ON $SU(3)$ TODA SYSTEM WITH MULTIPLE SINGULAR SOURCES

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Abstract. We consider the singular $SU(3)$ Toda system with multiple singular sources

$$\begin{cases} -\Delta w_1 = 2e^{2w_1} - e^{w_2} + 2\pi \sum_{\ell=1}^m \beta_{1,\ell}\delta_{P_\ell} & \text{in } \mathbb{R}^2 \\ -\Delta w_2 = 2e^{2w_2} - e^{w_1} + 2\pi \sum_{\ell=1}^m \beta_{2,\ell}\delta_{P_\ell} & \text{in } \mathbb{R}^2 \\ w_i(x) = -2\log|x| + O(1) \quad \text{as } |x| \to \infty, \ i = 1, 2, \end{cases}$$

with $m \geq 3$ and $\beta_{i,\ell} \in [0,1)$. We prove the existence and non-existence results under suitable assumptions on $\beta_{i,\ell}$. This generalizes Luo-Tian's [31] result for a singular Liouville equation in $\mathbb{R}^2$. We also study existence results for a higher order singular Liouville equation in $\mathbb{R}^n$.

1. Introduction

We consider the following singular $SU(3)$ Toda system with multiple singular sources

$$\begin{cases} -\Delta w_1 = 2e^{2w_1} - e^{w_2} + 2\pi \sum_{\ell=1}^m \beta_{1,\ell}\delta_{P_\ell} & \text{in } \mathbb{R}^2 \\ -\Delta w_2 = 2e^{2w_2} - e^{w_1} + 2\pi \sum_{\ell=1}^m \beta_{2,\ell}\delta_{P_\ell} & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $P_1, \dots, P_m$ are distinct points in $\mathbb{R}^2$, $\beta_{i,\ell} \in [0,1)$ and δ_P denotes the Dirac measure at P (notice that source terms are written with a plus sign). When $w_1 = w_2, \beta_{1,l} = \beta_{2,l} = \beta_l$, the above system reduces to the singular Liouville equation

$$-\Delta w = e^{2w} + 2\pi \sum_{\ell=1}^m \beta_\ell \delta_{P_\ell} \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

The Toda system (1.1) and the Liouville equation (1.2) have been widely studied in the literature due to its important role in geometry and mathematical physics. For instance, Eq. (1.2) is related to the problem of prescribing Gaussian curvature on surfaces with conical singularity, and abelian gauge in Chern-Simons theory [4,7,37,38]. The Toda system (1.1) appears in the description of holomorphic curves in $\mathbb{CP}^3$ [10,12,16,18], and in the non-abelian Chern-Simmon theory [19,36,42]. For classification and blow-up analysis to the (singular) Liouville equation and the $SU(n)$ Toda system we refer the reader to [5,7–9,11,13,14,21–30,33,35] and the references therein.

Luo-Tian [31] gave a necessary and sufficient condition for the existence of singular metric with three or more conical singularities on the 2-sphere, whose equivalent statement on $\mathbb{R}^2$ is the following theorem:

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Theorem A ([31]). *Let $m \geq 3$. Let $P_1, \dots, P_m$ be m distinct points in $\mathbb{R}^2$. Then there exist continuous functions h_ℓ around P_ℓ for $\ell = 1, \dots, m$, a bounded continuous function h_{m+1} outside a compact set, and a solution w to*

$$\begin{cases} -\Delta w = e^{2w} & \text{in } \mathbb{R}^2 \setminus \{P_1, P_2, \dots, P_m\} \\ w(x) = -\beta_\ell \log|x - P_\ell| + h_\ell(x) & \text{around each } P_\ell \\ w(x) = -2\log|x| + h_{m+1}(x) & \text{as } |x| \to \infty \\ \beta_\ell \in (0,1) & \ell = 1, \dots, m \end{cases} \tag{1.3}$$

if and only if

$$\sum_{\ell=1}^{m} \beta_i < 2 \quad \text{and} \quad \sum_{\ell \neq j} \beta_\ell > \beta_j \quad \text{for every } j = 1, 2, \dots, m. \tag{1.4}$$

Moreover, the solution is unique.

Troyanov [39] studied singular metrics with 2 singulirities (i.e., $m = 2$) and constant curvature 1 on the 2-sphere, and showed that the order of both singularities are equal (i.e., $\beta_1 = \beta_2 < 1$). A necessary and sufficient condition on $\{\beta_1, \beta_2, \beta_3\} \subset (-\infty, 1)$ for the existence of singular metric on the 2-sphere has been given in [20,40]. See also [5,6,32] and the references therein for various existence results on compact surfaces.

In this paper we study Problem (1.3) in the context of $SU(3)$ Toda system. More precisely, we prove existence and non-existence of solutions (w_1, w_2) to (1.1) satisfying

$$\begin{cases} w_i(x) = -\beta_{i,\ell} \log|x - P_\ell| + h_{i,\ell} & \text{around each point } P_\ell \\ w_i(x) = -2\log|x| + h_{i,m+1} & \text{as } |x| \to \infty \\ h_{i,\ell} \text{ is continuous in a neighborhood of } P_\ell, \end{cases} \tag{1.5}$$

for $i = 1, 2$ and $\ell = 1, \dots, m$, and $h_{i,m+1}$ is bounded outside a compact set. We write

$$u_i(x) = w_i(x) + \sum_{\ell=1}^{m} \beta_{i,\ell} \log|x - P_\ell|, \quad i = 1, 2.$$

Then w_i solves (1.1) if and only if u_i solves

$$\begin{cases} -\Delta u_1 = 2K_1 e^{2u_1} - K_2 e^{2u_2} & \text{in } \mathbb{R}^2 \\ -\Delta u_2 = 2K_2 e^{2u_2} - K_1 e^{2u_1} & \text{in } \mathbb{R}^2 \\ K_i(x) := \prod_{\ell=1}^{m} \frac{1}{|x - P_\ell|^{2\beta_{i,\ell}}} & i = 1, 2. \end{cases} \tag{1.6}$$

The condition (1.5) in terms of u_i is

$$\begin{cases} u_i(x) = -\beta_i \log|x| + \text{ a bounded continuous function} & \text{on } B_1^c \\ \beta_i := 2 - \sum_{\ell=1}^{m} \beta_{i,\ell}, & i = 1, 2, \end{cases} \tag{1.7}$$

provided u_i is continuous.

For Toda system with singular sources, the only complete result is [28] in which the case of single source, i.e., $m = 1$ is completely solved by PDE and integrable system theory. In [26], some special cases of $m = 2$ are classified using higher order hypergeometric equations. The following theorem gives the *first* existence result when $m \geq 3$:

Theorem 1.1. *Let $m \geq 3$. Let $\{\beta_{i,\ell} : i = 1, 2,\ \ell = 1, 2, \dots m\} \subset [0, 1)$ be such that*

$$3(1 + \beta_{i,j}) < 2\sum_{\ell=1}^{m} \beta_{i,\ell} + \sum_{\ell=1}^{m} \beta_{3-i,\ell}, \quad \sum_{\ell=1}^{m} \beta_{i,\ell} < 2, \quad \text{for } j = 1, 2, \dots, m,\ i = 1, 2. \tag{1.8}$$

Then given m distinct points $\{P_\ell\}_{\ell=1}^{m} \subset \mathbb{R}^2$ there exists continuous solution (u_1, u_2) to (1.6) such that (1.7) holds.

Note that if $\sum_\ell \beta_{1,\ell} = \sum_\ell \beta_{2,\ell}$, then the first condition of (1.8) reduces to

$$\sum_{\ell=1}^{m} \beta_{1,\ell} > 1 + \beta_{i,j} \quad \text{for every } i = 1, 2,\ j = 1, \dots, m,$$

which is stronger than (1.4). We shall show that an equivalent condition of (1.4) for the Toda system, namely a condition of the form

$$\sum_{\ell=1, \ell \neq j}^{m} \beta_{i,\ell} > \max\{\beta_{1,j}, \beta_{2,j}\} \quad \text{for every } j = 1, \dots, m,\ i = 1, 2, \tag{1.9}$$

is not sufficient for the existence of solutions to (1.6) satisfying the asymptotic behavior (1.7). See Lemma 3.2.

In [31], the existence of a solution to (1.2) is proved by a variational argument. In this paper we propose a new proof on the existence via fixed point theory. The crucial step in which we need condition (1.8) is Proposition 2.1 below, a compactness result which follows from the blow-up analysis of sequences of solutions (see Lemma 5.2). This compactness is used to prove the a priori bounds necessary to run the fixed point argument of [3,22,41]. Let us point out that condition (1.9) is sufficient to rule-out a "full blow-up" phenomena (that is, after a suitable rescaling, the limiting profile is a $SU(3)$ Toda system in $\mathbb{R}^2$) for a sequence of solutions to (1.6)-(1.7) (for "half blow-up" and "full blow-up" phenomena see e.g., [2, 17, 34]). In particular, condition (1.9) is sufficient to prove the a priori estimate when $\beta_{1,\ell} = \beta_{2,\ell} = \beta_\ell$ and $u_1 = u_2$, that is, a priori estimate for the singular Liouville problem (1.3) . Moreover, the same method also works for a higher order generalization of it.

Theorem 1.2. *Let $m \geq 3$ and $n \geq 2$. For $\ell = 1, 2, \dots, m$ let $\beta_\ell \in (0, 1)$ be such that (1.4) holds. Then given m distinct points $\{P_\ell\}_{\ell=1}^{m} \subset \mathbb{R}^n$ there exists a solution $w \in C^0(\mathbb{R}^n \setminus \{P_1, \dots, P_m\})$ to*

$$(-\Delta)^{\frac{n}{2}} w = e^{nw} + \gamma_n \sum_{\ell=1}^{m} \beta_\ell \delta_{P_\ell} \quad \text{in } \mathbb{R}^n$$

satisfying the asymptotic behavior

$$w(x) = -2\log|x| + O(1) \quad \text{as } |x| \to \infty.$$

Here $\gamma_n := \frac{(n-1)!}{2}|S^n|$ is such that

$$\frac{1}{\gamma_n}(-\Delta)^{\frac{n}{2}} \log\frac{1}{|x|} = \delta_0.$$

2. Proof of Theorem 1.1

It is well-know that if (u_1, u_2) is a solution to (1.6) with $\beta_{i,\ell} < 1$ and u_i, $K_i e^{2u_i} \in L^1_{loc}(\mathbb{R}^2)$, then u_i is continuous. On the other hand, if (u_1, u_2) is a continuous solution to (1.6)-(1.7) with $\beta_{i,\ell} < 1$, then $K_i e^{2u_i} = O(|x|^{-4})$ as $|x| \to \infty$. In particular, $\log|\cdot| \cdot |K_i e^{2u_i} \in L^1(\mathbb{R}^2)$, and u_i satisfies the integral equation

$$u_i(x) := \frac{1}{2\pi} \sum_{j=1}^{2} a_{i,j} \int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) K_j(y) e^{2(u_j(y))} dy + c_i, \quad i = 1, 2, \tag{2.1}$$

for some $c_i \in \mathbb{R}$, where $(a_{i,j})$ is the $SU(3)$ Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Moreover, the asymptotic behavior (1.7) implies that

$$\sum_{j=1}^{2} a_{i,j} \int_{\mathbb{R}^2} K_j e^{2u_j} dx = 2\pi\beta_i, \quad i = 1, 2,$$

that is

$$\int_{\mathbb{R}^2} K_i e^{2u_i} dx = 2\pi\bar{\beta}_i, \quad \bar{\beta}_i := \frac{1}{3}(2\beta_i + \beta_{3-i}), \quad i = 1, 2. \tag{2.2}$$

Thus, Theorem 1.1 is equivalent to the existence of solution (u_1, u_2) to (2.1)-(2.2). Moreover, (1.8) in terms of $\bar{\beta}_i$ is

$$\bar{\beta}_i > 0, \quad \bar{\beta}_i < 1 - \beta_{i,\ell} \quad \text{for every } i = 1, 2,\ \ell = 1, \ldots, m. \tag{2.3}$$

In order to prove existence of solution to (2.1)-(2.2) we use a fixed point argument on the space

$$X := C_0(\mathbb{R}^2) \times C_0(\mathbb{R}^2), \quad \|\mathbf{v}\| := \max\{\|v_1\|_{L^\infty(\mathbb{R}^2)}, \|v_2\|_{L^\infty(\mathbb{R}^2)}\} \quad \text{for } \mathbf{v} = (v_1, v_2) \in X,$$

where $C_0(\mathbb{R}^2)$ denotes the space of continuous functions vanishing at infinity. We fix $u_0 \in C^\infty(\mathbb{R}^2)$ such that

$$u_0(x) = -\log|x| \quad \text{on } B_1^c.$$

For $v \in C_0(\mathbb{R}^2)$ let $c_{i,v} \in \mathbb{R}$ be the unique number so that

$$\int_{\mathbb{R}^2} \bar{K}_i e^{2(v + c_{i,v})} dx = 2\pi\bar{\beta}_i, \quad \bar{K}_i := K_i e^{2\beta_i u_0}, \quad i = 1, 2, \tag{2.4}$$

where $\bar{\beta}_i$ is as in (2.2). Now we define $T : X \to X$, $(v_1, v_2) \mapsto (\bar{v}_1, \bar{v}_2)$, where we have set

$$\bar{v}_i(x) := \frac{1}{2\pi} \sum_{j=1}^{2} a_{i,j} \int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) \bar{K}_j(y) e^{2(v_j(y) + c_{j,v_j})} dy - \beta_i u_0(x), \quad i = 1, 2. \tag{2.5}$$

As $\beta_i = 2\bar{\beta}_i - \bar{\beta}_{3-i}$, for $x \in B_1^c$ this can be written as

$$\bar{v}_i(x) := \frac{1}{2\pi} \sum_{j=1}^{2} a_{i,j} \int_{\mathbb{R}^2} \log\left(\frac{|x|}{|x-y|}\right) \bar{K}_j(y) e^{2(v_j(y) + c_{j,v_j})} dy, \quad i = 1, 2.$$

Using that $\bar{K}_i = O(|x|^{-4})$ for $|x|$ large, one can show that $(\bar{v}_1, \bar{v}_2) \in X$. Moreover, the operator T is compact (see e.g. the proof of [22, Lemma 4.1]).

The following proposition is crucial in proving existence of fixed point of T.

Proposition 2.1. *There exists $C > 0$ such that*

$$\|\mathbf{v}\|_X \le C \quad \textit{for every } (\mathbf{v}, t) \in X \times [0,1] \textit{ satisfying } \mathbf{v} = tT(\mathbf{v}).$$

Proof. We assume by contradiction that the proposition is false. Then there exists $\mathbf{v}^k = (v_1^k, v_2^k)$ and $t^k \in (0,1]$ with $\mathbf{v}^k = t^k T(\mathbf{v}^k)$ such that $\|\mathbf{v}^k\| \to \infty$. We set

$$\psi_i^k := v_i^k + c_i^k, \quad c_i^k := c_{i,v_i^k} + \frac{1}{2}\log t^k.$$

Then we have

$$\begin{cases} \psi_1^k(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right)\left(2\bar{K}_1(y)e^{2\psi_1^k(y)} - \bar{K}_2(y)e^{2\psi_2^k(y)}\right)dy - t^k\beta_1 u_0(x) + c_1^k \\ \psi_2^k(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right)\left(2\bar{K}_2(y)e^{2\psi_2^k(y)} - \bar{K}_1(y)e^{2\psi_1^k(y)}\right)dy - t^k\beta_2 u_0(x) + c_2^k. \end{cases} \tag{2.6}$$

For $|x| \ge 1$ this is equivalent to

$$\begin{cases} \psi_1^k(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{|x|}{|x-y|}\right)\left(2\bar{K}_1(y)e^{2\psi_1^k(y)} - \bar{K}_2(y)e^{2\psi_2^k(y)}\right)dy + c_1^k \\ \psi_2^k(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{|x|}{|x-y|}\right)\left(2\bar{K}_2(y)e^{2\psi_2^k(y)} - \bar{K}_1(y)e^{2\psi_1^k(y)}\right)dy + c_2^k. \end{cases} \tag{2.7}$$

Since $\|\mathbf{v}^k\| \to \infty$, we necessarily have

$$\max\{\sup\psi_1^k, \sup\psi_2^k\} \to \infty.$$

Without any loss of generality we assume that $\sup\psi_1^k \ge \sup\psi_2^k$. We fix $x^k \in \mathbb{R}^2$ such that

$$\sup\psi_1^k < \psi_1^k(x^k) + 1.$$

If x^k is bounded then, up to a subsequence, $x^k \to x^\infty$.

We consider the following three cases.

Case 1 $x^\infty \in \mathbb{R}^2 \setminus \{P_\ell : \ell = 1, 2, \ldots, m\}$.

By Lemma 5.2 (see also [23, 30]) we have

$$\max\{\sigma_1(x^\infty), \sigma_2(x^\infty)\} \ge 1,$$

where the blow-up value at a point P is defined by

$$\sigma_i(P) := \lim_{r\to 0}\lim_{k\to\infty}\frac{1}{2\pi}\int_{B_r(P)} \bar{K}_i e^{2\psi_i^k}\,dx, \quad i = 1, 2.$$

This contradicts (2.3) as $\sigma_i(x^\infty) \le \bar{\beta}_i < 1$.

Case 2 $x^\infty \in \{P_\ell : \ell = 1, 2, \ldots, m\}$.

Without loss of generality we assume that $x^\infty = P_1$. Notice that

$$\bar{K}_i(x) = \frac{f_i(x)}{|x - P_1|^{2\beta_{i,1}}}, \quad i = 1, 2,$$

for some positive continuous functions f_1 and f_2 in a small neighborhood of the point P_1. In particular, the functions $w_i^k(x) := \psi_i^k(x - P_1)$ satisfies the conditions of Lemma 5.2 for some $R > 0$, and we get

$$\sigma_1(x^\infty) \ge 1 - \beta_{11}, \quad \text{or } \sigma_2(x^\infty) \ge 1 - \beta_{2,1},$$

a contradiction to (2.3).

Case 3 $|x^k| \to \infty$.

We set

$$\tilde\psi_i^k(x) = \psi_i^k(\frac{x}{|x|^2}), \quad \tilde K_i(x) = \frac{1}{|x|^4}\bar K_i(\frac{x}{|x|^2}) \quad \text{on } \mathbb{R}^2\setminus\{0\}, \quad i=1,2,$$

and extend them continuously at the origin. Then $\tilde\psi_i^k$ satisfies

$$\begin{cases} \tilde\psi_1^k(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left(\frac{|y|}{|x-y|}\right)\left(2\tilde K_1(y)e^{2\tilde\psi_1^k(y)} - \tilde K_2(y)e^{2\tilde\psi_2^k(y)}\right)dy + c_1^k \\ \tilde\psi_2^k(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log\left(\frac{|y|}{|x-y|}\right)\left(2\tilde K_2(y)e^{2\tilde\psi_2^k(y)} - \tilde K_1(y)e^{2\tilde\psi_1^k(y)}\right)dy + c_2^k, \end{cases} \tag{2.8}$$

for $x \in B_1$. Since $\tilde K_i(0) > 0$ for $i=1,2$, and

$$\tilde\psi_1^k(\tilde x_k) \to \infty, \quad \tilde x_k := \frac{x_k}{|x_k|^2} \to 0,$$

one obtains a contradiction as in Case 1.

We conclude the proposition.

□

Proof of Theorem 1.1 It follows from Proposition 2.1 and Schauder fixed point theorem that the operator T has a fixed point, say (v_1, v_2). Then setting

$$u_i := v_i + \beta_i u_0 + c_{i,v_i}, \quad i = 1,2,$$

one sees that (u_1,u_2) is a solution to (1.6)-(1.7).

3. Non-existence results

We show that Theorem 1.1 is not true if the assumption (1.8) is replaced by (1.9). Let us fix $\beta_1,\dots,\beta_7 \in (0,1)$ such that the assumptions $\mathcal{A}1)$ to $\mathcal{A}5)$ hold:

$\mathcal{A}1)$ $\beta_4 + \sum_{\ell=1}^4 \beta_\ell = 2$
$\mathcal{A}2)$ $\beta_2 + \beta_3 < \beta_1$
$\mathcal{A}3)$ $\beta_4 < \frac{1}{3}$
$\mathcal{A}4)$ $\beta_4 + \sum_{\ell=5}^7 \beta_\ell = 2$
$\mathcal{A}5)$ $\beta_4 + \beta_5 < 1$.

It is easy to see that $\mathcal{A}1)$ and $\mathcal{A}2)$ implies that

$\mathcal{A}6)$ $\beta_4 + \beta_1 > 1$ and $\beta_4 + \beta_\ell < 1$ for $\ell = 2,3$.

We shall show an non-existence result to the Toda system (1.1) satisfying (1.5) for the following choice of $\{\beta_{i,\ell}\}$:

$$\beta_{1,\ell} := \begin{cases} \beta_\ell & \text{for } \ell = 1,2,3,4 \\ 0 & \text{for } \ell = 5,6,7 \end{cases}, \quad \beta_{2,\ell} := \begin{cases} 0 & \text{for } \ell = 1,2,3,4 \\ \beta_\ell & \text{for } \ell = 5,6,7. \end{cases} \tag{3.1}$$

Let us point out that we can choose $\{\beta_\ell\}$ satisfying $\mathcal{A}1)$ to $\mathcal{A}5)$ in such a way that $\{\beta_{i,\ell}\}$ satisfy (1.9) with $m=7$, $i=1,2$. For instance, one can simply take

$$\beta_1 = 1-\varepsilon,\ \beta_2 = \beta_3 = \frac{1}{2} - \varepsilon,\ \beta_4 = \frac{3\varepsilon}{2},\ \beta_5 = 1 - \frac{5\varepsilon}{2},\ \beta_6 = \beta_7 = \frac{1+\varepsilon}{2},\ \varepsilon \in (0, \frac{2}{9}).$$

For these β_ℓ's one has

$$\sum_{\ell=1}^7 \beta_{1,\ell} = (1+\beta_{1,1}) - \frac{\varepsilon}{2},$$

and hence $\{\beta_{i,\ell}\}$ does not satisfy (1.8).

We begin with the following non-existence result for a singular Liouville equation.

Lemma 3.1. *Let $\beta_\ell \in (0,1)$ with $\ell = 1,2,3,4$ be such that $\mathcal{A}1)$ to $\mathcal{A}3)$ hold. Let P_1, P_2, P_3 be fixed three distinct points in $\mathbb{R}^2$. Then, for $|P_4|$ large enough, there exists no continuous solution to*

$$-\Delta u = \prod_{\ell=1}^{4} \frac{1}{|x-P_\ell|^{2\beta_\ell}} e^{2u} \quad \text{in } \mathbb{R}^2, \quad u(x) = -2\beta_4 \log|x| + O(1) \quad \text{as } |x| \to \infty. \tag{3.2}$$

Proof. Assume by contradiction that there exists a sequence of solutions (u^k) to (3.2) with

$$P_4 = P_{4,k}, \quad |P_4| \to \infty \quad \text{as } k \to \infty.$$

Notice that the assymptotic behavior

$$u^k(x) = -2\beta_4 \log|x| + O_k(1) \quad \text{as } |x| \to \infty$$

is equivalent to

$$\int_{\mathbb{R}^2} \frac{K_0(x)}{|x-P_4|^{2\beta_4}} e^{2u^k} dx = 4\pi\beta_4, \quad K_0(x) := \prod_{\ell=1}^{3} \frac{1}{|x-P_\ell|^{2\beta_\ell}}.$$

Step 1 We have

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{B_R^c} \frac{K_0(x)}{|x-P_4|^{2\beta_4}} e^{2u^k(x)} dx = 0.$$

To prove this we use Kelvin transform. Up to a small translation, we can assume that none of P_1, P_2, P_3 is the origin. We set

$$\tilde{u}^k(x) := u^k(\frac{x}{|x|^2}) - 2\beta_4 \log|x| + c^k, \quad x \neq 0,$$

for some $c^k \in \mathbb{R}$. Then setting $Q_\ell := \frac{P_\ell}{|P_\ell|^2}$ for $\ell = 1,2,3,4$ we see that

$$-\Delta \tilde{u}^k(x) = \frac{1}{|x|^4} \prod_{\ell=1}^{4} \frac{1}{|\frac{x}{|x|^2} - \frac{Q_\ell}{|Q_\ell|^2}|^{2\beta_\ell}} e^{2u^k(\frac{x}{|x|^2})} \quad \text{in } \mathbb{R}^2 \setminus \{0\}.$$

Using that $|x||y||\frac{x}{|x|^2} - \frac{y}{|y|^2}| = |x-y|$, $\mathcal{A}1)$, and for suitably chosen c^k, we obtain

$$-\Delta \tilde{u}^k(x) = |x|^{2\beta_4} \prod_{\ell=1}^{4} \frac{1}{|x-Q_\ell|^{2\beta_\ell}} e^{2\tilde{u}^k(x)} \quad \text{in } \mathbb{R}^2 \setminus \{0\}$$
$$\tilde{u}^k(x) = -2\beta_4 \log|x| + O_k(1) \quad \text{as } |x| \to \infty.$$

In fact, as $\tilde{u}^k = O_k(1)$ in B_1, it satisfies the above equation at the origin as well, that is,

$$-\Delta \tilde{u}^k(x) = \frac{|x|^{2\beta_4}}{|x-Q_4|^{2\beta_4}} f(x) e^{2\tilde{u}^k(x)} \quad \text{in } \mathbb{R}^2, \quad f(x) := \prod_{\ell=1}^{3} \frac{1}{|x-Q_\ell|^{2\beta_\ell}}.$$

As $|P_4| \to \infty$, we have that $Q_4 \to 0$. By $\mathcal{A}3)$ one gets

$$\int_{\mathbb{R}^2} \frac{|x|^{2\beta_4}}{|x-Q_4|^{2\beta_4}} f(x) e^{2\tilde{u}^k(x)} = 4\pi\beta_4 \leq 2\pi(1-\beta_4-\varepsilon) \tag{3.3}$$

for some $\varepsilon > 0$. Hence, by Lemma 5.1 we obtain

$$\tilde{u}^k \le C \quad \text{in } B_\delta \quad \text{for some } \delta > 0.$$

Step 1 follows immediately from the relation

$$\int_{B_R^c} \frac{K_0(x)}{|x-P_4|^{2\beta_4}} e^{2u^k(x)} dx = \int_{B_{\frac{1}{R}}} \frac{|x|^{2\beta_4}}{|x-Q_4|^{2\beta_4}} f^k(x) e^{2\tilde{u}^k(x)} dx.$$

Step 2 No blow-up occurs on bounded domains, that is, for every $R > 0$,

$$u^k - \beta_4 \log |P_4| \le C(R) \quad \text{on } B_R.$$

Writing $\bar{u}^k = u^k - \beta_4 \log |P_4|$ we see that

$$-\Delta \bar{u}^k = K_0 K_1 e^{2\bar{u}^k} \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} K_0 K_1 e^{2\bar{u}^k} dx = 4\pi\beta_4,$$

where

$$K_0(x) := \prod_{\ell=1}^{3} \frac{1}{|x-P_\ell|^{2\beta_\ell}}, \quad K_1 := \frac{|P_4|^{2\beta_4}}{|x-P_4|^{2\beta_4}}.$$

It follows that $K_1 \to 1$ in $C^0_{loc}(\mathbb{R}^2)$ as $k \to \infty$, and K_0 does not depend on k.

Assume by contradiction that $\bar{u}^k$ is not locally uniformly bounded from above. Then, as blow-up points are discrete, there exists $\delta > 0$ such that

$$\max_{B_\delta(x_0)} \bar{u}^k = \bar{u}^k(x^k) \to \infty, \quad x^k \to x_0,$$

for some $x_0 \in \mathbb{R}^2$. If $x_0 \notin \{P_2, P_3, P_4\}$, then one can show that

$$4\pi\beta_4 \ge \lim_{r \to 0} \lim_{k \to \infty} \int_{B_r(x_0)} K_0 K_1 e^{2\bar{u}^k} dx \ge 4\pi,$$

a contradiction as $\beta_4 < 1$. Thus, $x_0 = P_{\ell_0}$ for some $\ell_0 \in \{1,2,3\}$, and in fact, the set of all blow-up points is a subset of $\{P_1, P_2, P_3\}$. We fix $R > 0$ such that $\bar{B}_{2R}(x_0) \cap \{P_1, P_2, P_3\} = \{x_0\}$. Then $\bar{u}^k$ is uniformly bounded from above in $B_{2R}(x_0) \setminus B_{\frac{R}{2}}(x_0)$. Using this, and as $\bar{u}^k$ satisfies the integral equation

$$\bar{u}^k(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{1+|y|}{|x-y|}\right) K(y) e^{2\bar{u}^k(y)} dy + C^k, \quad K := K_0 K_1,$$

for some $C^k \in \mathbb{R}$, we get that

$$|\bar{u}^k(x) - \bar{u}^k(y)| \le C \quad \text{for every } x, y \in \partial B_R(x_0).$$

Hence, by the remark after Lemma 5.2 we have (this can be shown easily by a local Pohozaev type identity to the above integral equation satisfied by $\bar{u}_k$)

$$\sigma(x_0) = \lim_{r \to 0} \lim_{k \to \infty} \frac{1}{2\pi} \int_{B_r(x_0)} K_0 K_1 e^{2\bar{u}^k} dx = 2(1-\beta_{\ell_0}).$$

Thus $2\beta_4 \ge \sigma(x_0) = 2(1-\beta_{\ell_0})$. This and $\mathcal{A}6$) imply that $\ell_0 = 1$, that is, P_1 is the only blow-up point. In particular, $\bar{u}^k \to -\infty$ locally uniformly outside P_1. Therefore, by Step 1 and (3.3) we get

$$2\beta_4 = \frac{1}{2\pi} \lim_{k \to \infty} \int_{\mathbb{R}^2} K_0 K_1 e^{2\bar{u}^k} dx = \sigma(x_0) = 2(1-\beta_1),$$

a contradiction to $\mathcal{A}6$). This finishes Step 2.

Since $\bar{u}^k$ is locally uniformly bounded from above, up to a subsequence, either $\bar{u}^k \to \infty$ locally uniformly, or $\bar{u}^k \to \bar{u}$ in $C^0_{loc}(\mathbb{R}^2)$. In the first case we get a contradiction to

$$\int_{\mathbb{R}^2} K_0 K_1 e^{2\bar{u}^k} dx = 4\pi\beta_4, \quad \lim_{R\to\infty} \lim_{k\to\infty} \int_{B_R^c} K_0 K_1 e^{2\bar{u}^k} dx = 0,$$

thanks to Step 1. Therefore, only the later case can occur, and the limit function $\bar{u}$ satisfies

$$-\Delta \bar{u} = K_0 e^{2\bar{u}} \quad \text{in } \mathbb{R}^2, \quad K_0 = \prod_{\ell=1}^{3} \frac{1}{|x - P_\ell|^{2\beta_\ell}}.$$

Again by Step 1, we have that

$$\int_{\mathbb{R}^2} K_0 e^{2\bar{u}} dx = 4\pi\beta_4,$$

which is equivalent to

$$\bar{u}(x) = -2\beta_4 \log|x| + O(1) \quad \text{as } |x| \to \infty.$$

Thus,

$$w(x) := \bar{u}(x) - \sum_{\ell=1}^{3} \beta_\ell \log|x - P_\ell|$$

satisfies (1.3) with $m = 3$, where $\beta_1, \beta_2, \beta_3$ satisfy $\mathcal{A}2$). This contradicts the necessary condition (1.4) in Theorem A. ∎

Remark 1. *Problem (3.2) is super critical under the assumptions $\mathcal{A}1$) and $\mathcal{A}2$). To be more precise, if one uses fixed point arguments (as described in Section 4) to prove the lemma, then one would not be able to rule-out a blow-up phenomena around the point P_1. This is due to the fact that the energy of a singular bubble at P_1 is $4\pi(1-\beta_1)$, which is smaller than the total energy $4\pi\beta_4$.*

The super criticality of the Problem (3.2) under $\mathcal{A}1$) and $\mathcal{A}2$) can also be seen from the point of view of singular Moser-Trudinegr inequality, see e.g. [1,9,15,32,38] and the references therein.

Now we are in a position to prove non-existence of solution to the Toda system (1.1)-(1.5) for the choice of $\{\beta_{i,\ell}\}$ as in (3.1). More precisely, we have:

Lemma 3.2. *Let $\beta_\ell \in (0,1)$ with $\ell = 1, \dots, 7$ be such that $\mathcal{A}1$) to $\mathcal{A}5$) hold. Let $\{\beta_{i,\ell} : i = 1, 2,\ \ell = 1, \dots, 7\}$ be as in (3.1). Let $P_1, \dots, P_4$ be such that Problem (3.2) has no solution. Let P_5 be a fixed point (different from $P_1, \dots, P_4$). Then for $|P_6|, |P_7|$ large ($P_6 \neq P_7$) there exists no solution to (1.6) with $m = 7$ such that*

$$u_i(x) = -\beta_4 \log|x| + O(1) \quad \text{as } |x| \to \infty, \quad i = 1, 2.$$

Proof. We assume by contradiction that there is a sequence of solutions (u_i^k) with

$$P_\ell = P_{\ell,k}, \quad |P_\ell| \xrightarrow{k\to\infty} \infty \quad \text{for } \ell = 6, 7,$$

that is, u_i^k satisfies

$$\begin{cases} -\Delta u_1^k = 2K_1 e^{2u_1^k} - K_2 e^{2u_2^k} & \text{in } \mathbb{R}^2 \\ -\Delta u_2^k = 2K_2 e^{2u_2^k} - K_1 e^{2u_1^k} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} K_i e^{2u_i^k} dx = 2\pi\beta_4 & i = 1,2 \\ |P_\ell| \xrightarrow{k\to\infty} \infty & \ell = 6,7, \end{cases} \tag{3.4}$$

where

$$K_1(x) := \prod_{\ell=1}^{4} \frac{1}{|x-P_\ell|^{2\beta_\ell}}, \quad K_2(x) := \prod_{\ell=5}^{7} \frac{|P_6|^{2\beta_6}|P_7|^{2\beta_7}}{|x-P_\ell|^{2\beta_\ell}}.$$

Notice that K_1 does not depend on k, $K_1 \in L^1(\mathbb{R}^2)$, thanks to the assumption $\beta_4 < 1$, and

$$K_2 \to |x-P_5|^{-2\beta_5} \text{ locally uniformly in } \mathbb{R}^2 \setminus \{P_5\} \quad \text{as } k \to \infty.$$

We claim that $u_1^k \to u$ locally uniformly in $\mathbb{R}^2$, where u satisfies

$$-\Delta u = 2K_1 e^{2u} \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} K_1 e^{2u} dx = 2\pi\beta_4. \tag{3.5}$$

Then one can show that $u(x) = -2\beta_4 \log|x| + O(1)$ as $|x| \to \infty$. In particular, $\bar{u}(x) = u(x) + \frac{1}{2}\log 2$ is a solution to the Problem (3.2), a contradiction to our assumption on $P_1, \dots, P_4$ that the Problem (3.2) has no solution.

We prove the claim in few steps.

Step 1 We have

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{B_R^c} K_1 e^{2u_1} dx = 0.$$

The proof is very similar to that of Step 1 in Lemma 3.1. Here we give a sketch of it.

We set

$$\tilde{u}_1^k(x) = u_1^k\left(\frac{x}{|x|^2}\right) - \beta_4 \log|x| + c^k,$$

so that $\tilde{u}_1^k$ satisfies ($\tilde{K}$ does not depend on k)

$$-\Delta \tilde{u}_1^k = \tilde{K} e^{2\tilde{u}_1^k} - g^k \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \tilde{K} e^{2\tilde{u}^k} dx = 4\pi\beta_4, \quad \int_{\mathbb{R}^2} g^k dx = 2\pi\beta_4,$$

$$g^k, \tilde{K} > 0 \quad \text{in } \mathbb{R}^2, \quad \tilde{K}(x) \xrightarrow{|x|\to 0} 1.$$

Now we can apply Lemma 5.1 with $\beta = 0$, thanks to the assumption $\mathcal{A}3$), to get that $\tilde{u}_1^k \le C$ in a neighborhood of the origin. Step 1 follows.

Setting

$$S_i := \{x \in \mathbb{R}^2 : \text{ there is a sequence } x^k \to x \text{ such that } u_i^k(x^k) \to \infty\}, \quad i = 1,2,$$

we shall show that $S_1 \cup S_2 = \emptyset$. We start with:

Step 2 $S_1 \subseteq \{P_1, \dots, P_4\}$ and $S_2 \subseteq \{P_5\}$.

For $x_0 \in S_1 \cup S_2$ we can write

$$K_i(x) = \frac{c_i + o(1)}{|x-x_0|^{2\alpha_i}}, \quad c_i > 0, \quad o(1) \xrightarrow{x\to x_0} 0, \quad i = 1,2,$$

where $\alpha_1 \in \{0, \beta_1, \dots, \beta_4\}$, $\alpha_2 \in \{0, \beta_5\}$ and $\alpha_1\alpha_2 = 0$. By Lemma 5.1 and $\mathcal{A}3$) one gets $S_1 \subseteq \{P_1, \dots, P_4\}$ and $S_2 \subseteq \{P_5\}$.

Step 3 $S_1 \cup S_2 = \emptyset$.

It is well-known that u_i^k satisfies the integral equation

$$u_i^k(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(\frac{1+|y|}{|x-y|}\right) \left(2K_i(y)e^{2u_i^k(y)} - K_{3-i}(y)e^{2u_{3-i}^k(y)}\right) dy + C^k, \quad i = 1, 2.$$

For $x_0 \in S_1 \cup S_2$ let $R > 0$ be such that $\bar{B}_R(x_0) \cap (S_1 \cup S_2) = \{x_0\}$, and x_0 is the only singularity for K_1, K_2 on $\bar{B}_R(x_0)$. Then, from the above integral representation, one can show that

$$|u_i^k(x) - u_i^k(y)| \le C \quad \text{for every } x, y \in \partial B_R(x_0), \quad i = 1, 2.$$

In particular, u_i^k and K_i satisfy all the assumptions in Lemma 5.2. Therefore, if $S_2 = \{P_5\}$, then as $\sigma_1(P_5) = 0$, we must have $\sigma_2(P_5) = 1 - \beta_5$. This implies that

$$\beta_4 \ge \sigma_2(P_5) = 1 - \beta_5,$$

a contradiction to $\mathcal{A}5$). Hence, $S_2 = \emptyset$.

Now we assume that $\beta_{\ell_0} \in S_1$ for some $\ell_0 \in \{1, \ldots, 4\}$. Then, in a similar way we get that $\beta_4 \ge 1 - \beta_{\ell_0}$. In fact, by $\mathcal{A}6$), a strict inequality holds, that is, $\beta_4 > 1 - \beta_{\ell_0}$. Since

$$u_1^k \to -\infty \quad \text{locally uniformly in } \mathbb{R}^2 \setminus S_1,$$

we must have that the cardinality of S_1 is at least 2, thanks to Step 1. Taking $P_{\ell_1} \in S_1$ with $\ell_1 \in \{1, \ldots, 4\} \setminus \{\ell_0\}$, and again using that $\sigma(P_{\ell_1}) = 1 - \beta_{\ell_1}$, we obtain

$$\beta_4 \ge \sigma(P_{\ell_0}) + \sigma(P_{\ell_1}) = 2 - \beta_{\ell_0} - \beta_{\ell_1},$$

a contradiction to $\mathcal{A}1$).

We conclude Step 3.

Step 4 $u_1^k \to \bar{u}_1$ in $C^0_{loc}(\mathbb{R}^2)$ where $\bar{u}_1$ satisfies (3.5).

Since $S_1 \cup S_2 = \emptyset$, up to a subsequence, one of the following holds:

i) $u_i^k \to \bar{u}_i$ in $C^0_{loc}(\mathbb{R}^2)$ for $i = 1, 2$
ii) $u_1^k \to \bar{u}_1$ in $C^0_{loc}(\mathbb{R}^2)$ and $u_2^k \to -\infty$ locally uniformly in $\mathbb{R}^2$
iii) $u_2^k \to \bar{u}_2$ in $C^0_{loc}(\mathbb{R}^2)$ and $u_1^k \to -\infty$ locally uniformly in $\mathbb{R}^2$
iv) $u_i^k \to -\infty$ locally uniformly in $\mathbb{R}^2$ for $i = 1, 2$.

It follows from Step 1, and the integral condition $\int_{\mathbb{R}^2} K_1 e^{2u_1} dx = 2\pi\beta_4$ that either i) or ii) holds, and $\bar{u}_1$ satisfies the integral condition

$$\int_{\mathbb{R}^2} K_1 e^{2\bar{u}_1} dx = 2\pi\beta_4.$$

Now we assume by contradiction that i) holds. Then the limit functions $(\bar{u}_1, \bar{u}_2)$ satisfy the system

$$\begin{cases} -\Delta \bar{u}_1 = 2K_1 e^{2\bar{u}_1} - \bar{K}_2 e^{2\bar{u}_2} & \text{in } \mathbb{R}^2 \\ -\Delta \bar{u}_2 = 2\bar{K}_2 e^{2\bar{u}_2} - K_1 e^{2\bar{u}_1} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} K_1 e^{2\bar{u}_1} dx = 2\pi\beta_4, \quad \int_{\mathbb{R}^2} \bar{K}_2 e^{2\bar{u}_2} dx =: 2\pi\gamma \le 2\pi\beta_4, & \end{cases} \tag{3.6}$$

where $\bar{K}_2(x) := |x - P_5|^{-2\beta_5}$ is the limit of K_2 as $k \to \infty$. Then one has

$$\lim_{|x|\to\infty} \frac{\bar{u}_2(x)}{\log|x|} = -(2\gamma - \beta_4),$$

and together with $\bar{K}_2 e^{2\bar{u}_2} \in L^1(\mathbb{R}^2)$ we have $\beta_5 + 2\gamma - \beta_4 > 1$. Hence, $\beta_4 + \beta_5 > 1$, a contradiction to $\mathcal{A}5$).

Thus, ii) holds, and (3.6) reduces to a single equation (3.5).

We conclude the lemma. □

4. Higher order singular Liouville equation

The proof of Theorem 1.2 is very similar to that of Theorem 1.1 (see also [22]). Here we give a sketch of it.

Writing

$$w(x) = u(x) - \sum_{\ell=1}^{m} \beta_\ell \log |x - P_\ell|,$$

Theorem 1.2 is equivalent to prove the existence of solution $u \in C^0(\mathbb{R}^n)$ to

$$(-\Delta)^{\frac{n}{2}} u = K e^{nu} \quad \text{in } \mathbb{R}^n, \quad K(x) := \prod_{\ell=1}^{m} \frac{1}{|x - P_\ell|^{n\beta_\ell}}, \tag{4.1}$$

satisfying the asymptotic behavior

$$u(x) = -\beta \log |x| + O(1) \quad \text{as } |x| \to \infty, \quad \beta := 2 - \sum_{\ell=1}^{m} \beta_\ell. \tag{4.2}$$

As before we fix $u_0 \in C^\infty(\mathbb{R}^n)$ such that $u_0(x) = -\log |x|$ for $|x| \geq 1$, and we look for a solution u to (4.1) of the form

$$u = \beta u_0 + v + c,$$

where c is a normalizing constant and $v \in X$, where

$$X := C_0(\mathbb{R}^n) = \{v \in C^0(\mathbb{R}^n) : v(x) \xrightarrow{|x|\to\infty} 0\}, \quad \|v\| := \max_{x \in \mathbb{R}^n} |v(x)|.$$

Then u satisfies (4.1) if and only if $v = u - \beta u_0 - c$ satisfies

$$(-\Delta)^{\frac{n}{2}} v = \bar{K} e^{nv+c} - \beta(-\Delta)^{\frac{n}{2}} u_0 \quad \text{in } \mathbb{R}^n, \quad \bar{K} := K e^{n\beta u_0}. \tag{4.3}$$

The function $\bar{K}$ satisfies

$$\lim_{|x|\to\infty} |x|^{2n} \bar{K}(x) = 1. \tag{4.4}$$

For $v \in X$, we fix $c_v \in \mathbb{R}$ so that

$$\int_{\mathbb{R}^n} \bar{K}(x) e^{n(v(x)+c_v)} = \beta\gamma_n. \tag{4.5}$$

We define a compact operator

$$T : X \to X, \quad v \mapsto \bar{v},$$

$$\bar{v}(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) \bar{K}(y) e^{n(v(y)+c_v)} dy - \beta u_0(x), \quad x \in \mathbb{R}^n. \tag{4.6}$$

It follows that $\bar{v} \in C^0(\mathbb{R}^n)$ (in fact, Hölder continuous), and by (4.5)

$$\bar{v}(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{|x|}{|x-y|}\right) \bar{K}(y) e^{n(v(y)+c_v)} dy \quad \text{for } |x| > 1.$$

We claim that there exists $C > 0$ such that

$$\|v\|_X \leq C \quad \text{for every } (v, t) \in X \times [0, 1] \quad \text{satisfying } v = tT(v). \tag{4.7}$$

Then by Schauder fixed point theorem the operator T has a fixed point v in X, and consequently we get a continuous solution to (4.1) satisfying (4.2).

To prove (4.7) we assume by contradiction that there exists $(v^k, t^k) \in X \times [0,1]$ such that $\|v^k\|_X \to \infty$ and $v^k = t^k T(v^k)$, that is

$$v^k(x) = \frac{t^k}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) \bar{K}(y) e^{n(v^k(y)+c_{v^k})} dy - t^k \beta u_0(x). \tag{4.8}$$

Then we can choose $x^k \in \mathbb{R}^n$ so that

$$\sup_{x\in\mathbb{R}^n} \psi^k(x) \le \psi^k(x^k) + 1 \xrightarrow{k\to\infty} \infty, \quad \psi^k(x) := v^k(x) + c_{v^k} + \frac{1}{n}\log t^k.$$

The crucial ingredients to obtain a contradiction are Lemma 5.3, and the relation

$$\beta = 2 - \sum_{\ell=1}^{m} \beta_\ell = 2 - \beta_\ell - \sum_{\ell\neq j} \beta_\ell < 2(1-\beta_j) \quad \text{for every } j = 1, 2, \dots, m, \tag{4.9}$$

which follows from the second condition in (1.4). Up to a subsequence, we distinguish the following two cases:

Case 1 $x^k \to x^\infty \in \mathbb{R}^n \setminus \{P_\ell : \ell = 1, 2, \dots . m\}$.

In a small neighborhood of x^∞ we have for some $c_0 > 0$

$$\bar{K}(x) = \frac{c_0 + o(1)}{|x - x^\infty|^{n\alpha}}, \quad o(1) \xrightarrow{x\to x^\infty} 0,$$

where $\alpha \in \{0, \beta_1, \dots, \beta_m\}$. Using (4.8)-(4.9) one gets a contradiction as in [22], see also [3, 41].

Case 2 $|x^k| \to \infty$.

Setting

$$\tilde{\psi}^k(x) := \psi^k(\frac{x}{|x|^2}), \quad \tilde{x}^k := \frac{x^k}{|x^k|^2} \to 0,$$

we obtain $\tilde{\psi}_k(\tilde{x}^k) \to \infty$, and $\tilde{\psi}_k$ satisfies

$$\tilde{\psi}^k(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right) \tilde{K}(y) e^{n\tilde{\psi}^k(y)} dy + c^k \quad \text{in } B_1,$$

where

$$\tilde{K}(x) := \frac{1}{|x|^{2n}} \bar{K}(\frac{x}{|x|^2}), \quad c^k := c_{v^k} + \frac{1}{n}\log t^k.$$

Note that $\tilde{K}$ is smooth around the origin and

$$\tilde{K}(x) \xrightarrow{|x|\to 0} 1,$$

one can proceed as in Case 1. Thus, $\psi^k \le C$ on $\mathbb{R}^n$, and we have (4.7).

5. Some useful lemmas

The following lemma is a generalizations of Brezis-Merle [11] type results, compare [8, Theorem 5].

Lemma 5.1. *Let (u^k) be a sequence of solutions to*

$$-\Delta u^k = \frac{f^k(x)}{|x|^{2\alpha}}e^{2u^k} - g^k \quad \text{in } B_1, \quad \int_{B_1} \frac{f^k(x)}{|x|^{2\alpha}}e^{2u^k}dx \le 2\pi(1-\alpha-\varepsilon),$$

for some $\varepsilon > 0$ and $\alpha \in [0,1)$. Assume that $g^k \ge 0$, $\|g^k\|_{L^1(B_1)} \le C$, $0 \le f^k \le C$ and $\inf_{B_1\setminus B_\delta} f^k \ge C_\delta^{-1}$ for some $0 < \delta < \frac{1}{3}$. Then u^k is locally uniformly bounded from above in B_1.

Proof. We write $u^k = v^k + h^k$, where h^k is harmonic in B_1 and

$$v^k(x) := \frac{1}{2\pi}\int_{B_1} \log\left(\frac{2}{|x-y|}\right)\left(\frac{f^k(y)}{|y|^{2\alpha}}e^{2u^k(y)} - g^k(y)\right)dy.$$

Since $g_k \ge 0$, by Jensen's inequality one gets that

$$\int_{B_1} e^{2pv^k(x)}dx \le C(p), \quad p \in [1, \frac{1}{1-\alpha-\varepsilon/2}].$$

Notice that

$$\int_{B_1\setminus B_\delta} (h^k)^+ dx \le \int_{B_1\setminus B_\delta} ((u^k)^+ + |v^k|)dx \le C.$$

Since $\delta < \frac{1}{3}$, fixing $\delta + \frac{1}{3} < r_1 < r_2 < 1-\delta$ we see that

$$\partial B_t(x) \subset B_1 \setminus B_\delta \quad \text{for every } x \in \bar{B}_\delta,\ r_1 \le t \le r_2.$$

Therefore, by mean value theorem,

$$2\pi(r_2 - r_1)h^k(x) = \int_{r_1}^{r_2}\int_{\partial B_t(x)} h^k(y)d\sigma(y)dt \le \int_{B_1\setminus B_\delta} (h^k)^+dy \le C.$$

Thus, $\int_{B_1}(h^k)^+dx \le C$. If $\rho^k := \int_{B_{\frac{1}{2}}} |h^k|dx \le C$ then we have

$$h^k \to h \quad \text{in } C^2_{loc}(B_1), \quad \Delta h = 0 \quad \text{in } B_1.$$

In particular, (h^k) is bounded in $C^0_{loc}(B_1)$. If $\rho^k \to \infty$, then

$$\frac{h^k}{\rho^k} \to h \quad \text{in } C^2_{loc}(B_1), \quad \Delta h = 0, \quad h < 0 \quad \text{in } B_1.$$

This shows that (h^k) is locally uniformly bounded from above in B_1. This leads to

$$\int_{B_r} e^{2pu^k}dx \le C_r \int_{B_r} e^{2pv^k}dx \le C(p,r,\varepsilon,\alpha), \quad 0 < r < 1,\ p \in [1, \frac{1}{1-\alpha-\varepsilon/2}].$$

Using this uniform bound, and Hölder inequality with $p = \frac{1}{1-\alpha-\varepsilon/2}$, one gets $v^k \le C$ in B_r for $0 < r < 1$, and the lemma follows. □

A strong version (precise quantization value of σ_1, σ_2) of the following lemma is proven in [27, 29]. See [30] for a Pohozaev type identity for regular $SU(3)$ Toda system.

Lemma 5.2 ([27,29]). *Let (u_1^k, u_2^k) be a sequence of solutions to*

$$\begin{cases} -\Delta u_1^k = 2\frac{K_1^k}{|x|^{2\alpha_1}}e^{2u_1^k} - \frac{K_2^k}{|x|^{2\alpha_2}}e^{2u_2^k} & \text{in } B_1 \\ -\Delta u_2^k = 2\frac{K_2^k}{|x|^{2\alpha_2}}e^{2u_2^k} - \frac{K_1^k}{|x|^{2\alpha_1}}e^{2u_1^k} & \text{in } B_1 \\ \int_{B_1} \frac{K_i^k}{|x|^{2\alpha_i}}e^{2u_i^k}dx \le C & i = 1,2 \\ |u_i^k(x) - u_i^k(y)| \le C & \text{for every } x, y \in \partial B_1, \quad i = 1,2 \\ \|K_i^k\|_{C^3(B_1)} \le C, \quad 0 < \frac{1}{C} \le K_i^k & \text{in } B_1, \quad i = 1,2, \end{cases} \tag{5.1}$$

for some $\alpha_1, \alpha_2 < 1$, and B_1 is the unit ball in $\mathbb{R}^2$. Assume that 0 is the only blow-up point, that is,

$$\sup_{B_1 \setminus B_\varepsilon} u_i^k \le C(\varepsilon) \quad \text{for every } 0 < \varepsilon < 1, \quad i = 1, 2.$$

Then setting

$$\sigma_i := \lim_{r\to 0}\lim_{k\to\infty} \frac{1}{2\pi}\int_{B_r} \frac{K_i^k(x)}{|x|^{2\alpha_i}}e^{2u_i^k(x)}dx, \quad i = 1, 2,$$

we have

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_1(1-\alpha_1) + \sigma_2(1-\alpha_2).$$

In particular, if $(\sigma_1, \sigma_2) \neq (0,0)$ then

$$\sigma_1 \ge 1 - \alpha_1 \quad \text{or } \sigma_2 \ge 1 - \alpha_2.$$

Remark 2. *If $\alpha_1 = \alpha_2 = \alpha$, $K_1^k = K_2^k$ and $u_1^k = u_2^k$ in the above lemma, then $\sigma_1 = \sigma_2 = 2(1-\alpha)$.*

Theorem 5.3 ([22,35]). *Let u be a normal solution to*

$$(-\Delta)^{\frac{n}{2}} u = |x|^{n\alpha}e^{nu} \quad \text{in } \mathbb{R}^n, \quad \Lambda := \int_{\mathbb{R}^n} |x|^{n\alpha}e^{nu}dx < \infty, \tag{5.2}$$

for some $\alpha > -1$ and $n \ge 2$, that is, u satisfies the integral equation

$$u(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1+|y|}{|x-y|}\right)|y|^{n\alpha}e^{nu(y)}dy + C,$$

for some $C \in \mathbb{R}$. Then $\Lambda = \Lambda_1(1+\alpha)$, $\Lambda_1 := 2\gamma_n$.

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