$ or $\tau > \tau_3$.

Proof. Let us set

$$f(\tau\lambda) = \frac{2(K+\eta_0(1+\tau\lambda))}{(K+\eta_0)(1+\tau\lambda)}. \quad (2.15)$$

We note that

$$\lim_{\tau\lambda\to+\infty} f(\tau\lambda) = \frac{2\eta_0}{K+\eta_0} =: f_\infty.$$

If $\eta_0 < K$, then by Theorem 2.2 (2), problem (2.5) with $\mu = f_\infty$ has a positive eigenvalue α_1. Now by perturbation arguments (similar to those in [4]), for τ large, problem (2.14) has an eigenvalue near $\alpha_1 > 0$. This implies that for τ large, problem (2.14) is unstable.

Now we show that problem (2.14) has no nonzero eigenvalues with nonnegative real part, provided that either τ is small or $\eta_0 > K$ and τ is large. (It is immediately seen that $f(\tau\lambda) \to 2$ as $\tau\lambda \to 0$ and $f(\tau\lambda) \to \frac{2\eta_0}{\eta_0+K} > 1$ as $\tau\lambda \to +\infty$ if $\eta_0 > K$. Then Theorem 2.2 should apply. The problem is that we do not have control on $\tau\lambda$. Here we provide a rigorous proof.)

We apply the following inequality (Lemma 5.1 in [34]): For any (real-valued function) $\phi \in H_r^2(R^2)$, we have

$$\int_{R^2} (|\nabla\phi|^2 + \phi^2 - 2w\phi^2) + 2\frac{\int_{R^2} w\phi \int_{R^2} w^2\phi}{\int_{R^2} w^2} - \frac{\int_{R^2} w^3}{(\int_{R^2} w^2)^2}\left(\int_{R^2} w\phi\right)^2 \ge 0, \quad (2.16)$$

where equality holds if and only if ϕ is a multiple of w.

Now let $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$, $\phi = \phi_R + \sqrt{-1}\phi_I$ satisfy (2.14). Then we have

$$L_0\phi - f(\tau\lambda_0)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0\phi. \quad (2.17)$$

Multiplying (2.17) by $\bar{\phi}$—the conjugate function of ϕ—and integrating over R^2, we obtain that

$$\int_{R^2}(|\nabla\phi|^2+|\phi|^2-2w|\phi|^2)=-\lambda_0\int_{R^2}|\phi|^2-f(\tau\lambda_0)\frac{\int_{R^2}w\phi}{\int_{R^2}w^2}\int_{R^2}w^2\bar{\phi}. \tag{2.18}$$

Multiplying (2.17) by w and integrating over R^2, we obtain that

$$\int_{R^2}w^2\phi=\left(\lambda_0+f(\tau\lambda_0)\frac{\int_{R^2}w^3}{\int_{R^2}w^2}\right)\int_{R^2}w\phi. \tag{2.19}$$

Taking the conjugate of (2.19), we have

$$\int_{R^2}w^2\bar{\phi}=\left(\bar{\lambda_0}+f(\tau\bar{\lambda}_0)\frac{\int_{R^2}w^3}{\int_{R^2}w^2}\right)\int_{R^2}w\bar{\phi}. \tag{2.20}$$

Substituting (2.20) into (2.18), we have that

$$\begin{aligned}\int_{R^2}(|\nabla\phi|^2+|\phi|^2-2w|\phi|^2) \;=\; -\lambda_0\int_{R^2}|\phi|^2-f(\tau\lambda_0)\\ \times\left(\bar{\lambda}_0+f(\tau\bar{\lambda}_0)\frac{\int_{R^2}w^3}{\int_{R^2}w^2}\right)\frac{|\int_{R^2}w\phi|^2}{\int_{R^2}w^2}.\end{aligned} \tag{2.21}$$

We just need to consider the real part of (2.21). Now applying the inequality (2.16) and using (2.20), we arrive at

$$-\lambda_R\geq \mathrm{Re}\left(f(\tau\lambda_0)\left(\bar{\lambda}_0+f(\tau\bar{\lambda}_0)\frac{\int_{R^2}w^3}{\int_{R^2}w^2}\right)\right)-2\,\mathrm{Re}\left(\bar{\lambda}_0+f(\tau\bar{\lambda}_0)\frac{\int_{R^2}w^3}{\int_{R^2}w^2}\right)+\frac{\int_{R^2}w^3}{\int_{R^2}w^2},$$

where we recall $\lambda_0=\lambda_R+\sqrt{-1}\lambda_I$ with $\lambda_R,\ \lambda_I\in R$.

Assuming that $\lambda_R\geq 0$, then we have

$$\frac{\int_{R^2}w^3}{\int_{R^2}w^2}|f(\tau\lambda_0)-1|^2+\mathrm{Re}(\bar{\lambda}_0(f(\tau\lambda_0)-1))\leq 0. \tag{2.22}$$

By the usual Pohozaev's identity for (1.9) (multiplying (1.9) by $y\cdot\nabla w(y)$ and integrating by parts), we obtain that

$$\int_{R^2}w^3=\frac{3}{2}\int_{R^2}w^2. \tag{2.23}$$

Substituting (2.23) and the expression (2.15) for $f(\tau\lambda)$ into (2.22), we have

$$\frac{3}{2}|\eta_0+K+(\eta_0-K)\tau\lambda|^2+\mathrm{Re}\,((\eta_0+K)(1+\tau\bar{\lambda}_0)((\eta_0+K)\bar{\lambda}_0+(\eta_0-K)\tau|\lambda_0|^2))\leq 0,$$

which is equivalent to

$$\frac{3}{2}(1+\mu_0\tau\lambda_R)^2+\lambda_R+(\mu_0\tau+\tau+\mu_0\tau^2|\lambda_0|^2)\lambda_R+\left(\frac{3}{2}\mu_0^2\tau^2+\mu_0\tau-\tau\right)\lambda_I^2\le 0, \tag{2.24}$$

where we have introduced $\mu_0 := \frac{\eta_0-K}{\eta_0+K}$.

If $\eta_0 > K$ (i.e., $\mu_0 > 0$) and τ is large, then

$$\frac{3}{2}\mu_0^2\tau^2+\mu_0\tau-\tau\ge 0. \tag{2.25}$$

So (2.24) does not hold for $\lambda_R \ge 0$.

To consider the case when τ is small, we have now derived an upper bound for λ_I.

From (2.18), we have

$$\lambda_I\int_{R^2}|\phi|^2=\mathrm{Im}\left(-f(\tau\lambda_0)\frac{\int_{R^2}w\phi}{\int_{R^2}w^2}\int_{R^2}w^2\bar{\phi}\right).$$

Hence,

$$|\lambda_I|\le|f(\tau\lambda_0)|\sqrt{\frac{\int_{R^2}w^4}{\int_{R^2}w^2}}\le C, \tag{2.26}$$

where C is independent of λ_0.

Substituting (2.26) into (2.24), we see that (2.24) cannot hold for $\lambda_R \ge 0$, if τ is small. □

Remark. (2.2) From the proof of Theorem 2.5, it is possible to obtain explicit values for τ_2 and τ_3. (In fact, from (2.25), we obtain a value for τ_3. From (2.26) and (2.24), we obtain a value for τ_2.)

3. Preliminaries II: Calculating the Heights of the Peaks

In this section we formally calculate the heights of the peaks as needed in the sections below. In particular, we introduce the scale ξ_ϵ given in (1.10). It is found that in the leading order the heights depend on the number of peaks but not on their locations. This is a leading order asymptotic statement that is valid for $\epsilon \to 0$ and $D \to \infty$.

For $\beta > 0$, let $G_\beta(x,\xi)$ be the Green's function given by

$$\begin{cases}\Delta G_\beta-\beta^2G_\beta+\delta_\xi=0 & \text{in } \Omega,\\ \dfrac{\partial G_\beta}{\partial\nu}=0 & \text{on } \partial\Omega.\end{cases} \tag{3.1}$$

Let $G_0(x,\xi)$ be the Green's function given by (1.1). Then we can derive a relation between G_β and G_0 as follows. From (3.1), we get

$$\int_\Omega G_\beta(x,\xi)\,dx=\beta^{-2}.$$

Set

$$G_\beta(x,\xi) = \frac{\beta^{-2}}{|\Omega|} + \overline{G}_\beta(x,\xi). \tag{3.2}$$

Then

$$\begin{cases} \Delta \bar{G}_\beta - \beta^2 \bar{G}_\beta - \dfrac{1}{|\Omega|} + \delta_\xi = 0 & \text{in } \Omega, \\ \displaystyle\int_\Omega \bar{G}_\beta(x,\xi)\,dx = 0, \\ \dfrac{\partial \bar{G}_\beta}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

(1.1) and (3.3) imply that

$$\bar{G}_\beta(x,\xi) = G_0(x,\xi) + O(\beta^2)$$

in the operator norm of $L^2(\Omega) \to H^2(\Omega)$. (Note that the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ is compact.) Hence,

$$G_\beta(x,\xi) = \frac{\beta^{-2}}{|\Omega|} + G_0(x,\xi) + O(\beta^2) \tag{3.4}$$

in the operator norm of $L^2(\Omega) \to H^2(\Omega)$.

We define cut-off functions as follows: Let χ be a smooth cut-off function which is equal to 1 in $B_1(0)$ and equal to 0 in $R^2 \setminus \overline{B_2(0)}$. Let $\mathbf{P} \in \overline{\Lambda_\delta}$. Introduce

$$\chi_{\epsilon,P_j}(x) = \chi\left(\frac{x-P_j}{\delta}\right), \qquad x \in \Omega, \qquad j = 1,\ldots,K. \tag{3.5}$$

Let us assume that a multiple spike solution (A_ϵ, H_ϵ) of (1.7) is given by the following ansatz:

$$\begin{cases} A_\epsilon(x) \sim \sum_{i=1}^K \xi_{\epsilon,i}\, w\left(\dfrac{x-P_i^\epsilon}{\epsilon}\right) \chi_{\epsilon,P_i^\epsilon}(x), \\ H_\epsilon(P_i^\epsilon) \sim \xi_{\epsilon,i}, \end{cases} \tag{3.6}$$

where w is the unique solution of (1.9), $\xi_{\epsilon,i}$, $i = 1,\ldots,K$ are the heights of the peaks, to be determined later, and $\mathbf{P}^\epsilon = (P_1^\epsilon,\ldots,P_K^\epsilon) \in \overline{\Lambda_\delta}$ are the locations of K peaks. Then we can make the following calculations. In Sections 4 and 5, we will rigorously prove Theorem 1.1, which includes the asymptotic relations given in (3.6) with suitable error estimates.

Then from the equation for H_ϵ,

$$\Delta H_\epsilon - \beta^2 H_\epsilon + \beta^2 A_\epsilon^2 = 0,$$

we get by using (3.4),

$$\begin{aligned} H_\epsilon(P_i^\epsilon) &= \int_\Omega G_\beta(P_i^\epsilon,\xi)\beta^2 A_\epsilon^2(\xi)\,d\xi \\ &= \int_\Omega \left(\frac{\beta^{-2}}{|\Omega|} + G_0(P_i^\epsilon,\xi) + O(\beta^2)\right)\beta^2\left(\sum_{j=1}^K \xi_{\epsilon,j}^2 w^2\left(\frac{\xi-P_j^\epsilon}{\epsilon}\right) + e.s.t.\right) d\xi \\ &= \int_\Omega \left(\frac{1}{|\Omega|} + \beta^2 G_0(P_i^\epsilon,\xi) + O(\beta^4)\right)\left(\sum_{j=1}^K \xi_{\epsilon,j}^2 w^2\left(\frac{\xi-P_j^\epsilon}{\epsilon}\right) + e.s.t.\right) d\xi. \end{aligned}$$

Thus,

$$\begin{aligned} \xi_{\epsilon,i} &= \sum_{j=1}^K \xi_{\epsilon,j}^2 \frac{\epsilon^2}{|\Omega|}\int_{R^2} w^2(y)\,dy + \xi_{\epsilon,i}^2\beta^2\int_\Omega G_0(P_i^\epsilon,\xi)w^2\left(\frac{\xi-P_i^\epsilon}{\epsilon}\right)d\xi \\ &\quad + \sum_{j=1}^K \xi_{\epsilon,j}^2 O(\beta^2\epsilon^2). \end{aligned} \tag{3.7}$$

Using the expansion for G_0 in (3.7) gives

$$\begin{aligned} \xi_{\epsilon,i} &= \sum_{j=1}^K \xi_{\epsilon,j}^2 \frac{\epsilon^2}{|\Omega|}\int_{R^2} w^2(y)\,dy + \xi_{\epsilon,i}^2\beta^2\int_\Omega \left(\frac{1}{2\pi}\log\frac{1}{|P_i^\epsilon-\xi|} - H_0(P_i^\epsilon,\xi)\right) \\ &\quad \times w^2\left(\frac{\xi-P_i^\epsilon}{\epsilon}\right)d\xi + \sum_{j=1}^K \xi_{\epsilon,j}^2 O(\beta^2\epsilon^2) \\ &= \sum_{j=1}^K \xi_{\epsilon,j}^2 \frac{\epsilon^2}{|\Omega|}\int_{R^2} w^2(y)\,dy + \xi_{\epsilon,i}^2\frac{\beta^2}{2\pi}\epsilon^2\log\frac{1}{\epsilon}\int_{R^2} w^2(y)\,dy \\ &\quad + \sum_{j=1}^K \xi_{\epsilon,j}^2 O(\beta^2\epsilon^2). \end{aligned} \tag{3.8}$$

Note that $H_0 \in C^2(\bar{\Omega}\times\Omega)$.

Define

$$\xi_{\epsilon,i} = \frac{\hat{\xi}_{\epsilon,i}|\Omega|}{\epsilon^2\int_{R^2} w^2}. \tag{3.9}$$

Then (3.8) is equivalent to

$$\hat{\xi}_{\epsilon,i} = \sum_{j=1}^K \hat{\xi}_{\epsilon,j}^2 + \hat{\xi}_{\epsilon,i}^2\eta_\epsilon + \sum_{j=1}^K \hat{\xi}_{\epsilon,j}^2 O(\beta^2), \qquad i = 1,\ldots,K, \tag{3.10}$$

where we recall from (1.6) that

$$\eta_\epsilon = \frac{\beta^2|\Omega|}{2\pi}\log\frac{1}{\epsilon}.$$

We assume that as $\epsilon \to 0$, the heights of the spikes are asymptotically equal, i.e.,

$$\lim_{\epsilon \to 0} \frac{\xi_{\epsilon,i}}{\xi_{\epsilon,j}} = 1, \qquad \text{for } i \neq j. \tag{3.11}$$

(The case of **asymmetric patterns** will be discussed elsewhere.)

We solve (3.10) in three cases.

Case 1: $\eta_\epsilon \to 0$:

Then from (3.10), we get

$$\hat{\xi}_{\epsilon,i} = \frac{1}{K} + O(\eta_\epsilon), \qquad i = 1, \ldots, K. \tag{3.12}$$

This is clearly equivalent to

$$\xi_{\epsilon,i} = \frac{1}{K} \frac{|\Omega|}{\epsilon^2 \int_{R^2} w^2(y)\, dy} (1 + O(\eta_\epsilon)), \qquad i = 1, \ldots, K. \tag{3.13}$$

Case 2: $\eta_\epsilon \to \infty$:

Then from (3.10) we get

$$\hat{\xi}_{\epsilon,i} = \eta_\epsilon \hat{\xi}^2_{\epsilon,i} + \sum_{j=1}^{K} \hat{\xi}^2_{\epsilon,j} O(1),$$

and so, in the same way as in Case 1, it follows that

$$\xi_{\epsilon,i} = \frac{|\Omega|}{\eta_\epsilon \epsilon^2 \int_{R^2} w^2(y)\, dy} \left(1 + O\left(\frac{1}{\eta_\epsilon}\right)\right), \qquad i = 1, \ldots, K. \tag{3.14}$$

Case 3: $\eta_\epsilon \to \eta_0 \quad (0 < \eta_0 < \infty)$:

Then from (3.10) we get

$$\hat{\xi}_{\epsilon,i} = (1 + \eta_0) \hat{\xi}^2_{\epsilon,i} + \sum_{j \neq i} \hat{\xi}^2_{\epsilon,j} + \sum_{j=1}^{K} \hat{\xi}^2_{\epsilon,j} O(\beta^2).$$

This implies

$$\hat{\xi}_{\epsilon,1} = \cdots = \hat{\xi}_{\epsilon,K} = \frac{1}{K + \eta_0} (1 + O(\beta^2)), \qquad i = 1, \ldots, K$$

or, equivalently,

$$\xi_{\epsilon,i} = \frac{1}{K + \eta_0} \frac{|\Omega|}{\epsilon^2 \int_{R^2} w^2} (1 + O(\beta^2)), \qquad i = 1, \ldots, K. \tag{3.15}$$

Note that in all three cases the heights satisfy the relation

$$\xi_{\epsilon,i} = \xi_\epsilon (1 + O(h(\epsilon, \beta))), \qquad i = 1, \ldots, K,$$

where ξ_ϵ is given in (1.10) of Theorem 1.1 and

$$h(\epsilon,\beta)=\begin{cases}\eta_\epsilon & \text{if } \eta_\epsilon\to 0,\\ \eta_\epsilon^{-1} & \text{if } \eta_\epsilon\to\infty,\\ \beta^2 & \text{if } \eta_\epsilon\to\eta_0.\end{cases} \tag{3.16}$$

The analysis in this section calculates the height of the peaks under the assumption that their shape is given. In the next two sections, we provide a rigorous proof for the existence of equilibrium states.

4. Existence I: Reduction to Finite Dimensions

Let us start to prove Theorem 1.1.

The first step is to choose a good approximation to an equilibrium state. The second step is to use the Liapunov-Schmidt process to reduce the problem to a finite dimensional problem. The last step is to solve the reduced problem. Such a procedure has been used in the study of Gierer-Meinhardt system in the **strong coupling** case [39], [40].

Motivated by the results in Section 3, we rescale

$$x=\epsilon y,\qquad x\in\Omega,\qquad y\in\Omega_\epsilon=\{y|\epsilon y\in\Omega\}, \tag{4.1}$$

$$\hat{A}(y)\;=\;\frac{1}{\xi_\epsilon}A(\epsilon y),\qquad y\in\Omega_\epsilon,$$
$$\hat{H}(x)\;=\;\frac{1}{\xi_\epsilon}H(x),\qquad x\in\Omega,$$

where ξ_ϵ is given in (1.10).

Then an equilibrium solution $(\hat{A},\,\hat{H})$ has to solve the following rescaled Gierer-Meinhardt system:

$$\begin{cases}\Delta_y\hat{A}-\hat{A}+\dfrac{\hat{A}^2}{\hat{H}}=0, & y\in\Omega_\epsilon,\\ \Delta_x\hat{H}-\beta^2\hat{H}+\beta^2\xi_\epsilon\hat{A}^2=0, & x\in\Omega.\end{cases} \tag{4.2}$$

(This rescaling is chosen to achieve $\hat{A}=O(1)$, $\hat{H}=O(1)$ in terms of the maximum values.)

For a function $\hat{A}\in H^1(\Omega_\epsilon)$, let $T[\hat{A}]$ be the unique solution of the following problem:

$$\Delta T[\hat{A}]-\beta^2T[\hat{A}]+\beta^2\xi_\epsilon\hat{A}^2=0\quad\text{in }\Omega,\qquad \frac{\partial T[\hat{A}]}{\partial\nu}=0\quad\text{on }\partial\Omega. \tag{4.3}$$

In other words, we have

$$T[\hat{A}](x)=\int_\Omega G_\beta(x,\xi)\beta^2\xi_\epsilon\hat{A}^2\left(\frac{\xi}{\epsilon}\right)d\xi. \tag{4.4}$$

System (4.2) is equivalent to the following equation in operator form:

$$S_\epsilon(\hat{A}, \hat{H}) = \begin{pmatrix} S_1(\hat{A}, \hat{H}) \\ S_2(\hat{A}, \hat{H}) \end{pmatrix} = 0, \qquad H_N^2(\Omega_\epsilon) \times H_N^2(\Omega) \to L^2(\Omega_\epsilon) \times L^2(\Omega), \quad (4.5)$$

where

$$S_1(\hat{A}, \hat{H}) = \Delta_y \hat{A} - \hat{A} + \frac{\hat{A}^2}{\hat{H}}: \quad H_N^2(\Omega_\epsilon) \times H_N^2(\Omega) \to L^2(\Omega_\epsilon),$$
$$S_2(\hat{A}, \hat{H}) = \Delta_x \hat{H} - \beta^2 \hat{H} + \beta^2 \xi_\epsilon \hat{A}^2: \quad H_N^2(\Omega_\epsilon) \times H_N^2(\Omega) \to L^2(\Omega).$$

Here the index N indicates that the functions satisfy the Neumann boundary conditions

$$\frac{\partial \hat{A}}{\partial \nu} = 0, \quad y \text{ on } \partial\Omega_\epsilon, \qquad \frac{\partial \hat{H}}{\partial \nu} = 0, \quad x \text{ on } \partial\Omega.$$

Let $\mathbf{P} \in \Lambda_\delta$ and

$$w_{\epsilon,j}(y) := w\left(y - \frac{P_j}{\epsilon}\right) \chi_{\epsilon,P_j}(\epsilon y), \qquad y \in \Omega_\epsilon, \quad (4.6)$$

where w is the unique solution of (1.9) and χ_{ϵ,P_j} was defined in (3.5).

We choose our approximate solutions as follows:

$$A_{\epsilon,\mathbf{P}}(y) := \sum_{j=1}^{K} w_{\epsilon,j}(y), \qquad H_{\epsilon,\mathbf{P}}(x) := T[A_{\epsilon,\mathbf{P}}](x), \qquad x = \epsilon y \in \Omega. \quad (4.7)$$

Note that $H_{\epsilon,\mathbf{P}}$ satisfies

$$0 = \Delta_x H_{\epsilon,\mathbf{P}} - \beta^2 H_{\epsilon,\mathbf{P}} + \beta^2 \xi_\epsilon A_{\epsilon,\mathbf{P}}^2$$
$$= \Delta_x H_{\epsilon,\mathbf{P}} - \beta^2 H_{\epsilon,\mathbf{P}} + \beta^2 \xi_\epsilon \sum_{j=1}^{K} w_{\epsilon,j}^2 + e.s.t.$$

Hence,

$$H_{\epsilon,\mathbf{P}}(P_j) = \beta^2 \xi_\epsilon \int_\Omega G_\beta(x, \xi) \sum_{j=1}^{K} w_{\epsilon,j}^2 \left(\frac{\xi}{\epsilon}\right) d\xi + e.s.t.$$

Similar to the computation in Section 2 (using the definition (1.10) of ξ_ϵ), we obtain

$$H_{\epsilon,\mathbf{P}}(P_j) = 1 + O(h(\epsilon, \beta)), \qquad j = 1, \ldots, K. \quad (4.8)$$

We insert our ansatz (4.7) into (4.5) and calculate

$$S_2(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}}) = 0, \quad (4.9)$$

$$
\begin{aligned}
S_1(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}}) &= \Delta_y A_{\epsilon,\mathbf{P}} - A_{\epsilon,\mathbf{P}} + \frac{A^2_{\epsilon,\mathbf{P}}}{H_{\epsilon,\mathbf{P}}} \\
&= \sum_{j=1}^{K} \left[\Delta_y w \left(y - \frac{P_j}{\epsilon} \right) - w \left(y - \frac{P_j}{\epsilon} \right) \right] \\
&\quad + \sum_{j=1}^{K} w^2 \left(y - \frac{P_j}{\epsilon} \right) H^{-1}_{\epsilon,\mathbf{P}} + e.s.t. \\
&= \sum_{j=1}^{K} w^2 \left(y - \frac{P_j}{\epsilon} \right) (H^{-1}_{\epsilon,\mathbf{P}} - 1) + e.s.t. \\
&= \sum_{j=1}^{K} w^2 \left(y - \frac{P_j}{\epsilon} \right) (H^{-1}_{\epsilon,\mathbf{P}}(P_j) - 1) \\
&\quad + \sum_{j=1}^{K} w^2 \left(y - \frac{P_j}{\epsilon} \right) (H^{-1}_{\epsilon,\mathbf{P}}(x) - H^{-1}_{\epsilon,\mathbf{P}}(P_j)) + e.s.t. \qquad (4.10)
\end{aligned}
$$

On the other hand, we calculate for $j = 1, \ldots, K$ and $x = P_j + \epsilon z$, $|\epsilon z| < \delta$:

$$
\begin{aligned}
H_{\epsilon,\mathbf{P}}(P_j + \epsilon z) - H_{\epsilon,\mathbf{P}}(P_j) &= \beta^2 \int_\Omega [G_\beta(P_j + \epsilon z, \xi) - G_\beta(P_j, \xi)] \xi_\epsilon A^2_{\epsilon,\mathbf{P}} d\xi \\
&= \beta^2 \xi_\epsilon \int_\Omega [G_\beta(P_j + \epsilon z, \xi) - G_\beta(P_j, \xi)] w^2_{\epsilon,j} d\xi \\
&\quad + \beta^2 \xi_\epsilon \int_\Omega [G_\beta(P_j + \epsilon z, \xi) - G_\beta(P_j, \xi)] \\
&\quad \times \sum_{l \neq j} w^2_{\epsilon,l} d\xi + e.s.t. \\
&= k(\epsilon, \beta) \int_{R^2} \frac{1}{2\pi} \log \frac{|\zeta|}{|z - \zeta|} w^2(\zeta) d\zeta \\
&\quad - k(\epsilon, \beta) \left(\sum_{k=1}^{2} \frac{\partial F(\mathbf{P})}{\partial P_{j,k}} \epsilon z_k \int_{R^2} w^2 \right) \\
&\quad + O(\epsilon \beta^2 k(\epsilon, \beta) |z|), \qquad (4.11)
\end{aligned}
$$

where $k(\epsilon, \beta)$ is given by (1.11), and $F(\mathbf{P})$ is defined at (1.4).

Substituting (4.11) into (4.10), we have the following key estimate:

Lemma 4.1. *For $x = P_j + \epsilon z$, $|\epsilon z| < \delta$, we have*

$$
S_1(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}}) = S_{1,1} + S_{1,2}, \qquad (4.12)
$$

where

$$
S_{1,1}(z) = k(\epsilon, \beta)(H_{\epsilon,P_j}(P_j))^{-2} \left(\int_{R^2} w^2 \right) w^2(z) \left(\epsilon \nabla_{\mathbf{P}_j} F(\mathbf{P}) \cdot z + O(\epsilon \beta^2 |z|) \right), \qquad (4.13)
$$

and

$$S_{1,2}(z) = k(\epsilon, \beta) w^2(z) R(|z|) + O(\epsilon k(\epsilon, \beta) \beta^2 |z|), \tag{4.14}$$

where $R(|z|)$ *is a radially symmetric function with the property that* $R(|z|) = O(\log(1 + |z|))$.

Furthermore, $S_1(A_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}}) = e.s.t.$ *for* $|x - P_j| \geq \delta,\ j = 1, 2, \ldots, K$.

The above estimates will be very important in the following calculations, where (4.5) is solved exactly.

Now we study the linearized operator defined by

$$\tilde{L}_{\epsilon,\mathbf{P}} := S'_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{pmatrix},$$

$$\tilde{L}_{\epsilon,\mathbf{P}} : H^2_N(\Omega_\epsilon) \times H^2_N(\Omega) \to L^2(\Omega_\epsilon) \times L^2(\Omega),$$

where $\epsilon > 0$ is small, $\mathbf{P} \in \bar{\Lambda}_\delta$.

Set

$$K_{\epsilon,\mathbf{P}} := \operatorname{span}\left\{ \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \ldots, K,\ l = 1, 2 \right\} \subset H^2_N(\Omega_\epsilon),$$

and

$$C_{\epsilon,\mathbf{P}} := \operatorname{span}\left\{ \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \ldots, K,\ l = 1, 2 \right\} \subset L^2(\Omega_\epsilon).$$

$\tilde{L}_{\epsilon,\mathbf{P}}$ is not uniformly invertible in ϵ and β due to the approximate kernel,

$$\mathcal{K}_{\epsilon,\mathbf{P}} := K_{\epsilon,\mathbf{P}} \oplus \{0\} \subset H^2_N(\Omega_\epsilon) \times H^2_N(\Omega). \tag{4.15}$$

We choose the approximate cokernel as follows:

$$\mathcal{C}_{\epsilon,\mathbf{P}} := C_{\epsilon,\mathbf{P}} \oplus \{0\} \subset L^2(\Omega_\epsilon) \times L^2(\Omega). \tag{4.16}$$

We then define

$$\mathcal{K}^\perp_{\epsilon,\mathbf{P}} := K^\perp_{\epsilon,\mathbf{P}} \oplus H^2_N(\Omega) \subset H^2_N(\Omega_\epsilon) \times H^2_N(\Omega), \tag{4.17}$$

$$\mathcal{C}^\perp_{\epsilon,\mathbf{P}} := C^\perp_{\epsilon,\mathbf{P}} \oplus L^2(\Omega) \subset L^2(\Omega_\epsilon) \times L^2(\Omega), \tag{4.18}$$

where $C^\perp_{\epsilon,\mathbf{P}}$ and $K^\perp_{\epsilon,\mathbf{P}}$ denote the orthogonal complement with the scalar product of $L^2(\Omega_\epsilon)$ in $H^2_N(\Omega_\epsilon)$ and $L^2(\Omega_\epsilon)$, respectively.

Let $\pi_{\epsilon,\mathbf{P}}$ denote the projection in $L^2(\Omega_\epsilon) \times L^2(\Omega)$ onto $\mathcal{C}^\perp_{\epsilon,\mathbf{P}}$. (Here the second component of the projection is the identity map.) We are going to show that the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} = 0 \tag{4.19}$$

has the unique solution $\Sigma_{\epsilon,\mathbf{P}} = \begin{pmatrix} \Phi_{\epsilon,\mathbf{P}}(y) \\ \Psi_{\epsilon,\mathbf{P}}(x) \end{pmatrix} \in \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp}$ if $\epsilon,\ \beta$ are small enough.

Set

$$\mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon,\mathbf{P}} : \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp} \to \mathcal{C}_{\epsilon,\mathbf{P}}^{\perp}. \tag{4.20}$$

As a preparation, in the following two propositions we show the invertibility of the corresponding linearized operator $\mathcal{L}_{\epsilon,\mathbf{P}}$.

Proposition 4.2. *Assume that (1.8) holds. Let $\mathcal{L}_{\epsilon,\mathbf{P}}$ be given in (4.20). There exist positive constants $\overline{\epsilon}, \overline{\beta}, C$ such that for all $\epsilon \in (0, \overline{\epsilon}),\ \beta \in (0, \overline{\beta})$,*

$$\|\mathcal{L}_{\epsilon,\mathbf{P}}\Sigma\|_{L^2(\Omega_\epsilon)\times L^2(\Omega)} \geq C\|\Sigma\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)}, \tag{4.21}$$

for arbitrary $\mathbf{P} \in \overline{\Lambda_\delta}$, $\Sigma \in \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp}$.

Proposition 4.3. *Assume that (1.8) holds. There exist positive constants $\overline{\overline{\epsilon}}, \overline{\overline{\beta}}$ such that for all $\epsilon \in (0, \overline{\overline{\epsilon}}),\ \beta \in (0, \overline{\overline{\beta}})$, the map $\mathcal{L}_{\epsilon,\mathbf{P}}$ is surjective for arbitrary $\mathbf{P} \in \bar{\Lambda}_\delta$.*

The proofs of Propositions 4.2 and 4.3 are delayed to Appendix A.

Now we are in a position to solve the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \phi \\ H_{\epsilon,\mathbf{P}} + \psi \end{pmatrix} = 0. \tag{4.22}$$

Since $\mathcal{L}_{\epsilon,\mathbf{P}}|_{\mathcal{K}_{\epsilon,\mathbf{P}}^{\perp}}$ is invertible (call the inverse $\mathcal{L}_{\epsilon,\mathbf{P}}^{-1}$), we can rewrite (4.22) as

$$\Sigma = -(\mathcal{L}_{\epsilon,\mathbf{P}}^{-1} \circ \pi_{\epsilon,\mathbf{P}}) \left(S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{pmatrix} \right) - (\mathcal{L}_{\epsilon,\mathbf{P}}^{-1} \circ \pi_{\epsilon,\mathbf{P}})(N_{\epsilon,\mathbf{P}}(\Sigma)) \equiv M_{\epsilon,\mathbf{P}}(\Sigma), \tag{4.23}$$

where

$$\Sigma = \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

$$N_{\epsilon,\mathbf{P}}(\Sigma) = S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \phi \\ H_{\epsilon,\mathbf{P}} + \psi \end{pmatrix} - S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{pmatrix} - S_\epsilon' \begin{pmatrix} A_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} \end{pmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix},$$

and the operator $M_{\epsilon,\mathbf{P}}$ is defined by (4.23) for $\Sigma \in H_N^2(\Omega_\epsilon) \times H_N^2(\Omega)$. We are going to show that the operator $M_{\epsilon,\mathbf{P}}$ is a contraction on

$$B_{\epsilon,\eta} \equiv \{\Sigma \in H^2(\Omega_\epsilon) \times H^2(\Omega) | \|\Sigma\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} < \eta\}, \tag{4.24}$$

if η is small enough. We have by Lemma (4.1) and Propositions 4.2 and 4.3 that

$$\begin{aligned}\|M_{\epsilon,\mathbf{P}}(\Sigma)\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} &\le C\Bigg(\|\pi_{\epsilon,\mathbf{P}}\circ N_{\epsilon,\mathbf{P}}(\Sigma)\|_{L^2(\Omega_\epsilon)\times L^2(\Omega)} \\ &\qquad + \left\|\pi_{\epsilon,\mathbf{P}}\circ S_\epsilon\begin{pmatrix}A_{\epsilon,\mathbf{P}}\\ H_{\epsilon,\mathbf{P}}\end{pmatrix}\right\|_{L^2(\Omega_\epsilon)\times L^2(\Omega)}\Bigg) \\ &\le C(c(\eta)\eta + k(\epsilon,\beta)),\end{aligned}$$

where $C > 0$ is independent of $\eta > 0$ and $c(\eta) \to 0$ as $\eta \to 0$. Similarly we show

$$\|M_{\epsilon,\mathbf{P}}(\Sigma) - M_{\epsilon,\mathbf{P}}(\Sigma')\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} \le Cc(\eta)\|\Sigma - \Sigma'\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)},$$

where $c(\eta) \to 0$ as $\eta \to 0$. If we choose η small enough, then $M_{\epsilon,\mathbf{P}}$ is a contraction on $B_{\epsilon,\eta}$. The existence of a fixed point $\Sigma_{\epsilon,\mathbf{P}}$ plus an error estimate now follows from the Contraction Mapping Principle. Moreover, $\Sigma_{\epsilon,\mathbf{P}}$ is a solution of (4.23).

We have thus proved

Lemma 4.4. *There exist $\overline{\epsilon} > 0, \overline{\beta} > 0$ such that for every triple $(\epsilon, \beta, \mathbf{P})$ with $0 < \epsilon < \overline{\epsilon}$, $0 < \beta < \overline{\beta}$, $\mathbf{P} \in \overline{\Lambda_\delta}$ there exists a unique $(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}}) \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ satisfying $S_\epsilon\left(\begin{pmatrix}A_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}}\\ H_{\epsilon,\mathbf{P}}+\Psi_{\epsilon,\mathbf{P}}\end{pmatrix}\right) \in \mathcal{C}_{\epsilon,\mathbf{P}}$ and*

$$\|(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}})\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} \le Ck(\epsilon,\beta). \tag{4.25}$$

More refined estimates for $\Phi_{\epsilon,\mathbf{P}}$ are needed. We recall that S_1 can be decomposed into the two parts $S_{1,1}$ and $S_{1,2}$, where $S_{1,1}$ is in leading order an odd function and $S_{1,2}$ is in leading order a radially symmetric function. Similarly, we can decompose $\Phi_{\epsilon,\mathbf{P}}$:

Lemma 4.5. *Let $\Phi_{\epsilon,\mathbf{P}}$ be defined in Lemma 4.4. Then for $x = P_i + \epsilon z$, $|\epsilon z| < \delta$, we have*

$$\Phi_{\epsilon,\mathbf{P}} = \Phi_{\epsilon,\mathbf{P},1} + \Phi_{\epsilon,\mathbf{P},2}, \tag{4.26}$$

where $\Phi_{\epsilon,\mathbf{P},2}$ is a radially symmetric function in z and

$$\Phi_{\epsilon,\mathbf{P},1} = O(\epsilon k(\epsilon,\beta)) \quad \text{in } H^2_N(\Omega_\epsilon). \tag{4.27}$$

Proof. Let $S[v] := S_1(v, T[v])$. We first solve

$$S[A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P},2}] - S[A_{\epsilon,\mathbf{P}}] + \sum_{j=1}^K S_{1,2}\left(y - \frac{P_j}{\epsilon}\right) \in C_{\epsilon,\mathbf{P}}, \tag{4.28}$$

for $\Phi_{\epsilon,\mathbf{P},2} \in K^\perp_{\epsilon,\mathbf{P}}$.

Then we solve

$$S[A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P},2} + \Phi_{\epsilon,\mathbf{P},1}] - S[A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P},2}] + \sum_{j=1}^{K} S_{1,1}\left(y - \frac{P_j}{\epsilon}\right) \in C_{\epsilon,\mathbf{P}}, \quad (4.29)$$

for $\Phi_{\epsilon,\mathbf{P},1} \in K^{\perp}_{\epsilon,\mathbf{P}}$.

Using the same proof as in Lemma 4.4, both equations (4.28) and (4.29) have unique solutions for $\epsilon \ll 1$. By uniqueness, $\Phi_{\epsilon,\mathbf{P}} = \Phi_{\epsilon,\mathbf{P},1} + \Phi_{\epsilon,\mathbf{P},2}$. Since $S_{1,1} = S^0_{1,1} + S^{\perp}_{1,1}$, where $\|S^0_{1,1}\|_{H^2(\Omega_\epsilon)} = O(\epsilon k(\epsilon, \beta))$ and $S^{\perp}_{1,1} \in C^{\perp}_{\epsilon,\mathbf{P}}$, it is easy to see that $\Phi_{\epsilon,\mathbf{P},1}$ and $\Phi_{\epsilon,\mathbf{P},2}$ have the required properties. □

5. Existence II: The Reduced Problem

In this section, we solve the reduced problem and prove Theorem 1.1.

Let $\mathbf{P}^0$ be a nondegenerate critical point of $F(\mathbf{P})$.

By Lemma 4.4, for each $\mathbf{P} \in B_\delta(\mathbf{P}^0)$, there exists a unique solution $(\Phi_{\epsilon,\mathbf{P}}, \psi_{\epsilon,\mathbf{P}}) \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}$ such that

$$S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} = \begin{pmatrix} v_{\epsilon,\mathbf{P}} \\ 0 \end{pmatrix} \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Our idea is to find $\mathbf{P} = \mathbf{P}^\epsilon \in B_\delta(\mathbf{P}^0)$ such that

$$S_\epsilon \begin{pmatrix} A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} \perp \mathcal{C}_{\epsilon,\mathbf{P}}. \quad (5.1)$$

Let

$$W_{\epsilon,j,i}(\mathbf{P}) := \frac{1}{k(\epsilon,\beta)} \int_{\Omega_\epsilon} \left(S_1(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}) \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \right), \quad (5.2)$$

$$j = 1, \ldots, K, \quad i = 1, 2,$$

$$W_\epsilon(\mathbf{P}) := (W_{\epsilon,1,1}(\mathbf{P}), \ldots, W_{\epsilon,K,2}(\mathbf{P})). \quad (5.3)$$

Here we recall $k(\epsilon, \beta) = \epsilon^2 \beta \xi_\epsilon$.

Note that $W_\epsilon(\mathbf{P})$ is a map which is continuous in $\mathbf{P}$, and our problem is reduced to finding a zero of the vector field $W_\epsilon(\mathbf{P})$.

Let

$$\Omega_{\epsilon,P_j} = \{y | \epsilon y + P_j \in \Omega\}. \quad (5.4)$$

We calculate the asymptotic expansion of $W_{\epsilon,j,i}(\mathbf{P})$,

$$\frac{1}{k(\epsilon,\beta)} \int_{\Omega_\epsilon} S_1(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}) \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$
$$= \frac{1}{k(\epsilon,\beta)} \int_{\Omega_\epsilon} \left[\Delta(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - (A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) + \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}} \right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$

$$
\begin{aligned}
&= \frac{1}{k(\epsilon,\beta)} \int_{\Omega_\epsilon} \left[\Delta(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - (A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) + \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}} \right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \\
&\quad + \frac{1}{k(\epsilon,\beta)} \int_{\Omega_\epsilon} \left[\frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}} - \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}} \right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \\
&= I_1 + I_2,
\end{aligned}
$$

where I_1 and I_2 are defined at the last equality.

For I_1, we have by Lemma 4.5,

$$
\begin{aligned}
I_1 &= \frac{1}{k(\epsilon,\beta)} \left(\int_{\Omega_\epsilon} \left[\Delta(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - (A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) + \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}(P_j)} \right] \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \right. \\
&\qquad \left. - \int_{\Omega_\epsilon} \frac{(A_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2}{H^2_{\epsilon,\mathbf{P}}(P_j)} (H_{\epsilon,\mathbf{P}} - H_{\epsilon,\mathbf{P}}(P_j)) \frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \right) + o(1) \\
&= \frac{1}{\epsilon k(\epsilon,\beta)} \left(- \int_{\Omega_{\epsilon,P_j}} [\Delta(w_{\epsilon,j} + \Phi_{\epsilon,\mathbf{P}}) - (w_{\epsilon,j} + \Phi_{\epsilon,\mathbf{P}}) + (w_{\epsilon,j} + \Phi_{\epsilon,\mathbf{P}})^2] \frac{\partial w_{\epsilon,j}}{\partial y_i} \right. \\
&\qquad \left. + \int_{\Omega_{\epsilon,P_j}} \frac{(w_{\epsilon,j} + \Phi_{\epsilon,\mathbf{P},2})^2(y)}{(H_{\epsilon,\mathbf{P}}(P_j))^2} (H_{\epsilon,\mathbf{P}}(P_j + \epsilon y) - H_{\epsilon,\mathbf{P}}(P_j)) \frac{\partial w_{\epsilon,j}(y)}{\partial y_i}\, dy \right) \\
&\quad + o(1).
\end{aligned}
$$

Note that by Lemma 4.5,

$$
\begin{aligned}
&\int_{\Omega_{\epsilon,P_j}} [\Delta \Phi_{\epsilon,\mathbf{P}} - \Phi_{\epsilon,\mathbf{P}} + 2 w_{\epsilon,j} \Phi_{\epsilon,\mathbf{P}}] \frac{\partial w_{\epsilon,j}}{\partial y_i} \\
&\quad = \int_{\Omega_{\epsilon,P_j}} \Phi_{\epsilon,\mathbf{P},1} \frac{\partial}{\partial y_i} [\Delta w - w + w^2] + o(\epsilon k(\epsilon,\beta)) \\
&\quad = o(\epsilon k(\epsilon,\beta)),
\end{aligned}
\tag{5.5}
$$

$$
\int_{\Omega_{\epsilon,P_j}} (\Phi_{\epsilon,\mathbf{P}})^2 \frac{\partial w_{\epsilon,j}}{\partial y_i} = \int_{\Omega_{\epsilon,P_j}} (\Phi_{\epsilon,\mathbf{P},1})^2 \frac{\partial w_{\epsilon,j}}{\partial y_i} = o(\epsilon k(\epsilon,\beta)). \tag{5.6}
$$

Now by (4.11), (5.5), and (5.6),

$$
\begin{aligned}
I_1 &= o(1) - \frac{1}{\epsilon k(\epsilon,\beta)} \int_{\Omega_{\epsilon,P_j}} w^2_{\epsilon,j}(y) (H_{\epsilon,\mathbf{P}}(P_j + \epsilon y) - H_{\epsilon,\mathbf{P}}(P_j)) \frac{\partial w_{\epsilon,j}(y)}{\partial y_i}\, dy \\
&= o(1) + \sum_{k=1}^{2} \frac{\partial F(\mathbf{P})}{\partial P_{j,k}} \int_{R^2} w^2 y_k \frac{\partial w}{\partial y_i} \int_{R^2} w^2 \\
&= o(1) + \frac{\partial F(\mathbf{P})}{\partial P_{j,i}} \int_{R^2} w^2 y_i \frac{\partial w}{\partial y_i} \int_{R^2} w^2 \\
&= o(1) - \frac{1}{3} \int_{R^2} w^3 \int_{R^2} w^2 \frac{\partial F(\mathbf{P})}{\partial P_{j,i}}.
\end{aligned}
\tag{5.7}
$$

Similar to the estimate for I_1, we obtain that for I_2,

$$\begin{aligned} I_2 &= \frac{1}{k(\epsilon,\beta)}\int_{\Omega_\epsilon}\left[\frac{(A_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}+\Psi_{\epsilon,\mathbf{P}}}-\frac{(A_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})^2}{H_{\epsilon,\mathbf{P}}}\right]\frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}} \\ &= -\frac{1}{k(\epsilon,\beta)}\int_{\Omega_\epsilon}\frac{(A_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})^2}{H^2_{\epsilon,\mathbf{P}}}\Psi_{\epsilon,\mathbf{P}}\frac{\partial A_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}+o(1) \\ &= -\frac{1}{\epsilon k(\epsilon,\beta)}\int_{\Omega_{\epsilon,P_j}}\frac{1}{3}\frac{\partial w^3_{\epsilon,j}}{\partial y_i}(\Psi_{\epsilon,\mathbf{P}}-\Psi_{\epsilon,\mathbf{P}}(P_j))+o(1). \end{aligned} \tag{5.8}$$

Now we recall that $\Psi_{\epsilon,\mathbf{P}}$ satisfies

$$\Delta\Psi_{\epsilon,\mathbf{P}}-\beta^2\Psi_{\epsilon,\mathbf{P}}+2\beta^2\xi_\epsilon A_{\epsilon,\mathbf{P}}\Phi_{\epsilon,\mathbf{P}}+\beta^2\xi_\epsilon\Phi^2_{\epsilon,\mathbf{P}}=0.$$

Similar computations as those leading to (4.11) show that

$$\begin{aligned} \Psi_{\epsilon,\mathbf{P}}(P_j+\epsilon y)-\Psi_{\epsilon,\mathbf{P}}(P_j) &= \int_\Omega (G_\beta(P_j+\epsilon y,\xi)-G_\beta(P_j,\xi))\beta^2\xi_\epsilon \\ &\quad\times\left(2A_{\epsilon,\mathbf{P}}\left(\frac{\xi}{\epsilon}\right)\Phi_{\epsilon,\mathbf{P}}\left(\frac{\xi}{\epsilon}\right)+\Phi^2_{\epsilon,\mathbf{P}}\left(\frac{\xi}{\epsilon}\right)\right)d\xi \\ &= o(\epsilon k(\epsilon,\beta)|\nabla_{P_j}F(\mathbf{P})|\,|y|)+k(\epsilon,\beta)R_1(|y|), \end{aligned} \tag{5.9}$$

where $R_1(|y|)$ is a radially symmetric function.

Substituting (5.9) into (5.8), we obtain that

$$I_2 = o(1). \tag{5.10}$$

Combining the estimates for I_1 and I_2, we obtain

$$W_\epsilon(\mathbf{P}) = c_0\nabla_{\mathbf{P}}F(\mathbf{P})+o(1),$$

where $c_0=-\frac{1}{3}\int_{R^2}w^3\int_{R^2}w^2\neq 0$. Here $o(1)$ is a continuous function of $\mathbf{P}$, which goes to 0 as $\epsilon\to 0$.

At $\mathbf{P}^0$, we have $\nabla_{\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^0}F(\mathbf{P}^0)=0$, $\det(\nabla_{\mathbf{P}}\nabla_{\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^0}(F(\mathbf{P}^0)))\neq 0$. Then, since W_ϵ is continuous and for $\epsilon,\ \beta$ small enough maps balls $B_\delta(\mathbf{P}^0)$ into (possibly larger) balls, the standard Brouwer's fixed point theorem shows that for $\epsilon\ll 1$ there exists a $\mathbf{P}^\epsilon$ such that $W_\epsilon(\mathbf{P}^\epsilon)=0$ and $\mathbf{P}^\epsilon\to\mathbf{P}^0$.

Thus we have proved the following proposition.

Proposition 5.1. *For ϵ sufficiently small, there exist points $\mathbf{P}^\epsilon$ with $\mathbf{P}^\epsilon\to\mathbf{P}^0$ such that $W_\epsilon(\mathbf{P}^\epsilon)=0$.*

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 5.1, there exists $\mathbf{P}^\epsilon\to\mathbf{P}^0$ such that $W_\epsilon(\mathbf{P}^\epsilon)=0$. In other words, $S_1(A_{\epsilon,\mathbf{P}^\epsilon}+\Phi_{\epsilon,\mathbf{P}^\epsilon}, H_{\epsilon,\mathbf{P}^\epsilon}+\Psi_{\epsilon,\mathbf{P}^\epsilon})=0$. Let $A_\epsilon=\xi_\epsilon(A_{\epsilon,\mathbf{P}^\epsilon}+\Phi_{\epsilon,\mathbf{P}^\epsilon})$, $H_\epsilon=\xi_\epsilon(H_{\epsilon,\mathbf{P}^\epsilon}+\Psi_{\epsilon,\mathbf{P}^\epsilon})$. It is easy to see that $H_\epsilon=\xi_\epsilon T[A_{\epsilon,\mathbf{P}^\epsilon}+\Phi_{\epsilon,\mathbf{P}^\epsilon}]>0$. Hence $A_\epsilon\geq 0$. By the Maximum Principle, $A_\epsilon>0$. Therefore (A_ϵ,H_ϵ) satisfies Theorem 1.1. □

6. Stability Analysis I: Study of Large Eigenvalues

We consider the stability of (A_ϵ, H_ϵ) constructed in Theorem 1.1.

Linearizing the system (GM) around the equilibrium states (A_ϵ, H_ϵ), we obtain the following eigenvalue problem:

$$\begin{cases} \Delta_y \phi_\epsilon - \phi_\epsilon + 2\dfrac{A_\epsilon}{H_\epsilon}\phi_\epsilon - \dfrac{A_\epsilon^2}{H_\epsilon^2}\psi_\epsilon = \lambda_\epsilon \phi_\epsilon, \\ \dfrac{1}{\beta^2}\Delta \psi_\epsilon - \psi_\epsilon + 2A_\epsilon \phi_\epsilon = \tau \lambda_\epsilon \psi_\epsilon. \end{cases} \tag{6.1}$$

Here $D = \frac{1}{\beta^2}$, λ_ϵ is some complex number and

$$\phi_\epsilon \in H_N^2(\Omega_\epsilon), \qquad \psi_\epsilon \in H_N^2(\Omega). \tag{6.2}$$

Let

$$\hat{A}_\epsilon = \xi_\epsilon^{-1} A_\epsilon = A_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon},\ \hat{H}_\epsilon = \xi_\epsilon^{-1} H_\epsilon = H_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon}. \tag{6.3}$$

Then (6.1) becomes

$$\begin{cases} \Delta_y \phi_\epsilon - \phi_\epsilon + 2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon - \dfrac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2}\psi_\epsilon = \lambda_\epsilon \phi_\epsilon, \\ \dfrac{1}{\beta^2}\Delta \psi_\epsilon - \psi_\epsilon + 2\xi_\epsilon \hat{A}_\epsilon \phi_\epsilon = \tau \lambda_\epsilon \psi_\epsilon. \end{cases} \tag{6.4}$$

In this section, we study the large eigenvalues, i.e., we assume that $|\lambda_\epsilon| \ge c > 0$ for ϵ small. Furthermore, we may assume that $(1+\tau)c < \frac{1}{2}$. If $\mathrm{Re}(\lambda_\epsilon) \le -c$, we are done. (Then λ_ϵ is a stable large eigenvalue.) Therefore we may assume that $\mathrm{Re}(\lambda_\epsilon) \ge -c$ and for a subsequence $\epsilon \to 0$, $\lambda_\epsilon \to \lambda_0 \ne 0$. We shall derive the limiting eigenvalue problem which reduces to NLEPs.

The key references are Theorem 2.2 and Theorem 2.5.

The second equation in (6.4) is equivalent to

$$\Delta \psi_\epsilon - \beta^2(1+\tau\lambda_\epsilon)\psi_\epsilon + 2\beta^2 \xi_\epsilon \hat{A}_\epsilon \phi_\epsilon = 0. \tag{6.5}$$

We introduce the following:

$$\beta_{\lambda_\epsilon} = \beta\sqrt{1+\tau\lambda_\epsilon}, \tag{6.6}$$

where in $\sqrt{1+\tau\lambda_\epsilon}$ we take the principal part of the square root. (This means that the real part of $\sqrt{1+\tau\lambda_\epsilon}$ is positive, which is possible because Re $(1+\tau\lambda_\epsilon) \ge \frac{1}{2}$.)

Let us assume that

$$\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1. \tag{6.7}$$

We cut off ϕ_ϵ as follows: Introduce

$$\phi_{\epsilon,j}(y) = \phi_\epsilon(y)\chi_{\epsilon,P_j^\epsilon}(\epsilon y), \tag{6.8}$$

where $\chi_{\epsilon, P_j^\epsilon}(x)$ was introduced in (3.5).

From (6.4) using Lemma 4.4 and $\mathrm{Re}(\lambda_\epsilon) \geq -c$ and the exponential decay of w (see (1.12)), it follows that

$$\phi_\epsilon = \sum_{j=1}^{K} \phi_{\epsilon,j} + e.s.t. \quad \text{in } H^2(\Omega_\epsilon). \tag{6.9}$$

Then by a standard procedure, we extend $\phi_{\epsilon,j}$ to a function defined on R^2 such that

$$\|\phi_{\epsilon,j}\|_{H^2(R^2)} \leq C\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)}, \qquad j = 1, \ldots, K.$$

Since $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1$, $\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)} \leq C$. By taking a subsequence of ϵ, we may also assume that $\phi_{\epsilon,j} \to \phi_j$ as $\epsilon \to 0$ in $H^1(R^2)$ for $j = 1, \ldots, K$.

We have by (6.5)

$$\psi_\epsilon(x) = \int_\Omega 2\beta^2 \xi_\epsilon G_{\beta_{\lambda_\epsilon}}(x, \xi) \hat{A}_\epsilon\left(\frac{\xi}{\epsilon}\right) \phi_\epsilon\left(\frac{\xi}{\epsilon}\right) d\xi. \tag{6.10}$$

At $x = P_i^\epsilon$, $i = 1, \ldots, K$, we calculate

$$\begin{aligned}
\psi_\epsilon(P_j^\epsilon) &= 2\beta^2 \int_\Omega G_{\beta_{\lambda_\epsilon}}(P_i^\epsilon, \xi) \sum_{j=1}^{K} \xi_\epsilon w\left(\frac{\xi - P_j^\epsilon}{\epsilon}\right) \phi_{\epsilon,j}\left(\frac{\xi}{\epsilon}\right) d\xi + e.s.t. \\
&= 2\beta^2 \int_\Omega \left(\frac{(\beta_{\lambda_\epsilon})^{-2}}{|\Omega|} + G_0(P_i^\epsilon, \xi) + O(|\beta_{\lambda_\epsilon}|^2) \right) \\
&\qquad \times \sum_{j=1}^{K} \xi_\epsilon w\left(\frac{\xi - P_j^\epsilon}{\epsilon}\right) \phi_{\epsilon,j}\left(\frac{\xi}{\epsilon}\right) d\xi + e.s.t. \\
&= 2 \int_\Omega \left(\frac{1}{|\Omega|(1 + \tau\lambda_\epsilon)} + \beta^2 G_0(P_i^\epsilon, \xi) + O(|\beta_{\lambda_\epsilon}|^4) \right) \\
&\qquad \times \xi_\epsilon w\left(\frac{x - P_i^\epsilon}{\epsilon}\right) \phi_{\epsilon,i}\left(\frac{\xi}{\epsilon}\right) d\xi \\
&\quad + 2\sum_{j \neq i} \int_\Omega \left(\frac{1}{|\Omega|(1 + \tau\lambda_\epsilon)} + \beta^2 G_0(P_i^\epsilon, P_j^\epsilon) + O(|\beta_{\lambda_\epsilon}|^4) \right) \\
&\qquad \times \xi_\epsilon w\left(\frac{\xi - P_j^\epsilon}{\epsilon}\right) \phi_{\epsilon,j}\left(\frac{\xi}{\epsilon}\right) d\xi \\
&= \Bigg(2\sum_{j=1}^{K} \frac{1}{|\Omega|(1 + \tau\lambda_\epsilon)} \xi_\epsilon \epsilon^2 \int_{R^2} w(y)\phi_{\epsilon,j}(y)\, dy \\
&\quad + 2\xi_\epsilon \frac{\beta^2}{2\pi} \epsilon^2 \log\frac{1}{\epsilon} \int_{R^2} w(y)\phi_{\epsilon,i}(y)\, dy + O(|\beta_{\lambda_\epsilon}|^2 \xi_\epsilon \epsilon^2) \Bigg).
\end{aligned} \tag{6.11}$$

We distinguish the same three cases as in Section 3.

Case 1: $\eta_\epsilon \to 0$
We get from (6.11):

$$\psi_\epsilon(P_i^\epsilon) = 2\sum_{j=1}^{K} \frac{1}{|\Omega|(1+\tau\lambda_\epsilon)} \xi_\epsilon \epsilon^2 \int_{R^2} w\phi_{\epsilon,j}(1+o(1)). \tag{6.12}$$

Substituting (6.12) into the first equation (6.4), letting $\epsilon \to 0$, and using (3.13), we arrive at the following nonlocal eigenvalue problem (NLEP):

$$\Delta\phi_i - \phi_i + 2w\phi_i - \frac{2\sum_{j=1}^{K}\int_{R^2} w\phi_j}{K(1+\tau\lambda_0)\int_{R^2} w^2} w^2 = \lambda_0\phi_i, \qquad i = 1, \ldots, K. \tag{6.13}$$

If $K = 1$, by Theorem 2.2, problem (6.13) is stable if $\tau < \tau_1$, which implies that the large eigenvalues of (6.4) are stable.

If $\tau > \tau_1$, by Theorem 2.2, problem (6.13) has an eigenvalue λ_0 with Re $(\lambda_0) \geq a_0 > 0$ for some a_0. We now claim that problem (6.4) also admits an eigenvalue λ_ϵ with $\lambda_\epsilon = \lambda_0 + o(1)$, which implies that problem (6.4) is unstable. To this end, we follow the argument given in Section 2 of [4], where the following eigenvalue problem was studied:

$$\begin{cases} \epsilon^2\Delta h - h + pu_\epsilon^{p-1}h - \dfrac{qr}{s+1+\tau\lambda_\epsilon}\dfrac{\int_\Omega u_\epsilon^{r-1}h}{\int_\Omega u_\epsilon^r} u_\epsilon^p = \lambda_\epsilon h \quad \text{in } \Omega, \\ h = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.14}$$

where u_ϵ is a solution of the single equation

$$\begin{cases} \epsilon^2\Delta u_\epsilon - u_\epsilon + u_\epsilon^p = 0 \quad \text{in } \Omega, \\ u_\epsilon > 0 \text{ in } \Omega,\ u_\epsilon = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, and $1 < p < +\infty$ if $N = 1, 2$, $\frac{qr}{(s+1)(p-1)} > 1$, and $\Omega \subset R^N$ is a smooth bounded domain.

If u_ϵ is a single interior peak solution, then it can be shown ([34]) that the limiting eigenvalue problem is a NLEP,

$$\Delta\phi - \phi + pw^{p-1}\phi - \frac{qr}{s+1+\tau\lambda_0}\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p = \lambda_0\phi, \tag{6.15}$$

where w is the corresponding ground state solution in R^N:

$$\Delta w - w + w^p = 0, \qquad w > 0 \quad \text{in } R^N, \qquad w = w(|y|) \in H^1(R^N).$$

Dancer in [4] showed that if $\lambda_0 \neq 0,\ \text{Re}(\lambda_0) > 0$ is an unstable eigenvalue of (6.15), then there exists an eigenvalue λ_ϵ of (6.14) such that $\lambda_\epsilon \to \lambda_0$.

We now follow his idea. Let $\lambda_0 \neq 0$ be an eigenvalue of problem (6.13) with $\text{Re}(\lambda_0) > 0$. We first note that from the equation for ψ_ϵ, we can express ψ_ϵ in terms of ϕ_ϵ. Now we write the first equation for ϕ_ϵ as follows:

$$\phi_\epsilon = \mathcal{R}_\epsilon(\lambda_\epsilon)\left[2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon - \frac{\hat{A}_\epsilon^2}{\hat{H}_\epsilon^2}\psi_\epsilon\right], \tag{6.16}$$

where $\mathcal{R}_\epsilon(\lambda_\epsilon)$ is the inverse of $-\Delta + (1+\lambda_\epsilon)$ in $H^2_N(\Omega_\epsilon)$ (which exists if $\text{Re}(\lambda_\epsilon) > -1$ or $\text{Im}(\lambda_\epsilon) \neq 0$) and $\psi_\epsilon = \mathcal{F}[\phi_\epsilon]$ is given by (6.10), where $\mathcal{F}$ is a compact operator of ϕ_ϵ. The important thing is that $\mathcal{R}_\epsilon(\lambda_\epsilon)$ is a compact operator if ϵ is sufficiently small. The rest of the argument follows exactly that in [4]. For the sake of limited space, we omit the details here.

This finishes the case $K = 1$.

If $K > 1$, problem (6.13) admits a positive eigenvalue: We can choose, for example,

$$\phi_1 = -\phi_2 = \Phi_0, \qquad \phi_3 = \cdots = \phi_K = 0, \qquad \lambda_0 = \mu_1,$$

where Φ_0 is the principal eigenfunction of L_0 given in Lemma 2.1.

By the same argument as in the unstable eigenvalue case for $K = 1$, we conclude that there is an eigenvalue of (6.4) with positive real part. Thus this corresponds to the "shadow" system case: All multiple-peaked solutions are unstable.

Case 2: $\eta_\epsilon \to \infty$

In this case, similar to Case 1, we get from (6.11),

$$\psi_\epsilon(P_i^\epsilon) = 2\xi_\epsilon \frac{\eta_\epsilon}{|\Omega|}\epsilon^2 \int_{R^2} w\phi_{\epsilon,i}(1+o(1)), \tag{6.17}$$

and, for any $\tau \geq 0$, in the limit $\epsilon \to 0$ we obtain the following NLEP:

$$\Delta\phi_i - \phi_i + 2w\phi_i - \frac{2\int_{R^2} w\phi_i}{\int_{R^2} w^2} w^2 = \lambda_0\phi_i, \qquad i = 1, \ldots, K. \tag{6.18}$$

By Theorem 2.2, (6.18) has only stable eigenvalues.

In conclusion, if $\eta_\epsilon \to \infty$, then the large eigenvalues of a K-peaked solution are all stable. This is similar to the "strong coupling" system case.

Case 3: $\eta_\epsilon \to \eta_0$

Similar to Case 1, we get from (6.11),

$$\begin{aligned} \psi_\epsilon(P_i^\epsilon) \ = \ &\left(2\sum_{j=1}^K \frac{1}{|\Omega|(1+\tau\lambda_0)}\xi_\epsilon\epsilon^2\int_{R^2} w\phi_{\epsilon,j}\right. \\ &\left. + 2\xi_\epsilon\frac{\eta_0}{|\Omega|}\epsilon^2\int_{R^2} w\phi_{\epsilon,i}\right)(1+o(1)), \end{aligned} \tag{6.19}$$

and in the limit $\epsilon \to 0$, we obtain the following nonlocal eigenvalue problem (NLEP):

$$\begin{aligned} &\Delta\phi_i - \phi_i + 2w\phi_i \\ &- \frac{2[(1+\eta_0(1+\tau\lambda_0))\int_{R^2} w\phi_i + \sum_{j\neq i}\int_{R^2} w\phi_j]}{(K+\eta_0)(1+\tau\lambda_0)\int_{R^2} w^2} w^2 = \lambda_0\phi_i, \qquad i = 1, \ldots, K. \end{aligned} \tag{6.20}$$

Let

$$\mathcal{G} = \begin{pmatrix} 1+\eta_0(1+\tau\lambda_0) & 1 & \cdots & 1 \\ 1 & 1+\eta_0(1+\tau\lambda_0) & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1+\eta_0(1+\tau\lambda_0) \end{pmatrix}.$$

$\mathcal{G}$ is symmetric and the eigenvalues of $\mathcal{G}$ are given by

$$\rho_1 = \ldots = \rho_{K-1} = \eta_0(1+\tau\lambda_0), \qquad \rho_K = K + \eta_0(1+\tau\lambda_0).$$

Let P be an orthogonal matrix such that

$$P\mathcal{G}P^{-1} = \begin{pmatrix} \eta_0(1+\tau\lambda_0) & 0 & \cdots & 0 \\ 0 & \eta_0(1+\tau\lambda_0) & \cdots & 0 \\ 0 & \cdots & \eta_0(1+\tau\lambda_0) & 0 \\ 0 & \cdots & 0 & K+\eta_0(1+\tau\lambda_0) \end{pmatrix}.$$

From (6.20), using the notation

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_K \end{pmatrix},$$

we get

$$\Delta\Phi - \Phi + 2w\Phi - \frac{2\mathcal{G}\int_{R^2}\Phi w}{(K+\eta_0)(1+\tau\lambda_0)\int_{R^2} w^2} w^2 = \lambda_0\Phi.$$

Let $P\Phi = \bar{\Phi}$. Then we get

$$\Delta\bar{\Phi} - \bar{\Phi} + 2w\bar{\Phi} - \frac{2}{(K+\eta_0)(1+\tau\lambda_0)\int_{R^2} w^2}$$
$$\times \begin{pmatrix} \eta_0(1+\tau\lambda_0) & 0 & \cdots & 0 \\ 0 & \eta_0(1+\tau\lambda_0) & \cdots & 0 \\ 0 & \cdots & \eta_0(1+\tau\lambda_0) & 0 \\ 0 & \cdots & 0 & K+\eta_0(1+\tau\lambda_0) \end{pmatrix}$$
$$\times \left[\int_{R^2} w\bar{\Phi}\right] w^2 = \lambda_0\bar{\Phi},$$

and, written in components,

$$\Delta\bar{\Phi}_i - \bar{\Phi}_i + 2w\bar{\Phi}_i - \frac{2\rho_i}{(K+\eta_0)(1+\tau\lambda_0)\int_{R^2} w^2}\left[\int_{R^2} w(y)\bar{\Phi}_i(y)\,dy\right] w^2 = \lambda_0\bar{\Phi}_i,$$
$$i = 1, \ldots, K. \tag{6.21}$$

For $i = 1, \ldots, K-1$, (6.21) becomes

$$\Delta\bar{\Phi}_i - \bar{\Phi}_i + 2w\bar{\Phi}_i - \frac{2\eta_0}{(K+\eta_0)\int_{R^2} w^2}\left[\int_{R^2} w(y)\bar{\Phi}_i(y)\,dy\right] w^2 = \lambda_0\bar{\Phi}_i,$$
$$i = 1, \ldots, K-1. \tag{6.22}$$

For $i = K$, (6.21) becomes

$$\Delta\bar{\Phi}_K - \bar{\Phi}_K + 2w\bar{\Phi}_K - \frac{2(K+\eta_0(1+\tau\lambda_0))}{(K+\eta_0)(1+\tau\lambda_0)\int_{R^2} w^2}\left[\int_{R^2} w(y)\bar{\Phi}_K(y)\,dy\right] w^2 = \lambda_0\bar{\Phi}_K. \tag{6.23}$$

If $K > 1$ and $\frac{2\eta_0}{K+\eta_0} < 1$ (i.e., $\eta_0 < K$), then by Theorem 2.2, problem (6.22) is unstable for all $\tau \geq 0$, which implies that problem (6.4) is linearly unstable for all $\tau \geq 0$.

If $K \geq 1$ and $\frac{2\eta_0}{K+\eta_0} > 1$, or what is equivalent, $\eta_0 > K$, then by Theorem 2.2, problem (6.22) is stable. By Theorem 2.5, problem (6.23) is stable if $0 \leq \tau < \tau_2$ or $\tau > \tau_3$ for suitable $\tau_2 \leq \tau_3$.

If $K = 1$ and $\eta_0 < 1$, we only have problem (6.23). By Theorem 2.5, problem (6.23) is stable if $0 \leq \tau < \tau_4$ and unstable for $\tau > \tau_5$, for suitable $\tau_4 \leq \tau_5$.

This finishes the proof of Theorem 1.2 in the large eigenvalue case.

7. Stability Analysis II: Study of Small Eigenvalues

We now study (6.4) for small eigenvalues. Namely, we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. We will show that the small eigenvalues are related to the matrix $M(\mathbf{P}^0)$ given in (1.5).

Let us assume that condition (*) holds true. That is, all eigenvalues of the matrix $M(\mathbf{P}^0)$ are negative. Our main result in this section says that if $\lambda_\epsilon \to 0$, then

$$\lambda_\epsilon \sim \epsilon^2 k(\epsilon, \beta)\sigma_0, \tag{7.1}$$

where σ_0 is an eigenvalue of $M(\mathbf{P}^0)$. From (7.1), we see that all small eigenvalues of $\mathcal{L}_\epsilon$ are stable, provided that condition (*) holds.

Again let (A_ϵ, H_ϵ) be the equilibrium state of (1.7) which has been rigorously constructed in Theorem 1.1 and $(\hat{A}_\epsilon, \hat{H}_\epsilon)$ be the rescaled solution given by (6.3).

We cut off $\hat{A}_\epsilon$ as follows:

$$\hat{A}_{\epsilon,j}(y) = \chi_{\epsilon,P_j^\epsilon}(\epsilon y)\hat{A}_\epsilon(y), \qquad j = 1,\ldots,K, \tag{7.2}$$

where $\chi_{\epsilon,P_j^\epsilon}$ was defined in (3.5).

Then it is easy to see that

$$\hat{A}_\epsilon(y) = \sum_{j=1}^K \hat{A}_{\epsilon,j}(y) + e.s.t. \quad \text{in } H^2(\Omega_\epsilon). \tag{7.3}$$

We now give a formal argument which should explain to the reader our choice of decomposition of ϕ_ϵ, which will be given in (7.8) below. Later, in Step 1 of the proof, it will be shown that this choice gives the correct answer in leading order.

In Section 6, we have derived three NLEPs: (6.13), (6.18), (6.21). Let us now set $\lambda_0 = 0$ in (6.13). We have that

$$\Delta\phi_i - \phi_i + 2w\phi_i - \frac{2\sum_{j=1}^K \int_{R^2} w\phi_j}{K\int_{R^2} w^2} w^2 = 0, \qquad i = 1,\ldots,K,$$

which is equivalent to

$$L_0\left(\phi_i - \frac{2\sum_{j=1}^K \int_{R^2} w\phi_j}{K\int_{R^2} w^2} w\right) = 0, \qquad i = 1, \ldots, K,$$

where L_0 is defined at (2.1). By Lemma 2.1, we have

$$\phi_i - \frac{2\sum_{j=1}^K \int_{R^2} w\phi_j}{K\int_{R^2} w^2} w \in \text{ span } \left\{\frac{\partial w}{\partial y_j}, j = 1, 2\right\}, \qquad i = 1, \ldots, K. \tag{7.4}$$

Multiplying (7.4) by w and integrating over R^2 and summing up, we have

$$\sum_{j=1}^K \int_{R^2} w\phi_j = 0,$$

and hence

$$\phi_j \in K_0 = \text{span } \left\{\frac{\partial w}{\partial y_k}, k = 1, 2\right\}, \qquad j = 1, \ldots, K. \tag{7.5}$$

Setting $\lambda_0 = 0$ in (6.18) and (6.21) and using the technical condition (1.8), we also obtain (7.5). We omit the details. (Please see Appendix A for similar arguments.)

(7.5) suggests that, at least formally, we should have

$$\phi_\epsilon \sim \sum_{j=1}^K \sum_{k=1}^2 a_{j,k} \frac{\partial w}{\partial y_k}\left(y - \frac{P_j^\epsilon}{\epsilon}\right), \tag{7.6}$$

where $a_{j,k}$ are some constant coefficients.

Next we find a good approximation of $\frac{\partial w}{\partial y_k}(y - \frac{P_j^\epsilon}{\epsilon})$.

Note that $\hat{A}_{\epsilon,j}(y) \sim w(y - \frac{P_j^\epsilon}{\epsilon})$ in $H^2(\Omega_\epsilon)$, and $\hat{A}_{\epsilon,j}$ satisfies

$$\Delta_y \hat{A}_{\epsilon,j} - \hat{A}_{\epsilon,j} + \frac{(\hat{A}_{\epsilon,j})^2}{\hat{H}_\epsilon} + e.s.t. = 0.$$

Thus $\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k}$ satisfies

$$\Delta_y \frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} - \frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} + \frac{2\hat{A}_{\epsilon,j}}{\hat{H}_\epsilon}\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} - \epsilon\frac{(\hat{A}_{\epsilon,j})^2}{\hat{H}_\epsilon^2}\frac{\partial \hat{H}_\epsilon}{\partial x_k} + e.s.t. = 0, \tag{7.7}$$

and we have $\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} = (1 + o(1))\frac{\partial w}{\partial y_k}(y - \frac{P_j^\epsilon}{\epsilon})$.

We now decompose

$$\phi_\epsilon = \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} + \phi_\epsilon^\perp \tag{7.8}$$

with complex numbers $a^\epsilon_{j,k}$, where

$$\phi^\perp_\epsilon \perp \tilde{\mathcal{K}}_\epsilon := \text{span}\left\{\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} | j = 1, \ldots, K,\ k = 1, 2\right\} \subset H^2_N(\Omega_\epsilon). \tag{7.9}$$

Our main idea is to show that this is a good choice because the error $\phi^\perp_\epsilon$ is small in a suitable norm and thus can be neglected. Then we obtain algebraic equations for $a^\epsilon_{j,k}$ that are related to the matrix $M(\mathbf{P}^0)$.

Accordingly, we decompose ψ_ϵ

$$\psi_\epsilon(x) = \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k}\psi_{\epsilon,j,k} + \psi^\perp_\epsilon, \tag{7.10}$$

where $\psi_{\epsilon,j,k}$ is the unique solution of the problem

$$\begin{cases} \dfrac{1}{\beta^2}\Delta_x\psi_{\epsilon,j,k} - (1+\tau\lambda_\epsilon)\psi_{\epsilon,j,k} + 2\xi_\epsilon \hat{A}_{\epsilon,j}\dfrac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} = 0 \quad \text{in } \Omega, \\ \dfrac{\partial \psi_{\epsilon,j,k}}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{7.11}$$

and $\psi^\perp_\epsilon$ satisfies

$$\begin{cases} \dfrac{1}{\beta^2}\Delta_x\psi^\perp_\epsilon - (1+\tau\lambda_\epsilon)\psi^\perp_\epsilon + 2\xi_\epsilon \hat{A}_\epsilon\phi^\perp_\epsilon = 0 \quad \text{in } \Omega, \\ \dfrac{\partial \psi^\perp_\epsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{7.12}$$

Suppose that $\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1$. Then $|a^\epsilon_{j,k}| \le C$, since

$$a^\epsilon_{j,k} = \frac{\int_{\Omega_\epsilon} \phi_\epsilon \frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k}}{\int_{R^2}\left(\frac{\partial w}{\partial y_1}\right)^2} + o(1).$$

Substituting the decompositions of ϕ_ϵ and ψ_ϵ into (6.4), we have

$$\begin{aligned} &\epsilon \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k}\frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H}_\epsilon)^2}\left[-\frac{1}{\epsilon}\psi_{\epsilon,j,k} + \frac{\partial \hat{H}_\epsilon}{\partial x_k}\right] + e.s.t. \\ &\quad + \Delta_y\phi^\perp_\epsilon - \phi^\perp_\epsilon + 2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi^\perp_\epsilon - \frac{(\hat{A}_\epsilon)^2}{(\hat{H}_\epsilon)^2}\psi^\perp_\epsilon - \lambda_\epsilon\phi^\perp_\epsilon \\ &= \lambda_\epsilon \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k}\frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k} \quad \text{in } \Omega_\epsilon. \end{aligned} \tag{7.13}$$

Set

$$I_3 := \epsilon \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k}\frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H}_\epsilon)^2}\left[-\frac{1}{\epsilon}\psi_{\epsilon,j,k} + \frac{\partial \hat{H}_\epsilon}{\partial x_k}\right], \tag{7.14}$$

and

$$I_4 := \Delta_y \phi_\epsilon^\perp - \phi_\epsilon^\perp + 2\frac{\hat{A}_\epsilon}{\hat{H}_\epsilon}\phi_\epsilon^\perp - \frac{(\hat{A}_\epsilon)^2}{(\hat{H}_\epsilon)^2}\psi_\epsilon^\perp - \lambda_\epsilon \phi_\epsilon^\perp. \tag{7.15}$$

We divide our proof into two steps.

Step 1: Estimates for $\phi_\epsilon^\perp$.

The main contribution of this step is to obtain good error bounds for $\phi_\epsilon^\perp$.

We use equation (7.13). Since $\phi_\epsilon^\perp \perp \tilde{\mathcal{K}}_\epsilon$, then similar to the proof of Proposition 4.2, it follows that

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} \le C\|I_3\|_{L^2(\Omega_\epsilon)}. \tag{7.16}$$

Let us now compute I_3.

Let ξ_ϵ and $k(\epsilon, \beta)$ be the same as in Theorem 1.1, then we calculate that for $x \in B_\delta(P_l^\epsilon)$:

$$\begin{aligned}
\frac{\partial \hat{H}_\epsilon}{\partial x_k}(x) &= \xi_\epsilon \beta^2 \int_\Omega \frac{\partial}{\partial x_k} G_\beta(x,\xi)\left(\hat{A}_\epsilon\left(\frac{\xi}{\epsilon}\right)\right)^2 d\xi \\
&= \xi_\epsilon \beta^2 \Bigg(\int_\Omega \frac{\partial}{\partial x_k}(K_0(|x-\xi|) - H_0(x,\xi))\left(\hat{A}_{\epsilon,l}\left(\frac{\xi}{\epsilon}\right)\right)^2 d\xi \\
&\quad + \int_\Omega \sum_{s\neq l} \frac{\partial}{\partial x_k} G_0(x,\xi)\left(\hat{A}_{\epsilon,s}\left(\frac{\xi}{\epsilon}\right)\right)^2 d\xi + O(\beta^4\epsilon^2)\Bigg),
\end{aligned}$$

and by (3.4),

$$\begin{aligned}
\psi_{\epsilon,l,k}(x) &= 2\beta^2 \xi_\epsilon \int_\Omega G_{\beta_{\lambda_\epsilon}}(x,z)\hat{A}_{\epsilon,l}\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_k} dz \\
&= \epsilon \xi_\epsilon \beta^2 \int_\Omega (K_0(|x-\xi|) - H_0(x,\xi) + O(\beta^2))\frac{\partial}{\partial \xi_k}(\hat{A}_{\epsilon,l})^2\, d\xi.
\end{aligned}$$

Thus for $x \in B_\delta(P_l^\epsilon)$, we have

$$\begin{aligned}
&\frac{\partial \hat{H}_\epsilon}{\partial x_k}(x) - \frac{1}{\epsilon}\psi_{\epsilon,l,k}(x) \\
&= \xi_\epsilon \beta^2 \Bigg[\Bigg(\int_\Omega \Bigg[\frac{\partial}{\partial x_k}K_0(|x-\xi|)\left(\hat{A}_{\epsilon,l}\left(\frac{\xi}{\epsilon}\right)\right)^2 - K_0(|x-\xi|)\frac{\partial}{\partial \xi_k}\left(\hat{A}_{\epsilon,l}\left(\frac{\xi}{\epsilon}\right)\right)^2\Bigg] d\xi\Bigg) \\
&\quad - \int_\Omega \Bigg[\frac{\partial}{\partial x_k}H_0(x,\xi)\left(\hat{A}_{\epsilon,l}\left(\frac{\xi}{\epsilon}\right)\right)^2 - H_0(x,\xi)\frac{\partial}{\partial \xi_k}\left(\hat{A}_{\epsilon,l}\left(\frac{\xi}{\epsilon}\right)\right)^2\Bigg] d\xi \\
&\quad + \int_\Omega \sum_{s\neq l}\frac{\partial}{\partial x_k}G_0(x,\xi)\left(\hat{A}_{\epsilon,s}\left(\frac{\xi}{\epsilon}\right)\right)^2 d\xi + O(\epsilon^2\beta^4)\Bigg].
\end{aligned}$$

Using the fact that

$$\frac{\partial}{\partial x_k}K_0(|x-\xi|) + \frac{\partial}{\partial \xi_k}K_0(|x-\xi|) = 0, \quad \text{for } x \neq \xi, \tag{7.17}$$

and integrating by parts, we get

$$\frac{\partial \hat{H}_\epsilon}{\partial x_k}(x) - \frac{1}{\epsilon}\psi_{\epsilon,l,k}(x) = k(\epsilon,\beta)\int_{R^2} w^2 \left(-\frac{\partial}{\partial x_k}F_l(x) + o(\epsilon)\right), \tag{7.18}$$

where

$$F_l(x) = H_0(x, P_l^\epsilon) - \sum_{j\neq l} G_0(x, P_j^\epsilon). \tag{7.19}$$

Observe that

$$\frac{\partial}{\partial x_m}F_l(x)|_{x=P_l^\epsilon} = o(1),$$

since $\mathbf{P}^\epsilon \to \mathbf{P}^0$ and $\mathbf{P}^0$ is a critical point of $F(\mathbf{P})$.

Hence, we have

$$\|I_3\|_{L^2(\Omega_\epsilon)} = o\left(\epsilon k(\epsilon,\beta)\sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^\epsilon|\right), \tag{7.20}$$

and

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} \le C\|I_3\|_{L^2(\Omega_\epsilon)} = o\left(\epsilon k(\epsilon,\beta)\sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^\epsilon|\right). \tag{7.21}$$

Using the equation for $\psi_\epsilon^\perp$ and (7.21), we obtain that

$$\psi_\epsilon^\perp(x) = o\left(\epsilon k(\epsilon,\beta)\sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^\epsilon|\right). \tag{7.22}$$

We calculate

$$\begin{aligned}
\int_{\Omega_\epsilon}\left(I_4\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}\right)d\xi &= \int_{\Omega_\epsilon}\left(\frac{\hat{A}_{\epsilon,l}^2}{H_\epsilon^2}\left(\epsilon\frac{\partial \hat{H}_\epsilon}{\partial x_m}\phi_\epsilon^\perp - \frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}\psi_\epsilon^\perp\right)\right)d\xi - \lambda_\epsilon\int_{\Omega_\epsilon}\phi_\epsilon^\perp\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m} \\
&= \int_{\Omega_{\epsilon,P_l^\epsilon}}\frac{\hat{A}_{\epsilon,l}^2}{\hat{H}_\epsilon^2}\left(\epsilon\frac{\partial \hat{H}_\epsilon}{\partial x_m}(P_l^\epsilon+\epsilon y) - \epsilon\frac{\partial \hat{H}_\epsilon}{\partial x_m}(P_l^\epsilon)\right)\phi_\epsilon^\perp \\
&\quad + \int_{\Omega_{\epsilon,P_l^\epsilon}}\frac{\hat{A}_{\epsilon,l}^2}{\hat{H}_\epsilon^2}\left(\epsilon\frac{\partial \hat{H}_\epsilon}{\partial x_m}(P_l^\epsilon)\right)\phi_\epsilon^\perp \\
&\quad - \int_{\Omega_{\epsilon,P_l^\epsilon}}\frac{\hat{A}_{\epsilon,l}^2}{\hat{H}_\epsilon^2}\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}(\psi_\epsilon^\perp(P_l^\epsilon+\epsilon y) - \psi_\epsilon^\perp(P_l^\epsilon))\,d\xi \\
&\quad - \lambda_\epsilon\int_{\Omega_\epsilon}\phi_\epsilon^\perp\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m} \\
&= o\left(\epsilon^2 k(\epsilon,\beta)\sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^\epsilon|\right),
\end{aligned} \tag{7.23}$$

by using (7.12) and the estimate

$$\frac{\partial \hat{H}_\epsilon}{\partial x_m} = O(k(\epsilon,\beta)) \quad \text{in } \Omega.$$

Step 2: Algebraic equations for $a^\epsilon_{j,k}$.

This step gives us algebraic equations for $a^\epsilon_{j,k}$.

Multiplying both sides of (7.13) by $\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}$ and integrating over Ω_ϵ, we obtain

$$\begin{aligned} r.h.s. &= \lambda_\epsilon \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k} \int_{\Omega_\epsilon} \frac{\partial \hat{A}_{\epsilon,j}}{\partial y_k}\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m} \\ &= \lambda_\epsilon \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k}\delta_{jl}\delta_{km} \int_{R^2} \left(\frac{\partial w}{\partial y_1}\right)^2 dy(1+o(1)) \\ &= \lambda_\epsilon a^\epsilon_{l,m} \int_{R^2} \left(\frac{\partial w}{\partial y_1}\right)^2 dy(1+o(1)), \end{aligned}$$

and by (7.18) and (7.23),

$$\begin{aligned} l.h.s. &= \epsilon \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k} \int_{\Omega_{\epsilon,P^\epsilon_l}} \frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H}_\epsilon)^2}\left[-\frac{1}{\epsilon}\psi_{\epsilon,j,k} + \frac{\partial \hat{H}_\epsilon}{\partial x_k}\right]\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m} \\ &\quad + \int_{\Omega_\epsilon}\left(I_4 \frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m}\right) d\xi \\ &= \epsilon \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k} \int_{\Omega_{\epsilon,P^\epsilon_l}} \frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H}_\epsilon)^2}\left[-\frac{1}{\epsilon}\psi_{\epsilon,j,k} + \frac{\partial \hat{H}_\epsilon}{\partial x_k}\right]\frac{\partial \hat{A}_{\epsilon,l}}{\partial y_m} \\ &\quad + o\left(\epsilon^2 k(\epsilon,\beta)\sum_{j=1}^{K}\sum_{k=1}^{2}|a^\epsilon_{j,k}|\right). \end{aligned} \tag{7.24}$$

Using (7.18), we obtain

$$\begin{aligned} l.h.s. &= \epsilon k(\epsilon,\beta) \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k} \int_{\Omega_\epsilon} \frac{(\hat{A}_{\epsilon,j})^2}{(\hat{H})^2}\left(-\frac{\partial}{\partial x_k}F_j(x)\right)\frac{\partial \hat{A}_{\epsilon,l}}{\partial x_m} \\ &\quad + o\left(\epsilon^2 k(\epsilon,\beta)\sum_{j=1}^{K}\sum_{k=1}^{2}|a^\epsilon_{j,k}|\right) \\ &= \epsilon^2 k(\epsilon,\beta)\int_{R^2} w^2 \frac{\partial w}{\partial y_m} y_m \sum_{j=1}^{K}\sum_{k=1}^{2} a^\epsilon_{j,k}\left(-\frac{\partial}{\partial P^\epsilon_{l,m}}\frac{\partial}{\partial P^\epsilon_{j,k}}F(\mathbf{P}^\epsilon)\right) \\ &\quad + o\left(\epsilon^2 k(\epsilon,\beta)\sum_{j=1}^{K}\sum_{k=1}^{2}|a^\epsilon_{j,k}|\right). \end{aligned} \tag{7.25}$$

Note that

$$\int_{R^2} w^2 \frac{\partial w}{\partial y_m} y_m = -\frac{1}{3} \int_{R^2} w^3.$$

Thus we have

$$l.h.s. = \frac{\epsilon^2 k(\epsilon,\beta)}{3} \left(\int_{R^2} w^3 \right) \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^{\epsilon} \left(\frac{\partial}{\partial P_{l,m}^{\epsilon}} \frac{\partial}{\partial P_{j,k}^{\epsilon}} F(\mathbf{P}^{\epsilon}) \right) + o\left(\epsilon^2 k(\epsilon,\beta) \sum_{j=1}^{K} \sum_{k=1}^{2} |a_{j,k}^{\epsilon}| \right). \tag{7.26}$$

Combining the $l.h.s.$ and $r.h.s$, we have

$$\frac{\epsilon^2 k(\epsilon,\beta)}{3} \left(\int_{R^2} w^3 \right) \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^{\epsilon} \left(\frac{\partial}{\partial P_{l,m}^{\epsilon}} \frac{\partial}{\partial P_{j,k}^{\epsilon}} F(\mathbf{P}^{\epsilon}) \right) + o\left(\epsilon^2 k(\epsilon,\beta) \sum_{j=1}^{K} \sum_{k=1}^{2} |a_{j,k}^{\epsilon}| \right) = \lambda_\epsilon a_{l,m}^{\epsilon} \int_{R^2} \left(\frac{\partial w}{\partial y_1} \right)^2 dy (1 + o(1)). \tag{7.27}$$

From (7.27), we see that the small eigenvalues with $\lambda_\epsilon \to 0$ satisfy $|\lambda_\epsilon| \sim \epsilon^2 k(\epsilon,\beta)$. Furthermore,

$$\frac{\lambda_\epsilon}{\epsilon^2 k(\epsilon,\beta)} \to \frac{\int_{R^2} w^3}{3 \int_{R^2} (\frac{\partial w}{\partial y_1})^2 dy} \sigma_0,$$

as $\epsilon \to 0$, where σ_0 is an eigenvalue of the matrix $M(\mathbf{P}^0)$, and $\mathbf{P}^\epsilon \to \mathbf{P}^0$ as $\epsilon \to 0$. (The vector $\vec{a}^\epsilon = (a_{1,1}^\epsilon, a_{1,2}^\epsilon, \ldots, a_{K,2}^\epsilon)^T$ approaches an eigenvector of $M(\mathbf{P}^0)$ corresponding to σ_0.) By condition (*), the matrix $M(\mathbf{P}^0)$ is negative definite, and it follows that $\mathrm{Re}(\lambda_\epsilon) < 0$. Therefore, the small eigenvalues λ_ϵ are stable for (6.4) if ϵ is small enough.

Completion of the proofs of Theorem 1.2. Theorem 1.2 now follows from Section 6 and Section 7. □

8. Discussion

Let us discuss what has been achieved in this paper and which important questions are still left open. We have investigated the Gierer-Meinhardt system, which is a very important reaction-diffusion system within the class of Turing systems. We study the weak coupling case, i.e., the diffusion coefficient D of the inhibitor tends to infinity, for small diffusion coefficient ϵ^2 of the activator. In a bounded domain, we rigorously prove existence of multipeaked solutions and are able to locate the peaks in terms of the Green's function and its derivatives.

Furthermore, we derive rigorous results on linear stability. There are $o(1)$ eigenvalues which are given to leading order in terms of the Green's function and its derivatives and are implicitly linked to the spike locations. It would be desirable to find conditions on the small eigenvalues which are not given in terms of the Green's function and its derivatives but explicitly in terms of the domain Ω.

On the other hand, there are also $O(1)$ eigenvalues which are given as eigenvalues of related nonlocal eigenvalue problems in R^2. For many cases, we can show that these $O(1)$ eigenvalues lie on the left or right half of the complex plane. Some of the cases—in particular, in the borderline case $\eta_\epsilon \to K$ and in the case that τ is finite—are still missing.

There are no results in either the weak or the strong coupling case on the dynamics of the full Gierer-Meinhardt system in a two-dimensional domain. Furthermore, there are no results at all about the existence or stability of K-peaked solutions in a three-dimensional domain. These important questions are still open.

9. Appendix A: Invertibility of the Linearized Operator and the Proofs of Propositions 4.2 and 4.3

In this appendix we prove Propositions 4.2 and 4.3. This establishes the invertibility of the linearized operator.

Proof of Proposition 4.2. We follow the Liapunov-Schmidt reduction method, which has been used in [7] and [37]. Suppose that (4.21) is false. Then there exist sequences $\{\epsilon_k\}$, $\{\beta_k\}$, $\{\mathbf{P}_k\}$, and $\{\Sigma_k\}$ with

$$\epsilon_k > 0, \qquad \epsilon_k \to 0, \qquad \beta_k > 0, \qquad \beta_k \to 0, \qquad \mathbf{P}_k \in \overline{\Lambda_\delta},$$

$$\Sigma_k = \begin{pmatrix} \phi_k(y) \\ \psi_k(x) \end{pmatrix} \in \mathcal{K}^\perp_{\epsilon_k,\mathbf{P}_k}$$

such that

$$\|\mathcal{L}_{\epsilon_k,\mathbf{P}_k}\Sigma_k\|_{L^2(\Omega_{\epsilon_k})\times L^2(\Omega)} \to 0, \tag{9.1}$$

$$\|\Sigma_k\|_{H^2(\Omega_{\epsilon_k})\times H^2(\Omega)} = 1, \qquad k = 1,\ 2,\ \dots . \tag{9.2}$$

Written explicitly, we have the following situation:

$$\Delta_y\phi_k - \phi_k + 2A_{\epsilon_k,\mathbf{P}_k}H^{-1}_{\epsilon_k,\mathbf{P}_k}\phi_k - A^2_{\epsilon_k,\mathbf{P}_k}H^{-2}_{\epsilon_k,\mathbf{P_k}}\psi_k = f_k^1 + f_k^2, \tag{9.3}$$

where

$$\|f_k^1\|_{L^2(\Omega_{\epsilon_k})} \to 0, \qquad f_k^2 \in \mathcal{C}^\perp_{\epsilon_k,\mathbf{P}_k},$$

$$\Delta_x\psi_k - \beta_k^2\psi_k + 2\beta_k^2\xi_{\epsilon_k}A_{\epsilon_k,\mathbf{P}_k}\phi_k = g_k, \tag{9.4}$$

$$\|g_k\|_{L^2(\Omega)} \to 0, \qquad \phi_k \in K^\perp_{\epsilon_k,\mathbf{P}_k}, \tag{9.5}$$

$$\|\phi_k\|^2_{H^2(\Omega_{\epsilon_k})} + \|\psi_k\|^2_{H^2(\Omega)} = 1. \tag{9.6}$$

We now show that this is impossible. To simplify notation, we set $A_k = A_{\epsilon_k,\mathbf{P}_k}$, $\Omega_k = \Omega_{\epsilon_k}$, $\xi_k = \xi_{\epsilon_k}$.

In the first step of the proof, we show that the linearized problem given by (9.3), (9.4) tends to a limit problem as $\epsilon \to 0$. This analysis is very similar to the one given in Section 6 in the case $\lambda_0 = 0$. In fact, the analysis in Section 6 also covers this case (but does not give the leading order of the $o(1)$ eigenvalues and their eigenfunctions). Therefore we may introduce $\phi_{\epsilon_k,j}$, $j = 1, \dots, K$ as before by cut-off and extension.

If we decompose

$$\phi_k = \sum_{j=1}^{K} \phi_{k,j} + \phi_{k,K+1},$$

it is easy to see that $\phi_{k,K+1} = o(1)$ in $H^2(\Omega_k)$ because it satisfies the equation

$$\Delta_y \phi_{k,K+1} - \phi_{k,K+1} = o(1) \quad \text{in } H^2(\Omega_k).$$

This implies

$$\phi_{k,K+1} = o(1) \quad \text{in } H^2(\Omega_k).$$

We define $\Psi_{k,i}$ for $i = 1, \dots, K+1$ by

$$\Delta_x \Psi_{k,i} - \beta_k^2 \Psi_{k,i} + 2\beta_k^2 \xi_{\epsilon,k} A_{\epsilon_k,\mathbf{P}_k} \phi_{k,i} = 0, \qquad \frac{\partial \Psi_{k,i}}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Note that as $\|g_k\|_{L^2(\Omega)} \to 0$, we have

$$\|\psi_k - \sum_{k=1}^{K+1} \psi_{k,i}\|_{H^2(\Omega)} \to 0.$$

Since $\phi_{k,K+1} = o(1)$ in $H^2(\Omega_k)$, we also have $\|\psi_{k,K+1}\|_{H^2(\Omega)} = o(1)$.

Letting $k \to \infty$, it can be shown as in Section 6 that

$$\phi_{\epsilon_k,j} \to \phi_j \quad \text{in } H^2(R^2).$$

Then for $i = 1, \dots, K$ we have

$$\phi_i \in \left\{ \phi \in H^2(R^2) \;\middle|\; \int_{R^2} \phi \frac{\partial w}{\partial y_j} dy = 0,\ j = 1, 2 \right\} = K_0^{\perp},$$

and ϕ_i has to satisfy the following nonlocal linear problem:

Case 1: $\eta_\epsilon \to 0$

$$\Delta \phi_i - \phi_i + 2w\phi_i - \frac{2\sum_{j=1}^{K} \int_{R^2} w\phi_j}{K \int_{R^2} w^2} w^2 \in C_0^{\perp}. \tag{9.7}$$

Case 2: $\eta_\epsilon \to \infty$

$$\Delta \phi_i - \phi_i + 2w\phi_i - \frac{2\int_{R^2} w\phi_i}{\int_{R^2} w^2} w^2 \in C_0^{\perp}. \tag{9.8}$$

Case 3: $\eta_\epsilon \to \eta_0$

$$\Delta\phi_i - \phi_i + 2w\phi_i - \frac{2[(1+\eta_0)\int_{R^2} w\phi_i + \sum_{j\neq i}\int_{R^2} w\phi_j]}{(K+\eta_0)\int_{R^2} w^2} w^2 \in C_0^\perp, \tag{9.9}$$

where

$$C_0 := \text{ span } \left\{\frac{\partial w}{\partial y_j},\ j = 1, 2\right\},$$

and $K_0^\perp$, $C_0^\perp$ denotes the orthogonal complement with respect to the scalar product of $L^2(R^2)$ in the space $H^2(R^2)$ and $L^2(R^2)$, respectively.

After transforming the functions $(\phi_1, \ldots, \phi_K)$ in Case 3 in the same way as in Section 6 (i.e., diagonalizing the matrix $\mathcal{G}$) and in Case 1 diagonalizing the matrix

$$H = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

we get the following decoupled equations for ϕ_i:

$$\Delta_y\phi_i - \phi_i + 2w\phi_i - 2\rho_i \frac{\int_{R^2} w\phi_i}{\int_{R^2} w^2} w \in C_0^\perp, \tag{9.10}$$

where

$$\rho_i = \begin{cases} 0, \ldots, 0, K & \text{in Case 1,} \\ 1, \ldots, 1 & \text{in Case 2,} \\ \dfrac{\eta_0}{K+\eta_0}, \ldots, \dfrac{\eta_0}{K+\eta_0},\ 1 & \text{in Case 3.} \end{cases}$$

Since $L_0 w = w^2$, (9.10) can be written as

$$(\Delta_y - 1 + 2w)\left(\phi_i - 2\rho_i \frac{\int_{R^2} w\phi_i}{\int_{R^2} w^2} w\right) \in C_0^\perp.$$

Since the operator

$$L_0 = \Delta_y - 1 + 2w : K_0^\perp \to C_0^\perp$$

is one-to-one map with bounded inverse (by Lemma 2.1), we have

$$\phi_i - 2\rho_i \frac{\int_{R^2} w\phi_i}{\int_{R^2} w} = 0. \tag{9.11}$$

Now we multiply by w and integrate. This gives

$$(1 - 2\rho_i)\int_{R^2} w\phi_i = 0. \tag{9.12}$$

If $\rho_i \neq \frac{1}{2}$, then by (9.12),

$$\int_{R^2} w\phi_i = 0,$$

which implies that

$$L_0\phi_i = 0, \qquad i = 1, \ldots, K,$$

and by Lemma 2.1 that

$$\phi_i \in K_0, \qquad i = 1, \ldots, K.$$

Therefore by (9.11),

$$\phi_i = 0, \qquad i = 1, \ldots, K.$$

Now we can explain why Remark (1.1) is important: It is easy to see that $\rho_i = \frac{1}{2}$ for some i if and only $K > 1$ and $\eta_0 = K$. In this case, the method of Liapunov-Schmidt reduction is not readily applicable.

By taking the limit in (9.4), we see that this implies $\psi_i \to 0$ in $H^2(\Omega)$.

Furthermore, the assumption (9.6) implies that

$$\sum_{i=1}^{K}(\|\phi_i\|^2_{H^2(R^2)} + \|\psi_i\|^2_{H^2(\Omega)}) = 1.$$

This contradicts $\phi_i = \psi_i = 0$, and the proof of Proposition 4.2 is completed. □

Proof of Proposition 4.3. We just need to show that the conjugate operator of $\mathcal{L}_{\epsilon,\mathbf{P}}$ (denoted by $\mathcal{L}^*_{\epsilon,\mathbf{P}}$) is injective from $\mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ to $\mathcal{C}^\perp_{\epsilon,\mathbf{P}}$. Suppose not. Then there exist $\phi \in K^\perp_{\epsilon,\mathbf{P}}$, $\psi \in W^{2,t}_N(\Omega)$ such that

$$\begin{aligned}\Delta_y\phi - \phi + 2A_{\epsilon,\mathbf{P}}H^{-1}_{\epsilon,\mathbf{P}}\phi + 2\xi_\epsilon\beta^2 A_{\epsilon,\mathbf{P}}\psi &\in C^\perp_{\epsilon,\mathbf{P}},\\ \Delta_x\psi - \beta^2\psi - A^2_{\epsilon,\mathbf{P}}H^{-2}_{\epsilon,\mathbf{P}}\phi &= 0,\\ \|\phi\|^2_{H^2(\Omega_\epsilon)} + \|\psi\|^2_{H^2(\Omega)} &= 1.\end{aligned}$$

Similar to the proof of Proposition 4.2, we obtain

$$L_{\epsilon,\mathbf{P}}\phi + o(1) \in C^\perp_{\epsilon,\mathbf{P}}, \qquad \phi \in K^\perp_{\epsilon,\mathbf{P}}.$$

By Proposition 4.2, $\|\phi\|_{H^2(\Omega_\epsilon)} = o(1)$, and hence $\|\psi\|_{H^2(\Omega)} = o(1)$. This is a contradiction, and the proof of Proposition 4.3 is finished. □

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