FINITE MORSE INDEX IMPLIES FINITE ENDS

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Abstract. We prove that finite Morse index solutions to the Allen-Cahn equation in $\mathbb{R}^2$ have **finitely many ends** and **linear energy growth**. The main tool is a **curvature decay estimate** on level sets of these finite Morse index solutions, which in turn is reduced to a problem on the uniform second order regularity of clustering interfaces for the singularly perturbed Allen-Cahn equation in $\mathbb{R}^n$. Using an indirect blow-up technique, in the spirit of the classical Colding-Minicozzi theory in minimal surfaces, we show that the **obstruction** to the uniform second order regularity of clustering interfaces in $\mathbb{R}^n$ is associated to the existence of nontrivial entire solutions to a (finite or infinite) **Toda system** in $\mathbb{R}^{n-1}$. For finite Morse index solutions in $\mathbb{R}^2$, we show that this obstruction does not exist by using information on stable solutions of the Toda system.

1. Introduction

The intricate connection between the Allen-Cahn equation and minimal surfaces is best illustrated by the following famous De Giorgi's Conjecture [21].

Conjecture. *Let $u \in C^2(\mathbb{R}^n)$ be a solution to the Allen-Cahn equation*

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n \tag{1.1}$$

satisfying $\partial_{x_n} u > 0$. If $n \leq 8$, all level sets $\{u = \lambda\}$ of u must be hyperplanes.

In the last twenty years, great advances in De Giorgi's conjecture have been achieved, having been fully established in dimensions $n = 2$ by Ghoussoub and Gui [37] and for $n = 3$ by Ambrosio and Cabre [2]. A celebrated result by Savin [67] established its validity for $4 \leq n \leq 8$ under an extra assumption that

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{1.2}$$

On the other hand, Del Pino, Kowalczyk and Wei [24] constructed a counterexample in dimensions $n \geq 9$.

After the classification of monotone solutions, it is natural to consider **stable solutions**. Unfortunately this has been less successful. The arguments in [2, 22, 37] imply that all stable solutions in $\mathbb{R}^2$ are one-dimensional. On the other hand, Pacard and Wei [63] found a nontrivial stable solution in $\mathbb{R}^8$. This is later shown to be also global minimizer [58]. (For local minimizers or stable solutions in bounded domains we refer to Modica [62], Kohn-Sterberg [49], Le [55], Sternberg-Zumbrun [72], Tonegawa-Wickramasekera [75] and the references therein.)

In this paper we consider a more difficult problem of classification of **finite Morse index solutions** in $\mathbb{R}^2$. Finite Morse index is a spectrum condition which is hard to use to obtain energy estimate. In the literature, another condition–**finite-ended solutions**–is used. Roughly speaking a solution is called finite-ended if the number of components of the nodal set $\{u = 0\}$ is finite outside a ball. (In fact more restrictions are needed, see del Pino, Kowalczyk and Pacard [23], Gui [41].) Analogous to the structure of minimal surfaces with finite Morse index ([35, 43, 44]), a long standing conjecture is that finite Morse index solutions to the Allen-Cahn equation in $\mathbb{R}^2$ have linear energy growth and hence finitely many ends (see [41, 25]). In this paper we will prove this conjecture by establishing a **curvature decay estimate** on level sets of these finite Morse index solutions.

This curvature estimate is similar to the one for stable minimal surfaces established by Schoen in [70]. However, the key tool used in minimal surfaces is the so-called Simons type inequality [71] which has no analogue for semilinear elliptic equations. (The closest one may be the so-called Sternberg-Zumbrun inequality [72] for stable solutions.) Here an indirect blow up method will be employed in this paper. Our blow-up procedure is inspired by the groundbreaking work of Colding and Minicozzi on the structure of limits of sequences of embedded minimal surfaces of fixed genus in a ball in $\mathbb{R}^3$ ([14, 15, 16, 17, 18, 19]).

1991 *Mathematics Subject Classification.* 35B08, 35J62.

Key words and phrases. Allen-Cahn equation; Toda system; Morse index; Minimal surfaces; Clustering interfaces .

We prove the curvature estimate by studying the uniform second order regularity of clustering interfaces in the singularly perturbed Allen-Cahn equation. It turns out the uniform second order regularity does not always hold true and the obstruction is associated to the existence of nontrivial entire solutions to the **Toda system**

$$-\Delta f_\alpha = e^{-\sqrt{2}(f_{\alpha+1}-f_\alpha)} - e^{-\sqrt{2}(f_\alpha - f_{\alpha-1})}, \qquad \text{in } \mathbb{R}^{n-1}. \tag{1.3}$$

This connection between the Allen-Cahn equation and the Toda system was previously used in [23, 25, 27] to construct solutions to the Allen-Cahn equation with clustering interfaces. The analysis of clustering interfaces started in Hutchinson-Tonegawa [46]. It is shown that the energy at the clustered interfaces is quantized. In [73, 75], the convergence of clustering interfaces as well as regularity of their limit varifolds were studied. However, the uniform regularity of clustering interfaces (see [74]) and precise behavior of the solutions near the interfaces and the connection to Toda system (except some special cases such as two end solutions in $\mathbb{R}^3$ studied in [42]) are still missing. In this paper we give precise second order estimates and show that when clustering interfaces appear, then a suitable rescaling of these interfaces converge to the graphs of a solution to the Toda system. It is through this blow up procedure we reduce the uniform second order regularity of interfaces to the non-existence of nontrivial entire stable solutions to the Toda system. We also show that the stability condition is preserved in this blow up procedure. Then using results on stable solutions of the Toda system, we establish the uniform second order regularity of interfaces for stable solutions of the singularly perturbed Allen-Cahn equation, and then the curvature estimate for finite Morse index solutions in $\mathbb{R}^2$.

For other related results on De Giorgi conjecture for Allen-Cahn equation, we refer to [1, 8, 31, 32, 33, 34, 38, 48, 69, 76] and the references therein.

2. Main results

We consider general Allen-Cahn equation

$$\Delta u = W'(u), \quad |u| < 1, \qquad \text{in } \mathbb{R}^n \tag{2.1}$$

where $W(u)$ is a double well potential, that is, $W \in C^3([-1,1])$ satisfying

- $W > 0$ in $(-1,1)$ and $W(\pm 1) = 0$;
- $W'(\pm 1) = 0$ and $W''(-1) = W''(1) = 2$;
- there exists only one critical point of W in $(-1,1)$, which is assumed to be 0.

A typical model is given by $W(u) = (1-u^2)^2/4$.

Under these assumptions on W, it is known that there exists a unique solution (up to a translation) to the following one dimensional problem

$$g''(t) = W'(g(t)), \quad g(0) = 0 \quad \text{and} \quad \lim_{t \to \pm\infty} g(t) = \pm 1. \tag{2.2}$$

After a scaling $u_\varepsilon(x) := u(\varepsilon^{-1}x)$, we obtain the singularly perturbed version of the Allen-Cahn equation:

$$\varepsilon \Delta u_\varepsilon = \frac{1}{\varepsilon} W'(u_\varepsilon) \qquad \text{in } \mathbb{R}^n. \tag{2.3}$$

2.1. Finite Morse index solutions.

We say a solution $u \in C^2(\mathbb{R}^n)$ has finite Morse index if there is a finite upper bound on its Morse index in any compact set. By [28], this is equivalent to the condition that u is stable outside a compact set, that is, there is a compact set $K \subset \mathbb{R}^n$ such that

$$\mathcal{Q}(\varphi) := \int_{\mathbb{R}^n} |\nabla \varphi|^2 + W''(u)\varphi^2 \geq 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n \setminus K).$$

Classifying finite Mores index solutions is in general a difficult task, even in dimension $n = 2$. In $\mathbb{R}^2$ we know that stable solutions (Morse index 0) are one dimensional, i.e. after rigid motions in $\mathbb{R}^2$, $u(x_1, x_2) \equiv g(x_2)$. Since the finite Morse index is a difficult condition to use, another class of solutions–finite-ended solutions–has been introduced by del Pino, Kowalczyk and Pacard [23], which we recall here.

Definition 2.1. *A solution u is said to be a finite-ended solution to the Allen-Cahn equation* (2.1) *in $\mathbb{R}^2$ if there exist k oriented half lines $\{a_j \cdot x + b_j = 0\}$, $j = 1, ..., k$, (for some choices of $a_j \in \mathbb{R}^2, |a_j| = 1$*

and $b_j \in \mathbb{R}$*), such that along these half lines and away from a compact set* K *containing the origin,* u *is asymptotic to* $g(a_j x + b_j)$*, that is, there exist positive constants* $C, c > 0$ *such that*

$$\|u(x) - \sum_{j=1}^{k}(-1)^{j+1} g(a_j x + b_j)\|_{L^\infty(\mathbb{R}^2\setminus K)} \leq Ce^{-c|x|}. \tag{2.4}$$

The set of $k-$ended solutions is denoted by $\mathcal{M}_k$. A simple counting of nodal domains shows that k must be even. In [23], it is shown that L^2 convergence implies (2.4). Furthermore it was shown [23] that $\mathcal{M}_k$ is a smooth k-dimensional Banach manifold in neighborhoods of u satisfying nondegeneracy conditions.

Gui ([40, 41]) showed that if the nodal sets $\{u = 0\}$ is finite outside a compact set, and each component is contained by a non-overlapping cone, then $u \in \mathcal{M}_k$. Moreover he also derived the Halmitonian identity and proved that the following balancing condition holds

$$\sum_{j=1}^{k} a_j = 0. \tag{2.5}$$

All two-ended solutions are one-dimensional. Near each end the solution approaches to the one-dimensional profile exponentially, see Del Pino-Kowalczyk-Pacard [23], Gui [40] and Kowalczyk-Liu-Pacard [50, 51, 52]. The existence of multiple-ended solutions and infinite-ended solutions to Allen-Cahn equation in $\mathbb{R}^2$ have been constructed in [3, 26, 53, 54]. The structure and classification of four end solutions have been studied extensively in [41, 23, 50, 51, 52]. It is shown that the four-ended solutions have even symmetries and the moduli space of four-ended solutions is one-dimensional.

A long standing and important question is

Question: *Does finite Morse index solution have finite ends*?

Our first main result gives a positive answer to the above question:

Theorem 2.2. *Suppose* u *is a finite Morse index solution of* (2.1) *in* $\mathbb{R}^2$*. Then there exists* $k \in \mathbb{N}$ *such that* $u \in \mathcal{M}_k$*, i.e.,* u *is a finite-ended solution. Moreover,* u *has linear energy growth, i.e., there exists a constant* C *such that*

$$\int_{B_R(0)} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \leq CR, \quad \forall R \geq 1. \tag{2.6}$$

As a byproduct of our analysis, for solutions with Morse index 1 we can show that

Theorem 2.3. *Any solution to* (2.1) *in* $\mathbb{R}^2$ *with Morse index* 1 *has four ends.*

Remark 2.4. *The linear growth condition (2.6) implies that the nodal set* $\{u = 0\}$ *has finite length at* ∞*. In* $\mathbb{R}^n$ *the analogue energy bound*

$$\int_{B_R(0)} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \leq C\ R^{n-1}, \quad \forall R \geq 1 \tag{2.7}$$

is a classical assumption in the setting of semilinear elliptic equations (see e.g. Hutchinson-Tonegawa [46]*). It is satisfied by minimizers or monotone solutions satisfying* (1.2)*. This is precisely the use of* (1.2) *in Savin's proof of De Giorgi's conjecture (*[67]*). (See also Ambrosio-Cabre* [2] *and Alberti-Ambrosio-Cabre* [1]*.) In dimensions* 4 *and* 5*, condition (2.7) is also an essential estimate in Ghoussoub-Gui* [38]*. A similar area bound for minimal hypersurfaces seems to be also crucial for the study of* Stable Bernstein Conjecture *when the dimension is larger than* 3*. (Only three dimension case has been solved in* [29, 36]*. See also* [11, 56, 57]*.)*

Remark 2.5. *In a recent paper* [59]*, Mantoulidis showed that for* $2m$*-ended solutions the Morse index is at least* $m - 1$*.*

The main tool to prove Theorem 2.2 is the following curvature estimate on level sets of u (see Theorem 3.5 and Theorem 3.8 below):

Key Curvature Estimates (Theorem 3.5): For any solution of (2.1) in $\mathbb{R}^2$ with finite Morse index and $b \in (0, 1)$, there exist a constant C and $R = R(b)$ such that

$$|B(u)(x)| \leq \frac{C}{|x|} \quad \text{for } x \in \{|u| \leq 1 - b\} \cap (B_{R(b)}(0))^c, \quad \text{where } |B(u)(x)| = \sqrt{\frac{|\nabla^2 u(x)|^2 - |\nabla|\nabla u(x)||^2}{|\nabla u(x)|^2}}. \tag{2.8}$$

This curvature decay is similar to Schoen's curvature estimate for stable minimal surfaces [70], however the proof is quite different. This is mainly due to the lack of a suitable Simons type inequality for semilinear elliptic equations. Hence an indirect method is employed, by introducing a blow up procedure and reducing the curvature decay estimate to a second order estimate on interfaces of solutions to (2.3), see Theorem 3.6 below.

2.2. **Second order estimates on interfaces.** It turns out that our analysis on the uniform second order regularity of level sets of solutions to the singularly perturbed Allen-Cahn equation (2.3) works in a more general setting and any dimension $n \geq 2$. In Part II of this paper we give precise analysis in the case of clustering interfaces. More precisely we assume that

- **(H1)** u_ε is a sequence of solutions to (2.3) in $\mathcal{C}_2 = B_2^{n-1} \times (-1,1) \subset \mathbb{R}^n$, where $\varepsilon \to 0$;
- **(H2)** there exists $Q \in \mathbb{N}$, $b \in (0,1)$ and $t_\varepsilon \in (-1+b, 1-b)$ such that $\{u_\varepsilon = t_\varepsilon\}$ consists of Q connected components
$$\Gamma_{\alpha,\varepsilon} = \{x_n = f_{\alpha,\varepsilon}(x'), \quad x' := (x_1, \cdots, x_n) \in B_2^{n-1}\}, \quad \alpha = 1, \cdots, Q,$$
where $-1/2 < f_{1,\varepsilon} < f_{2,\varepsilon} < \cdots < f_{Q,\varepsilon} < 1/2$;
- **(H3)** for each α, $f_{\alpha,\varepsilon}$ are uniformly bounded in $Lip(B_2^{n-1})$ and they converge to the same limit f_∞ in $C_{loc}(B_2^{n-1})$.

Here Q is called the multiplicity of the interfaces. Analyzing clustering interfaces is one of main difficulties in the study of singularly perturbed Allen-Cahn equations. See e.g. [46, 73, 75, 74]. In particular, it is not known if flatness implies uniform $C^{1,\theta}$ regularity when there are clustering interfaces (i.e. the Lipschitz regularity in the above hypothesis **(H3)**).

Under these assumptions, it can be shown that f_∞ satisfies the minimal surface equation (see [46])

$$\operatorname{div}\left(\frac{\nabla f_\infty}{\sqrt{1+|\nabla f_\infty|^2}}\right) = 0 \quad \text{in } \mathbb{R}^{n-1}. \tag{2.9}$$

Because f_∞ is Lipschitz, by standard elliptic estimates on the minimal surface equation [39, Chapter 16], $f_\infty \in C_{loc}^\infty(B_2^{n-1})$.

We want to study whether $f_{\alpha,\varepsilon}$ converges to f_∞ in $C_{loc}^2(B_2^{n-1})$. It turns out this may not be true and the obstruction is related to a Toda system

$$\Delta f_\alpha(x') = A_1 e^{-\sqrt{2}\left(f_\alpha(x') - f_{\alpha-1}(x')\right)} - A_2 e^{-\sqrt{2}\left(f_{\alpha+1}(x') - f_\alpha(x')\right)}, \quad x' \in \mathbb{R}^{n-1}, \quad 1 \leq \alpha \leq Q', \tag{2.10}$$

where $Q' \leq Q$, A_1 and A_2 are positive constants.

More precisely, we show

Theorem 2.6. *If $f_{\alpha,\varepsilon}$ does not converge to f_∞ in $C_{loc}^2(B_2^{n-1})$, then a suitable rescaling of them converge to a nontrivial entire solution to the Toda system* (2.10).

For the multiplicity one case $Q = 1$, we get the following uniform $C^{2,\theta}$ estimate.

Theorem 2.7. *If $\{u_\varepsilon = 0\} = \{x_n = f_\varepsilon(x_1, \cdots, x_{n-1})\}$, then for any $\theta \in (0,1)$, f_ε are uniformly bounded in $C_{loc}^{2,\theta}(B_2^{n-1})$.*

These results answer partly a question of Tonegawa [74] and improves the uniform $C^{1,\theta}$ estimate in Caffarelli-Cordoba [10] to the second order $C^{2,\theta}$ estimate.

The main idea in the proof of these two theorems relies on the determination of the **interaction** between $\Gamma_{\alpha,\varepsilon}$. To this aim, we introduce the Fermi coordinates with respect to $\Gamma_{\alpha,\varepsilon}$ and near each $\Gamma_{\alpha,\varepsilon}$ we find the optimal approximation of u_ε along the normal direction using the one dimensional profile g and the distance to $\Gamma_{\alpha,\varepsilon}$. More precisely, we use an approximate solution in the form

$$g\left(\frac{\operatorname{dist}_{\Gamma_{\alpha,\varepsilon}} - h_{\alpha,\varepsilon}}{\varepsilon}\right).$$

Here $h_{\alpha,\varepsilon}$ is introduced to make sure that this is the optimal approximation along the normal direction with respect to $\Gamma_{\alpha,\varepsilon}$. With this construction, using the nondegeneracy of g, we can get a good estimate on the error between u_ε and these approximate solutions, which in turn shows that the interaction between $\Gamma_{\alpha,\varepsilon}$ is exactly through the Toda system

$$\Delta f_{\alpha,\varepsilon} = \frac{A_1}{\varepsilon} e^{-\sqrt{2}\frac{f_{\alpha,\varepsilon} - f_{\alpha-1,\varepsilon}}{\varepsilon}} - \frac{A_2}{\varepsilon} e^{-\sqrt{2}\frac{f_{\alpha+1,\varepsilon} - f_{\alpha,\varepsilon}}{\varepsilon}} + \text{high order terms}.$$

Using this representation, we show that the uniform second order regularity of $f_{\alpha,\varepsilon}$ does not hold only if the lower bound of intermediate distances between $\Gamma_{\alpha,\varepsilon}$ is of the order

$$\frac{\sqrt{2}}{2}\varepsilon|\log\varepsilon| + O(\varepsilon). \tag{2.11}$$

(Here the constant $\sqrt{2} = \sqrt{W''(1)}$.) Moreover, if this is the case, the rescalings

$$\widetilde{f}_{\alpha,\varepsilon}(x') := \frac{1}{\varepsilon} f_{\alpha,\varepsilon}\left(\varepsilon^{\frac{1}{2}}x'\right) - \frac{\sqrt{2}\alpha}{2}|\log\varepsilon|$$

converges to a solution of (2.10).

In other words, if intermediate distances between $\Gamma_{\alpha,\varepsilon}$ are large (compared with (2.11)), the interaction between different interfaces is so weak enough that it does not affect the second order regularity of $f_{\alpha,\varepsilon}$. In particular, if there is only one component and hence no interaction between different components, we get Theorem 2.7.

In Theorem 3.6, the situation is a little different where more and more connected components of $\{u_\varepsilon = 0\}$ could appear. However, the above discussion still applies. This is because, by using the stability condition, we can get an explicit lower bound on intermediate distance between different components of $\{u_\varepsilon = 0\}$ which is just a little smaller than (2.11). To get a lower bound higher than (2.11), we use the stability of $f_{\alpha,\varepsilon}$ (as a solution to the approximate Toda system) inherited from u_ε. By this stability and a classical estimate of Choi-Schoen [13], we get a decay estimate of $e^{-\sqrt{2}\frac{f_{\alpha,\varepsilon}-f_{\alpha-1,\varepsilon}}{\varepsilon}}$ in the interior. In some sense $e^{-\sqrt{2}\frac{f_{\alpha,\varepsilon}-f_{\alpha-1,\varepsilon}}{\varepsilon}}$ replaces the role of the curvature in minimal surface theory.

We also would like to call readers' attention to the resemblance of pictures here (especially when we consider $\mathbb{R}^3$ and not only $\mathbb{R}^2$) with the multi-valued graph construction in seminal Colding-Minicozzi theory [14, 15, 16, 17, 20]. When the number of connected components of $\{u_\varepsilon = 0\}$ goes to infinity and we do not assume the stability condition, the blow up procedure as in Theorem 2.6 produces a solution to the Toda lattice (i.e. in (2.10) the index α runs over integers $\mathbb{Z}$). The difference is that, different sheets of minimal surfaces do not interact (in other words, interact only when they touch) while different sheets of interfaces in the Allen-Cahn equation have an exponential interaction. It is this exponential interaction leading to the Toda system. We notice that in a recent paper [47], Jerison and Kambrunov also performed a similar blow-up procedure for the one-phase free boundary problem in $\mathbb{R}^2$. The difference is again that different sheets of free boundaries do not interact.

Organization of the paper. This paper is divided into three parts. Part I is devoted to the analysis of finite Morse index solutions, by assuming the curvature decay estimate. In Part II we study the second order regularity of interfaces and prove Theorem 2.6 and Theorem 2.7. Techniques in Part II are modified in Part III to prove the curvature decay estimate needed in Part I. Some technical calculations in Part II are collected in the Appendix.

Acknowledgement. The research of J. Wei is partially supported by NSERC of Canada and the Cheung-Kong Chair Professorship. K. Wang is supported by NSFC no. 11631011 and "the Fundamental Research Funds for the Central Universities". We thank Professor Changfeng Gui for useful discussions and Dr. Mantoulidis for some suggestions. K. Wang is also grateful to Yong Liu for several enlightening discussions.

Part 1. Finite Morse index solutions

In this part we study finite Morse index solutions of (2.1) in $\mathbb{R}^2$ and prove Theorem 2.2 and Theorem 2.3, by assuming the curvature decay estimates Theorem 3.5 and Theorem 3.8, which is based on Theorem 3.6 whose proof is given in Part III. Throughout this section we always assume that $n = 2$ and that u is a finite Morse index solution to (2.1).

3. Curvature decay

The following characterization of stable solutions is well known (see for example [2, 22, 37]).

Theorem 3.1. *Let u be a stable solution of* (2.1) *in $\mathbb{R}^2$, then there exists a unit vector $\xi \in \mathbb{R}^2$ and $t \in \mathbb{R}$ such that $u(x) \equiv g(x \cdot \xi - t), \quad \forall x \in \mathbb{R}^2$.*

Since u has finite Morse index, u is stable outside a compact set. As a consequence we then obtain

Lemma 3.2. *For any $b \in (0,1)$, there exist $c(b) > 0$, $R(b) > 0$ such that, for $x \in \{|u| \leq 1-b\} \setminus B_{R(b)}(0)$,*

$$|\nabla u(x)| \geq c(b).$$

Proof. If the claim were false, there would exist a sequence of $x_i \in \{|u| \leq 1-b\}$, $x_i \to \infty$, but

$$|\nabla u(x_i)| \to 0. \tag{3.1}$$

Let $u_i(x) := u(x_i + x)$. By standard elliptic estimates and the Arzela-Ascoli theorem, up to a subsequence, u_i converges to a limit u_∞ in $C^2_{loc}(\mathbb{R}^2)$. Because u is stable outside a compact set, u_∞ is stable in $\mathbb{R}^2$. Then by Theorem 3.1, u_∞ is one dimensional. In particular, $|\nabla u_\infty| \neq 0$ everywhere. However, by passing to the limit in (3.1), we get

$$|\nabla u_\infty(0)| = \lim_{i\to+\infty} |\nabla u_i(0)| = \lim_{i\to+\infty} |\nabla u(x_i)| = 0.$$

This is a contradiction. □

The proof also shows that u is close to one dimensional solutions at infinity.

The following lemma shows that the nodal set $\{u = 0\}$ cannot be contained in any bounded set.

Lemma 3.3. *For any solution of (2.1) in $\mathbb{R}^2$ with finite Morse index, if u is not constant, then $\{u = 0\}$ is unbounded.*

Proof. Assume by the contradiction, $u > 0$ outside a ball $B_R(0)$. By Lemma 3.2 and the fact that the only positive solution to the one dimensional Allen-Cahn equation is the constant function 1, we see $u(x) \to 1$ uniformly as $|x| \to +\infty$. For u near 1, $W'(u) \leq -c(1-u)$ for some positive constant $c > 0$. Thus by comparison principle we get two constants C and R such that, for any $x \in B_R^c$,

$$u(x) \geq 1 - Ce^{-\frac{|x|-R}{C}}. \tag{3.2}$$

Then by standard elliptic estimates or Modica's estimates ([61]),

$$|\nabla u(x)| \leq \sqrt{2W(u)} \leq Ce^{-\frac{|x|-R}{C}}. \tag{3.3}$$

Recall that the Pohozaev type equality on ball $B_r(0)$ is

$$\int_{B_r} 2W(u) = r\int_{\partial B_r} \frac{1}{2}|\nabla u|^2 + W(u) - \left(\frac{\partial u}{\partial r}\right)^2.$$

For $r > R$, substituting (3.2) and (3.3) into the right hand side, we get

$$\int_{B_r} 2W(u) \leq Cre^{-\frac{r-R}{C}}.$$

Letting $r \to +\infty$ leads to

$$\int_{\mathbb{R}^2} W(u) = 0.$$

Hence either $u \equiv 1$ or $u \equiv -1$. □

If $|\nabla u(x)| \neq 0$, denote

$$\nu(x) := \frac{\nabla u(x)}{|\nabla u(x)|}, \quad \text{and} \quad B(u)(x) = \nabla \nu(x). \tag{3.4}$$

If $|\nabla u(x)| \neq 0$, locally $\{u = u(x)\}$ is a C^2 curve, thus its curvature H is well defined. Then

$$|B(u)(x)|^2 = \frac{|\nabla^2 u(x)|^2 - |\nabla^2 u(x) \cdot \nu(x)|^2}{|\nabla u(x)|^2} = H(x)^2 + |\nabla_T \log |\nabla u(x)||^2, \tag{3.5}$$

where ∇_T is the tangential derivative along the level set of u, see [72, 73].

Corollary 3.4. *For any $b \in (0,1)$, $|B(u)(x)|^2$ is bounded in $(\mathbb{R}^2 \setminus B_{R(b)}(0)) \cap \{|u| \leq 1-b\}$. Moreover, for $x \in (\mathbb{R}^2 \setminus B_{R(b)}(0)) \cap \{|u| \leq 1-b\}$, if $x \to \infty$, $|B(u)(x)|^2 \to 0$.*

Proof. The first claim follows from the fact that $|\nabla^2 u|^2$ is bounded in $\mathbb{R}^2$ and the lower bound on $|\nabla u|$ in Lemma 3.2. The second claim also follows from Lemma 3.2, by noting that for one dimensional solutions $|B(g)| \equiv 0$. □

Now we give the following key estimate on the decay rate on $|B(u)(x)|$ at infinity.

Theorem 3.5. *For any solution of* (2.1) *in* $\mathbb{R}^2$ *with finite Morse index and* $b \in (0,1)$*, there exists a constant* C *such that*

$$|B(u)(x)| \le \frac{C}{|x|}, \quad \text{for } x \in \{|u| \le 1-b\} \cap B_{R(b)}(0)^c.$$

To prove this theorem, we argue by contradiction. Take $\mathcal{X}$ to be the complete metric space $\{|u| \le 1-b\}$ with the extrinsic distance and $\Gamma := \mathcal{X} \cap \overline{B_{R(b)}(0)}$. Assume there exists a sequence of $X_k \in \mathcal{X} \setminus \Gamma$, $|B(X_k)|\text{dist}(X_k, \Gamma) \ge 2k$. By the doubling lemma in [64], there exist $Y_k \in \mathcal{X} \setminus \Gamma$ such that

$$|B(Y_k)| \ge |B(X_k)|, \qquad |B(Y_k)|\text{dist}(Y_k, \Gamma) \ge 2k,$$
$$|B(Z)| \le 2|B(Y_k)| \qquad \text{for} \quad Z \in B_{k|B(Y_k)|^{-1}}(Y_k).$$

Let $\varepsilon_k := |B(Y_k)|$ and define $u_k(x) := u(y_k + \varepsilon_k^{-1} x)$. Note that

$$\text{dist}(Y_k, \Gamma) \ge 2k|B(Y_k)|^{-1}. \tag{3.6}$$

By Corollary 3.4, $|Y_k| \to +\infty$ and $\varepsilon_k \to 0$.

In $B_k(0)$, u_k is a solution of (2.3) with the parameter ε_k. By (3.6), u_k is stable in $B_k(0)$.

For $X \in B_k(0) \cap \{|u_k| < 1-b\}$,

$$|B(u_k)(X)| \le 2. \tag{3.7}$$

On the other hand, by the above construction we have

$$|B(u_k)(0)| = 1. \tag{3.8}$$

The bound on $|B_k|$ implies that, for any $X \in \{|u_k| < 1-b\} \cap B_k(0)$, $\{u_k = u_k(X)\} \cap B_{1/8}(X)$ can be represented by the graph of a function with a uniform $C^{1,1}$ bound, cf. [20, Chapter 2, Lemma 2.4].

The following theorem leads to a contradiction with (3.8) and the proof of Theorem 3.5 is thus finished.

Theorem 3.6. *Suppose* u_ε *is a sequence of stable solutions to* (2.3) *in* $\mathcal{C}_1(0)$ *satisfying for some constant* $b \in (0,1)$ *and* $C > 0$ *independent of* ε*,*

$$|B(u_\varepsilon)| \le C, \quad \text{in } \{|u_\varepsilon| < 1-b\} \cap B_1(0).$$

Then for all ε *small,*

$$\sup_{\{|u_\varepsilon|<1-b\}\cap B_{1/2}(0)} |B(u_\varepsilon)| \le C\varepsilon^{1/7}.$$

The proof will be postponed to Part III. Here we only note that under the assumptions of this theorem, locally the level set of u_ε is a family of graphs. For example, after a rotation, assume that the connected component of $\{u_\varepsilon = u_\varepsilon(0)\} \cap B_{1/8}(0)$ passing through 0 (denoted by Σ_ε) is represented by the graph $\{x_2 = f_\varepsilon(x_1)\}$, where $f_\varepsilon(0) = f_\varepsilon'(0) = 0$. By the curvature bound (3.7), $|f_\varepsilon''| \le 32$ in $[-1/8, 1/8]$. By these bound, after passing to a subsequence, we can assume f_ε converges to f_∞ in $C^1([-1/8, 1/8])$.

There are two cases.

- **Case 1.** $\{x_2 = f_\varepsilon(x_1)\}$ is an isolated component of $\{u_\varepsilon = u_\varepsilon(0)\}$. In other words, there exists an $h > 0$ independent of ε such that $\{u_\varepsilon = u_\varepsilon(0)\} \cap B_h(0) = \{x_2 = f_\varepsilon(x_1)\}$.
- **Case 2.** There exists a sequence of points on other components of $\{u_\varepsilon = u_\varepsilon(0)\} \cap B_{1/8}(0)$ disjoint from Σ_ε, converging to a point on Σ_ε.

The following simple lemma can be proved by combining the curvature bound (3.7) with the fact that different connected components of $\{u_\varepsilon = u_\varepsilon(0)\}$ are disjoint. (This fact has been used a lot in minimal surface theory, in particular, in [15].)

Lemma 3.7. *There exist two universal constants* h *and* $C(h)$ *such that if a connected component* Γ *of* $\{u_\varepsilon = u_\varepsilon(0)\} \cap B_{1/8}(0)$ *(other than* Σ_ε*) intersects* $B_h(0)$*, then* $\Gamma \cap B_{2h}(0)$ *can be represented by the graph* $\{x_2 = \widetilde{f}_\varepsilon(x_1)\}$*, where* $\|\widetilde{f}_\varepsilon\|_{C^{1,1}([-2h,2h])} \le C(h)$*.*

Using this lemma, we deduce that under the assumptions in Theorem 3.6, the nodal set of u_ε is given by $\cup_\alpha \{x_2 = f_{\alpha,\varepsilon}(x_1)\}$, where $|f''_{\alpha,\varepsilon}(x_1)| \le 64$ for every α and $x_1 \in (-1, 1)$. Here the cardinality of the index set α could remain uniformly bounded or go to infinite.

In the above we do not use the full power of Theorem 3.6. In fact, we can improve Theorem 3.5 to a higher order decay rate.

Theorem 3.8. *There exists a constant* C *such that*

$$|B(u)(x)| \le \frac{C}{|x|^{8/7}}, \quad \text{for } x \in \{|u| < 1-b\} \cap B_{R(b)}(0)^c.$$

Proof. Take an arbitrary sequence $X_k \in \{|u| < 1-b\} \to \infty$. Denote $\varepsilon_k := |B(u)(X_k)|$, which converges to 0 as $k \to \infty$. Let $u_k(x) := u(X_k + \varepsilon_k^{-1}x)$, which is a solution of (2.3) with parameter ε_k.

By Theorem 3.5, $\varepsilon_k |X_k| \le C$ and for any $x \in \{|u| < 1-b\} \cap B_{|X_k|/2}(X_k)$,

$$|B(u)(x)| \le \frac{C}{|x|} \le \frac{2C}{|X_k|} \le 2C\varepsilon_k.$$

Thus after a scaling, there exists a constant $\rho \in (0, 1/2)$ independent of k such that in $B_\rho(0)$, u_k satisfies the assumptions of Theorem 3.6. Note that for all k large, u is stable in $B_{|X_k|/2}(X_k)$. Hence u_k is stable in $B_\rho(0)$. Applying Theorem 3.6 gives $|B(u_k)(0)| \le C\varepsilon_k^{1/7}$. Rescaling back we get the desired bound on $|B(u)(X_k)|$. □

4. Lipschitz regularity of nodal sets at infinity

First using Theorem 3.8 and proceeding as in [77], we can show that there are at most finitely many connected components of $\{u = 0\}$. This is achieved by choosing the smallest ball centered at the origin which contains a bounded connected component of $\{u = 0\}$ and comparing their curvatures at the contact point.

In the following we take a constant $R_0 > R(1/2)$ so that u is stable outside $B_{R_0}(0)$. We first give a chord-arc bound on $\{u = 0\}$.

Lemma 4.1. *Let Σ be a unbounded connected component of $\{u = 0\} \setminus B_{R_0}(0)$ and $X(t)$ be an arc length parametrization of Σ, where $t \in [0, +\infty)$. Then there exists a constant c such that for any t large,*

$$|X(t)| \ge c|t|.$$

Proof. Because Σ is a smooth embedded curve diffeomorphic to $[0, +\infty)$, if $t \to +\infty$, $|X(t)| \to +\infty$.

By direct differentiation and applying Theorem 3.8, we obtain

$$\frac{d^2}{dt^2}\big|X(t)\big|^2 = 2\Big|\frac{dX}{dt}\Big|^2 + 2X(t) \cdot \frac{d^2X}{dt^2}(t) \ge 2 - \frac{C}{|X(t)|^{1/8}} \ge 1,$$

for all t large. Integrating this differential inequality we finish the proof. □

Keeping assumptions as in this lemma, we can further show that

Proposition 4.2. *The limit*

$$e_\infty := \lim_{t \to +\infty} X'(t)$$

exists. Moreover, for all t large,

$$|X'(t) - e_\infty| \le \frac{C}{t^{1/7}}.$$

Proof. Combining the previous lemma with Theorem 3.8 we obtain

$$|X''(t)| \le \frac{C}{t^{8/7}}.$$

Integrating this in t we finish the proof. □

The direction e_∞ obtained in this proposition is called the limit direction of the connected component Σ.

5. Energy growth bound: Proof of Theorem 2.2

First using the stability of u outside $B_{R_0}(0)$, we study the structure of nodal set of direction derivatives of u at infinity. The following method can be compared with those in [22, 68].

Proposition 5.1. *For any unit vector e, every connected component of $\{u_e := e \cdot \nabla u \neq 0\}$ intersects with $B_{R_0}(0)$.*

Proof. Assume by the contrary there exists a unit vector e and a connected component Ω of $\{u_e \neq 0\}$ contained in $B_{R_0}(0)^c$. Let ψ be the restriction of $|u_e|$ to Ω, with zero extension outside it. Hence ψ is continuous, and in Ω it satisfies the linearized equation

$$\Delta \psi = W''(u)\psi. \tag{5.1}$$

For any $R > R_0$, let

$$\eta_R(x) := \begin{cases} 1, & x \in B_R(0), \\ 2 - \frac{\log|x|}{\log R}, & x \in B_{R^2}(0) \setminus B_R(0), \\ 0, & x \in B_{R^2}(0)^c. \end{cases}$$

Multiplying (5.1) by $\psi\eta_R^2$ and integrating by parts leads to

$$\int_{\mathbb{R}^2} |\nabla (\psi\eta_R)|^2 + W''(u)(\psi\eta_R)^2 = \int_{\mathbb{R}^2} \psi^2|\nabla\eta_R|^2 \le \frac{C}{\log R}, \tag{5.2}$$

where we have used the fact that $|\psi| \le |\nabla u| \le C$.

Take an $X \in \partial\Omega$ such that $\partial\Omega$ is smooth in a neighborhood of X. By a suitable compact modification of ψ in a small ball $B_h(X)$, we get a new function $\tilde{\psi}$ and a constant $\delta > 0$ so that

$$\int_{B_h(X)} \frac{1}{2}|\nabla\tilde{\psi}|^2 + W''(u)\tilde{\psi}^2 \le \left[\int_{B_h(X)} \frac{1}{2}|\nabla\psi|^2 + W''(u)\psi^2\right] - \delta. \tag{5.3}$$

Combining (5.2) and (5.3) we get an R such that

$$\int_{\mathbb{R}^2} |\nabla\left(\tilde{\psi}\eta_R\right)|^2 + W''(u)\left(\tilde{\psi}\eta_R\right)^2 < 0.$$

This is a contradiction with the stability condition of u outside $B_{R_0}(0)$. □

The following finiteness result on the ends of u can be proved by the same method in [77], using Proposition 5.1 and Proposition 4.2.

Proposition 5.2. *By taking a large enough $R_1 > 0$, there are only finitely many connected components of $\{u = 0\} \cap B_{R_1}(0)^c$.*

The main idea is as follows.

(i) By choosing a generic direction e, using Proposition 4.2 we can show that for each connected component of $\{u = 0\} \cap B_{R_1}(0)^c$, u_e has fixed sign in an $O(1)$ neighborhood of it.

(ii) If two connected components of $\{u = 0\} \cap B_{R_1}(0)^c$ are neighboring and the angle between their limit directions are small, u_e has different sign near these two connected components.

(iii) If there are too many connected components of $\{u = 0\} \cap B_{R_1}(0)^c$, we can construct as many connected components of $\{u_e \neq 0\} \cap B_{R_1}(0)^c$ as we want. On the other hand, by Proposition 5.1, the number of connected components of $\{u_e \neq 0\} \cap B_{R_1}(0)^c$ is controlled by the number of connected components of $\{u_e \neq 0\} \cap \partial B_{R_1}(0)$. This leads to a contradiction.

With this proposition in hand, we can proceed as in [40, 77] to obtain the linear energy growth bound in Theorem 2.2. The main idea is to divide $\mathbb{R}^2 \setminus B_{R_0}(0)$ into a number of cones with their angles strictly smaller than π and $\{u = 0\}$ is strictly contained in the interior of these cones, and then apply the Hamiltonian identity of Gui [40] in these cones separately.

Once we have this linear energy growth bound, there are many ways to show that the solution has finitely many ends in the sense of [23] and the refined asymptotic behavior of u at infinity, see for example [23, 30, 40, 78].

6. Morse index 1 solutions: Proof of Theorem 2.3

In this section we study solutions with Morse index 1 in detail. We use nodal set information to show that these solutions have only one critical point of saddle type.

First we establish a general estimate on the number of nodal domains for direction derivatives of u, in terms of the Morse index bound.

Proposition 6.1. *Suppose the Morse index of u equals N. For any unit vector e, the number of connected components of $\{u_e \neq 0\}$ is not larger than $2N$.*

Proof. First recall some basic facts about the nodal set $\{u_e = 0\}$ (see [7]). Because u_e satisfies the linearized equation (5.1), it can be decomposed into $\text{sing}(u_e) \cup \text{reg}(u_e)$, where $\text{sing}(u_e)$ consists of isolated points and $\text{reg}(u_e)$ is a family of embedded smooth curves with their endpoints in $\text{sing}(u_e)$ or at infinity.

Assume by the contrary, the number of connected components of $\{u_e \neq 0\}$ is larger than $2N$. Without loss of generality, assume $\{u_e > 0\}$ has at least $N+1$ connected components, $\Omega_i, i = 1, \cdots, N+1$. By the strong maximum principle, $u_e > 0$ on the other side of regular parts of $\partial\Omega_i$.

Let ψ_i be the restriction of $|u_e|$ to Ω_i, with zero extension outside Ω_i. Hence ψ_i is continuous, and in $\{\psi_i > 0\}$, it satisfies the linearized equation (5.1).

For any $R > R_0$, choose the cut-off function η_R as in the previous section. Multiplying (5.1) by $\psi_i\eta_R^2$ and integrating by parts on Ω_i leads to

$$\int_{\mathbb{R}^2} |\nabla(\psi_i\eta_R)|^2 + W''(u)(\psi_i\eta_R)^2 = \int_{\mathbb{R}^2} \psi_i^2|\nabla\eta_R|^2 \leq \frac{C}{\log R}. \tag{6.1}$$

Take an x_i belonging to the regular part of $\partial\Omega_i$. There exists $h_i > 0$ so that $B_{h_i}(x_i)$ is disjoint from Ω_j, for any $j \neq i$. (For example, $u_e < 0$ in $B_{h_i}(x_i)\setminus\Omega_i$.) Let $\tilde{\psi}_i$ equal ψ_i outside $B_{h_i}(x_i)$, while in $B_{h_i}(x_i)$ it solves (5.1). By this choice, we get a constant $\delta_i > 0$ such that

$$\int_{B_{h_i}(x_i)} \frac{1}{2}|\nabla\tilde{\psi}_i|^2 + W''(u)\tilde{\psi}_i^2 \leq \left[\int_{B_{h_i}(x_i)} \frac{1}{2}|\nabla\psi_i|^2 + W''(u)\psi_i^2\right] - \delta_i. \tag{6.2}$$

Combining (6.1) and (6.2) we get an R such that

$$\int_{\mathbb{R}^2} |\nabla\left(\tilde{\psi}_i\eta_R\right)|^2 + W''(u)\left(\tilde{\psi}_i\eta_R\right)^2 < 0, \quad \forall i = 1,\cdots,N+1. \tag{6.3}$$

Note that $\tilde{\psi}_i\eta_R \in H_0^1(B_R)$ are continuous functions satisfying

$$\tilde{\psi}_i\eta_R\tilde{\psi}_j\eta_R \equiv 0, \quad \forall 1 \leq i \neq j \leq N+1.$$

Hence they form an orthogonal basis of an $(N+1)$-dimensional subspaces of $H_0^1(B_R)$. By (6.3), $\mathcal{Q}$ is negative definite on this subspace. This is a contradiction with the Morse index bound on u. □

Remark 6.2. *It seems to be more interesting to establish a relation between the number of ends and the Morse index, as in minimal surfaces* [12, 45, 65]. *Recently Mantoulidis* [59]*, by a combinatorial analysis of the nodal domain structure of u_e, showed that the number of ends is at most 2N+2.*

As a corollary we get

Corollary 6.3. *Given a solution u with Morse index* 1*, for any direction e, the nodal set $\{u_e = 0\}$ is a single smooth curve. In particular, $\nabla u_e \neq 0$ on $\{u_e = 0\}$.*

Proof. First recall that $\text{reg}(u_e)$ are smooth embedded curves where $\nabla u_e \neq 0$, and $\text{sing}(u_e) = \{u_e = 0, \nabla u_e = 0\}$. Moreover, for any $X \in \{u_e = 0, \nabla u_e = 0\}$, in a neighborhood of X, $\{u_e = 0\}$ consists of at least 4 smooth curves emanating from X. See [7]. Hence if there is a singular point on $\{u_e = 0\}$, by Jordan curve theorem there exist at least three connected components of $\{u_e \neq 0\}$, a contradiction with Proposition 6.1. Therefore there is no singular point on $\{u_e = 0\}$ and they are smooth curves.

If there are two connected components of $\{u_e = 0\}$, they are smooth, properly embedded curves. Hence they are either closed or unbounded. By Jordan curve theorem, there are at least three components of $\{u_e \neq 0\}$, still a contradiction with Proposition 6.1. □

Corollary 6.4. *Given a solution u with Morse index* 1*, any critical point of u is nondegenerate.*

Proof. Suppose X is a critical point of u. For any direction e, we have $u_e(X) = \nabla u(X)\cdot e = 0$, that is, $X \in \{u_e = 0\}$. By the previous corollary, $\nabla^2 u(X)\cdot e = \nabla u_e(X) \neq 0$. Since e is arbitrary, this means $\nabla^2 u(X)$ is invertible. □

Denote

$$P := W(u) - \frac{1}{2}|\nabla u|^2. \tag{6.4}$$

By the Modica's inequality [61], $P > 0$ in $\mathbb{R}^2$. By the proof of Lemma 3.2, we also have

$$\lim_{|X|\to+\infty} P(X) = 0.$$

Lemma 6.5. $\nabla P = 0$ *if and only if* $\nabla u = 0$.

Proof. Since

$$\nabla P = W'(u)\nabla u - \nabla^2 u \cdot \nabla u,$$

we see that $\nabla P = 0$ if $\nabla u = 0$.

On the other hand assume that $\nabla u(X) \neq 0$. Without loss of generality, take two orthonormal basis $\{e_1, e_2\}$ and assume $u_{e_2}(X) = |\nabla u(X)|$, $u_{e_1}(X) = 0$. Note that locally $\{u_{e_1}/u_{e_2} = 0\}$ coincides with

$\{u_{e_1} = 0\}$, which is a smooth curve by Corollary 6.3. Since both u_{e_1} and u_{e_2} satisfy the linearized equation (5.1), we infer that

$$\mathrm{div}\left(u_{e_2}^2 \nabla \frac{u_{e_1}}{u_{e_2}}\right) = 0,$$

which implies that $\nabla \frac{u_{e_1}}{u_{e_2}}(X) \neq 0$.

By a direct calculation we get

$$\nabla P = u_{e_2}^2 J \nabla \frac{u_{e_1}}{u_{e_2}},$$

where J is the $\pi/2$-rotation in the anti-clockwise direction. Therefore $\nabla P(X) \neq 0$. □

At a critical point of P, since $\nabla u = 0$, we have

$$\nabla^2 P = W'(u)\nabla^2 u - \nabla^2 u \cdot \nabla^2 u = \Delta u \nabla^2 u - \nabla^2 u \cdot \nabla^2 u,$$

where $\cdot$ denotes matrix multiplication. Since $\nabla^2 u$ is invertible at this point, by a direct calculation we see both of the eigenvalues of $\nabla^2 P$ equal $\det \nabla^2 u$. Thus every critical point of P is either a strict maximal or a strict minimal point.

Proposition 6.6. *There is only one critical point of P.*

Proof. Since $P > 0$ and $P \to 0$ at infinity, the maxima of P is attained, which is a critical point of P. Denote this point by X_1.

Assume there exists a second critical point of P, X_2. By the previous analysis, X_2 is either a strict maximal or minimal point.

Case 1. If X_2 is a strict maximal point, take

$$\Upsilon := \{\gamma \in H^1([0,1], \mathbb{R}^2) : \gamma(0) = X_1, \gamma(1) = X_2\}.$$

Define

$$c_* := \max_{\gamma \in \Upsilon} \min_{t \in [0,1]} P(\gamma(t)).$$

Clearly $c_* < \min\{u(X_1), u(X_2)\}$. Since $P \to 0$ at infinity, by constructing a competitor curve, we see $c_* > 0$. By the Mountain Pass Theorem, c_* is a critical value of P. Moreover, there exists a curve $\gamma_* \in \Upsilon$ and $t_* \in (0,1)$ such that

$$P(\gamma_*(t_*)) = \min_{t \in [0,1]} P(\gamma_*(t)) = c_*$$

and $\nabla P(\gamma_*(t_*)) = 0$. Therefore $\gamma_*(t_*)$ cannot be a strict local maxima. If it is a strict local minima, by deforming γ_* in a small neighborhood of $\gamma_*(t_*)$, we get a contradiction with the definition of c_*. This contradiction implies that X_2 cannot be a strict maximal point of P.

Case 2. If X_2 is a strict local minimal point, take

$$\Upsilon := \{\gamma \in H^1([0,+\infty), \mathbb{R}^2) : \gamma(0) = X_2, \lim_{t \to +\infty} \gamma(t) = +\infty\}.$$

Define

$$c_* := \min_{\gamma \in \Upsilon} \max_{t \in [0,+\infty)} P(\gamma(t)).$$

As in the first case we get a critical point of P, which is of mountain pass type. This leads to the same contradiction as before.

These contradictions show that X_1 is the only critical point of P. □

By Lemma 6.5, u has only one critical point, too. Denote this point by X. Since this point is the maximal point of P, $\det \nabla^2 u(X) < 0$. Thus it is a nondegenerate saddle point of u.

Remark 6.7. *Let $\Psi := g^{-1} \circ u$ be the distance type function. The Modica inequality* [61] *is equivalent to the condition that $|\nabla \Psi| < 1$. The above method can be further developed to show that $\nabla \Psi$ is a diffeomorphism from $\mathbb{R}^2$ to $B_1(0)$. In particular, for any $r > 0$,*

$$deg\left(\frac{\nabla u}{|\nabla u|}, \partial B_r(X)\right) = 1 \quad or \ -1,$$

Compare this with [9].

Lemma 6.8. *$\{u = u(X)\}$ is composed by two smooth curves diffeomorphic to $\mathbb{R}$, intersecting exactly at X.*

Proof. Since X is the only critical point of u, $\{u = u(X)\}$ is a smooth embedded curve outside X. Because X is nondegenerate and of saddle type, in a small neighborhood of X this level set consists of two smooth curves intersecting transversally at X.

Denote this connected component of $\{u = u(X)\}$ by Σ. Σ does not enclose any bounded domain, because otherwise u would have a local maximal or minimal point in this bounded domain, which is a contradiction with Proposition 6.6. Hence we can write $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are smooth properly embedded curves diffeomorphic to $\mathbb{R}$. Moreover, Σ_1 and Σ_2 intersect at and only at X.

If there exists a second connected component of $\{u = u(X)\}$. Denote it by $\widetilde{\Sigma}$. Similar to the above discussion, $\widetilde{\Sigma}$ is a smooth embedded curve diffeomorphic to $\mathbb{R}$. $\widetilde{\Sigma}$ and Σ bound a domain Ω. Without loss of generality, assume $u > u(X)$ in Ω.

Let

$$\Upsilon := \{\gamma \in H^1([0,1], \mathbb{R}^2) : \gamma(0) \in \Sigma, \gamma(1) \in \widetilde{\Sigma}\},$$

and

$$c_* := \min_{\gamma \in \Upsilon} \max_{t \in [0,1]} u(\gamma(t)).$$

By choosing a competitor curve, we see $c_* < 1$. Hence by Lemma 3.2, c_* is attained by a curve $\gamma_* \in \Upsilon$. Because Σ and $\widetilde{\Sigma}$ are separated,

$$\max_{t \in [0,1]} u(\gamma_*(t)) > u(X).$$

By the Mountain Pass Theorem, there exists $t_* \in (0,1)$ such that $u(\gamma_*(t_*)) = c_*$ and $\gamma_*(t_*)$ is a critical point of u. This is a contradiction with Proposition 6.6. Therefore $\{u = u(X)\} = \Sigma$. □

Combining this lemma with Theorem 2.2, we see there are exactly four ends of u. This completes the proof of Theorem 2.3.

Part 2. Second order estimate on interfaces

In this part we study second order regularity of clustering interfaces and prove Theorem 2.6 and Theorem 2.7. Recall that u_ε is a sequence of solutions to (2.3) satisfying (**H1**) – (**H3**) in Section 2.2.

7. The case of unbounded curvatures

By standard elliptic regularity theory, $u_\varepsilon \in C^4_{loc}(\mathcal{C}_2)$. Concerning the regularity of $f_{\alpha,\varepsilon}$, we first prove that different components are at least $O(\varepsilon)$ apart.

Lemma 7.1. *For any $\alpha \in \{1, \cdots, Q\}$ and $x_\varepsilon \in \Gamma_{\alpha,\varepsilon} \cap \mathcal{C}_{3/2}$, as $\varepsilon \to 0$, $\tilde{u}_\varepsilon(x) := u_\varepsilon(x_\varepsilon + \varepsilon x)$ converges to a one dimensional solution in $C^2_{loc}(\mathbb{R}^n)$. In particular, for any $\alpha \in \{1, \cdots, Q\}$,*

$$\frac{f_{\alpha+1,\varepsilon} - f_{\alpha,\varepsilon}}{\varepsilon} \to +\infty \quad \text{uniformly in } B^{n-1}_{3/2}. \tag{7.1}$$

Proof. In $B_{\varepsilon^{-1}/2}$, $\tilde{u}_\varepsilon(x)$ satisfies the Allen-Cahn equation (2.1). By standard elliptic regularity theory, $\tilde{u}_\varepsilon(x)$ is uniformly bounded in $C^{2,\theta}_{loc}(\mathbb{R}^n)$. Using Arzela-Ascoli theorem, as $\varepsilon \to 0$, it converges to a limit function u_∞ in $C^2_{loc}(\mathbb{R}^n)$. For each $\beta \in \{1, \cdots, Q\}$, either $\left(f_{\beta,\varepsilon}(x'_* + \varepsilon x') - f_{\alpha,\varepsilon}(x'_*)\right)/\varepsilon$ converges to a limit function $f_{\beta,\infty}$ in $C_{loc}(\mathbb{R}^{n-1})$ or it converges to $\pm\infty$ uniformly on any compact set of $\mathbb{R}^{n-1}$.

Assume $t_\varepsilon \to t_\infty$. Then $\{u_\infty = t_\infty\}$ consists of $Q' \le Q$ connected components, $\Gamma_{\alpha,\infty}$, $1 \le \alpha \le Q'$. Each $\Gamma_{\alpha,\infty}$ is represented by the graph $\{x_n := f_{\alpha,\infty}(x')\}$. In $\mathbb{R}^{n-1}$, $|\nabla f_{\alpha,\infty}| \le C$ for a universal constant C and $f_{1,\infty} \le \cdots \le f_{Q',\infty}$.

By applying the sliding method in [6], $u_\infty(x) = g(x \cdot e)$ for some unit vector e. In particular, $Q' = 1$ and for any $\beta \ne \alpha$, $\left(f_{\beta,\varepsilon}(x'_* + \varepsilon x') - f_{\alpha,\varepsilon}(x'_*)\right)/\varepsilon$ goes to $\pm\infty$ uniformly on any compact set of $\mathbb{R}^{n-1}$. □

A consequence of this lemma is

Corollary 7.2. *Given a constant $b \in (0,1)$,*

(i) *there exists a constant $c(b) > 0$ depending only on b such that*

$$\frac{\partial u_\varepsilon}{\partial x_n} > \frac{c(b)}{\varepsilon}, \quad \text{in } \{|u_\varepsilon| < 1 - b\} \cap \mathcal{C}_{3/2};$$

(ii) *for any $t \in [-1+b, 1-b]$ and all ε small, $\{u_\varepsilon = t\}$ is composed by Q Lipschitz graphs*

$$\{x_n = f^t_{\alpha,\varepsilon}(x')\}, \quad \alpha = 1, \cdots, Q.$$

By the implicit function theorem, $f_{\alpha,\varepsilon}$ belongs to $C^{2,\theta}_{loc}(B_2^{n-1})$, although we do not have any uniform bound on their $C^{2,\theta}$ norm but only a uniform Lipschitz bound.

Now

$$\nu_\varepsilon(x) := \frac{\nabla u_\varepsilon(x)}{|\nabla u_\varepsilon(x)|}$$

is well defined and smooth in $\{|u_\varepsilon| \leq 1-b\}$. Recall that $B(u_\varepsilon)(x) = \nabla \nu_\varepsilon(x)$. We have

$$|B(u_\varepsilon)(x)|^2 = |A_\varepsilon(x)|^2 + |\nabla_T \log |\nabla u_\varepsilon(x)||^2,$$

where $A_\varepsilon(x)$ is the second fundamental form of the level set $\{u_\varepsilon = u_\varepsilon(x)\}$ and ∇_T denotes the tangential derivative along the level set $\{u_\varepsilon = u_\varepsilon(x)\}$.

Assume as $\varepsilon \to 0$,

$$\sup_{\mathcal{C}_1 \cap \{|u_\varepsilon| \leq 1-b\}} |B(u_\varepsilon)(x)| \to +\infty.$$

Let $x_\varepsilon \in \mathcal{C}_1 \cap \{|u_\varepsilon| \leq 1-b\}$ attain the following maxima (we denote $x = (x', x_n)$)

$$\max_{\mathcal{C}_{3/2} \cap \{|u_\varepsilon| \leq 1-b\}} \left(\frac{3}{2} - |x'|\right) |B(u_\varepsilon)(x)|. \tag{7.2}$$

Denote

$$L_\varepsilon := |B(u_\varepsilon)(x_\varepsilon)|, \qquad r_\varepsilon := \left(\frac{3}{2} - |x'_\varepsilon|\right)/2. \tag{7.3}$$

Then by definition

$$L_\varepsilon r_\varepsilon \geq \frac{1}{2} \sup_{\mathcal{C}_1 \cap \{|u_\varepsilon| \leq 1-b\}} |B(u_\varepsilon)(x)| \to +\infty. \tag{7.4}$$

In particular, $L_\varepsilon \to +\infty$.

By the choice of r_ε at (7.3), we have (here $\mathcal{C}_{r_\varepsilon}(x'_\varepsilon) := B^{n-1}_{r_\varepsilon}(x'_\varepsilon) \times (-1,1)$)

$$\max_{x \in \mathcal{C}_{r_\varepsilon}(x'_\varepsilon) \cap \{|u_\varepsilon| \leq 1-b\}} |B(u_\varepsilon)(x)| \leq 2L_\varepsilon. \tag{7.5}$$

Let $\epsilon := L_\varepsilon \varepsilon$ and define $u_\epsilon(x) := u_\varepsilon(x_\varepsilon + L_\varepsilon^{-1} x)$. Then u_ϵ satisfies (2.3) with parameter ϵ in $B_{L_\varepsilon r_\varepsilon}(0)$. For any $t \in [-1+b, 1-b]$, the level set $\{u_\epsilon = t\}$ consists of Q Lipschitz graphs

$$\left\{x_n = f^t_{\beta,\epsilon}(x') := L_\varepsilon \left[f^t_{\beta,\varepsilon}(x'_\varepsilon + L_\varepsilon^{-1} x') - f^t_{\alpha,\varepsilon}(x'_\varepsilon)\right]\right\},$$

where α is chosen so that x_ε lies in the connected component of $\{|u_\varepsilon| \leq 1-b\}$ containing $\Gamma_{\alpha,\varepsilon}$.

By (7.5), we also have

$$|B(u_\epsilon)| \leq 2, \quad \text{for } x \in \mathcal{C}_1 \cap \{|u_\epsilon| \leq 1-b\}.$$

Without loss of generality, by abusing notations, we will assume in the following

(H4) There exist two constants $b \in (0,1)$ and $C > 0$ independent of ε such that $|B(u_\varepsilon)| \leq C$ for any $x \in \mathcal{C}_2 \cap \{|u_\varepsilon| \leq 1-b\}$.

8. Fermi coordinates

8.1. **Definition.** For simplicity of presentation, we now work in the stretched version and do not write the dependence on ε explicitly.

By denoting $R = \varepsilon^{-1}$, $u(x) = u_\varepsilon(\varepsilon x)$ satisfies the Allen-Cahn equation (2.1) in $\mathcal{C}_{2R} := B^{n-1}_{2R} \times (-R, R)$.

Its nodal set $\{u = 0\}$ consists of Q connected components, Γ_α, $1 \leq \alpha \leq Q$, which is represented by the graph $\{x_n := f_\alpha(x')\}$. In B^{n-1}_{2R}, there is a constant C independent of ε such that

$$|\nabla f_\alpha| \leq C, \quad |\nabla^2 f_\alpha| \leq C\varepsilon. \tag{8.1}$$

By **(H2)**, $-R/2 < f_1 < \cdots < f_Q < R/2$.

The second fundamental form of Γ_α with respect to the parametrization $y \mapsto (y, f_\alpha(y))$ is given by

$$A_{ij}(y,0) = \frac{1}{\sqrt{1+|\nabla f_\alpha(y)|^2}} \frac{\partial}{\partial y_i}\left[\frac{1}{\sqrt{1+|\nabla f_\alpha(y)|^2}} \frac{\partial f_\alpha}{\partial y_j}(y)\right].$$

The Fermi coordinate is defined by $(y,z) \mapsto x$ as $x = (y, f_\alpha(y)) + zN_\alpha(y)$, where

$$N_\alpha(y) = \frac{(-\nabla' f_\alpha(x'), 1)}{\sqrt{1+|\nabla' f_\alpha(x')|^2}}.$$

Note that here z is nothing else but the signed distance to Γ_α which is positive above Γ_α. By (8.1), there exists a constant $\delta \in (0, 1/2)$ independent of ε such that, the Fermi coordinate is well defined and C^4 in the open set $\{|y| < 3R/2, |z| < \delta R\}$.

Define the vector field

$$X_i := \frac{\partial}{\partial y^i} + z\frac{\partial N_\alpha}{\partial y^i} = \sum_{j=1}^{n-1} (\delta_{ij} - zA_{ij}) \frac{\partial}{\partial y^j}.$$

For any $z \in (-\delta R, \delta R)$, let $\Gamma_{\alpha,z} := \{dist(x, \Gamma_\alpha) = z\}$. The Euclidean metric restricted to $\Gamma_{\alpha,z}$ is denoted by $g_{ij}(y, z)dy^i \otimes dy^j$, where

$$\begin{aligned} g_{ij}(y,z) &= < X_i(y,z), X_j(y,z) > \\ &= g_{ij}(y,0) - 2z\sum_{k=1}^{n-1} A_{ik}(y,0)g_{jk}(y,0) + z^2 \sum_{k,l=1}^{n} g_{kl}(y,0)A_{ik}(y,0)A_{jl}(y,0). \end{aligned} \tag{8.2}$$

Here

$$g_{ij}(y,0) = \frac{1}{1+|\nabla f_\alpha(y)|^2}\left[\delta_{ij} + \frac{\partial f_\alpha}{\partial y_i}(y)\frac{\partial f_\alpha}{\partial y_j}(y)\right]. \tag{8.3}$$

The second fundamental form of Γ_z has the form

$$A(y,z) = (I - zA(y,0))^{-1} A(y,0). \tag{8.4}$$

8.2. **Error in z.** In this subsection we collect several estimates on the error of various terms in z. Recall that ε is the upper bound on curvatures of level sets of u, see (8.1).

By (8.1), $|A(y, 0)| \lesssim \varepsilon$. Thus for $|z| < \delta R$, $|A(y, z)| \lesssim \varepsilon$. We also have

Lemma 8.1. *In* $B^{n-1}_{3R/2}$,

$$|\nabla A(y,0)| + |\nabla^2 A(y,0)| \lesssim \varepsilon. \tag{8.5}$$

Proof. By Corollary 7.2, $|\nabla u| \geq c(b) > 0$ in $\{|u| < 1 - b\}$, where $c(b)$ is a constant depending only on b. Hence $\nu = \nabla u/|\nabla u|$ is well defined and smooth in $\{|u| < 1 - b\}$.

By direct calculation, we have

$$-\text{div}\left(|\nabla u|^2 \nabla \nu\right) = |\nabla u|^2 |\nabla \nu|^2 \nu. \tag{8.6}$$

Recall that $B = \nabla\nu$. Differentiating (8.6) gives the following Simons type equation

$$-\text{div}\left(|\nabla u|^2 \nabla B\right) = |\nabla u|^2|B|^2 B + |\nabla u|^2 \nabla |B|^2 \otimes \nu + |B|^2 \nabla |\nabla u|^2 \otimes \nu + |\nabla u|^2 \nabla^2 \log |\nabla u|^2 \cdot B. \tag{8.7}$$

For any $x \in \{|u| < 1 - 2b\}$, there exists a constant $h(b)$ such that $B_{2h(b)}(x) \subset \{|u| < 1 - b\}$. Because $|\nabla u|^2$ has a positive lower and upper bound and it is uniformly continuous in $B_{2h(b)}(x)$, by standard interior gradient estimate,

$$\sup_{B_{h(b)}} |\nabla B| \lesssim \sup_{B_{2h(b)}} |B| + \sup_{B_{2h(b)}} |\text{div}\left(|\nabla u|^2 \nabla B\right)| \lesssim \varepsilon.$$

The bound on $|\nabla^2 B|$ is obtained by bootstrapping elliptic estimates. □

By (8.4),

$$|A(y,z) - A(y,0)| \lesssim |z||A(y,0)|^2 \lesssim \varepsilon^2|z|. \tag{8.8}$$

Similarly, by (8.2), the error of metric tensors is

$$|g_{ij}(y,z) - g_{ij}(y,0)| \lesssim \varepsilon|z|, \tag{8.9}$$

$$|g^{ij}(y,z) - g^{ij}(y,0)| \lesssim \varepsilon|z|. \tag{8.10}$$

As a consequence, the error of mean curvature is

$$|H(y,z) - H(y,0)| \lesssim \varepsilon^2|z|. \tag{8.11}$$

By (8.1) and (8.5), for any $|z| < \delta R$,

$$|\nabla_y g_{ij}(y,z)| + |\nabla_y g^{ij}(y,z)| \lesssim \varepsilon. \tag{8.12}$$

The Laplacian operator in Fermi coordinates has the form

$$\Delta_{\mathbb{R}^N} = \Delta_z - H(y,z)\partial_z + \partial_{zz},$$

where

$$\begin{aligned}\Delta_z &= \sum_{i,j=1}^{n-1} \frac{1}{\sqrt{\det(g_{ij}(y,z))}} \frac{\partial}{\partial y_j}\left(\sqrt{\det(g_{ij}(y,z))}g^{ij}(y,z)\frac{\partial}{\partial y_i}\right)\\ &= \sum_{i,j=1}^{n-1} g^{ij}(y,z)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{n-1} b^i(y,z)\frac{\partial}{\partial y_i}\end{aligned}$$

with

$$b^i(y,z) = \frac{1}{2}\sum_{j=1}^{n-1} g^{ij}(y,z)\frac{\partial}{\partial y_j}\log\det(g_{ij}(y,z)).$$

By (8.10) and (8.12), we get

Lemma 8.2. *For any function $\varphi \in C^2(B^{n-1}_{3R/2})$,*

$$|\Delta_z\varphi(y) - \Delta_0\varphi(y)| \lesssim \varepsilon|z|\left(|\nabla^2\varphi(y)| + |\nabla\varphi(y)|\right). \tag{8.13}$$

8.3. **Comparison of distance functions.** For each α, the local coordinates on Γ_α is fixed to be the same one, $y \in B^{n-1}_{3R/2}$, which represents the point $(y, f_\alpha(y))$. The singed distance to Γ_α, which is positive in the above, is denoted by d_α. Given a point X, if $(y, f_\beta(y))$ is the nearest point on Γ_β to X, we then define $\Pi_\beta(X) = y$.

If $\alpha \neq \beta$, we cannot expect $\Pi_\alpha(X) = \Pi_\beta(X)$. However, the following estimates on their distance hold, when Γ_α and Γ_β are close in some sense.

Lemma 8.3. *For any $X \in B^{n-1}_{3R/2}\times(-\delta R, \delta R)$ and $\alpha \neq \beta$, if $|d_\alpha(X)| \le K|\log\varepsilon|$ and $|d_\beta(X)| \le K|\log\varepsilon|$, then we have*

$$dist_{\Gamma_\beta}\left(\Pi_\beta\circ\Pi_\alpha(X), \Pi_\beta(X)\right) \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2}, \tag{8.14}$$

$$|d_\beta\left(\Pi_\alpha(X)\right) + d_\alpha\left(\Pi_\beta(X)\right)| \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2}, \tag{8.15}$$

$$|d_\alpha(X) - d_\beta(X) + d_\beta\left(\Pi_\alpha(X)\right)| \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2}, \tag{8.16}$$

$$|d_\alpha(X) - d_\beta(X) - d_\alpha\left(\Pi_\beta(X)\right)| \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2}, \tag{8.17}$$

$$1 - \nabla d_\alpha(X)\cdot\nabla d_\beta(X) \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2}, \tag{8.18}$$

Proof. We divide the proof into three steps.

Step 1. After a rotation and a translation, assume $\Pi_\alpha(X) = 0$, the tangent plane of Γ_α at $(0,0)$ is the horizontal hyperplane and $X = (0,T)$. Since the curvature of Γ_α is of the order $O(\varepsilon)$, $\Gamma_\alpha \cap \mathcal{C}_{\delta R}$ is a Lipschitz graph $\{x_n = f_\alpha(x')\}$. By the above choice, $f_\alpha(0) = \nabla f_\alpha(0) = 0$.

Because $|d_\beta(0)| \le K|\log\varepsilon|$, Γ_α and Γ_β are disjoint and their curvature is of the order $O(\varepsilon)$, we can show that $\Gamma_\beta \cap \mathcal{C}_{\delta R}$ is also a Lipschitz graph $\{x_n = f_\beta(x')\}$, see Lemma 3.7.

By this Lipschitz property of f_α and f_β,

$$|f_\beta(0) - f_\alpha(0)| \le C|d_\beta(0)| \le C\left(|d_\alpha(X)| + |d_\beta(X)|\right) \le 2CK|\log\varepsilon|.$$

Since $f_\beta - f_\alpha \neq 0$ and $|\nabla^2(f_\beta - f_\alpha)| \lesssim \varepsilon$ in $B^{n-1}_{\delta R}(0)$, by an interpolation inequality we get

$$|\nabla f_\beta(0)| = |\nabla(f_\beta - f_\alpha)(0)| \lesssim \sqrt{\varepsilon|\log\varepsilon|}.$$

Step 2. Because $\Gamma_\beta \cap \mathcal{C}_{2K|\log\varepsilon|}$ belongs to an $O(\varepsilon|\log\varepsilon|^2)$ neighborhood of the hyperplane $\mathcal{P}_\beta := \{x_n = f_\beta(0) + \nabla f_\beta(0)\cdot x'\}$,

$$\begin{aligned} d_\beta(X) &= T - f_\beta(0) + O(\sqrt{\varepsilon|\log\varepsilon|}|T|) + O(\varepsilon|\log\varepsilon|^2) \\ &= T - f_\beta(0) + O(\varepsilon^{1/2}|\log\varepsilon|^{3/2}). \end{aligned} \tag{8.19}$$

Similarly,

$$d_\beta(\Pi_\alpha(X)) = f_\alpha(0) - f_\beta(0) + O(\varepsilon^{1/2}|\log\varepsilon|^{3/2}). \tag{8.20}$$

Interchanging the position of α, β gives

$$d_\alpha(\Pi_\beta(X)) = f_\beta(0) - f_\alpha(0) + O(\varepsilon^{1/2}|\log\varepsilon|^{3/2}). \tag{8.21}$$

Combining (8.19), (8.20) and (8.21), we obtain (8.15)-(8.17).

Step 3. In our setting, we have

$$1 - \nabla d_\alpha(X)\cdot\nabla d_\beta(X) = 1 - \frac{\partial}{\partial x_n}d_\beta(0,T).$$

For any $t \in (0, f_\beta(0))$, let $(x'(t), f_\beta(x'(t))$ be the unique nearest point on Γ_β to $(0,t)$. By definition, we have

$$x'(t) + [f_\beta(x'(t)) - t]\,\nabla f_\beta(x'(t)) = 0. \tag{8.22}$$

Differentiating this identity in t leads to

$$\left[1 + |\nabla f_\beta(x'(t))|^2\right] \frac{d}{dt}x'(t) + [f_\beta(x'(t)) - t]\,\nabla^2 f_\beta(x'(t)) \cdot \frac{d}{dt}x'(t) = \nabla f_\beta(x'(t)).$$

As in Step 1, we still have $|\nabla f_\beta(x'(t))| \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{1/2}$. Together with the fact that $|\nabla^2 f_\beta| \lesssim \varepsilon$, we get

$$\left|\frac{d}{dt}x'(t)\right| \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{1/2}. \tag{8.23}$$

Integrating this in t on $[0,T]$ gives (8.14).

Note that (8.22) also implies that

$$|x'(t)| \le C(K)\varepsilon^{1/2}|\log\varepsilon|^{3/2}. \tag{8.24}$$

Because

$$d_\beta(0,t) = \sqrt{|x'(t)|^2 + (f_\beta(x'(t)) - t)^2},$$

we have

$$\begin{aligned}
\frac{d}{dt}d_\beta(0,t) &= -\frac{f_\beta(x'(t)) - t}{\sqrt{|x'(t)|^2 + (f_\beta(x'(t)) - t)^2}} + \frac{x'(t) + (f_\beta(x'(t)) - t)\,\nabla f_\beta(x'(t))}{\sqrt{|x'(t)|^2 + (f_\beta(x'(t)) - t)^2}} \cdot \frac{d}{dt}d_\beta(0,t) \\
&= -1 + O\left(|x'(t)|\right) + O\left(\left|\frac{d}{dt}x'(t)\right|\right) \\
&= -1 + O\left(\varepsilon^{1/2}|\log\varepsilon|^{3/2}\right),
\end{aligned}$$

which gives (8.18). □

8.4. **Some notations.** In the remaining part of this paper the following notations will be employed.

- Given a point on Γ_α with local coordinates $(y,0)$ in the Fermi coordinates, denote
$$D_\alpha(y) := \min\{|d_{\alpha-1}(y,0)|, |d_{\alpha+1}(y,0)|\}.$$
- For $\lambda \ge 0$, let
$$\mathcal{M}_\alpha^\lambda := \{|d_\alpha| < |d_{\alpha-1}| + \lambda \quad \text{and} \quad |d_\alpha| < |d_{\alpha+1}| + \lambda\}.$$
In this Part II we take the convention that $d_0 = -\delta R$ and $d_{Q+1} = \delta R$.
- In the Fermi coordinates with respect to Γ_α, there exist two smooth functions $\rho_\alpha^\pm(y)$ such that
$$\mathcal{M}_\alpha^0 = \{(y,z) : \rho_\alpha^-(y) < z < \rho_\alpha^+(y)\}.$$
- For any $r > 0$, let
$$\mathcal{M}_\alpha^0(r) := \{(y,z) \in \mathcal{M}_\alpha^0, \quad |y| < r\}.$$
- In this Part II we denote
$$\mathcal{D}(r) = \cup_{\alpha=1}^Q \mathcal{M}_\alpha^0(r).$$
- The covariant derivative on Γ_z with respect to the induced metric is denoted by ∇_z.

9. The approximate solution

9.1. **Optimal approximation.** Fix a function $\zeta \in C_0^\infty(-2,2)$ with $\zeta \equiv 1$ in $(-1,1)$, $|\zeta'| + |\zeta''| \le 16$. Let

$$\bar{g}(x) = \zeta(3|\log\varepsilon|x)g(x) + (1 - \zeta(3|\log\varepsilon|x))\,\mathrm{sgn}(x), \quad x \in (-\infty, +\infty).$$

Then $\bar{g}$ is an approximate solution to the one dimensional Allen-Cahn equation, that is,

$$\bar{g}'' = W'(\bar{g}) + \bar{\xi}, \tag{9.1}$$

where $\mathrm{spt}(\bar{\xi}) \in \{3|\log\varepsilon| < |x| < 6|\log\varepsilon|\}$, and $|\bar{\xi}| + |\bar{\xi}'| + |\bar{\xi}''| \lesssim \varepsilon^3$.

We also have (for the definition of σ_0 see Appendix A)

$$\int_{-\infty}^{+\infty} \bar{g}'(t)^2 dt = \sigma_0 + O(\varepsilon^3). \tag{9.2}$$

Without loss of generality assume $u<0$ below Γ_1. Given a tuple of functions $\mathbf{h}:=(h_1(y),\cdots,h_Q(y))$, for each α, in the Fermi coordinates with respect to Γ_α, let

$$g_\alpha(y,z):=\bar{g}\left((-1)^{\alpha-1}(z-h_\alpha(y))\right)=\bar{g}\left((-1)^{\alpha-1}\left(d_\alpha(y,z)-h_\alpha(y)\right)\right).$$

Then we define

$$g(y,z;\mathbf{h}):=\sum_\alpha g_\alpha+\frac{(-1)^Q+1}{2}.$$

For simplicity of notation, denote

$$g'_\alpha=\bar{g}'\left((-1)^{\alpha-1}(z-h_\alpha(y))\right),\quad g''_\alpha=\bar{g}''\left((-1)^{\alpha-1}(z-h_\alpha(y))\right),\quad\cdots.$$

Proposition 9.1. *There exists $\boldsymbol{h}(y)=(h_\alpha(y))$ with $|h_\alpha|\ll 1$ for each α, such that for any α and $y\in B_R^{n-1}$,*

$$\int_{-\delta R}^{\delta R}\left[u(y,z)-g(y,z;\boldsymbol{h})\right]\bar{g}'\left((-1)^{\alpha-1}(z-h_\alpha(y))\right)dz=0, \tag{9.3}$$

where (y,z) denotes the Fermi coordinates with respect to Γ_α.

Proof. Denote

$$F(h_1,\cdots,h_Q):=\left(\int_{-\delta R}^{\delta R}\left[u(y,z)-g(y,z;\mathbf{h})\right]\bar{g}'\left((-1)^{\alpha-1}(z-h_\alpha(y))\right)dz\right),$$

which is viewed as a map from the Banach space $\mathcal{X}:=C^0(B_R^{n-1}(0))^Q$ to itself.

Clearly F is a C^1 map. Furthermore,

$$\begin{aligned}(DF(\mathbf{h})\xi)_\alpha &= (-1)^\alpha\xi_\alpha(y)\int_{-\delta R}^{\delta R}\left[g'_\alpha(y,z)^2-\left(u(y,z)-g(y,z;\mathbf{h})\right)g''_\alpha(y,z)\right]dz\\ &+\sum_{\beta\neq\alpha}(-1)^\beta\xi_\beta(\Pi_\beta(y,z))\int_{-\delta R}^{\delta R}g'_\alpha(y,z)\,g'_\beta(y,z)\,\nabla d_\beta(y,z)\cdot\nabla d_\alpha(y,z)\,dz.\end{aligned}$$

By Lemma 7.1, there exists a $\delta>0$ such that for any $\|\mathbf{h}\|_{\mathcal{X}}<\delta$,

$$\int_{-\delta R}^{\delta R}\left[g'_\alpha(y,z)^2-\left(u(y,z)-g(y,z;\mathbf{h})\right)g''_\alpha(y,z)\right]dz\geq\frac{\sigma_0}{2},$$

$$\Big|\int_{-\delta R}^{\delta R}g'_\alpha(y,z)\,g'_\beta(y,z)\,\nabla d_\beta(y,z)\cdot\nabla d_\alpha(y,z)\,dz\Big|\ll 1.$$

Thus in this ball $DF(\mathbf{h})$ is diagonal dominated and invertible with $\|DF(\mathbf{h})^{-1}\|_{\mathcal{X}\mapsto\mathcal{X}}\leq C$. By Lemma 7.1, for all ε small enough, $\|F(0)\|_{\mathcal{X}}<<1$. The existence of $\mathbf{h}$ then follows from the inverse function theorem. □

Remark 9.2. *The proof shows that $\|\boldsymbol{h}\|_{L^\infty(B_R^{n-1})}=o(1)$. By differentiating* (9.3)*, we can show that $\|\boldsymbol{h}\|_{C^3(B_R^{n-1})}=o(1)$.*

Denote $g_*(y,z):=g(y,z;\mathbf{h}(y))$, where $\mathbf{h}$ is as in the previous lemma. Let

$$\phi:=u-g_*.$$

In the Fermi coordinates with respect to Γ_α,

$$\begin{aligned}\Delta g_\alpha &= g''_\alpha-(-1)^{\alpha-1}g'_\alpha H^\alpha-(-1)^{\alpha-1}g'_\alpha\Delta_z h_\alpha+g''_\alpha|\nabla_z h_\alpha|^2\\ &= W'(g_\alpha)+\xi_\alpha+(-1)^\alpha g'_\alpha\mathcal{R}_{\alpha,1}+g''_\alpha\mathcal{R}_{\alpha,2},\end{aligned}$$

where

$$\xi_\alpha(y,z)=\bar{\xi}\left((-1)^{\alpha-1}(z-h_\alpha(y))\right),$$

$$\mathcal{R}_{\alpha,1}(y,z):=H^\alpha(y,z)+\Delta_z h_\alpha(y)\quad\text{and}\quad\mathcal{R}_{\alpha,2}(y,z):=|\nabla_z h_\alpha(y)|^2.$$

In the Fermi coordinates with respect to Γ_α, the equation for ϕ reads as

$$\begin{aligned}&\Delta_z\phi-H^\alpha(y,z)\partial_z\phi+\partial_{zz}\phi\\ =\;& W'(g_*+\phi)-\sum_{\beta=1}^Q W'(g_\beta)-(-1)^\alpha g'_\alpha\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]-g''_\alpha|\nabla_z h_\alpha|^2\end{aligned} \tag{9.4}$$

$$
\begin{aligned}
&- \sum_{\beta\neq\alpha}\left[(-1)^\beta g_\beta'\mathcal{R}_{\beta,1}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)+g_\beta''\mathcal{R}_{\beta,2}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)\right]-\sum_\beta\xi_\beta\\
&= W''(g_*)\phi+\mathcal{R}(\phi)+\left[W'(g_*)-\sum_{\beta=1}^{Q}W'(g_\beta)\right]-(-1)^\alpha g_\alpha'\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]-g_\alpha''|\nabla_z h_\alpha|^2\\
&- \sum_{\beta\neq\alpha}\left[(-1)^\beta g_\beta'\mathcal{R}_{\beta,1}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)+g_\beta''\mathcal{R}_{\beta,2}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)\right]-\sum_\beta\xi_\beta.
\end{aligned}
$$

In the above we have denoted

$$\mathcal{R}(\phi):=W'(g_*+\phi)-W'(g_*)-W''(g_*)\phi=O(\phi^2).$$

9.2. **Interaction terms.** In this subsection we establish several estimates on the interaction term between different components, $W'(g_*)-\sum_{\beta=1}^{Q}W'(g_\beta)$.

Lemma 9.3. *In $\mathcal{M}_\alpha^4$,*

$$
\begin{aligned}
W'(g_*)-\sum_\beta W'(g_\beta) &= \left[W''(g_\alpha)-2\right]\left[g_{\alpha-1}-(-1)^\alpha\right]+\left[W''(g_\alpha)-2\right]\left[g_{\alpha+1}+(-1)^\alpha\right] \qquad (9.5)\\
&+ O\left(e^{-2\sqrt{2}d_{\alpha-1}}+e^{2\sqrt{2}d_{\alpha+1}}\right)+O\left(e^{-\sqrt{2}d_{\alpha-2}-\sqrt{2}|d_\alpha|}+e^{\sqrt{2}d_{\alpha+2}-\sqrt{2}|d_\alpha|}\right).
\end{aligned}
$$

Proof. In $\mathcal{M}_\alpha^4$,

$$g_*=g_\alpha+\sum_{\beta<\alpha}\left[g_\beta-(-1)^{\beta-1}\right]+\sum_{\beta>\alpha}\left[g_\beta+(-1)^{\beta-1}\right].$$

By Lemma 7.1, $g_\beta-(-1)^{\beta-1}$ (for $\beta<\alpha$) and $g_\beta+(-1)^{\beta-1}$ (for $\beta>\alpha$) are all small quantities.

By the Taylor expansion,

$$
\begin{aligned}
W'(g_*) &= W'(g_\alpha)+W''(g_\alpha)\left[\sum_{\beta<\alpha}\left(g_\beta-(-1)^{\beta-1}\right)+\sum_{\beta>\alpha}\left(g_\beta+(-1)^{\beta-1}\right)\right]\\
&+\sum_{\beta<\alpha}O\left(|g_\beta-(-1)^{\beta-1}|^2\right)+\sum_{\beta>\alpha}O\left(|g_\beta+(-1)^{\beta-1}|^2\right).
\end{aligned}
$$

On the other hand, for $\beta<\alpha$,

$$W'(g_\beta)=2\left(g_\beta-(-1)^{\beta-1}\right)+O\left(|g_\beta-(-1)^{\beta-1}|^2\right),$$

and for $\beta>\alpha$,

$$W'(g_\beta)=2\left(g_\beta+(-1)^{\beta-1}\right)+O\left(|g_\beta+(-1)^{\beta-1}|^2\right).$$

Combining these expansions we get

$$
\begin{aligned}
W'(g_*)-\sum_{\beta=1}^{Q}W'(g_\beta) &= \sum_{\beta<\alpha}\left[W''(g_\alpha)-2\right]\left(g_\beta-(-1)^{\beta-1}\right)+\sum_{\beta<\alpha}O\left(|g_\beta-(-1)^{\beta-1}|^2\right)\\
&+ \sum_{\beta>\alpha}\left[W''(g_\alpha)-2\right]\left(g_\beta+(-1)^{\beta-1}\right)+\sum_{\beta>\alpha}O\left(|g_\beta+(-1)^{\beta-1}|^2\right).
\end{aligned}
$$

Using the fact that

$$|W''(g_\alpha)-W''(1)|\lesssim 1-g_\alpha^2\lesssim e^{-\sqrt{2}|d_\alpha|}$$

and similar estimates on g_β, we get the main order terms and estimates on remainder terms in (9.5) . ☐

The following upper bound on the interaction term will be used a lot in the below.

Lemma 9.4. *In $\mathcal{M}_\alpha^4$,*

$$\left|W'(g_*(y,z))-\sum_{\beta=1}^{Q}W'(g_\beta(y,z))\right|\lesssim e^{-\sqrt{2}D_\alpha(y)}+\varepsilon^2.$$

Proof. We need to estimate those terms in the right hand side of (9.5). To simplify notations, assume $(-1)^{\alpha-1}=1$.

- There exists a constant C depending only on W such that
$$\Big|\left[W''(g_\alpha)-2\right]\left(g_{\alpha-1}+1\right)\Big| \lesssim e^{-\sqrt{2}d_{\alpha-1}-\sqrt{2}|d_\alpha|}.$$
Note that $d_{\alpha-1}>0$ in $\mathcal{M}^4_\alpha$. If one of $d_{\alpha-1}$ and $|d_\alpha|$ is larger than $\sqrt{2}|\log\varepsilon|$, we have
$$e^{-\sqrt{2}d_{\alpha-1}-\sqrt{2}|d_\alpha|}\leq \varepsilon^2,$$
and we are done.
If both $d_{\alpha-1}$ and $|d_\alpha|$ are not larger than $\sqrt{2}|\log\varepsilon|$, by Lemma 8.3,
$$d_{\alpha-1}(y,z)=d_{\alpha-1}(y,0)+d_\alpha(y,z)+O(\varepsilon^{1/3}).$$
Therefore
$$e^{-\sqrt{2}d_{\alpha-1}-\sqrt{2}|d_\alpha|}\leq 2e^{-\sqrt{2}d_{\alpha-1}(y,0)}.$$
- In the same way we get
$$\Big|\left[W''(g_\alpha)-W''(1)\right]\left(g_{\alpha+1}-1\right)\Big| \lesssim e^{\sqrt{2}d_{\alpha+1}-\sqrt{2}|d_\alpha|}\lesssim \varepsilon^2+e^{\sqrt{2}d_{\alpha+1}(y,0)}.$$
- If $|d_{\alpha+1}(y,z)|\geq|\log\varepsilon|$, then $e^{-2\sqrt{2}d_{\alpha+1}}\leq\varepsilon^2$. If $|d_{\alpha+1}(y,z)|\leq|\log\varepsilon|$, we also have $|z|=|d_\alpha(y,z)|\leq|\log\varepsilon|+4$. Hence by Lemma 8.3,
$$d_{\alpha+1}(y,z)=d_{\alpha+1}(y,0)+d_\alpha(y,z)+O(\varepsilon^{1/3}).$$
Because $|d_\alpha(y,z)|<|d_{\alpha+1}(y,z)|+4$, we get
$$d_{\alpha+1}(y,z)\leq\frac{1}{2}d_{\alpha+1}(y,0)+4.$$
Therefore
$$e^{2\sqrt{2}d_{\alpha+1}(y,z)}\leq e^4e^{\sqrt{2}d_{\alpha+1}(y,0)}.$$
- Similarly
$$e^{-2\sqrt{2}d_{\alpha-1}(y,z)}\leq\varepsilon^2+e^4e^{-\sqrt{2}d_{\alpha-1}(y,0)}.$$
- As in the first two cases,
$$e^{-\sqrt{2}d_{\alpha-2}-\sqrt{2}|d_\alpha|}+e^{\sqrt{2}d_{\alpha+2}-\sqrt{2}|d_\alpha|}\lesssim\varepsilon^2+e^{-\sqrt{2}d_{\alpha-1}(y,0)}+e^{\sqrt{2}d_{\alpha+1}(y,0)}.$$

Putting all of these together we finish the proof. □

The Hölder norm of interaction terms can also be estimated in the following way.

Lemma 9.5. *For any* $(y,z)\in\mathcal{M}^3_\alpha$*,*
$$\Big\|W'(g_*)-\sum_{\beta=1}^{Q}(-1)^{\beta-1}W'(g_\beta)\Big\|_{C^\theta(B_1(y,z))}\lesssim\sup_{B_1(y)}e^{-\sqrt{2}D_\alpha}+\varepsilon^2.$$

Proof. We only need to notice that, for any $(y,z)\in\mathcal{M}^3_\alpha$ and any $\beta\in\{1,\cdots,Q\}$,
$$\|g_\beta^2-1\|_{C^\theta(B_1(y,z))}\lesssim\|g_\beta^2-1\|_{Lip(B_1(y,z))}\lesssim e^{-\sqrt{2}|d_\beta(y,z)|}. \tag{9.6}$$
Then we can proceed as in the previous lemma to conclude the proof. □

9.3. **Controls on h using ϕ.** The choice of optimal approximation in Subsection 9.1 has the advantage that h is controlled by ϕ. This will allow us to iterate various elliptic estimates in the below.

Lemma 9.6. *For each* α*,*
$$|h_\alpha(y)|\lesssim|\phi(y,0)|+e^{-\sqrt{2}D_\alpha(y)},$$
$$|\nabla h_\alpha(y)|\lesssim|\nabla\phi(y,0)|+o\left(e^{-\sqrt{2}D_\alpha(y)}\right),$$
$$|\nabla^2 h_\alpha(y)|\lesssim|\nabla^2\phi(y,0)|+o\left(e^{-\sqrt{2}D_\alpha(y)}\right),$$
$$\|\nabla^2 h_\alpha\|_{C^\theta(B_1(y))}\lesssim\|\nabla^2\phi\|_{C^\theta(B_1(y,0))}+\sup_{B_1(y)}e^{-\sqrt{2}D_\alpha}.$$

Proof. Fix an $\alpha \in \{1, \cdots, Q\}$. In the Fermi coordinates with respect to Γ_α, because $u(y,0)=0$,

$$\begin{aligned}\phi(y,0) = -\bar{g}\left((-1)^\alpha h_\alpha(y)\right) &- \sum_{\beta<\alpha}\left[\bar{g}\left((-1)^{\beta-1}\left(d_\beta(y,0)-h_\beta(\Pi_\beta(y,0))\right)\right)-(-1)^{\beta-1}\right] \\ &- \sum_{\beta>\alpha}\left[\bar{g}\left((-1)^{\beta-1}\left(d_\beta(y,0)-h_\beta(\Pi_\beta(y,0))\right)\right)+(-1)^{\beta-1}\right]. \end{aligned} \tag{9.7}$$

Note that for $\beta\neq\alpha$, $|h_\beta(\Pi_\beta(y,0))|\ll 1$. Thus

$$|h_\alpha(y)| \lesssim |\phi(y,0)| + \sum_{\beta\neq\alpha} e^{-\sqrt{2}d_\beta(y,0)} \lesssim |\phi(y,0)| + e^{-\sqrt{2}D_\alpha(y)}. \tag{9.8}$$

Differentiating (9.7), we get

$$\nabla_0\phi(y,0) = (-1)^{\alpha+1}\bar{g}'\left((-1)^\alpha h_\alpha(y)\right)\nabla_0 h_\alpha(y) + \sum_{\beta\neq\alpha}(-1)^\beta g_\beta'(y,0)\nabla_0\left[d_\beta(y,0)-h_\beta(\Pi_\beta(y,0))\right],$$

and

$$\begin{aligned}\nabla_0^2\phi(y,0) &= (-1)^{\alpha+1}\bar{g}'\left((-1)^\alpha h_\alpha(y)\right)\nabla_0^2 h_\alpha(y) - \bar{g}''\left((-1)^\alpha h_\alpha(y)\right)\nabla_0 h_\alpha(y)\otimes\nabla_0 h_\alpha(y) \\ &+ \sum_{\beta\neq\alpha}(-1)^\beta g_\beta'(y,0)\nabla_0^2\left[d_\beta(y,0)-h_\beta(\Pi_\beta(y,0))\right] \\ &- \sum_{\beta\neq\alpha} g_\beta''(y,0)\nabla_0\left[d_\beta(y,0)-h_\beta(\Pi_\beta(y,0))\right]\otimes\nabla_0\left[d_\beta(y,0)-h_\beta(\Pi_\beta(y,0))\right].\end{aligned}$$

Note that $|\nabla h_\beta| = o(1)$ and by Lemma 8.3, if $\bar{g}'(d_\beta(y,0)-h_\beta(\Pi_\beta(y,0))\neq 0$,

$$|\nabla_0 d_\beta| = \sqrt{1-\nabla d_\beta\cdot\nabla d_\alpha} = O(\varepsilon^{1/6}).$$

Thus

$$\begin{aligned}|\nabla_0 h_\alpha(y)| &\lesssim |\nabla_0\phi(y,0)| + O(\varepsilon^{1/6}+|\nabla h_\beta(\Pi_\beta(y,0))|)O\left(e^{\sqrt{2}d_{\alpha+1}(y,0)}+e^{-\sqrt{2}d_{\alpha-1}(y,0)}\right) \\ &\lesssim |\nabla_0\phi(y,0)| + o\left(e^{-\sqrt{2}D_\alpha(y)}\right).\end{aligned} \tag{9.9}$$

Similarly, because $|\nabla^2 h_\beta| = o(1)$ and recalling that $\nabla^2 d_\beta$ is the second fundamental form of $\Gamma_{\beta,z}$,

$$|\nabla_0^2 d_\beta| \le |\nabla^2 d_\beta| = O(\varepsilon),$$

we have

$$\begin{aligned}|\nabla_0^2 h_\alpha(y)| &\lesssim |\nabla_0^2\phi(y,0)| + |\nabla_0 h_\alpha(y)|^2 + e^{-\sqrt{2}D_\alpha(y)}\left[\varepsilon^{\frac{1}{3}} + \sum_\beta \sup_{B_{\varepsilon^{1/3}}}\left(|\nabla^2 h_\beta| + |\nabla h_\beta|\right)\right] \\ &\lesssim |\nabla_0^2\phi(y,0)| + |\nabla_0 h_\alpha(y)|^2 + o\left(e^{-\sqrt{2}D_\alpha(y)}\right).\end{aligned} \tag{9.10}$$

Finally, by the above formulation and (9.6), we get a control on $\|\nabla^2 h_\alpha\|_{C^\theta(B_1(y))}$ using $\|\nabla^2\phi\|_{C^\theta(B_1(y,0))}$ and $\sup_{B_1(y)} e^{-\sqrt{2}D_\alpha}$. □

10. A Toda system

In the Fermi coordinates with respect to Γ_α, multiplying (9.4) by g_α' and integrating in z leads to

$$\begin{aligned}&\int_{-\delta R}^{\delta R} g_\alpha'\Delta_z\phi - H^\alpha(y,z)g_\alpha'\partial_z\phi + g_\alpha'\partial_{zz}\phi \\ =& \int_{-\delta R}^{\delta R}\left[W'(g_*+\phi)-\sum_\beta W'(g_\beta)\right]g_\alpha' - (-1)^\alpha\int_{-\delta R}^{\delta R}\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]g_\alpha'(z)^2 \\ -& \int_{-\delta R}^{\delta R} g_\alpha''g_\alpha'|\nabla_z h_\alpha|^2 - \sum_{\beta\neq\alpha}(-1)^\beta\int_{-\delta R}^{\delta R} g_\alpha' g_\beta'\mathcal{R}_{\beta,1} - \sum_{\beta\neq\alpha}\int_{-\delta R}^{\delta R} g_\beta'' g_\alpha'\mathcal{R}_{\beta,2} - \sum_\beta\int_{-\delta R}^{\delta R}\xi_\beta g_\alpha'.\end{aligned} \tag{10.1}$$

By the calculation in Appendix B, we obtain

$$H^\alpha(y,0)+\Delta_0 h_\alpha(y) = \frac{4}{\sigma_0}\left[A_{(-1)^\alpha}^2 e^{-\sqrt{2}d_{\alpha-1}(y,0)} - A_{(-1)^{\alpha-1}}^2 e^{\sqrt{2}d_{\alpha+1}(y,0)}\right] + O(\varepsilon^2)$$

$$
\begin{aligned}
&+ \quad O\left(|h_\alpha(y)| + |h_{\alpha-1}(\Pi_{\alpha-1}(y,z))| + \varepsilon^{1/3}\right) e^{-\sqrt{2}d_{\alpha-1}(y,0)} \\
&+ \quad O\left(|h_\alpha(y)| + |h_{\alpha+1}(\Pi_{\alpha+1}(y,z))| + \varepsilon^{1/3}\right) e^{\sqrt{2}d_{\alpha+1}(y,0)} \qquad (10.2)\\
&+ \quad O(e^{-\frac{3\sqrt{2}}{2}d_{\alpha-1}(y,0)}) + O(e^{\frac{3\sqrt{2}}{2}d_{\alpha+1}(y,0)}) + O(e^{-\sqrt{2}d_{\alpha-2}(y,0)}) + O(e^{\sqrt{2}d_{\alpha+2}(y,0)}) \\
&+ \quad \sum_{\beta\neq\alpha} |d_\beta(y,0)| e^{-\sqrt{2}|d_\beta(y,0)|} \left[\sup_{B_{\varepsilon^{1/3}}(y)} |H^\beta + \Delta_0^\beta h_\beta| + \sup_{B_{\varepsilon^{1/3}}(y)} |\nabla h_\beta|^2\right] \\
&+ \quad \sup_{(-6|\log\varepsilon|, 6|\log\varepsilon|)} \left(|\nabla_y^2\phi(y,z)|^2 + |\nabla_y\phi(y,z)|^2 + |\phi(y,z)|^2\right).
\end{aligned}
$$

By this equation we get an upper bound on $H^\alpha(y,0) + \Delta_0 h_\alpha(y)$.

Lemma 10.1.

$$
\sup_{B_r} \left|H^\alpha(y,0) + \Delta_0 h_\alpha(y)\right| \lesssim \sup_{B_{r+1}} e^{-\sqrt{2}D_\alpha} + \varepsilon^2 + \|\phi\|^2_{C^{2,\theta}(\mathcal{D}_{r+1})} + \sum_{\beta\neq\alpha} \sup_{B_{r+1}} \left[|H^\beta + \Delta^\beta h_\beta|^2 + e^{-2\sqrt{2}D_\beta}\right]. \qquad (10.3)
$$

Proof. In the right hand side of (10.2), those terms in the first four lines are bounded by $O\left(e^{-\sqrt{2}D_\alpha} + \varepsilon^2\right)$.

If $d_\beta(y,0) > 2|\log\varepsilon|$, the terms in the fifth line is controlled by $O(\varepsilon^2)$. If $d_\beta(y,0) < 2|\log\varepsilon|$, using the Cauchy inequality, they are controlled by

$$
\begin{aligned}
& |d_\beta(y,0)|^2 e^{-2\sqrt{2}|d_\beta(y,0)|} + \sup_{B_{\varepsilon^{1/3}}(y)} |H^\beta + \Delta_0^\beta h_\beta|^2 + \sup_{B_{\varepsilon^{1/3}}(y)} |\nabla h_\beta|^4 \\
\lesssim \; & e^{-\sqrt{2}|d_\beta(y,0)|} + \sup_{B_{\varepsilon^{1/3}}(y)} |H^\beta + \Delta_0^\beta h_\beta|^2 + \sup_{\mathcal{M}_\beta^0(r+1)} |\nabla\phi|^4,
\end{aligned}
$$

where we have used the fact that $|d_\beta(y,0)| \gg 1$, Lemma 9.6 and the fact that $B^\beta_{\varepsilon^{1/3}}(y,0) \subset \mathcal{M}^0_\beta(r+1)$ (by Lemma 8.3).

Finally, the term in the last line of (10.2) is controlled by $\|\phi\|^2_{C^{2,\theta}(\mathcal{D}_{r+1})}$. □

11. $C^{1,\theta}$ ESTIMATE ON ϕ

In this section we prove the following $C^{1,\theta}$ estimate on ϕ.

Proposition 11.1. *There exist constants $L > 0$, $\sigma(L) \ll 1$ and $C(L)$ such that*

$$
\begin{aligned}
\|\phi\|_{C^{1,\theta}(\mathcal{D}_\alpha(r))} \;\leq\; & \sigma(L)\|\phi\|_{C^{2,\theta}(\mathcal{D}(r+4L)} + C(L)\varepsilon^2 + C(L)\sup_{B_{r+4L}} e^{-\sqrt{2}D_\alpha(y)} \qquad (11.1)\\
+ \; & \sigma(L)\sum_{\beta=1}^{Q} \sup_{B_{r+4L}(y)} \left|H^\beta + \Delta_0^\beta h_\beta\right| + C(L)\sum_{\beta\neq\alpha}\sup_{B_{r+4L}} e^{-4\sqrt{2}D_\beta}.
\end{aligned}
$$

To prove this proposition, fix a large constant $L > 0$ and define

$$
\mathcal{N}^1_\alpha(r) := \{-L < d_\alpha < L\} \cap \mathcal{M}^0_\alpha(r), \quad \text{and} \quad \mathcal{N}^2_\alpha(r) := \{d_\alpha > L/2\} \cap \mathcal{M}^0_\alpha(r).
$$

We will estimate the $C^{1,\theta}$ norm of ϕ in $\mathcal{N}^1_\alpha(r)$ and $\mathcal{N}^2_\alpha(r)$ separately.

11.1. $C^{1,\theta}$ estimate in $\mathcal{N}^2_\alpha(r)$.

In $\mathcal{N}^2_\alpha(r)$, by using (8.8)-(8.13) and Lemma 9.4, the equation for ϕ can be written in the following way.

Lemma 11.2. *In $\mathcal{N}^2_\alpha(r)$,*

$$
\Delta_z\phi - H^\alpha(y,z)\partial_z\phi + \partial_{zz}\phi = \left(2 + O(e^{-cL})\right)\phi + E^2_\alpha,
$$

where

$$
\begin{aligned}
|E^2_\alpha(y,z)| \;\lesssim\; & \varepsilon^2 + e^{-\sqrt{2}D_\alpha(y)} + |\nabla_0^2 h_\alpha(y)|^2 + |\nabla_0 h_\alpha(y)|^2 + e^{-cL}\left|H^\alpha(y,0) + \Delta_0 h_\alpha(y)\right| \\
+ \; & \sum_{\beta\neq\alpha} \sup_{B_{\varepsilon^{1/3}}(y)} \left[|H^\beta + \Delta_0^\beta h_\beta|^2 + |\nabla h_\beta|^4 + |\nabla^2 h_\beta|^4\right].
\end{aligned}
$$

By standard interior elliptic estimate, we deduce that, for any $r > 1$,

$$\begin{aligned}
&\|\phi\|_{C^{1,\theta}(\mathcal{N}^2_\alpha(r))}\\
\lesssim\ & e^{-cL}\|\phi\|_{L^\infty(\mathcal{M}^L_\alpha(r+L)\cap\{|y|=r+L\})} + e^{-cL}\|\phi\|_{L^\infty(\{|y|<r+L,z=L/4\})} + e^{-cL}\|\phi\|_{L^\infty(\{|y|<r+L,z=\rho^+_\alpha(y)+L\})}\\
+\ & \sup_{B_{r+L}} e^{-\sqrt{2}D_\alpha} + \varepsilon^2 + \sup_{B_{r+L}}\left(|\nabla_0^2 h_\alpha(\tilde{y})|^2 + |\nabla_0 h_\alpha(\tilde{y})|^2\right)\\
+\ & e^{-cL}\sup_{B_{r+L}}\left|H^\alpha + \Delta_0 h_\alpha\right| + \sup_{B_{r+L}}\left[|H^\beta + \Delta_0^\beta h_\beta|^2 + |\nabla h_\beta|^4 + |\nabla^2 h_\beta|^4\right].
\end{aligned}$$

Substituting (10.3) into this estimate, after simplification we obtain

$$\begin{aligned}
\|\phi\|_{C^{1,\theta}(\mathcal{N}^2_\alpha(r))} \leq\ & \sigma(L)\|\phi\|_{C^{2,\theta}(\mathcal{D}(r+L))} + C(L)\sup_{B_{r+L}(y)} e^{-\sqrt{2}D_\alpha} + C(L)\varepsilon^2 \qquad (11.2)\\
+\ & \sigma(L)\sum_{\beta=1}^{Q}\sup_{B_{r+L}(y)}\left|H^\beta + \Delta_0^\beta h_\beta\right| + C(L)\sum_{\beta\neq\alpha}\sup_{B_{r+L}} e^{-4\sqrt{2}D_\beta}.
\end{aligned}$$

Here $\sigma(L) \lesssim e^{-cL} + \max_\beta \|H^\beta + \Delta_0^\beta h_\beta\|_{L^\infty} \ll 1$.

11.2. $C^{1,\theta}$ **estimate in** $\mathcal{N}^1_\alpha(r)$**.** In $\mathcal{N}^1_\alpha(r)$, similar to Lemma 11.2, the equation for ϕ can be written in the following way.

Lemma 11.3. *In* $\mathcal{N}^1_\alpha(r)$*,*

$$\Delta_z\phi - H^\alpha(y,z)\partial_z\phi + \partial_{zz}\phi = W''(g_\alpha)\phi + O(\phi^2) - (-1)^\alpha g'_\alpha\left[H^\alpha(y,0) + \Delta_0 h_\alpha(y)\right] + E^1_\alpha,$$

where

$$|E^1_\alpha(y,z)| \lesssim e^{-\sqrt{2}D_\alpha(y)} + \varepsilon^2 + \left(|\nabla^2 h_\alpha(y)|^2 + |\nabla h_\alpha(y)|^2\right)e^{-\sqrt{2}|z|}.$$

Take a function $\eta \in C_0^\infty(-2L, 2L)$ satisfying $\eta \equiv 1$ in $(-L, L)$, $|\eta'| \lesssim L^{-1}$ and $|\eta''| \lesssim L^{-2}$. Let $\phi_\alpha(y,z) := \phi(y,z)\eta(z)$ and $\widetilde{\phi}_\alpha(y,z) := \phi_\alpha(y,z) - c_\alpha(y)g'_\alpha(y,z)$, where

$$\begin{aligned}
c_\alpha(y) &= \int_{-\delta R}^{\delta R}\phi_\alpha(y,z)g'_\alpha(y,z)dz \qquad (11.3)\\
&= \int_{-\delta R}^{\delta R}\phi(y,z)\left(\eta(z) - 1\right)g'_\alpha(y,z)dz. \qquad (\text{ by } (9.3))
\end{aligned}$$

Therefore we still have the orthogonal condition

$$\int_{-\delta R}^{\delta R}\widetilde{\phi}_\alpha(y,z)g'_\alpha(y,z)dz = 0, \quad \forall\ y \in B_R^{n-1}. \qquad (11.4)$$

Lemma 11.4 (Estimates on c_α)**.** *There exists a constant* $\sigma > 0$ *small such that*

$$|c_\alpha(y)| \lesssim e^{-\sigma L}\sup_{L<|z|<6|\log\varepsilon|} e^{-(\sqrt{2}-\sigma)|z|}|\phi(y,z)|, \qquad (11.5)$$

$$|\nabla c_\alpha(y)| \lesssim e^{-\sigma L}\sup_{L<|z|<6|\log\varepsilon|} e^{-(\sqrt{2}-\sigma)|z|}\left(|\phi(y,z)| + |\nabla_y\phi(y,z)|\right), \qquad (11.6)$$

$$|\nabla^2 c_\alpha(y)| \lesssim e^{-\sigma L}\sup_{L<|z|<6|\log\varepsilon|} e^{-(\sqrt{2}-\sigma)|z|}\left(|\phi(y,z)| + |\nabla_y\phi(y,z)| + |\nabla_y^2\phi(y,z)|\right). \qquad (11.7)$$

Proof. By (11.3) and the definition of η,

$$\begin{aligned}
|c_\alpha(y)| &\lesssim \left(\sup_{L<|z|<6|\log\varepsilon|} e^{-(\sqrt{2}-\sigma)|z|}|\phi(y,z)|\right)\int_L^{+\infty} e^{-\sigma z}dz\\
&\lesssim e^{-\sigma L}\sup_{L<|z|<6|\log\varepsilon|} e^{-(\sqrt{2}-\sigma)|z|}|\phi(y,z)|.
\end{aligned}$$

Differentiating (11.3) gives

$$\nabla c_\alpha(y) = \int_{-\delta R}^{\delta R}\nabla_y\phi(y,z)\left(\eta(z)-1\right)g'_\alpha(y,z)dz + (-1)^\alpha\nabla h_\alpha(y)\int_{-\delta R}^{\delta R}\phi(y,z)\left(\eta(z)-1\right)g''_\alpha(y,z)dz.$$

(11.6) follows as above. The derivation of (11.7) is similar. ∎

In the Fermi coordinates with respect to Γ_α, the equation satisfied by $\widetilde{\phi}_\alpha$ reads as

$$\Delta_z\widetilde{\phi}_\alpha - H^\alpha(y,z)\partial_z\widetilde{\phi}_\alpha + \partial_{zz}\widetilde{\phi}_\alpha = W''(g_\alpha)\widetilde{\phi}_\alpha + o(\widetilde{\phi}_\alpha) + \widetilde{c}_\alpha(y)g'_\alpha + \widetilde{E}_\alpha, \tag{11.8}$$

where

$$\widetilde{c}_\alpha(y) = (-1)^{\alpha-1}\left[H^\alpha(y,0) + \Delta_0 h_\alpha(y)\right] - \Delta_0 c_\alpha(y),$$

while

$$\begin{aligned}|\widetilde{E}_\alpha(y,z)| \quad\lesssim\quad & \varepsilon^2\eta + e^{-\sqrt{2}D_\alpha(y)}\eta + e^{-\sqrt{2}|z|}\left(|\nabla^2 h_\alpha(y)|^2 + |\nabla h_\alpha(y)|^2\right)\eta \\ + \quad & |\phi(y,z)||c_\alpha(y)|g'_\alpha + g'_\alpha\big|1-\eta\big|\big|H^\alpha(y,0) + \Delta_0 h_\alpha(y)\big| \\ + \quad & |\phi|\left[\varepsilon|\eta'| + |\eta''|\right] + |\phi_z||\eta'| \\ + \quad & \varepsilon L\left[|c_\alpha(y)| + |\nabla c_\alpha(y)| + |\nabla^2 c_\alpha(y)|\right]e^{-\sqrt{2}|z|} \\ + \quad & |c_\alpha(y)||\xi_\alpha| + |c_\alpha(y)|\left[|\nabla^2 h_\alpha(y)| + |\nabla h_\alpha(y)|\right]e^{-\sqrt{2}|z|}.\end{aligned}$$

Combining this expression with Lemma 11.2 and Lemma 11.4, we obtain

Lemma 11.5 (L^2 estimates on $\widetilde{E}_\alpha$)**.** *For any y,*

$$\begin{aligned}\|\widetilde{E}_\alpha(y,\cdot)\|^2_{L^2(-\delta R,\delta R)} \quad\lesssim\quad & L\varepsilon^4 + Le^{-2\sqrt{2}D_\alpha(y)} + |\nabla^2 h_\alpha(y)|^4 + |\nabla h_\alpha(y)|^4 \\ + \quad & e^{-2\sqrt{2}L}\big|H^\alpha(y,0) + \Delta_0 h_\alpha(y)\big|^2 + \frac{1}{L}\sup_{L<|z|<2L}\left(|\phi(y,z)|^2 + |\phi_z(y,z)|^2\right) \\ + \quad & e^{-2\sigma L}\sup_{L<|z|<6|\log\varepsilon|} e^{-2(\sqrt{2}-\sigma)|z|}\left(|\phi(y,z)| + |\nabla_y\phi(y,z)| + |\nabla_y^2\phi(y,z)|\right)^2.\end{aligned}$$

Next we prove an L^2 estimate on $\widetilde{\phi}$.

Lemma 11.6. *For any $r>0$,*

$$\begin{aligned}\sup_{\tilde{y}\in B_r}\|\widetilde{\phi}_\alpha(\tilde{y},\cdot)\|^2_{L^2(-\delta R,\delta R)} \quad\lesssim\quad & e^{-cL}\sup_{\tilde{y}\in B_{r+L}}\|\widetilde{\phi}_\alpha(\tilde{y},\cdot)\|^2_{L^2(-\delta R,\delta R)} + L\varepsilon^4 + L\sup_{\tilde{y}\in B_{r+L}} e^{-2\sqrt{2}D_\alpha} \\ + \quad & e^{-2\sqrt{2}L}\sup_{\tilde{y}\in B_{r+L}}\big|H^\alpha + \Delta_0 h_\alpha\big|^2 \\ + \quad & \frac{1}{L}\sup_{B_{r+L}\times\{L<|z|<2L\}}\left(|\phi|^2 + |\nabla\phi|^2\right) + L^5\sup_{B_{r+L}\times\{|z|<2L\}}\left(|\nabla_y^2\phi|^4 + |\nabla_y\phi|^4\right) \\ + \quad & e^{-2\sigma L}\sup_{B_{r+L}\times\{L<|z|<6|\log\varepsilon|\}} e^{-2(\sqrt{2}-\sigma)|z|}\left(|\phi| + |\nabla_y\phi| + |\nabla_y^2\phi|\right)^2.\end{aligned} \tag{11.9}$$

Proof. Multiplying (11.8) by $\widetilde{\phi}_\alpha$ and integrating in z, we obtain

$$\int_{-\infty}^{+\infty}\widetilde{\phi}_\alpha\Delta_z\widetilde{\phi}_\alpha + H^\alpha(y,z)\partial_z\widetilde{\phi}_\alpha\widetilde{\phi}_\alpha + \partial_{zz}\widetilde{\phi}_\alpha\widetilde{\phi}_\alpha = \int_{-\infty}^{+\infty} W''(g_\alpha)\widetilde{\phi}_\alpha^2 + o(\widetilde{\phi}_\alpha^2) + \widetilde{E}_\alpha\widetilde{\phi}_\alpha.$$

Integrating by parts and applying Theorem A.2 leads to

$$\begin{aligned}\int_{-\infty}^{+\infty}\widetilde{\phi}_\alpha\Delta_z\widetilde{\phi}_\alpha \quad=\quad & \int_{-\infty}^{+\infty}|\partial_z\widetilde{\phi}_\alpha|^2 + W''(g_\alpha)\widetilde{\phi}_\alpha^2 + o(\widetilde{\phi}_\alpha^2) + \widetilde{E}_\alpha\widetilde{\phi}_\alpha + \frac{1}{2}\frac{\partial H^\alpha}{\partial z}\widetilde{\phi}_\alpha^2 \\ \geq\quad & \frac{3\mu}{4}\int_{-\infty}^{+\infty}\widetilde{\phi}_\alpha^2 - C\int_{-\infty}^{+\infty}\widetilde{E}_\alpha^2.\end{aligned}$$

On the other hand, by direct differentiation we also have

$$\begin{aligned}\frac{1}{2}\Delta_0\int_{-\infty}^{+\infty}\widetilde{\phi}_\alpha^2 \quad=\quad & \int_{-\infty}^{+\infty}\widetilde{\phi}_\alpha(y,z)\Delta_0\widetilde{\phi}_\alpha(y,z) + |\nabla_0\widetilde{\phi}_\alpha(y,z)|^2 dz \\ \geq\quad & \int_{-\infty}^{+\infty}\left(\Delta_0\widetilde{\phi}_\alpha - \Delta_z\widetilde{\phi}_\alpha\right)\widetilde{\phi}_\alpha + \frac{3\mu}{4}\int_{-\infty}^{+\infty}\widetilde{\phi}_\alpha^2 - C\int_{-\infty}^{+\infty}\widetilde{e}_\alpha^2 \\ \geq\quad & \frac{\mu}{2}\int_{-\infty}^{+\infty}\widetilde{\phi}_\alpha^2 - C\int_{-\infty}^{+\infty}\widetilde{E}_\alpha^2 - C\varepsilon^2\int_{-\infty}^{+\infty}z^2\left(|\nabla_y^2\widetilde{\phi}_\alpha(y,z)|^2 + |\nabla_y\widetilde{\phi}_\alpha(y,z)|^2\right)dz.\end{aligned}$$

This inequality implies that

$$\sup_{\tilde{y}\in B_r}\|\widetilde{\phi}_\alpha(\tilde{y},\cdot)\|^2_{L^2(-\delta R,\delta R)} \quad\lesssim\quad e^{-cL}\sup_{\tilde{y}\in B_{r+L}}\|\widetilde{\phi}_\alpha(\tilde{y},\cdot)\|^2_{L^2(-\delta R,\delta R)} + \sup_{\tilde{y}\in B_{r+L}}\|\widetilde{E}_\alpha(\tilde{y},\cdot)\|^2_{L^2(-\delta R,\delta R)}$$

$$+\quad \varepsilon^2 \sup_{B_{r+L}(y)} \int_{-\infty}^{+\infty} z^2 \left(|\nabla^2 \widetilde{\phi}_\alpha|^2 + |\nabla \widetilde{\phi}_\alpha|^2\right) dz. \tag{11.10}$$

Note that

$$\begin{aligned}
|\nabla_y^2 \widetilde{\phi}_\alpha(y,z)| + |\nabla_y \widetilde{\phi}_\alpha(y,z)| &\lesssim |\nabla_y^2 \phi(y,z)|\eta(z) + |\nabla_y \phi(y,z)|\eta(z) + \left(|c_\alpha(y)| + |\nabla c_\alpha(y)| + |\nabla^2 c_\alpha(y)|\right) e^{-\sqrt{2}|z|} \\
&\lesssim |\nabla_y^2 \phi(y,z)|\eta(z) + |\nabla_y \phi(y,z)|\eta(z) \\
&+ e^{-\sigma L - \sqrt{2}|z|} \sup_{L<|z|<6|\log\varepsilon|} e^{-(\sqrt{2}-\sigma)|z|} \left(|\phi(y,z)| + |\nabla_y \phi(y,z)| + |\nabla_y^2 \phi(y,z)|\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{-\infty}^{+\infty} z^2 \left(|\nabla_y^2 \widetilde{\phi}_\alpha(y,z)|^2 + |\nabla_y \widetilde{\phi}_\alpha(y,z)|^2\right) dz \\
\lesssim\ & L^3 \sup_{|z|<2L} \left(|\nabla_y^2 \phi(y,z)|^2 + |\nabla_y \phi(y,z)|^2\right) \\
+\ & e^{-2\sigma L} \sup_{L<|z|<6|\log\varepsilon|} e^{-2(\sqrt{2}-\sigma)|z|} \left(|\phi(y,z)|^2 + |\nabla_y \phi(y,z)|^2 + |\nabla_y^2 \phi(y,z)|^2\right).
\end{aligned}$$

Substituting this and Lemma 11.5 into (11.10) gives

$$\begin{aligned}
\sup_{\tilde{y}\in B_r} \|\widetilde{\phi}_\alpha(\tilde{y},\cdot)\|^2_{L^2(-\delta R,\delta R)} &\lesssim e^{-cL} \sup_{\tilde{y}\in B_{r+L}} \|\widetilde{\phi}_\alpha(\tilde{y},\cdot)\|^2_{L^2(-\delta R,\delta R)} + L\varepsilon^4 + L \sup_{B_{r+L}} e^{-2\sqrt{2}D_\alpha} \\
&+ \sup_{B_{r+L}} \left[|\nabla^2 h_\alpha|^4 + |\nabla h_\alpha|^4\right] + e^{-2\sqrt{2}L} \sup_{B_{r+L}} \left|H^\alpha + \Delta_0 h_\alpha\right|^2 \qquad (11.11) \\
&+ \frac{1}{L} \sup_{B_{r+L}\times\{L<|z|<2L\}} \left(|\phi|^2 + |\phi_z|^2\right) + L^3\varepsilon^2 \sup_{B_{r+L}\times\{|z|<2L\}} \left(|\nabla_y^2 \phi|^2 + |\nabla_y \phi|^2\right) \\
&+ e^{-2\sigma L} \sup_{B_{r+L}\times\{L<|z|<6|\log\varepsilon|\}} e^{-2(\sqrt{2}-\sigma)|z|} \left(|\phi| + |\nabla_y \phi| + |\nabla_y^2 \phi|\right)^2.
\end{aligned}$$

The terms involving h_α can be estimated by using Lemma 9.6, while by the Cauchy inequality we have

$$L^3\varepsilon^2 \sup_{|z|<2L} \left(|\nabla_y^2 \phi(y,z)|^2 + |\nabla_y \phi(y,z)|^2\right) \lesssim L\varepsilon^4 + L^5 \sup_{|z|<2L} \left(|\nabla_y^2 \phi(y,z)|^4 + |\nabla_y \phi(y,z)|^4\right).$$

Substituting these into (11.11) we get (11.9). □

By standard elliptic estimates we deduce that

$$\begin{aligned}
&\|\widetilde{\phi}_\alpha\|_{C^{1,\theta}(B_1(y)\times(-3L/4,3L/4))} \\
\lesssim\ & \|\widetilde{\phi}_\alpha\|_{L^2(B_L(y)\times(-L,L))} + \|\Delta\widetilde{\phi}_\alpha\|_{L^\infty(B_L(y)\times(-L,L))} \\
\lesssim\ & L^{\frac{n-1}{2}} e^{-cL} \sup_{\tilde{y}\in B_{2L}(y)} \|\widetilde{\phi}_\alpha(\tilde{y},\cdot)\|_{L^2(-\delta R,\delta R)} + L^{\frac{n+2}{2}}\varepsilon^2 + L^{\frac{n+2}{2}} \sup_{B_{2L}(y)} e^{-\sqrt{2}D_\alpha} \\
+\ & L^{\frac{n-1}{2}} \sup_{B_{2L}(y)} \left|H^\alpha + \Delta_0 h_\alpha\right| + L^{\frac{n-2}{2}} \sup_{B_{2L}(y)\times\{L<|z|<2L\}} \left(|\phi| + |\nabla\phi|\right) \\
+\ & L^{\frac{n+6}{2}} \sup_{B_{r+L}\times\{|z|<2L\}} \left(|\nabla_y^2 \phi|^2 + |\nabla_y \phi|^2\right) \\
+\ & L^{\frac{n}{2}} e^{-\sigma L} \sup_{B_{2L}(y)\times\{L<|z|<6|\log\varepsilon|\}} e^{-(\sqrt{2}-\sigma)|z|} \left(|\phi(y,z)| + |\nabla_y \phi(y,z)| + |\nabla_y^2 \phi(y,z)|\right).
\end{aligned}$$

By using (10.3) we get a bound on $\sup_{B_{2L}(y)} \left|H^\alpha + \Delta_0 h_\alpha\right|$. Hence we have

$$\begin{aligned}
&\|\widetilde{\phi}_\alpha\|_{C^{1,\theta}(B_1(y)\times(-3L/4,3L/4))} \\
\lesssim\ & L^{\frac{n+1}{2}} e^{-cL} \sup_{B_{3L}(y)\times(-2L,2L)} |\phi| + L^{\frac{n+2}{2}}\varepsilon^2 + L^{\frac{n+2}{2}} \sup_{B_{3L}(y)} e^{-\sqrt{2}D_\alpha} \\
+\ & L^{\frac{n-1}{2}} \sup_{B_{3L}(y)} \sum_{\beta\neq\alpha} D_\alpha e^{-\sqrt{2}D_\alpha} \left[|H^\beta + \Delta_0^\beta h_\beta| + |\nabla h_\beta|^2\right] \\
+\ & L^{\frac{n-2}{2}} \sup_{B_{3L}(y)\times\{L<|z|<2L\}} \left(|\phi| + |\nabla\phi|\right) + L^{\frac{n+6}{2}} \sup_{B_{3L}(y)\times(-2L,2L)} \left(|\nabla^2\phi|^2 + |\nabla\phi|^2\right) \\
+\ & L^{\frac{n}{2}} e^{-\sigma L} \sup_{B_{3L}(y)\times\{L<|z|<6|\log\varepsilon|\}} e^{-(\sqrt{2}-\sigma)|z|} \left(|\phi| + |\nabla_y \phi| + |\nabla_y^2 \phi|\right).
\end{aligned}$$

Since this estimate holds for any y, it implies that

$$\begin{aligned}
\|\phi\|_{C^{1,\theta}(\mathcal{N}^1_\alpha(r))} &\leq Ce^{-cL}\sup_{B_{r+3L}\times(-2L,2L)}|\phi|+CL^{\frac{n+2}{2}}\varepsilon^2+L^{\frac{n+2}{2}}\sup_{B_{3L}(y)}e^{-\sqrt{2}D_\alpha(y)}\\
&+ CL^{\frac{n-1}{2}}\sup_{B_{r+3L}}\sum_{\beta\neq\alpha}D_\alpha e^{-\sqrt{2}D_\alpha}\left[|H^\beta+\Delta_0^\beta h_\beta|+|\nabla h_\beta|^2\right]\\
&+ CL^{\frac{n-2}{2}}\sup_{B_{r+3L}\times\{L<|z|<2L\}}\left(|\phi(y,z)|+|\nabla\phi_z(y,z)|\right)\\
&+ CL^{\frac{n+6}{2}}\sup_{B_{r+3L}\times(-2L,2L)}\left(|\nabla^2\phi|^2+|\nabla\phi|^2\right)\\
&+ CL^{\frac{n}{2}}e^{-\sigma L}\sup_{B_{r+3L}\times\{L<|z|<6|\log\varepsilon|\}}e^{-(\sqrt{2}-\sigma)|z|}\left(|\phi|+|\nabla_y\phi|+|\nabla_y^2\phi|\right).
\end{aligned}$$

As before, this can be written as

$$\begin{aligned}
\|\phi\|_{C^{1,\theta}(\mathcal{N}^1_\alpha(r))} &\leq \sigma(L)\|\phi\|_{C^{2,\theta}(\mathcal{D}(r+4L)}+C(L)\|\phi\|^2_{C^{2,\theta}(\mathcal{D}(r+4L)}\\
&+ C(L)\varepsilon^2+C(L)\sup_{B_{r+4L}}e^{-\sqrt{2}D_\alpha(y)}\\
&+ CL^{\frac{n-1}{2}}\sup_{B_{3L}(y)}\sum_{\beta\neq\alpha}D_\alpha e^{-\sqrt{2}D_\alpha}\left[|H^\beta+\Delta_0^\beta h_\beta|+|\nabla h_\beta|^2\right]\\
&+ CL^{\frac{n-2}{2}}\sup_{L<|z|<2L}\left(|\phi(y,z)|+|\nabla\phi_z(y,z)|\right).
\end{aligned}$$

The last term can be estimated by (11.2). After simplification this estimate is rewritten as

$$\begin{aligned}
\|\phi\|_{C^{1,\theta}(\mathcal{N}^1_\alpha(r))} &\leq \sigma(L)\|\phi\|_{C^{2,\theta}(\mathcal{D}(r+4L)}+C(L)\varepsilon^2+C(L)\sup_{B_{r+4L}}e^{-\sqrt{2}D_\alpha(y)}\\
&+ \sigma(L)\sum_{\beta=1}^{Q}\sup_{B_{r+4L}(y)}\left|H^\beta+\Delta_0^\beta h_\beta\right|+C(L)\sum_{\beta\neq\alpha}\sup_{B_{r+4L}}e^{-4\sqrt{2}D_\beta}.
\end{aligned}\tag{11.12}$$

Combining (11.2) with (11.12) we obtain (11.1).

12. $C^{2,\theta}$ estimate on ϕ

In the equation of ϕ, (9.4), the coefficients before ϕ have a universal Lipschitz bound. Concerning the Hölder bounds on the right hand side of (9.4), we have the following estimates.

Lemma 12.1. *For any* $(y,z)\in\mathcal{M}_\alpha$,

$$\begin{aligned}
\|\Delta\phi-W''(g_*)\phi\|_{C^\theta(B_{2/3}(y,z))} &\lesssim \varepsilon^2+\sup_{B_1(y)}e^{-\sqrt{2}D_\alpha}+\|\phi\|^2_{C^{2,\theta}(B_1(y,z))}\\
&+ e^{-\sqrt{2}|z|}\|H^\alpha+\Delta_0h_\alpha\|_{C^\theta(B_1(y,0))}\\
&+ e^{-\sqrt{2}|d_\beta(y,z)|}\left(\|\phi\|^2_{C^{2,\theta}(B_2^\beta(y,0))}+\sup_{B_2(y)}e^{-2\sqrt{2}D_\beta}\right)\\
&+ e^{-\sqrt{2}|d_\beta(y,z)|}\|H^\beta+\Delta_0^\beta h_\beta\|_{C^\theta(B_2^\beta(y,0))}.
\end{aligned}$$

The proof is similar to the one for Lemma 11.2 and Lemma 11.3, but now we use Lemma 9.5 instead of Lemma 9.4.

By Schauder estimates, for any $(y,z)\in\mathcal{M}^0_\alpha(r)$,

$$\begin{aligned}
\|\phi\|_{C^{2,\theta}(B_{1/2}(y,z))} &\lesssim \|\phi\|_{C^\theta(B_{2/3}(y,z))}+\|\Delta\phi-W''(g_*)\phi\|_{C^\theta(B_{2/3}(y,z))}\\
&\lesssim \varepsilon^2+\sup_{B_1(y)}e^{-\sqrt{2}D_\alpha}+\|\phi\|^2_{C^{2,\theta}(B_1(y,z))}+\|\phi\|^2_{C^{2,\theta}(B_1(y,0))}\\
&+ e^{-\sqrt{2}|z|}\|H^\alpha+\Delta_0h_\alpha\|_{C^\theta(B_1(y,0))}\\
&+ \sum_{\beta\neq\alpha}e^{-\sqrt{2}|d_\beta(y,z)|}\left(\|\phi\|^2_{C^{2,\theta}(B_2^\beta(y,0))}+\sup_{B_2(y)}e^{-2\sqrt{2}D_\beta}\right)
\end{aligned}$$

$$+ \sum_{\beta\neq\alpha} e^{-\sqrt{2}|d_\beta(y,z)|}\|H^\beta+\Delta_0^\beta h_\beta\|_{C^\theta(B_2^\beta(y,0))}.$$

Because in $\mathcal{M}_\alpha^0$, either $e^{-\sqrt{2}|d_\beta(y,z)|}\lesssim\varepsilon$ or $e^{-\sqrt{2}|d_\beta(y,z)|}\lesssim e^{-\frac{\sqrt{2}}{2}D_\alpha(y)}$, from this we deduce that

$$\begin{aligned}\|\phi\|_{C^{2,\theta}(\mathcal{M}_\alpha^0(r))} &\lesssim \varepsilon^2+\sup_{B_{r+L}} e^{-\sqrt{2}D_\alpha}+\|\phi\|^2_{C^{2,\theta}(\mathcal{M}_\alpha^0(r))}+\|H^\alpha+\Delta_0 h_\alpha\|_{C^\theta(B_{r+L})} \\ &+ \sum_{\beta\neq\alpha}\left[\|H^\beta+\Delta_0^\beta h_\beta\|^2_{C^\theta(B_{r+L})}+\sup_{B_{r+L}} e^{-4\sqrt{2}D_\beta}+\|\phi\|^4_{C^{2,\theta}(\mathcal{M}_\beta^0(r+L))}\right].\end{aligned} \tag{12.1}$$

Adding in α leads to

$$\begin{aligned}\|\phi\|_{C^{2,\theta}(\mathcal{D}(r))} &\lesssim \varepsilon^2+\sum_{\alpha=1}^{Q}\sup_{B_{r+L}} e^{-\sqrt{2}D_\alpha}+\sigma\|\phi\|_{C^{2,\theta}(\mathcal{D}_{r+L})} \\ &+ \|H^\alpha+\Delta_0 h_\alpha\|_{C^\theta(B_{r+L})}+\sum_{\beta\neq\alpha}\|H^\beta+\Delta_0^\beta h_\beta\|^2_{C^\theta(B_{r+L})}.\end{aligned} \tag{12.2}$$

Concerning the Hölder norm of $H^\alpha+\Delta_0 h_\alpha$, we have

Lemma 12.2. *There exist $\sigma\ll 1$ and $L\gg 1$ such that*

$$\|H^\alpha+\Delta_0 h_\alpha\|_{C^\theta(B_r)}\le C\varepsilon^2+C\sum_\beta\sup_{B_{r+L}} e^{-\sqrt{2}D_\beta}+\sigma\|\phi\|_{C^{2,\theta}(\mathcal{D}_{r+L})}+\sigma\sum_{\beta=1}^{Q}\|H_\beta+\Delta_0^\beta h_\beta\|_{C^\theta(B_{r+L})}. \tag{12.3}$$

Proof. First by (10.1) and (B.1)-(B.3), we have

$$\begin{aligned}&\int_{-\delta R}^{\delta R}(\Delta_z\phi-\Delta_0\phi)\,g_\alpha'+(-1)^{\alpha-1}\left(\int_{-\delta R}^{\delta R}\phi g_\alpha''\right)\Delta_0 h_\alpha+2(-1)^{\alpha-1}\int_{-\delta R}^{\delta R} g_\alpha'' g^{ij}(y,0)\frac{\partial\phi}{\partial y_i}\frac{\partial h_\alpha}{\partial y_j} \\ &- \left(\int_{-\delta R}^{\delta R}\phi g_\alpha'''\right)|\nabla_0 h_\alpha(y)|^2+\int_{-\delta R}^{\delta R}H^\alpha(y,z)g_\alpha'\phi_z+\int_{-\delta R}^{\delta R}\xi_\alpha'\phi \\ &= \int_{-\delta R}^{\delta R}\left[W''(g_*)-W''(g_\alpha)\right]g_\alpha'\phi+\int_{-\delta R}^{\delta R}\mathcal{R}(\phi)g_\alpha' \\ &+ \int_{-\delta R}^{\delta R}\left[W'(g_*)-\sum_{\beta=1}^{Q}W'(g_\beta)\right]g_\alpha'-(-1)^\alpha\left(\int_{-\delta R}^{\delta R}|g_\alpha'|^2\right)\left[H^\alpha(y,0)+\Delta_0 h_\alpha(y)\right] \\ &- (-1)^\alpha\int_{-\delta R}^{\delta R}|g_\alpha'|^2\left[H^\alpha(y,z)-H^\alpha(y,0)\right]-(-1)^\alpha\int_{-\delta R}^{\delta R}|g_\alpha'|^2\left[\Delta_0 h_\alpha(y)-\Delta_z h_\alpha(y)\right] \\ &+ \frac{1}{2}\left(\int_{-\delta R}^{\delta R}|g_\alpha'|^2\frac{\partial}{\partial z}g^{ij}(y,z)\right)\frac{\partial h_\alpha}{\partial y_i}\frac{\partial h_\alpha}{\partial y_j} \\ &- \sum_{\beta\neq\alpha}(-1)^\beta\int_{-\delta R}^{\delta R}g_\alpha' g_\beta'\mathcal{R}_{\beta,1}-\sum_{\beta\neq\alpha}\int_{-\delta R}^{\delta R}g_\alpha' g_\beta''\mathcal{R}_{\beta,2}-\sum_{\beta\neq\alpha}\int_{-\delta R}^{\delta R}g_\alpha'\xi_\beta.\end{aligned}$$

We can estimate the Hölder norms of these term one by one, by using (8.8)-(8.13) and Lemma 9.5, which gives

$$\begin{aligned}\|H^\alpha+\Delta_0 h_\alpha\|_{C^\theta(B_1(y))} &\lesssim \varepsilon^2+\sup_{B_2(y)} e^{-\sqrt{2}D_\alpha}+\|\phi\|^2_{C^{2,\theta}(B_2(y)\times(-6|\log\varepsilon|,6|\log\varepsilon|)} \\ &+ \sum_{\beta\neq\alpha}\left[\|\phi\|^4_{C^{2,\theta}(B_2^\beta(y,0))}+\|H^\beta+\Delta_0^\beta h_\beta\|^2_{C^\theta(B_2^\beta(y,0))}+\sup_{B_2^\beta(y,0)} e^{-4\sqrt{2}D_\beta}\right].\end{aligned}$$

Hence

$$\begin{aligned}\|H^\alpha+\Delta_0 h_\alpha\|_{C^\theta(B_r)} &\lesssim \varepsilon^2+\sup_{B_{r+L}} e^{-\sqrt{2}D_\alpha}+\|\phi\|^2_{C^{2,\theta}(\mathcal{D}_{r+L})}+\sum_{\beta\neq\alpha}\|H^\beta+\Delta_0^\beta h_\beta\|^2_{C^\theta(B_{r+L})}+\sum_{\beta\neq\alpha}\sup_{B_2(y)} e^{-4\sqrt{2}D_\beta} \\ &\le C\varepsilon^2+C\sum_\beta\sup_{B_{r+L}} e^{-\sqrt{2}D_\beta}+\sigma\|\phi\|_{C^{2,\theta}(\mathcal{D}_{r+L})}+\sigma\sum_\beta\|H^\beta+\Delta_0^\beta h_\beta\|_{C^\theta(B_{r+L})}.\end{aligned} \tag{12.4}$$

(12.3) follows after some simplification. □

Combining (12.2) with (12.3) we obtain

$$\|\phi\|_{C^{2,\theta}(\mathcal{D}_r)}+\|H^\alpha+\Delta_0 h_\alpha\|_{C^\theta(B_r)} \leq C\varepsilon^2+C\sum_\beta \sup_{B_{r+L}} e^{-\sqrt{2}D_\beta}+\sigma\left(\|\phi\|_{C^{2,\theta}(\mathcal{D}_{r+L})}+\sum_\beta \|H^\beta+\Delta_0^\beta h_\beta\|_{C^\theta(B_{r+L})}\right).$$

An iteration of this inequality from $r+K|\log\varepsilon|$ to r (with K large but depending only on L and σ) gives

$$\|\phi\|_{C^{2,\theta}(\mathcal{D}_r)}+\sum_\beta \|H^\beta+\Delta_0^\beta h_\beta\|_{C^\theta(B_r)} \leq C\varepsilon^2+C\sum_\beta \sup_{B_{r+K|\log\varepsilon|}} e^{-\sqrt{2}D_\beta}. \tag{12.5}$$

13. Improved estimates on horizontal derivatives

In this section we prove an improvement on the $C^{1,\theta}$ estimates of horizontal derivatives of ϕ, $\phi_i := \partial\phi/\partial y_i$. $1 \leq i \leq n-1$.

Differentiating (9.4) in y_i, we obtain an equation for $\phi_i := \phi_{y_i}$,

$$\Delta_z\phi_i+\partial_{zz}\phi_i = W''(g_\alpha)\phi_i-(-1)^\alpha g_\alpha'\left[H_i(y,0)+\Delta_0 h_{\alpha,i}(y)\right]+E_i, \tag{13.1}$$

where $h_{\alpha,i}(y) := \frac{\partial h_\alpha}{\partial y_i}$ and the remainder term

$$\begin{aligned}
E_i &= (\Delta_z\phi_i-\partial_{y_i}\Delta_z\phi)+\frac{\partial H}{\partial y_i}(y,z)\phi_z+H^\alpha(y,z)\partial_z\phi_{y_i}+\left[W''(g_*+\phi)-W''(g_\alpha)\right]\phi_i\\
&+ (-1)^\alpha\left[W''(g_*+\phi)-W''(g_\alpha)\right]g_\alpha' h_{\alpha,i}(y)\\
&+ \sum_{\beta\neq\alpha}(-1)^\beta\left[W''(g_*+\phi)-W''(g_\beta)\right]g_\beta'\left[\frac{\partial d_\beta}{\partial y_i}-\sum_{j=1}^n h_{\beta,j}\left(\Pi_\beta(y,z)\right)\frac{\partial\Pi_\beta^j}{\partial y_i}(y,z)\right]\\
&- g_\alpha'' h_{\alpha,i}(y)\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]-(-1)^\alpha g_\alpha'\left[\frac{\partial H}{\partial y_i}(y,z)-\frac{\partial H}{\partial y_i}(y,0)+\frac{\partial}{\partial y_i}\left(\Delta_z h_\alpha(y)\right)-\Delta_0 h_{\alpha,i}(y)\right]\\
&- (-1)^\alpha g_\alpha'''|\nabla_z h_\alpha|^2 h_{\alpha,i}(y)-g_\alpha''\frac{\partial}{\partial y_i}|\nabla_z h_\alpha|^2\\
&- \sum_{\beta\neq\alpha}\frac{\partial}{\partial y_i}\left[(-1)^\beta g_\beta'\mathcal{R}_{\beta,1}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)+g_\beta''\mathcal{R}_{\beta,2}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)\right]\\
&- \sum_\beta \xi_\beta'\sum_{j=1}^{n-1}h_{\beta,j}\left(\Pi_\beta(y,z)\right)\frac{\partial\Pi_\beta^j}{\partial y_i}(y,z).
\end{aligned}$$

Compared with the orthogonal part in the equation of ϕ, the order of E_i is increased by one due to the appearance of one more term involving horizontal derivatives of ϕ. More precisely, we have

Lemma 13.1. *In $\mathcal{M}_\alpha^2(r)$,*

$$\begin{aligned}
|E_i| &\lesssim \varepsilon^2+\|\phi\|^2_{C^{2,\theta}(\mathcal{D}(r+1))}+\sup_{B_{r+1}} e^{-2\sqrt{2}D_\alpha}+\sum_{\beta=1}^Q \sup_{B_{r+1}^\beta}\left|H^\beta+\Delta_0^\beta h_\beta\right|^2+\varepsilon^{1/5}\sup_{B_{r+1}} e^{-\sqrt{2}D_\alpha}\\
&+ \left(\sup_{B_{r+1}} e^{-\frac{\sqrt{2}}{2}D_\alpha}\right)\left[\sum_{\beta\neq\alpha}\sup_{B_{r+1}^\beta}\left|\nabla H_\beta+\Delta_0^\beta\nabla h_\beta\right|+\sum_{\beta\neq\alpha}\sup_{B_{r+1}^\beta} e^{-2\sqrt{2}D_\beta}+\|\phi\|_{C^{2,\theta}(\mathcal{D}(r+1))}\right].
\end{aligned}$$

We do not give detailed calculations here but only show a commutator estimate needed in the proof of this lemma.

Lemma 13.2. *For any $\varphi \in C^2(B_R^{n-1})$,*

$$\frac{\partial}{\partial y_i}\Delta_z\varphi-\Delta_z\varphi_i = O(\varepsilon)\left(|\nabla^2\varphi(y)|+|\nabla\varphi(y)|\right).$$

Proof. Because $|\nabla f_\alpha| \leq C$, $|\nabla^2 f_\alpha| \lesssim \varepsilon$ and $g_{ij}(y,0) = \delta_{ij}+f_{\alpha,i}(y)f_{\alpha,j}(y)$, we have

$$\left|\nabla_y g_{ij}(y,0)\right| \lesssim \varepsilon.$$

By Lemma 8.5, we see

$$|\nabla_y g^{ij}(y,z)|+|\nabla_y^2 g^{ij}(y,z)| = O(\varepsilon).$$

Then

$$\begin{aligned}\frac{\partial}{\partial y_i}\Delta_z\varphi &= \Delta_z\varphi_i+\sum_{k,l=1}^{n-1}\left(\frac{\partial}{\partial y_i}g^{kl}(y,z)\right)\frac{\partial^2\varphi}{\partial y_k\partial y_l}(y)+\sum_{k=1}^{n-1}\left(\frac{\partial}{\partial y_i}b^k(y,z)\right)\frac{\partial\varphi}{\partial y_k}(y)\\ &= \Delta_z\varphi_i+O(\varepsilon)\left(|\nabla^2\varphi(y)|+|\nabla\varphi(y)|\right). \quad \square\end{aligned}$$

By differentiating (9.3) we obtain

$$\int_{-\delta R}^{\delta R}\phi_i g_\alpha' dz=h_{\alpha,i}(y)\int_{-\delta R}^{\delta R}\phi g_\alpha'' dz=O\left(\|\phi\|^2_{C^1(\mathcal{D}(r))}+\sup_{B_r}e^{-2\sqrt{2}D_\alpha}\right),\quad \forall y\in B_R^{n-1}. \tag{13.2}$$

Take a large constant p so that $W^{2,p}$ embeds into $C^{1,\theta}$. By noting that

$$\begin{aligned}\Big|\int_{-\delta R}^{\delta R}\left(\Delta_z\phi_i-\Delta_0\phi_i\right)g_\alpha'\Big| &\lesssim \varepsilon\int_{-\delta R}^{\delta R}\left(|\nabla^2\phi_i(y,z)|+|\nabla\phi_i(y,z)|\right)|z|g_\alpha'\\ &\lesssim \varepsilon\left[\int_{-\delta R}^{\delta R}\left(|\nabla^2\phi_i(y,z)|+|\nabla\phi_i(y,z)|\right)^2e^{-\sqrt{2}|z|}dz\right]^{\frac12},\end{aligned}$$

proceeding as in Section 10 we obtain for any $y\in B_r$,

$$\begin{aligned}\|H_i^\alpha+\Delta_0h_{\alpha,i}\|_{L^p(B_1(y))} &\lesssim \|E_i\|_{L^\infty(\mathcal{M}^2_\alpha(r+2))}\\ &+ \varepsilon\left[\int_{B_2(y)}\int_{-\delta R}^{\delta R}\left(|\nabla^2\phi_i(y,z)|+|\nabla\phi_i(y,z)|\right)^pe^{-\sigma|z|}dz\right]^{\frac1p}\\ &+ \left[\int_{B_2(y)}\int_{-\delta R}^{\delta R}\left(|\nabla^2\phi_i(y,z)|+|\nabla\phi_i(y,z)|\right)^pe^{-\sigma|z|}dz\right]^{\frac2p}.\end{aligned} \tag{13.3}$$

On the other hand, for any $(y,z)\in\mathcal{M}^1_\alpha(r)$, by standard elliptic estimates, we have

$$\begin{aligned}\|\phi_i\|_{W^{2,p}(B_2(y,z))} &\lesssim \|\phi_i\|_{L^p(B_{5/2}(y,z))}+\|\Delta_z\phi_i+\partial_{zz}\phi_i\|_{L^p(B_{5/2}(y,z))}\\ &\lesssim \|\phi_i\|_{L^\infty(B_3(y,z))}+e^{-\sqrt{2}|z|}\|H_i^\alpha+\Delta_0^\alpha h_{\alpha,i}\|_{L^p(B_3(y))}+\|E_i\|_{L^\infty(\mathcal{M}^2_\alpha(r+3))}.\end{aligned}$$

Substituting this into (13.3) leads to, for any $y\in B_r$,

$$\begin{aligned}\sum_\beta\|H_i^\beta+\Delta_0^\beta h_{\beta,i}\|_{L^p(B_1(y))} &\le \sigma\sup_{\tilde{y}\in B_2(y)}\sum_\beta\|H_i^\beta+\Delta_0^\beta h_{\beta,i}\|_{L^p(B_1(\tilde{y}))}+C\varepsilon^2\\ &+ C\sum_{\beta=1}^Q\sup_{B_4(y)}e^{-2\sqrt{2}D_\beta}+C\sum_{\beta=1}^Q\sup_{B_4^\beta(y)}\left|H^\beta+\Delta_0^\beta h_\beta\right|^2\\ &+ C\varepsilon^{1/5}\sum_{\beta=1}^Q\sup_{B_4(y)}e^{-\sqrt{2}D_\beta}+C\|\phi\|_{C^{2,\theta}(\mathcal{D}(r+4))}\sup_{B_{r+4}}e^{-\frac{\sqrt{2}}{2}D_\alpha}.\end{aligned} \tag{13.4}$$

An iteration of this estimate gives

$$\begin{aligned}\sup_{y\in B_r}\sum_\beta\|H_i^\beta+\Delta_0^\beta h_{\beta,i}\|_{L^p(B_1(y))} &\lesssim \varepsilon^2+\sum_{\beta=1}^Q\sup_{B_{r+K|\log\varepsilon|}}e^{-2\sqrt{2}D_\beta}+\sum_{\beta=1}^Q\sup_{B_{r+K|\log\varepsilon|}}\left|H^\beta+\Delta_0^\beta h_\beta\right|^2\\ &+ \varepsilon^{1/5}\sum_{\beta=1}^Q\sup_{B_{r+K|\log\varepsilon|}}e^{-\sqrt{2}D_\beta}+\|\phi\|_{C^{2,\theta}(\mathcal{D}(r+K|\log\varepsilon|))}\sup_{B_{r+K|\log\varepsilon|}}e^{-\frac{\sqrt{2}}{2}D_\alpha}.\end{aligned}$$

Substituting (12.5) into this we obtain

$$\sup_{y\in B_r}\sum_\beta\|H_i^\beta+\Delta_0^\beta h_{\beta,i}\|_{L^p(B_1(y))}\lesssim\varepsilon^2+\sum_{\beta=1}^Q\sup_{B_{r+2K|\log\varepsilon|}}e^{-\frac{3\sqrt{2}}{2}D_\beta}+\varepsilon^{1/5}\sum_{\beta=1}^Q\sup_{B_{r+2K|\log\varepsilon|}}e^{-\sqrt{2}D_\beta}. \tag{13.5}$$

Then using Lemma 13.1 and (13.2) and proceeding as in Section 11, we obtain

$$\|\phi_i\|_{C^{1,\theta}(\mathcal{M}^0_\alpha(r))}\lesssim\varepsilon^2+\sum_\beta\sup_{B_{r+2K|\log\varepsilon|}}e^{-\frac{3\sqrt{2}}{2}D_\beta}+\varepsilon^{1/5}\sum_\beta\sup_{B_{r+2K|\log\varepsilon|}}e^{-\sqrt{2}D_\beta}. \tag{13.6}$$

14. A lower bound on D_α

Define

$$A_\alpha(r) := \sup_{B_r} e^{-\sqrt{2}D_\alpha} \quad \text{and} \quad A(r) = \sum_{\alpha=1}^{Q} A_\alpha(r).$$

By (9.10) and (13.6),

$$\sup_{B_r} |\Delta_0 h^\alpha(y)| \lesssim \varepsilon^2 + A(r + K|\log\varepsilon|)^{\frac{3}{2}} + \varepsilon^{1/5} A(r + K|\log\varepsilon|).$$

By (10.2), in B_r

$$H^\alpha(y,0) = \frac{4}{\sigma_0}\left[A^2_{(-1)^\alpha} e^{-\sqrt{2}d_{\alpha-1}(y,0)} - A^2_{(-1)^{\alpha-1}} e^{\sqrt{2}d_{\alpha+1}(y,0)}\right] + o\left(A_\alpha(r + K|\log\varepsilon|)\right) + O(\varepsilon^{4/3}).$$

Because $H^\alpha = O(\varepsilon)$, an induction on α from 1 to Q gives

$$A(r) \le C\varepsilon + \frac{1}{2}A(r + K|\log\varepsilon|).$$

An iteration of this estimate from $r = R$ to $r = 5R/6$ gives

$$A(5R/6) \lesssim \varepsilon.$$

In particular, for any $y \in B_{5R/6}$ and $\alpha = 1, \cdots, Q$,

$$D_\alpha(y) \ge \frac{\sqrt{2}}{2}|\log\varepsilon| - C.$$

With this lower bound at hand, (12.5) can now be written as

$$\|\phi\|_{C^{2,\theta}(\mathcal{D}(R/2))} + \sum_{\alpha=1}^{Q} \|H^\alpha + \Delta_0 h_\alpha\|_{C^\theta(B_{R/2})} \lesssim \varepsilon,$$

and (13.6) reads as

$$\|\phi_i\|_{C^{1,\theta}(\mathcal{D}(R/2))} \lesssim \varepsilon^{\frac{6}{5}}, \quad \forall i = 1, \cdots, n-1.$$

Therefore by Lemma 9.6 we get

$$\sum_{\alpha=1}^{Q} \|\Delta_0 h_\alpha\|_{C^\theta(B_{R/2})} \lesssim \varepsilon^{\frac{6}{5}}.$$

Now (10.2) reads as

$$\operatorname{div}\left(\frac{\nabla f_\alpha(y)}{\sqrt{1+|\nabla f_\alpha(y)|^2}}\right) = \frac{4}{\sigma_0}\left[A^2_{(-1)^{\alpha-1}} e^{-\sqrt{2}d_{\alpha-1}(y)} - A^2_{(-1)^\alpha} e^{-\sqrt{2}d_{\alpha+1}(y)}\right] + O\left(\varepsilon^{\frac{6}{5}}\right). \tag{14.1}$$

A remark is in order concerning whether we can improve this lower bound.

Remark 14.1. *If there exists a constant M, $\alpha \in \{1, \cdots, Q\}$ and $y_\varepsilon \in B_{R/2}$ such that*

$$d_{\alpha+1}(y_\varepsilon, 0) \le \frac{\sqrt{2}}{2}|\log\varepsilon| + M.$$

After a rotation and translation, we may assume $y_\varepsilon = 0$ and $f_\alpha(0) = 0$, $\nabla f_\alpha(0) = 0$.

Define

$$\tilde{f}_\beta(y) := f_\beta(\varepsilon^{-1/2}y) - \frac{\sqrt{2}}{2}(\beta - \alpha)|\log\varepsilon|, \quad \forall \beta \in \{1, \cdots, Q\}.$$

By the curvature bound on Γ_α and Lemma 8.3, for any $\beta \in \{1, \cdots, Q\}$, if $\tilde{f}_\beta(0)$ does not go to $\pm\infty$, then in $B^{n-1}_{R^{2/3}}$,

$$|\nabla f_\beta| \lesssim \varepsilon^{\frac{1}{6}}.$$

Subsisting this into (14.1) and performing a rescaling we obtain

$$\Delta\tilde{f}_\beta(y) = \frac{4}{\sigma_0}\left[A^2_{(-1)^{\alpha-1}} e^{-\sqrt{2}\left(\tilde{f}_\beta(y) - \tilde{f}_{\beta-1}(y)\right)} - A^2_{(-1)^\alpha} e^{-\sqrt{2}\left(\tilde{f}_{\beta+1}(y) - \tilde{f}_\beta(y)\right)}\right] + O(\varepsilon^{1/6}), \quad \text{in } B_{R^{1/2}}.$$

Moreover, as $\varepsilon \to 0$, $\tilde{f}_\beta$ converges in $C^2_{loc}(\mathbb{R}^{n-1})$ to $\bar{f}_\beta$, which is a nontrivial entire solution to (2.10). This blow up procedure will be employed in Section 20.

15. Multiplicity one case

If there is only one connected component of $\{u = 0\}$, the estimates can be simplified a lot. For example, now (12.5) reads as

$$\|\phi\|_{C^{2,\theta}(\mathcal{D}(3R/4))} + \|H + \Delta_0 h\|_{C^\theta(B^{n-1}_{3R/4})} \lesssim \varepsilon^2.$$

On the other hand, by Lemma 9.6 we get

$$\|h\|_{C^{2,\theta}(B^{n-1}_{3R/4})} \lesssim \varepsilon^2.$$

Hence

$$\|H\|_{C^\theta(B^{n-1}_{3R/4})} \lesssim \varepsilon^2.$$

After a scaling, this is equivalent to the condition that

$$\left\|\operatorname{div}\left(\frac{\nabla f_\varepsilon}{\sqrt{1+|\nabla f_\varepsilon|^2}}\right)\right\|_{C^\theta(B^{n-1}_{3/4})} \lesssim \varepsilon^{1-\theta}.$$

Because $\sup_{B_{3/4}} |\nabla f_\varepsilon| \le C$, by standard elliptic estimates on mean curvature equations [39, Chapter 16], we get

$$\|f_\varepsilon\|_{C^{2,\theta}(B^{n-1}_{2/3})} \le C,$$

where the constant C is independent of ε. This completes the proof of Theorem 2.7.

16. Arbitrary Riemannian metric

In the previous analysis the background metric is an Euclidean one. Now we consider an arbitrary Riemannian metric. Since we are concerned with local problems, we will work in the following setting. Assume $B_1(0) \subset \mathbb{R}^n$ is equipped with a C^3 Riemannian metric $g = g_{ij}(x)dx^i \otimes dx^j$. We assume the exponential map is a globally defined diffeomorphism.

Assume

- $u_\varepsilon \in C^3(B_1)$ is a sequence of solutions to the Allen-Cahn equation
$$\varepsilon\Delta_g u_\varepsilon = \frac{1}{\varepsilon}W'(u_\varepsilon).$$
- The nodal set $\{u_\varepsilon = 0\}$ consists of Q components,
$$\Gamma_{\alpha,\varepsilon} = \{(x', f_{\alpha,\varepsilon}(x'))\}, \quad \alpha = 1, \cdots, Q,$$
where $f_{1,\varepsilon} < f_{2,\varepsilon} < \cdots < f_{Q,\varepsilon}$.
- For each α, the curvature of $\Gamma_{\alpha,\varepsilon}$ is uniformly bounded as $\varepsilon \to 0$.

By this curvature bound, the Fermi coordinates with respect to $\Gamma_{\alpha,\varepsilon}$ is well defined and C^2 in a δ-neighborhood of $\Gamma_{\alpha,\varepsilon}$. In the Fermi coordinates, the Laplace-Beltrami operator Δ_g has the same expansion as in Section 8, for more details see [27, Section 2]. By denoting $H_{\alpha,\varepsilon}$ the mean curvature of $\Gamma_{\alpha,\varepsilon}$, we get the following Toda system

$$H_{\alpha,\varepsilon} = \frac{4}{\sigma_0\varepsilon}\left[A^2_{(-1)^{\alpha-1}}e^{-\frac{\sqrt{2}}{\varepsilon}d_{\alpha-1,\varepsilon}} - A^2_{(-1)^\alpha}e^{-\frac{\sqrt{2}}{\varepsilon}d_{\alpha+1,\varepsilon}}\right] + O\left(\varepsilon^{\frac{1}{6}}\right). \tag{16.1}$$

If $\Gamma_{\alpha,\varepsilon}$ collapse to a same minimal hypersurface Γ_∞, this system can be written as a Jacobi-Toda system on Γ_∞ as in [27].

Now we come to the proof of Theorem 2.6.

Proof of Theorem 2.6. First by results in Section 7, we can assume $f_{\alpha,\varepsilon}$ are uniformly bounded in $C^{1,1}(B^{n-1}_2)$. Hence we can assume they converge to f_∞ in $C^{1,\theta}(B^{n-1}_2)$ for any $\theta \in (0,1)$. Assume there exists $\alpha \in \{1, \cdots, Q\}$ such that $f_{\alpha,\varepsilon}$ do not converge to f_∞ in $C^2(B^{n-1}_1)$.

Using the Fermi coordinates (y, z) with respect to $\Gamma_\infty := \{x_n = f_\infty(x')\}$, $\Gamma_{\alpha,\varepsilon}$ is represented by the graph $\{z = f_{\alpha,\varepsilon}(y)\}$, where $f_{\alpha,\varepsilon}$ converges to 0 in C^1 but not in C^2. Assume $|\nabla^2 f_{\alpha,\varepsilon}(y_\varepsilon)|$ does not converge to 0, then we can preform the same blow up analysis as in Remark 14.1, with the base point at y_ε. This procedure results in a nontrivial solution of (2.10). □

Part 3. Second order estimate for stable solutions: Proof of Theorem 3.6

This part is devoted to the proof of Theorem 3.6 which in turn implies Theorem 3.5 and Theorem 3.8. Throughout this part we are in dimension two and u_ε denotes a solution satisfying the hypothesis of Theorem 3.6. We shall use the stability condition of u_ε to prove the uniform second order estimates.

17. A lower bound on the intermediate distance

In this section we use the stability condition to prove

Proposition 17.1. *For any $\sigma > 0$, there exists a universal constant $C(\sigma)$ such that for any α, $x_1 \in (-5/6, 5/6)$ and $f_{\alpha,\varepsilon}(x_1) \in (-5/6, 5/6)$,*

$$dist\left((x_1, f_{\alpha,\varepsilon}(x_1)), \Gamma_{\alpha+1,\varepsilon}\right) \geq \frac{\sqrt{2}-\sigma}{2}\varepsilon|\log\varepsilon| - C(\sigma)\varepsilon.$$

The idea of proof is to choose a direction derivative of u_ε to construct a subsolution to the linearized equation and perform a surgery as in Lemma 5.1.

17.1. **An upper bound on $\mathcal{Q}(\varphi_\varepsilon)$.** Without loss of generality, assume $u_\varepsilon > 0$ in $\{f_{\alpha,\varepsilon}(x_1) < x_2 < f_{\alpha+1,\varepsilon}(x_1)\} \cap \mathcal{C}_{6/7}$. Recall that Lemma 7.1 still holds. Hence near $\{x_2 = f_{\alpha,\varepsilon}(x_1)\}$, $\frac{\partial u_\varepsilon}{\partial x_2} > 0$, while near $\{x_2 = f_{\alpha+1,\varepsilon}(x_1)\}$, $\frac{\partial u_\varepsilon}{\partial x_2} < 0$.

Let $\mathcal{D}_{\alpha,\varepsilon}$ be the connected component of $\{\frac{\partial u_\varepsilon}{\partial x_2} > 0\} \cap \mathcal{C}_{6/7}$ containing $\{x_2 = f_{\alpha,\varepsilon}(x_1)\}$. Let φ_ε be the restriction of $\frac{\partial u_\varepsilon}{\partial x_2}$ to this domain, extended to be 0 outside. After such an extension, φ_ε is a nonnegative continuous function and in $\{\varphi_\varepsilon > 0\}$ it satisfies the linearized equation

$$\varepsilon\Delta\varphi_\varepsilon = \frac{1}{\varepsilon}W''(u_\varepsilon)\varphi_\varepsilon. \tag{17.1}$$

Concerning $\mathcal{D}_{\alpha,\varepsilon}$ we have

Lemma 17.2. *$\mathcal{D}_{\alpha,\varepsilon} \cap \{|u_\varepsilon| < 1-b\} \cap \mathcal{C}_{6/7}$ belongs to an $O(\varepsilon)$ neighborhood of $\{x_2 = f_{\alpha,\varepsilon}(x_1)\}$.*

Proof. By Lemma 7.1, $\{|u_\varepsilon| < 1-b\} \cap \mathcal{C}_{6/7}$ belongs to an $O(\varepsilon)$ neighborhood of $\{x_2 = f_{\alpha,\varepsilon}(x_1)\}$, where $\frac{\partial u_\varepsilon}{\partial x_2} > 0$. On the other hand, since $\frac{\partial u_\varepsilon}{\partial x_2} < 0$ in a neighborhood of $\{x_2 = f_{\alpha\pm1,\varepsilon}(x_1)\}$, $\mathcal{D}_{\alpha,\varepsilon} \cap \mathcal{C}_{6/7}$ belongs to the set $\{f_{\alpha-1,\varepsilon} < x_2 < f_{\alpha+1,\varepsilon}\}$. □

Choose an arbitrary point $x_\varepsilon \in \{x_2 = f_{\alpha,\varepsilon}(x_1), |x_1| < 5/6, |x_2| < 5/6\}$. Take an $\eta_1 \in C_0^\infty(B_{1/100}(x_\varepsilon))$, satisfying $\eta_1 \equiv 1$ in $B_{1/200}(x_\varepsilon)$ and $|\nabla\eta_1| \leq 1000$. Multiplying (17.1) by $\varphi_\varepsilon\eta_1^2$ and integrating by parts leads to

$$\begin{aligned}\int_{B_{1/100}(x_\varepsilon)} \varepsilon|\nabla(\varphi_\varepsilon\eta_1)|^2 + \frac{1}{\varepsilon}W''(u_\varepsilon)\varphi_\varepsilon^2\eta_1^2 = \int_{B_{1/100}(x_\varepsilon)} \varepsilon\varphi_\varepsilon^2|\nabla\eta_1|^2 &\leq C\int_{B_{1/100}(x_\varepsilon)} \varepsilon\varphi_\varepsilon^2 \\ &\leq C.\end{aligned} \tag{17.2}$$

In the above we have used the following fact.

Lemma 17.3. *There exists a universal constant C such that*

$$\int_{B_{1/100}(x_\varepsilon)} \varepsilon\varphi_\varepsilon^2 \leq C.$$

Proof. We divide the estimate into two parts: $\{|u_\varepsilon| < 1-b\}$ and $\{|u_\varepsilon| > 1-b\}$.

Step 1. There exists a universal constant C such that

$$\int_{\mathcal{D}_{\alpha,\varepsilon}\cap\{|u_\varepsilon|<1-b\}\cap B_{1/50}(x_\varepsilon)} \varepsilon\left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 \leq C. \tag{17.3}$$

Because $|\nabla u_\varepsilon| \lesssim \varepsilon^{-1}$, this estimate follows from the fact that

$$\left|\mathcal{D}_{\alpha,\varepsilon} \cap \{|u_\varepsilon| < 1-b\} \cap \mathcal{C}_{6/7}\right| \leq C\varepsilon,$$

which in turn is a consequence of Lemma 17.2, the co-area formula and the following two facts: (i) for any $t \in [-1+b, 1-b]$, $\{u_\varepsilon = t\}$ is a smooth curve with uniformly bounded curvature and hence its length is uniformly bounded; (ii) by Lemma 7.1, $\frac{\partial u_\varepsilon}{\partial x_2} \geq c\varepsilon^{-1}$ in $\{|u_\varepsilon| < 1-b\}$.

Step 2. There exists a universal constant C such that

$$\int_{\{|u_\varepsilon|>1-b\}\cap B_{1/100}(x_\varepsilon)} \varepsilon\varphi_\varepsilon^2 \leq C(b). \tag{17.4}$$

In order to prove this estimate, take a cut-off function $\eta_2 \in C_0^\infty(B_{1/50}(x_\varepsilon))$ with $\eta_2 \equiv 1$ in $B_{1/100}(x_\varepsilon)$ and $|\nabla\eta_2| \leq 1000$, and $\zeta \in C^\infty(-1,1)$ with $\zeta \equiv 1$ in $(-1,-1+b)\cup(1-b,1)$, $\zeta \equiv 0$ in $(-1+2b,1-2b)$ and $|\zeta'| \leq 2b^{-1}$. Multiplying (17.1) by $\varphi_\varepsilon\eta_2^2\zeta(u_\varepsilon)^2$ and integrating by parts leads to

$$\begin{aligned}
&\int_{B_{1/50}(x_\varepsilon)} \varepsilon|\nabla\left(\varphi_\varepsilon\eta_2\zeta(u_\varepsilon)\right)|^2 + \frac{1}{\varepsilon}W''(u_\varepsilon)\varphi_\varepsilon^2\eta_2^2\zeta(u_\varepsilon)^2 \\
= &\int_{B_{1/50}(x_\varepsilon)} \varepsilon\varphi_\varepsilon^2|\nabla\left(\eta_2\zeta(u_\varepsilon)\right)|^2 \\
\lesssim &\int_{B_{1/50}(x_\varepsilon)} \varepsilon\varphi_\varepsilon^2\left[|\nabla\eta_2|^2\zeta(u_\varepsilon)^2 + 2\eta_2\zeta(u_\varepsilon)|\zeta'(u_\varepsilon)||\nabla\eta_2||\nabla u_\varepsilon| + \eta_2^2\zeta'(u_\varepsilon)^2|\nabla u_\varepsilon|^2\right] \\
\lesssim &\ \varepsilon\int_{B_{1/50}(x_\varepsilon)} \varphi_\varepsilon^2 + \frac{1}{\varepsilon}\int_{\{1-2b<|u_\varepsilon|<1-b\}\cap B_{1/50}(x_\varepsilon)} \varphi_\varepsilon^2,
\end{aligned}$$

where we have substituted the estimate $|\nabla u_\varepsilon| \lesssim \varepsilon^{-1}$ in the last line.

Since $W''(u_\varepsilon) \geq c(b) > 0$ in $\{|u_\varepsilon| > 1-b\}$, we obtain

$$\begin{aligned}
\int_{\{|u_\varepsilon|>1-b\}\cap B_{1/100}(x_\varepsilon)} \varepsilon\varphi_\varepsilon^2 &\leq C\varepsilon\int_{B_{1/50}(x_\varepsilon)} W''(u_\varepsilon)\varphi_\varepsilon^2\eta_2^2\zeta(u_\varepsilon)^2 \\
&\leq C\varepsilon^3\int_{B_{1/50}(x_\varepsilon)} |\nabla u_\varepsilon|^2 + C\varepsilon\int_{\{1-2b<|u_\varepsilon|<1-b\}\cap B_{1/50}(x_\varepsilon)} \varphi_\varepsilon^2 \\
&\leq C.
\end{aligned}$$

Combining Step 1 and Step 2 we finish the proof. □

17.2. **A surgery on φ_ε.** Next we use the smoothing modification in the proof of Proposition 6.1 to decrease the left hand side of (17.2).

Without loss of generality, assume $f_{\alpha,\varepsilon}(0) = 0$, $f_{\alpha+1,\varepsilon}(0) = \rho_\varepsilon$ and $\rho_\varepsilon \leq \varepsilon|\log\varepsilon|$. By Lemma 7.1, $\rho_\varepsilon \gg \varepsilon$. For any fixed constant $L > 0$, $u_\varepsilon > 1-b$ in $\Omega_{\alpha,\varepsilon} := \{|x_1| < L\varepsilon, L\varepsilon < x_2 < \rho_\varepsilon - L\varepsilon\}$. Let $\widetilde{\varphi}_\varepsilon$ be the solution of

$$\begin{cases} \varepsilon\Delta\widetilde{\varphi}_\varepsilon = \frac{1}{\varepsilon}W''(u_\varepsilon)\widetilde{\varphi}_\varepsilon, & \text{in } \Omega_{\alpha,\varepsilon}, \\ \widetilde{\varphi}_\varepsilon = \varphi_\varepsilon, & \text{on } \partial\Omega_{\alpha,\varepsilon}. \end{cases}$$

By the stability of u_ε, such an $\widetilde{\varphi}_\varepsilon$ exists uniquely.

A direct integration by parts gives

$$\begin{aligned}
&\left[\int_{\Omega_{\alpha,\varepsilon}} \varepsilon|\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}W''(u_\varepsilon)\varphi_\varepsilon^2\right] - \left[\int_{\Omega_{\alpha,\varepsilon}} \varepsilon|\nabla\widetilde{\varphi}_\varepsilon|^2 + \frac{1}{\varepsilon}W''(u_\varepsilon)\widetilde{\varphi}_\varepsilon^2\right] \\
= &\int_{\Omega_{\alpha,\varepsilon}} \varepsilon|\nabla\left(\varphi_\varepsilon - \widetilde{\varphi}_\varepsilon\right)|^2 + \frac{1}{\varepsilon}W''(u_\varepsilon)\left(\varphi_\varepsilon - \widetilde{\varphi}_\varepsilon\right)^2 \qquad (17.5) \\
\geq &\ \frac{c}{\varepsilon}\int_{\Omega_{\alpha,\varepsilon}} \left(\varphi_\varepsilon - \widetilde{\varphi}_\varepsilon\right)^2.
\end{aligned}$$

Because $u_\varepsilon > 1-b$ in $\Omega_{\alpha,\varepsilon}$, $2-\delta(b) < W''(u_\varepsilon) < 2+\delta(b)$ in $\Omega_{\alpha,\varepsilon}$, where $\delta(b)$ is constant satisfying $\lim_{b\to0}\delta(b) = 0$. Therefore,

$$\Delta\widetilde{\varphi}_\varepsilon \leq \frac{2+\delta(b)}{\varepsilon^2}\widetilde{\varphi}_\varepsilon, \quad \text{in } \Omega_{\alpha,\varepsilon}.$$

On $\partial\Omega_{\alpha,\varepsilon}\cap\{x_2 = L\varepsilon\}$, by Lemma 7.1,

$$\widetilde{\varphi}_\varepsilon = \varphi_\varepsilon = \frac{\partial u_\varepsilon}{\partial x_2} \geq \frac{c}{\varepsilon}.$$

By constructing an explicit subsolution, we obtain

$$\widetilde{\varphi}_\varepsilon(x_1,x_2) \geq \frac{c}{\varepsilon}e^{-\sqrt{2+\delta(b)+\frac{C}{L^2}}\frac{x_2-L\varepsilon}{\varepsilon}}, \quad \text{in } \left\{|x_1| < \frac{L\varepsilon}{2}, \frac{\rho_\varepsilon}{4} < x_2 < \frac{3\rho_\varepsilon}{4}\right\}. \tag{17.6}$$

Lemma 17.4. *For any δ fixed, if ε is small enough, $\varphi_\varepsilon = 0$ in $\{|x_1| < \frac{L\varepsilon}{2}, \frac{(1+\delta)\rho_\varepsilon}{2} < x_2 < \frac{3\rho_\varepsilon}{4}\}$.*

Proof. Let $\epsilon := \varepsilon/\rho_\varepsilon \ll 1$ and $u_\epsilon(x) := u_\varepsilon(\rho_\varepsilon^{-1}x)$, which is a solution of (2.3) with parameter ϵ.

The nodal set of u_ϵ has the form $\cup_\beta\{x_2 = \tilde{f}_{\beta,\epsilon}(x_1)\}$, where $\tilde{f}_{\beta,\epsilon}(x_1) = f_{\beta,\varepsilon}(\rho_\varepsilon x_1)/\rho_\varepsilon$. Thus for any $x_1 \in (-\rho_\varepsilon^{-1}, \rho_\varepsilon^{-1})$,

$$|\tilde{f}''_{\alpha,\epsilon}(x_1)| \le 4\rho_\varepsilon, \quad |\tilde{f}''_{\alpha+1,\epsilon}(x_1)| \le 4\rho_\varepsilon.$$

and $\tilde{f}_{\alpha,\epsilon}(0) = \tilde{f}'_{\alpha,\epsilon}(0) = 0$, $\tilde{f}_{\alpha+1,\epsilon}(0) = 1$.

Since $\rho_\varepsilon \to 0$, $\tilde{f}_{\alpha,\epsilon} \to 0$ uniformly on any compact set of $\mathbb{R}$. Because different components of $\{u_\epsilon = 0\}$ do not intersect, $\tilde{f}_{\alpha+1,\epsilon} \to 1$ uniformly on any compact set of $\mathbb{R}$.

Consider the distance type function Ψ_ϵ, which is defined by the relation

$$u_\epsilon = g\left(\frac{\Psi_\epsilon}{\epsilon}\right).$$

By the vanishing viscosity method, in any compact set of $\{|x_2| < 1\}$, Ψ_ϵ converges uniformly to

$$\Psi_\infty(x_1, x_2) := \begin{cases} 1 - |x_2|, & x_2 \ge 1/2, \\ x_2, & -1/2 \le x_2 \le 1/2, \\ -1 - |x_2|, & x_2 \le -1/2. \end{cases}$$

Moreover, because Ψ_∞ is C^1 in $\{|x_1| < 1, (1+\delta)/2 < x_2 < 3/4\}$, Ψ_ϵ converges in $C^1(\{|x_1| < 1, (1+\delta)/2 < x_2 < 3/4\})$. In particular, for all ϵ small,

$$\frac{\partial u_\epsilon}{\partial x_2} = \frac{1}{\epsilon} g'\left(\frac{\Psi_\epsilon}{\epsilon}\right)\frac{\partial \Psi_\epsilon}{\partial x_2} > 0, \quad \text{in } \left\{|x_1| < 1, \frac{1+\delta}{2} < x_2 < \frac{3}{4}\right\}.$$

Rescaling back we finish the proof. □

Remark 17.5. *The above proof also shows that*

$$dist\left((x_1, f_{\alpha,\varepsilon}(x_1)), \Gamma_{\alpha+1,\varepsilon}\right) = (1 + o(1))\left(f_{\alpha+1,\varepsilon}(x_1) - f_{\alpha,\varepsilon}(x_1)\right).$$

By this lemma and (17.6), we obtain

$$\int_{\Omega_{\alpha,\varepsilon}} (\varphi_\varepsilon - \widetilde{\varphi}_\varepsilon)^2 \ge \int_{\{|x_1| < \frac{L\varepsilon}{2}, \frac{(1+\delta)\rho_\varepsilon}{2} < x_2 < \frac{3\rho_\varepsilon}{4}\}} \widetilde{\varphi}_\varepsilon^2 \ge \frac{c(L)}{\varepsilon} e^{-\frac{(1+\delta)}{2}\sqrt{2+\delta(b)+\frac{C}{L^2}}\frac{\rho_\varepsilon}{\varepsilon}}. \tag{17.7}$$

As in the proof of Proposition 6.1, by combining (17.2), (17.7) and the stability of u_ε, we obtain

$$\frac{c(L)}{\varepsilon} e^{-\frac{(1+\delta)}{2}\sqrt{2+\delta(b)+\frac{C}{L^2}}\frac{\rho_\varepsilon}{\varepsilon}} \le C.$$

By choosing δ, $\delta(b)$ sufficiently small, L sufficiently large (depending only on σ), this implies that

$$\rho_\varepsilon \ge \frac{\sqrt{2} - \sigma}{2}\varepsilon|\log\varepsilon| - C(\sigma)\varepsilon,$$

which in view of Remark 17.5 finishes the proof of Proposition 17.1.

18. Toda system

18.1. **Optimal approximation.** As in Section 8, we still work in the stretched version, i.e. after the rescaling $x \mapsto \varepsilon^{-1}x$. The analysis in Section 8 still holds, although now $\{u = 0\} = \cup_\alpha \Gamma_\alpha$, where the cardinality of the index set α could go to infinity.

Given a sequence of functions $h_\alpha \in C^2(-R, R)$, let (note here a sign difference with Section 9)

$$g_\alpha(y, z) := \bar{g}\left((-1)^\alpha\left(z - h_\alpha(y)\right)\right),$$

where (y, z) is the Fermi coordinates with respect to Γ_α. Define the function $g(y, z; h_\alpha)$ in the following way:

$$g(y, z; h_\alpha) := g_\alpha + \sum_{\beta<\alpha}\left(g_\beta + (-1)^\beta\right) + \sum_{\beta>\alpha}\left(g_\beta - (-1)^\beta\right) \quad \text{in } \mathcal{M}_\alpha.$$

By the definition of $\bar{g}$ and Proposition 17.1, the above sum involves only finitely many terms (at most 25 terms).

Similar to Proposition 9.1, we have

Proposition 18.1. *There exists (h_α) such that for any $|\alpha| > 100$, $h_\alpha \equiv 0$, while for any $|\alpha| \le 100$, in the Fermi coordinates with respect to Γ_α,*

$$\int_{-\delta R}^{\delta R} [u(y,z) - g(y,z;h_\alpha)]\, \bar{g}'\left((-1)^\alpha \left(z - h_\alpha(y)\right)\right) dz = 0, \quad \forall y \in (-5R/6, 5R/6). \tag{18.1}$$

Denote $g_*(y,z) := g(y,z;\mathbf{h}(y))$, where $\mathbf{h}$ is as in the previous lemma. Let $\phi := u - g_*$. As in Subsection 9.1, denote

$$g_\alpha'(y,z) = \bar{g}'\left((-1)^\alpha \left(z - h_\alpha(y)\right)\right), \quad g_\alpha''(y,z) = \bar{g}''\left((-1)^\alpha \left(z - h_\alpha(y)\right)\right), \quad \cdots.$$

In the Fermi coordinates with respect to Γ_α, ϕ satisfies the following equation

$$\begin{aligned} & \Delta_z \phi - H^\alpha(y,z)\partial_z\phi + \partial_{zz}\phi \\ = \; & W'(g_* + \phi) - \sum_\beta W'(g_\beta) + (-1)^\alpha g_\alpha' \left[H^\alpha(y,z) + \Delta_z h_\alpha(y)\right] - g_\alpha'' |\nabla_z h_\alpha|^2 \\ + \; & \sum_{\beta \ne \alpha} \left[(-1)^\beta g_\beta' \mathcal{R}_{\beta,1}\left(\Pi_\beta(y,z), d_\beta(y,z)\right) - g_\beta'' \mathcal{R}_{\beta,2}\left(\Pi_\beta(y,z), d_\beta(y,z)\right)\right] - \sum_\beta \xi_\beta, \end{aligned} \tag{18.2}$$

where $\mathcal{R}_{\beta,1}$, $\mathcal{R}_{\beta,2}$ and ξ_β are defined as in Subsection 9.1.

18.2. **Estimates on ϕ.** By (12.5) and Proposition 17.1, for any $\sigma > 0$ and $|\alpha| \le 90$,

$$\|\phi\|_{C^{2,1/2}(\mathcal{M}_\alpha(r))} + \|H^\alpha + \Delta_0^\alpha h_\alpha\|_{C^{1/2}(-r,r)} \lesssim \varepsilon^{2-2\sigma} + \sup_{(-r-K|\log\varepsilon|, r+K|\log\varepsilon|)} e^{-\sqrt{2}D_\alpha}. \tag{18.3}$$

Substituting Proposition 17.1 into (18.3) gives a first (non-optimal) bound

$$\|\phi\|_{C^{2,1/2}(\mathcal{M}_\alpha(6R/7))} + \|H^\alpha + \Delta_0^\alpha h_\alpha\|_{C^{1/2}(B_{6R/7})} \lesssim \varepsilon^{1-\sigma}, \quad \forall |\alpha| \le 90. \tag{18.4}$$

By (13.6), we can improve the estimates on $\phi_y := \partial\phi/\partial y$ to

$$\|\phi_y\|_{C^{1,1/2}(\mathcal{M}_\alpha(6R/7))} \lesssim \varepsilon^{7/6}. \tag{18.5}$$

18.3. **A Toda system.** Denote

$$A_\alpha(r) := \sup_{(-r,r)} e^{-\sqrt{2}D_\alpha}.$$

By Proposition 17.1, for any $r < 5R/6$, $A_\alpha(r) \lesssim \varepsilon^{1-\sigma}$.

In $(-6R/7, 6R/7)$, (10.2) reads as

$$\frac{f_\alpha''(x_1)}{\left[1 + |f_\alpha'(x_1)|^2\right]^{3/2}} = \frac{4}{\sigma_0}\left[A_{(-1)^{\alpha-1}}^2 e^{-\sqrt{2}d_{\alpha-1}(x_1, f_\alpha(x_1))} - A_{(-1)^\alpha}^2 e^{-\sqrt{2}d_{\alpha+1}(x_1, f_\alpha(x_1))}\right] + O\left(\varepsilon^{7/6}\right). \tag{18.6}$$

19. Reduction of the stability condition

In this section we show that the blow up procedure in Remark 14.1 preserves the stability condition. More precisely, if u is stable, (f_α) satisfies a kind of stability condition related to Toda system.

Fix a smooth function η_3 defined on $\mathbb{R}$ satisfying $\eta_3 \equiv 1$ in $(-\infty, 0)$, $\eta_3 \equiv 0$ in $(1, +\infty)$ and $|\eta_3'| + |\eta_3''| \le 16$. Take a large constant L and define

$$\chi(y,z) := \begin{cases} \eta_3\left(\frac{z - \rho_\alpha^+(y)}{L}\right), & \text{if } z > 0, \\ \eta_3\left(\frac{-z + \rho_\alpha^-(y)}{L}\right), & \text{if } z < 0. \end{cases}$$

Clearly we have $\chi \equiv 1$ in $\mathcal{M}_\alpha$, $\chi \equiv 0$ outside $\{\rho_\alpha^-(y) - L < z < \rho_\alpha^+(y) + L\}$. Moreover, $|\nabla\chi| \lesssim L^{-1}$, $|\nabla^2\chi| \lesssim L^{-2}$.

For any $\eta \in C_0^\infty(-5R/6, 5R/6)$, let

$$\varphi(y,z) := \eta(y) g_\alpha'(y,z) \chi(y,z).$$

The stability condition for u implies that

$$\int_{\mathcal{C}_{5R/6}} |\nabla\varphi|^2 + W''(u)\varphi^2 \ge 0.$$

The purpose of this section is to rewrite this inequality as a stability condition for the Toda system (18.6).

In the Fermi coordinates with respect to Γ_α, we have (Recall that now in (8.2), the metric tensor g_{ij} has only one component, which is denoted by $\lambda(y,z)$ here)

$$|\nabla\varphi(y,z)|^2 = \Big|\frac{\partial\varphi}{\partial z}(y,z)\Big|^2 + \lambda(y,z)\Big|\frac{\partial\varphi}{\partial y}(y,z)\Big|^2.$$

19.1. **The horizontal part.** A direct differentiation leads to

$$\frac{\partial\varphi}{\partial y} = \eta'(y)g'_\alpha\chi - \eta(y)g''_\alpha\chi h'_\alpha(y) + \eta(y)g'_\alpha\chi_y.$$

Since $c \le \lambda(y,z) \le C$,

$$\begin{aligned}
\int_{\mathcal{C}_{5R/6}} \Big|\frac{\partial\varphi}{\partial y}(y,z)\Big|^2\lambda(y,z)dzdy &\lesssim \int_{-5R/6}^{5R/6}\int_{-\delta R}^{\delta R} \eta'(y)^2|g'_\alpha|^2\chi^2 + \eta(y)^2|g''_\alpha|^2\chi^2 h'_\alpha(y)^2 + \eta^2|g'_\alpha|^2\chi_y^2 \\
&\lesssim \int_{-5R/6}^{5R/6}\eta'(y)^2dy + \varepsilon^2\int_{-5R/6}^{5R/6}\eta(y)^2dy \\
&\quad + \frac{1}{L}\int_{-5R/6}^{5R/6}\eta(y)^2\left[e^{-2\sqrt{2}\rho_\alpha^+(y)} + e^{2\sqrt{2}\rho_\alpha^-(y)}\right]dy.
\end{aligned}$$

Here the last term follows from the following three facts:

- in $\{\chi_y \neq 0\}$, which is exactly $\{\rho_\alpha^+(y) < z < \rho_\alpha^+(y) + L\} \cup \{\rho_\alpha^-(y) - L < z < \rho_\alpha^-(y)\}$, $|\chi_y| \lesssim L^{-1}$;
- in $\{\rho_\alpha^+(y) < z < \rho_\alpha^+(y)+L\}$ (respectively, $\{\rho_\alpha^-(y)-L < z < \rho_\alpha^-(y)\}$), $g'_\alpha \lesssim e^{-\sqrt{2}\rho_\alpha^+(y)}$ (respectively, $g'_\alpha \lesssim e^{\sqrt{2}\rho_\alpha^-(y)}$);
- By (18.4) and Lemma 9.6, for $y \in (-6R/7, 6R/7)$, $h'_\alpha(y)^2 \lesssim \varepsilon^{2-2\sigma}$.

19.2. **The vertical part.** As before we have

$$\varphi_z = \eta g''_\alpha\chi + \eta g'_\alpha\chi_z.$$

Thus by a direct expansion and integrating by parts, we have

$$\begin{aligned}
\int_{\mathcal{C}_{5R/6}} \varphi_z^2\lambda(y,z)dzdy &= \int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}|g''_\alpha|^2\chi^2\lambda + 2g'_\alpha g''_\alpha\chi\chi_z\lambda + |g'_\alpha|^2\chi_z^2\lambda dz\right]dy \\
&= -\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}W''(g_\alpha)|g'_\alpha|^2\chi^2\lambda + g'_\alpha\xi'_\alpha\chi^2\lambda dz\right]dy \\
&\quad - \int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}g'_\alpha g''_\alpha\chi^2\lambda_z - |g'_\alpha|^2\chi_z^2\lambda dz\right]dy.
\end{aligned}$$

In the right hand side, except the first term, other terms can be estimated in the following way.

- Concerning the second term, because $\xi'_\alpha = O(\varepsilon^3)$,
$$\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}g'_\alpha(y,z)\xi'_\alpha(y,z)\chi(y,z)^2\lambda(y,z)dz\right]dy = O(\varepsilon^2)\int_{-5R/6}^{5R/6}\eta(y)^2dy.$$
- Concerning the third term, an integration by parts in z leads to
$$\begin{aligned}
& -\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}g'_\alpha g''_\alpha\chi^2\lambda_z dz\right]dy \\
=\ & 2\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}|g'_\alpha|^2\chi\chi_z\lambda_z dz\right]dy + \int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}|g'_\alpha|^2\chi^2\lambda_{zz}dz\right]dy \\
\lesssim\ & \varepsilon\int_{-5R/6}^{5R/6}\eta(y)^2\left[e^{-2\sqrt{2}\rho_\alpha^+(y)} + e^{2\sqrt{2}\rho_\alpha^-(y)}\right]dy + \varepsilon^2\int_{-5R/6}^{5R/6}\eta(y)^2dy,
\end{aligned}$$
where in the last line for the first term we have used the same facts as in Subsection 19.1 and the estimate
$$\lambda_z = -2\lambda(y,0)H^\alpha(y,0)\left(1 - zH^\alpha(y,0)\right) = O(\varepsilon).$$
For the second term we have used the fact that
$$\lambda_{zz} = 2H^\alpha(y,0)^2\lambda(y,0) = O(\varepsilon^2).$$

- By the same reasoning as in Subsection 19.1,

$$\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}|g_\alpha'|^2\chi_z^2\lambda dz\right]dy\lesssim\frac{1}{L}\int\eta(y)^2\left[e^{-2\sqrt{2}\rho_\alpha^+(y)}+e^{2\sqrt{2}\rho_\alpha^-(y)}\right]dy.$$

In conclusion, we get

$$\begin{aligned}\int\varphi_z^2\lambda(y,z)dzdy &= -\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}W''(g_\alpha)|g_\alpha'|^2\chi^2\lambda dz\right]dy\\ &+ O(\varepsilon^2)\int\eta(y)^2dy+O\left(\frac{1}{L}+\varepsilon\right)\int\eta(y)^2\left[e^{-2\sqrt{2}\rho_\alpha^+(y)}+e^{2\sqrt{2}\rho_\alpha^-(y)}\right]dy.\end{aligned}$$

Now the stability condition for u is transformed into

$$\begin{aligned}0 &\leq C\int_{-5R/6}^{5R/6}\eta'(y)^2dy+C\varepsilon^{2-2\sigma}\int_{-5R/6}^{5R/6}\eta(y)^2dy+C\left(\frac{1}{L}+\varepsilon\right)\int_{-5R/6}^{5R/6}\eta(y)^2\left[e^{-2\sqrt{2}\rho_\alpha^+(y)}+e^{2\sqrt{2}\rho_\alpha^-(y)}\right]dy\\ &+ \int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\left(W''(u)-W''(g_\alpha)\right)|g_\alpha'|^2\chi^2\lambda dz\right]dy. \qquad (19.1)\end{aligned}$$

It remains to rewrite the last integral.

19.3. **The interaction part.** Differentiating (18.2) in z leads to

$$\begin{aligned}&\frac{\partial}{\partial z}\Delta_z^\alpha\phi-\frac{\partial}{\partial z}\left(H^\alpha(y,z)\partial_z^\alpha\phi\right)+\partial_{zzz}^\alpha\phi\\ =\;& W''(u)\left[(-1)^\alpha g_\alpha'+\phi_z+\sum_{\beta\neq\alpha}(-1)^\beta g_\beta'\frac{\partial d_\beta}{\partial z}\right]-(-1)^\alpha W''(g_\alpha)g_\alpha'-\sum_{\beta\neq\alpha}(-1)^\beta W''(g_\beta)g_\beta'\frac{\partial d_\beta}{\partial z}\\ -\;& (-1)^\alpha\frac{\partial}{\partial z}\left[g_\alpha'\left(H^\alpha(y,z)+\Delta_z h_\alpha(y)\right)\right]-\frac{\partial}{\partial z}\left(g_\alpha''|\nabla_z h_\alpha|^2\right) \qquad (19.2)\\ -\;& \sum_{\beta\neq\alpha}\frac{\partial}{\partial z}\left[(-1)^\beta g_\beta'\mathcal{R}_{\beta,1}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)+g_\beta''\mathcal{R}_{\beta,2}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)\right]+\sum_\beta\frac{\partial\xi_\beta}{\partial z}.\end{aligned}$$

Multiplying this equation by $\eta^2g_\alpha'\chi^2\lambda$ and then integrating in y and z gives

$$\begin{aligned}&\int_{-5R/6}^{5R/6}\eta(y)^2\int_{-\delta R}^{\delta R}\left[\frac{\partial}{\partial z}\Delta_z^\alpha\phi-\frac{\partial}{\partial z}\left(H^\alpha(y,z)\partial_z^\alpha\phi\right)\right]g_\alpha'\chi^2\lambda dzdy+\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\partial_{zzz}^\alpha\phi g_\alpha'\eta^2\chi^2\lambda dz\right]dy\\ =\;& (-1)^\alpha\int_{-5R/6}^{5R/6}\eta(y)^2\int_{-\delta R}^{\delta R}\left[W''(u)-W''(g_\alpha)\right]|g_\alpha'|^2\chi^2\lambda dzdy+\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}W''(u)\phi_z g_\alpha'\chi^2\lambda dz\right]dy\\ +\;& \sum_{\beta\neq\alpha}(-1)^\beta\int_{-5R/6}^{5R/6}\eta(y)^2\int_{-\delta R}^{\delta R}\left[W''(u)-W''(g_\beta)\right]g_\alpha'g_\beta'\frac{\partial d_\beta}{\partial z}\chi^2\lambda dzdy\\ -\;& (-1)^\alpha\int_{-5R/6}^{5R/6}\eta(y)^2\int_{-\delta R}^{\delta R}\frac{\partial}{\partial z}\left[g_\alpha'\left(H^\alpha(y,z)-\Delta_z h_\alpha(y)\right)\right]g_\alpha'\chi^2\lambda dzdy\\ -\;& \int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\frac{\partial}{\partial z}\left(g_\alpha''|\nabla_z h_\alpha|^2\right)g_\alpha'\chi^2\lambda dz\right]dy\\ -\;& \sum_{\beta\neq\alpha}(-1)^\beta\int_{-5R/6}^{5R/6}\eta(y)^2\int_{-\delta R}^{\delta R}g_\alpha'\chi^2\lambda\frac{\partial}{\partial z}\left[g_\beta'\mathcal{R}_{\beta,1}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)\right]dzdy\\ -\;& \sum_{\beta\neq\alpha}\int_{-5R/6}^{5R/6}\eta(y)^2\int_{-\delta R}^{\delta R}g_\alpha'\chi^2\lambda\frac{\partial}{\partial z}\left[g_\beta''\mathcal{R}_{\beta,2}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)\right]dzdy+\sum_\beta\int_{-5R/6}^{5R/6}\eta(y)^2\int_{-\delta R}^{\delta R}g_\alpha'\chi^2\lambda\frac{\partial\xi_\beta}{\partial z}dzdy.\end{aligned}$$

We need to estimate each term.

(1) Integrating by parts in z leads to

$$\int_{-\delta R}^{\delta R}\frac{\partial}{\partial z}\Delta_z\phi g_\alpha'\chi^2\lambda(y,z)dz=-\int_{-\delta R}^{\delta R}\Delta_z\phi\frac{\partial}{\partial z}\left(g_\alpha'\chi^2\lambda(y,z)\right)dz.$$

Using (18.5) and the exponential decay of g'_α and g''_α, we get

$$\Big|\int_{-\delta R}^{\delta R}\frac{\partial}{\partial z}\Delta_z\phi g'_\alpha\chi^2\lambda(y,z)dz\Big|\lesssim\varepsilon^{\frac{7}{6}}.$$

(2) Integrating by parts and using the exponential decay of g'_α and g''_α, we get

$$\begin{aligned}\Big|\int_{-\delta R}^{\delta R}\frac{\partial}{\partial z}\left(H^\alpha(y,z)\phi_z\right)g'_\alpha\chi^2\lambda\Big|=\Big|\int_{-\delta R}^{\delta R}H^\alpha(y,z)\phi_z\frac{\partial}{\partial z}\left(g'_\alpha\chi^2\lambda\right)\Big| &\lesssim \varepsilon\sup_{\rho_\alpha^-(y)-L<z<\rho_\alpha^+(y)+L}|\phi_z(y,z)|\\ &\lesssim \varepsilon^{2-\sigma}.\end{aligned}$$

(3) The second term in the left hand side and the second one in the right hand side can be canceled with a remainder term of higher order. More precisely,

$$\begin{aligned}&\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\phi_{zzz}g'_\alpha\chi^2\lambda dz\right]dy\\
=&\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\phi_z\left(g'''_\alpha\chi^2\lambda+2g'_\alpha\left(\chi_z^2+\chi\chi_{zz}\right)\lambda+g'_\alpha\chi^2\lambda_{zz}\right)dz\right]dy\\
+&\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\phi_z\left(4g''_\alpha\chi\chi_z\lambda+2g''_\alpha\chi^2\lambda_z+4g'_\alpha\chi\chi_z\lambda_z\right)dz\right]dy\\
=&\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\phi_z g'''_\alpha\chi^2\lambda dz+h.o.t.\right]dy\\
=&\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}W''(g_\alpha)\phi_z g'_\alpha\chi^2\lambda dz+h.o.t\right]dy.\end{aligned}$$

In the above those higher order terms can be bounded by $O(\varepsilon^{\frac{3}{2}-2\sigma})$. We only show how to prove

$$\int_{-\delta R}^{\delta R}\phi_z g''_\alpha\chi\chi_z\lambda=O(\varepsilon^{\frac{3}{2}-2\sigma}).\tag{19.3}$$

In $\mathrm{spt}(\chi_z)$, $|\chi_z|\lesssim L^{-1}$,

$$|g''_\alpha|\lesssim e^{-\sqrt{2}|z|}\lesssim e^{-\frac{\sqrt{2}}{2}D_\alpha(y)}+\varepsilon\lesssim\varepsilon^{\frac{1-\sigma}{2}},$$
$$|\phi_z|\lesssim\|\phi\|_{C^{2,1/2}(\mathcal{M}_\alpha(r))}\lesssim\varepsilon^{1-\sigma}.$$

Combining these three estimates we get (19.3). Similarly, other terms are bounded by $O(\varepsilon^2)+O\left(|\nabla\phi(y,z)|^2_{L^\infty(\mathcal{M}_\alpha)}\right)=O(\varepsilon^{2-2\sigma})$.

Next we show that

$$\int_{-\delta R}^{\delta R}\left[W''(u)-W''(g_\alpha)\right]\phi_z g'_\alpha\chi^2\lambda dz=O\left(\varepsilon^{2-3\sigma}\right).$$

This is because in $\{\chi\neq 0\}$,

$$u=g_\alpha+\phi+\sum_{\beta<\alpha}\left(g_\beta-(-1)^\beta\right)+\sum_{\beta>\alpha}\left(g_\beta+(-1)^\beta\right),$$

hence this integral is bounded by

$$\begin{aligned}\int_{-\delta R}^{\delta R}\left(|\phi||\phi_z||g'_\alpha\chi^2\lambda+\sum_{\beta\neq\alpha}|\phi_z|g'_\beta g'_\alpha\chi^2\lambda\right)dz &\lesssim \|\phi\|^2_{C^{1,1/2}(\mathcal{M}_\alpha(6R/7))}+\sup_{(-r,r)}D_\alpha^2e^{-2\sqrt{2}D_\alpha}\\ &\lesssim \varepsilon^{2-3\sigma}.\end{aligned}$$

(4) In $\{\chi\neq 0\}$,

$$W''(u)=W''(g_\alpha)+O(|\phi|)+\sum_{\beta\neq\alpha}O\left(g'_\beta\right),$$

and for $\beta\neq\alpha$,

$$W''(g_\beta)=W''(1)+O\left(g'_\beta\right).$$

Hence

$$\int_{-\delta R}^{\delta R}\left[W''(u)-W''(g_\beta)\right]g_\alpha' g_\beta'\frac{\partial d_\beta}{\partial z}\chi^2\lambda dz=\int_{-\delta R}^{\delta R}\left[W''(g_\alpha)-W''(1)\right]g_\alpha' g_\beta'\lambda(y,0)dz+h.o.t.,$$

where higher order terms are controlled by

$$\begin{aligned}
&\int_{-\delta R}^{\delta R}|\phi|g_\alpha' g_\beta'\chi^2+\sum_{\beta\neq\alpha}\int_{-\delta R}^{\delta R}g_\alpha'\left|g_\beta'\right|^2\chi^2+\sum_{\beta\neq\alpha}\int_{-\delta R}^{\delta R}\left|g_\alpha'\right|^2 g_\beta'\left(1-\chi^2\right)\\
+&\left\|\frac{\partial d_\beta}{\partial z}-1\right\|_{L^\infty(\mathcal{M}_\alpha(r))}\int_{-\delta R}^{\delta R}\left|g_\alpha'\right|^2 g_\beta'\chi^2+\varepsilon\int_{-\delta R}^{\delta R}|z|\left|g_\alpha'\right|^2 g_\beta'\chi^2 dz\\
\lesssim&\ \varepsilon^{1-\sigma}D_\alpha(y)e^{-\sqrt{2}D_\alpha(y)}+e^{-\frac{3\sqrt{2}}{2}D_\alpha(y)}+\varepsilon e^{-\sqrt{2}D_\alpha(y)}\\
\lesssim&\ \varepsilon^{\frac{3}{2}-3\sigma}.
\end{aligned}$$

(5) Integrating by parts gives

$$\begin{aligned}
&\int_{-\delta R}^{\delta R}\frac{\partial}{\partial z}\left[g_\alpha'\left(H^\alpha(y,z)+\Delta_z h_\alpha(y)\right)\right]g_\alpha'\chi^2\lambda dz\\
=&-\int_{-\delta R}^{\delta R}g_\alpha'\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]\left[g_\alpha''\chi^2\lambda+2g_\alpha'\chi\chi_z\lambda+g_\alpha'\chi^2\lambda_z\right]dz\\
=&\ \frac{1}{2}\int_{-\delta R}^{\delta R}\left|g_\alpha'\right|^2\frac{\partial}{\partial z}\left[\left(H^\alpha(y,z)+\Delta_z h_\alpha(y)\right)\chi^2\lambda\right]\\
-&\ 2\int_{-\delta R}^{\delta R}\left|g_\alpha'\right|^2\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]\chi\chi_z\lambda-\int_{-\delta R}^{\delta R}\left|g_\alpha'\right|^2\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]\chi^2\lambda_z dz\\
=&\ \frac{1}{2}\int_{-\delta R}^{\delta R}\left|g_\alpha'\right|^2\chi^2\lambda\frac{\partial}{\partial z}\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]\\
-&\int_{-\delta R}^{\delta R}\left|g_\alpha'\right|^2\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]\chi\chi_z\lambda-\frac{1}{2}\int_{-\delta R}^{\delta R}\left|g_\alpha'\right|^2\left[H^\alpha(y,z)+\Delta_z h_\alpha(y)\right]\chi^2\lambda_z dz.
\end{aligned}$$

Because

$$\frac{\partial}{\partial z}H^\alpha(y,z)=\frac{H^\alpha(y,0)^2}{1-zH^\alpha(y,0)}=O\left(\varepsilon^2\right),$$

$$\left|\frac{\partial}{\partial z}\Delta_z h_\alpha(y)\right|\lesssim\varepsilon\left(\left|h_\alpha''(y)\right|+\left|h_\alpha'(y)\right|\right)\lesssim\varepsilon^{2-\sigma},$$

the first integral is bounded by $O\left(\varepsilon^{2-2\sigma}\right)$.

In $\{\chi_z\neq 0\}$,

$$\left|g_\alpha'\right|^2\lesssim\varepsilon^2+e^{-\sqrt{2}D_\alpha}\lesssim\varepsilon^{1-\sigma}.$$

Because

$$H^\alpha(y,z)+\Delta_z h_\alpha(y)=O(\varepsilon^{1-\sigma}),$$

the second integral is bounded by $O(\varepsilon^{2-2\sigma})$.

Because $\lambda_z=O(\varepsilon)$, the third integral is bounded by $O(\varepsilon^{2-\sigma})$.

(6) Integrating by parts in z and using (18.4) leads to

$$\begin{aligned}
\int_{-\delta R}^{\delta R}\frac{\partial}{\partial z}\left(g_\alpha''|\nabla_z h_\alpha|^2\right)g_\alpha'\chi^2\lambda dz=-\int_{-\delta R}^{\delta R}g_\alpha''|\nabla_z h_\alpha|^2\frac{\partial}{\partial z}\left(g_\alpha'\chi^2\lambda\right)dz\ &\lesssim\ |h_\alpha'(y)|^2\\
&\lesssim\ \varepsilon^{2-2\sigma}.
\end{aligned}$$

(7) For $\beta\neq\alpha$,

$$\int_{-\delta R}^{\delta R}g_\alpha'\chi^2\lambda\frac{\partial}{\partial z}\left[g_\beta'\mathcal{R}_{\beta,1}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)\right]dz=-\int_{-\delta R}^{\delta R}\frac{\partial}{\partial z}\left[g_\alpha'\chi^2\lambda\right]g_\beta'\mathcal{R}_{\beta,1}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)dz.$$

Because

$$\left|\mathcal{R}_{\beta,1}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)\right|\lesssim\varepsilon^{1-\sigma},$$

the above integral can be controlled by

$$\varepsilon^{1-\sigma}\int_{-\delta R}^{\delta R}\chi e^{-\sqrt{2}(|d_\alpha(y,z)|+|d_\beta(y,z)|)}dz \lesssim \varepsilon^{1-\sigma}D_\alpha(y)e^{-\sqrt{2}D_\alpha(y)} \lesssim \varepsilon^{2-3\sigma}.$$

(8) Similarly, because $|\mathcal{R}_{\beta,2}|\lesssim \varepsilon^{2-2\sigma}$, we have

$$\begin{aligned}\int_{-\delta R}^{\delta R} g_\alpha'\chi^2\lambda\frac{\partial}{\partial z}\left[g_\beta''\mathcal{R}_{\beta,2}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)\right]dz &= -\int_{-\delta R}^{\delta R}\frac{\partial}{\partial z}\left[g_\alpha'\chi^2\lambda\right]g_\beta''\mathcal{R}_{\beta,2}\left(\Pi_\beta(y,z),d_\beta(y,z)\right)dz\\ &= O(\varepsilon^{2-2\sigma}).\end{aligned}$$

(9) Finally, by the defintion of ξ_β,

$$\int_{-5R/6}^{5R/6}\eta(y)^2\int_{-\delta R}^{\delta R} g_\alpha'\chi^2\lambda\frac{\partial \xi_\beta}{\partial z}dzdy = O(\varepsilon^2)\int_{-5R/6}^{5R/6}\eta(y)^2dy.$$

Combining all of these estimates together, we obtain

$$\begin{aligned}&\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\left[W''(u)-W''(g_\alpha)\right]|g_\alpha'|^2\chi^2\lambda dz\right]dy\\ = &\sum_{\beta\neq\alpha}\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\left[W''(g_\alpha)-W''(1)\right]g_\alpha'g_\beta'dz\right]\lambda(y,0)dy + O(\varepsilon^{\frac{4}{3}})\int_{-5R/6}^{5R/6}\eta(y)^2dy.\end{aligned}$$

The last integral can be computed by applying Lemma A.1, which leads to

$$\begin{aligned}&\int_{-5R/6}^{5R/6}\eta(y)^2\left[\int_{-\delta R}^{\delta R}\left[W''(u)-W''(g_\alpha)\right]|g_\alpha'|^2\chi^2\lambda dz\right]dy \qquad (19.4)\\ = &-4\int_{-5R/6}^{5R/6}\eta(y)^2\left[A^2_{(-1)^{\alpha-1}}e^{-\sqrt{2}d_{\alpha-1}(y,0)}+A^2_{(-1)^\alpha}e^{\sqrt{2}d_{\alpha+1}(y,0)}\right]\lambda(y,0)dy + O(\varepsilon^{\frac{4}{3}})\int_{-5R/6}^{5R/6}\eta(y)^2dy.\end{aligned}$$

19.4. **A stability condition for the Toda system.** Substituting (19.4) into (19.1) we get

$$\begin{aligned}0 \leq\ & C\int_{-5R/6}^{5R/6}\eta'(y)^2dy + C\varepsilon^{\frac{4}{3}}\int_{-5R/6}^{5R/6}\eta(y)^2dy + C\left(\frac{1}{L}+\varepsilon\right)\int_{-5R/6}^{5R/6}\eta(y)^2\left[e^{-2\sqrt{2}\rho_\alpha^+(y)}+e^{2\sqrt{2}\rho_\alpha^-(y)}\right]dy\\ -\ & c\int_{-5R/6}^{5R/6}\eta(y)^2\left[A^2_{(-1)^{\alpha-1}}e^{-\sqrt{2}d_{\alpha-1}(y,0)}+A^2_{(-1)^\alpha}e^{\sqrt{2}d_{\alpha+1}(y,0)}\right]\lambda(y,0)dy. \qquad (19.5)\end{aligned}$$

First we have the following estimates. Because $d_\alpha(y,\rho_\alpha^+(y)) = d_{\alpha+1}(y,\rho_\alpha^+(y))$, if they are smaller than $\sqrt{2}|\log\varepsilon|$, by Lemma 8.3,

$$d_\alpha(y,\rho_\alpha^+(y)) = d_{\alpha+1}(y,\rho_\alpha^+(y)) = -\frac{1}{2}d_{\alpha+1}(y,0)+o(1).$$

Hence

$$e^{-2\sqrt{2}\rho_\alpha^+(y)}\lesssim \varepsilon^2 + e^{\sqrt{2}d_{\alpha+1}(y,0)}.$$

A similar estimate holds for $e^{2\sqrt{2}\rho_\alpha^-(y)}$. From these we deduce that

$$\int_{-5R/6}^{5R/6}\eta(y)^2\left[e^{-2\sqrt{2}\rho_\alpha^+(y)}+e^{2\sqrt{2}\rho_\alpha^-(y)}\right]dy \leq C\varepsilon^2\int_{-5R/6}^{5R/6}\eta(y)^2dy + C\int_{-5R/6}^{5R/6}\eta(y)^2e^{-\sqrt{2}D_\alpha(y)}dy.$$

Substituting these estimates into (19.5) leads to

$$\begin{aligned}&\left(c-\frac{C}{L}\right)\int_{-5R/6}^{5R/6}\eta(y)^2\left[A^2_{(-1)^{\alpha-1}}e^{-\sqrt{2}d_{\alpha-1}(y,0)}+A^2_{(-1)^\alpha}e^{\sqrt{2}d_{\alpha+1}(y,0)}\right]\lambda(y,0)dy \qquad (19.6)\\ \leq\ & C\int_{-5R/6}^{5R/6}\eta'(y)^2dy + C\varepsilon^{\frac{4}{3}}\int_{-5R/6}^{5R/6}\eta(y)^2dy.\end{aligned}$$

By choosing L large enough, we get

$$\int_{-5R/6}^{5R/6}\eta(y)^2\left[e^{-\sqrt{2}d_{\alpha-1}(y,0)}+e^{\sqrt{2}d_{\alpha+1}(y,0)}\right]dy \leq C\int_{-5R/6}^{5R/6}\eta'(y)^2dy + C\varepsilon^{\frac{4}{3}}\int_{-5R/6}^{5R/6}\eta(y)^2dy. \qquad (19.7)$$

Remark 19.1. *With a little more work and passing to the blow up limit as in Remark 14.1, we get exactly the stability condition for the Toda system* (2.10).

20. Proof of Theorem 3.6

In this section we prove

Proposition 20.1. *For any α and $y \in (-R/2, R/2)$, if $|f_\alpha(y)| < 2\delta R$, then*

$$D_\alpha(y) \geq \frac{4\sqrt{2}}{7}|\log\varepsilon|.$$

First let us use this proposition to prove Theorem 3.6.

Proof of Theorem 3.6. Substituting Proposition 20.1 into (18.3), we get

$$\|\phi\|_{C^{2,1/2}(\mathcal{M}_\alpha(R/2))} + \|H^\alpha + \Delta_0^\alpha h_\alpha\|_{C^{1/2}(B_{R/2})} \lesssim \varepsilon^{8/7}. \tag{20.1}$$

By Lemma 9.6,

$$\|H^\alpha\|_{L^\infty(-R/2,R/2)} \lesssim \|\phi\|_{C^{2,1/2}(\mathcal{M}_\alpha(R/2))} + \|H^\alpha + \Delta_0^\alpha h_\alpha\|_{C^{1/2}(B_{R/2})} \lesssim \varepsilon^{8/7}.$$

Then for any $|y| < R/2$ and $|z| < \delta R$,

$$|H^\alpha(y,z)| \lesssim |H^\alpha(y,0)| \lesssim \varepsilon^{8/7}.$$

In $\mathcal{M}_\alpha(R/2)$,

$$\nabla u = (-1)^\alpha g'_\alpha\left(\frac{\partial}{\partial z} - h'_\alpha(y)\frac{\partial}{\partial y}\right) + \nabla\phi + \sum_{\beta\neq\alpha}(-1)^\beta g'_\beta\left(\nabla d_\beta - h'_\beta(\Pi_\beta(y,z))\nabla\Pi_\beta(y,z)\right),$$

$$\begin{aligned}
\nabla^2 u \;&=\; -(-1)^\alpha g'_\alpha h''_\alpha(y)\frac{\partial}{\partial y}\otimes\frac{\partial}{\partial y} - (-1)^\alpha g'_\alpha h'_\alpha(y)\nabla\frac{\partial}{\partial y} + (-1)^\alpha g'_\alpha\nabla\frac{\partial}{\partial z}\\
&+\; g''_\alpha\left(\frac{\partial}{\partial z} - h'_\alpha(y)\frac{\partial}{\partial y}\right)\otimes\left(\frac{\partial}{\partial z} - h'_\alpha(y)\frac{\partial}{\partial y}\right)\\
&+\; \sum_{\beta\neq\alpha}(-1)^\beta g'_\beta(y,z)(\Pi_\beta(y,z))\mathcal{R}_{\beta,3} + \sum_{\beta\neq\alpha} g''_\beta(y,z)\mathcal{R}_{\beta,4} + \nabla^2\phi,
\end{aligned}$$

where in the Fermi coordinates with respect to Γ_β,

$$\mathcal{R}_{\beta,3} = -h''_\beta\frac{\partial}{\partial y}\otimes\frac{\partial}{\partial y} - h'_\beta\nabla\frac{\partial}{\partial y} + \nabla\frac{\partial}{\partial z},$$

$$\mathcal{R}_{\beta,4} = \left(\frac{\partial}{\partial z} - h'_\alpha(y)\frac{\partial}{\partial y}\right)\otimes\left(\frac{\partial}{\partial z} - h'_\alpha(y)\frac{\partial}{\partial y}\right).$$

By the estimates on ϕ and h_α we obtain

$$\nabla u = (-1)^\alpha g'_\alpha\frac{\partial}{\partial z} + O\left(\varepsilon^{8/7}\right),$$

$$\nabla^2 u = (-1)^\alpha g''_\alpha\frac{\partial}{\partial z}\otimes\frac{\partial}{\partial z} + O\left(\varepsilon^{8/7}\right).$$

Using these forms we obtain, for any $L > 0$, in $\mathcal{M}_0(R/2) \cap \{|z| < L\}$

$$\frac{|\nabla^2 u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2} \leq C(L)\varepsilon^{16/7}.$$

For any $b \in (0,1)$, there exists an $L(b)$ such that $\mathcal{M}_0(R/2) \cap \{|u| < 1-b\} \subset \mathcal{M}_0(R/2) \cap \{|z| < L(b)\}$. Therefore this estimates holds for this domain, too.

After a rescaling we obtain $|B(u_\varepsilon)| \leq C\varepsilon^{1/7}$ in $\{|u_\varepsilon| < 1-b\} \cap \{|x_1| < 1/2, |x_2| < 1/2\}$, and Theorem 3.6 is proven. □

Recalling the definition of $A_\alpha(r)$ and $D_\alpha(y)$ in Section 8 and Section 14. Note that $A_\alpha(r)$ is non-decreasing in r while $D_\alpha(r)$ is non-increasing in r.

To prove Proposition 20.1, we assume $\alpha = 0$ and by the contrary

$$A_0(R/2) \geq \varepsilon^{8/7}. \tag{20.2}$$

This implies that for any $r \in [R/2, 4R/5]$, $A_0(r) \geq \varepsilon^{8/7}$.

We will establish the following decay estimate.

Lemma 20.2. *There exists a constant K such that for any $r \in [R/2, 4R/5]$, we have*

$$A_0\left(r - KR^{\frac{4}{7}}\right) \le \frac{1}{2} A_0(r).$$

An iteration of this decay estimate from $r = 4R/5$ to $R/2$ leads to

$$A_0(R/2) \le 2^{-CK^{-1}R^{-4/7}} A_0(4R/5) \le Ce^{-c\varepsilon^{-4/7}} \ll \varepsilon^2.$$

This is a contradiction with the assumption (20.2). Thus we finish the proof of Proposition 20.1, provided that Lemma 20.2 holds true.

Now let us prove Lemma 20.2. Fix an $r \in [R/2, 4R/5]$ and denote $\epsilon := A_0(r)$. We will prove

$$A_0\left(r - K\epsilon^{-1/2}\right) \le \frac{\epsilon}{2}. \tag{20.3}$$

By (20.2), $\epsilon \ge \varepsilon^{8/7}$. Thus

$$K\epsilon^{-1/2} \le K\varepsilon^{-\frac{4}{7}} = KR^{\frac{4}{7}},$$

and

$$A_0\left(r - KR^{\frac{4}{7}}\right) \le A_0\left(r - K\epsilon^{-1/2}\right) \le \frac{\epsilon}{2},$$

which is Lemma 20.2.

To prove (20.3), we need to prove that for any $x_* \in [-r + K\epsilon^{1/2}, r - K\epsilon^{1/2}]$,

$$e^{-\sqrt{2}D_0(x_*)} \le \frac{\epsilon}{2}. \tag{20.4}$$

After a rotation and a translation, we may assume $x_* = 0$ and

$$f_0(0) = f_0'(0) = 0. \tag{20.5}$$

By the Toda system (10.2), for any $y \in [-K\epsilon^{-1/2}, K\epsilon^{-1/2}]$,

$$|f_0''(y)| \lesssim e^{-\sqrt{2}D_0(y)} + \varepsilon^{7/6} \lesssim \epsilon. \tag{20.6}$$

We also have a semi-bound on $f_\pm$.

Lemma 20.3.

$$f_{-1}''(y) \gtrsim -e^{-\sqrt{2}d_{-1}(y)} - \varepsilon^{7/6} \gtrsim -\epsilon, \tag{20.7}$$

$$f_1''(y) \lesssim e^{\sqrt{2}d_1(y)} + \varepsilon^{7/6} \lesssim \epsilon. \tag{20.8}$$

Proof. By (18.6),

$$\begin{aligned}\frac{f_1''}{\left(1+|f_1'|^2\right)^{3/2}} &= \frac{4}{\sigma_0}\left[A_1^2 e^{-\sqrt{2}|d_0^1|} - A_{-1}^2 e^{-\sqrt{2}|d_2^1|}\right] + O(\varepsilon^{7/6}) \\ &\le \frac{4A_1^2}{\sigma_0} e^{-\sqrt{2}|d_0^1|} + O(\varepsilon^{7/6}).\end{aligned}$$

By Lemma 8.3, either $|d_0^1(y)| \le \sqrt{2}|\log\varepsilon|$ or

$$d_0^1(y) = d_1(y) + O(\varepsilon^{1/3}).$$

The bound (20.8) then follows from (20.6). In the same way we get (20.7). □

By (20.5) and (20.6), for any $y \in [-K\epsilon^{-1/2}, K\epsilon^{-1/2}]$,

$$|f_0'(y)| \le C\epsilon|y| \le CK\epsilon^{1/2}. \tag{20.9}$$

Substituting these into the Toda system (18.6) we obtain in $(-K\epsilon^{-1/2}, K\epsilon^{-1/2})$

$$\begin{aligned}f_0'' &= \frac{f_0''}{\left(1+|f_0'|^2\right)^{3/2}} + O(\epsilon^2) \\ &= \frac{4}{\sigma_0}\left(A_{-1}^2 e^{-\sqrt{2}d_{-1}} - A_1^2 e^{\sqrt{2}d_1}\right) + O(\epsilon^2) + O(\varepsilon^{7/6}) \\ &= \frac{4}{\sigma_0}\left(A_{-1}^2 e^{-\sqrt{2}d_{-1}} - A_1^2 e^{\sqrt{2}d_1}\right) + O(\epsilon^{49/48}).\end{aligned} \tag{20.10}$$

Lemma 20.4. *For $y \in [-2\epsilon^{-1/2}, 2\epsilon^{-1/2}]$, if $|d_{-1}(y)| \leq \sqrt{2}|\log \varepsilon|$, then we have*

$$e^{-\sqrt{2}|d_{-1}(y)|} = e^{-\sqrt{2}(f_0(y)-f_{-1}(y))} + O(\epsilon^{49/48});$$

if $|d_1(y)| \leq \sqrt{2}|\log \varepsilon|$, then we have

$$e^{-\sqrt{2}|d_1(y)|} = e^{-\sqrt{2}(f_1(y)-f_0(y))} + O(\epsilon^{49/48}).$$

Proof. We only prove the second identity. The first one can be proved in the same way.

As in Lemma 8.3, if $|d_1(y)| \leq \sqrt{2}|\log \varepsilon|$, we have

$$\sup_{(y-1,y+1)} \left|f_1' - f_0'\right| \lesssim \varepsilon^{1/2}|\log \varepsilon|^2 \lesssim \epsilon^{1/4}.$$

Because $\left|f_0'\right| \lesssim \epsilon^{1/2}$ in $[-K\epsilon^{-1/2}, K\epsilon^{-1/2}]$, this implies that

$$\sup_{(y-1,y+1)} \left|f_1'\right| \lesssim \epsilon^{1/4}.$$

As in Lemma 8.3, from this we deduce that

$$\bar{d}_1(y) = f_0(y) - f_1(y) + O(\epsilon^{1/16}).$$

Then

$$e^{\sqrt{2}\bar{d}_1(y)} = e^{-\sqrt{2}(f_1(y)-f_0(y))} + O(\epsilon^{17/16}).$$

This finishes the proof. □

By this lemma and the fact that $e^{-\sqrt{2}D_0(y)} \leq \epsilon$, in $(-2\epsilon^{-1/2}, 2\epsilon^{-1/2})$, (20.10) can be rewritten as

$$f_0''(y) = \frac{4}{\sigma_0}\left(A_{-1}^2 e^{-\sqrt{2}[f_0(y)-f_{-1}(y)]} - A_1^2 e^{\sqrt{2}[f_1(y)-f_0(y)]}\right) + O(\epsilon^{1/48}). \tag{20.11}$$

Now define the functions in $[-K, K]$,

$$\tilde{f}_\alpha(y) := f_\alpha\left(\epsilon^{-1/2}y\right) - \alpha\frac{\sqrt{2}}{2}|\log \epsilon|, \quad \alpha = -1, 0, 1.$$

They satisfy

- $\tilde{f}_0(0) = \tilde{f}_0'(0) = 0$.
- In $(-K, K)$, $|\tilde{f}_0''| \leq C$, $\tilde{f}_1'' \leq C$ and $\tilde{f}_{-1}'' \geq -C$.
- In $(-2, 2)$,

$$\tilde{f}_0'' = \frac{4}{\sigma_0}\left[A_{-1}^2 e^{-\sqrt{2}\left(\tilde{f}_0-\tilde{f}_{-1}\right)} - A_1^2 e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)}\right] + O(\epsilon^{1/48}). \tag{20.12}$$

By the stability we get

Lemma 20.5.

$$\int_{-2}^{2}\left[e^{-\sqrt{2}\left(\tilde{f}_0-\tilde{f}_{-1}\right)} + e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)}\right] \leq \frac{C}{K} + CK\epsilon^{1/16}. \tag{20.13}$$

Proof. Take a function $\bar{\eta} \in C_0^\infty(-K, K)$ satisfying $\bar{\eta} \equiv 1$ in $(-2, 2)$ and $|\bar{\eta}'| \lesssim K^{-1}$. Taking the test function η in (19.7) to be $\bar{\eta}(\epsilon^{-1/2}y)$, we obtain

$$\begin{aligned}
\int_{-2}^{2}\left(e^{-\sqrt{2}d_{-1}(\epsilon^{-1/2}y)} + e^{\sqrt{2}d_1(\epsilon^{-1/2}y)}\right)dy &\leq \int_{-K}^{K}\bar{\eta}(y)^2\left(e^{-\sqrt{2}d_{-1}(\epsilon^{-1/2}y)} + e^{\sqrt{2}d_1(\epsilon^{-1/2}y)}\right)dy \\
&\leq C\epsilon\int_{-K}^{K}\bar{\eta}'(y)^2 + C\epsilon^{16/15}\int_{-K}^{K}\bar{\eta}(y)^2 \\
&\leq \frac{C}{K}\epsilon + CK\epsilon^{16/15}.
\end{aligned}$$

After using Lemma 20.4 and a rescaling, the left hand side can be transformed into the required form. □

The following lemma establishes (20.3), thus completes the proof of Lemma 20.2.

Lemma 20.6. *If K is large enough (but independent of ϵ), then*

$$\max_{[-1,1]}\left(e^{-\sqrt{2}\left(\tilde{f}_0-\tilde{f}_{-1}\right)} + e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)}\right) \leq \frac{1}{2}.$$

Proof. By (18.6), in $(-2\epsilon^{-1/2}, 2\epsilon^{-1/2})$,

$$f_1'' \leq \frac{4A_{-1}^2}{\sigma_0} e^{-\sqrt{2}(f_1-f_0)} \left(1+|f_1'|^2\right)^{3/2} + O\left(\varepsilon^{7/6}\right).$$

By the proof of Lemma 20.4,

- either $f_1 - f_0 \geq \sqrt{2}|\log\varepsilon|$, which implies that
$$e^{-\sqrt{2}(f_1-f_0)} \lesssim \varepsilon^2;$$
- or $|f_1' - f_0'| \leq \epsilon^{1/4}$, which together with (20.9) implies that
$$\left|f_1'\right| \leq 2\epsilon^{1/4}.$$

Therefore, because $e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)} \leq \epsilon$, we obtain

$$f_1'' \leq \frac{4A_{-1}^2}{\sigma_0} e^{-\sqrt{2}(f_1-f_0)} + O(\epsilon^{7/6}).$$

After a rescaling this gives

$$\tilde{f}_1'' \leq \frac{4A_{-1}^2}{\sigma_0} e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)} + O(\epsilon^{1/16}), \quad \text{in } (-2,2).$$

By (20.12),

$$\left(\tilde{f}_1 - \tilde{f}_0\right)'' \leq \frac{8A_{-1}^2}{\sigma_0} e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)} + O(\epsilon^{1/16}), \quad \text{in } (-2,2). \tag{20.14}$$

Then

$$\begin{aligned}
\frac{d^2}{dy^2} e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)} &\geq -\sqrt{2} e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)} \left(\tilde{f}_1 - \tilde{f}_0\right)'' \\
&\geq -Ce^{-2\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)} - C\epsilon^{1/16} e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)}.
\end{aligned}$$

By the estimate of Choi-Schoen [13], there exists a universal constant η_* such that if

$$\int_{-2}^{2} e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)} \leq \eta_*, \tag{20.15}$$

then

$$\sup_{[-1,1]} e^{-\sqrt{2}\left(\tilde{f}_1-\tilde{f}_0\right)} \leq \frac{1}{4}.$$

In (20.13), we can first choose K small and then let ϵ be small enough so that (20.15) holds. Then the claim follows by proving the same bound on $\sup_{[-1,1]} e^{-\sqrt{2}\left(\tilde{f}_0-\tilde{f}_{-1}\right)}$. □

Appendix A. Some facts about the one dimensional solution

It is known that the following identity holds for g,

$$g'(t) = \sqrt{2W(g(t))} > 0, \quad \forall t \in \mathbb{R}. \tag{A.1}$$

Moreover, as $t \to \pm\infty$, $g(t)$ converges exponentially to ± 1 and the following total energy is well defined

$$\sigma_0 := \int_{-\infty}^{+\infty} \left[\frac{1}{2} g'(t)^2 + W(g(t))\right] dt \in (0, +\infty).$$

In fact, as $t \to \pm\infty$, the following expansions hold. There exists a positive constants A_1 such that for all $t > 0$ large,

$$g(t) = 1 - A_1 e^{-\sqrt{2}t} + O(e^{-2\sqrt{2}t}), \tag{A.2}$$

and a similar expansion holds as $t \to -\infty$ with A_1 replaced by another positive constant A_{-1}. Furthermore expansion (A.2) can also be differentiated.

The following result describes the interaction between two one dimensional profiles.

Lemma A.1. *For all $T > 0$ large, we have the following expansion:*

$$\int_{-\infty}^{+\infty} \left[W''(g(t)) - W''(1)\right]\left[g(-t-T)+1\right] g'(t)dt = -4A_{-1}^2 e^{-\sqrt{2}T} + O\left(e^{-\frac{4\sqrt{2}}{3}T}\right). \tag{A.3}$$

$$\int_{-\infty}^{+\infty} \left[W''(g(t)) - W''(1)\right]\left[g(T-t)-1\right] g'(t)dt = 4A_{1}^2 e^{-\sqrt{2}T} + O\left(e^{-\frac{4\sqrt{2}}{3}T}\right). \tag{A.4}$$

Proof. We only prove the first expansion.

Step 1. Note that

$$\left|W''(g(t)) - 2\right| \lesssim g'(t).$$

Therefore the integral in $(-\infty, -3T/4)$ is controlled by

$$\int_{-\infty}^{-3T/4} g'(t)^2 g'(-t-T)dt \lesssim \int_{-\infty}^{-3T/4} e^{2\sqrt{2}t}dt \lesssim e^{-\frac{3\sqrt{2}}{2}T}.$$

Step 2. Similarly the integral in $(3T/4, +\infty)$ is controlled by

$$\int_{3T/4}^{+\infty} g'(t)^2 g'(-t-T)dt \lesssim \int_{3T/4}^{+\infty} e^{-3\sqrt{2}t-\sqrt{2}T}dt \lesssim e^{-\frac{3\sqrt{2}}{2}T}.$$

Step 3. In $(-3T/4, 3T/4)$,

$$g(-t-T)+1 = A_{-1}e^{-\sqrt{2}t-\sqrt{2}T} + O\left(e^{-2\sqrt{2}t-2\sqrt{2}T}\right).$$

Because

$$\begin{aligned}\Big|\int_{-3T/4}^{3T/4} \left[W''(g(t)) - 2\right] g'(t)e^{-2\sqrt{2}t-2\sqrt{2}T}dt\Big| \lesssim e^{-2\sqrt{2}T}\int_{-3T/4}^{3T/4} g'(t)^2 e^{-2\sqrt{2}t}dt &\lesssim Te^{-2\sqrt{2}T}\\ &\lesssim e^{-\frac{3\sqrt{2}}{2}T},\end{aligned}$$

we have

$$\int_{-\infty}^{+\infty} \left[W''(g(t)) - 2\right] g'(t)g'(t+T)dt = A_{-1}e^{-\sqrt{2}T}\int_{-3T/4}^{3T/4} \left[W''(g(t)) - 2\right] g'(t)e^{-\sqrt{2}t}dt + O\left(e^{-\frac{3\sqrt{2}}{2}T}\right).$$

As in Step 1 and Step 2, we have

$$\Big|\int_{-\infty}^{-3T/4} \left[W''(g(t)) - 2\right] g'(t)e^{-\sqrt{2}t}dt\Big| \lesssim e^{-\frac{3\sqrt{2}}{4}T},$$

$$\Big|\int_{3T/4}^{+\infty} \left[W''(g(t)) - 2\right] g'(t)e^{-\sqrt{2}t}dt\Big| \lesssim e^{-\frac{3\sqrt{2}}{4}T}.$$

Therefore

$$\int_{-\infty}^{+\infty} \left[W''(g(t)) - 2\right]\left[g(-t-T)+1\right] g'(t)dt = A_{-1}e^{-\sqrt{2}T}\int_{-\infty}^{+\infty} \left[W''(g(t)) - 2\right] g'(t)e^{-\sqrt{2}t}dt + O\left(e^{-\frac{3\sqrt{2}}{2}T}\right).$$

Step 4. It remains to determine the integral

$$\int_{-\infty}^{+\infty} \left[W''(g(t)) - 2\right] g'(t)e^{-\sqrt{2}t}dt.$$

Note that g' satisfies

$$g''' - 2g' = [W''(g(t)) - 2]g'.$$

As in Step 1 and Step 2, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \left[W''(g(t)) - 2\right] g'(t)e^{-\sqrt{2}t}dt &= \lim_{L\to+\infty}\int_{-L}^{L} \left[W''(g(t)) - 2\right] g'(t)e^{-\sqrt{2}t}dt\\ &= g''(L)e^{-\sqrt{2}L} + \sqrt{2}g'(L)e^{-\sqrt{2}L} - \left[g''(-L)e^{\sqrt{2}L} + \sqrt{2}g'(-L)e^{\sqrt{2}L}\right]\\ &= -4A_{-1} + O(e^{-2\sqrt{2}L}).\end{aligned}$$

Letting $L \to +\infty$ we finish the proof. □

Next we discuss the spectrum of the linearized operator at g, i.e. $\mathcal{L} = -\frac{d^2}{dt^2} + W''(g(t))$. By a direct differentiation we see $g'(t)$ is an eigenfunction of $\mathcal{L}$ corresponding to eigenvalue 0. By (A.1), 0 is the lowest eigenvalue. In other words, g is stable. By a contradiction argument, we have

Theorem A.2. *There exists a constant $\mu > 0$ such that for any $\varphi \in H^1(\mathbb{R})$ satisfying*

$$\int_{-\infty}^{+\infty} \varphi(t)g'(t)dt = 0, \tag{A.5}$$

we have

$$\int_{-\infty}^{+\infty} \left[\varphi'(t)^2 + W''(g(t))\varphi(t)^2\right] dt \geq \mu \int_{-\infty}^{+\infty} \varphi(t)^2 dt.$$

Appendix B. Derivation of (10.2)

We only give estimates on a couple of terms in (10.1). Other terms can be estimated by integrating by parts in z and applying (8.8)-(8.13).

B.1. **Horizontal terms.** Differentiating (13.2) twice leads to

$$\int_{-\delta R}^{\delta R} \frac{\partial \phi}{\partial y^i} g'_\alpha + (-1)^\alpha \phi g''_\alpha \frac{\partial h_\alpha}{\partial y^i} = 0, \tag{B.1}$$

$$\int_{-\delta R}^{\delta R} \frac{\partial^2 \phi}{\partial y^i \partial y^j} g'_\alpha + (-1)^\alpha \frac{\partial \phi}{\partial y^i} g''_\alpha \frac{\partial h_\alpha}{\partial y^j} + (-1)^\alpha \frac{\partial \phi}{\partial y^j} g''_\alpha \frac{\partial h_\alpha}{\partial y^i} + (-1)^\alpha \phi g''_\alpha \frac{\partial^2 h_\alpha}{\partial y^i \partial y^j} + \phi g'''_\alpha \frac{\partial h_\alpha}{\partial y^i}\frac{\partial h_\alpha}{\partial y^j} = 0. \tag{B.2}$$

Therefore

$$\int_{-\delta R}^{\delta R} \Delta_0 \phi(y,z) g'_\alpha = (-1)^{\alpha-1}\Delta_0 h_\alpha \int_{-\delta R}^{\delta R} \phi g''_\alpha + 2(-1)^\alpha \int_{-\delta R}^{\delta R} g^{ij}(y,0)\frac{\partial \phi}{\partial y^i}\frac{\partial h_\alpha}{\partial y^j} g''_\alpha - |\nabla_0 h_\alpha|^2 \int_{-\delta R}^{\delta R} \phi g'''_\alpha. \tag{B.3}$$

Then by (8.13),

$$\begin{aligned}
\int_{-\delta R}^{\delta R} \Delta_z \phi(y,z) g'_\alpha &= \int_{-\delta R}^{\delta R} \Delta_0 \phi(y,z) g'_\alpha + O(\varepsilon) \int_{-\delta R}^{\delta R} \left(|\nabla_y^2 \phi(y,z)| + |\nabla_y \phi(y,z)|\right) |z| e^{-\sqrt{2}|z|} dz \\
&= (-1)^{\alpha-1}\Delta_0 h_\alpha \int_{-6|\log\varepsilon|}^{6|\log\varepsilon|} \phi g''_\alpha + O(|\nabla h_\alpha(y)|^2) \int_{-6|\log\varepsilon|}^{6|\log\varepsilon|} |\phi(y,z)| e^{-\sqrt{2}|z|} dz \\
&+ O(|\nabla h_\alpha(y)| + \varepsilon) \int_{-6|\log\varepsilon|}^{6|\log\varepsilon|} \left(|\nabla_y^2 \phi(y,z)| + |\nabla_y \phi(y,z)|\right)(1+|z|) e^{-\sqrt{2}|z|} dz \\
&= (-1)^{\alpha-1}\Delta_0 h_\alpha \int_{-6|\log\varepsilon|}^{6|\log\varepsilon|} \phi g''_\alpha \\
&+ O(|\nabla h_\alpha(y)| + \varepsilon) \sup_{(-6|\log\varepsilon|, 6|\log\varepsilon|)} \left(|\nabla_y^2 \phi(y,z)| + |\nabla_y \phi(y,z)| + |\phi(y,z)|\right) e^{-\left(\sqrt{2}-\sigma\right)|z|}.
\end{aligned} \tag{B.4}$$

B.2. **Interaction terms.** To determine the integral

$$\int_{-\delta R}^{\delta R} \left[W'(g_*) - \sum_\beta W'(g_\beta)\right] g'_\alpha,$$

consider for each β, the integral on $(-\delta R, \delta R) \cap \mathcal{M}_\beta$, which is an interval $(\rho_\beta^-(y), \rho_\beta^+(y))$. If $\beta \neq \alpha$, by Lemma 9.4, in $(\rho_\beta^-(y), \rho_\beta^+(y))$,

$$\left|W'(g_*) - \sum_\beta W'(g_\beta)\right| \lesssim e^{-\sqrt{2}(|d_\beta|+|d_{\beta-1}|)} + e^{-\sqrt{2}(|d_\beta|+|d_{\beta+1}|)} + \varepsilon^2.$$

We only consider the case $\beta > \alpha$ and estimate

$$\int_{\rho_\beta^-(y)}^{\rho_\beta^+(y)} e^{-\sqrt{2}(|d_\beta|+|d_{\beta-1}|)} g'_\alpha.$$

If $|z|$, $|d_\beta|$ and $|d_{\beta-1}|$ are all smaller than $6|\log\varepsilon|$, by Lemma 8.3,

$$d_\beta(y,z) = z + d_\beta(y,0) + O(\varepsilon^{1/3}), \tag{B.5}$$

$$d_{\beta-1}(y,z) = z + d_{\beta-1}(y,0) + O(\varepsilon^{1/3}). \tag{B.6}$$

Note that since $\beta > \alpha$, $z > 0$ while $d_\beta(y,0) < d_{\beta-1}(y,0) \leq 0$.

We have

$$\begin{aligned}\int_{\rho_\beta^-(y)}^{\rho_\beta^+(y)} e^{-\sqrt{2}(|d_\beta|+|d_{\beta-1}|)}g_\alpha' &\lesssim \int_{\rho_\beta^-(y)}^{\rho_\beta^+(y)} e^{-\sqrt{2}(|z|+|z+d_{\beta-1}(y,0)|+|z+d_\beta(y,0)|)}\\ &\lesssim \int_{\rho_\beta^-(y)}^{-d_\beta(y,0)} e^{-\sqrt{2}(z+d_{\beta-1}(y,0)-d_\beta(y,0))}+\int_{-d_\beta(y,0)}^{\rho_\beta^+(y)} e^{-\sqrt{2}(3z+d_{\beta-1}(y,0)+d_\beta(y,0))}\\ &\lesssim e^{-\sqrt{2}(d_{\beta-1}(y,0)-d_\beta(y,0))-\sqrt{2}\rho_\beta^-(y)}+e^{-\sqrt{2}(d_{\beta-1}(y,0)-2d_\beta(y,0))}.\end{aligned}$$

By definition, $-d_\beta(y,\rho_\beta^-(y)) = d_{\beta-1}(y,\rho_\beta^-(y))$. Thus by (B.5) and (B.6),

$$\rho_\beta^-(y) = -\frac{d_{\beta-1}(y,0)+d_\beta(y,0)}{2}+O(\varepsilon^{1/3}).$$

Substituting this into the above estimate gives

$$\int_{\rho_\beta^-(y)}^{\rho_\beta^+(y)} e^{-\sqrt{2}(|d_\beta|+|d_{\beta-1}|)}g_\alpha' \lesssim e^{-\frac{\sqrt{2}}{2}(d_{\beta-1}(y,0)-3d_\beta(y,0))}+e^{-\sqrt{2}(d_{\beta-1}(y,0)-2d_\beta(y,0))}.$$

If $\beta=\alpha+1$, because $d_{\beta-1}(y,0)=0$, this is bounded by $O(e^{\frac{3\sqrt{2}}{2}d_{\alpha+1}(y,0)})$.

If $\beta\geq\alpha+2$, this is bounded by $O(e^{\sqrt{2}d_{\alpha+2}(y,0)})$.

It remains to consider the integration in $(\rho_\alpha^-(y),\rho_\alpha^+(y))$. In this case we use Lemma 9.3, which gives

$$\begin{aligned}&\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)}\left[W'(g_*)-\sum_\beta W'(g_\beta)\right]g_\alpha' \qquad\text{(B.7)}\\ =&\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)}\left[W''(g_\alpha)-2\right]\left[g_{\alpha-1}-(-1)^{\alpha-1}\right]g_\alpha'+\left[W''(g_\alpha)-2\right]\left[g_{\alpha+1}+(-1)^{\alpha}\right]g_\alpha'\\ +&\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)}\left[O\left(e^{-2\sqrt{2}d_{\alpha-1}}+e^{2\sqrt{2}d_{\alpha+1}}\right)+O\left(e^{-\sqrt{2}d_{\alpha-2}-\sqrt{2}|z|}+e^{\sqrt{2}d_{\alpha+2}-\sqrt{2}|z|}\right)\right]g_\alpha'.\end{aligned}$$

Because $g_\alpha'\lesssim e^{-\sqrt{2}|z|}$ and $e^{-2\sqrt{2}d_{\alpha-1}}\lesssim e^{-2\sqrt{2}d_{\alpha-1}(y,0)-2\sqrt{2}z}+\varepsilon^2$, we get

$$\begin{aligned}\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} e^{-2\sqrt{2}d_{\alpha-1}}g_\alpha' &\lesssim \varepsilon^2+e^{-2\sqrt{2}d_{\alpha-1}(y,0)}\left[\int_{\rho_\alpha^-(y)}^0 e^{-\sqrt{2}z}dz+\int_0^{\rho_\alpha^+(y)} e^{-3\sqrt{2}z}dz\right]\\ &\lesssim \varepsilon^2+e^{-2\sqrt{2}d_{\alpha-1}(y,0)-\sqrt{2}\rho_\alpha^-(y)}\\ &\lesssim \varepsilon^2+e^{-\frac{3}{2}\sqrt{2}d_{\alpha-1}(y,0)}.\end{aligned}$$

Similarly, we have

$$\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} e^{2\sqrt{2}d_{\alpha+1}}g_\alpha' \lesssim \varepsilon^2+e^{\frac{3}{2}\sqrt{2}d_{\alpha+1}(y,0)},$$

$$\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)} O\left(e^{-\sqrt{2}d_{\alpha-2}-\sqrt{2}|z|}+e^{\sqrt{2}d_{\alpha+2}-\sqrt{2}|z|}\right)g_\alpha' \lesssim e^{-\sqrt{2}d_{\alpha-2}}+e^{\sqrt{2}d_{\alpha+2}}.$$

To determine the first integral in (B.7), arguing as above, if both g_α' and $g_{\alpha-1}-(-1)^{\alpha-1}$ are nonzero, then

$$g_{\alpha-1}(y,z)=\bar{g}\left((-1)^\alpha(z+d_{\alpha-1}(y,0)+h_{\alpha-1}(\Pi_{\alpha-1}(y,z))+O(\varepsilon^{1/3}))\right).$$

Therefore

$$\begin{aligned}&\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)}\left[W''(g_\alpha)-2\right]\left(g_{\alpha-1}-(-1)^{\alpha-1}\right)g_\alpha'\\ =&\int_{\rho_\alpha^-(y)}^{\rho_\alpha^+(y)}\left[W''\left(\bar{g}\left((-1)^{\alpha-1}(z-h_\alpha(y))\right)\right)-2\right]\bar{g}'\left((-1)^{\alpha-1}(z-h_\alpha(y))\right)\\ &\times\left[\bar{g}\left((-1)^\alpha(z+d_{\alpha-1}(y,0)+h_{\alpha-1}(\Pi_{\alpha-1}(y,z))+O(\varepsilon^{1/3}))\right)-(-1)^{\alpha-1}\right]dz\\ =&\int_{-\infty}^{+\infty}\left[W''\left(\bar{g}\left((-1)^{\alpha-1}(z-h_\alpha(y))\right)\right)-2\right]\bar{g}'\left((-1)^{\alpha-1}(z-h_\alpha(y))\right)\end{aligned}$$

$$
\begin{aligned}
& \times\left[\bar g\left((-1)^\alpha(z+d_{\alpha-1}(y,0)+h_{\alpha-1}(\Pi_{\alpha-1}(y,z))+O(\varepsilon^{1/3}))\right)-(-1)^{\alpha-1}\right]dz\\
+ \quad & O(e^{-\frac{3\sqrt2}{2}d_{\alpha-1}(y,0)})\\
= \quad & (-1)^\alpha 4A^2_{(-1)^\alpha}e^{-\sqrt2 d_{\alpha-1}(y,0)}+O\left(|h_\alpha(y)|+|h_{\alpha-1}(\Pi_{\alpha-1}(y,z))|+\varepsilon^{1/3}\right)e^{-\sqrt2 d_{\alpha-1}(y,0)}\\
+ \quad & O(e^{-\frac{3\sqrt2}{2}d_{\alpha-1}(y,0)}).
\end{aligned}
$$

In conclusion we get

$$
\begin{aligned}
& \int_{-\delta R}^{\delta R}\left[W'(g_*)-\sum_\beta W'(g_\beta)\right]g'_\alpha\\
= \quad & (-1)^\alpha\left[4A^2_{(-1)^\alpha}e^{-\sqrt2 d_{\alpha-1}(y,0)}-4A^2_{(-1)^{\alpha-1}}e^{\sqrt2 d_{\alpha+1}(y,0)}\right]+O(\varepsilon^2)\\
+ \quad & O\left(|h_\alpha(y)|+|h_{\alpha-1}(\Pi_{\alpha-1}(y,z))|+\varepsilon^{1/3}\right)e^{-\sqrt2 d_{\alpha-1}(y,0)}\\
+ \quad & O\left(|h_\alpha(y)|+|h_{\alpha+1}(\Pi_{\alpha+1}(y,z))|+\varepsilon^{1/3}\right)e^{\sqrt2 d_{\alpha+1}(y,0)}\\
+ \quad & O(e^{-\frac{3\sqrt2}{2}d_{\alpha-1}(y,0)})+O(e^{\frac{3\sqrt2}{2}d_{\alpha+1}(y,0)})+O(e^{-\sqrt2 d_{\alpha-2}(y,0)})+O(e^{\sqrt2 d_{\alpha+2}(y,0)}).
\end{aligned}
$$

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