ON THE STABILITY OF CAFFARELLI-KOHN-NIRENBERG INEQUALITY

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Abstract. In this paper, we consider the Caffarelli-Kohn-Nirenberg (CKN) inequality:

$$\left(\int_{\mathbb{R}^N}|x|^{-b(p+1)}|u|^{p+1}dx\right)^{\frac{2}{p+1}}\leq C_{a,b,N}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2dx$$

where $N\geq 3$, $a<\frac{N-2}{2}$, $a\leq b\leq a+1$ and $p=\frac{N+2(1+a-b)}{N-2(1+a-b)}$. It is well-known that up to dilations $\tau^{\frac{N-2}{2}-a}u(\tau x)$ and scalar multiplications $Cu(x)$, the CKN inequality has a unique extremal function $W(x)$ which is positive and radially symmetric in the parameter region $b_{FS}(a)\leq b<a+1$ with $a<0$ and $a\leq b<a+1$ with $a\geq 0$ and $a+b>0$, where $b_{FS}(a)$ is the Felli-Schneider curve. We prove that in the above parameter region the following stabilities hold:

(1) stability of CKN inequality in the functional inequality setting

$$dist^2_{D^{1,2}_a}(u,\mathcal{Z})\lesssim\|u\|^2_{D^{1,2}_a(\mathbb{R}^N)}-C^{-1}_{a,b,N}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}$$

where $\mathcal{Z}=\{cW_\tau\mid c\in\mathbb{R}\backslash\{0\},\tau>0\}$;

(2) stability of CKN inequality in the critical point setting (in the class of nonnegative functions)

$$dist_{D^{1,2}_a}(u,\mathcal{Z}^\nu_0)\lesssim\begin{cases}\Gamma(u), & p>2\text{ or }\nu=1,\\ \Gamma(u)|\log\Gamma(u)|^{\frac{1}{2}}, & p=2\text{ and }\nu\geq 2,\\ \Gamma(u)^{\frac{p}{2}}, & 1<p<2\text{ and }\nu\geq 2,\end{cases}$$

where $\Gamma(u)=\|div(|x|^{-a}\nabla u)+|x|^{-b(p+1)}|u|^{p-1}u\|_{(D^{1,2}_a)'}$ and

$$\mathcal{Z}^\nu_0=\{(W_{\tau_1},W_{\tau_2},\cdots,W_{\tau_\nu})\mid\tau_i>0\}.$$

Our results generalize the recent works in [7, 11, 15] on the sharp stability of profile decompositions for the the special case $a=b=0$ (the Sobolev inequality) to the above parameter region of the CKN inequality. This parameter region is optimal for such stabilities in the sense that in the region $a<b<b_{FS}(a)$ with $a<0$, the nonnegative solution of the Euler-Lagrange equation of CKN inequality is no longer unique.

Keywords: Caffarelli-Kohn-Nirenberg inequality; Sharp stability; Profile decomposition.

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1. Introduction

In this paper, we consider the following Caffarelli-Kohn-Nirenberg (CKN for short) inequality:

$$\left(\int_{\mathbb{R}^N}|x|^{-b(p+1)}|u|^{p+1}dx\right)^{\frac{2}{p+1}}\leq C_{a,b,N}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2dx,\tag{1.1}$$ eq0001

where

$$N \geq 3, \quad -\infty < a < \frac{N-2}{2}, \quad a \leq b \leq a+1 \quad \text{and} \quad p = \frac{N+2(1+a-b)}{N-2(1+a-b)}, \tag{1.2}$$ eq0003

$u \in D^{1,2}_a(\mathbb{R}^N)$ and $D^{1,2}_a(\mathbb{R}^N)$ is the Hilbert space given by

$$D^{1,2}_a(\mathbb{R}^N) = \{u \in D^{1,2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx < +\infty\} \tag{1.3}$$ eqn886

with the inner product

$$\langle u, v\rangle_{D^{1,2}_a(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \nabla v dx$$

and $D^{1,2}(\mathbb{R}^N) = \dot{W}^{1,2}(\mathbb{R}^N)$ being the usual homogeneous Sobolev space (cf. [15, Definition 2.1]).

(1.1) is established in the celebrated paper [4] by Caffarelli, Kohn and Nirenberg, as it is named, for a much more general version. Moreover, as pointed out in [5], a fundamental task in understanding the CKN inequality (1.1) is to study the best constants, existence (and nonexistence) of extremal functions, as well as their qualitative properties for parameters a and b in the full region (1.2), since (1.1) contains the classical Sobolev inequality ($a = b = 0$) and the classical Hardy inequality ($a = 0, b = 1$) as special cases, which have played important roles in many applications by virtue of the complete knowledge about the best constants, extremal functions, and their qualitative properties.

Under the condition (1.2), it is well-known (cf. [1, 5, 6, 17, 23]) that (1.1) has extremal functions if and only if either for $a < b < a + 1$ with $a < 0$ or for $a \leq b < a + 1$ with $a \geq 0$. Moreover, let

$$b_{FS}(a) = \frac{N(a_c - a)}{2\sqrt{(a_c - a)^2 + (N-1)}} + a - a_c > a \tag{1.4}$$ eqn991

be the Felli-Schneider curve found in [16], then it is also well-known (cf. [1, 6, 12–14, 16, 17, 23]) that up to dilations $\tau^{a_c - a} u(\tau x)$ and scalar multiplications $Cu(x)$ (also up to translations $u(x + y)$ in the special case $a = b = 0$), (1.1) has a unique extremal function

$$W(x) = (2(p+1)(a_c - a)^2)^{\frac{1}{(p-1)}} \left(1 + |x|^{(a_c - a)(p-1)}\right)^{-\frac{2}{p-1}} \tag{1.5}$$ eq0004

either for $b_{FS}(a) \leq b < a + 1$ with $a < 0$ or for $a \leq b < a + 1$ with $a \geq 0$ while, extremal functions of (1.1) must be non-radial for $a < b < b_{FS}(a)$ with $a < 0$. Here, for the sake of simplicity, we denote $a_c = \frac{N-2}{2}$, as that in [12–14]. On the other hand, it has been proved in [18] that extremal functions of (1.1) must have $\mathcal{O}(N-1)$ symmetry for $a < b < b_{FS}(a)$ with $a < 0$, that is, extremal functions of (1.1) must depend on the radius r and the angle θ_N between the positive x_N-axis and $\overrightarrow{Ox}$ for $a < b < b_{FS}(a)$ with $a < 0$ up to rotations. To our best knowledge, whether the extremal function of (1.1) is unique or not for $a < b < b_{FS}(a)$ with $a < 0$ is still unknown.

Once the extremal functions of (1.1) are well understood, it is natural to study quantitative stability for the CKN inequality (1.1) by asking whether the deviation of a given function from attaining equality in (1.1) controls its distance to the family of extremal functions. These studies were initialed by Brezís and Lieb in [3] by raising an open question for the classical Sobolev inequality ($a = b = 0$) which was settled by Bianchi and Egnell in [2] while, quantitative stability for the Hardy-Sobolev inequality ($a = 0$, $0 < b < 1$) was studied in [20]. Since the extremal function of (1.1) is unique up to dilations $\tau^{a_c - a}u(\tau x)$ and scalar multiplications $Cu(x)$ either for $b_{FS}(a) \leq b < a+1$ with $a < 0$ or for $a \leq b < a+1$ with $a \geq 0$ and $a + b > 0$, the smooth manifold

$$\mathcal{Z} = \{cW_\tau(x) \mid c \in \mathbb{R}\backslash\{0\} \text{ and } \tau > 0\}$$

is all extremal functions of (1.1) in the above parameter region. Thus, it is natural to extend the quantitative stability for the Sobolev inequality ($a = b = 0$) and the Hardy-Sobolev inequality ($a = 0$, $0 < b < 1$) to the CKN inequality (1.1) in the above parameter region. Our first result in this paper, which devoted to this aspect, can be stated as follows.

⟨thm0001⟩ **Theorem 1.1.** *Let $b_{FS}(a)$ be the Felli-Schneider curve given by (1.4) and assume that either*

(1) $b_{FS}(a) \leq b < a+1$ *with* $a < 0$ *or*
(2) $a \leq b < a+1$ *with* $a \geq 0$ *and* $a + b > 0$.

Then

$$dist^2_{D^{1,2}_a}(u, \mathcal{Z}) \lesssim \|u\|^2_{D^{1,2}_a(\mathbb{R}^N)} - C^{-1}_{a,b,N}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}$$

for all $u \in D^{1,2}_a(\mathbb{R}^N)$, where $L^{p+1}(|x|^{-b(p+1)}, \mathbb{R}^N)$ is the usual weighted Lebesgue space and its usual norm is given by

$$\|u\|_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |x|^{-b(p+1)}|u|^{p+1}dx\right)^{\frac{1}{p+1}}.$$

Remark 1.1. *The conditions $b_{FS}(a) \leq b < a+1$ with $a < 0$ and $a \leq b < a+1$ with $a \geq 0$ and $a + b > 0$ in Theorem 1.1 is sharp in the sense that, for $a < b < b_{FS}(a)$ with $a < 0$, the extremal function of (1.1) is no longer cW_τ.*

On the other hand, it is well-known that the Euler-Lagrange equation of the Sobolev inequality ($a = b = 0$) is given by

$$-\Delta u = |u|^{\frac{4}{N-2}}u, \quad \text{in } \mathbb{R}^N \tag{1.6}$$ eqn880

and the Aubin-Talanti bubbles, given by

$$U[z, \lambda] = [N(N-2)]^{\frac{N-2}{4}}\left(\frac{\lambda}{\lambda^2 + |x-z|^2}\right)^{\frac{N-2}{2}},$$

are the only poaitive solutions of (1.6), where $z \in \mathbb{R}^N$ and $\lambda > 0$. Thus, the smooth manifold (except $c = 0$)

$$\mathcal{M} = \{cU[z, \lambda] \mid c \in \mathbb{R}, z \in \mathbb{R}^N, \lambda > 0\}$$

is all nonnagetive solutions of (1.6). Moreover, Struwe proved in [21] the following well-known stability of profile decompositions to (1.6) for nonnegative functions.

⟨thm0003⟩ **Theorem 1.2.** *Let $N \geq 3$ and $\nu \geq 1$ be positive integers. Let $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ be a nonnegative sequence with*

$$(\nu - \frac{1}{2})S^{\frac{N}{2}} < \|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} < (\nu + \frac{1}{2})S^{\frac{N}{2}},$$

where S is the best Sobolev constant. Assume that $\|\Delta u_n + |u_n|^{\frac{4}{N-2}}u_n\|_{H^{-1}} \to 0$ as $n \to \infty$, then there exist a sequence $(z_1^{(n)}, z_2^{(n)}, \cdots, z_\nu^{(n)})$ of ν-tuples of points in $\mathbb{R}^N$ and a sequence of $(\lambda_1^{(n)}, \lambda_2^{(n)}, \cdots, \lambda_\nu^{(n)})$ of ν-tuples of positive real numbers such that

$$\|\nabla u_n - \sum_{i=1}^{\nu} \nabla U[z_i^{(n)}, \lambda_i^{(n)}]\|_{L^2(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

In the recent papers [7, 15], Figalli et al. initialed a study on the quantitative version of Theorem 1.2 by proving

(1) (Ciraolo-Figalli-Maggi [7]) Let $N \geq 3$ and $u \in D^{1,2}(\mathbb{R}^N)$ be positive such that $\|\nabla u\|^2_{L^2(\mathbb{R}^N)} \leq \frac{3}{2}S^{\frac{N}{2}}$ and $\|\Delta u + |u|^{\frac{4}{N-2}}u\|_{H^{-1}} \leq \delta$ for some $\delta > 0$ sufficiently small, then

$$dist_{D^{1,2}}(u, \mathcal{M}_0) \lesssim \|\Delta u + |u|^{\frac{4}{N-2}}u\|_{H^{-1}},$$

where $\mathcal{M}_0 = \{U[z, \lambda] \mid z \in \mathbb{R}^N, \lambda > 0\}$.

(2) (Figalli-Glaudo [15]) Let $u \in D^{1,2}(\mathbb{R}^N)$ be nonnegative such that

$$(\nu - \frac{1}{2})S^{\frac{N}{2}} < \|u\|^2_{D^{1,2}(\mathbb{R}^N)} < (\nu + \frac{1}{2})S^{\frac{N}{2}}$$

and $\|\Delta u + |u|^{\frac{4}{N-2}}u\|_{H^{-1}} \leq \delta$ for some $\delta > 0$ sufficiently small, then for $3 \leq N \leq 5$,

$$dist_{D^{1,2}}(u, \mathcal{M}_0^\nu) \lesssim \|\Delta u + |u|^{\frac{4}{N-2}}u\|_{H^{-1}}$$

where

$$\mathcal{M}_0^\nu = \{\sum_{i=1}^{\nu} U[z_i, \lambda_i] \mid z_i \in \mathbb{R}^N, \lambda_i > 0\}.$$

Remark 1.2. *The stability obtained in [15] is more general than the conclusion* (2) *stated here in the sense that, u could be sign-changing if u is close to the sum of $U[z_i, \lambda_i]$ in $D^{1,2}(\mathbb{R}^N)$ where $U[z_i, \lambda_i]$ are weakly interacting (the definition of weakly interaction can be found in [15, Definition 3.1]). We choose to state the conclusion* (2) *since it is more close to Struwe's theorem on the stability of profile decompositions to* (1.6) *for nonnegative functions.*

As pointed out in [15], it is rather surprisingly that the conclusion (2) is false for $N \geq 6$. Figalli and Glaudo constructed a counterexample of the conclusion (2) for $N \geq 6$ with two bubbles and conjectured in [15] that the quantitative version of Theorem 1.2 for $N \geq 6$ and $\nu > 1$ will be

$$dist_{D^{1,2}}(u, \mathcal{M}_0^\nu) \lesssim \begin{cases} \|\Delta u + |u|u\|_{H^{-1}}|\ln(\|\Delta u + |u|u\|_{H^{-1}})|, & N = 6; \\ \|\Delta u + |u|^{\frac{4}{N-2}}u\|_{H^{-1}}^{\gamma(N)}, & N \geq 7 \end{cases}$$

with $0 < \gamma(N) < 1$ under the same assumptions of the conclusion (2). In the very recent work [11], the first author of the current paper, together with Deng and Sun, proved that the quantitative version of Theorem 1.2 for $N \geq 6$ and $\nu > 1$ is actually

$$dist_{D^{1,2}}(u, \mathcal{M}_0^\nu) \lesssim \begin{cases} \|\Delta u + |u|u\|_{H^{-1}} |\ln(\|\Delta u + |u|u\|_{H^{-1}})|^{\frac{1}{2}}, & N = 6; \\ \|\Delta u + |u|^{\frac{4}{N-2}}u\|_{H^{-1}}^{\frac{N+2}{2(N-2)}}, & N \geq 7 \end{cases}$$

under the same assumptions of the conclusion (2), where the orders of the right hand sides in above estimates are optimal. We would like to refer the recent works [7,15] once more for more motivations, discussions and applications of the study on the quantitative version of Theorem 1.2.

Since the Sobolev inequality ($a = b = 0$) is a special case of the CKN inequality (1.1) and the Euler-Lagrange equation of (1.1) is given by

$$-div(|x|^{-a}\nabla u) = |x|^{-b(p+1)}|u|^{p-1}u, \quad \text{in } \mathbb{R}^N, \tag{1.7}$$ eq0018

it is natural to ask whether the stability of profile decompositions to (1.7) for nonnegative functions which is similar to that of (1.6) holds or not. Our second main result of this paper, which devoted to this natural question, can be stated as follows.

⟨thm0002⟩ **Theorem 1.3.** *Let $N \geq 3$ and $\nu \geq 1$ be positive integers. Let $b_{FS}(a)$ be the Felli-Schneider curve given by (1.4) and assume that either $b_{FS}(a) \leq b < a+1$ with $a < 0$ or $a \leq b < a+1$ with $a \geq 0$ and $a+b > 0$. Then for any nonnegative $u \in D_a^{1,2}(\mathbb{R}^N)$ such that*

$$(\nu - \frac{1}{2})(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}} < \|u\|^2_{D_a^{1,2}(\mathbb{R}^N)} < (\nu + \frac{1}{2})(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}} \tag{1.8}$$ eqn99

and $\Gamma(u) \leq \delta$ with some $\delta > 0$ sufficiently small, we have

$$dist_{D_a^{1,2}}(u, \mathcal{Z}_0^\nu) \lesssim \begin{cases} \Gamma(u), & p > 2 \text{ or } \nu = 1, \\ \Gamma(u)|\log\Gamma(u)|^{\frac{1}{2}}, & p = 2 \text{ and } \nu \geq 2, \\ \Gamma(u)^{\frac{p}{2}}, & 1 < p < 2 \text{ and } \nu \geq 2. \end{cases}$$

where $\Gamma(u) = \|div(|x|^{-a}\nabla u) + |x|^{-b(p+1)}|u|^{p-1}u\|_{(D_a^{1,2})'}$ and

$$\mathcal{Z}_0^\nu = \{\sum_{j=1}^{\nu} W_{\tau_j} \mid \tau_j > 0\}.$$

Moreover, this stability is sharp in the sense that, there exists nonnegative $u_* \in D_a^{1,2}(\mathbb{R}^N)$, satisfying (1.8) and $\Gamma(u_*) \leq \delta$ for some $\delta > 0$ sufficiently small, such that

$$dist_{D_a^{1,2}}(u_*, \mathcal{Z}_0^\nu) \gtrsim \begin{cases} \Gamma(u_*), & p > 2 \text{ or } \nu = 1, \\ \Gamma(u_*)|\log\Gamma(u_*)|^{\frac{1}{2}}, & p = 2 \text{ and } \nu \geq 2, \\ \Gamma(u_*)^{\frac{p}{2}}, & 1 < p < 2 \text{ and } \nu \geq 2. \end{cases}$$

Remark 1.3. (1) *The conditions $b_{FS}(a) \leq b < a+1$ with $a < 0$ and $a \leq b < a+1$ with $a \geq 0$ and $a+b > 0$ in Theorem 1.3 is optimal in the sense that, for $a < b < b_{FS}(a)$ with $a < 0$, the nonnegative solutions of (1.7) is no longer unique.*

(2) *(2) of Theorem 1.3 can be generalized to more general class of functions as that in [15, Theorem 3.3] by introducing the concept of weakly interaction for the nonnegative solutions of (1.7) as that in [15, Definition 3.1].*

Notations. Throughout this paper, C and C' are indiscriminately used to denote various absolutely positive constants. $a \sim b$ means that $C'b \leq a \leq Cb$ and $a \lesssim b$ means that $a \leq Cb$.

2. Preliminaries

Let $D_a^{1,2}(\mathbb{R}^N)$ be the Hilbert space given by (1.3) with the norm $\|\cdot\|_{D_a^{1,2}(\mathbb{R}^N)}$. By [5, Proposition 2.2], $D_a^{1,2}(\mathbb{R}^N)$ is isomorphic to the Hilbert space $H^1(\mathcal{C})$ by the transformation

$$u(x) = |x|^{-(a_c-a)}v(-\ln|x|, \frac{x}{|x|}), \tag{2.1}$$ $\boxed{\text{eq0007}}$

where $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{N-1}$ is the standard cylinder, the inner product in $H^1(\mathcal{C})$ is given by

$$\langle w, v\rangle_{H^1(\mathcal{C})} = \int_{\mathcal{C}} \nabla w \nabla v + (a_c - a)^2 uv d\mu$$

with $d\mu$ being the volume element on $\mathcal{C}$, $u \in D_a^{1,2}(\mathbb{R}^N)$ and $w, v \in H^1(\mathcal{C})$.

The CKN inequality (1.1) can be rewritten as the following minimizing problem:

$$C_{a,b,N}^{-1} = \inf_{u \in D_a^{1,2}(\mathbb{R}^N)\backslash\{0\}} \frac{\|u\|^2_{D_a^{1,2}(\mathbb{R}^N)}}{\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}}, \tag{2.2}$$ $\boxed{\text{eq0002}}$

where $L^{p+1}(|x|^{-b(p+1)}, \mathbb{R}^N)$ is the usual weighted Lebesgue space and its usual norm is given by $\|u\|_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |x|^{-b(p+1)}|u|^{p+1}dx\right)^{\frac{1}{p+1}}$. The Euler-Lagrange equation of the minimizing problem (2.2) is given by (1.7). By (2.1), (2.2) is equivalent to the following minimizing problem:

$$C_{a,b,N}^{-1} = \inf_{v \in H^1(\mathcal{C})\backslash\{0\}} \frac{\|v\|^2_{H^1(\mathcal{C})}}{\|u\|^2_{L^{p+1}(\mathcal{C})}}, \tag{2.3}$$ $\boxed{\text{eq0009}}$

where $\|\cdot\|_{L^{p+1}(\mathcal{C})}$ is the usual norm in the Lebesgue space $L^{p+1}(\mathcal{C})$. Let $t = -\ln|x|$ and $\theta = \frac{x}{|x|}$ for $x \in \mathbb{R}^N\backslash\{0\}$, then by [5, Proposition 2.2], (1.7) is equivalent to the following equation of v:

$$-\Delta_\theta v - \partial_t^2 v + (a_c - a)^2 v = |v|^{p-1}v, \quad \text{in } \mathcal{C} \tag{2.4}$$ $\boxed{\text{eq0006}}$

where Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$.

Clearly, minimizers of (2.2) are ground states of (1.7). It is also well-known (cf. [5, 6, 16]) that up to dilations $u_\tau(x) = \tau^{a_c-a}u(\tau x)$ and scalar multiplications $Cu(x)$ (also up to translations $u(x+y)$ for the spacial case $a = b = 0$), the radial function $W(x)$ given by (1.5) is also the unique minimizer of the following minimizing problem

$$C_{a,b,N,rad}^{-1} = \inf_{u \in D_{a,rad}^{1,2}(\mathbb{R}^N)\backslash\{0\}} \frac{\|u\|^2_{D_a^{1,2}(\mathbb{R}^N)}}{\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}}$$

under the condition (1.2), where

$$D^{1,2}_{a,rad}(\mathbb{R}^N) = \{u \in D^{1,2}_a(\mathbb{R}^N) \mid u \text{ is radially symmetric}\}.$$

Thus, $W(x)$ is always a solution of (1.7) under the condition (1.2). It has been proved in [6,14] that $W(x)$ is the unique nonnegative solution of (1.7) in $D^{1,2}_a(\mathbb{R}^N)$ either for $b_{FS}(a) \le b < a+1$ with $a < 0$ or for $a \le b < a+1$ with $a \ge 0$. Moreover, it has also been proved in [16] that $W(x)$ is nondegenerate in $D^{1,2}_a(\mathbb{R}^N)$ under the condition (1.2). That is, up to scalar multiplications $CV(x)$,

$$V(x) := \nabla W(x)\cdot x - (a_c - a)W(x) = \frac{\partial}{\partial\lambda}(\lambda^{-(a_c-a)}W(\lambda x))|_{\lambda=1} \tag{2.5}$$ eq0010

is the only nonzero solution in $D^{1,2}_a(\mathbb{R}^N)$ to the linearization of (1.7) around W which is given by

$$-div(|x|^{-a}\nabla u) = p|x|^{-b(p+1)}W^{p-1}u, \quad \text{in } \mathbb{R}^N. \tag{2.6}$$ eq0017

By the transformation (2.1), the linear equation (2.6) can be rewritten as follows:

$$-\Delta_\theta v - \partial_t^2 v + (a_c - a)^2 v = p\Psi^{p-1}v, \quad \text{in } \mathcal{C}, \tag{2.7}$$ eq0016

where Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$, $t = -\ln|x|$ and $\theta = \frac{x}{|x|}$ for $x \in \mathbb{R}^N\backslash\{0\}$, and

$$\Psi(t) = \left(\frac{(p+1)(a_c-a)^2}{2}\right)^{\frac{1}{p-1}}\left(cosh(\frac{(a_c-a)(p-1)}{2}t)\right)^{-\frac{2}{p-1}}. \tag{2.8}$$ eq0026

It follows from the transformation (2.1) that

$$\Psi_s'(t) = \Psi'(t-\log s) = \frac{\partial}{\partial t}\Psi(t-\log s) = -s\frac{\partial}{\partial s}\Psi(t-\log s)$$

is the only nonzero solution of (2.8) in $H^1(\mathcal{C})$.

3. Profile decompositions of nonnegative functions

It is well-known that all minimizers of (2.2) are positive in $\mathbb{R}^N\backslash\{0\}$. Indeed, let

$$\mathcal{E}(u) = \frac{1}{2}\|u\|^2_{D^{1,2}_a(\mathbb{R}^N)} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)},$$

then by (1.1), $\mathcal{E}(u)$ is of class C^2 in $D^{1,2}_a(\mathbb{R}^N)$. Since it is well-known that extremal functions of (2.2) are ground states of (1.7), extremal functions of (2.2) are also minimizers of the minimizing problem

$$c = \inf_{u\in\mathcal{N}}\mathcal{E}(u),$$

where

$$\mathcal{N} = \{u \in D^{1,2}_a(\mathbb{R}^N)\backslash\{0\} \mid \mathcal{E}'(u)u = 0\}$$

is the usual Nehari manifold. Since $p > 1$ for $a \le b < a+1$, it is standard to use the fibering maps to show the double-energy property of $\mathcal{E}(u)$, that is, $c_{sg} \ge 2c$, where $c_{sg} = \inf_{u\in\mathcal{N}_{sg}}\mathcal{E}(u)$ with

$$\mathcal{N}_{sg} = \{u \in D^{1,2}_a(\mathbb{R}^N)\backslash\{0\} \mid \mathcal{E}'(u^\pm)u^\pm = 0\}$$

and $u^\pm = \max\{\pm u, 0\}$. Thus, by the the double-energy property of $\mathcal{E}(u)$, all minimizers of $\mathcal{E}(u)$ in $\mathcal{N}$ at the energy level c are nonnegative which implies that all extremal functions of (2.2) are nonnegative. It follows from the maximum principle that all extremal functions of (2.2) are positive in $\mathbb{R}^N\backslash\{0\}$.

As that of the Sobolev and Hardy-Sobolev inequalities, we have the following relatively compactness of minimizing sequences of (2.2).

⟨prop0001⟩ **Proposition 3.1.** *Suppose that* $\{u_n\}\subset D^{1,2}_a(\mathbb{R}^N)$ *be a minimizing sequence of* (2.2) *either for* $a<b<a+1$ *with* $a<0$ *or for* $a\leq b<a+1$ *with* $a\geq0$ *and* $a+b>0$. *Then there exists* $\{\tau_n\}\subset(0,+\infty)$ *such that* $(u_n)_{\tau_n}\to u_0$ *strongly in* $D^{1,2}_a(\mathbb{R}^N)$ *as* $n\to\infty$ *up to a subsequence, where* u_0 *is a minimizer of* (2.2). *Moreover, we have* $u_0=CW_{\tau_0}$ *with some* $\tau_0>0$ *and* $C\in\mathbb{R}\backslash\{0\}$ *either for* $b_{FS}(a)\leq b<a+1$ *with* $a<0$ *or for* $a\leq b<a+1$ *with* $a\geq0$ *and* $a+b>0$, *where* $b_{FS}(a)$ *is the Felli-Schneider curve given by* (1.4).

Proof. Since the case $a\geq0$ is considered in [24, Theorem 4], we shall only give the proof for $a<0$. Moreover, the proof is rather standard nowadays (cf. [22]), so we only sketch it here. Without loss of generality, we may assume that

$$\|u_n\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}=1.$$

Then $\{u_n\}$ is bounded in $D^{1,2}_a(\mathbb{R}^N)$ and thus, $u_n\rightharpoonup\widehat{u}_0$ weakly in $D^{1,2}_a(\mathbb{R}^N)$ as $n\to\infty$ up to a subsequence. Moreover, by the double-energy property of $\mathcal{E}(u)$, we may also assume that $\widehat{u}_0\geq0$ without loss of generality. By (2.1), the corresponding $v_n\rightharpoonup v_0$ weakly in $H^1(\mathcal{C})$ as $n\to\infty$ up to a subsequence with $v_0\geq0$. Since $a<b<a+1$ for $a<0$, we have $1<p<\frac{N+2}{N-2}$ by (1.2). Thus, by [5, Lemma 4.1], there exists $\{\tau_n\}\subset\mathbb{R}$ such that

$$\overline{v}_n=v_n(t-\tau_n,\theta)\rightharpoonup\overline{v}_0\neq0\quad\text{weakly in } H^1(\mathcal{C}) \text{ as } n\to\infty.$$

It follows from the Brezís-Lieb lemma and the concavity of the function $t^{\frac{2}{p+1}}$ for $0<t<1$ with $p>1$ that

$$\begin{aligned}1+o_n(1)&=C_{a,b,N}(\|\overline{v}_n-\overline{v}_0\|^2_{H^1(\mathcal{C})}+\|\overline{v}_0\|^2_{H^1(\mathcal{C})})\\&\geq(1-\|\overline{v}_0\|^{p+1}_{L^{p+1}(\mathcal{C})}+o_n(1))^{\frac{2}{p+1}}+(\|\overline{v}_0\|^{p+1}_{L^{p+1}(\mathcal{C})})^{\frac{2}{p+1}}\\&\geq1+o_n(1),\end{aligned}$$

which implies that $\overline{v}_n\to\overline{v}_0$ strongly in $L^{p+1}(\mathcal{C})$ as $n\to\infty$. Correspondingly, by (2.1), we have $(u_n)_{\tau_n}\to u_0$ strongly in $L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)$ as $n\to\infty$. It is then easy to show that u_0 is a minimizer of (2.2). In the cases $b_{FS}(a)\leq b<a+1$ with $a<0$ or $a\leq b<a+1$ with $a\geq0$ and $a+b>0$, W is the unique minimizer of (2.2) up to dilations $u_\tau(x)=\tau^{a_c-a}u(\tau x)$ and scalar multiplications $Cu(x)$. Thus, we must have $u_0=CW_{\tau_0}$ with some $\tau_0>0$ and $C\in\mathbb{R}\backslash\{0\}$. □

As the well-known results of profile decompositions to the Sobolev inequality due to Struwe (cf. [21, 22]), we have the following profile decompositions of (1.7) for nonnegative functions which, to our best knowledge, is new.

⟨prop0002⟩ **Proposition 3.2.** *Let* $\{w_n\}$ *be a nonnegative* (PS) *sequence of* $\mathcal{E}(u)$ *with*

$$(\nu-\frac{1}{2})(C^{-1}_{a,b,N})^{\frac{p+1}{p-1}}<\|w_n\|^2_{D^{1,2}_a(\mathbb{R}^N)}<(\nu+\frac{1}{2})(C^{-1}_{a,b,N})^{\frac{p+1}{p-1}}$$

with some $\nu\in\mathbb{N}$ *either for* $b_{FS}(a)\leq b<a+1$ *with* $a<0$ *or for* $a\leq b<a+1$ *with* $a\geq0$ *and* $a+b>0$ *where* $b_{FS}(a)$ *is the Felli-Schneider curve given by* (1.4). *Then there exists* $\{\tau_{i,n}\}\subset\mathbb{R}_+$, *satisfying*

$$\min_{i\neq j}\left\{\max\left\{\frac{\tau_{i,n}}{\tau_{j,n}},\frac{\tau_{j,n}}{\tau_{i,n}}\right\}\right\}\to+\infty$$

as $n \to \infty$ *for* $\nu \geq 2$*, such that*

(1) $w_n = \sum_{i=1}^{\nu}(W)_{\tau_{i,n}} + o_n(1)$ *in* $D^{1,2}_a(\mathbb{R}^N)$.

(2) $\|w_n\|^2_{D^{1,2}_a(\mathbb{R}^N)} = \nu\|W\|^2_{D^{1,2}_a(\mathbb{R}^N)} + o_n(1)$.

Proof. Since $p = \frac{N+2}{N-2}$ for $a = b$ and $p < \frac{N+2}{N-2}$ for $a < b$, We shall divide the proof into two parts which is devoted to the case $p < \frac{N+2}{N-2}$ and $p = \frac{N+2}{N-2}$, respectively.

The case $p < \frac{N+2}{N-2}(a < b)$.

In this case, we use the transformation (2.1) to w_n. Then the related $\widetilde{w}_n(t,\theta)$ satisfy

$$(\nu - \frac{1}{2})(C^{-1}_{a,b,N})^{\frac{p+1}{p-1}} < \|\widetilde{w}_n\|^2_{H^1(\mathcal{C})} < (\nu + \frac{1}{2})(C^{-1}_{a,b,N})^{\frac{p+1}{p-1}}$$

and $\mathcal{J}'(\widetilde{w}_n) \to 0$ in $H^{-1}(\mathcal{C})$ as $n \to \infty$, where $H^{-1}(\mathcal{C})$ is the dual space of $H^1(\mathcal{C})$ and

$$\mathcal{J}(u) = \frac{1}{2}\|u\|^2_{H^1(\mathcal{C})} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathcal{C})}.$$

Since $p < \frac{N+2}{N-2}$ and $\Psi(t)$ is the unique nonnegative solution of (2.8) in $H^1(\mathcal{C})$ either for $b_{FS}(a) \leq b < a+1$ with $a < 0$ or for $a < b < a+1$ with $a \geq 0$ and $a+b > 0$, the conclusion then follows from (2.1) and adapting [5, Lemma 4.1] in a standard way.

The case $p = \frac{N+2}{N-2}(a = b)$.

In this case, we have $a > 0$ by the assumptions. Moreover, [5, Lemma 4.1] is invalid to drive the conclusion and thus, we shall mainly follows Struwe's idea in proving [22, Theorem 3.1]. However, according to the singular potential $|x|^{-2a}$, the argument is more involved. Let U_ε be the standard Aubin-Talanti bubbles, that is,

$$U_\varepsilon = [N(N-2)]^{\frac{N-2}{4}}\left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N-2}{2}}.$$

By [24, Lemma 1],

$$C^{-1}_{a,a,N} < S \quad \text{for } a > 0. \tag{3.1}$$ eqn992

Thus, there exists $R_\varepsilon > 0$ such that

$$\int_{B_{R_\varepsilon}(0)}|\nabla U_\varepsilon|^2dx > \frac{1}{L_{R_\varepsilon}}(C^{-1}_{a,b,N})^{\frac{N}{2}}, \tag{3.2}$$ eq0054

where L_{R_ε} is the number such that the ball $B_{2R_\varepsilon}(0)$ is covered by L_{R_ε} balls with radius R_ε. Here, we have used the fact that $2 \leq L_{R_\varepsilon} \leq 2^N$. Let

$$Q_n(r) = \sup_{y\in\mathbb{R}^N}\int_{B_r(y)}|x|^{-2a}|\nabla w_n|^2dx$$

be the well-known concentration function of w_n. Since

$$(\nu - \frac{1}{2})(C^{-1}_{a,a,N})^{\frac{N}{2}} < \|w_n\|^2_{D^{1,2}_a(\mathbb{R}^N)} < (\nu + \frac{1}{2})(C^{-1}_{a,a,N})^{\frac{N}{2}}$$

for some $\nu \in \mathbb{N}$, we can choose $r_n > 0$ and $y_n \in \mathbb{R}^N$ such that

$$Q_n(r_n) = \int_{B_{r_n}(y_n)}|x|^{-2a}|\nabla w_n|^2dx = \frac{1}{2L_{R_\varepsilon}}(C^{-1}_{a,a,N})^{\frac{N}{2}}.$$

Let

$$v_n=(r_nR_\varepsilon^{-1})^{-(a_c-a)}w_n(r_nR_\varepsilon^{-1}x),$$

then

$$\sup_{y\in\mathbb{R}^N}\int_{B_{R_\varepsilon}(y)}|x|^{-2a}|\nabla v_n|^2dx=\int_{B_{R_\varepsilon}(\frac{R_\varepsilon y_n}{r_n})}|x|^{-2a}|\nabla v_n|^2dx=\frac{1}{2L_{R_\varepsilon}}(C_{a,a,N}^{-1})^{\frac{N}{2}}. \tag{3.3}$$

eq0050

Since $\|\cdot\|_{D_a^{1,2}}$ is invariant under the dilation $u_\tau(x)=\tau^{a_c-a}u(\tau x)$, $\{v_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and thus, $v_n\rightharpoonup v_0$ weakly in $D^{1,2}(\mathbb{R}^N)$ as $n\to\infty$ up to a subsequence. Clearly, $v_0\geq 0$. We define $\varpi_n=(v_n-v_0)\varphi$, where φ is a smooth cut-off function such that $\varphi=1$ in $B_{R_\varepsilon}(z)$ and $\varphi=0$ in $B^c_{\frac{3}{2}R_\varepsilon}(z)$ for any $z\in\mathbb{R}^N$. Then by [24, Lemma 2],

$$\|\varpi_n\|^2_{D_a^{1,2}(\mathbb{R}^N)}\lesssim\|v_n-v_0\|^2_{D_a^{1,2}(\mathbb{R}^N)}+\int_{B_{2R_\varepsilon}(z)\backslash B_{R_\varepsilon}(z)}|x|^{-2a}|v_n-v_0|^2dx\lesssim 1$$

and

$$\int_{B_{2R_\varepsilon}(z)\backslash B_{R_\varepsilon}(z)}|x|^{-2a}|v_n-v_0|^2dx\to 0\quad\text{as } n\to\infty.$$

Note that by the fact that $\mathcal{E}'(w_n)\to 0$ in $D_a^{-1,2}(\mathbb{R}^N)$ as $n\to\infty$ and the invariance of $\mathcal{E}(w_n)$ under the the dilation $u_\tau(x)=\tau^{a_c-a}u(\tau x)$, $\mathcal{E}'(v_n)\to 0$ in $D_a^{-1,2}(\mathbb{R}^N)$ as $n\to\infty$. It follows that $\mathcal{E}'(v_0)=0$, which, together with the Brezís-Lieb lemma, implies

$$\begin{aligned}
o_n(1) &= \int_{\mathbb{R}^N}(|x|^{-2a}\nabla(v_n-v_0)\nabla(\varpi_n\varphi)-|x|^{-\frac{2Na}{N-2}}(v_n^{\frac{N+2}{N-2}}-v_0^{\frac{N+2}{N-2}})\varpi_n\varphi)dx\\
&= \int_{\mathbb{R}^N}(|x|^{-2a}|\nabla\varpi_n|^2-|x|^{-\frac{2Na}{N-2}}|v_n-v_0|^{\frac{4}{N-2}}|\varpi_n|^2)dx+o_n(1)\\
&\geq \|\varpi_n\|^2_{D_a^{1,2}(\mathbb{R}^N)}-\|\varpi_{n,*}\|^{\frac{4}{N-2}}_{L^{\frac{2N}{N-2}}(|x|^{-\frac{2Na}{N-2}},\mathbb{R}^N)}\|\varpi_n\|^2_{L^{\frac{2N}{N-2}}(|x|^{-\frac{2Na}{N-2}},\mathbb{R}^N)}\\
&\quad +o_n(1).
\end{aligned}\tag{3.4}$$

eq0051

Here, $\varpi_{n,*}=(v_n-v_0)\varphi_*$, where φ_* is a smooth cut-off function such that $\varphi_*=1$ in $B_{\frac{3}{2}R_\varepsilon}(z)$ and $\varphi_*=0$ in $B^c_{2R_\varepsilon}(z)$. By the Brezís-Lieb lemma once more,

$$\begin{aligned}
\|\varpi_{n,*}\|^{\frac{2N}{N-2}}_{L^{\frac{2N}{N-2}}(|x|^{-\frac{2Na}{N-2}},\mathbb{R}^N)} &\leq \int_{B_{2R_\varepsilon}(z)}|x|^{-\frac{2Na}{N-2}}|v_n-v_0|^{\frac{2N}{N-2}}dx\\
&\leq \int_{B_{2R_\varepsilon}(z)}|x|^{-\frac{2Na}{N-2}}|v_n|^{\frac{2N}{N-2}}dx+o_n(1).
\end{aligned}$$

It follows from the CKN inequality (1.1), (3.3) and (3.4) that $\varpi_n\to 0$ strongly in $D_a^{1,2}(\mathbb{R}^N)$ as $n\to\infty$, which implies that $v_n\to v_0$ strongly in $D_a^{1,2}(B_{R_0}(z))$ as $n\to\infty$ for any $z\in\mathbb{R}^N$. Thus, by a standard covering argument, $v_n\to v_0$ strongly $D_{a,loc}^{1,2}(\mathbb{R}^N)$ as $n\to\infty$. We claim that $v_0\neq 0$. Assume the contrary that $v_0=0$, then by (3.3), $|\frac{R_\varepsilon y_n}{r_n}|\to+\infty$ as $n\to\infty$. It follows that

$$\int_{B_{\frac{1}{2}|\frac{R_\varepsilon y_n}{r_n}|}(\frac{R_\varepsilon y_n}{r_n})}|\nabla(\frac{v_n}{|\frac{R_\varepsilon y_n}{r_n}|^a})|^2\sim\int_{B_{\frac{1}{2}|\frac{R_\varepsilon y_n}{r_n}|}(\frac{R_\varepsilon y_n}{r_n})}|x|^{-2a}|\nabla v_n|^2\lesssim\|v_n\|^2_{D_a^{1,2}(\mathbb{R}^N)},$$

which implies that $\{\overline{v}_n\}$ is bounded in $D^{1,2}_{loc}(\mathbb{R}^N)$ and thus, $\overline{v}_n \rightharpoonup \overline{v}_0$ weakly in $D^{1,2}_{loc}(\mathbb{R}^N)$ as $n\to\infty$, where $\overline{v}_n = \frac{v_n(x+\frac{R_\varepsilon y_n}{r_n})}{|\frac{R_\varepsilon y_n}{r_n}|^a}$. Now, by $\mathcal{E}'(v_n)\to 0$ in $D^{-1,2}_a(\mathbb{R}^N)$ as $n\to\infty$, we know that

$$-\Delta\overline{v}_n = \overline{v}_n^{\frac{N+2}{N-2}} + o_n(1) \quad \text{in } D^{1,2}_{loc}(\mathbb{R}^N). \tag{3.5}$$

By (3.1), (3.3) and $|\frac{R_\varepsilon y_n}{r_n}|\to+\infty$ as $n\to\infty$,

$$\begin{aligned}\int_{B_{R_\varepsilon}(0)}|\nabla\overline{v}_n|^2dx &= \sup_{y\in\mathbb{R}^N}\int_{B_{R_\varepsilon}(y)}|\nabla\overline{v}_n|^2dx\\ &= \frac{1}{2L_{R_\varepsilon}}(C^{-1}_{a,b,N})^{\frac{N}{2}} + o_n(1)\\ &< \frac{1}{2L_{R_\varepsilon}}S^{\frac{N}{2}} + o_n(1)\end{aligned} \tag{3.6}$$

Thus, by applying similar arguments as that used for (3.4) to (3.5), we can show that $\overline{v}_n\to\overline{v}_0$ strongly in $D^{1,2}(B_{R_\varepsilon}(0))$ as $n\to\infty$. By (3.6), $\overline{v}_0\neq 0$ and thus, $\overline{v}_0 = U_\varepsilon$ for some $\varepsilon>0$ by (3.5). It is impossible since (3.2) and (3.6) hold at the same time now. Therefore, we must have $v_0\neq 0$. Since $v_0\geq 0$, by [6, Theorem B and Proposition 4.4] and [14, Theorem 1.2], we have $v_0 = W$ either for $b_{FS}(a)\leq b<a+1$ with $a<0$ or for $a\leq b<a+1$ with $a\geq 0$ and $a+b>0$. Thus, $w_n\rightharpoonup (W)_{r_nR_\varepsilon^{-1}}$ weakly in $D^{1,2}_a(\mathbb{R}^N)$ as $n\to\infty$. By running the above argument to $w_n-(W)_{r_nR_\varepsilon^{-1}}$, we will arrive at that $w_n\rightharpoonup (W)_{r_{n,1}r_nR_\varepsilon^{-1}R_{\varepsilon_1}^{-1}} + (W)_{r_{n,1}R_{\varepsilon_1}^{-1}}$ weakly in $D^{1,2}_a(\mathbb{R}^N)$ as $n\to\infty$ for some $r_{n,1}>0$ and $\varepsilon_1>0$. The conclusion then follows from iterating the above arguments for ν times and using the fact that $W(|x|)$ is the unique nonnegative solution of (1.7) in $D^{1,2}_a(\mathbb{R}^N)$ either for $b_{FS}(a)\leq b<a+1$ with $a<0$ or for $a\leq b<a+1$ with $a\geq 0$ and $a+b>0$. □

4. Stability of CKN inequality in the functional inequality setting

It is well-known that the minimizing problem (2.2) and the equation (1.7) are invariant under the dilation $u_\tau(x)=\tau^{\frac{N-2-2a}{2}}u(\tau x)$. Thus, the smooth manifold

$$\mathcal{Z}=\{cW_\tau(x)\mid c\in\mathbb{R}\backslash\{0\} \text{ and } \tau>0\}$$

is all extremal functions of the minimizing problem (2.2). Let

$$d^2(u)=\inf_{c\in\mathbb{R},\ \ \tau>0}\|u-cW_\tau\|^2_{D^{1,2}_a(\mathbb{R}^N)},$$

where $u\in D^{1,2}_a(\mathbb{R}^N)$. Then we have the following stability for the CKN inequality (1.1).

Proposition 4.1. *Let $e(u):=\|u\|^2_{D^{1,2}_a(\mathbb{R}^N)}-C^{-1}_{a,b,N}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}$. Then $e(u)\gtrsim d^2(u)$ for all $u\in D^{1,2}_a(\mathbb{R}^N)$ in the following two cases:*

(1) $b_{FS}(a)\leq b<a+1$ *with* $a<0$,
(2) $a\leq b<a+1$ *with* $a\geq 0$ *and* $a+b>0$.

Proof. The proof mainly follows the arguments in [2] for the stability of the Sobolev inequality. It is easy to see that $d^2(u)$ can be attained by some $c_0\neq 0$ and $\tau_0>0$. Indeed,

$$\|u-cW_\tau\|^2_{D^{1,2}_a(\mathbb{R}^N)}=\|u\|^2_{D^{1,2}_a(\mathbb{R}^N)}+c^2\|W_\tau\|^2_{D^{1,2}_a(\mathbb{R}^N)}-c\langle u,W_\tau\rangle_{D^{1,2}_a(\mathbb{R}^N)}.$$

Thus, by taking $(c,\tau)\in(\mathbb{R},\mathbb{R}^+)$ such that $c\langle u,W_\tau\rangle_{D^{1,2}_a(\mathbb{R}^N)}>0$ with $|c|>0$ sufficiently small, we have $d^2(u)<\|u\|^2_{D^{1,2}_a(\mathbb{R}^N)}$. By the invariance of the norm $\|\cdot\|_{D^{1,2}_a(\mathbb{R}^N)}$ under the dilation $u_\tau(x)=\tau^{a_c-a}u(\tau x)$,

$$\begin{aligned}\|u-cW_\tau\|^2_{D^{1,2}_a(\mathbb{R}^N)} &= \|u\|^2_{D^{1,2}_a(\mathbb{R}^N)}+c^2\|W_\tau\|^2_{D^{1,2}_a(\mathbb{R}^N)}-c\langle u,W_\tau\rangle_{D^{1,2}_a(\mathbb{R}^N)} \\ &\geq \|u\|^2_{D^{1,2}_a(\mathbb{R}^N)}+c^2\|W\|^2_{D^{1,2}_a(\mathbb{R}^N)}-|c|\|u\|_{D^{1,2}_a(\mathbb{R}^N)}\|W\|_{D^{1,2}_a(\mathbb{R}^N)}.\end{aligned}\tag{4.1}$$

Thus, the minimizing sequence of $d^2(u)$, say $\{(c_n,\tau_n)\}$, must satisfy $|c_n|\sim 1$. On the other hand,

$$\begin{aligned}|\int_{|\tau x|\leq\rho}|x|^{-2a}\nabla u\nabla W_\tau| &\leq \int_{|y|\leq\rho}|y|^{-2a}|\nabla u_{\frac{1}{\tau}}(y)\nabla W(y)| \\ &\leq \|u\|_{D^{1,2}_a(\mathbb{R}^N)}\left(\int_{|y|\leq\rho}|y|^{-2a}|\nabla W|^2\right)^{\frac12} \\ &= o_\rho(1)\end{aligned}$$

as $\rho\to 0$ which is uniformly for $\tau>0$ and

$$\begin{aligned}|\int_{|\tau x|\geq\rho}|x|^{-2a}\nabla u\nabla W_\tau| &\leq \|W\|_{D^{1,2}_a(\mathbb{R}^N)}\left(\int_{|x|\geq\frac{\rho}{\tau}}|x|^{-2a}|\nabla u|^2\right)^{\frac12} \\ &= o_\tau(1)\end{aligned}$$

as $\tau\to 0$ for any fixed $\rho>0$. By taking $\tau\to 0$ first and $\rho\to 0$ next, we have $|\int_{\mathbb{R}^N}|x|^{-2a}\nabla u\nabla W_\tau|\to 0$ as $\tau\to 0$. Note that $1+|\tau x|\sim 1$ for $|\tau x|\lesssim 1$, by (1.5),

$$|\int_{|\tau x|\leq R}|x|^{-2a}\nabla u\nabla W_\tau|\lesssim\tau^{a_c-a}\|u\|_{D^{1,2}_a(\mathbb{R}^N)}\left(\int_0^{\frac{R}{\tau}}r^{-2a+N-1}\right)^{\frac12}=o_\tau(1)$$

as $\tau\to+\infty$ for any fixed $R>0$ and

$$\begin{aligned}|\int_{|\tau x|\geq R}|x|^{-2a}\nabla u\nabla W_\tau| &\leq \int_{|y|\geq R}|y|^{-2a}|\nabla u_{\frac{1}{\tau}}(y)\nabla W(y)| \\ &\leq \|u\|_{D^{1,2}_a(\mathbb{R}^N)}\left(\int_{|y|\geq R}|y|^{-2a}|\nabla W|^2\right)^{\frac12} \\ &= o_R(1)\end{aligned}$$

as $R\to+\infty$ which is uniformly for $\tau>0$. Thus, by taking $\tau\to+\infty$ first and $R\to+\infty$ next, we also have $|\int_{\mathbb{R}^N}|x|^{-2a}\nabla u\nabla W_\tau|\to 0$ as $\tau\to+\infty$. It follows from (4.1) and $d^2(u)<\|u\|^2_{D^{1,2}_a(\mathbb{R}^N)}$ that the minimizing sequence $\{(c_n,\tau_n)\}$ must satisfy $|\tau_n|\sim 1$. Thus, $d^2(u)$ can be attained by some $c_0\neq 0$ and $\tau_0>0$, which implies

$$\langle u,c_0W_{\tau_0}\rangle_{D^{1,2}_a(\mathbb{R}^N)}=\|c_0W_{\tau_0}\|^2_{D^{1,2}_a(\mathbb{R}^N)}\quad\text{and}\quad\langle u,\partial_\tau W_\tau|_{\tau=\tau_0}\rangle_{D^{1,2}_a(\mathbb{R}^N)}=0.$$

Note that

$$\mathcal{T}_{W_{\tau_0}}\mathcal{Z}=\mathrm{span}\{\partial_\tau W_\tau|_{\tau=\tau_0}\},$$

and $\partial_\tau W_\tau|_{\tau=\tau_0}=V_{\tau_0}$, where $V(x)$ is given by (2.5). Thus, by the nondegeneracy of W_{τ_0} in $D^{1,2}_a(\mathbb{R}^N)$,

$$u=c_0W_{\tau_0}+\phi_{\tau_0}\tag{4.2}$$

in $D^{1,2}_a(\mathbb{R}^N)$, where

$$\langle \phi_{\tau_0}, W_{\tau_0}\rangle_{D^{1,2}_a(\mathbb{R}^N)} = \langle \phi_{\tau_0}, V_{\tau_0}\rangle_{D^{1,2}_a(\mathbb{R}^N)} = 0. \tag{4.3}$$

eq0058

It follows that $d^2(u) = \|\phi_{\tau_0}\|^2_{D^{1,2}_a(\mathbb{R}^N)}$. Since W_{τ_0} is the ground state, the Morse index of W_{τ_0} is equal to 1. It follows from the nondegeneracy of W_{τ_0} in $D^{1,2}_a(\mathbb{R}^N)$ that

$$\|\phi_{\tau_0}\|^2_{D^{1,2}_a(\mathbb{R}^N)} > p\int_{\mathbb{R}^N} |x|^{-b(p+1)} W^{p-1}_{\tau_0}\phi^2_{\tau_0}. \tag{4.4}$$

eq0011

Let us first consider the case that $d(u) > 0$ is sufficiently small, then by the elementary inequality

$$\left| |\alpha+\beta|^q - |\alpha|^q - q|\alpha|^{q-2}\alpha\beta - \frac{q(q-1)}{2}|\alpha|^{q-2}\beta^2\right| \lesssim |\beta|^q + |\alpha|^{q-3}|\beta|^3\chi_{q\geq 3}$$

for $q > 2$, where $\chi_{q\geq3} = 1$ for $q \geq 3$ and $\chi_{q\geq3} = 0$ for $2 < q < 3$, and the CKN inequality (1.1),

$$\begin{aligned}\|W_{\tau_0} + \frac{\phi_{\tau_0}}{c_0}\|^{p+1}_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)} &= \|W_{\tau_0}\|^{p+1}_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)} + o(d^2(u))\\ &+(p+1)\int_{\mathbb{R}^N}|x|^{-b(p+1)}W^p_{\tau_0}\frac{\phi_{\tau_0}}{c_0}\\ &+\frac{p(p+1)}{2}\int_{\mathbb{R}^N}|x|^{-b(p+1)}W^{p-1}_{\tau_0}(\frac{\phi_{\tau_0}}{c_0})^2,\end{aligned}$$

which, together with $\langle\phi_{\tau_0}, W_{\tau_0}\rangle_{D^{1,2}_a(\mathbb{R}^N)} = 0$ and the fact that W_{τ_0} is a solution of (1.7), implies that

$$\begin{aligned}\|W_{\tau_0} + \frac{\phi_{\tau_0}}{c_0}\|^{p+1}_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)} &= \|W_{\tau_0}\|^{p+1}_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)} + o(d^2(u))\\ &+\frac{p(p+1)}{2}\int_{\mathbb{R}^N}|x|^{-b(p+1)}W^{p-1}_{\tau_0}(\frac{\phi_{\tau_0}}{c_0})^2.\end{aligned}$$

On the other hand, by the fact that W_{τ_0} is a solution of (1.7) and it is also a minimizer of (2.2), we have

$$C^{-1}_{a,b,N} = \frac{\|W_{\tau_0}\|^2_{D^{1,2}_a(\mathbb{R}^N)}}{\|W_{\tau_0}\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}} = \|W_{\tau_0}\|^{p-1}_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}.$$

It follows from (4.2), (4.3) and (4.4) that for $d(u) > 0$ is sufficiently small,

$$\begin{aligned}e(u) &:= \|u\|^2_{D^{1,2}_a(\mathbb{R}^N)} - C^{-1}_{a,b,N}\|u\|^2_{L^{p+1}(|x|^{-b(p+1)},\mathbb{R}^N)}\\ &= c_0^2\left(\|\frac{\phi_{\tau_0}}{c_0}\|^2_{D^{1,2}_a(\mathbb{R}^N)} - p\int_{\mathbb{R}^N}|x|^{-b(p+1)}W^{p-1}_{\tau_0}(\frac{\phi_{\tau_0}}{c_0})^2 + o(d^2(u))\right)\\ &\gtrsim d^2(u).\end{aligned}$$

It remains to consider the case $d(u) \gtrsim 1$. Assume that $e(u) \gtrsim d^2(u)$ does not hold for all $u \in D^{1,2}_a(\mathbb{R}^N)$. Then there exists $\{u_n\} \subset D^{1,2}_a(\mathbb{R}^N)\backslash\{0\}$ such that $e(u_n) = o(d^2(u_n))$. Thus, $e(u_n) \to 0$ as $n \to \infty$ in this case. It follows that $\{u_n\}$ is a minimizing sequence of (2.2). By Proposition 3.1, we have $d(u_n) \to 0$ as $n \to \infty$, which is a contradiction. □

We close this section by the proof of Theorem 1.1.

Proof of Theorem 1.1: It follows immediately from Proposition 4.1. □

5. Stability of profile decompositions to nonnegative functions

5.1. **The one-bubble case.** In this section, we will consider the one-bubble case and prove the following result.

⟨propn0001⟩ **Proposition 5.1.** *Let $v \in H^1(\mathcal{C})$ be nonnegative such that*

$$\frac{1}{2}(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}} < \|v\|^2_{H^1(\mathcal{C})} < \frac{3}{2}(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}} \quad \textit{and} \quad \|f\|_{H^{-1}(\mathcal{C})} \leq \delta$$

for $\delta > 0$ sufficiently small, where $f = -\Delta_\theta v - \partial_t^2 v + (a_c - a)^2 v - v^p$. Then either for $b_{FS}(a) \leq b < a+1$ with $a < 0$ or for $a \leq b < a+1$ with $a \geq 0$ and $a+b > 0$, we have

$$d_0(v) \lesssim \|f\|_{H^{-1}(\mathcal{C})} \tag{5.1}$$ eqn20001

where $d_0^2(v) = \inf_{s\in\mathbb{R}} \|v - \Psi_s\|^2_{H^1(\mathcal{C})}$.

Proof. We shall mainly adapt the ideas in [7] to prove this proposition. As that in the proof of Proposition 4.1,

$$\widetilde{d}_0(v) = \inf_{c\in\mathbb{R},\quad s\in\mathbb{R}} \|v - c\Psi_s\|_{H^1(\mathcal{C})}$$

is attained by some $c_0 \neq 0$ and $s_0 \in \mathbb{R}$, which implies that $v = c_0\Psi_{s_0} + \psi_0$ and $\widetilde{d}_0(v) = \|\psi_0\|_{H^1(\mathcal{C})}$, where

$$\langle \Psi_{s_0}, \psi_0\rangle_{H^1(\mathcal{C})} = 0 \quad \text{and} \quad \langle \Psi'_{s_0}, \psi_0\rangle_{H^1(\mathcal{C})} = 0. \tag{5.2}$$ eqn20000

By Proposition 3.2, we also have that $\|\psi_0\|_{H^1(\mathcal{C})} \to 0$ and $c_0 = 1 + \alpha_0$ with $\alpha_0 \to 0$ as $\delta \to 0$. Now, by the orthogonal condition (5.2),

$$\|\psi_0\|^2_{H^1(\mathcal{C})} = \langle \psi_0, v\rangle_{H^1(\mathcal{C})} = \int_{\mathcal{C}} v^p\psi_0 + \int_{\mathcal{C}} f\psi_0.$$

By the Taylor expansion and some elementary inequalities,

$$\int_{\mathcal{C}} v^p\psi_0 = c_0^p\int_{\mathcal{C}} \Psi_{s_0}^p\psi_0 + pc_0^{p-1}\int_{\mathcal{C}} \Psi_{s_0}^{p-1}\psi_0^2 + O(\|\psi_0\|^{\sigma_p+1}_{H^1(\mathcal{C})}),$$

where $\sigma_p = 2$ for $p \geq 2$ and $\sigma_p = p$ for $1 < p < 2$. Since Ψ is a solution of (2.4),

$$\int_{\mathcal{C}} v^p\psi_0 = pc_0^{p-1}\int_{\mathcal{C}} \Psi_{s_0}^{p-1}\psi_0^2 + O(\|\psi_0\|^{\sigma_p+1}_{H^1(\mathcal{C})}).$$

Note that Ψ is the ground state of (2.4), thus, the Morse index of Ψ is equal to 1. It follows from the orthogonal condition (5.2) and the nondegeneracy of Ψ in $H^1(\mathcal{C})$ that

$$p\int_{\mathcal{C}} \Psi_{s_0}^{p-1}\psi_0^2 < \|\psi_0\|^2_{H^1(\mathcal{C})}.$$

Thus, by $\|\psi_0\|_{H^1(\mathcal{C})} \to 0$ and $c_0 = 1 + \alpha_0$ with $\alpha_0 \to 0$ as $\delta \to 0$,

$$\|\psi_0\|_{H^1(\mathcal{C})} \lesssim \|f\|_{H^{-1}(\mathcal{C})} \tag{5.3}$$ eqn5555

for $\delta > 0$ sufficiently small. On the other hand, we have

$$\|v\|^2_{H^1(\mathcal{C})} = \|v\|^{p+1}_{L^{p+1}(\mathcal{C})} + \int_{\mathcal{C}} fv.$$

Since $c_0 = 1 + \alpha_0$ with $\alpha_0 \to 0$ as $\delta \to 0$, by the orthogonal condition (5.2),

$$\|v\|^2_{H^1(\mathcal{C})} = (1 + 2\alpha_0 + O(\alpha_0^2))\|\Psi\|^2_{H^1(\mathcal{C})} + \|\psi_0\|^2_{H^1(\mathcal{C})}.$$

By the Taylor expansion, the orthogonal condition (5.2) and some elementary inequalities,

$$\|v\|^{p+1}_{L^{p+1}(\mathcal{C})} = (1 + (p+1)\alpha_0 + O(\alpha_0^2))\|\Psi\|^{p+1}_{L^{p+1}(\mathcal{C})} + O(\|\psi_0\|^2_{H^1(\mathcal{C})}),$$

which, together with (5.3) and the fact that Ψ is a solution of (2.4), implies that

$$(p-1)|\alpha_0| \lesssim \|f\|^2_{H^{-1}(\mathcal{C})} + \|f\|_{H^{-1}(\mathcal{C})} \sim \|f\|_{H^{-1}(\mathcal{C})}$$

for $\delta > 0$ sufficiently small. Now, (5.1) then follows from rewriting $v = \Psi_{s_0} + \alpha_0\Psi_{s_0} + \psi_0$. $\square$

5.2. **The multi-bubble case.** Let us first compute the interaction of two bubbles, which plays an important role in the stability for the multi-bubble case.

⟨lem0001⟩ **Lemma 5.1.** *Let W_{τ_1} and W_{τ_2} be two bubbles such that $\tau_1 \neq \tau_2$. Then*

$$\langle W_{\tau_1}, W_{\tau_2}\rangle_{D^{1,2}_a(\mathbb{R}^N)} \sim \left(\frac{\min\{\tau_1,\tau_2\}}{\max\{\tau_1,\tau_2\}}\right)^{a_c - a}.$$

Moreover, by the transformation (2.1), we also have

$$\langle \Psi_{s_1}, \Psi_{s_2}\rangle_{H^1(\mathcal{C})} \sim e^{-(a_c - a)|s_1 - s_2|}, \tag{5.4}$$ eq0025

where $\Psi(t)$ is given by (2.8), $\Psi_s(t) = \Psi(t-s)$ and $s_i = \ln \tau_i$.

Proof. Without loss of generality, we may assume that $\tau_1 = 1$ and $\tau_2 := \tau < 1$ by the invariance of $\langle \cdot, \cdot \rangle_{D^{1,2}_a(\mathbb{R}^N)}$ under the dilation $u_\tau = \tau^{a_c - a}u(\tau x)$. Since W is a solution of (1.7),

$$\begin{aligned}
\langle W, W_\tau\rangle_{D^{1,2}_a(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |x|^{-b(p+1)} W^p W_\tau \\
&= \int_{|x|\leq 1} |x|^{-b(p+1)} W^p W_\tau + \int_{1<|x|\leq \frac{1}{\tau}} |x|^{-b(p+1)} W^p W_\tau \\
&\quad + \int_{\frac{1}{\tau}<|x|} |x|^{-b(p+1)} W^p W_\tau.
\end{aligned}$$

Since $\tau < 1$, by (1.5), $W(x) \sim 1$ and $W_\tau(x) \sim \tau^{a_c - a}$ in the region $\{x \in \mathbb{R}^N \mid |x| \leq 1\}$. It follows that

$$\int_{|x|\leq 1} |x|^{-b(p+1)} W^p W_\tau \sim \tau^{a_c - a}\int_0^1 r^{N-1-b(p+1)} \sim \tau^{a_c - a}.$$

Here, we have used the fact that $N - b(p+1) = (p+1)(a_c - a) > 0$. In the region $\{x \in \mathbb{R}^N \mid 1 < |x| \leq \frac{1}{\tau}\}$, $W_\tau(x) \sim \tau^{a_c - a}$ and $W(x) \sim |x|^{-2(a_c - a)}$ by (1.5). Thus,

$$\int_{1<|x|\leq \frac{1}{\tau}} |x|^{-b(p+1)} W^p W_\tau \sim \tau^{a_c - a}\int_1^{\frac{1}{\tau}} r^{N-1-b(p+1)-2p(a_c - a)} \sim \tau^{a_c - a},$$

where we have used the fact that $N - b(p+1) - 2p(a_c - a) = (1-p)(a_c - a) < 0$ and $\tau < 1$. In the region $\{x \in \mathbb{R}^N \mid \frac{1}{\tau} < |x|\}$, $W_\tau(x) \sim \tau^{-(a_c - a)}|x|^{-2(a_c - a)}$ and $W(x) \sim |x|^{-2(a_c - a)}$ by (1.5). Therefore,

$$\int_{\frac{1}{\tau}<|x|} |x|^{-b(p+1)} W^p W_\tau \sim \tau^{-(a_c - a)}\int_{\frac{1}{\tau}}^{+\infty} r^{N-1-b(p+1)-2(p+1)(a_c - a)} \sim \tau^{p(a_c - a)},$$

where we have used the fact that $N-b(p+1)-2(p+1)(a_c-a) = -(p+1)(a_c-a) < 0$. Thus, by $\tau < 1$ and $p > 1$,

$$\langle W, W_\tau\rangle_{D_a^{1,2}(\mathbb{R}^N)} \sim \tau^{a_c-a}. \tag{5.5}$$ eqn994

By (2.1), we have $\langle \Psi, \Psi_s\rangle_{H^1(\mathcal{C})} = \langle W, W_\tau\rangle_{D_a^{1,2}(\mathbb{R}^N)}$, where $\Psi(t)$ is given by (2.8), $\Psi_s(t) = \Psi(t-s)$ and $s = \ln\tau$. Then (5.4) follows immediately from (5.5). □

Let $v \in H^1(\mathcal{C})$ be nonnegative such that

$$(\nu - \frac{1}{2})(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}} < \|v\|_{H^1(\mathcal{C})}^2 < (\nu + \frac{1}{2})(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}}$$

for some positive integer $\nu \geq 2$ and denote

$$f := -\Delta_\theta v - \partial_t^2 v + (a_c - a)^2 v - v^p. \tag{5.6}$$ eq0060

Then $f \in H^{-1}(\mathcal{C})$. As that in [7, 11, 15], we consider the following minimizing problem:

$$d_*^2(v) = \min_{s_j\in\mathbb{R}} \|v - \sum_{j=1}^{\nu} \Psi_{s_j}\|_{H^1(\mathcal{C})}^2.$$

By similar arguments as that used in the proof of Proposition 4.1, we can show that $d_*^2(v)$ is attained at some $\{s_j\} \in \mathbb{R}^\nu$ and thus, we can write $v = \sum_{j=1}^{\nu} \Psi_{s_j} + \rho$, where ρ satisfies the following orthogonal conditions:

$$\langle \Psi_{s_j}', \rho\rangle_{H^1(\mathcal{C})} = 0 \quad \text{for all } j = 1, 2, \cdots, \nu. \tag{5.7}$$ eq0013

Clearly, $d_*^2(v) = \|\rho\|_{H^1(\mathcal{C})}^2$. Moreover, by Proposition 3.2, we know that $d_*(v) \to 0$ as $\|f\|_{H^{-1}(\mathcal{C})} \to 0$ either for $b_{FS}(a) \leq b < a+1$ with $a < 0$ or for $a \leq b < a+1$ with $a \geq 0$ and $a+b > 0$. Thus, if $\|f\|_{H^{-1}(\mathcal{C})} \leq \delta$ for $\delta > 0$ sufficiently small, we have $\|\rho\|_{H^1(\mathcal{C})} \leq \delta'$ either for $b_{FS}(a) \leq b < a+1$ with $a < 0$ or for $a \leq b < a+1$ with $a \geq 0$ and $a+b > 0$, where $\delta' \to 0$ as $\delta \to 0$.

Since Ψ_{s_j} are solutions of (2.4), by (5.7), we can rewrite (5.6) as follows:

$$\begin{cases} -\Delta_\theta\rho - \partial_t^2\rho + (a_c-a)^2\rho = (\sum_{j=1}^{\nu} \Psi_{s_j} + \rho)^p - \sum_{j=1}^{\nu} \Psi_{s_j}^p + f, \quad \text{in } \mathcal{C}, \\ \langle \Psi_{s_j}', \rho\rangle_{H^1(\mathcal{C})} = 0 \quad \text{for all } j = 1, 2, \cdots, \nu. \end{cases} \tag{5.8}$$ eq0014

In what follows, for the sake of simplicity, we denote

$$R = \min_{i\neq j} |s_i - s_j| \quad \text{and} \quad Q = e^{-(a_c-a)\min_{i\neq j}|s_i-s_j|}$$

as that in [11]. Moreover, we also assume that $s_1 < s_2 < \cdots < s_\nu$ without loss of generality. For the sake of simplicity, we also denote $s_0 = -\infty$ and $s_{\nu+1} = +\infty$.

⟨lemn0001⟩ **Lemma 5.2.** *Let $b_{FS}(a) \leq b < a+1$ for $a < 0$ and $a \leq b < a+1$ for $a \geq 0$ and $a+b > 0$. Then*

$$\|f\|_{H^{-1}(\mathcal{C})} \gtrsim Q + O(Q^{\frac{1}{2}}\|\rho\|_{H^1(\mathcal{C})} + \|\rho\|_{H^1(\mathcal{C})}^{p+1}) \tag{5.9}$$ eq1146

for $\|f\|_{H^{-1}(\mathcal{C})} \leq \delta$ with $\delta > 0$ sufficiently small.

Proof. Suppose that $R = s_{j_0+1} - s_{j_0}$ for some $j_0 \in \{1, 2, \cdots, \nu - 1\}$. Multiplying (5.8) with $-\Psi'_{s_{j_0}}$ and integrating by parts, we have

$$\|f\|_{H^{-1}(\mathcal{C})} \gtrsim \langle (\sum_{j=1}^{\nu} \Psi_{s_j} + \rho)^p - \sum_{j=1}^{\nu} \Psi_{s_j}^p, -\Psi'_{s_{j_0}} \rangle_{L^2(\mathcal{C})}.$$

In the region $\{3|\rho| < \sum_{j=1}^{\nu} \Psi_{s_j}\}$, by the Taylor expansion,

$$\begin{aligned}(\sum_{j=1}^{\nu} \Psi_{s_j} + \rho)^p - (\sum_{j=1}^{\nu} \Psi_{s_j})^p - p(\sum_{j=1}^{\nu} \Psi_{s_j})^{p-1}\rho &= p(p-1)(\sum_{j=1}^{\nu} \Psi_{s_j} + \xi\rho)^{p-2}\rho^2 \\ &\sim (\sum_{j=1}^{\nu} \Psi_{s_j})^{p-2}\rho^2 > 0, \end{aligned} \quad (5.10) \ \boxed{\texttt{eqn19985}}$$

where $\xi \in (0, 1)$. In the region $\{\sum_{j=1}^{\nu} \Psi_{s_j} \leq 3|\rho|\}$, since Ψ_{s_j} are all positive, we also have $\Psi_{s_{j_0}} \leq 3|\rho|$ which, together with $-\Psi'_{s_{j_0}} \sim \Psi_{s_{j_0}}$, implies that

$$\left((\sum_{j=1}^{\nu} \Psi_{s_j} + \rho)^p - (\sum_{j=1}^{\nu} \Psi_{s_j})^p - p(\sum_{j=1}^{\nu} \Psi_{s_j})^{p-1}\rho \right) \Psi'_{s_{j_0}} \lesssim \rho^{p+1}. \quad (5.11) \ \boxed{\texttt{eqn19984}}$$

By Proposition 3.2 and Lemma 5.1, $R \to +\infty$ as $\delta \to 0$. Thus,

$$e^{-(a_c - a)|t - s_i|} \gtrsim e^{-(a_c - a)|t - s_{i-1}|} + e^{-(a_c - a)|t - s_{i+1}|} \sim \sum_{j \neq i} e^{-(a_c - a)|t - s_j|}$$

in $(s_i - \frac{R}{2} + O(1), s_i + \frac{R}{2} + O(1))$ for all $i = 1, 2, \cdots, \nu$, which, together with (2.8), implies that $\{\Psi_{s_i} \gtrsim \sum_{j \neq i} \Psi_{s_j}\}$ in the region $(s_i - \frac{R}{2} + O(1), s_i + \frac{R}{2} + O(1))$ for all $i = 1, 2, \cdots, \nu$. It follows from the Taylor expansion that

$$(\sum_{j=1}^{\nu} \Psi_{s_j})^p - \sum_{j=1}^{\nu} \Psi_{s_j}^p = p(\Psi_{s_i} + \xi_i \sum_{j \neq i} \Psi_{s_j})^{p-1} \sum_{j \neq i} \Psi_{s_j} \sim \Psi_{s_i}^{p-1} \sum_{j \neq i} \Psi_{s_j} \quad (5.12) \ \boxed{\texttt{eqn19999}}$$

in the region $(s_i - \frac{R}{2} + O(1), s_i + \frac{R}{2} + O(1))$ for all $i = 1, 2, \cdots, \nu$, where $\xi_i \in (0, 1)$. In the region $\mathbb{R} \backslash (\cup_{i=1}^{\nu} (s_i - \frac{R}{2} + O(1), s_i + \frac{R}{2} + O(1)))$, by (2.8),

$$|(\sum_{j=1}^{\nu} \Psi_{s_j})^p - \sum_{j=1}^{\nu} \Psi_{s_j}^p| \lesssim \sum_{j=1}^{\nu} \Psi_{s_j}^p \sim \sum_{j=1}^{\nu} e^{-p(a_c - a)|t - s_j|}. \quad (5.13) \ \boxed{\texttt{eqn19998}}$$

Thus, by $-\Psi'_{s_{j_0}}\sim\Psi_{s_{j_0}}$ once more, (5.13), $p>1$ and the orthogonal conditions in (5.8),

$$
\begin{aligned}
\|f\|_{H^{-1}(\mathcal{C})} &\gtrsim -\int_{\mathcal{C}}((\sum_{j=1}^{\nu}\Psi_{s_j})^p-\sum_{j=1}^{\nu}\Psi_{s_j}^p+p((\sum_{j=1}^{\nu}\Psi_{s_j})^{p-1}-\Psi_{s_{j_0}}^{p-1})\rho)\Psi'_{s_{j_0}})\\
&\quad+O(\|\rho\|_{H^1(\mathcal{C})}^{p+1})\\
&\gtrsim \sum_{i=1}^{\nu}\int_{(s_i-\frac{R}{2}+O(1),s_i+\frac{R}{2}+O(1))}\Psi_{s_i}^{p-1}\Psi_{s_{j_0}}\sum_{j\neq i}^{\nu}\Psi_{s_j}+o(Q)\\
&\quad-\int_{(s_{j_0}-\frac{R}{2}+O(1),s_{j_0}+\frac{R}{2}+O(1))}\Psi_{s_{j_0}}^{p-1}\sum_{j\neq j_0}\Psi_{s_j}|\rho|\\
&\quad-\int_{(s_{j_0}-\frac{R}{2}+O(1),s_{j_0}+\frac{R}{2}+O(1))^c}\Psi_{s_{j_0}}\sum_{j=1}^{\nu}\Psi_{s_j}^{p-1}|\rho|+O(\|\rho\|_{H^1(\mathcal{C})}^{p+1})\\
&\sim \sum_{i=1}^{\nu}\int_{(s_i-\frac{R}{2}+O(1),s_i+\frac{R}{2}+O(1))}\Psi_{s_i}^{p-1}\Psi_{s_{j_0}}\sum_{j\neq i}^{\nu}\Psi_{s_j}+o(Q)\\
&\quad+O(Q^{\frac12}\|\rho\|_{H^1(\mathcal{C})}+\|\rho\|_{H^1(\mathcal{C})}^{p+1}).
\end{aligned}
$$

By (2.8), $p>1$ and [25, Lemma 4.1],

$$
\begin{aligned}
&\sum_{i=1}^{\nu}\int_{(s_i-\frac{R}{2}+O(1),s_i+\frac{R}{2}+O(1))}\Psi_{s_i}^{p-1}\Psi_{s_{j_0}}\sum_{j\neq i}^{\nu}\Psi_{s_j}\\
\sim\ &\int_{(s_{j_0}-\frac{R}{2}+O(1),s_{j_0}+\frac{R}{2}+O(1))}\Psi_{s_{j_0}}^{p}(\Psi_{s_{j_0+1}}+\Psi_{s_{j_0-1}})\\
&+\int_{(s_{j_0+1}-\frac{R}{2}+O(1),s_{j_0+1}+\frac{R}{2}+O(1))}\Psi_{s_{j_0}}^{2}\Psi_{s_{j_0+1}}^{p-1}\\
&+\int_{(s_{j_0-1}-\frac{R}{2}+O(1),s_{j_0-1}+\frac{R}{2}+O(1))}\Psi_{s_{j_0}}^{2}\Psi_{s_{j_0-1}}^{p-1}+o(Q)\\
\sim\ &\int_0^{\frac{R}{2}}e^{-(a_c-a)pr}e^{-(a_c-a)(R-r)}+\int_0^{\frac{R}{2}}e^{-2(a_c-a)(R-r)}e^{-(a_c-a)(p-1)r}+o(Q)\\
\sim\ &Q.
\end{aligned}
$$

It follows that (5.9) holds for $\delta>0$ sufficiently small. □

As that in [11], we want to drive the precise behavior of first approximation of ρ by considering the following equation:

$$
\begin{cases}
-\Delta_\theta\phi-\partial_t^2\phi+(a_c-a)^2\phi\\
=|\sum_{j=1}^{\nu}\Psi_{s_j}+\phi|^{p-1}(\sum_{j=1}^{\nu}\Psi_{s_j}+\phi)\\
-\sum_{j=1}^{\nu}\Psi_{s_j}^p+\sum_{j=1}^{\nu}c_j\Psi_{s_j}^{p-1}\Psi'_{s_j},\quad\text{in }\mathcal{C},\\
\langle\Psi'_{s_j},\phi\rangle_{H^1(\mathcal{C})}=0\quad\text{for all }j=1,2,\cdots,\nu,
\end{cases}
\tag{5.14}
$$

eq0015

where c_j and ϕ are all unknowns. By (2.8) and some elementary inequalities, we can rewrite

$$
\begin{aligned}
|\sum_{j=1}^{\nu}\Psi_{s_j}+\phi|^{p-1}(\sum_{j=1}^{\nu}\Psi_{s_j}+\phi)-\sum_{j=1}^{\nu}\Psi_{s_j}^{p} \quad &= \quad p(\sum_{j=1}^{\nu}\Psi_{s_j})^{p-1}\phi+(\sum_{j=1}^{\nu}\Psi_{s_j})^{p}\\
&\quad -\sum_{j=1}^{\nu}\Psi_{s_j}^{p}+O(|\phi|^{\sigma_p}),
\end{aligned}
$$

where $\sigma_p=2$ for $p\geq 2$ and $\sigma_p=p$ for $1<p<2$. Thus, (5.14) can be rewritten as follows:

$$
\begin{cases}
\mathcal{L}(\phi)=E+N(\phi)+\sum_{j=1}^{\nu}c_j\Psi_{s_j}^{p-1}\Psi_{s_j}', \quad \text{in } \mathcal{C},\\
\langle\Psi_{s_j}',\phi\rangle_{H^1(\mathcal{C})}=0 \quad \text{for all } j=1,2,\cdots,\nu,
\end{cases}
\tag{5.15}
$$
eq0020

where the linear operator $\mathcal{L}(\phi)$ is given by

$$
\mathcal{L}(\phi):=-\Delta_\theta\phi-\partial_t^2\phi+(a_c-a)^2\phi-p(\sum_{j=1}^{\nu}\Psi_{s_j})^{p-1}\phi,
\tag{5.16}
$$
eq0062

$E=(\sum_{j=1}^{\nu}\Psi_{s_j})^p-\sum_{j=1}^{\nu}\Psi_{s_j}^p$ is the error and $N(\phi)=O(|\phi|^{\sigma_p})$ is the nonlinear part.

⟨lemn0002⟩ **Lemma 5.3.** *For $\delta>0$ sufficiently small, we have*

$$
\|E\|_{\natural}:=\sum_{i=1}^{\nu}\sup_{t\in(\frac{s_i+s_{i-1}}{2},\frac{s_{i+1}+s_i}{2})}\frac{|E|}{Qe^{-(a_c-a)(p-2)|t-s_i|}}\lesssim 1
\tag{5.17}
$$
eqn19997

for $1<p<3$ and

$$
\|E\|_{\sharp}:=\sum_{i=1}^{\nu}\sup_{t\in(\frac{s_i+s_{i-1}}{2},\frac{s_{i+1}+s_i}{2})}\frac{|E|}{Qe^{-(1-\varsigma)(a_c-a)|t-s_i|}}\lesssim 1
\tag{5.18}
$$
eqn19996

for $p\geq 3$ with $\varsigma>0$ sufficiently small.

Proof. By (2.8) and similar arguments as that used for (5.12),

$$
\begin{aligned}
E \quad &\sim \quad \Psi_{s_i}^{p-1}\Psi_{s_{i+1}}\\
&\sim \quad e^{-(p-1)(a_c-a)|t-s_i|}e^{-(a_c-a)|t-s_{i+1}|}\\
&\sim \quad e^{-(a_c-a)|s_i-s_{i+1}|}e^{-(a_c-a)(p-2)|t-s_i|}
\end{aligned}
\tag{5.19}
$$
eqn19990

in the region $(s_i,\frac{s_{i+1}+s_i}{2})$ for all $i=1,2,\cdots,\nu-1$ and

$$
\begin{aligned}
E \quad &\sim \quad \Psi_{s_i}^{p-1}\Psi_{s_{i-1}}\\
&\sim \quad e^{-(p-1)(a_c-a)|t-s_i|}e^{-(a_c-a)|t-s_{i-1}|}\\
&\sim \quad e^{-(a_c-a)|s_i-s_{i-1}|}e^{-(a_c-a)(p-2)|t-s_i|}
\end{aligned}
\tag{5.20}
$$
eqn19989

in the region $(\frac{s_{i-1}+s_i}{2},s_i)$ for all $i=2,3,\cdots,\nu$. In the region $(-\infty,s_1)$, since $s_1<s_2<\cdots<s_{\nu-1}<s_\nu$,

$$
\begin{aligned}
E \quad &\sim \quad \Psi_{s_1}^{p-1}\Psi_{s_2}\\
&\sim \quad e^{-(p-1)(a_c-a)|t-s_1|}e^{-(a_c-a)|t-s_2|}\\
&\sim \quad e^{-(a_c-a)|s_1-s_2|}e^{-p(a_c-a)|t-s_1|}.
\end{aligned}
\tag{5.21}
$$
eqn19970

In the region $(s_\nu, +\infty)$, since $s_1 < s_2 < \cdots < s_{\nu-1} < s_\nu$,

$$\begin{aligned} E &\sim \Psi_{s_\nu}^{p-1}\Psi_{s_{\nu-1}} \\ &\sim e^{-(p-1)(a_c-a)|t-s_\nu|}e^{-(a_c-a)|t-s_{\nu-1}|} \\ &\sim e^{-(a_c-a)|s_\nu-s_{\nu-1}|}e^{-p(a_c-a)|t-s_i|}. \end{aligned} \tag{5.22}$$

eqn19969

(5.17) and (5.18) then follow immediately from (5.19), (5.20) and (5.21), (5.22). □

To solve (5.14), we shall use the fix point argument, which leads us to establish a good linear theory by considering the following linear equation:

$$\begin{cases} \mathcal{L}(\phi) = g, \quad \text{in } \mathcal{C}, \\ \langle \Psi_{s_j}', \phi\rangle_{H^1(\mathcal{C})} = 0 \quad \text{for all } j = 1, 2, \cdots, \nu, \end{cases}$$

where g satisfies $\langle \Psi_{s_j}', g\rangle_{L^2(\mathcal{C})} = 0$ for all $j = 1, 2, \cdots, \nu$ and $\mathcal{L}(\phi)$ is given by (5.16). Based on Lemma 5.3, we shall introduce the following spaces:

$$X = \{\phi \in H^1(\mathcal{C}) \mid \|\phi\|_\natural < +\infty\} \quad , Y = \{\phi \in L^2(\mathcal{C}) \mid \|\phi\|_\natural < +\infty\},$$

and

$$\widehat{X} = \{\phi \in H^1(\mathcal{C}) \mid \|\phi\|_\sharp < +\infty\} \quad , \widehat{Y} = \{\phi \in L^2(\mathcal{C}) \mid \|\phi\|_\sharp < +\infty\}.$$

Clearly, X, Y and $\widehat{X}$, $\widehat{Y}$ are all Banach spaces. Let

$$\begin{aligned} X^\perp &= \{\phi \in X \mid \langle \Psi_{s_j}', \phi\rangle_{H^1(\mathcal{C})} = 0 \quad \text{for all } j = 1, 2, \cdots, \nu\}, \\ Y^\perp &= \{\phi \in Y \mid \langle \Psi_{s_j}', \phi\rangle_{L^2(\mathcal{C})} = 0 \quad \text{for all } j = 1, 2, \cdots, \nu\} \end{aligned}$$

and

$$\begin{aligned} \widehat{X}^\perp &= \{\phi \in \widehat{X} \mid \langle \Psi_{s_j}', \phi\rangle_{H^1(\mathcal{C})} = 0 \quad \text{for all } j = 1, 2, \cdots, \nu\}, \\ \widehat{Y}^\perp &= \{\phi \in \widehat{Y} \mid \langle \Psi_{s_j}', \phi\rangle_{L^2(\mathcal{C})} = 0 \quad \text{for all } j = 1, 2, \cdots, \nu\}, \end{aligned}$$

then we have the following.

⟨lem0002⟩ **Lemma 5.4.** *Let $b_{FS}(a) \le b < a+1$ for $a < 0$ and $a \le b < a+1$ for $a \ge 0$ and $a + b > 0$.*

(1) *If $p \ge 3$, then for $\delta > 0$ sufficiently small, there exists a unique $\phi \in \widehat{X}^\perp$ such that $\mathcal{L}(\phi) = g$ and $\|\phi\|_\sharp \lesssim \|g\|_\sharp$ for every $g \in \widehat{Y}^\perp$.*

(2) *If $1 < p < 3$, then for $\delta > 0$ sufficiently small, there exists a unique $\phi \in X^\perp$ such that $\mathcal{L}(\phi) = g$ and $\|\phi\|_\natural \lesssim \|g\|_\natural$ for every $g \in Y^\perp$.*

Here $\mathcal{L}(\phi)$ is given by (5.16).

Proof. Since the proof is rather standard nowadays (cf. [8–10, 26, 27]), we only sketch it here. We start by proving the a-priori estimates $\|\phi\|_\sharp \lesssim \|g\|_\sharp$ for $p \ge 3$ and $\|\phi\|_\natural \lesssim \|g\|_\natural$ for $1 < p < 3$. Assuming the contrary, that is, there exist $\{g_n\}$ and $\{\delta_n\}$ such that $\|g_n\|_\sharp \to 0$ and $\delta_n \to 0$ as $n \to \infty$ and $\|\phi_n\|_\sharp = 1$ for $p \ge 3$ while, $\|g_n\|_\natural \to 0$ and $\delta_n \to 0$ as $n \to \infty$ and $\|\phi_n\|_\natural = 1$ for $1 < p < 3$. Since $\delta_n \to 0$ as $n \to \infty$, by proposition 3.2,

$$R_n = \min_{i \neq j} |s_{i,n} - s_{j,n}| \to +\infty \quad \text{as } n \to \infty.$$

By the definition of the norms $\|\cdot\|_\natural$ and $\|\cdot\|_\sharp$ given by (5.17) and (5.18),

$$|\mathcal{L}(\phi_n)| \lesssim \begin{cases} \|g_n\|_\natural \sum_{i=1}^{\nu} Q_n\varphi_{i,n}(t)\chi_{i,n}(t), & 1<p<3,\\ \|g_n\|_\sharp \sum_{i=1}^{\nu} Q_n\varphi_{i,n}(t)\chi_{i,n}(t), & p\geq 3,\end{cases}$$

where

$$\varphi_{i,n}(t)=\begin{cases} e^{-(1-\varsigma)(a_c-a)|t-s_{i,n}|}, & p\geq 3,\\ e^{-(p-2)(a_c-a)|t-s_{i,n}|}, & 1<p<3,\end{cases} \tag{5.23}$$ eqn19993

and $\chi_{i,n}$ is a cut-off function such that

$$\chi_{i,n}(t)=\begin{cases} 1, & t\in(\frac{s_{i,n}+s_{i-1,n}}{2},\frac{s_{i+1,n}+s_{i,n}}{2}),\\ 0, & t\in(\frac{s_{i,n}+s_{i-1,n}}{2},\frac{s_{i+1,n}+s_{i,n}}{2})^c.\end{cases} \tag{5.24}$$ eqn19992

By (2.8), it is easy to see that

$$\mathcal{L}(\varphi_{i,n}) \gtrsim \begin{cases} e^{-(1-\varsigma)(a_c-a)|t-s_{i,n}|}, & p\geq 3,\\ e^{-(p-2)(a_c-a)|t-s_{i,n}|}, & 1<p<3\end{cases}$$

in $(\frac{s_{i,n}+s_{i-1,n}}{2},\frac{s_{i+1,n}+s_{i,n}}{2})\backslash(s_i-T,s_i+T)$ for a sufficiently large $T>0$. Thus, by the maximum principle,

$$|\phi_n| \lesssim \begin{cases} \|g_n\|_\natural Q_n\varphi_{i,n}(t), & 1<p<3,\\ \|g_n\|_\sharp Q_n\varphi_{i,n}(t), & p\geq 3\end{cases} \tag{5.25}$$ eqn19995

in $(\frac{s_{i,n}+s_{i-1,n}}{2},\frac{s_{i+1,n}+s_{i,n}}{2})\backslash(s_{i,n}-T,s_{i,n}+T)$ for all $i=1,2,\cdots,\nu$. On the other hand, by $\|\phi_n\|_\sharp=1$ and $\|g_n\|_\sharp=o_n(1)$ for $p\geq 3$ while $\|\phi_n\|_\natural=1$ and $\|g_n\|_\natural=o_n(1)$ for $1<p<3$, it is standard to use the Moser iteration and the Sobolev embedding theorem to show that $Q_n^{-1}\phi_n(\cdot+s_{i,n})\to\widehat{\phi}$ uniformly in every compact set of $\mathcal{C}$ as $n\to\infty$ for all $i=1,2,\cdots,\nu$, where $\widehat{\phi}$ is a solution of (2.7). We recall that by the nondegeneracy of Ψ in $H^1(\mathcal{C})$, Ψ' is the only nonzero solution of (2.7). Thus, we must have that $\widehat{\phi}=C\Psi'$, which together with the orthogonal condition in $X^\perp$ for $1<p<3$ and the orthogonal condition in $\widehat{X}^\perp$ for $p\geq 3$, implies that $\widehat{\phi}=0$. Since $\varphi_{i,n}(t)\sim 1$ in $[s_{i,n}-T,s_{i,n}+T]$ for fixed $T>0$, $\frac{|\phi_n|}{Q_n\varphi_{i,n}(t)}=o_n(1)$ in $[s_{i,n}-T,s_{i,n}+T]$ for fixed $T>0$. Thus, by (5.25), $\|\phi_n\|_\sharp=o_n(1)$ for $p\geq 3$ and $\|\phi_n\|_\natural=o_n(1)$ for $1<p<3$. It contradicts $\|\phi_n\|_\sharp=1$ for $p\geq 3$ and $\|\phi_n\|_\natural=1$ for $1<p<3$. The a-priori estimates $\|\phi\|_\sharp\lesssim\|g\|_\sharp$ for $p\geq 3$ and $\|\phi\|_\natural\lesssim\|g\|_\natural$ for $1<p<3$ imply that $\mathcal{L}:X^\perp\to Y^\perp$ for $1<p<3$ and $\mathcal{L}:\widehat{X}^\perp\to\widehat{Y}^\perp$ for $p\geq 3$ are injective for $\delta>0$ sufficiently small. Since $(\sum_{j=1}^{\nu}\Psi_{s_j})^{p-1}\to 0$ as $|t|\to+\infty$ by (2.8), it is standard to use the the Fredholm alternative to show that for $\delta>0$ sufficiently small, $\mathcal{L}(\phi)=g$ is unique solvable in $X^\perp$ for every $g\in Y^\perp$ in the case of $1<p<3$ and $\mathcal{L}(\phi)=g$ is unique solvable in $\widehat{X}^\perp$ for every $g\in\widehat{Y}^\perp$ in the case of $p\geq 3$. □

Let us go back to (5.15), then we have the following.

⟨lem0003⟩ **Lemma 5.5.** *Let $b_{FS}(a) \leq b < a+1$ for $a < 0$ and $a \leq b < a+1$ for $a \geq 0$ and $a+b > 0$. Then (5.15) has a unique solution $(\psi, c_1, c_2, \cdots, c_\nu)$ for $\delta > 0$ sufficiently small. Moreover,*

$$\|\phi\|_{H^1(\mathcal{C})} \lesssim \begin{cases} Q, & p > 2, \\ Q|\log(Q)|^{\frac{1}{2}}, & p = 2, \\ Q^{\frac{p}{2}}, & 1 < p < 2. \end{cases} \tag{5.26}$$
eq0027

and $\sum_{j=1}^{\nu} |c_l| \lesssim Q$.

Proof. Since $R \to +\infty$ as $\delta \to 0$ and $p > 1$, by [19, Lemma 6] and (2.8),

$$\langle \Psi_{s_j}^{p-1}\Psi'_{s_j}, \Psi'_{s_i}\rangle_{L^2(\mathcal{C})} \sim Q \tag{5.27}$$
eq0033

for $i \neq j$ for $\delta > 0$ sufficiently small. Thus, $\{c_j\}$ in (5.15) can be chosen to be the unique solution of the following equation:

$$(\langle \Psi_{s_j}^{p-1}\Psi'_{s_j}, \Psi'_{s_i}\rangle_{L^2(\mathcal{C})})_{i,j=1,2,\cdots,\nu} \bullet (c_j)_{j=1,2,\cdots,\nu} = -(\langle E + N(\phi), \Psi'_{s_i}\rangle_{L^2(\mathcal{C})})_{i=1,2,\cdots,\nu}.$$

By Lemmas 5.3 and 5.4, and adapting the fix point arguments in a standard way (cf. [8–10, 26, 27]), (5.15) is unique solvable in the set $\widehat{B} = \{\phi \in \widehat{X}^\perp \mid \|\phi\|_\sharp \leq C\}$ in the case of $p \geq 3$ and in the set $B = \{\phi \in X^\perp \mid \|\phi\|_\natural \leq C\}$ in the case of $1 < p < 3$ for a sufficiently large $C > 0$. Note that $N(\phi) = O(\phi^2)$ for $p \geq 2$, by $-\Psi' \sim \Psi$ and [19, Lemma 6],

$$|\langle N(\phi), \Psi'_{s_j}\rangle| \lesssim \int_{\mathcal{C}} \sum_{i=1}^{\nu} (Q\varphi_i(t)\chi_i(t))^2 \Psi_{s_j} \lesssim Q^2,$$

where φ_i is given by (5.23) and χ_i is a cut-off function given by (5.24). For $1 < p < 2$, $N(\phi) = O(|\phi|^p)$. Thus, by $-\Psi' \sim \Psi$ and (2.8),

$$|\langle N(\phi), \Psi'_{s_j}\rangle| \lesssim \int_{\mathcal{C}} \sum_{i=1}^{\nu} (Q\varphi_i(t)\chi_i(t))^p \Psi_{s_j} \sim Q^p \int_0^{\frac{R}{2}} e^{(2-p)p(a_c-a)r-r} = O(Q^{\frac{p^2+1}{2}}).$$

On the other hand, by (2.8), (5.19), (5.20), (5.21), (5.22) and $-\Psi'_{s_j} \sim \Psi_{s_j}$,

$$|\langle E, \Psi'_{s_j}\rangle_{L^2(\mathcal{C})}| \sim Q \sum_{i=1}^{\nu} \int_{(\frac{s_{i-1}+s_i}{2}, \frac{s_i+s_{i+1}}{2})} \Psi_{s_i}^{p-2}\Psi_{s_j} \sim Q \int_{(\frac{s_{j-1}+s_j}{2}, \frac{s_j+s_{j+1}}{2})} \Psi_{s_j}^{p-1} \sim Q.$$

It follows from $p > 1$ that $\sum_{j=1}^{\nu} |c_l| \lesssim Q$ and

$$\begin{aligned} \|\phi\|_{H^1(\mathcal{C})}^2 &= \langle E + N(\phi) + \sum_{j=1}^{\nu} c_j \Psi_{s_j}^{p-1}\Psi'_{s_j}, \phi\rangle_{L^2(\mathcal{C})} \\ &\leq \langle E, \phi\rangle_{L^2(\mathcal{C})} + O(Q\|\phi\|_{H^1(\mathcal{C})} + \|\phi\|_{H^1(\mathcal{C})}^{\sigma_p+1}), \end{aligned} \tag{5.28}$$
eqn19988

where $\sigma_p = 2$ for $p \geq 2$ and $\sigma_p = p$ for $1 < p < 2$. Since $\|\phi\|_\sharp \leq C$ for $p \geq 3$, by (5.18),

$$\langle E, \phi\rangle_{L^2(\mathcal{C})} \lesssim \sum_{i=1}^{\nu} Q^2 \int_{(\frac{s_{i-1}+s_i}{2}, \frac{s_i+s_{i+1}}{2})} \Psi_{s_i}^{(2-2\varsigma)} \sim Q^2$$

for $p \geq 3$. For $1 < p < 3$, $\|\phi\|_\natural \leq C$. Thus, by (5.17),

$$\langle E, \phi\rangle_{L^2(\mathcal{C})} \lesssim \sum_{i=1}^{\nu} Q^2 \int_{(\frac{s_{i-1}+s_i}{2}, \frac{s_i+s_{i+1}}{2})} \Psi_{s_i}^{2(p-2)} \sim \begin{cases} Q^2, & p > 2, \\ Q^2 \log(Q), & p = 2, \\ Q^p, & 1 < p < 2. \end{cases}$$

(5.26) then follows from (5.28). □

Let $\varphi = \rho - \phi$, then by (5.8) and (5.14),

$$\begin{cases} -\Delta_\theta \varphi - \partial_t^2 \varphi + (a_c - a)^2 \varphi \\ = (\sum_{j=1}^{\nu} \Psi_{s_j} + \phi + \varphi)^p - |\sum_{j=1}^{\nu} \Psi_{s_j} + \phi|^{p-1}(\sum_{j=1}^{\nu} \Psi_{s_j} + \phi) \\ - \sum_{j=1}^{\nu} c_j \Psi_{s_j}^{p-1} \Psi_{s_j}' + f, \quad \text{in } \mathcal{C}, \\ \langle \Psi_{s_j}', \varphi\rangle_{H^1(\mathcal{C})} = 0 \quad \text{for all } j = 1, 2, \cdots, \nu. \end{cases} \tag{5.29}$$
eq0028

Let $M_0 = \mathrm{span}\{\Psi_{s_j}\}$ and $M = \mathrm{span}\{\Psi_{s_j}'\}$. Then by the orthogonal conditions satisfied by φ, we can decompose $\varphi = \sum_{j=1}^{\nu} \beta_j \Psi_{s_j} + \Psi^\perp$, where $\Psi^\perp \in (M_0 \oplus M)^\perp$ in $H^1(\mathcal{C})$.

⟨lem0004⟩ **Lemma 5.6.** *Let $b_{FS}(a) \leq b < a+1$ for $a < 0$ and $a \leq b < a+1$ for $a \geq 0$ and $a + b > 0$. Then for $\delta > 0$ sufficiently small, we have*

$$|\beta_j| \lesssim \|f\|_{H^{-1}(\mathcal{C})} + Q^2 \tag{5.30}$$
eq0035

and

$$\|\Psi^\perp\|_{H^1(\mathcal{C})} \lesssim \|f\|_{H^{-1}(\mathcal{C})} + Q^2. \tag{5.31}$$
eq0036

Proof. Since Ψ is the minimizer of (2.3), the Morse index of Ψ is equal to 1. It follows from the nondegeneracy of Ψ that

$$\int_{\mathcal{C}} |\nabla_\theta v|^2 + |\partial_t v|^2 + (a_c - a)^2 v^2 > (p + 2\varepsilon) \int_{\mathcal{C}} \Psi^{p-1} v^2 \tag{5.32}$$
eq0029

for all $v \in \mathrm{span}\{\Psi, \Psi'\}^\perp$ with some $\varepsilon > 0$ sufficiently small. Since $R \to +\infty$ as $\delta \to 0$ by Proposition 3.2 and $p > 1$, for $\delta > 0$ sufficiently small, it is standard to use (5.32) and the exponential decay of Ψ at infinity given by (2.8) to show that

$$\int_{\mathcal{C}} |\nabla_\theta v|^2 + |\partial_t v|^2 + (a_c - a)^2 v^2 > (p + \varepsilon) \int_{\mathcal{C}} (\sum_{j=1}^{\nu} \Psi_{s_j})^{p-1} v^2 \tag{5.33}$$
eq0032

for all $v \in (M_0 \oplus M)^\perp$. By (2.8), $\|\phi\|_\sharp \leq C$ for $p \geq 3$ and $\|\phi\|_\natural \leq C$ for $1 < p < 3$,

$$\begin{aligned} &(\sum_{j=1}^{\nu} \Psi_{s_j} + \phi + \varphi)^p - |\sum_{j=1}^{\nu} \Psi_{s_j} + \phi|^{p-1}(\sum_{j=1}^{\nu} \Psi_{s_j} + \phi) \\ = \;& p|\sum_{j=1}^{\nu} \Psi_{s_j} + \phi|^{p-1} \varphi + O(|\varphi|^{\sigma_p}) \\ = \;& p(\sum_{j=1}^{\nu} \Psi_{s_j})^{p-1} \varphi + q(t, \theta)\varphi + O(|\varphi|^{\sigma_p}), \end{aligned}$$

where $\|q\|_{L^\infty(\mathcal{C})} \leq \widehat{\delta}$ with $\widehat{\delta} \to 0$ as $\delta \to 0$, $\sigma_p = 2$ for $p \geq 2$ and $\sigma_p = p$ for $1 < p < 2$. Now, multiplying (5.29) with Ψ_{s_j} for all j and $\Psi^\perp$, respectively, and integrating by parts,

$$\begin{aligned}(1-p)\beta_j\|\Psi\|_{H^1(\mathcal{C})}^2 &= \int_{\mathcal{C}}(p(\sum_{l=1}^{\nu}\Psi_{s_l})^{p-1}+q(t,\theta))(\sum_{i\neq j}\beta_i\Psi_{s_i}+\Psi^\perp)\Psi_{s_j}\\ &\quad+\beta_j\int_{\mathcal{C}}(p(\sum_{l=1}^{\nu}\Psi_{s_l})^{p-1}-p\Psi_{s_j}^{p-1}+q(t,\theta))\Psi_{s_j}^2\\ &\quad+\langle f,\Psi_{s_j}\rangle_{L^2(\mathcal{C})}+\sum_{i\neq j}c_i\langle\Psi_{s_i}^{p-1}\Psi_{s_i}',\Psi_{s_j}\rangle_{L^2(\mathcal{C})}\\ &\quad+O(\sum_{i=1}^{\nu}\beta_i^{\sigma_p+1}+\|\Psi^\perp\|_{H^1(\mathcal{C})}^{\sigma_p+1})\end{aligned}$$

and

$$\begin{aligned}\|\Psi^\perp\|_{H^1(\mathcal{C})}^2 &= \int_{\mathcal{C}}(p(\sum_{l=1}^{\nu}\Psi_{s_l})^{p-1}+q(t,\theta))(\sum_{i=1}^{\nu}\beta_i\Psi_{s_i}+\Psi^\perp)\Psi^\perp\\ &\quad+\langle f,\Psi^\perp\rangle_{L^2(\mathcal{C})}+O(\sum_{i=1}^{\nu}\beta_i^{\sigma_p+1}+\|\Psi^\perp\|_{H^1(\mathcal{C})}^{\sigma_p+1}).\end{aligned}$$

By (5.19), (5.20), (5.21), (5.22) and [19, Lemma 6],

$$|\beta_j| \lesssim \widetilde{\delta}(\sum_{i\neq j}|\beta_i|+\|\Psi^\perp\|_{H^1(\mathcal{C})})+\|f\|_{H^{-1}(\mathcal{C})}+\sum_{i\neq j}|c_i\langle\Psi_{s_i}^{p-1}\Psi_{s_i}',\Psi_{s_j}\rangle_{L^2(\mathcal{C})}|$$

for all $j = 1, 2, \cdots, \nu$ and by (5.33),

$$\|\Psi^\perp\|_{H^1(\mathcal{C})} \lesssim \widetilde{\delta}\sum_{i=1}^{\nu}|\beta_i|+\|f\|_{H^{-1}(\mathcal{C})},$$

where $\widetilde{\delta} \to 0$ as $\delta \to 0$. It follows from Lemma 5.5 and (5.27) that (5.30) and (5.31) hold for $\delta > 0$ sufficiently small. □

We are now in the position to prove the following stability.

⟨prop0005⟩ **Proposition 5.2.** *Let $v \in H^1(\mathcal{C})$ be nonnegative such that*

$$(\nu-\frac{1}{2})(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}} < \|v\|_{H^1(\mathcal{C})}^2 < (\nu+\frac{1}{2})(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}}$$

with $\nu \geq 2$ and $\|f\|_{H^{-1}(\mathcal{C})} \leq \delta$ for $\delta > 0$ sufficiently small, where f is given by (5.6)*. Then either for $b_{FS}(a) \leq b < a+1$ with $a < 0$ or for $a \leq b < a+1$ with $a \geq 0$ and $a+b > 0$, we have*

$$d_*(v) \lesssim \begin{cases}\|f\|_{H^{-1}(\mathcal{C})}, & p > 2,\\ \|f\|_{H^{-1}(\mathcal{C})}|\log(\|f\|_{H^{-1}(\mathcal{C})})|^{\frac{1}{2}}, & p = 2,\\ \|f\|_{H^{-1}(\mathcal{C})}^{\frac{p}{2}}, & 1 < p < 2.\end{cases}$$

Proof. We recall that $d_*^2(v) = \|\rho\|_{H^1(\mathcal{C})}^2$ and $\rho = \phi+\varphi$. The conclusion then follows immediately from Lemmas 5.2, 5.5 and 5.6. □

6. Optimality of the stability to profile decompositions

In this section, we will construct examples, as that in [7, 11], to show that the orders in Propositions 5.1 and 5.2 are sharp. Let us begin with the examples for $\nu = 1$.

⟨propn0002⟩ **Proposition 6.1.** *Let $b_{FS}(a) \leq b < a+1$ for $a < 0$ and $a \leq b < a+1$ for $a \geq 0$ and $a+b>0$. Then the stability stated in Proposition 5.1 is sharp in the sense that, there exists nonnegative $v_* \in H^1(\mathcal{C})$, with*

$$\frac{1}{2}(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}} < \|v_*\|_{H^1(\mathcal{C})}^2 < \frac{3}{2}(C_{a,b,N}^{-1})^{\frac{p+1}{p-1}} \quad \text{and} \quad \|f_*\|_{H^{-1}(\mathcal{C})} \leq \delta$$

for $\delta > 0$ sufficiently small, such that $d_0(v_*) \gtrsim \|f_*\|_{H^{-1}(\mathcal{C})}$.

Proof. Let $v_\varepsilon = \Psi + \varepsilon\varphi$ where $\varphi \in C_0^\infty(\mathcal{C})$ is positive and even such that $\langle \Psi', \varphi\rangle_{H^1(\mathcal{C})} = 0$. Then as that in the proof of Proposition 5.1, we have

$$\begin{aligned} f_\varepsilon &= -\Delta_\theta v - \partial_t^2 v + (a_c - a)^2 v - v^p \\ &= \varepsilon(-\Delta_\theta\varphi - \partial_t^2\varphi + (a_c-a)^2\varphi - p\Psi^{p-1}\varphi) + O((\varepsilon\varphi)^{\sigma_p}). \end{aligned}$$

It follows that $\|f_\varepsilon\|_{H^{-1}(\mathcal{C})} \lesssim \varepsilon$ for $\varepsilon > 0$ sufficiently small. As that in the proof of Proposition 5.1, it is easy to see that $d_0(v_\varepsilon) \lesssim \varepsilon$ is attained by some $s_\varepsilon \in \mathbb{R}$. Thus, we can rewrite $v_\varepsilon = \Psi_{s_\varepsilon} + \widetilde{\varphi}_\varepsilon$, where $d_0(v_\varepsilon) = \|\widetilde{\varphi}_\varepsilon\|_{H^1(\mathcal{C})} \lesssim \varepsilon$ and

$$\langle \Psi_{s_\varepsilon}, \widetilde{\varphi}_\varepsilon\rangle_{H^1(\mathcal{C})} = \langle \Psi'_{s_\varepsilon}, \widetilde{\varphi}_\varepsilon\rangle_{H^1(\mathcal{C})} = 0.$$

Note that

$$\|\Psi_{s_\varepsilon} - \Psi\|_{H^1(\mathcal{C})} = \|\widetilde{\varphi}_\varepsilon - \varepsilon\varphi\|_{H^1(\mathcal{C})} \lesssim \varepsilon,$$

we have $s_\varepsilon = o_\varepsilon(1)$. Clearly, $\widetilde{\varphi}_\varepsilon$ satisfies

$$-\Delta_\theta\widetilde{\varphi}_\varepsilon - \partial_t^2\widetilde{\varphi}_\varepsilon + (a_c-a)^2\widetilde{\varphi}_\varepsilon = (\Psi_{s_\varepsilon} + \widetilde{\varphi}_\varepsilon)^p - \Psi_{s_\varepsilon}^p + f_\varepsilon.$$

Let $\widetilde{f}_\varepsilon$ be the projection of f_ε in $H^1(\mathcal{C})$, then by the Taylor expansion and some elementary inequalities,

$$|\langle \widetilde{\varphi}_\varepsilon, \widetilde{f}_\varepsilon\rangle_{H^1(\mathcal{C})}| \gtrsim -|\langle \widetilde{\varphi}_\varepsilon, \widetilde{f}_\varepsilon\rangle_{H^1(\mathcal{C})}| - |\langle \widetilde{\varphi}_\varepsilon^{\sigma_p}, \widetilde{f}_\varepsilon\rangle_{L^2(\mathcal{C})}| + \|f_\varepsilon\|_{H^{-1}}^2.$$

It follows that $d_0(v_\varepsilon) = \|\widetilde{\varphi}_\varepsilon\|_{H^1(\mathcal{C})} \gtrsim \|f_\varepsilon\|_{H^{-1}}$. □

We next construct examples for $\nu \geq 2$.

⟨prop0006⟩ **Proposition 6.2.** *Let $b_{FS}(a) \leq b < a+1$ for $a < 0$ and $a \leq b < a+1$ for $a \geq 0$ and $a+b>0$. Then the stability stated in Proposition 5.2 is sharp in the sense that, there exists nonnegative $v_* \in H^1(\mathcal{C})$ such that*

$$d_*(v_*) \gtrsim \begin{cases} \|f_*\|_{H^{-1}(\mathcal{C})}, & p > 2, \\ \|f_*\|_{H^{-1}(\mathcal{C})}|\log(\|f_*\|_{H^{-1}(\mathcal{C})})|^{\frac{1}{2}}, & p = 2, \\ \|f_*\|_{H^{-1}(\mathcal{C})}^{\frac{p}{2}}, & 1 < p < 2. \end{cases}$$

Proof. Let us consider the following equation:

$$\begin{cases} -\Delta_\theta\widetilde{\phi}_R-\partial_t^2\widetilde{\phi}_R+(a_c-a)^2\widetilde{\phi}_R \\ =|\sum_{j=1}^2\Psi_{s_{j,R}}+\widetilde{\phi}_R|^{p-1}(\sum_{j=1}^2\Psi_{s_{j,R}}+\widetilde{\phi}_R) \\ -\sum_{j=1}^2\Psi_{s_{j,R}}^p+\sum_{j=1}^2c_{j,R}\Psi_{s_{j,R}}^{p-1}\Psi'_{s_{j,R}}, \quad \text{in } \mathcal{C}, \\ \langle\Psi'_{s_{j,R}},\widetilde{\phi}_R\rangle_{H^1(\mathcal{C})}=0 \quad \text{for all } j=1,2, \end{cases} \tag{6.1}$$ eq0045

where $s_{1,R}=-\frac{R}{2}$ and $s_{2,R}=\frac{R}{2}$. By Lemma 5.5, (6.1) is solvable for $R>0$ sufficiently large with $|c_{1,R}|+|c_{2,R}|\lesssim Q=e^{-(a_c-a)R}$ either for $b_{FS}(a)\leq b<a+1$ with $a<0$ or for $a\leq b<a+1$ with $a\geq 0$ and $a+b>0$. Let $v_R=\sum_{j=1}^2\Psi_{s_{j,R}}+\widetilde{\phi}_R$, then

$$\begin{aligned} f_R &:= -\Delta_\theta v_R-\partial_t^2v_R+(a_c-a)^2v_R-|v_R|^{p-1}v_R \\ &= \sum_{j=1}^2c_{j,R}\Psi_{s_{j,R}}^{p-1}\Psi'_{s_{j,R}}. \end{aligned} \tag{6.2}$$ eqn4444

which, together with Lemma 5.2 and Proposition 5.2, implies that

$$\|f_R\|_{H^{-1}(\mathcal{C})}\sim\sum_{j=1}^2|c_{j,R}|\sim Q \tag{6.3}$$ eq0047

for $R>0$ sufficiently large. Note that as that in the proof of Proposition 4.1, we can show that $d_*^2(v_R)\leq\|\widetilde{\phi}_R\|_{H^1(\mathcal{C})}^2$ is attained at $\sum_{j=1}^2\Psi_{s'_{j,R}}$ for some $s'_{1,R}$ and $s'_{2,R}$. Thus, we can rewrite $v_R=\sum_{j=1}^2\Psi_{s'_{j,R}}+\widetilde{\varphi}_R$, where $\widetilde{\varphi}_R\in(\text{span}\{\Psi'_{s'_{j,R}}\})^\perp$ in $H^1(\mathcal{C})$. It follows that $d_*^2(v_R)=\|\widetilde{\varphi}_R\|_{H^1(\mathcal{C})}^2\leq\|\widetilde{\phi}_R\|_{H^1(\mathcal{C})}^2$. Since

$$\|\sum_{j=1}^2\Psi_{s'_{j,R}}-\sum_{j=1}^2\Psi_{s_{j,R}}\|_{H^1(\mathcal{C})}\lesssim\|\widetilde{\phi}_R\|_{H^1(\mathcal{C})}\to 0 \quad \text{as } R\to+\infty,$$

we have $s'_{j,R}=s_{j,R}+o_R(1)$. Clearly, by (6.1), $\widetilde{\varphi}_R$ satisfies the following equation:

$$\begin{cases} -\Delta_\theta\widetilde{\varphi}_R-\partial_t^2\widetilde{\varphi}_R+(a_c-a)^2\widetilde{\varphi}_R \\ =|\sum_{j=1}^2\Psi_{s'_{j,R}}+\widetilde{\varphi}_R|^{p-1}(\sum_{j=1}^2\Psi_{s'_{j,R}}+\widetilde{\varphi}_R) \\ -\sum_{j=1}^2\Psi_{s'_{j,R}}^p+\sum_{j=1}^2c_{j,R}\Psi_{s_{j,R}}^{p-1}\Psi'_{s_{j,R}}, \quad \text{in } \mathcal{C}, \\ \langle\Psi'_{s'_{j,R}},\widetilde{\varphi}_R\rangle_{H^1(\mathcal{C})}=0 \quad \text{for all } j=1,2. \end{cases} \tag{6.4}$$ eq0145

Let $\varrho_R:[0,1]\to[0,1]$ be a smooth cut-off function such that

$$\varrho_R(t)=\begin{cases} 1, & s'_{1,R}+\frac{R}{2}-3\leq t\leq s'_{1,R}+\frac{R}{2}-2, \\ 0, & t\leq s'_{1,R}+\frac{R}{2}-4 \text{ or } t\geq s'_{1,R}+\frac{R}{2}-1. \end{cases} \tag{6.5}$$ eqn29980

Then $\|\varrho_R\|_{H^1(\mathcal{C})} \lesssim 1$. Similar to that of (5.10), (5.11) and (5.12),

$$|\sum_{j=1}^{2}\Psi_{s'_{j,R}}+\widetilde{\varphi}_R|^{p-1}(\sum_{j=1}^{2}\Psi_{s'_{j,R}}+\widetilde{\varphi}_R)-\sum_{j=1}^{2}\Psi^{p}_{s'_{j,R}} \gtrsim \Psi^{p-1}_{s'_{1,R}}\Psi_{s'_{2,R}}+p\Psi^{p-1}_{s'_{1,R}}\widetilde{\varphi}_R+O(|\widetilde{\varphi}_R|^{\sigma_p})$$

in the region $[s'_{1,R}+\frac{R}{2}-4, s'_{1,R}+\frac{R}{2}-1]$. Thus, by multiplying (6.4) with ϱ_R and integrating by parts,

$$\begin{aligned}\|\widetilde{\varphi}_R\|_{H^1(\mathcal{C})} &\gtrsim \int_{\mathcal{C}}(\Psi^{p-1}_{s'_{1,R}}\Psi_{s'_{2,R}}+p\Psi^{p-1}_{s'_{1,R}}\widetilde{\varphi}_R+O(|\widetilde{\varphi}_R|^{\sigma_p}))\varrho_R+\int_{\mathcal{C}}\sum_{j=1}^{2}c_{j,R}\Psi^{p-1}_{s_{j,R}}\Psi'_{s_{j,R}}\varrho_R\\ &\gtrsim \int_{\mathcal{C}}\Psi^{p-1}_{s'_{1,R}}\Psi_{s'_{2,R}}\varrho_R-\sum_{j=1}^{2}|c_{j,R}|\int_{\mathcal{C}}\Psi^{p-1}_{s_{j,R}}\Psi'_{s_{j,R}}\varrho_R-\|\widetilde{\varphi}_R\|_{H^1(\mathcal{C})}.\end{aligned}$$

By (2.8) and (6.5),

$$\sum_{j=1}^{2}|c_{j,R}|\int_{\mathcal{C}}\Psi^{p-1}_{s_{j,R}}\Psi'_{s_{j,R}}\varrho_R=o(\sum_{j=1}^{2}|c_{j,R}|)\quad\text{as } R\to+\infty$$

and

$$\int_{\mathcal{C}}\Psi^{p-1}_{s'_{1,R}}\Psi_{s'_{2,R}}\varrho_R \gtrsim \int_{\frac{R}{2}-3}^{\frac{R}{2}-2}e^{-(p-1)(a_c-a)r}e^{-(a_c-a)(R-r)}\sim\begin{cases}Q, & p\geq 2,\\ Q^{\frac{p}{2}}, & 1<p<2\end{cases}$$

for $R>0$ sufficiently large. It follows from (6.3) that

$$d_*(v_R)\gtrsim\begin{cases}\|f_R\|_{H^{-1}(\mathcal{C})}, & p>2,\\ \|f_R\|^{\frac{p}{2}}_{H^{-1}(\mathcal{C})}, & 1<p<2\end{cases}$$

for $R>0$ sufficiently large. For $p=2$, we shall modify the test function ϱ_R by

$$\widetilde{\varrho}_R(t)=\begin{cases}1, & s'_{1,R}+\dfrac{R}{4}\leq t\leq s'_{1,R}+\dfrac{R}{2}-2,\\ 0, & t\leq s'_{1,R}+\dfrac{R}{4}-1 \text{ or } t\geq s'_{1,R}+\dfrac{R}{2}-1.\end{cases}\qquad(6.6)\ \boxed{\texttt{eqn19980}}$$

Then $\|\widetilde{\varrho}_R\|_{H^1(\mathcal{C})}\lesssim\sqrt{R}$ for $R>0$ sufficiently large. Thus, by multiplying (6.4) with $\widetilde{\varrho}_R$ and integrating by parts,

$$\begin{aligned}\sqrt{R}\|\widetilde{\varphi}_R\|_{H^1(\mathcal{C})} &\gtrsim \int_{\mathcal{C}}(\Psi_{s'_{1,R}}\Psi_{s'_{2,R}}+p\Psi_{s'_{1,R}}\widetilde{\varphi}_R+O(\widetilde{\varphi}_R^2))\widetilde{\varrho}_R+\int_{\mathcal{C}}\sum_{j=1}^{2}c_{j,R}\Psi_{s_{j,R}}\Psi'_{s_{j,R}}\widetilde{\varrho}_R\\ &\gtrsim \int_{\mathcal{C}}\Psi_{s'_{1,R}}\Psi_{s'_{2,R}}\widetilde{\varrho}_R-\sum_{j=1}^{2}|c_{j,R}|\int_{\mathcal{C}}\Psi_{s_{j,R}}\Psi'_{s_{j,R}}\widetilde{\varrho}_R-\|\widetilde{\varphi}_R\|_{H^1(\mathcal{C})}\sqrt{R}.\end{aligned}$$

By (2.8) and (6.6),

$$\sum_{j=1}^{2}|c_{j,R}|\int_{\mathcal{C}}\Psi_{s_{j,R}}\Psi'_{s_{j,R}}\widetilde{\varrho}_R\sim\sum_{j=1}^{2}|c_{j,R}|\int_{\frac{R}{4}}^{\frac{R}{2}}e^{-2(a_c-a)r}\sim\sum_{j=1}^{2}|c_{j,R}|e^{-(a_c-a)\frac{R}{2}}$$

as $R\to+\infty$. On the other hand,

$$\int_{\mathcal{C}}\Psi_{s'_{1,R}}\Psi_{s'_{2,R}}\widetilde{\varrho}_R\gtrsim\int_{\frac{R}{4}}^{\frac{R}{2}-1}e^{-(a_c-a)r}e^{-(a_c-a)(R-r)}\sim Re^{-(a_c-a)R}$$

for $R>0$ sufficiently large. It follows from (6.3) that

$$d_*(v_R)\gtrsim\|f_R\|_{H^{-1}(\mathcal{C})}|\log(\|f_R\|_{H^{-1}(\mathcal{C})})|^{\frac12}$$

for $p=2$ and $R>0$ sufficiently large. Now, we take $v_*=v_R^+$ then $v_*=v_R+v_R^-$, where $v_R^\pm=\max\{\pm v_R,0\}$. Clearly, we have $0\leq v_R^-\leq|\widetilde{\phi}_R|$ since $\sum_{j=1}^2\Psi_{s_{j,R}}$ is positive. It follows from (6.2) that

$$\|v_R^-\|_{H^1(\mathcal{C})}^2=\langle v_R^-,V_R\rangle_{H^1(\mathcal{C})}=\int_{\mathcal{C}}|v_R|^{p-1}v_Rv_R^-+\int_{\mathcal{C}}f_Rv_R^-\lesssim\|v_R^-\|_{H^1(\mathcal{C})}^{p+1}+\int_{\mathcal{C}}|f_R||v_R^-|.$$

For $1<p\leq2$, by Lemma 5.5 and (6.3),

$$\|v_R^-\|_{H^1(\mathcal{C})}^2\lesssim\int_{\mathcal{C}}|f_R||\widetilde{\phi}_R|\lesssim\begin{cases}Q^2|\log(Q)|^{\frac12}, & p=2,\\ Q^{1+\frac{p}{2}}, & 1<p<2.\end{cases}$$

For $p>2$, recall that by Lemma 5.5, $\|\widetilde{\phi}_R\|_\sharp\lesssim1$. Thus, by (2.8) and (5.18), $v_R^-=0$ for $|t-s_{j,R}|\leq\frac{R}{2}$. It follows that

$$\|v_R^-\|_{H^1(\mathcal{C})}^2\lesssim\int_{\mathcal{C}}|f_R||v_R^-|\lesssim\int_{(\cup_{j=1}^2\{|t-s_{j,R}|\leq R\})^c}|f_R||\widetilde{\phi}_R|=o(Q^2).$$

The conclusion then follows from $d_*(v_*)\geq d_*(v_R)-\|v_R^-\|_{H^1(\mathcal{C})}$. □

We close this section by the proof of Theorem 1.3.

Proof of Theorem 1.3: It follows immediately from (2.1) and Propositions 5.1, 5.2, 6.1 and 6.2. □

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